TUTORIAL 07 — SOLUTIONS

7CCMCS04
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Problem 7.1

Point **a.** proceeds as in the other exercises. To answer point **b.**, let us start multipling times n the master equation, and sum over n, to obtain:

$$\frac{\mathrm{d}\,\mathbb{E}[\mathsf{N}_t]}{\mathrm{d}\,t} = -c_2\mathbb{E}[\mathsf{N}_t^2] + (c_1 + c_2)\mathbb{E}[\mathsf{N}_t]$$

In such a mean-field approach, we neglect the fluctuations, i.e., $\mathbb{E}[\mathsf{N}_t^2] \simeq \mathbb{E}[\mathsf{N}_t]^2$. This yields the answer of point **c.** and **d.**

$$\frac{\mathrm{d}\,\mathbb{E}[\mathsf{N}_t]}{\mathrm{d}\,t} = -c_2\mathbb{E}[\mathsf{N}_t]^2 + (c_1 + c_2)\mathbb{E}[\mathsf{N}_t]$$

which has two fixed points at $\mathbb{E}[\mathsf{N}_t] = 0$ and $\mathbb{E}[\mathsf{N}_t] = \frac{c_1+c_2}{c_2}$. Inspecting the velocity of the phase flow, the former is unstable and the latter is strongly stable for $c_2 > c_1$. For a long time the system will settle to

$$\frac{c_1 + c_2}{c_2} \to \begin{cases} 1 & \text{if } c_2 \gg c_1 \\ \gg 1 & \text{if } c_1 \gg c_2. \end{cases}$$

To answer point **e.**, observe that the obtained ODE deals with continuous variables: this is an approximation that is acceptable for large numbers of molecules. However, for small numbers, fluctuations around the deterministic behavior due to the discrete nature of molecules become more and more important.

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Problem 7.2

Let us start with the master equation

$$\dot{P}_n = P_{n+1}\mu(n+1) + P_{n-1}\lambda(n-1) - P_n(\lambda+\mu)n$$

The equation for the generating function $F(z,t) = \sum_n P_n z^n$ can be obtained as usual. The result is

$$\partial_t F(z,t) = \mu(1-z)\left(1 - \frac{\lambda}{\mu}z\right)\partial_z F(z,t).$$

To solve this equation, we use, once again, the method of characteristics: we seek for a parametrisation $z(s)$ $t(s)$ and $F(s)$ such that

$$\frac{\mathrm{d}\,F}{\mathrm{d}\,s} = \frac{\partial F}{\partial t}\frac{\mathrm{d}\,t}{\mathrm{d}\,s} + \frac{\partial F}{\partial z}\frac{\mathrm{d}\,z}{\mathrm{d}\,s} = 0$$

that, by comparison, gives the equations

$$\frac{\mathrm{d}\,t}{\mathrm{d}\,s} = 1, \qquad \frac{\mathrm{d}\,z}{\mathrm{d}\,s} = -\mu(1-z)\left(1 - \frac{\lambda}{\mu}z\right), \qquad \frac{\mathrm{d}\,F}{\mathrm{d}\,s} = 0$$

which imply

$$\frac{\mathrm{d}\,z}{\mathrm{d}\,t} = -\mu(1-z)\left(1 - \frac{\lambda}{\mu}z\right) \qquad \frac{\mathrm{d}\,F}{\mathrm{d}\,z} = 0.$$

For $\mu \neq \lambda$, this means that

$$\frac{\mu - \lambda z}{1-z}\,\mathrm{e}^{(\mu-\lambda)t} = a \qquad F = \phi \equiv \phi(a)$$

where a is some constant. Therefore

$$F = \phi\left(\frac{\mu - \lambda z}{1-z}\,\mathrm{e}^{(\mu-\lambda)t}\right)$$

and ϕ is determined from the initial conditions. One has $F(z,0) = z^{n_0}$ so

$$z^{n_0} = \phi\left(\frac{\mu-\lambda z}{1-z}\right) \Rightarrow \phi(x) = \left(\frac{\mu - x}{\lambda - x}\right)^{n_0}$$

and finally

$$F(z,t) = \left(\frac{(\mu-\lambda z)\,\mathrm{e}^{(\mu-\lambda)t} - \mu(1-z)}{(\mu-\lambda z)\,\mathrm{e}^{(\mu-\lambda)t} - \lambda(1-z)}\right)^{n_0}, \qquad \text{if } \lambda \neq \mu$$

For $\mu = \lambda$, instead, the procedure is different. We obtain

$$\frac{1}{1-z} + \mu t = a$$

so

$$F = \phi\left(\frac{1}{1-z} + \mu t\right),$$

and again from the initial condition

$$z^{n_0} = \phi(\frac{1}{1-z}) \Rightarrow \phi(x) = \left(\frac{x-1}{x}\right)^{n_0}$$

yielding

$$F(z,t) = \left(\frac{1 + (1-z)(\mu t - 1)}{1 + (1-z)\mu t}\right)^{n_0}, \qquad \text{if } \lambda = \mu.$$

Writing the extinction probability as $P_0(t) = F(0,t)$, one has

$\mu \neq \lambda$**:** $P_0(t) = \left(\frac{e^{(\mu-\lambda)t}-1}{e^{(\mu-\lambda)t}-\lambda/\mu}\right)^{n_0} \to \begin{cases} (\mu/\lambda)^{n_0} & \text{if } \mu < \lambda \\ 1 & \text{if } \mu > \lambda \end{cases}.$

$\mu = \lambda$**:** $P_0(t) = \left(\frac{\mu t}{1+\mu t}\right)^{n_0} \to 1$

From the full expressions of $P_0(t)$ we see that such probability, for $\mu \neq \lambda$, converges at its asymptotic at exponential rate $\tau = |\mu - \lambda|^{-1}$. We note that the relaxation time $\tau \to \infty$ as $\mu \to \lambda$, denoting critical slowing down. For $\mu = \lambda$, instead, $P_0(t) \to 1$ as a power law.