# CHAPTER ONE

# INDRODUCTION

## 1.1 GENERAL REMARKS

An *ordinary differential equation* is a relationship between an unknown function of a single variable and one or more of its derivatives. Such relationships often results when expressing scientific laws connecting physical quantities and their rates of change in mathematical terms. For example, Newton's laws stating that the rate of change of the momentum of a particle is equal to the force acting on it or the statement that a population is growing exponential can both be translated into mathematical language as ordinary differential equations.

Our goal to study differential equations is to determine both the qualitative and quantitative properties of those functions which satisfy them; such functions are called *solutions*. Three differential equations that arise quite often in applications are:

**1**. $\frac{dy}{dx} = k(t)F(y)$, the growth equation;

**2**. $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t)$, the LCR oscillator equations; and

**3.** $\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$, the pendulum equation.

In all these three equations $t$ is called the *independent variable* and the unknown functions $y, Q$ and $\theta$ whose derivatives explicitly appear, are called the *dependent variables.* There are several methods in solving these differential equations, the method we will use in this research will the ***Laplace Transform***.

## 1.2 LAPLACE TRANSFORMS

The *transform method* is used to solve initial value problems for constant coefficient linear differential equations. Transform methods are used extensively in the analysis of linear systems; the common feature of these methods are:

1. The solution of the system $y(t)$, which exists in the $t-$domain, is transformed into a function $Y(s)$ of another independent variable $s$; call this the $s-domain$. The initial conditions are usually incorporated in the transformation.
2. The system of differential or integral equations, which $y(t)$ satisfies in the $t-domain$, is in turn transformed into a system of algebraic equations in the $s-domain$. If one can solve these equations for $Y(s)$, one obtains a relation of the form $Y(s)=\phi(s)$.
3. If one can find a function $\phi(t)$ whose transform is $\phi(s)$, then one can apply the inverse transformation that takes us from the $s-domain$ to the $t-domain$ and assert that $y(t)=\phi(t)$ is the required solution.

The two most commonly used transforms are the *Fourier and* the *Laplace transforms* and we will study the latter *(i.e. the Laplace Transform)*. It was first introduce by the *great French mathematician Pierre Laplace (1749-1827)*, but its application and the techniques associated with it were not developed until about a hundred years later.

The *Laplace transform,* used as a technique for solving constant coefficient linear ordinary differential equations, is particularly effective for problems where the forcing function is discontinuous or has corners.

But in addition to this, the *Laplace transform* is used extensively to study $input-output$ in relations to systems analysis, to analyze feedback control systems, and to solve certain classes of partial differential equations of mathematical physics. It is one of the important tools of applied mathematics.

# CHAPTER TWO

# LAPLACE TRANSFORMS AND ITS APPLICATIONS

## 2.1 THE LAPLACE TRANSFORM OF $e^{at}$

Given a function $f(t)$, defined for $0 \le t < \infty$, *define* the *Laplace transform* of $f(t)$ by

$$L\{f\} = \int_0^\infty e^{-st} f(t)dt = F(s) \qquad (2.1.1)$$

where $s$ is a complex variable, whenever the improper integral exists. Recall that the existence of the integral means that the limit

$$\lim_{R\to\infty} \int_0^R e^{-st} f(t)dt \ ,$$

exits. The fact that the variable $s$ is complex is not significant in many applications, but it should be noted that the *Laplace transform* takes functions $f(t)$ defined on the half line $0 \le t < \infty$ into functions $F(s)$ defined on the complex pane. Proceeding formally from the definition (2.1.1) above, we get

$$L\left\{e^{at}\right\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt.$$

For the transform to exists, the improper integral must exist, so we must determine whether

$$\lim_{R\to\infty}\int_0^R e^{-(s-a)t}dt = \lim_{R\to\infty}\left[\frac{1}{s-a}\left(1-e^{-(s-a)R}\right)\right] \qquad (2.1.2)$$

exists. Since $a$ is real and $s$ is complex, let $s = x + iy$ and therefore $s-a = (x-a)+iy$ and

$$e^{-(s-a)R} = e^{-(x-a)R}e^{-iyR} = e^{-(s-a)R}\left[\cos(yR) - i\sin(yR)\right].$$

Also note that the modulus of $e^{-(s-a)R}$ is

$$\left|e^{-(s-a)R}\right| = e^{-(x-a)R}\left|\cos(yR)-i\sin(yR)\right| = e^{-(x-a)R}$$

since $|u+iv| = \sqrt{u^2+v^2}$. One can now immediately see that the limit in (2.1.2)

a) does not exist if $x-a>0, \sin ce \lim_{R\to\infty} e^{-(x-a)R} = \infty$

b) does not exist if $x-a=0,$ since

$$\lim_{R\to\infty}\left\{\frac{1}{iy}\left[1-\cos(yR)+i\sin(yR)\right]\right\}$$

does not exist for any value of $y$.

c) does exist if $x-a>0$, since $\lim_{R\to\infty} e^{-(x-a)R} = 0$.

We conclude that if $x = \mathrm{Re}(s) > a, then$

$$\lim_{R\to\infty}\left\{\frac{1}{s-a}\left[1-e^{-(s-a)R}\right]\right\}=\frac{1}{s-a},$$

and our *Laplace transform* is

$$L\left\{e^{at}\right\}=F(s)=\frac{1}{s-a},\operatorname{Re}(s)>a \qquad (2.1.3)$$

Therefore, $L$ takes the real function $e^{at}$ into the complex rational function $1/_{s-a}$.

NB: If it was assumed that $s, was$ real, we would have easily derived the formula

$$L\left\{e^{at}\right\}=F(s)=\frac{1}{s-a},\operatorname{Re}(s)>a,$$

since the limit in (1.3.2) exists only if $s-a>0$.

**EXAMPLES 2.1**

1. Find the Laplace transforms of each of the following functions.

   a) $e^{2t}$  b) $e^{-7t}$

2. Using the definition of the integral and the formula for $L\left\{e^{at}\right\}$, establish the formula

$$L\left\{Ae^{\alpha t}+Be^{\beta t}\right\}=\frac{A}{s-\alpha}+\frac{B}{s-\beta},$$

valid for $\text{Re}(s)>\max(\alpha,\beta)$, where $\alpha,\beta$ are real and $A,B$ are arbitrary constants.

