TUTORIAL 06

7CCMCS04
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Problem 6.1 A walker on the integer numbers takes steps left or right with a rate μ. Let X_t be its position at time t.

a. Write down an expression for the probability $\mathbb{P}_{1|1}[\mathsf{X}_{t+\Delta t} = n | \mathsf{X}_t = m]$ for a transition from m to n in a short time Δt neglecting $O(\Delta t^2)$ contributions, and the master equation for the evolution of the probability $P_n(t) = \mathbb{P}[\mathsf{X}_t = n]$.

b. Show, by using generating function, that the solution of the master equation, for the initial condition $P_n(0) = \delta_{n\,0}$ is

$$P_n(t) = \mathrm{e}^{-2t\mu}\, I_n(2t\mu),$$

where $I_n(x)$ is the n-th modified Bessel function of the first kind

$$I_n(x) \coloneqq \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{n+2k} \frac{1}{\ell!(n+k)!}.$$

c. Deduce the properties of the modified Bessel function $I_n(0) = \delta_{n,0}$, $I_n(t) = I_{-n}(t)$ and

$$\frac{\mathrm{d}\, I_n(x)}{\mathrm{d}\, x} = \frac{I_{n-1}(x) + I_{n+1}(x)}{2},$$

and show that the generating function of modified Bessel functions of the first kind is indeed

$$G(z, x) = \sum_{n=-\infty}^{\infty} I_n(x) z^n = \mathrm{e}^{\frac{z+z^{-1}}{2} x}$$

d. Verify that $\mathbb{E}[\mathsf{X}_t] = 0$ and $\mathbb{E}[\mathsf{X}_t^2] = 2\mu t$ (the scaling $\mathbb{E}[\mathsf{X}_t^2] \sim t$ is a typical feature of diffusion processes).

Problem 6.2 A radioactive sample consists of a number of identical nuclei, each with a decay probability γ per unit time. We regard the number N_t of undecayed nuclei at time t as a stochastic process.

a. Write down an expression for the probability $\mathbb{P}_{1|1}[\mathsf{N}_{t+\Delta t} = n | \mathsf{N} = m]$ of a transition from m to n undecayed nuclei in a short time interval Δt.
Hint: If there are m nuclei, the probability *per unit time* of a decay is $m\gamma$.

b. Write down the master equation for the evolution of the probability $P_n(t) = \mathbb{P}[\mathsf{N}_t = n]$.

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c. Find the equation of motion for the generating function

$$F(z,t) = \sum_{n=0}^{\infty} z^n P_n(t).$$

Assuming that at time $t = 0$ there are n_0 nuclei in the sample, show that for $t > 0$

$$F(z,t) = \big((z-1)\,\mathrm{e}^{-\gamma t} + 1\big)^{n_0}$$

and thereby find $P_n(t)$ by expanding $F(z,t)$ in powers of z.

d. Calculate the mean number of undecayed nuclei as a function of time and show that the sample becomes extinct in the large time limit, i.e. $\lim_{t\to\infty} P_n(t) = 0 \ \forall\ n \geq 1$, and $\lim_{t\to\infty} P_0(t) = 1$. What is the half-life of this process?