TUTORIAL 09 — SOLUTIONS

7CCMCS04
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Problem 9.1

a. Multiplying the master equation by σ_i and summing over i

$$\begin{aligned}\partial_t \mathbb{E}[\sigma_i] &= \sum_{\boldsymbol{\sigma}} \sum_j [P_{F_j\boldsymbol{\sigma}}(t) W_j(F_j\boldsymbol{\sigma}) - P_{\boldsymbol{\sigma}}(t) W_j(\boldsymbol{\sigma})]\sigma_i \\ &= \sum_{\boldsymbol{\sigma}} \sum_j P_{\boldsymbol{\sigma}}(t) W_j(\boldsymbol{\sigma})\,(F_j\sigma_i - \sigma_i) = -2\sum_{\boldsymbol{\sigma}} P_{\boldsymbol{\sigma}}(t) W_i(\boldsymbol{\sigma})\sigma_i \\ &= -2\mathbb{E}[\sigma_i W_i(\boldsymbol{\sigma})] = -\mathbb{E}[\sigma_i] + \mathbb{E}[\tanh \beta h_i(\boldsymbol{\sigma})] = -\mathbb{E}[\sigma_i] + \mathbb{E}[\tanh \beta J m_{\boldsymbol{\sigma}}].\end{aligned}$$

b. Summing over i and dividing by N the equation for $\mathbb{E}[\sigma_i]$, we get

$$\partial_t m = -m + \frac{1}{N}\sum_i \mathbb{E}[\tanh \beta J m_{\boldsymbol{\sigma}}] = -m + \mathbb{E}[\tanh \beta J m_{\boldsymbol{\sigma}}].$$

Note that $m_{\boldsymbol{\sigma}} = N^{-1}\sum_i \sigma_i$ depends on the configuration $\boldsymbol{\sigma}$, while m does not: m is the thermodynamic average of $m_{\boldsymbol{\sigma}}$, i.e. it is the average of $m_{\boldsymbol{\sigma}}$ over all the possible configurations $\boldsymbol{\sigma}$, drawn from their distribution $P_{\boldsymbol{\sigma}}(t)$. The above equation expresses the evolution of m in terms of $m^\star$, hence, in principle, it does not close.

However, since $m_{\boldsymbol{\sigma}}$ is the sum of a large number of variables, by the law of large numbers, when the variables are independent, their sum will converge to the thermodynamic average (i.e. average over all realizations). Hence, one can assume that the deviations of $m_{\boldsymbol{\sigma}}$ (in a given realization $\boldsymbol{\sigma}$) from the thermodynamics average m over all realizations are small, i.e. $\delta m_{\boldsymbol{\sigma}} = m_{\boldsymbol{\sigma}} - m \ll 1$. This assumption is reasonable when the variables σ_i are independent, i.e., away from criticality: at criticality, we have seen that long-correlations develop in the system and this assumption is no longer valid. Expanding the last term on the RHS of the above equation, about m, to orders Δ, gives

$$\mathbb{E}[\tanh \beta J m_{\boldsymbol{\sigma}}] = \tanh \beta J m + \mathbb{E}[\delta m_{\boldsymbol{\sigma}}]\beta J\big(1 - \tanh^2 \beta J m\big) = \tanh \beta J m.$$

as $\mathbb{E}[\delta m_{\boldsymbol{\sigma}}] = 0$ by definition. This leads to a closed relation for m

$$\partial_t m = -m + \tanh \beta J m.$$

c. The stationary state is found from

$$m = \tanh \beta J m$$

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A sketch of $m = \tanh x$ and $m = x/\beta J$ shows that $m = 0$ is the only solution for $\beta J < 1$, whereas for $\beta J > 1$ two non-zero solutions appear at $\pm m_T$, with $m_T \neq 0$. Linearizing the dynamical equation about $m = 0$

$$\partial_t m = m(-1 + \beta J)$$

shows that $m(t) = m_0\, \mathrm{e}^{-(1-\beta J)t}$, so $m = 0$ is stable for $1 - \beta J > 0$ (i.e. $T > J$) and unstable for $T < J$.

