TUTORIAL 02 — SOLUTIONS

7CCMCS04
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Problem 2.1

To answer all questions it is useful to find eigenvalues and eigenvectors of $\boldsymbol{Q}$ first. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1 - \alpha - \beta$ (as it follows from the fact that $\mathrm{tr}[\boldsymbol{Q}] = \lambda_1 + \lambda_2 = 2 - \alpha - \beta$. Doing the calculations for the eigenvectors, you will find

$$\begin{aligned} \lambda_1 &= 1, & |\boldsymbol{\psi}^1\rangle &= \begin{pmatrix} \frac{\beta}{\alpha+\beta} \\ \frac{\alpha}{\alpha+\beta} \end{pmatrix}, & \langle \boldsymbol{\phi}^1| &= \begin{pmatrix} 1 & 1 \end{pmatrix} \\ \lambda_2 &= 1-\alpha-\beta, & |\boldsymbol{\psi}^2\rangle &= \begin{pmatrix} -1 \\ 1 \end{pmatrix}, & \langle \boldsymbol{\phi}^2| &= \begin{pmatrix} -\frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \end{pmatrix}. \end{aligned} \tag{1}$$

In the decomposition above, $|\boldsymbol{\psi}^1\rangle \equiv |\boldsymbol{\Pi}\rangle$. Also, observe that the $(t+1)$th day correspond to time t (the first day is our $t = 0$).

a. The probability of having rain the second day is simply $Q_{11} = 1 - \alpha$. At the generic $(t+1)$th day, the probability of having rain is $(\boldsymbol{Q}^t)_{11} = \sum_a \lambda_a^t \psi_1^a \phi_1^a = \frac{\beta}{\alpha+\beta} + (1-\alpha-\beta)^t \frac{\alpha}{\alpha+\beta}$.

b. In this case, using that $|\boldsymbol{P}(t)\rangle = \boldsymbol{Q}^t |\boldsymbol{P}(0)\rangle$ and the fact that here $|\boldsymbol{P}(0)\rangle = \binom{p}{1-p}$, then the evolution is

$$\begin{aligned} |\boldsymbol{P}(t)\rangle &= |\boldsymbol{\psi}^1\rangle + (1-\alpha-\beta)^t |\boldsymbol{\psi}^2\rangle\langle \boldsymbol{\phi}^2|\boldsymbol{P}(0)\rangle \\ &= |\boldsymbol{\psi}^1\rangle + \frac{(1-\alpha-\beta)^t(\beta - p(\alpha+\beta))}{\alpha+\beta} |\boldsymbol{\psi}^2\rangle \end{aligned}$$

so that $P_2(t) = \frac{\alpha}{\alpha+\beta} + \frac{(1-\alpha-\beta)^t(\beta-p(\alpha+\beta))}{\alpha+\beta}$ gives the probability of observing sunshine the second $(t = 1)$ and third $(t = 2)$ day.

The question also asks the probability of observing sunshine the first three days,

$$\begin{aligned} \mathbb{P}_3[\mathsf{X}_0 = 2, \mathsf{X}_1 = 2, \mathsf{X}_2 = 2] &= \mathbb{P}_{1|1}[\mathsf{X}_2 = 2|\mathsf{X}_1 = 2]\mathbb{P}_{1|1}[\mathsf{X}_1 = 2|\mathsf{X}_0 = 2]\mathbb{P}_1[\mathsf{X}_0 = 2] \\ &= Q_{22}Q_{22}P_2(0) = (1-\beta)^2(1-p). \end{aligned}$$

Also, the probability of observing sunshine the second and third day is

$$\begin{aligned} \mathbb{P}_2[\mathsf{X}_1 = 2, \mathsf{X}_2 = 2] &= \mathbb{P}_{1|1}[\mathsf{X}_2 = 2|\mathsf{X}_1 = 2]\mathbb{P}_1[\mathsf{X}_1 = 2] \\ &= Q_{22}P_2(1) = (1-\beta)(1-\beta-p(1-\alpha-\beta)). \end{aligned}$$

c. See above.

