TUTORIAL 08 — SOLUTIONS

7CCMCS04
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Problem 8.1

a. We have

$$\sum_{\boldsymbol{\sigma}} Q_{\boldsymbol{\sigma}\,\boldsymbol{\sigma}'} = \sum_{\sigma_1\ldots\sigma_N} \prod_{i=1}^{N} \frac{\mathrm{e}^{\beta\sigma_i h_i(\boldsymbol{\sigma}')}}{2\cosh(\beta h_i(\boldsymbol{\sigma}'))} = \prod_{i=1}^{N} \frac{\sum_{\sigma_i} \mathrm{e}^{\beta\sigma_i h_i(\boldsymbol{\sigma}')}}{2\cosh(\beta h_i(\boldsymbol{\sigma}'))} \equiv 1.$$

b. First, we note that $Q_{\boldsymbol{\sigma}\,\boldsymbol{\sigma}'}$ describes an ergodic system, as it has all elements strictly non-negative, i.e. for any initial configuration $\boldsymbol{\sigma}$ it can reach any final state $\boldsymbol{\sigma}'$ with nonzero probability. We can derive

$$\begin{aligned} Q_{\boldsymbol{\sigma}\,\boldsymbol{\sigma}'}\Pi_{\boldsymbol{\sigma}'} &= \prod_{i=1}^{N} \frac{\mathrm{e}^{\beta\sigma_i h_i(\boldsymbol{\sigma}')}}{2\cosh(\beta h_i(\boldsymbol{\sigma}'))} \frac{\exp\left(\beta\sum_j \theta_j\sigma'_j + \sum_j \ln\cosh(\beta h_j(\boldsymbol{\sigma}'))\right)}{\sum_{\hat{\boldsymbol{\sigma}}} \mathrm{e}^{-\beta H(\hat{\boldsymbol{\sigma}})}} \\ &= \frac{\exp\left(\beta\sum_{ij}\sigma_i J_{ij}\sigma'_j + \beta\sum_i \theta_i(\sigma_i+\sigma'_i)\right)}{2^N \sum_{\hat{\boldsymbol{\sigma}}} \mathrm{e}^{-\beta H(\hat{\boldsymbol{\sigma}})}} \end{aligned}$$

Similarly one obtaines

$$Q_{\boldsymbol{\sigma}'\,\boldsymbol{\sigma}}\Pi_{\boldsymbol{\sigma}} = \frac{\exp\left(\beta\sum_{ij}\sigma'_i J_{ij}\sigma_j + \beta\sum_i \theta_i(\sigma'_i+\sigma_i)\right)}{2^N \sum_{\hat{\boldsymbol{\sigma}}} \mathrm{e}^{-\beta H(\hat{\boldsymbol{\sigma}})}}$$

If we compute

$$\frac{Q_{\boldsymbol{\sigma}'\,\boldsymbol{\sigma}}\Pi_{\boldsymbol{\sigma}}}{Q_{\boldsymbol{\sigma}\,\boldsymbol{\sigma}'}\Pi_{\boldsymbol{\sigma}'}} = \exp\left(\beta\sum_{ij}\sigma'_i(J_{ij}-J_{ji})\sigma_j\right)$$

which gives 1 for $J_{ij} = J_{ji}$.

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Problem 8.2

a. The configuration $\boldsymbol{\sigma}$ can occur at time $t+\tau$ if at the earlier time t the system was either in one of the N configurations $F_i\boldsymbol{\sigma}$ (with spin i being sampled probability N^{-1} and flipped with probability $W_i(F_i\boldsymbol{\sigma})$) or in the same configuration $\boldsymbol{\sigma}$ with no transition occurring after any site is sampled. We can write therefore

$$P_{\boldsymbol{\sigma}}(t+\tau) = \frac{1}{N}\sum_i P_{F_i\boldsymbol{\sigma}}(t)W_i(F_i\boldsymbol{\sigma}) + P_{\boldsymbol{\sigma}}(t)\left(1-\frac{1}{N}\sum_i W_i(\boldsymbol{\sigma})\right).$$

