TUTORIAL 02

7CCMCS04
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Problem 2.1 Assume that the weather of a city can be described by a variable X_t which takes values in the set $\Omega = \{1, 2\}$, where 1 denotes rain and 2 denotes sunshine. Its value is updated on a daily basis. We denote Q_{ij} the probability that the weather changes from state j to state i in one day. Assume that

$$\boldsymbol{Q} = \begin{pmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{pmatrix}.$$

a. Assume that the first day is rainy. What is the probability to have rain on the second day? And on the t-th day?

b. Assume that the probability to have rain on the first day is p. What is the probability to observe rain on the first day and sunshine on the second and third day? What is the probability to observe sunshine on the first three days? What is the probability to observe sunshine on the second and third day?

c. Find the eigenvalues λ_1 and λ_2, with $|\lambda_1| > |\lambda_2|$, and the right and left eigenvectors of $\boldsymbol{Q}$, given respectively by $|\boldsymbol{\psi}^a\rangle$ and $\langle\boldsymbol{\phi}^a|$, with $a = 1, 2$. Show that they are bio-orthogonal. Write them in orthonormal form.

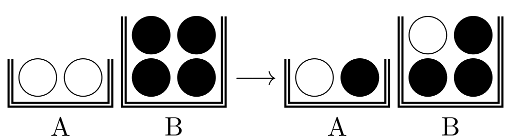
d. If the probability to have rain on day 1 is $\frac{\beta}{\alpha+\beta}$, what is the probability to have rain on day 2? Why?

e. Write $\boldsymbol{Q}$ in spectral form. Show that the relaxation time is a function of $\alpha + \beta$.

f. Does the stationary distribution satisfy detailed balance?

(g) What is the value of λ_2 when $\alpha = \beta = 1$? does the dynamics converge to the stationary distribution in this case? Why? What is the value of λ_2 when $\alpha = \beta = 0$? What happens in this case?

Problem 2.2 There are two white marbles is urn A and four black marbles in urn B which can be interchanged. At each step of the process a marble is selected at random from each urn and the two marbles selected are interchanged, for example



a. Find the transition matrix $\boldsymbol{Q}$, and obtain its eigenvalues and its left and right eigenvectors. Make a list of the properties of eigenvalues and

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eigenvectors which are a consequence of the fact that $\boldsymbol{Q}$ is a stochastic matrix.

b. Express the probability vector $|\boldsymbol{P}(t)\rangle$ in terms of the left and right eigenvectors of the transition matrix.

c. What is the probability that there are two red marbles in urn A after three steps? And after many steps?