TECHNICAL NOTE

Solution of Stochastic Differential Equations by Random Time Change

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1. Introduction

In this paper, we try to solve stochastic differential equations of the form

$$dX_t = \alpha(t, X_s, s \leq t)dB_t \tag{1}$$

by means of random time change. Such a problem was studied by Yershov [4] and an interesting example was given by Nisio [2] (Example 3). We will discuss this problem more systematically and give some examples. Once the equation (1) can be solved, then, if α is non-degenerate, the equation of the form

$$dX_t = \alpha(t, X_s, s \leq t)dB_t + \beta(t, X_s, s \leq t)dt$$

can be solved by the well known transformation of measures.

2. Solution of the equation (1) *by a random time change*

Let $W = C([0,\infty) \to \mathbb{R})$ be the space of all continuous, real functions defined on $[0,\infty)$, $\mathscr{B}(W)$ be the σ-field generated by Borel cylinder sets and $\mathscr{B}_t(W)$ be the σ-field generated by Borel cylinder sets up to time t.

Let $\alpha(t,w)$ be a mapping

$$\alpha(t,w):(t,w) \in [0,\infty) \times W \rightsquigarrow \alpha(t,w) \in \mathbb{R}$$

such that it is bounded, $\mathscr{B}[0,\infty) \times \mathscr{B}(W)/\mathscr{B}(\mathbb{R})$-measurable and for each $t \geqq 0$, the mapping $w \rightsquigarrow \alpha(t,w)$ is $\mathscr{B}_t(W)/\mathscr{B}(\mathbb{R})$-measurable. We will consider the following one-dimensional stochastic differential equation

$$dX_t = \alpha(t,X)dB_t. \tag{1}$$

A precise formulation is as follows:

By a solution of (1), we mean a family $\{B = B(t), X = X(t)\}$ of stochastic processes defined on a probability space $(\Omega,\mathscr{F},P;\mathscr{F}_t)$ with an increasing family $\{\mathscr{F}_t\}$ of sub σ-fields of $\mathscr{F}$, such that

(i) B is an $\mathscr{F}_t$-Brownian motion,
(ii) X is an $\mathscr{F}_t$-adapted continuous process,

i.e. X is a W-valued random variable such that for each t, it is $\mathscr{F}_t/\mathscr{B}_t(W)$-measurable,
(iii) with probability one,

$$X(t)-X(0) = \int_0^t \alpha(s,X)dB_s \tag{1)'$$

where the integral is understood in the sense of Itô's stochastic integral.

In the following, we assume that, for some positive constant c,

$$c \leq \alpha(t,w), \text{ for all } t,w.$$

Of course, well-known condition for the existence and uniqueness of solutions is the *Lipschitz condition*: for every $T > 0$, a constant $K(T) > 0$ exists such that $\int_0^T |\alpha(t,w)-\alpha(t,w')|^2 dt \leq K(T) \int_0^T |w(t)-w'(t)|^2 dt$.

Here, we will try to solve the equation (1) by means of a random time change. For this, we will introduce several notations.

Let $\varphi_t : t \in [0,\infty) \mapsto \varphi_t \in \mathbb{R}$ be a strictly increasing and continuous function such that $\varphi_0 = 0$ and $\lim_{t\uparrow\infty} \varphi_t = \infty$. The set of all such functions is denoted by $\mathscr{I}$. Clearly $\mathscr{I} \subset W$ and let $\mathscr{B}(\mathscr{I}) = \mathscr{B}(W)|\mathscr{I}, \mathscr{B}_t(\mathscr{I}) = \mathscr{B}_t(W)|\mathscr{I}$. For $\varphi \in \mathscr{I}$, its inverse function is denoted by φ^{-1}. Clearly $\varphi^{-1} \in \mathscr{I}$ and $\varphi^{-1}_{\varphi_t} = \varphi_{\varphi_t-1} = t$. For $\varphi \in \mathscr{I}$, let $T^\varphi : W \to W$ be defined by $(T^\varphi w)(t) = w(\varphi_t^{-1})$, $w \in W$, $t \in [0,\infty)$.