Setting $\tau = N^{-1}$ and taking the limit $N \gg 1$

$$\partial_t P_{\boldsymbol{\sigma}}(t) = \lim_{N\to\infty}\frac{P_{\boldsymbol{\sigma}}(t+N^{-1}) - P_{\boldsymbol{\sigma}}(t)}{N^{-1}} = \sum_i \left(P_{F_i\boldsymbol{\sigma}}(t)W_i(F_i\boldsymbol{\sigma}) - P_{\boldsymbol{\sigma}}(t)W_i(\boldsymbol{\sigma})\right)$$

b. Detailed balance with the Gibbs-Boltzmann distribution requires

$$\frac{W_i(F_i\boldsymbol{\sigma})}{W_i(\boldsymbol{\sigma})} = \frac{\mathrm{e}^{-\beta H(\boldsymbol{\sigma})}}{\mathrm{e}^{-\beta H(F_i\boldsymbol{\sigma})}} = \mathrm{e}^{\beta[H(F_i\boldsymbol{\sigma})-H(\boldsymbol{\sigma})]}.$$

Write

$$H(\boldsymbol{\sigma}) = -\frac{1}{2}\sum_i \sigma_i h_i(\boldsymbol{\sigma}), \quad \text{with } h_i(\boldsymbol{\sigma}) = \frac{1}{N}\sum_{j\neq i} J_{ij}\sigma_j.$$

We have then

$$\begin{aligned} H(\boldsymbol{\sigma}) &= -\frac{1}{2}\sum_{j(\neq i)} \sigma_j h_j(\boldsymbol{\sigma}) - \frac{1}{2}\sigma_i h_i(\boldsymbol{\sigma}) \\ H(F_i\boldsymbol{\sigma}) &= -\frac{1}{2}\sum_{j(\neq i)} \sigma_j h_j(F_i\boldsymbol{\sigma}) + \frac{1}{2}\sigma_i h_i(F_i\boldsymbol{\sigma}). \end{aligned}$$

Use now

$$h_i(F_i\boldsymbol{\sigma}) = h_i(\boldsymbol{\sigma}) \qquad h_j(F_i\boldsymbol{\sigma}) = -2\frac{J_{ji}}{N}\sigma_j$$

so

$$\begin{aligned} H(F_i\boldsymbol{\sigma}) - H(\boldsymbol{\sigma}) &= -\frac{1}{2}\sum_{j(\neq i)} \sigma_j(h_j(F_i\boldsymbol{\sigma}) - h_j(\boldsymbol{\sigma})) + \frac{1}{2}\sigma_i h_i(\boldsymbol{\sigma}) \times 2 \\ &= -\frac{1}{2}\sum_{j(\neq i)} \sigma_j\left(-2\frac{J_{ji}}{N}\sigma_i\right) + \sigma_i h_i(\boldsymbol{\sigma}) \\ &= \sigma_i \frac{1}{N}\sum_{j(\neq i)} J_{ij}\sigma_j + \sigma_i h_i(\boldsymbol{\sigma}) \\ &= h_i(\boldsymbol{\sigma})\sigma_i + \sigma_i h_i(\boldsymbol{\sigma}) = 2h_i(\boldsymbol{\sigma})\sigma_i \end{aligned}$$

Hence the detail balance condition becomes

$$\frac{W_i(F_i\boldsymbol{\sigma})}{W_i(\boldsymbol{\sigma})} = \frac{\mathrm{e}^{\beta\sigma_i h_i(\boldsymbol{\sigma})}}{\mathrm{e}^{-\beta\sigma_i h_i(\boldsymbol{\sigma})}} = \frac{1+\sigma_i\tanh\beta h_i(\boldsymbol{\sigma})}{1-\sigma_i\tanh\beta h_i(\boldsymbol{\sigma})}$$

where we used $\frac{e^{\pm x}}{\cosh x} = 1 \pm \tanh x$, $\tanh\sigma x = \sigma\tanh x$ for $\sigma = \pm 1$ and $\cosh x = \cosh(-x)$. The above condition is satisfied by the Glauber rates.