## SOLUTIONS TO EXAMPLE 2.1

1. a) $L\left\{e^{2t}\right\}=\int_0^\infty e^{2t}e^{-st}dt=\int_0^\infty e^{-(s-2)t}dt \;=\lim_{R\to\infty}\left\{\frac{1}{s-2}\left[1-e^{-(s-2)R}\right]\right\}=\frac{1}{s-2}$

Therefore $L\left\{e^{2t}\right\}=\frac{1}{s-2}, R(s)>2$

b) Similarly, $L\left\{e^{-7t}\right\}=\frac{1}{s+7}$

2. $L\left\{Ae^{\alpha t}+Be^{\beta t}\right\}=\int_0^\infty e^{-st}\left(Ae^{\alpha t}+Be^{\beta t}\right)=A\int_0^\infty e^{-(s-\alpha)t}dt+B\int_0^\infty e^{-(s-\beta)t}dt$

$$=A\lim_{R\to\infty}\left[\frac{1}{s-\beta}\left(1-e^{-(s-\alpha)R}\right)\right]+B\lim_{R\to\infty}\left[\frac{1}{s-\beta}\left(1-e^{-(s-\beta)R}\right)\right]$$

$$=A\left(\frac{1}{s-\alpha}\right)+B\left(\frac{1}{s-\beta}\right)$$

Hence $L\left\{Ae^{\alpha t}+Be^{\beta t}\right\}=\frac{A}{s-\alpha}+\frac{B}{s-\beta}$, valid for $\mathrm{Re}(s)>\max(\alpha,\beta)$ and $A, B$ are arbitrary constants.

## 2.2 AN APPLICATION TO FIRST ODER EQUATIONS

At this point, rather than developing a collection of formulas for the *Laplace transform* of various functions, I will state two of its important properties and then proceed directly to use it to solve a first order differential equation.

**2.2.a Property of linearity:** If $f(t)=Ag(t)+Bh(t),$ where A and B are constants and $g(t)$ and $h(t)$ have Laplace transforms $G(s)$ and $H(s)$, respectively, valid for $\mathrm{Re}(s)>a$, then

$$L\left\{f(t)\right\}=F(s)=AL\left\{g(t)\right\}+BL\left\{h(t)\right\}=AG(s)+BH(s) \qquad (2.2.1)$$

$$\mathrm{Re}(s)>a$$

***Proof of property***

By the definition of (2.1.1)

$$L\left\{Ag(t)+Bh(t)\right\}=\int_0^\infty e^{-st}\left[Ag(t)+Bg(t)\right]dt=A\int_0^\infty e^{-st}g(t)dt+B\int_0^\infty e^{-st}h(t)dt=AG(s)+BH(s)$$

**Example 1 on property**

Find the Laplace transform of $\cosh at$ and $\sinh at$

**Solution**

Since $\cosh at = \frac{1}{2}\left(e^{at} + e^{-at}\right)$ and $\sinh at = \frac{1}{2}\left(e^{at} - e^{-at}\right)$, then it follows that;

$$L\{\cosh at\} = \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)}\right] = \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]$$

Therefore $L\{\cosh at\} = \dfrac{s}{s^2-a^2}$

Similarly

$$L\{\sinh at\} = \frac{1}{2}\left[L\{e^{at} - e^{-at}\}\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$$

Therefore $L\{\sinh at\} = \dfrac{a}{s^2-a^2}$

**Example two on Property**

Derive the formulas

$$L\{\cos at\} = \frac{s}{s^2+a^2}, \qquad L\{\sin at\} = \frac{a}{a^2+s^2}$$

One can find these by computing

$L\{\cos at\} = \int_0^\infty e^{-st}\cos at dt$ *and* $L\{\sin at\} = \int_0^\infty e^{-st}\sin at dt$ through integration by parts. However, these integrals and the transforms of $\cos(at)$ and $\sin(at)$ can be found by using complex exponentials. Therefore, from the previous formula for $L\{e^{at}\}$ it follows that

$$L\{e^{iat}\} = \frac{1}{s-ia},\ \mathrm{Re}(s) > 0,$$

and furthermore

$$\frac{1}{s-ia} = \frac{1}{s-ia}\frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + \frac{ia}{s^2+a^2} \text{-------------------- (1)}$$

We now use linearity property and the fact that

$$L\{e^{iat}\} = L\{\cos(at) + i\sin(at)\} \text{------------------------------------------ (2)}$$

then equating the real and imaginary parts of equations (1) and (2) gives:

$$L\{\cos(at)\} = \frac{s}{s^2+a^2} \text{ and } L\{\sin(at)\} = \frac{a}{s^2+a^2}$$

**2.2. b. The differential formula.**

$$L\{f'(t)\} = sL\{f(t)\} - f(0) = F(s) - F(0), \qquad (2.2.2)$$

Note that if $F(s)$ exists for $\mathrm{Re}(s) > a$, then so does $L\{f'(t)\}$, and the above result is obtained under the assumption that $f'(t)$ is continuous or piecewise continuous.

With our meager collection of tools, consisting of the formula $L\left\{e^{at}\right\}$, of the property of linearity (1.4.1), and of the differentiation formula (1.4.2), we will show how to use the *Laplace transform* to solve the initial value problem

$$y'+ky=be^{at},\, y(0)=A, a\neq -k \qquad (2.2.3)$$

Let $y(t)$ be the solution of (1.4.3) and $Y(s)=L\{y(t)\}$ be its *Laplace transform*. Applying the transform to both sides of the equation, using the property of linearity and the formula for $L\left\{e^{at}\right\}$ to obtain

$$L\{y'+ky\}=L\{y'\}+KL\{y\}=L\left\{be^{at}\right\}=\frac{b}{s-a}.$$

Now applying the Differentiation Formula to the first term on the left, we get

$$sL\{y\}-y(0)+kL\{y\}=\frac{b}{s-a},$$

since $L\{y\}=Y(s)$ and $y(0)=A$, it now follows that;

$$sY(s)-A+kY(s)=\frac{b}{s-a}$$

Solving the last expression for $Y(s)$ gives;

$$Y(s)=\frac{A}{s+k}+\frac{b}{(s+k)(s-a)} \qquad (2.2.4)$$

Thus, the first step is done, namely, by using, by using the properties of the transform and algebraic operations an expression for Y(s), the transform of the solution y (t) has been obtained.

