TUTORIAL 06 — SOLUTIONS

7CCMCS04
A. ANNIBALE AND G. SICURO

Problem 6.1

We have
$$\partial_t P_n(t) = \mu P_{n+1}(t) + \mu P_{n-1}(t) - 2\mu P_n(t).$$
Let us introduce the generating function $F(z,t) = \sum_n P_n(t) z^n$. Using the master equation
$$\partial_t F(z,t) = \frac{\mu}{z} F(z,t) + \mu z F(z,t) - 2\mu F(z,t)$$
solved by
$$F(z,t) = F(z,0)\,\mathrm{e}^{t\mu(z+1/z-2)}\,.$$
For $P_n(0) = \delta_{n\,0}$, we have $F(z,0) = 1$. Expanding in powers of z
$$F(z,t) = \mathrm{e}^{t\mu(z+1/z-2)} = \mathrm{e}^{-2\mu t} \sum_{k,n=0}^{\infty} \frac{(t\mu)^{k+n} z^{k-n}}{k!n!} = \mathrm{e}^{-2\mu t} \sum_{n=-\infty}^{\infty} z^n \sum_{k=0}^{\infty} \frac{(t\mu)^{n+2k}}{(n+k)!k!}$$
so that
$$P_n(t) = \mathrm{e}^{-2t\mu}\, I_n(2t\mu).$$
The properties of $I_n(x)$ follow from properties of $P_n(t)$ and from the master equation for $P_n(t)$.

Date: February 13, 2025.

Problem 6.2

a. Let $P_n(t)$ be the probability that there are n surviving nuclei at time t. If γ is the decay probability per unit time for one nucleus, then the probability that one nucleus decay during an infinitesimal time interval Δt is $\gamma\Delta t$ at leading order. Assuming that there are n nuclei, the probability that there is one decay is $\gamma n\Delta t$ (and the probability that there is no decay $1-\gamma n\Delta t$) at the leading order. The transition probability from m to n in a short time Δt is therefore

$$\mathbb{P}_{1|1}[\mathsf{N}_{t+\Delta t}=n|\mathsf{N}=m]=\delta_{m,n}\underbrace{(1-\gamma m\Delta t)}_{\text{No one of } m \text{ nuclei decay}}+\delta_{m-1,n}\underbrace{\gamma m\Delta t}_{\text{One of } m \text{ nuclei decays}}+O(\Delta t^2).$$

b. Summing over all possible states at time t and weighting with the probability of occurrence of each configuration we get

$$P_n(t+\mathrm{d}\,t)=\sum_m \mathbb{P}_{1|1}[\mathsf{N}_{t+\mathrm{d}\,t}=n|\mathsf{N}=m]P_m(t)=(n+1)\gamma\,\mathrm{d}\,tP_{n+1}(t)+(1-n\gamma\,\mathrm{d}\,t)P_n(t),$$

which leads to the master equation

$$\dot{P}_n(t)=\gamma(n+1)P_{n+1}(t)-\gamma nP_n(t).$$

c. In order to get an equation for the generating function $F(z,t)=\sum_{n=0}^{\infty}P_n(t)z^n$, we multiply the master equation by z^n and sum over n

$$\begin{aligned}\frac{\partial F}{\partial t}&=\gamma\sum_n(n+1)z^nP_{n+1}(t)-\gamma\sum_n nz^nP_n(t)\\&=\gamma\frac{\partial}{\partial z}\sum_n z^{n+1}P_{n+1}(t)-\gamma z\frac{\partial}{\partial z}\sum_n z^nP_n(t)\\&=\gamma(1-z)\frac{\partial F}{\partial z}\end{aligned}$$

By using the method of characteristics,

$$\frac{\mathrm{d}\,t}{1}=\frac{\mathrm{d}\,z}{\gamma(z-1)}=\frac{\mathrm{d}\,F}{0}.$$

Two integrals are easily obtained considering the systems t, z and z, F:

$$t=\frac{1}{\gamma}\ln|z-1|+c\to(z-1)\,\mathrm{e}^{-\gamma t}=a;\qquad F(z,t)=\phi\equiv\phi(a).$$

Then, the general solution is, in terms of a yet unknown function ϕ

$$F(z,t)=\phi\left((z-1)\,\mathrm{e}^{-\gamma t}\right).$$

From the initial condition $P_1(n,0)=\delta_{n\,n_0}$ we get

$$F(z,0)=z^{n_0}=\phi(z-1)$$

and the functional form of ϕ follows as

$$\phi(z)=(z+1)^{n_0}$$

For $t>0$ we have therefore

$$F(z,t)=\left((z-1)\,\mathrm{e}^{-\gamma t}+1\right)^{n_0}=\sum_{m=0}^{n_0}\binom{n_0}{m}z^m\,\mathrm{e}^{-\gamma mt}(1-\mathrm{e}^{-\gamma t})^{n_0-m}$$

which leads to

$$P_m(t) = \binom{n_0}{m} \mathrm{e}^{-\gamma m t}(1 - \mathrm{e}^{-\gamma t})^{n_0 - m}$$

d. The average number of undecayed nuclei at time t can be obtained using the generating function found in point **c.**,

$$F(z,t) = \sum_n P_n(t) z^n = \big((z-1)\,\mathrm{e}^{-\gamma t} + 1\big)^{n_0}$$

$$\Rightarrow \partial_z F(z,t) = \sum_n n P_n(t) z^{n-1} = n_0\,\mathrm{e}^{-\gamma t}\,\big((z-1)\,\mathrm{e}^{-\gamma t} + 1\big)^{n_0 - 1}.$$

Indeed

$$\mathbb{E}[\mathsf{N}_t] = \sum_n n P_n(t) = \partial_z F(z,t)|_{z=1} = n_0\,\mathrm{e}^{-\gamma t}$$

so that the half-life of the process is $1/\gamma$. For large t

$$\lim_{t\to\infty} F(z,t) = 1 = P(0,\infty) + \sum_{m\geq 1} z^m P_m(\infty),$$

but $P_m(\infty) = 0\ \forall m \geq 1$. So $P_0(\infty) = 1$. Alternatively, one can multiply both sides of the master equation by n and sum over n, thus obtaining

$$\begin{aligned}\frac{\mathrm{d}\,\mathbb{E}[\mathsf{N}_t]}{\mathrm{d}\,t} &= \sum_{n=0}^{\infty} n\dot{P}_n = \gamma \sum_{n=0}^{\infty} n(n+1)P_{n+1} - \gamma \sum_{n=0}^{\infty} n^2 P_n \\ &= \gamma \sum_{n=0}^{\infty}(n-1)nP_n - \gamma \sum_{n=0}^{\infty} n^2 P_n = -\gamma \sum_{n=0}^{\infty} nP_n = -\gamma\mathbb{E}[\mathsf{N}_t].\end{aligned}$$

Solving the above equation for $\mathbb{E}[\mathsf{N}_0] = n_0$ gives

$$\mathbb{E}[\mathsf{N}_t] = n_0\,\mathrm{e}^{-\gamma t}.$$