Problem 9.2

a. Multiplying the master equation by σ_i and summing over i

$$\begin{aligned}
\partial_t \mathbb{E}[\sigma_i] &= \sum_{\boldsymbol{\sigma}} \sum_j [P_{F_j\boldsymbol{\sigma}}(t) W_j(F_j\boldsymbol{\sigma}) - P_{\boldsymbol{\sigma}}(t) W_j(\boldsymbol{\sigma})]\sigma_i \\
&= \sum_{\boldsymbol{\sigma}} \sum_j P_{\boldsymbol{\sigma}}(t) W_j(\boldsymbol{\sigma}) \big(F_j\sigma_i - \sigma_i\big) = -2 \sum_{\boldsymbol{\sigma}} P_{\boldsymbol{\sigma}}(t) W_i(\boldsymbol{\sigma})\sigma_i \\
&= -2\mathbb{E}[\sigma_i W_i(\boldsymbol{\sigma})] = -\mathbb{E}[\sigma_i] + \mathbb{E}[\tanh \beta h_i(\boldsymbol{\sigma})] \\
&= -\mathbb{E}[\sigma_i] + \mathbb{E}[\tanh \beta J(\sigma_{i+1} + \sigma_{i-1})] = -\mathbb{E}[\sigma_i] + \mathbb{E}\left[\tanh\left(2\beta J \frac{\sigma_{i+1}+\sigma_{i-1}}{2}\right)\right] \\
&= -\mathbb{E}[\sigma_i] + \tanh(2\beta J)\mathbb{E}\left[\frac{\sigma_{i+1}+\sigma_{i-1}}{2}\right] = -m_i + \gamma \frac{m_{i+1}+m_{i-1}}{2}
\end{aligned}$$

where in the last line we used that $\frac{\sigma_{i+1}+\sigma_{i-1}}{2} \in \{0, \pm 1\}$, $\tanh(\epsilon x) = \epsilon \tanh x \;\forall\, \epsilon \in \{0, \pm 1\}$ and $\gamma = \tanh 2\beta J$.

b. Multiplying the equation for z^i and summing over i, we get an equation for the generating function $G(z,t) = \sum_n z^n m_n(t)$

$$\partial_t G(z,t) = \left(\gamma \frac{z + {}^1\!/\!_z}{2} - 1\right) G(z,t) \Rightarrow G(z,t) = G(z,0)\, \mathrm{e}^{\gamma \frac{z+{}^1\!/\!_z}{2} t - t}$$

From $m_n(0) = \delta_{n,0}$, $G(z,0) = 1$ so $G(z,t) = \mathrm{e}^{\gamma\frac{z+{}^1\!/\!_z}{2}t-t}$. Using the definition $\sum_{n=-\infty}^{\infty} I_n(t) z^n = \mathrm{e}^{\frac{z+{}^1\!/\!_z}{2}t}$, we get

$$G(z,t) = \sum_{n=-\infty}^{\infty} I_n(t) z^n\, \mathrm{e}^{-t}$$

hence $m_n(t) = I_n(\gamma t)\, \mathrm{e}^{-t}$.

c. Using the asymptotics of $I_n(t)$ for large t $I_n(t) \sim \frac{\mathrm{e}^t}{\sqrt{2\pi t}}\, \mathrm{e}^{-\frac{n^2}{2t}}$, we finally get $m_n(t) \sim \frac{\mathrm{e}^{t(\gamma-1)}}{\sqrt{2\pi t\gamma}}\, \mathrm{e}^{-\frac{n^2}{2t\gamma}}$. At $T = 0$, where $\gamma = 1$,

$$m_n(t) \sim \frac{1}{\sqrt{2\pi t}}\, \mathrm{e}^{-\frac{n^2}{2t}} \Rightarrow m_n(t) \sim t^{-1/2} f\left(\frac{n}{L(t)}\right)$$

displays the typical self-similarity of coarsening, due to the growth of a length-scale $L(t) \sim \sqrt{t}$.