It is important to note in (1.4.4) that the initial value y(0) = A is incorporated into the expression for Y(s). The fact that the initial data are a part of the transform is an attractive feature of the end to find values of arbitrary constants. Of course the initial data enter through the Differentiation Formula.

As it stands, the expression for Y(s) is not in the form we recognize because of the second term. But if this term is decomposed by using partial fraction, then

$$Y(s) = \frac{A}{s+k} + \frac{b}{k+a}\left[\frac{1}{s-a} - \frac{1}{s+k}\right]$$

or

$$Y(s) = \left(A - \frac{b}{k+a}\right)\frac{1}{s+k} + \frac{b}{k+a}\frac{1}{s-a} \qquad (2.2.5)$$

Now using the formula for $L\left\{e^{at}\right\}$ and linearity, we see that

$$y(t) = \left(A - \frac{b}{k+a}\right)e^{-kt} + \frac{b}{k+a}e^{at} \qquad (2.2.6)$$

is a function whose Laplace transform is Y(s) and that it is the solution of the initial value problem.

The astute reader may well ask how did we get from the expression (1.4.5) to the solution (1.3.6) since the Laplace transform only tells us how to get y(t) to Y(s) = $L\{y(t)\}$. The answer is that associated with the operator L is an operator $L^{-1}$, called the inverse Laplace transform, and that it has the following properties:

**Uniqueness property:** If $f(t)$ is continuous and has a transform $F(s) = L\{f(t)\}$, then

$$L^{-1}\{F(s)\} = f(t)$$

**Linearity property:** If $g(t)$ and $h(t)$ are continuous and have Laplace transform $G(s)$ and $H(s)$, respectively, and A and B are constants, then

$$L^{-1}\{AG(s) + BH(s)\} = Ag(t) + Bh(t)$$

One sees now that applying $L^{-1}$ to both sides of (2.2.5) gives us the solution $y(t)$.

We finally remark that, given a certain general class of functions $F(s)$ in the complex plane, there is an integral that permits us to calculate $f(t) = L^{-1}\{F(s)\}$.It is a rather complicated line integral in the complex plane, and if it had to be used each time to get back to the $t-domain$, then the Laplace transform method would not be worth using. The Uniqueness property above is what saves matters, for it means that for continuous functions $f(t)$ one can go back and forth via a table once $F(s)$ has been calculated. Our table now has only one entry

| $f(t)$ | $F(s) = L\{f(t)\}$ |
|---|---|

| | |
|---|---|
| $e^{at}$ | $\frac{1}{s-a}$ |

Or equivalently

| $F(s)$ | $f(t)=L^{-1}\{F(s)\}$ |
|---|---|
| $\frac{1}{s-a}$ | $e^{at}$ |

*In the next section the table will be expanded to include most functions that will arise in the analysis of linear constant coefficient differential equations.*

## 2.3 FURTHER PROPERTIES AND TRANSFORM FORMULAS

In the previous sections, when the Differential Formula was derived, we mentioned the class of functions of exponential order.

> The class of functions $E_\alpha$ of exponential order $\alpha$ is the set of all continuous functions on $0 \le t \le \infty$ having the property that, if $f(t)$ is in $E_\alpha$ there exist a constant $\alpha$ and a positive constant $M$ such the
>
> $$\left| f(t) \right| \le Me^{at}, 0 \le t \le \infty.$$

In a later section the requirement that members of $E_\alpha$ be continuous will be dropped, but for now such generality is not-needed.

In a nutshell, the function of exponential order $\alpha$ are functions which grow no faster than the exponential $e^{at}$. For instance, *a)* $f(t) = 3e^{2t}$ belongs to $E_2$ since $|f(t)| \le 3e^{2t}$, so M = 3 and $\alpha = 2$.

b) $f(t) = t^n$ belongs to $E_\alpha$ for any $\alpha > 0$ since

$$\frac{\alpha^n t^n}{n!} \le 1 + \alpha t + \ldots + \frac{\alpha^n t^n}{n!} + \ldots = e^{\alpha t}$$, which implies that $|t^n| \le \left( n!/\alpha^n \right) e^{\alpha t}$ *and* so that $M = n!/\alpha^n$

c) $f(t) = 4e^{-t}\cos 3t$ belongs to $E_{-1}$ since $|4e^{-t}\cos 3t| \le 4e^{-t}$

A little thought shows that the set of all possible solutions of homogeneous constant coefficient linear differential equations is of exponential order. These solutions are linear combinations of functions like $e^{at}, e^{at}\cos bt, e^{at}\sin bt,$ and integral powers of $t$ times them.

The important property of functions of exponential order is that they are Lapalce transformable.

If $f(t)$ belongs to $E_{\alpha,}$ then $L\{f(t)\} = F(s)$ exists for $\mathrm{Re}(s) > \alpha$ .

The proof of this statement follows from these simple estimates:

$$\left| \int_0^\infty f(t) e^{-st} dt \right| \le \int_0^\infty |f(t)| \left| e^{-st} \right| dt$$

$$\le M \int_0^\infty e^{\alpha t} e^{-\mathrm{Re}(s)t} dt = M \int_0^\infty e^{-[\mathrm{Re}(s) - \alpha]t} dt.$$

The last improper integral converges if $\mathrm{Re}(s) - \alpha > 0$, which implies that

$$L\{f\} = \int_0^\infty f(t)e^{-st}dt\,,$$

converges also. Since the last integral equals $M/[\mathrm{Re}(s)-\alpha]$, this also shows that:

If $f(t)$ belongs to $E_\alpha$, then $F(s)$ approaches to zero as $\mathrm{Re}(s)$ approaches infinity.

Functions like $1, s^n, n>0, s^2/s^2-3$ or $e^{2s}$ cannot be Laplace transforms of functions of exponential order.

We next state a property that allows the table of transforms to be extended considerably. It is called the ***shift property***

If $L\{f(t)\} = F(s)$, then $L\{e^{at} f(t)\} = F(s-\alpha)$.

Therefore the computation of the Laplace transform of $e^{\alpha t} f(t)$ is simply accomplished by shifting the argument of $F(s)$ by $\alpha$ units. If $F(s)$ is valid for $\mathrm{Re}(s) > a$, then $F(s-\alpha)$ is valid for $\mathrm{Re}(s) > a+\alpha$.

