TUTORIAL 07

7CCMCS04
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Problem 7.1 Consider the chemical reaction

$$\mathrm{X} \xrightarrow{c_1} \mathrm{X}+\mathrm{X}, \qquad \mathrm{X}+\mathrm{X} \xrightarrow{c_2} \varnothing$$

where c_1 and c_2 are rates of the reactions. Let us denote N_t the number of components X at time t.

a. Show that the master equation is

$$\dot{P}_n = c_2 \frac{(n+2)(n+1)}{2} P_{n+2} + c_1(n-1)P_{n-1} - c_1 n P_n - c_2 \frac{n(n-1)}{2} P_n.$$

b. Derive the equation for the first moment $\mathbb{E}[\mathsf{N}_t]$ and explain why this does not close.

c. Show that closure of the equation is attained by assuming that fluctuations are negligible, i.e. $\mathbb{E}[\mathsf{N}_t^2] \simeq \mathbb{E}[\mathsf{N}_t]^2$.

d. Find the fixed points of the dynamics and characterize their stability. Discuss your result in the limit $c_1 \ll c_2$ and $c_1 \gg c_2$. Explain why the behaviour of the system around the unstable fixed point resulting from the deterministic analysis is not accurate.

e. Explain how you would close the equations assuming that fluctuations are Gaussian.

Problem 7.2 Consider a large population where offspring production occurs with rate λ and spontaneous death occurs at rate μ. Assume the initial size of the population at time $t=0$ is n_0. Write the master equation governing the evolution of the probability density $P_n(t)$ to have $\mathsf{N}_t = n$ individuals at time t. Use the master equation to write a dynamical equation for the generating function $F(z,t) = \sum_{n=0}^{\infty} z^n P_n(t)$ and solve it by using the method of characteristics, for $\mu \neq \lambda$ and $\mu = \lambda$. Show that the extinction probability $P_0(t)$ decays to its equilibrium value exponentially for $\mu \neq \lambda$ and as a power law for $\mu = \lambda$. This phenomenon is known as the "critical slowing down" taking place when a system is close to its critical point $\rho = \frac{\mu}{\lambda} = 1$.

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