d. Multiplying the master equation by $\sigma_i\sigma_j$ and summing over i

$$\begin{aligned}
\partial_t\mathbb{E}[\sigma_i\sigma_j] &= \sum_{\boldsymbol{\sigma}}\sum_k [P_{F_k\boldsymbol{\sigma}}(t)W_k(F_k\boldsymbol{\sigma}) - P_{\boldsymbol{\sigma}}(t)W_k(\boldsymbol{\sigma})]\sigma_i\sigma_j \\
&= \sum_{\boldsymbol{\sigma}}\sum_k P_{\boldsymbol{\sigma}}(t)W_k(\boldsymbol{\sigma})[F_k(\sigma_i\sigma_j - \sigma_i\sigma_j] \\
&= -2\sum_{\boldsymbol{\sigma}} P_{\boldsymbol{\sigma}}(t)[W_i(\boldsymbol{\sigma})\sigma_i\sigma_j + W_j(\boldsymbol{\sigma})\sigma_j\sigma_i] \\
&= -2\mathbb{E}[\sigma_i W_i(\boldsymbol{\sigma})\sigma_j] - \mathbb{E}[\sigma_i W_j(\boldsymbol{\sigma})\sigma_j] \\
&= -2\mathbb{E}[\sigma_i\sigma_j] + \mathbb{E}[\sigma_j \tanh\beta h_i(\boldsymbol{\sigma})] + \mathbb{E}[\sigma_i \tanh\beta h_j(\boldsymbol{\sigma})] \\
&= -2\mathbb{E}[\sigma_i\sigma_j] + \mathbb{E}\left[\sigma_j\frac{\sigma_{i-1}+\sigma_{i+1}}{2}\right]\tanh(2\beta J) + \mathbb{E}\left[\sigma_i\frac{\sigma_{j-1}+\sigma_{j+1}}{2}\right]\tanh(2\beta J) \\
&= -2C_{ij} + \gamma\frac{C_{i-1,j}+C_{i+1,j}+C_{i,j-1}+C_{i,j+1}}{2}.
\end{aligned}$$

Use now $C_{ij} = C_k$ with $k = |i-j|$

$$\partial_t C_k = -2C_k + \gamma(C_{k-1} + C_{k+1}).$$

Since $\sigma_i^2(t) = 1$, $C_0(t) = 1 \ \forall\ t$.

e. Substituting $C_k = \eta^k$ in the steady state equation we get

$$2 = \gamma\left(\eta + \frac{1}{\eta}\right) \Rightarrow \eta^2 - \frac{2}{\gamma}\eta + 1 = 0$$

Solving for η, we obtain

$$\eta = \frac{1\pm\sqrt{1-\gamma^2}}{\gamma} = \frac{1-\frac{1}{\cosh(2\beta J)}}{\tanh(2\beta J)} = \frac{\cosh(2\beta J)-1}{\sinh(2\beta J)} = \frac{2\sinh^2(\beta J)}{2\sinh(\beta J)\cosh(\beta J)} = \tanh\beta J.$$

Hence,

$$C_k(\infty) = \tanh^k(\beta J) = \mathrm{e}^{k\ln\tanh\beta J} = \mathrm{e}^{-k\ln\coth\beta J} = \mathrm{e}^{-\frac{k}{\xi}} \qquad \text{with} \quad \xi = \frac{1}{\ln\coth\beta J}$$

or $\xi^{-1} = -\ln\tanh\beta J$. This gives $C_k \simeq 0$ for $k \gg \xi$ and $C_k \simeq 1$ for $k \ll \xi$. So ξ is the typical lengthscale over which spins are correlated. As $T \to 0$, $\xi \to \infty$ and we have one domain.