A proof of the ***shift property*** is immediate since

$L\{e^{\alpha t} f(t)\} = \int_0^\infty e^{-st}e^{\alpha t} f(t)dt = \int_0^\infty e^{-(s-\alpha)t} f(t)dt,$ and the last expression is merely the Laplace transform of $f(t)$ evaluated at $s-\alpha$.

**Example:** Find $L^{-1}\left\{\dfrac{3s-2}{s^2+6s+25}\right\}$

**Solution**

Completing the square in the denominator and applying a little algebra gives

$$\frac{3s-2}{s^2+6s+25}=\frac{3s-2}{(s+3)^2+16}=\frac{3(s+3)}{(s+3)^2+16}-\frac{11}{(s+3)^2+16}$$

$$=3\frac{s+3}{(s+3)^2+16}-\frac{11}{4}\frac{4}{(s+3)^2+16}$$

Therefore,

$$L^{-1}\left\{\frac{3s-2}{s^2+6s+25}\right\}=3e^{-3t}\cos 4t-\frac{11}{4}e^{-3t}\sin 4t$$

**<u>Two other properties of the Laplace transform, which are occasionally useful are:</u>**

1. If $f(t)$ belongs to $E_\alpha$ and its Laplace transform is $F(s)$, then

$$L\left\{\int_0^t f(r)dr\right\} = \frac{1}{s}F(s), \mathrm{Re}(s) > \alpha.$$

2. If $f(t)$ belongs to $E_\alpha$, then its Laplace transform $F(s)$ has derivatives of all order given by the formula

$$\frac{d^n}{ds^n}F(s) = L\left\{(-t)^n f(t)\right\}.$$

Finally, we state a more general form of the differential property of Laplace transforms. Its proof is a repeated application of integration by parts combined with the argument given in the previous derivation.

**The differential formula:** If $f(t), f^{'}(t), \ldots, f^{n}(t)$ belong to $E_\alpha$ and $L\left\{f(t)\right\} = F(s), then$ for $\mathrm{Re}(s) > \alpha$, the following is valid:

$L\left\{f^{'}(t)\right\} = sF(s) - f(0)$

$L\left\{f^{''}(t)\right\} = s^2F(s) - sf(0) - f^{'}(0)$

$L\left\{f^{n}(t)\right\} = s^nF(s) - s^{n-1}f(0) - \ldots - sf^{n-2}(0) - f^{n-1}(0)$

One sees, therefore that the transform of the nth derivative depends only on the transform of the function and n initial data values.

**Example:** If $y(t)$ is the solution of a third order differential equation with initial conditions

$y(0)=1, y'(0)=-3, y''(0)=2,$ and $Y(s)$ is its Laplace transforms, then

$$L\{y'(t)\}=sY(s)-1$$
$$L\{y''(t)\}=s^2Y(s)-s+3$$
$$L\{y'''(t)\}=s^3Y(s)-s^2+3s-2$$

We conclude this section with a short table for Laplace transforms (Table 1.1) and then proceed to solve some differential equations.

**TABLE 2.1 Short Table of Laplace Transforms**

| | $f(t)$ | $F(s)$ | **Region of validity** |
|---|---|---|---|
| 1. | 1 | $\frac{1}{s}$ | Re(s)>0 |
| 2. | $t^n, n=0,1,2,\ldots$ | $\frac{n!}{s^{n+1}}$ | Re(s)>0 |

| | | | |
|---|---|---|---|
| 3. | $e^{at}$ | $\frac{1}{s-a}$ | Re(s)>0 |
| 4. | $\sin bt$ | $\frac{b}{s^2+b^2}$ | Re(s)>0 |
| 5. | $\cos bt$ | $\frac{s}{s^2+b^2}$ | Re(s)>0 |
| 6. | $\sinh bt$ | $\frac{b^2}{s^2-b^2}$ | Re(s)>$\lvert b \rvert$ |
| 7. | $\cosh bt$ | $\frac{s^2}{s^2-b^2}$ | Re(s)>$\lvert b \rvert$ |
| 8. | $e^{at}\sin bt$ | $\frac{b}{(s-a)^2+b^2}$ | Re(s)>$a$ |

9. $e^{at}\cos bt$ $\dfrac{s-a}{(s-a)^2+b^2}$ Re(s)> $a$

10. $t\sin bt$ $\dfrac{2bs}{\left(s^2+b^2\right)^2}$ $\operatorname{Re}(s)>0$

11. $t\cos bt$ $\dfrac{s^2-b^2}{\left(s^2+b^2\right)^2}$ $\operatorname{Re}(s)>0$

12. $e^{at}f(t)$ $F(s-a)$ $\operatorname{Re}(s)>\alpha+a$ if F(s) valid for Re(s)> $\alpha$

13. $\int_0^t f(r)dr$ $\dfrac{1}{s}F(s)$

$\operatorname{Re}(s)>\max(a,0)$ if $F(s)$ valid for Re(s)> $a$

14. $(-t)^n\, f(t)$ $\dfrac{d^n}{ds^n}F(s)$ Valid for same region as $F(s)$

15. $\dfrac{1}{b}\sin bt - t\cos bt$ $\dfrac{2b^2}{\left(s^2+b^2\right)^2}$ $\operatorname{Re}(s)>0$

16. $f(t-c)u(t-c)$ $e^{-sc}F(s)$ Valid for same region as $F(s)$

| | | | |
|---|---|---|---|
| 17. | $f(t+T)=f(t)$ | $\dfrac{\int_0^T e^{-sr} f(r)dr}{1-e^{-sT}}$ | $\mathrm{Re}(s)>0$ |

## 2.6. SOLVING CONSTANT COEFFICIENT LINEAR EQUATIONS:

**Example 1** Solve the initial value problem $y''+9y=\cos 3t,\ y(0)=1,\ y'(0)=-1$

If $Y(s)=L\{y(t)\}$, then from the initial data and the differentiation formulas it follows that

$$L\{y''(t)\}=s^2Y(s)-s+1.$$

Applying the Laplace transform to both sides of the differential equation and using entry (5) from table 1.1 gives

$$s^2Y(s)-s+1+9Y(s)=L\{\cos 3t\}=\frac{s}{s^2+9}.$$

Now solve for $Y(s)$ to obtain

$$Y(s)=\frac{s-1}{s^2+9}+\frac{s}{\left(s^2+9\right)^2}.$$

The denominator is $s^2+9=s^2+3^2$, and an examination of the last term tells us that the key entries in Table 1.1 are (4), (5) and (10). A little rewriting gives $Y(s)=\dfrac{s}{s^2+9}-\dfrac{1}{3}\dfrac{3}{s^2+9}+\dfrac{s}{\left(s^2+9\right)^2}$, and now the inverse transform can be applied (by using table 1.1 directly!) to obtain

$y(t) = \cos 3t - \frac{1}{3}\sin 3t + \frac{1}{6}t\sin 3t,$ the desired solution.

