Linear Algebra Thm Archive - for Mid-Term Exam

by Gyeonggi Science High School for the Gifted 'Linear Algebra' Participants
Main Author : 14129 황동욱
Revised by : 14121 하석민
LaTeX Technician : 14041 박승원

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Preface and License

This work is a theorem archive for 'Linear Algebra' subject in Gyeonggi Science High School for the Gifted. It is initiated and mainly written by 황동욱, and being revised by 하석민, 박승원.

Each 'Theorem' is identically numbered as textbook. (Except Chapter 3.5) On the other hand, 'Extra Theorem' is things that aren't discussed or proved in textbook.

Anyway, good luck on your mid-term exam on Friday!

Theorems in this archive can have some errors. Please come to us if you find some of them, then we will revise them.

경기과학고 TEX사용자협회에서 알려드립니다 : 책의 본문에 관련된 내용은 책의 구성에 관한 저작권으로 인해 삭제되었습니다. Terminology 관련 부분을 제외한 모든 부분을 Lorem Ipsum 으로 대체하였습니다.

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Chapter 1

Vectors

1.1 Terminology relating to vectors

A vector can be represented either in geometric way or algebraic way. In geometric definition of vectors, a vector is a **directed line segment**. A vector from point A (**initial point**, or **tail**) to point B (**terminal point**, or **head**) is denoted as $\overrightarrow{AB}$. Vectors with their tails in the origin is called **position vectors**, and they are at **standard position**.
In algebraic view of vectors, a vector is an **ordered pair** of **components**. We denote the set of all vectors containing n components in $\mathbb{R}$ as $\mathbb{R}^n$. Similarly, set of all vectors containing n integer components is $\mathbb{Z}^n$.
A vector is written in form of **column vectors** and **row vectors**. We use square brackets for denoting vectors, such as

$$\mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 4, & 1, & 6 \end{bmatrix}$$

A **zero vector** is a vector which components are all zero. A zero vector is denoted as $\mathbf{0}$.
Two vectors are equal if and only if all the components of two vectors are equal. (Of course, the number of components should be same.)
Standard unit Vectors have components which one of them is 1 and rest of them are all 0. Unit vector which has 1 in ith component is denoted as $\mathbf{e}_i$, and

$$\mathbf{e}_i = \begin{bmatrix} 0, & 0, & \cdots & 1, & \cdots & 0 \end{bmatrix}$$

* *Note* We should always denote vector either with arrows ($\vec{v}$), or with boldface letters ($\mathbf{v}$). Scalar denotations (v) are not allowed.
A set of vectors with n components taken from finite set of integers $\mathbb{Z}_m = \{0, 1, 2, \cdots, m-1\}$ is denoted as $\mathbb{Z}_m^n$. $\mathbb{Z}_m^n$ is closed with respect to operations of vector addition and scalar multiplication (which is defined later). We can perform those operations in same way, but with modulo operations. Vectors in $\mathbb{Z}_2^n$ (all components are 0 or 1) are called **binary vectors**.

1.2 Basic Operations of Vectors

Theorem 1.1 : Algebraic Properties of Vectors : Basic Vector Operations
Let $\mathbf{u}$,$\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$and let c and d be scalars.

a. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutative Property of Vector Addition)

b. $(\mathbf{u} + \mathbf{v}) + \mathbf{u} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associative Property of Vector Addition)

c. $\mathbf{u} + \mathbf{0} = \mathbf{u}$

d. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

e. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (Left-Distributive Property of Scalar Multiplication over Vector Addition)

f. $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$ (Right-Distributive Property of Scalar Multiplication over Vector Addition)

g. $c(d\mathbf{u}) = (cd)\mathbf{u}$

h. $1\mathbf{u} = \mathbf{u}$

Theorem 1.2
This theorem has no name.

Extra Theorem : ASDF
This is extra theorem.

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Chapter 2

Systems of Linear Equations

2.1 Terminology

Definition. A vector $\mathbf{v}$ is a **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ if there exist scalars $c_1, c_2, \cdots, c_n$ such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

The scalars $c_1, c_2, \cdots, c_n$ are called **coefficients** of linear combination.

