TUTORIAL 01

7CCMCS04
A. ANNIBALE AND G. SICURO

Problem 1.1 Consider the bi-variate Gaussian distribution for the random variable $\mathsf{X} = \binom{\mathsf{X}_1}{\mathsf{X}_2} \in \mathbb{R}^2$

$$p_2(\boldsymbol{x}) = \frac{\sqrt{1-\lambda^2}}{2\pi\sigma^2} \exp\left(-\frac{x_1^2 - 2\lambda x_1 x_2 + x_2^2}{2\sigma^2}\right) \tag{1}$$

where the parameter $-1 < \lambda < 1$ is such to ensure that the quadratic form in the exponent is positive definite.

a. Verify that this is well normalized by direct integration or by comparing our distribution with the form of a general zero-mean bivariate Gaussian distribution

$$p_2(\boldsymbol{x}) = \sqrt{\frac{\det \boldsymbol{A}}{(2\pi)^N}}\, \mathrm{e}^{-\frac{1}{2}\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x}}, \tag{2}$$

where $\boldsymbol{A} \in \mathbb{R}^2$ is positive definite.

b. Verify that the marginal probability of the individual variables are Gaussian with variance $\sigma_\lambda^2 \geq \sigma^2$. Verify that $\sigma_\lambda^2 = \langle\!\langle \mathsf{X}_1^2 \rangle\!\rangle \coloneqq \mathbb{E}[\mathsf{X}_1^2] - \mathbb{E}[\mathsf{X}_1]^2$.

c. Show that the covariance of X_1 and X_2 is

$$\langle\!\langle \mathsf{X}_1\mathsf{X}_2 \rangle\!\rangle \coloneqq \mathbb{E}[\mathsf{X}_1\mathsf{X}_2] - \mathbb{E}[\mathsf{X}_1]\mathbb{E}[\mathsf{X}_2] = \frac{\lambda}{1-\lambda^2}\sigma^2 \tag{3}$$

d. It is often convenient to calculate the *Pearson's correlation coefficient*,

$$\rho \coloneqq \frac{\langle\!\langle \mathsf{X}_1\mathsf{X}_2 \rangle\!\rangle}{\sqrt{\langle\!\langle \mathsf{X}_1^2 \rangle\!\rangle \langle\!\langle \mathsf{X}_2^2 \rangle\!\rangle}} \tag{4}$$

Show that this is merely given by λ: in other words, the parameter λ in the distribution is a measure of how correlated the variables X_1 and X_2 are.

Note. In the limit $\lambda \to 0$ the variables are not correlated at all and the distribution factorizes,

$$p_2(\boldsymbol{x})|_{\lambda=0} = \frac{1}{\sqrt{2\pi\sigma^2}}\, \mathrm{e}^{-\frac{x_1^2}{2\sigma^2}}\, \frac{1}{\sqrt{2\pi\sigma^2}}\, \mathrm{e}^{-\frac{x_2^2}{2\sigma^2}}\,. \tag{5}$$

In the limit $\lambda \to 1$, on the other hand, the variables are maximally correlated, $p = 1$. The distribution becomes a function of $x_1 - x_2$, so it is favoured that x_1 and x_2 take similar values

$$p_2(\boldsymbol{x})|_{\lambda=1} \to \mathrm{e}^{-\frac{(x_1-x_2)^2}{2\sigma^2}}\,. \tag{6}$$

Date: February 13, 2025.

We can now interpret the increase of the variance with λ: the correlation between the variables allow them to take arbitrarily large values, with the only restriction of their difference being small.

e. By using Bayes rule show that

$$p_{1|1}(x_1|x_2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_1-\lambda x_2)^2}{2\sigma^2}\right). \tag{7}$$

Note. At $\lambda = 0$ (no correlation) the values taken by x_1 are independent of x_2, while for $\lambda \to 1$ they are centered around those taken by x_2, and hence strongly conditioned by them.

