TUTORIAL 01 — SOLUTIONS

7CCMCS04
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Problem 1.1

Instead of tackling directly the problem, let us summarise some fundamental properties of Gaussian integrals that might help us to answer straight away to the questions in it. A Gaussian integral takes the form

$$\mathcal{I}[f] = \int_{-\infty}^{\infty} \mathrm{d}\,x\, f(x)\,\mathrm{e}^{-\frac{ax^2}{2}}$$

where $f(x)$ is an arbitrary function of x. We start with the easiest case where $f(x) = 1$ and consider the integral $\mathcal{I}[1]$. This is equal to

$$\mathcal{I}[1] = \sqrt{\frac{2\pi}{a}}.$$

This fundamental fact can be shown by calculating

$$\begin{aligned}\mathcal{I}[1] = \sqrt{\mathcal{I}^2[1]} &= \sqrt{\int_{-\infty}^{\infty} \mathrm{d}\,x\ \mathrm{e}^{-\frac{ax^2}{2}} \int_{-\infty}^{\infty} \mathrm{d}\,y\ \mathrm{e}^{-\frac{ay^2}{2}}} = \sqrt{\iint_{-\infty}^{\infty} \mathrm{d}\,x\,\mathrm{d}\,y\ \mathrm{e}^{-a\frac{x^2+y^2}{2}}} \\ &= \sqrt{\int_0^{2\pi} \mathrm{d}\,\theta \int_0^{\infty} r\,\mathrm{d}\,r\ \mathrm{e}^{-a\frac{r^2}{2}}} = \sqrt{\frac{2\pi}{a}}.\end{aligned}$$

In the second line, we have switched polar coordinates using $x = r\cos\theta$ and $y = r\sin\theta$. We can now easily calculate Gaussian integrals where an additional linear term is present in the exponent, i.e., something of the type

$$\int \mathrm{d}\,x\ \mathrm{e}^{-\frac{ax^2}{2}+bx}.$$

By completing the square in the exponent, one finds

$$\int \mathrm{d}\,x\ \mathrm{e}^{-\frac{ax^2}{2}+bx} = \sqrt{\frac{2\pi}{a}}\,\mathrm{e}^{\frac{b^2}{2a}}$$

Note now that if we consider instead $f(x) = x^n$, we can evaluate $\mathcal{I}[x^n]$ as

$$\mathcal{I}[x^n] = \frac{\partial^n}{\partial b^n} \int \mathrm{d}\,x\ \mathrm{e}^{-\frac{ax^2}{2}+bx}\Big|_{b=0} = \sqrt{\frac{2\pi}{a}}\frac{\partial^n\,\mathrm{e}^{\frac{b^2}{2a}}}{\partial b^n}\Big|_{b=0}.$$

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Recall now that a random variable X has a Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$ $p(x)$ if

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

which is of course normalised to $1 = \int \mathrm{d}\,x\,p(x)$. Similarly, a multi-variate Gaussian distribution $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for a vectorial random variable $\mathbf{X}^\top = (\mathsf{X}_1, \ldots, \mathsf{X}_n)$ has the form

$$p(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^n \det \boldsymbol{\Sigma}}}\,\mathrm{e}^{-\frac{1}{2}\boldsymbol{x}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{x}}$$

where $\boldsymbol{\Sigma}$ is positive-definite and symmetric. Let us check that this is correctly normalised, i.e., $\int \mathrm{d}\,\boldsymbol{x} p(\boldsymbol{x}) = 1$. The matrix $\boldsymbol{\Sigma}$ is real and symmetric, so there exists a base of orthogonal eigenvectors. Let $\boldsymbol{O}$ the matrix whose columns are the eigenvectors of $\boldsymbol{\Sigma}$, so that $\boldsymbol{O}\boldsymbol{\Sigma}\boldsymbol{O}^{-1} \equiv= \boldsymbol{\Lambda}$ where $\boldsymbol{\Lambda}$ is the diagonal matrix of the eigenvalues,