Problem 9.3

a. At stationarity m solves the equation $m = \tanh(\beta m)$. Consider

$$m^2 = m\tanh(\beta m) = |m||\tanh(\beta m)| \le \beta m^2$$

so for $\beta < 1$ one only has the solution $m = 0$. Linearised equation about $m = 0$:

$$\frac{\mathrm{d}\, m}{\mathrm{d}\, t} \simeq \beta m - m \Rightarrow m = m_0\,\mathrm{e}^{(\beta-1)t}$$

which gives $m \propto \mathrm{e}^{-\frac{t}{\tau}}$ with $\tau = \frac{1}{1-\beta} = \frac{T}{T-1}$. The critical slowing down arises when the approach to the stationary state is no longer exponential, i.e. when the characteristic timescale diverges. This occurs in the limit $T \to 1$.

b. A qualitative sketch of the curves $m = \frac{x}{\beta}$ and $m = \tanh x$ show that for $\beta > 1$ these intersects in three points, $0, \pm m_\star$. For $T < 1$, $\tau < 0$, so $m \propto \mathrm{e}^{\frac{t}{|\tau|}}$ and the system moves away from $m = 0$ (unstable). To linearise about $m_\star$, set $m = m_\star + \Delta$, and expand about $\Delta = 0$

$$\begin{aligned}\frac{\mathrm{d}\,\Delta}{\mathrm{d}\, t} &= \tanh(\beta(m_\star + \Delta)) - m_\star - \Delta \\ &\simeq \tanh(\beta m_\star) + \beta\Delta[1 - \tanh^2(\beta m_\star)] - m_\star - \Delta \\ &= \Delta[\beta(1 - m_\star^2) - 1] \Rightarrow \Delta = \Delta_0\,\mathrm{e}^{-\frac{t}{\tau}}\end{aligned}$$

with $\tau = \frac{1}{1-\beta(1-m_\star^2)}$. As $T \to 1^-$, $m_\star$ is small, so expand the stationary condition about $m = 0$ and set $T = 1 - \epsilon$ ($\beta = 1 + \epsilon$), where ϵ is small:

$$m = \tanh(\beta m) \simeq \beta m - \frac{1}{3}(\beta m)^3 \Rightarrow \epsilon m = \frac{1}{3} m^3 (1 + 3\epsilon)$$

and the non-zero (stable) solution is $m_\star \simeq \sqrt{3\epsilon}$, so

$$m_\star = \sqrt{3}\sqrt{1-T}.$$

Inserting in the definition of τ we have

$$\tau \simeq \frac{1}{1 - (1+\epsilon)(1-3\epsilon)} \simeq \frac{1}{2\epsilon} = \frac{1}{2(1-T)}.$$

c. At $T = 1$ the dynamical equation becomes

$$\frac{\mathrm{d}\, m}{\mathrm{d}\, t} = \tanh m - m \simeq m - \frac{m^3}{3} - m = -\frac{m^3}{3}$$

Solving for $m(0) = m_0$

$$\int_{m_0}^{m} \frac{\mathrm{d}\, m}{m^3} = -\int_0^t \frac{\mathrm{d}\, t}{3} \Rightarrow \frac{1}{2m_0^2} - \frac{1}{2m^2} = -\frac{1}{3}t \Rightarrow m^2 = \frac{m_0^2}{1 + \frac{2}{3}m_0^2 t}$$

For large times the constant in the denominator is negligible and the solution becomes independent on the initial condition

$$m^2 \simeq \frac{3}{2t} \Rightarrow m \simeq \sqrt{\frac{3}{2t}}.$$

Problem 9.4

a. From the assumed independence of the ξ_i^μ it follows from the definition of C_i^ν that

$$\mathbb{E}[C_i^\nu]=-\frac{1}{N-1}\sum_{k(\neq i)}\sum_{\mu(\neq\nu)}\mathbb{E}[\xi_i^\nu\xi_i^\mu\xi_k^\mu\xi_k^\nu]=-\frac{1}{N-1}\sum_{k(\neq i)}\sum_{\mu(\neq\nu)}\mathbb{E}[\xi_i^\nu]\mathbb{E}[\xi_i^\mu]\mathbb{E}[\xi_k^\mu]\mathbb{E}[\xi_k^\nu]=0$$

In order to square C_i^ν, we have to duplicate each sum:

$$\mathbb{E}\left[(C_i^\nu)^2\right]=\frac{1}{(N-1)^2}\sum_{\substack{k_1(\neq i)\\k_2(\neq i)}}\sum_{\substack{\mu_1(\neq\nu)\\\mu_2(\neq\nu)}}\mathbb{E}\left[(\xi_i^\nu)^2\xi_i^{\mu_1}\xi_i^{\mu_2}\xi_{k_1}^{\mu_1}\xi_{k_2}^{\mu_2}\xi_{k_1}^{\nu}\xi_{k_2}^{\nu}\right].$$

