TUTORIAL 04 — SOLUTIONS

7CCMCS04
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Problem 4.1

a. The matrix of the given chain is

$$\boldsymbol{Q} = \begin{pmatrix} 1 & 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 1/2 & 1/2 & 0 \\ 0 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 \end{pmatrix}$$

b. The absorbing states are 1 and 5: these are also the recurrent states. States 2, 3 and 4 are instead transient, because there is a finite probability of starting in one of these states and end up in 1 or 5, which are absorbing and we cannot leave anymore.

c. Using the notation of the lecture notes, from the matrix $\boldsymbol{Q}$ we can extract

$$\boldsymbol{Q}_{\mathcal{T}\mathcal{T}} = \begin{pmatrix} 1/3 & 1/2 & 1/2 \\ 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}.$$

Observing that the recurrent states are both absorbing, we can use the formula

$$\begin{aligned} \langle \boldsymbol{T}^a | &= \langle \mathbf{1} | \left(\boldsymbol{I} - \boldsymbol{Q}_{\mathcal{T}\mathcal{T}} \right)^{-1} \\ &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 12/5 & 9/5 & 6/5 \\ 4/5 & 8/5 & \frac{2}{5} \\ \frac{2}{5} & 4/5 & 6/5 \end{pmatrix} \\ &= \begin{pmatrix} \frac{18}{5} & \frac{21}{5} & \frac{14}{5} \end{pmatrix}. \end{aligned}$$

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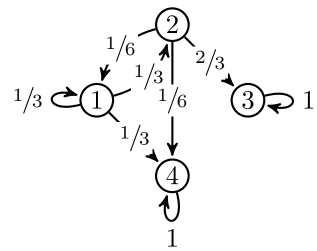
PROBLEM 4.2

There are four states a student can be:

(1) first year student;
(2) second year student;
(3) graduated student;
(4) dropped.

The Markov chain can be represented



corresponding to the matrix

$$\boldsymbol{Q} = \begin{pmatrix} 1/3 & 1/6 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 1/3 & 1/6 & 0 & 1 \end{pmatrix}$$

As in the previous exercise, there are two absorbing states, namely 3 (dropped) and 4 (graduated), the remaining being transient. The matrix of transitions probabilities from transient states to transient states is

$$\boldsymbol{Q}_{\mathcal{T}\mathcal{T}} = \begin{pmatrix} 1/3 & 1/6 \\ 1/3 & 0 \end{pmatrix}.$$

The mean number of years at university can be read from

$$\begin{aligned} \langle \boldsymbol{T}^a | &= \langle \mathbf{1} | \, (\boldsymbol{I} - \boldsymbol{Q}_{\mathcal{T}\mathcal{T}})^{-1} \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 2/3 & -1/6 \\ -1/3 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{24}{11} & \frac{15}{11} \end{pmatrix} \end{aligned}$$

and in particular from the first element of it, corresponding to initial state in 1: it is $\frac{24}{11} \simeq 2.18$. To compute the probability of graduating, we have to compute the array $\boldsymbol{A}$ of "absorption probabilities" and focus on the element corresponding to the absorption in 4 starting from 1. The general formula is

$$\boldsymbol{A} = \boldsymbol{Q}_{\mathcal{R}\mathcal{T}} (\boldsymbol{I} - \boldsymbol{Q}_{\mathcal{T}\mathcal{T}})^{-1},$$

where

$$\boldsymbol{Q}_{\mathcal{R}\mathcal{T}} = \begin{pmatrix} 0 & 2/3 \\ 1/3 & 1/6 \end{pmatrix}$$

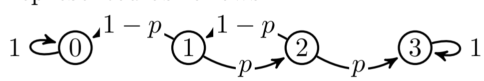
contains the transitions from the transient to recurrent states. So

$$\boldsymbol{A} = \boldsymbol{Q}_{\mathcal{R}\mathcal{T}} (\boldsymbol{I} - \boldsymbol{Q}_{\mathcal{T}\mathcal{T}})^{-1} = \begin{pmatrix} 4/11 & 8/11 \\ 7/11 & 3/11 \end{pmatrix}.$$

The probability of graduating starting from the first year is therefore $4/11$.

Problem 4.3

The chain can be represented as follows:



corresponding to the matrix

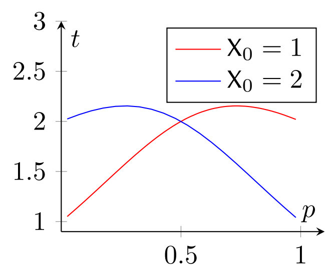
$$\boldsymbol{Q} = \begin{pmatrix} 1 & 1-p & 0 & 0 \\ 0 & 0 & 1-p & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 1 \end{pmatrix}$$

In particular note that, as the transient states are 1 and 2, and the only two recurrent states (0 and 3) are absorbing. We can write

$$\boldsymbol{Q}_{\mathcal{TT}} = \begin{pmatrix} 0 & 1-p \\ p & 0 \end{pmatrix}$$

The mean number of rounds until the game is over is therefore

$$\begin{aligned} \langle \boldsymbol{T}^a| &= \langle \mathbf{1}| \, (\boldsymbol{I} - \boldsymbol{Q}_{\mathcal{TT}})^{-1} \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & p-1 \\ -p & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1+p}{p^2-p+1} & \frac{2-p}{p^2-p+1} \end{pmatrix}. \end{aligned}$$



As a result, starting from $\mathsf{X}_0 = 1$ the game will end on average in $\frac{1+p}{p^2-p+1}$ steps. Note that this makes sense: for $p = 0$ the gambler would loose instantly in 1 step; for $p = 1$ would certainly win in 2 steps.

To compute the value $p^\star$ that makes the game last longer,

$$\frac{\mathrm{d}}{\mathrm{d}\,p} \frac{1+p}{p^2-p+1} = \frac{2-2p-p^2}{(p^2-p+1)^2} = 0 \Rightarrow p^\star = \sqrt{3} - 1.$$

To compute the probability of winning the game, we need to use

$$\boldsymbol{A} = \boldsymbol{Q}_{\mathcal{RT}}(\boldsymbol{I} - \boldsymbol{Q}_{\mathcal{TT}})^{-1},$$

where $\boldsymbol{A}$ is the matrix whose entry a_{1i} is the probability of loosing starting from i, and a_{2j} is the probability of winning starting from j. Here

$$\boldsymbol{Q}_{\mathcal{RT}} = \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}$$

is the matrix of transition probabilities from the transient to the recurrent states. Applying the formula,

$$\boldsymbol{A} = \boldsymbol{Q}_{\mathcal{RT}}(\boldsymbol{I} - \boldsymbol{Q}_{\mathcal{TT}})^{-1} = \frac{1}{p^2-p+1} \begin{pmatrix} 1-p & (1-p)^2 \\ p^2 & p \end{pmatrix}.$$

The probability of winning is therefore $\frac{p^2}{p^2-p+1}$.