Problem 1.2 The *Wiener* process was originally introduced to describe the behaviour of the *position* of a free Brownian particle in one dimension. On the other hand, it plays a central role in the rigorous foundation of the stochastic differential equations and occurs often in applied mathematics, physics and economics. It is an absolutely central stochastic process and it is defined through the law

$$p_{1|1}(x_2,t_2|x_1,t_1)=\frac{1}{\sqrt{2\pi(t_2-t_1)}}\exp\left(-\frac{(x_2-x_1)^2}{2(t_2-t_1)}\right) \qquad \text{with} \quad p_1(x,0)=\delta(x_1). \tag{8}$$

under the assumption $0 < t_1 < t_2$.

a. Show that the probability density for $t > 0$ is

$$p_1(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right). \tag{9}$$

Note. This is a non-stationary (p_1 depends on t), Gaussian process. A process is called a *Gaussian process* if all its joint densities p_n are mutivariate Gaussian distributions: in this case, all cumulants of order higher than 2 are zero. Gaussian processes are often used to approximate physical processes where it can be assumed that higher order cumulants are negligible.

b. Show that

$$\mathbb{E}[\mathsf{X}_{t_1}] = 0 \quad \text{and} \quad \mathbb{E}[\mathsf{X}_{t_1}\mathsf{X}_{t_2}] = \min(t_1,t_2) \tag{10}$$

c. Show that (9) satisfies the following *diffusion equation*

$$\frac{\partial p_1(x,t)}{\partial t} = \frac{\partial^2 p_1(x,t)}{\partial x^2}. \tag{11}$$

d. Use Eq. (11) to find the equation of motion for the moments $\mathbb{E}[\mathsf{X}_t]$ and $\mathbb{E}[\mathsf{X}_t^2]$.

e. Find $p_1(x,t)$ for the generic initial condition $p_1(x,0)$.

Problem 1.3 The *Ornstein–Uhlenbeck* process was introduced to describe the *velocity* of a free Brownian particle in one dimension. It also describes the position of an overdamped particle in a harmonic potential. Denoting $\tau > 0$, it is defined by

$$p_2(x_2, t+\tau|x_1, t) = \frac{1}{\sqrt{2\pi(1-\mathrm{e}^{-2\tau})}} \exp\left(-\frac{(x_2 - x_1\,\mathrm{e}^{-\tau})^2}{2(1-\mathrm{e}^{-2\tau})}\right), \tag{12}$$

with

$$p_1(x,t) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{x^2}{2}} \equiv p_1(x) \quad \forall\, t. \tag{13}$$

The Ornstein–Uhlenbeck process is stationary, Gaussian and Markovian.

a. The Gaussian property is clear for p_1. By using $p_2(x_2, t+\tau; x_1, t) = p_1(x_1)p_{1|1}(x_2, t+\tau|x_1, t)$ show that $p_2(x_2, t+\tau; x_1, t)$ can be written as the bivariate Gaussian distribution of Problem 1.1 with $\lambda = \mathrm{e}^{-\tau}$ and $\sigma^2 = 1-\lambda^2$.

b. Show that the process has an exponentially-decaying correlation function

$$\langle\!\langle \mathsf{X}_t \mathsf{X}_{t+\tau} \rangle\!\rangle := \mathbb{E}[\mathsf{X}_t\mathsf{X}_{t+\tau}] - \mathbb{E}[\mathsf{X}_t]\mathbb{E}[\mathsf{X}_{t+\tau}] = \mathrm{e}^{-\tau}\,. \tag{14}$$

The evolution with τ of the velocity correlation has a clear meaning. The values of the velocity of the Brownian particle at two times separated by τ are strongly correlated for small τ: here $\lambda \simeq 1$ and the variance σ^2 of the distribution shrinks to zero. As time elapses, λ decreases and for long time differences $\lambda \simeq 0$: here the velocity at time $t+\tau$ has lost all memory of its value at the initial time t due to the noise and hence the velocities at times t and $t+\tau$ become uncorrelated for $\tau \to +\infty$.