By an *$\mathscr{F}_t$-adapted increasing process* defined on a quadruplet $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$, we mean a mapping $\Omega \to \mathscr{I}$ such that for each $t \geqq 0$, it is $\mathscr{F}_t/\mathscr{B}_t(\mathscr{I})$-measurable.

We are interested in the solution $X = (X_t)$ of (1) such that $X(0) = x \in \mathbb{R}$ (constant). For $w \in W$ and $x \in \mathbb{R}$, let $x+w \in W$ be defined by $(x+w)(t) = x+w(t)$.

Theorem (i) *Let $b = (b(t))$ be an $\mathscr{F}_t$-Brownian motion ($b(0) = 0$) defined on a quadruplet $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$. Let φ_t be an $\mathscr{F}_t$-adapted increasing process such that, with probability one,*

$$\varphi_t = \int_0^t \frac{ds}{\alpha^2(\varphi_s, x+T^\varphi b)} \tag{2}$$

holds. We set $\tilde{\mathscr{F}}_t = \mathscr{F}_{\varphi_t^{-1}}$ and $X(t) = x+b(\varphi_t^{-1})$. Then, there exists an $\tilde{\mathscr{F}}_t$-Brownian motion $\tilde{B} = (\tilde{B}_t)$ such that $(\tilde{B},X)$ is a solution of (1) *on $(\Omega, \mathscr{F}, P; \tilde{\mathscr{F}}_t)$ with $X(0) = x$, a.s.* .
(ii) *Conversely, let (B,X) be a solution of* (1) *on a quadruplet $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ such that $X_0 = x$. Then, there exist an increasing family $\tilde{\mathscr{F}}_t$ of sub σ-fields of $\mathscr{F}$, an $\tilde{\mathscr{F}}_t$-Brownian motion $b = (b(t))$ such that $b(0) = 0$ and an $\tilde{\mathscr{F}}_t$-adapted increasing process φ_t such that, with probability one,*

$$\varphi_t = \int_0^t \frac{ds}{\alpha^2(\varphi_s, x+T^\varphi b)}$$

for all $t \geqq 0$ and $X(t) = x+b(\varphi_t^{-1})$. Namely, any solution (B,X) of (1) *is given as in* (i).

Corollary. If, for a given $\mathscr{F}_t$-Brownian motion $b = (b(t))$ such that $b(0) = 0$, there exists a unique $\mathscr{F}_t$-adapted increasing process φ_t such that (2) *holds, then a solution (B,X) of* (1) *such that $X(0) = x$ exists and is unique in the law sense. Thus, to solve the equation* (1) *is equivalent to solve the equation* (2) *for φ_t.*

Proof (i). Let $b = (b(t))$ $(b(0) = 0)$ be an $\mathscr{F}_t$-Brownian motion and φ_t be an $\mathscr{F}_t$-adapted increasing process such that (2) holds *a.s.*. Then, $M(t) = b(\varphi_t^{-1})$ is a continuous square integrable martingale with respect to $\tilde{\mathscr{F}}_t = \mathscr{F}_{\varphi_t^{-1}}$ such that $\langle M\rangle_t = \varphi_t^{-1}$. Let $X(t) = x+M(t) = x+b(\varphi_t^{-1})$. Then, by (2),

$$\begin{aligned} t &= \int_0^t \alpha^2(\varphi_s, x+T^{\varphi}b)d\varphi_s \\ &= \int_0^t \alpha^2(\varphi_s, X)d\varphi_s \text{ and hence,} \\ \langle M\rangle_t = \varphi_t^{-1} &= \int_0^{\varphi_t^{-1}} \alpha^2(\varphi_s, X)d\varphi_s = \int_0^t \alpha^2(\varphi_{\varphi_s^{-1}}, X)ds \\ &= \int_0^t \alpha^2(s, X)ds. \end{aligned}$$