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_N \end{pmatrix}$$

Setting now $\boldsymbol{x} = \boldsymbol{O}^\top \boldsymbol{z}$, since $\boldsymbol{O}$ is orthogonal, then $\mathrm{d}\,\boldsymbol{x} = \mathrm{d}\,\boldsymbol{z}$ and

$$\int \mathrm{d}\,\boldsymbol{x}\,\mathrm{e}^{-\frac{1}{2}\boldsymbol{x}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{x}} = \int \mathrm{d}\,\boldsymbol{z}\,\mathrm{e}^{-\frac{1}{2}\boldsymbol{z}^\top\boldsymbol{O}\boldsymbol{\Sigma}^{-1}\boldsymbol{O}^\top\boldsymbol{z}} = \int \mathrm{d}\,\boldsymbol{z}\,\mathrm{e}^{-\frac{1}{2}\boldsymbol{z}^\top\boldsymbol{\Lambda}^{-1}\boldsymbol{z}}$$
$$= \prod_i \int \mathrm{d}\,z_i\,\mathrm{e}^{-\frac{1}{2}\sum_i \lambda_i^{-1} z_i^2} = \prod_i \int \mathrm{d}\,z_i\,\mathrm{e}^{-\frac{1}{2}\lambda_i^{-1} z_i^2} = \prod_i \sqrt{2\pi\lambda_i} = \sqrt{(2\pi)^n \det \boldsymbol{\Sigma}}$$

All odd moments are zero by symmetry, $\mathbb{E}[\mathsf{X}_i] = 0$. Even moments are non-zero, but, remarkably, all even moments higher than the second, can be expressed in terms of second order moments by the so-called Wick theorem — see notes on prerequisites material. Here we just show that $\mathbb{E}[\mathsf{X}_i\mathsf{X}_j] = \Sigma_{ij}$. To make the expression lighter, let us call $\boldsymbol{A} = \boldsymbol{\Sigma}^{-1}$ and remember that $\det \boldsymbol{A} = \frac{1}{\det \boldsymbol{\Sigma}}$, so

$$\mathbb{E}[\mathsf{X}_i\mathsf{X}_j] = \sqrt{\frac{\det \boldsymbol{A}}{(2\pi)^n}} \int \mathrm{d}\,\boldsymbol{x}\,x_i x_j\,\mathrm{e}^{-\frac{1}{2}\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x}} = \sqrt{\frac{\det \boldsymbol{A}}{(2\pi)^n}} \int \mathrm{d}\,\boldsymbol{x}\,x_i x_j\,\mathrm{e}^{-\frac{1}{2}\sum_{uv} x_u A_{uv} x_v}$$
$$= -2\sqrt{\frac{\det \boldsymbol{A}}{(2\pi)^n}}\frac{\partial}{\partial A_{ij}} \int \mathrm{d}\,\boldsymbol{x}\,\mathrm{e}^{-\frac{1}{2}\sum_{uv} x_u A_{uv} x_v} = -2\sqrt{\frac{\det \boldsymbol{A}}{(2\pi)^n}}\frac{\partial}{\partial A_{ij}}\sqrt{\frac{(2\pi)^n}{\det \boldsymbol{A}}}$$
$$= -2\sqrt{\det \boldsymbol{A}}\frac{\partial}{\partial A_{ij}}\,\mathrm{e}^{-\frac{1}{2}\ln\det\boldsymbol{A}} = (\boldsymbol{A}^{-1})_{ji}$$

where we have used the identity

$$\frac{\partial}{\partial A_{ij}} \ln \det \boldsymbol{A} = (\boldsymbol{A}^{-1})_{ji}$$

But $\boldsymbol{A}^{-1} = \boldsymbol{\Sigma}$, hence the result. For this reason the matrix $\boldsymbol{\Sigma}$ is called the *co-variance matrix.* Problem 1.1 is easily solved by using the properties of Gaussian distributions summarised above.