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d. It is again $\frac{\beta}{\alpha+\beta}$. Assuming this value, means that we are on the steady state as this is the first entry of $|\boldsymbol{\psi}^1\rangle$ (and the fact that we have 2 states only means that it is enough to specify the entire vector).

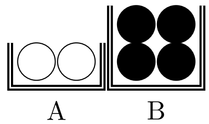
e. We have that $\boldsymbol{Q} = \sum_{a=1}^{2} \lambda_a |\boldsymbol{\psi}^a\rangle\langle\boldsymbol{\phi}^a| = \frac{1}{\alpha+\beta}\left(\begin{smallmatrix} \beta & \beta \\ \alpha & \alpha \end{smallmatrix}\right) + \frac{1-\alpha-\beta}{\alpha+\beta}\begin{pmatrix} \alpha & -\beta \\ -\alpha & \beta \end{pmatrix}$. Being $\lambda_2 = 1 - \alpha - \beta$, if $|\lambda_2| < 1$ the relaxation time is $\tau = -\frac{1}{\ln|\lambda_2|}$ which depends indeed on $\alpha + \beta$ only.

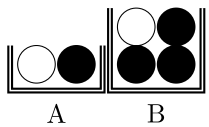
f. Yes: we have indeed that $Q_{21}\Pi_1 = \beta\frac{\alpha}{\alpha+\beta} = Q_{12}\Pi_2$.

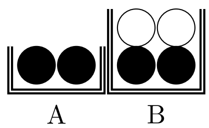
g. For $\alpha = \beta = 1$, $\lambda_2 = -1$ and in this case the dynamics does not converge in general to the stationary distribution as there are *two* eigenvalues with modulus equal to 1. Instead for $\alpha = \beta = 0$ we have $\lambda_2 = 1$. Again, there is not a unique asymptotic state: the chain is made of two isolated configurations, and starting in one of them, the system remains in it forever.

Problem 2.2

Let us denote by $1, 2, 3$ respectively, the following:

1:


2:


3:


Then

$$\boldsymbol{Q} = \begin{pmatrix} 0 & 1/8 & 0 \\ 1 & 1/2 & 1/2 \\ 0 & 3/8 & 1/2 \end{pmatrix}.$$

In principle we can get the eigenvalues by solving the equation for the eigenvalues, $\det(\boldsymbol{Q} - \lambda \boldsymbol{I}) = 0$ but it is easier to appeal to the properties of the stochastic matrix: we know indeed that $\lambda_1 = 1$ is always an eigenvalue. Hence from the trace and determinant of $\boldsymbol{Q}$ we can determine the other two. We know that

$$\operatorname{tr} \boldsymbol{Q} = \lambda_1 + \lambda_2 + \lambda_3 = 1 \Rightarrow \lambda_2 = -\lambda_3$$

but also

$$\det \boldsymbol{Q} = \lambda_1 \lambda_2 \lambda_3 = -\frac{1}{16} \Rightarrow \lambda_3 = -\frac{1}{16\lambda_2}$$

so $\lambda_2 = 1/4$ and $\lambda_3 = -1/4$. We know that the left eigenvector $\langle \boldsymbol{\phi}^1|$ associated to $\lambda_1 = 1$ is $\langle \mathbf{1}|$. The other eigenvectors are found, up to a global factor, by solving the equations for the right and left eigenvectors, respectively, i.e., $\boldsymbol{Q}|\boldsymbol{\psi}^a\rangle = \lambda_a |\boldsymbol{\psi}^a\rangle$ and $\langle \boldsymbol{\phi}^a|\boldsymbol{Q} = \langle \boldsymbol{\phi}^a|\lambda_a \ \forall \ a = 1, 2, 3$. By requiring the normalization $\langle \boldsymbol{\phi}^a|\boldsymbol{\psi}^b\rangle = \delta_{ab}$, which fixes the factor in front of $|\boldsymbol{\psi}^1\rangle$, we have