**Example 2** Solve the initial value problem $y'' + 3y' + 2y = 6, y(0) = 0, y'(0) = 2.$

Proceeding as above, first use the initial data to compute

$$L\{y''(t)\} = s^2Y(s) - 2, L\{y'(t)\} = sY(s);$$

Applying the Laplace transform to the equation gives

$$s^2Y(s) - 2 + 3sY(s) + 2Y(s) = L\{6\} = \frac{6}{s},$$

Now solve for $Y(s)$ to obtain

$$Y(s) = \frac{2}{s^2+3s+2} + \frac{6}{s\left(s^2+3s+2\right)}$$

or

$$Y(s) = \frac{2}{(s+1)(s+2)} + \frac{6}{s(s+1)(s+2)} = \frac{6}{s(s+1)(s+2)}$$

From the theory of partial fractions we know that

$$\frac{6}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2};$$

combining terms on the right side and equating the numerators gives the relation

$$2s + 6 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1).$$

Equating like powers of $s$ leads to the equations

$$A + B + C = 0, 3A + 2B + C = 2, 2A = 6.$$

These have the solution $A=3, B=-4, C=1,$ which could have also been obtained by letting $s=0, s=-1, and, s=-2$, successively, in the previous equation. Therefore,

$$Y(s)=\frac{3}{s}-\frac{4}{s+1}+\frac{1}{s+2},$$

and now Table 1.1 gives

$$y(t)=3-4e^{-t}+e^{-2t},$$

the required solution.

**Example 3** Find the solution $x(t), y(t)$ of the initial value problem

$$x'=2x+y, x(0)=1$$

$$y'=3x+4y, y(0)=0.$$

This is a constant coefficient system of two linear differential equations, and so we must apply the Laplace transforms to each equation. Letting $X(s)=L\{x(t)\}$ and $Y(s)=L\{y(t)\}$, we get by the differentiation formula and the initial data that

$$L\{x'(t)\}=sX(s)-1,$$

$$L\{y'(t)\}=sY(s).$$

Now apply the transform to each differential equation to get

$$sX(s)-1=2X(s)+Y(s),$$

$$sY(s)=3X(s)+4Y(s),$$

or equivalently,

$$(s-2)X(s)-Y(s)=1,$$

$$-3X(s)+(s-4)Y(s)=0.$$

This is a system of linear algebraic equations which must be solved for $X(s)$ and $Y(s)$; using Cramer's rule we obtain

$$X(s) = \frac{\left|\begin{pmatrix} 1 & -1 \\ 0 & s-4 \end{pmatrix}\right|}{\left|\begin{pmatrix} s-2 & -1 \\ -3 & s-4 \end{pmatrix}\right|}$$

$$= \frac{s-4}{s^2-6s+5} = \frac{s-4}{(s-1)(s-5)},$$

$$Y(s) = \frac{\left|\begin{pmatrix} s-2 & 1 \\ -3 & 0 \end{pmatrix}\right|}{\left|\begin{pmatrix} s-2 & -1 \\ -3 & s-4 \end{pmatrix}\right|}$$

$$= \frac{3}{(s-1)(s-5)}.$$

Finally, partial fractions decomposition gives

$$X(s) = \frac{s-4}{(s-1)(s-5)} = \frac{3/4}{s-1} + \frac{1/4}{s-5},$$

$$Y(s) = \frac{3}{(s-1)(s-5)}$$

$$= \frac{-3/4}{s-1} + \frac{3/4}{s-5},$$

and therefore

$$x(t) = \frac{3}{4}e^{t} + \frac{1}{4}e^{5t},\ y(t) = \frac{-3}{4}e^{t} + \frac{3}{4}e^{5t}.$$

## PRACTICE EXERCISES

1. Use the Laplace transform to solve the following initial value problems:

a) $y''+9y=0, y(0)=1, y'(0)=2$

b) $y''-y'-6y=0, y(0)=0, y'(0)=3.$

c) $y''+y=\sin t, y(0)=2, y'(0)=-1$

d) $y''+5y'+6y=4e^{t}, y(0)=y'(0)=0.$

e) $y''-4y=8, y(0)=2, y'(0)=0.$

3 Use Laplace transforms to find the solutions $x(t), y(t)$ of the following initial value problems:

a) $x'=x+y, x(0)=3,$

$y'=9x+y=y(0)=-3$

b) $x'=x+3y, x(0)=1,$

$y'=3x+y, y(0)=0.$

## 2.7 THE **CONVOLUTION INTEGRAL: WEIGHTING FUNCTION**

A natural question to ask is the following one: If $f(t)$ has a Laplace transform $F(s)$ and $g(t)$ has a Laplace transform $G(s)$, what function $h(t)$ has the Laplace transform $H(s)=F(s)G(s)$? A moment's reflection clearly shows the answer is not $h(t) = f(t)g(t)$. For instance, if $f(t)=g(t)=1,$ then $f(t)g(t)=1, but$

$$H(s)=\frac{1}{s}\frac{1}{s}=\frac{1}{s^2},$$

whose inverse Laplace transform is $t \neq f(t)g(t)$.

The answer to the question is that $h(t)$ is an expression involving the integral of $f(t)$ and $g(t)$, *called the convolution integral, or convolution of* $f$ *and* $g$:

$$f * g = \int_0^t f(t-T)dT.$$

Note that in the standard notation on the left the dependence on the variable $t$ is understood and is *omitted*. To emphasize the dependence, it is sometimes useful to write $(f * g)(t)$.