Problem 1.2

The process is non-stationary (p_1 depends on time); homogeneous (transition probabilities only depends on the system's state before and after the transition and the duration of the transition steps, not on the time at which the transition has occurred); Gaussian.

a. To get the answer, start from

$$p_1(x,t) = \int \mathrm{d}\, x' p_2(x,t;x',t') = \int \mathrm{d}\, x' p_{1|1}(x,t|x',t')p_1(x',t')$$

and, setting $t'=0$, impose $p_1(x,0)=\delta(x)$.

b. We have

$$\mathbb{E}[\mathsf{X}_{t_1}] = \int \mathrm{d}\, x_1\, x_1 p_1(x_1,t_1) = 0.$$

For $t_2 > t_1$, instead

$$\begin{aligned}\mathbb{E}[\mathsf{X}_{t_1}\mathsf{X}_{t_2}] &= \iint \mathrm{d}\, x_1\, \mathrm{d}\, x_2\, x_1 x_2 p_2(x_1,t_1;x_2,t_2) \\ &= \int \mathrm{d}\, x_1\, x_1 p_1(x_1,t_1) \int \mathrm{d}\, x_2\, x_2 p_{1|1}(x_2,t_2|x_1,t_1) \\ &= \int \mathrm{d}\, x_1 x_1^2 p_1(x_1,t_1) = t_1.\end{aligned}$$

For $t_1 > t_2$, we would have obtained $\mathbb{E}[\mathsf{X}_{t_1}\mathsf{X}_{t_2}] = t_2$.

c. The answer is obtained by direct inspection, substituting $p_1(x,t)$ in Eq. (**??**).

d. From Eq. (**??**)

$$\frac{\partial \mathbb{E}[\mathsf{X}_t]}{\partial t} = \int x \frac{\partial p_1(x,t)}{\partial t}\, \mathrm{d}\, x = \int x \frac{\partial^2 p_1(x,t)}{\partial x^2}\, \mathrm{d}\, x = 0$$

by integrating by parts. Similarly, by the same trick,

$$\frac{\partial \mathbb{E}[\mathsf{X}_t^2]}{\partial t} = \int x^2 \frac{\partial p_1(x,t)}{\partial t}\, \mathrm{d}\, x = \int x^2 \frac{\partial^2 p_1(x,t)}{\partial x^2}\, \mathrm{d}\, x = 1.$$

e. Check the prerequisite material, in particular the Chapter on PDEs and the solution of the heat equation therein.

1. Problem 1.3

Expanding,

$$p_2(x_2, t+\tau; x_1, t) = \frac{1}{\sqrt{2\pi(1-\mathrm{e}^{-2\tau})}}\,\mathrm{e}^{-\frac{x_1^2+x_2^2-2x_1x_1\,\mathrm{e}^{-\tau}}{2(1-\mathrm{e}^{-2\tau})}}$$

hence this is the expression of a bivariate Gaussian distribution with

$$\boldsymbol{\Sigma}^{-1} = \frac{1}{1-\mathrm{e}^{-2\tau}}\begin{pmatrix} 1 & -\mathrm{e}^{-\tau} \\ -\mathrm{e}^{-\tau} & 1 \end{pmatrix}$$

and therefore

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & \mathrm{e}^{-\tau} \\ \mathrm{e}^{-\tau} & 1 \end{pmatrix}.$$

So for $\tau > 0$, $C(t, t+\tau) \coloneqq \langle\!\langle \mathsf{X}_t \mathsf{X}_{t+\tau} \rangle\!\rangle = \mathrm{e}^{-\tau}$. Note that $C(t, t+\tau) \to 1$ as $\tau \ll 1$, while $C(t, t+\tau) \to 0$ as $\tau \gg 1$ (i.e., velocities are strongly correlated for short time differences, while all memory is lost for large time differences).