TUTORIAL 08

7CCMCS04
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✎ **Problem 8.1** A neural network of N neurons $\sigma_i = \pm 1 \quad i = 1, \dots, N$ evolves according to parallel Glauber dynamics, so that the probability $P_{\boldsymbol{\sigma}}(t)$ to find the system in configuration $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)$ evolves according to the Markov chain

$$P_{\boldsymbol{\sigma}}(t+1) = \sum_{\boldsymbol{\sigma}'} Q_{\boldsymbol{\sigma}\,\boldsymbol{\sigma}'} P_{\boldsymbol{\sigma}'}(t)$$

with

$$Q_{\boldsymbol{\sigma}\,\boldsymbol{\sigma}'} = \prod_{i=1}^{N} \frac{\mathrm{e}^{\beta \sigma_i h_i(\boldsymbol{\sigma}')}}{2\cosh(\beta h_i(\boldsymbol{\sigma}'))}, \qquad h_i(\boldsymbol{\sigma}) = \sum_j J_{ij}\sigma_j + \theta_i.$$

a. Show that $\sum_{\boldsymbol{\sigma}} Q_{\boldsymbol{\sigma}\,\boldsymbol{\sigma}'} = 1$.

b. Show that for symmetric interactions $J_{ij} = J_{ji}$ the system satisfies detailed balance with the Boltzmann distribution

$$\Pi_{\boldsymbol{\sigma}} = \frac{1}{Z}\,\mathrm{e}^{-\beta H(\boldsymbol{\sigma})} \qquad Z = \sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H(\boldsymbol{\sigma})},$$

with

$$H(\boldsymbol{\sigma}) = -\sum_i \theta_i \sigma_i - \beta^{-1} \sum_i \ln\cosh(\beta h_i(\boldsymbol{\sigma})).$$

✎ **Problem 8.2** A neural network of N neurons $\sigma_i = \pm 1$, $i = 1, \dots, N$ evolves according to sequential Glauber dynamics so that at every time step $\tau = N^{-1}$ a spin σ_i is picked at random and updated according to the stochastic rule

$$\sigma_i(t+\tau) = \begin{cases} -\sigma_i(t) & \text{with probability } W_i(\boldsymbol{\sigma}(t)) \\ \sigma_i(t) & \text{with probability } 1 - W_i(\boldsymbol{\sigma}(t)) \end{cases}$$

where

$$W_i(\boldsymbol{\sigma}) = \frac{1 - \sigma_i \tanh \beta h_i(\boldsymbol{\sigma})}{2}$$

and we have denoted $\beta = T^{-1}$ the inverse temperature and h_i the effective local field

$$h_i(\boldsymbol{\sigma}) = \sum_{j \neq i} J_{ij}\sigma_j.$$

(a) Show that for $N \gg 1$ the microstate distribution $P_{\boldsymbol{\sigma}}(t)$ evolves according to the master equation

$$\dot{P}_{\boldsymbol{\sigma}}(t) = \sum_i \left(W_i(F_i\boldsymbol{\sigma}) P_{F_i\boldsymbol{\sigma}}(t) - W_i(\boldsymbol{\sigma}) P_{\boldsymbol{\sigma}}(t) \right),$$

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where F_i is the i-spin flip operator $F_i\boldsymbol{\sigma}=(\sigma_1,...,\sigma_{i-1},-\sigma_i,\sigma_{i+1},...,\sigma_N)$.

(b) Prove that the Glauber rates $W_i(\boldsymbol{\sigma})$ satisfy detailed balance with the Gibbs–Boltzmann distribution

$$\Pi_{\boldsymbol{\sigma}} = \frac{\mathrm{e}^{-\beta H(\boldsymbol{\sigma})}}{\sum_{\boldsymbol{\sigma}'} \mathrm{e}^{-\beta H(\boldsymbol{\sigma}')}}.$$

with

$$H(\boldsymbol{\sigma}) = -\frac{1}{2}\sum_{k\neq\ell} J_{kl}\sigma_k\sigma_\ell$$

for symmetric synaptic interaction without self-interactions, i.e., $J_{ii} = 0 \ \forall\ i$ and $J_{ij} = J_{ji} \ \forall\ i, j$.