$$\langle \boldsymbol{\phi}^1| = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \qquad |\boldsymbol{\psi}^1\rangle = \begin{pmatrix} 1/15 \\ 8/15 \\ 2/5 \end{pmatrix}, \tag{2}$$

$$\langle \boldsymbol{\phi}^2| = \begin{pmatrix} 4 & 1 & -2 \end{pmatrix}, \qquad |\boldsymbol{\psi}^2\rangle = \begin{pmatrix} 1/12 \\ 1/6 \\ -1/4 \end{pmatrix}, \tag{3}$$

$$\langle \boldsymbol{\phi}^3| = \begin{pmatrix} 12 & -3 & 2 \end{pmatrix}, \qquad |\boldsymbol{\psi}^3\rangle = \begin{pmatrix} 1/20 \\ -1/10 \\ 1/20 \end{pmatrix}. \tag{4}$$

Let us derive these eigenvectors explicitly for the sake of completeness. We know that we *always* have $\langle \boldsymbol{\phi}^1| = (\,1\;1\;1\,)$. Then to compute $|\boldsymbol{\psi}^1\rangle$, we compute straightforwardly $\boldsymbol{Q}|\boldsymbol{\psi}^1\rangle = |\boldsymbol{\psi}^1\rangle$, i.e.,

$$\begin{pmatrix} 0 & 1/8 & 0 \\ 1 & 1/2 & 1/2 \\ 0 & 3/8 & 1/2 \end{pmatrix} \begin{pmatrix} \psi_1^1 \\ \psi_2^1 \\ \psi_3^1 \end{pmatrix} = \begin{pmatrix} \frac{1}{8}\psi_2^1 \\ \psi_1^1 + \frac{1}{2}\psi_2^1 + \frac{1}{2}\psi_3^1 \\ \frac{3}{8}\psi_2^1 + \frac{1}{2}\psi_3^1 \end{pmatrix} = \begin{pmatrix} \psi_1^1 \\ \psi_2^1 \\ \psi_3^1 \end{pmatrix}$$

The first equation implies $\psi_2^1 = 8\psi_1^1$, and from the second we get $\psi_1^1 + \frac{1}{2}\psi_2^1 + \frac{1}{2}\psi_3^1 = \psi_1^1 + 4\psi_1^1 + \frac{1}{2}\psi_3^1 = \psi_2^1 = 8\psi_1^1 \Rightarrow \psi_3^1 = 6\psi_1^1$. Note that if we plug what we got in the *third* equation, we will not manage to find ψ_1^1: this is always the case as eigenvectors are defined up to a multiplicative constant. To fix ψ_1^1 we need to use $\langle \mathbf{1}|\boldsymbol{\psi}^1\rangle = 1 \Rightarrow \psi_1^1 + \psi_2^1 + \psi_3^1 = 15\psi_1^1 = 1 \Rightarrow \psi_1^1 = \frac{1}{15}$, and from this $\psi_2^1 = \frac{8}{15}$ and $\psi_1^1 = \frac{6}{15} = \frac{2}{5}$. Let us now move to the *second* pair of eigenvalues, the one corresponding to $\lambda_2 = 1/4$. Again, to compute $|\boldsymbol{\psi}^2\rangle$, we compute straightforwardly $\boldsymbol{Q}|\boldsymbol{\psi}^1\rangle = 1/4|\boldsymbol{\psi}^1\rangle$, i.e.,

$$\begin{pmatrix} 0 & 1/8 & 0 \\ 1 & 1/2 & 1/2 \\ 0 & 3/8 & 1/2 \end{pmatrix} \begin{pmatrix} \psi_1^2 \\ \psi_2^2 \\ \psi_3^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{8}\psi_2^2 \\ \psi_1^2 + \frac{1}{2}\psi_2^2 + \frac{1}{2}\psi_3^2 \\ \frac{3}{8}\psi_2^2 + \frac{1}{2}\psi_3^2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \psi_1^2 \\ \psi_2^2 \\ \psi_3^2 \end{pmatrix}$$