**Example 1** Compute $t * \cos t.$

$$t * \cos t. = \int_0^t (t-T)\cos T dT = t\int_0^t \cos T dT - \int_0^t T\cos T dT$$

$$= t\sin t - \cos t - t\sin t + 1 = 1 - \cos t.$$

The simple change of variable $T = t - s$ in the convolution integral gives

$$f * g = -\int_t^0 f(s)g(t-s)ds = \int_0^t g(t-s)f(s)ds = g * f\ ,$$

so we see that the order of the function is immaterial.

The convolution integral, or, equivalently, the operation $(*)$, can be thought of as a generalized product between functions, and simple calculations show that $f * g$ satisfies

$$f * g = g * f \qquad \textit{(Commutatively)},$$

$$f * (g+h) = f * g + f * h \qquad \textit{(distributivity)},$$

$$f * (g * h) = (f * g) * h \qquad \textit{(associatively)}.$$

However, the important property is the one mentioned at the beginning of this section: If $L\{f(t)\} = F(s)$ and

$L\{g(t)\} = G(s)$ exist for $\operatorname{Re}(s) > a, then$

$$L\{f * g\} = F(s)G(s), \operatorname{Re}(s) > a,$$

or , equivalently,

$$f * g = L^{-1}\{F(s)G(s)\}.$$

A very useful application of the convolution property is in the calculation of inverse Laplace transforms
$L\{H(s)\}$, where $H(s)$ is recognized as the product of two simple functions $F(s)$ and $G(s)$. An integration will have to be performed at the end, but that may be easier than working out a complicated partial fraction expansion.

**Example 2** Compute $L^{-1}\left\{\dfrac{1}{s^3(s^2+1)}\right\}$

One can write

$$\frac{1}{s^3(s^2+1)} = \left(\frac{1}{s^3}\right)\left(\frac{1}{s^2+1}\right),$$

and so

$$F(s) = \frac{1}{s^2} = L^{-1}\left\{\frac{t^2}{2}\right\}$$

and

$$G(s) = \frac{1}{s^2+1} = L^{-1}\{\sin t\}.$$

Therefore,

$$L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \frac{t^2}{2} * \sin t = \frac{1}{2}\int_0^t (t-T)^2 \sin T dT$$

$$= \frac{1}{2}t^2\int_0^t \sin TdT - t\int_0^t T\sin TdT + \frac{1}{2}\int_0^t T^2 \sin TdT$$

$$= -1 + \frac{t^2}{2} + \cos t.$$

**Example 3** Compute $L^{-1}\left\{\frac{2b^2}{\left(s^2+b^2\right)}\right\}$.

This example, which was discussed in the pr4evious section, is ideally set up for using the convolution property. Just write

$$\frac{2b^2}{\left[s^2+b^2\right]^2} = 2\left(\frac{b}{s^2+b^2}\right)\left(\frac{b}{s^2+b^2}\right),$$

and since each of the terms in parentheses is the transform of $\sin bt,$ we get

$$L^{-1}\left\{\frac{2b^2}{s^2+b^2}\right\} = 2\left[\sin bt * \sin bt\right] = 2\int_0^t \sin b(t-T)\sin bTdT$$

$$= 2\sin bt\int_0^t \cos bT\sin bTdT - 2cobt\int_0^t \sin^2 bTdT$$

$$= \frac{1}{b}\sin^3 bt - t\cos bt + \frac{1}{2b}\sin 2bt\cos bt$$

Since $\sin 2bt = 2\sin bt\cos bt$, the answer previously given will be obtained after a little trigonometric manipulation.

The convolution integral approach is especially efficient when the denominator of $F(s)$ is a product of irreducible quadratic factors.

The real importance of the convolution integral is that it is one of the building blocks in the analysis of linear systems. This is best illustrated by an example. Consider the general initial value problem

$$y''+ay'+by = f(t), y(0)=0, y'(0)=0,$$

which can be regarded as a mechanical or electrical system at rest excited by an input or driving term $f(t)$. First apply the Laplace transform to obtain

$$s^2Y(s)+asY(s)+bY(s)=L\{f(t)\}=F(s)$$

and hence

$$Y(s)=\frac{F(s)}{s^2+as+b}=P(s)F(s)$$

The function $P(s)=\left(s^2+as+b\right)^{-1}$ is called the *transfer function* of the system describe by the differential equation above.

In the $s-domain$ the system could be described by the diagram in Fig. 1.1, where the box represents the action of the system at rest on the input, namely, multiplication by the transfer function. Now apply the inverse Laplace transform to get

$$y(t)=L^{-1}\{Y(s)\}=L^{-1}\{P(s)F(s)\}$$

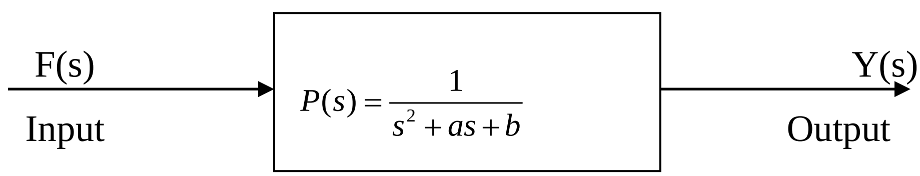


**Figure 2.1**

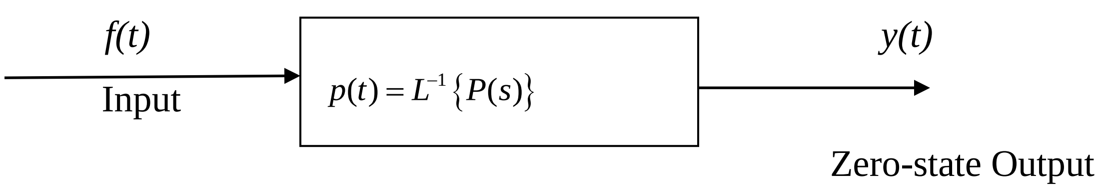


**Figure 2.2**

and by the convolution property

$$y(t)=p(t)*f(t),$$

where $p(t) = L^{-1}\{P(s)\}$ is called the *weighing function* of the system, and $y(t)$ is called the *zero-state output*.

In the *t*-domain the system could then be described by the diagram in Fig.1.2, where the box represents the action of the system at rest on the input, namely convolution with the weighing function.

