TUTORIAL 09

7CCMCS04
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Problem 9.1 Consider the infinite-range Ising ferromagnet where $\sigma_i = \pm 1$, with $i = 1, \dots, N$, denote spin variables, $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)$ is the microscopic state of the system and $J/N > 0$ is the strength of interactions between any pair of spins σ_i and σ_j. We assume that the system follows a random sequential Glauber dynamics, so that the microstate distribution $P_{\boldsymbol{\sigma}}(t)$ evolves according to the master equation

$$\dot{P}_{\boldsymbol{\sigma}}(t) = \sum_i [W_i(F_i\boldsymbol{\sigma})P_{F_i\boldsymbol{\sigma}}(t) - W_i(\boldsymbol{\sigma})P_{\boldsymbol{\sigma}}(t)],$$

where F_i is the i-spin flip operator $F_i\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_{i-1}, -\sigma_i, \sigma_{i+1}, \dots, \sigma_N)$, and

$$W_i(\boldsymbol{\sigma}) = \frac{1 - \sigma_i \tanh \beta h_i(\boldsymbol{\sigma})}{2}, \qquad \beta = \frac{1}{T},$$

and h_i is the effective local field defined as

$$h_i(\boldsymbol{\sigma}) = \frac{J}{N} \sum_{j \neq i} \sigma_j.$$

a. Show that for $N \gg 1$ the equation for the moment $\mathbb{E}[\sigma_i(t)] = \sum_{\boldsymbol{\sigma}} \sigma_i \, P_{\boldsymbol{\sigma}}(t)$ is given, to orders $O(1)$, by

$$\frac{\mathrm{d}\,\mathbb{E}[\sigma_i]}{\mathrm{d}\,t} = -\mathbb{E}[\sigma_i] + \mathbb{E}[\tanh \beta J m_{\sigma}],$$

where $m_{\boldsymbol{\sigma}} = \frac{1}{N}\sum_i \sigma_i$ is the instantaneous magnetization.

b. Define the mean magnetization as $m = \mathbb{E}[m_{\boldsymbol{\sigma}}] = \frac{1}{N}\sum_i \mathbb{E}[\sigma_i]$. Show that away from criticality, where the magnetization fluctuations $\delta m_{\boldsymbol{\sigma}} = m_{\boldsymbol{\sigma}} - m$ are small, the mean magnetization evolves, to linear orders in $\delta m_{\boldsymbol{\sigma}}$, according to

$$\frac{\mathrm{d}\,m}{\mathrm{d}\,t} = -m + \tanh \beta J m.$$

c. Show that the system undergoes a thermodynamic phase transition at $T = J$, with the equilibrium phase at $T > J$ given by the disordered paramagnetic state $m = 0$ and the phase at $T < J$ given by the ordered ferromagnetic state, where $m \neq 0$. Show in particular that $m = 0$ is a stable fixed point of the dynamics for $T > J$, while it is unstable for $T < J$.

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Problem 9.2 Consider the one-dimensional Ising model in which at each site $i = 1, \dots, N$ there is a spin $\sigma_i = \pm 1$ and periodic boundary conditions are imposed. Each spin interacts with spins on the two neighboring sites through a coupling $J > 0$, so that the Hamiltonian is

$$H(\boldsymbol{\sigma}) = -J \sum_i \sigma_i \sigma_{i+1}$$

where $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)$ is the microscopic state of the system. Assume that the system evolves according to random sequential Glauber dynamics, so that the master equation for the microstate distribution $P_{\boldsymbol{\sigma}}(t)$ is given by

$$\dot{P}_{\boldsymbol{\sigma}}(t) = \sum_i [W_i(F_i\boldsymbol{\sigma}) P_{F_i\boldsymbol{\sigma}}(t) - W_i(\boldsymbol{\sigma}) P_{\boldsymbol{\sigma}}(t)],$$

where F_i is the i-spin flip operator $F_i\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_{i-1}, -\sigma_i, \sigma_{i+1}, \dots, \sigma_N)$ and the transition rates are

$$W_i(\boldsymbol{\sigma}) = \frac{1 - \sigma_i \tanh \beta h_i(\boldsymbol{\sigma})}{2},$$

where $\beta = T^{-1}$ is the inverse temperature and h_i is the effective local field defined as

$$h_i(\boldsymbol{\sigma}) = J(\sigma_{i-1} + \sigma_{i+1}).$$

a. Show that the equation for the moments $m_i(t) = \mathbb{E}[\sigma_i] = \sum_{\boldsymbol{\sigma}} \sigma_i P_{\boldsymbol{\sigma}}(t)$, is given by

$$\partial_t m_i = -m_i + \gamma \frac{m_{i-1} + m_{i+1}}{2}, \qquad \gamma = \tanh 2\beta J.$$

b. Show that the solution of equation (), for the initial condition $m_i(0) = \delta_{i,0}$, is $m_i(t) = \mathrm{e}^{-t} I_i(\gamma t)$.
Hint: You may use that the generating function of the n-th modified Bessel function of the first kind $I_n(t)$ is $\sum_{n=-\infty}^{\infty} I_n(t) z^n = \mathrm{e}^{\frac{1}{2}(z + 1/z)t}$.

c. By using the asymptotics of $I_n(t)$ for large t, $I_n(t) \sim \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{t - \frac{n^2}{2t}}$, deduce that at $T = 0$, the system is coarsening through the growth of a lengthscale $L(t) \sim t^z$, where z should be found.