The first equation implies $\psi_2^2 = 2\psi_1^2$, and from the second we get $\psi_1^2 + \frac{1}{2}\psi_2^2 + \frac{1}{2}\psi_3^2 = \psi_1^2 + \psi_1^2 + \frac{1}{2}\psi_3^2 = \frac{1}{4}\psi_2^2 = \frac{1}{2}\psi_1^2 \Rightarrow \psi_3^2 = -3\psi_1^2$. Once again, the last equation does not give us information on the value of ψ_1^2. Let us compute $\langle \boldsymbol{\phi}^2|$, by using

$$\begin{pmatrix} \phi_1^2 & \phi_2^2 & \phi_3^2 \end{pmatrix} \begin{pmatrix} 0 & 1/8 & 0 \\ 1 & 1/2 & 1/2 \\ 0 & 3/8 & 1/2 \end{pmatrix} = \begin{pmatrix} \phi_2^2 & \frac{\phi_1^2 + 4\phi_2^2 + 3\phi_3^2}{8} & \frac{\phi_2^2 + \phi_3^2}{2} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \phi_1^2 & \phi_2^2 & \phi_3^2 \end{pmatrix}.$$

We immediately obtain $\phi_2^2 = \frac{1}{4}\phi_1^2$ and $\phi_3^2 = -\frac{1}{2}\phi_1^2$. Again, the last equation does not tell us anything about ϕ_1^2. If we use $\langle \boldsymbol{\phi}^2|\boldsymbol{\psi}^2\rangle = 1$ we obtain $\phi_1^2\psi_1^2 + \frac{1}{2}\phi_1^2\psi_1^2 + \frac{3}{2}\phi_1^2\psi_1^2 = 1$, i.e., $\psi_1^2 = \frac{1}{3\phi_1^2}$, so we expressed one constant in term of the other. However, ϕ_1^2 is left arbitrary and indeed *we can pick it at our choice*: the solution presented above has, for example, $\phi_1^2 = 4$, but a different choice (e.g., $\phi_1^2 = 1$) is acceptable as well. Repeating the arguments for the third eigenvalue we can identify the last pair: the same ambiguity will appear, and we will have to fix an arbitrary constant. Note that this ambiguity would not be there if $\boldsymbol{Q}$ were symmetric: in this case the constant is fixed by the condition that *left and right eigenvectors have the same entries*. Once we have eigenvectors and eigenvalues of $\boldsymbol{Q}$, we can decompose it as

$$\boldsymbol{Q} = \sum_{a=1}^{3} \lambda_a |\boldsymbol{\psi}^a\rangle\langle \boldsymbol{\phi}^a| = |\boldsymbol{\psi}^1\rangle\langle \boldsymbol{\phi}^1| + \frac{1}{4}|\boldsymbol{\psi}^2\rangle\langle \boldsymbol{\phi}^2| - \frac{1}{4}|\boldsymbol{\psi}^3\rangle\langle \boldsymbol{\phi}^3|$$

so that, for example, $Q_{ij} = \sum_{a=1}^{3} \lambda_a \psi_i^a \phi_j^a$. Note that $\sum_i \psi_i^a = \delta_{a,1}$.

Starting now from the definition of the Markov Chain, $|\boldsymbol{P}(t)\rangle = \boldsymbol{Q}^t|\boldsymbol{P}(0)\rangle$, assuming $P_i(0) = \delta_{i,1}$, then $P_i(t) = (\boldsymbol{Q}^t)_{i1}$. Therefore

$$P_3(3) = (\boldsymbol{Q}^3)_{31} = \sum_a \lambda_a^3 \psi_3^a \phi_1^a = 1\cdot 1\cdot \frac{2}{5} + \left(\frac{1}{4}\right)^3 \cdot 4\cdot \left(-\frac{1}{4}\right) + \left(-\frac{1}{4}\right)^3 \cdot 12\cdot \frac{1}{20} = \frac{3}{8}.$$

Moreover

$$\lim_{t\to\infty} P_3(t) = \psi_3^1 = \frac{2}{5}.$$