Definition. A **linear equation** in the n variables $x_1, x_2, \cdots, x_n$ is an equation that can be written in the form of

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where the **coefficients** $a_1, a_2, \cdots, a_n$ and the **constant term** b are constants.

A **set of linear equations** is a finite set of linear equations with same variables. A **solution** of a linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is a vector $\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$ which satisfies the equation. A solution of a set of linear equations is a vector which is simultaneously a solution of all linear equations in the system. A **solution set** of a system of linear equations is the set of all solutions of the system.

A system of linear equations is **consistent** if there exists a solution. **Inconsistent** set of linear equations has an empty solution set. Two linear systems are **equivalent** if they have same solution set.

The **coefficient matrix** contains the coefficients of variables in the set of linear equations. The **augmented matrix** is the coefficient matrix augmented by a vector containing constant terms. For the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

the coefficient matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and the augmented matrix is

$$[A|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Definition. A matrix is in **row echelon form** (REF) if:

a. All rows consisting entirely of zeros are at the bottom.

b. The **leading entry** (the first nonzero entry) of each rows is located to the left of any leading entries below it.

Definition. A matrix is in **reduced row echelon form** (RREF) if:

a. It is in REF.

b. All leading entries are 1. (**leading 1**)

c. Each columns containing a leading 1 has 0 everywhere else.

REF of a matrix is not unique, but all matrices have unique RREF.

Definition. A system of linear equations is **homogeneous** if the constant term in each equation is zero.

2.2 Solving Linear Systems

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Chapter 3

Matrices

3.1 Terminology

Definition. A **matrix** is a rectangular array of numbers, which are called as **entries** or **elements**. If the matrix has n rows and m columns, the **size** of the matrix is $n \times m$.

A $1 \times n$ matrix is called a **row matrix**, or **row vector**. A $n \times 1$ matrix is called a **column matrix**, or **column vector**. (*A vector is considered as a matrix*) We can denote matrices using row vectors or column vectors, such as

$$A = \begin{bmatrix} \mathbf{A}_1^C & \mathbf{A}_2^C & \cdots & \mathbf{A}_m^C \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^R \\ \mathbf{A}_2^R \\ \vdots \\ \mathbf{A}_n^R \end{bmatrix}$$

where $\mathbf{A}_i^C$ is the ith column of A and $\mathbf{A}_i^R$ is the ith row of A.

The element at ith row and jth column is denoted by A_{ij}. We can also denote matrices using elements, such as $A = [A_{ij}]$.

Definition. The **diagonal entries**of A are A_{ii}.

Definition. The **square matrix** is a matrix which has same number of rows and columns (so the size is $n \times n$). **Diagonal matrix** is a square matrix which has its nondiagonal entries as 0. A diagonal matrix with all of its diagonal entries are the same are **scalar matrix**. If the value of diagonal entries are all 1, it is **identity matrix**.
A $n \times n$ identity matrix is denoted as I_n, and

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

A **zero matrix** O is a matrix which all of its entires are zero.

Definition. Two matrices are **equal** if and only if
(i) The size of two matrices are the same.
(ii) The corresponding entries of the matrices are the same.

3.2 Matrix Operations

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Appendix A

Cautions on Exam

A.1 Notations

Wrong	Right	Explanation
$2 \cdot 3 = 6$	$2 \times 3 = 6$	$\cdot$ for dot product is only valid for vectors.
$2 \cdot \mathbf{v}$	$2\mathbf{v}$	
$\begin{bmatrix} 1 & 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 1, & 3, & 4 \end{bmatrix}$	When writing row vectors, commas are necessary.

A.2 Description

- Free variables like s, t and u must be indicated that they are arbitrary real number. *e.g.* $s, t \in \mathbb{R}$
- When proving $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \mathbb{R}^3$, you must prove each side, not only one.
- $A = B$ means not only same entries, but also same size.
- Write as '적어도 하나는 0이 아닌' instead of '모두 0은 아닌' or else.
- Abbreviation : only 4 things are allowed : 'REF', 'RREF', 'EMO', 'F.T.I.M.'. Each of them stands for (Reduced) Row Echelon Form, Elementary Matrix Operation, Fundamental Theorems of Invertible Matrices.