If the equation $y(t) = p(t) * f(t)$, which describes the zero-state output, is written in the integral form, then

$$y(t) = \int_0^t p(T) f(t-T) dT$$

since $p * f = f * p$. Since $0 \le T \le t$, the integral can be interpreted by saying that the input T units in the past, namely $f(t-T)$, is weighted by the value of the weighting function evaluated at time T. The idea of regarding the zero-state output as a weighted integral in turn allows us to discuss the memory of the system.

We conclude this section with a proof of the convolution property which depends on the following very useful formula for the change of order of integration in a double integral:

$$\int_a^b \left[ \int_a^t F(t,T) \right] dt = \int_a^b \left[ \int_T^b F(t,T) \right] dT$$

One can now proceed directly to prove the convolution property. By the definition of the convolution integral and of the Laplace transforms, we have

$$L\{f * g\} = \int_0^\infty e^{-st} \left[ \int_0^t f(t-T) g(T) dT \right] dt$$

$$= \int_0^\infty \left[ \int_0^t e^{-st} f(t-T) g(T) dT \right] dt.$$

Apply the formula for the interchange of the order of integration with $a=0, b=\infty$ to obtain

$$L\left\{f*g\right\}=\int_0^\infty\left[\int_T^\infty e^{-st}f(t-T)g(T)dt\right]dT.$$

In the bracketed integral make the change of variable $t=T+r$ to obtain finally

$$L\left\{f*g\right\}=\int_0^\infty\left[\int_0^\infty e^{-s(T+r)}f(r)g(T)dr\right]dT=\int_0^\infty e^{-sT}g(T)\left[\int_0^\infty e^{-sT}f(r)dr\right]dT$$

$$=\left[\int_0^\infty e^{-sr}f(r)dr\right]\left[\int_0^\infty e^{-sT}g(T)dT\right]=L\left\{f\right\}L\left\{g\right\}=F(s)G(s)$$ *,which is the desired result.*

**Example 1** Find the transfer function and weighting function of the system described by the following differential equation:

$$y''+16y=f(t)$$

**SOLUTION**

Taking the Laplace transform gives:

$$y''=s^2Y(s), y=Y(s) \text{ and } L\left\{f(t)\right\}=F(s)$$

Implies

$$s^2Y(s)+16Y(s)=F(s)$$

$$Y(s)\left(s^2+16\right)=F(s)$$

$\frac{F(s)}{Y(s)}=\frac{1}{s^2+16}$. The transfer function $P(s)=\frac{1}{s^2+16}$ *and the weighting function is*

$p(t)=L^{-1}\left\{P(s)\right\}=L^{-1}\left\{\frac{1}{s^2+16}\right\}=L^{-1}\left\{\frac{1}{s^2+4^2}\right\}$. From table 1.1, $p(t)=\sin 4t$. Hence the transfer function and weighting function are:

$$p(t) = \sin 4t$$

$$P(s) = \frac{1}{s^2 + 16}$$

EXERCISES

a) $y'' - 9y = f(t)$

b) $y'' - 3y' + 10y = f(t)$

### *2.8 The Method of Clearing Fractions*

The method of clearing fractions is the most general method for solving for the constants $k_1, k_2, ..., k_n$. The method is as follows:

- Write the expression for $F(s)$ in accordance
- Multiply both sides of the equation by the denominator of the left hand side $Q(s)$
- Write $n$ equations equating terms multiplied by $s^n$
- Solve the system of $n$ equations for the constants $k_1, k_2, ..., k_n$

**Example:** Find the Partial Fraction Expansion of $F(s) = \dfrac{s^3 + 3s^2 + 4s + 6}{(s+1)(s+2)(s+3)^2}$

As presented above, this expression can be expanded into

$$\frac{s^3 + 3s^2 + 4s + 6}{(s+1)(s+2)(s+3)^2} = \frac{k_1}{s+1} + \frac{k_2}{s+2} + \frac{k_3}{s+3} + \frac{k_4}{(s+3)^2}$$

After multiplying both sides by the denominator of the left hand side we obtain

$$\begin{aligned} s^3 + 3s^2 + 4s + 6 &= \left[\frac{k_1}{s+1} + \frac{k_2}{s+2} + \frac{k_3}{s+3} + \frac{k_4}{(s+3)^2}\right](s+1)(s+2)(s+3)^2 \\ &= k_1(s+2)(s+3)^2 + k_2(s+1)(s+3)^2 + k_3(s+1)(s+2)(s+3) + k_4(s+1)(s+2) \\ &= k_1\left(s^3 + 8s^2 + 21s + 18\right) + k_2\left(s^3 + 7s^2 + 15s + 9\right) + k_3\left(s^3 + 6s^2 + 11s + 6\right) \\ &\quad + k_4\left(s^2 + 3s + 2\right) \end{aligned}$$

This expression can be rewritten by grouping the right hand side by powers of $s$ as

$$s^3+3s^2+4s+6=s^3(k_1+k_2+k_3)+s^2(8k_1+7k_2+6k_3+k_4)+s(21k_1+15k_2+11k_3+3k_4)$$
$$+18k_1+9k_2+6k_3+2k_4$$

Once it is in this form, four equations can be written equating like powers of $s$ as

$$\begin{aligned}
s^3 &: \quad k_1+k_2+k_3=1\\
s^2 &: \quad 8k_1+7k_2+6k_3+k_4=3\\
s^1 &: \quad 21k_1+15k_2+11k_3+3k_4=4\\
s^0 &: \quad 18k_1+9k_2+6k_3+2k_4=6
\end{aligned}$$

This system of equations can then be placed in a matrix for solution as

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 8 & 7 & 6 & 1 \\ 21 & 15 & 11 & 3 \\ 18 & 9 & 6 & 2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}=\begin{bmatrix} 1 \\ 3 \\ 4 \\ 6 \end{bmatrix}$$

and the constants can be solved for by

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 & 0 \\ 8 & 7 & 6 & 1 \\ 21 & 15 & 11 & 3 \\ 18 & 9 & 6 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 3 \\ 4 \\ 6 \end{bmatrix}=\begin{bmatrix} -1/4 & 1/4 & -1/4 & 1/4 \\ 8 & -4 & 2 & -1 \\ -27/4 & 15/4 & -7/4 & 3/4 \\ -27/2 & 9/2 & -3/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 4 \\ 6 \end{bmatrix}=\begin{bmatrix} 1 \\ -2 \\ 2 \\ -3 \end{bmatrix}$$

thus $F(s)=\dfrac{1}{s+1}-\dfrac{2}{s+2}+\dfrac{2}{s+3}-\dfrac{3}{(s+3)^2}$.