d. Write the equation for the equal-time correlation function $C_{ij} = \mathbb{E}[\sigma_i \sigma_j]$. Show that for homogenous systems, where $C_{ij} = C_k$ for $|i - j| = k$, the correlation function evolves according to

$$\partial_t C_k = -2C_k + \gamma(C_{k+1} + C_{k-1})$$

with boundary condition $C_0(t) = 1 \ \forall \ t$.

e. By demanding that $C_k(\infty) = \eta^k$ is the stationary solution of equation for C_k, show that the equilibrium correlation function is $C_k(\infty) = \mathrm{e}^{-\frac{k}{\xi}}$ where $\xi^{-1} = -\ln \tanh \beta$. Comment on the role of ξ as a correlation length and its behavior at criticality.
Hint: You may use, without proof, the identities $\sinh 2x = 2 \sinh x \cosh x$ and $\cosh 2x - 1 = 2 \sinh^2 x$.

Problem 9.3 A single pattern $\boldsymbol{\xi} \in \{-1, 1\}^N$ is stored in a neural network of N binary neurons $\sigma_i \in \{-1, 1\}$, $i = 1, \dots, N$ evolving according to sequential dynamics. The macroscopic overlap $m = \frac{1}{N}\sum_i \xi_i^\mu \sigma_i$ of the system configuration with the stored pattern evolves, in the limit $N \to \infty$ according to

$$\frac{\mathrm{d}\, m}{\mathrm{d}\, t} = \tanh(\beta m) - m$$

where $\beta = T^{-1}$ is the inverse temperature.

a. For $T > 1$ show that the only stationary state is $m = 0$. By solving the linearized equation about this state, show that for long time $m(t)$ is proportional to $\exp(-{}^{t}\!/\!{}_{\tau})$ where the characteristic timescale τ of the exponential decay to the stationary state should be found. In which temperature limit does the critical slowing down arise?

b. Show, by graphical methods, that for $T < 1$ there are three stationary states $-m_\star, 0, m_\star$, where $m_\star$ is the positive solution of $m_\star = \tanh(\beta m_\star)$. Show that $m = 0$ is unstable for $T < 1$. Show, by linearizing the equation about the other two stationary states, that the convergence to the latter is again exponential, with characteristic time $\tau = \left(1 - \frac{1-m_\star^2}{T}\right)^{-1}$. By using the stationarity condition, show that $m_\star$ is proportional to $\sqrt{1-T}$ as $T \to 1^-$ where the proportionality constant should be found. Hence show that τ diverges as $\tau = \frac{1}{2(1-T)}$, as $T \to 1^{-1}$.

c. Show that at $T = 1$ the decay to the stationary state for long time is given by a power law that does not depend on any undetermined constant.

✎ **Problem 9.4** Consider random uniformly distributed binary vectors $\boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^p$, $\boldsymbol{\xi}^\mu = (\xi_1^\mu, \ldots, \xi_N^\mu)$, with $\mu = 1, \ldots, p$, where each component is distributed identically and independently according to $\rho(\xi_i^\mu) = \frac{1}{2}\delta_{\xi_i^\mu,+1} + \frac{1}{2}\delta_{\xi_i^\mu,-1}$ $\forall i, \mu$.

a. Show for the random variable

$$C_i^\nu = -\frac{\xi_i^\nu}{N-1}\sum_{k(\neq i)}^{N}\sum_{\mu(\neq\nu)}^{p} \xi_i^\mu \xi_k^\mu \xi_k^\nu$$

that the first two moments are $\mathbb{E}[C_i^\nu] = 0$ and $\mathbb{E}[(C_i^\nu)^2] = \frac{p-1}{N-1}$, where the expectation is taken with respect to the probability distribution $\rho(\boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^p) = \prod_{i,\mu} \rho(\xi_i^\mu)$.

b. The patterns $\boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^p$ are now stored in a noiseless network evolving via the parallel dynamics

$$\sigma_i(t+1) = \mathrm{sign}\left(\frac{1}{N}\sum_{k(\neq i)}^{N}\sum_{\mu=1}^{p} \xi_i^\mu \xi_k^\mu \sigma_k(t)\right), \ \forall i = 1, \ldots, N$$

so that if the network is at time t in the configuration $\boldsymbol{\sigma}(t) = \boldsymbol{\xi}^\nu$, the network configuration at time $t+1$ is given by

$$\sigma_i(t+1) = \xi_i^\nu \mathrm{sign}(1 - C_i^\nu), \ \forall i = 1, \ldots, N.$$

Find an expression, without evaluating it, for the probability P_{error} that an error will occur in the i-th bit of the retrieval pattern, when the the network is presented with the pattern $\boldsymbol{\xi}^\nu$, assuming that $N \gg 1$, $p \gg 1$

and that C_i^ν is Gaussian in this limit. Estimate this probability for $p \ll N$.

Hint: You may use, if you wish, the result of the Gaussian integral $\int_{-\infty}^{\infty} \mathrm{d}\,x \, \exp(-ax^2) = \sqrt{\frac{\pi}{a}}$ and the expansion $1 - \mathrm{erf}(x) \simeq \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}x}$, as $x \to \infty$, with $\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} \mathrm{d}\,u \; \mathrm{e}^{-u^2}$.