TUTORIAL 03 — SOLUTIONS

`7CCMCS04`
A. ANNIBALE AND G. SICURO

Problem 3.1

Point **a** is left to the student. On point **b**, a reversible chain satisfies Detailed balance, so $Q_{ij}\Pi_j = Q_{ji}\Pi_i$. This shows that, given the form of $\boldsymbol{Q}$, Π_i has to be proportional to d_i: normalization of $|\boldsymbol{\Pi}\rangle$ leads to the form provided. This implies that a random walker on a graph will visit nodes with a frequency that is proportional to their degrees, hence most of the time is spent on "hubs", i.e., nodes with very high degree.

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Problem 3.2

Let us solve the first two questions together, and move then to question **c**.

a.+b. Let us start writing down the matrix $\boldsymbol{Q}$

$$\boldsymbol{Q} = \begin{pmatrix} 0 & 3/5 & 1/3 \\ 3/4 & 0 & 2/3 \\ 1/4 & 2/5 & 0 \end{pmatrix}$$

The fraction of time spent by the mouse in each room can be extracted by the stationary distribution $|\boldsymbol{\Pi}\rangle$, found from

$$\boldsymbol{Q}|\boldsymbol{\Pi}\rangle = |\boldsymbol{\Pi}\rangle \Rightarrow |\boldsymbol{\Pi}\rangle = \begin{pmatrix} 1/3 \\ 5/12 \\ 1/4 \end{pmatrix}.$$

The mouse spends most of its time in room B (as expected from the fact that this is the most mobile configuration).

c. The transition matrix for the new dynamical process has entries $Q'_{x'x} = a_{x'x} Q_{x'x}$. Detailed balance with the uniform measure $|\boldsymbol{\Pi}\rangle = (1/3, 1/3, 1/3)^\top$ requires a symmetric $\boldsymbol{Q'}$ i.e.

$$a_{x'x} Q_{x'x} = a_{xx'} Q_{xx'}$$

The choice of the rates is not unique. Using Glauber prescription:

$$a_{x'x} = \frac{Q_{xx'}}{Q_{x'x} + Q_{xx'}}$$

This leads to $a_{\mathrm{A|B}} = 5/9$, $a_{\mathrm{A|C}} = 3/7$, $a_{\mathrm{B|A}} = 4/9$, $a_{\mathrm{B|C}} = 3/8$, $a_{\mathrm{C|A}} = 4/7$, $a_{\mathrm{C|B}} = 5/8$. The resulting transition matrix is symmetric as required

$$\boldsymbol{Q'} = \begin{pmatrix} 11/21 & 1/3 & 1/7 \\ 1/3 & 5/12 & 1/4 \\ 1/7 & 1/4 & 17/28 \end{pmatrix}.$$