We can see that $F(s)$ in this form lends easily to the inverse Laplace transform being performed from **Error! Reference source not found.** . In fact, the inverse Laplace transform of $F(s)$ can now be written as $f(t)=\left[e^{-t}-2e^{-2t}+2e^{-3t}-3te^{-3t}\right]u(t)$.

### *2.9 The Unit Step Function*

The unit step function $u(t)$ is defined as

$$u(t) = \begin{Bmatrix} 1, t \geq 0 \\ 0, else \end{Bmatrix}$$

and can be seen in 2.3 From the staircase-like nature of the function it is evident from whence it gets its name. Physically, this function is representative of a switch,

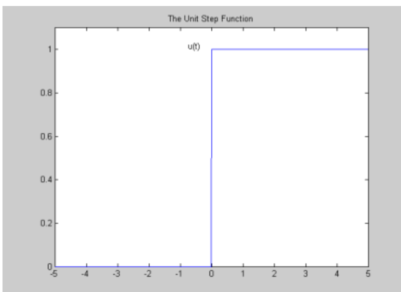


**Figure 2-3: The Unit Step Function**

being off initially and then turning on at time $t = 0$. The unit step function thus provides an easy method for forcing any function to be zero-valued before time $t = 0$. In other

$$f(t)u(t) = \begin{Bmatrix} f(t), t \geq 0 \\ 0, else \end{Bmatrix}$$

Looking at Table 2.1, we see that every function in the time domain is multiplied by the unit step function, forcing each to be zero before $t = 0$. Recall that the limits of integration for the single-sided Laplace transform went basically from $0 \to \infty$. The unit step function thus forces generic functions to meet this requirement.

### *2.10 The Time Shifted Unit Step Function*

An important variation of the unit step function is the time shifted unit step function. This can be defined as in equation below and can be seen in figure 2.4

$$u(t-t_0)=\begin{Bmatrix}1, t-t_0 \geq 0 \\ 0, else\end{Bmatrix}=\begin{Bmatrix}1, t \geq t_0 \\ 0, else\end{Bmatrix}$$

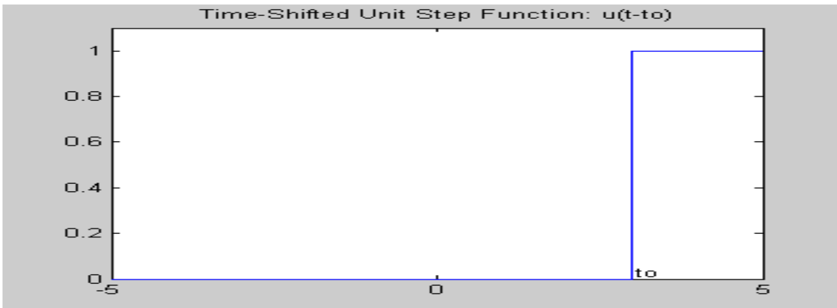


**Figure 2-4: Time-Shifted Unit Step Function**

It is evident that, since the shape is the same as before, this function is still physically representative of a switch. However, due to the time shifting property, this function is equivalent to not turning on until some time $t = t_o$.

# CHAPTER THREE

## RESULTS AND DISCUSSION

The *transform method* is used to solve initial value problems for constant coefficient linear differential equations. Transform methods are used extensively in the analysis of linear systems; the common feature of these methods are:

1. The solution of the system $y(t)$, which exists in the $t-$domain, is transformed into a function $Y(s)$ of another independent variable $s$; call this the $s-domain$. The initial conditions are usually incorporated in the transformation.
2. The system of differential or integral equations, which $y(t)$ satisfies in the $t-domain$, is in turn transformed into a system of algebraic equations in the $s-domain$. If one can solve these equations for $Y(s)$, one obtains a relation of the form $Y(s)=\phi(s)$.
3. If one can find a function $\phi(t)$ whose transform is $\phi(s)$, then one can apply the inverse transformation that takes us from the $s-domain$ to the $t-domain$ and assert that $y(t)=\phi(t)$ is the required solution.

The two most commonly used transforms are the *Fourier and* the *Laplace transforms* and we will study the latter *(i.e. the Laplace Transform)*. It was first introduce by the *great French mathematician Pierre Laplace (1749-1827)*, but its application and the techniques associated with it were not developed until about a hundred years later.

The *Laplace transform*, used as a technique for solving constant coefficient linear ordinary differential equations, is particularly effective for problems where the forcing function is discontinuous or has corners.

But in addition to this, the *Laplace transform* is used extensively to study $input-output$ in relations to systems analysis, to analyze feedback control systems, and to solve certain classes of partial differential equations of mathematical physics. It is one of the important tools of applied mathematics.

The Laplace Transform, when used in conjunction with tables of transform pairs and properties, allows complex systems to be accurately modeled while eliminating the requirement of Calculus for input-output analysis. As many students do not take Calculus until their senior year, if at all, the mathematics presented in Physics and Chemistry classes often severely limits the detail that systems may be modeled with. The target of this module is students in their second year of Algebra, thus removing the mathematical limitations for modeling before Calculus.

## REFERENCES


1 . R.V.Churchill, *Operational Mathematics*, 2nd ed., McGraw-Hill, New York, 1958

2 . W.Kaplan, *Operational Methods for Linear Systems*, Addison-Wesley,Reading, 1962

3 . J.J.DiStefano III, A.R.Stubberud, and I.J. Williams, *Feedback and Control Systems*, *Schaum*, New York, 1993.

4 . C.A.Desoer and L. A. Zadeh, *Linear Systems Theory*, McGraw-Hill, New York, 1993

5 . E.V.Bohn, *The transform Analysis of Linear Systems*, Addison-Wesley, Reading, 1963

6. Lathi, B.P., *Signal Processing & Linear Systems*. 1998, Carmichael, CA: Berkeley-Cambridge Press.

7. Chen, C.-T., *Linear System Theory and Design, 3rd Edition*. The Oxford Series in Electrical and Computer Engineering, ed. A.S. Sedra and M.R. Lightner. 1999, New York: Oxford University Press.