Using $(\xi_i^\nu)^2=1$ and the independence of ξ_i^μ, ξ_j^μ for $i\neq j$ (as well as of ξ_k^μ, ξ_k^ν for $\nu\neq\mu$)

$$\mathbb{E}\left[(C_i^\nu)^2\right]=\frac{1}{(N-1)^2}\sum_{\substack{k_1(\neq i)\\k_2(\neq i)}}\sum_{\substack{\mu_1(\neq\nu)\\\mu_2(\neq\nu)}}\mathbb{E}\left[\xi_i^{\mu_1}\xi_i^{\mu_2}\right]\mathbb{E}\left[\xi_{k_1}^{\mu_1}\xi_{k_2}^{\mu_2}\right]\mathbb{E}\left[\xi_{k_1}^{\nu}\xi_{k_2}^{\nu}\right]$$

We now separate the diagonal terms in which $\mu_1=\mu_2$ from the off-diagonal terms in which $\mu_1\neq\mu_2$. The off-diagonal terms do not contribute because they all factorize in averages of single fields, which are zero by definition, so we are simply left with

$$\mathbb{E}\left[(C_i^\nu)^2\right]=\frac{1}{(N-1)^2}\sum_{\substack{k_1(\neq i)\\k_2(\neq i)}}\sum_{\mu_1(\neq\nu)}\mathbb{E}\left[\xi_{k_1}^{\mu_1}\xi_{k_2}^{\mu_1}\right]\mathbb{E}\left[\xi_{k_1}^{\nu}\xi_{k_2}^{\nu}\right]$$

Repeating the reasoning above for k_1,k_2 we have

$$\mathbb{E}\left[(C_i^\nu)^2\right]=\frac{1}{(N-1)^2}\sum_{k_1(\neq i)}\sum_{\mu_1(\neq\nu)}1=\frac{p-1}{N-1}$$

b. The update equation for σ_i is

$$\sigma_i(t+1)=\xi_i^\nu\,\mathrm{sign}(1-C_i^\nu)$$

with

$$C_i^\nu=-\frac{\xi_i^\nu}{N-1}\sum_{k\neq i}\sum_{\mu\neq\nu}\xi_i^\mu\xi_k^\mu\xi_k^\nu.$$

It is clear, from the update equation for σ_i, that $\sigma_i(t+1)\neq\xi_i^\nu$, when $C_i^\nu>1$. In this case, an error will arise in the recall of the i-th entry of pattern $\boldsymbol{\xi}^\nu$. For $p\gg 1$ and $N\gg 1$,

$$P(C_i^\nu)=\frac{1}{\sqrt{2\pi}}\frac{N}{p}\,\mathrm{e}^{-\frac{NC_i^{\nu\,2}}{2p}}.$$

hence the probability of an error in a bit is

$$P_{\text{error}}=\int_1^\infty\frac{\mathrm{d}\,C_i^\nu}{\sqrt{2\pi}}\frac{N}{p}\,\mathrm{e}^{-NC_i^{\nu\,2}/2p}=\int_{N/\sqrt{2}p}^\infty\frac{\mathrm{d}\,x}{\sqrt{\pi}}\,\mathrm{e}^{-x^2}=\frac{1-\mathrm{erf}\left(\sqrt{N/2p}\right)}{2}$$

For $p\ll N$

$$P_{\text{error}}\simeq\frac{1}{2}\sqrt{\frac{2p}{N\pi}}\,\mathrm{e}^{-N/2p}.$$

The probability that no error is made in the recall of pattern $\boldsymbol{\xi}^\nu$ is equal to the probability that no error is made in recalling any of the components:

$$P_{\text{correct}} = (1 - P_{\text{error}})^N.$$

If one requires a confidence threshold $P_{\text{correct}} > p^\star$, one gets a bound on the ratio p/N.