Let $\tilde{B}(t) = \displaystyle\int_0^t \frac{dM_s}{\alpha^2(s, X)}$ by stochastic integral with respect to a martingale M (cf. [1]). Then, $\langle \tilde{B}\rangle_t = \displaystyle\int_0^t \frac{d\langle M\rangle_s}{\alpha^2(s, X)} = t$ and hence, $\tilde{B}$ is $\tilde{\mathscr{F}}_t$-Brownian motion. Also,

$$M(t) = X(t)-x = \int_0^t \alpha^2(s, X)d\tilde{B}_s.$$

(ii) Conversely, let (B, X) be a solution of (1) on $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ such that $X(0)=x$. Let $M(t) = X(t) - x$. Then, M is a square integrable martingale with respect to $\mathscr{F}_t$ such that

$$\langle M\rangle_t = \int_0^t \alpha^2(s, X)ds, \quad a.s..$$

Let $\psi_t = \langle M\rangle_t = \int_0^t \alpha^2(s, X)ds$, $\varphi_t = \psi_t^{-1}$, and $\tilde{\mathscr{F}}_t = \mathscr{F}_{\varphi_t}$. Then $b(t) = M(\varphi_t)$ is $\tilde{\mathscr{F}}_t$-Brownian motion and also, φ_t is $\tilde{\mathscr{F}}_t$-adapted increasing process. Since $t = \displaystyle\int_0^t \frac{d\psi_s}{\alpha^2(s, X)}$ and $X(t) = x+b(\psi_t) = x+(T^{\varphi}b)(t)$, we have

$$\varphi_t = \int_0^{\varphi_t} \frac{d\psi_s}{\alpha^2(s, X)} = \int_0^t \frac{du}{\alpha^2(\varphi_u, X)} = \int_0^t \frac{du}{\alpha^2(\varphi_u, x+T^{\varphi}b)}$$

This completes the proof. □

3. Examples

Example 1. $\alpha(t,w) = a(w(t))$, where $a(x)$ is a bounded, Borel measurable function on $\mathbb{R}$ such that $c \leq a(x)$ where, here and in the following, c is a positive constant. Then, the equation (2) is given as,

$$\varphi_t = \int_0^t \frac{ds}{a^2[(x+T^{\varphi}b)(\varphi_s)]} = \int_0^t \frac{ds}{a^2(x+b(s))} \tag{3}$$

Thus, the equation

$$\begin{cases} dX(t) = a(X(t))dB(t) \\ X(0) = x \end{cases} \tag{4}$$

is solved as $X(t) = x+b(\varphi_t^{-1})$ by a Brownian motion $b(t)$ where φ_t is given by (3). This is a well known construction in the diffusion theory.

Example 2. $\alpha(t, w) = a(t, w(t))$, where $a(t, x)$ is a bounded, Borel measurable function on $[0,\infty) \times \mathbb{R}$ such that $c \leqq a(t, x)$. Then, the equation (2) is given as,

$$\varphi_t = \int_0^t \frac{ds}{a^2[\varphi_s, x+(T^{\varphi}b)(\varphi_s)} = \int_0^t \frac{ds}{a^2(\varphi_s, x+b(s))} \tag{5}$$

This is equivalent to the following differential equation

$$\begin{cases} \dot{\varphi}_t = \dfrac{1}{a^2(\varphi_t, x+b(t))} \\ \\ \varphi_0 = 0. \end{cases} \tag{5'}$$

Thus, if, for given Brownian motion $b(t)$, (5′) has the unique solution, we can solve the equation

$$\begin{cases} dX(t) = a(t, X(t))dB(t) \\ X(0) = x \end{cases} \tag{6}$$

uniquely. One simple sufficient condition is that $a(t, x)$ is Lipschitz continuous in t as was already remarked by Yershov [4]. Assume now that $a(t, x)$ is continuous in (t, x). Then, by Stroock-Varadhan [3], (6) has the unique solution in law sense. Given a Brownian motion $b(t)$, the ordinary differential equation (5′) has the maximal and minimal solutions $\bar{\varphi}_t$ and $\underline{\varphi}_t$. Since $\overline{X}_t = x+b(\bar{\varphi}_t^{-1})$ and $\underline{X}_t = x+b(\underline{\varphi}_t^{-1})$ are solutions of (6), it is easy to see that $\bar{\varphi}_t = \underline{\varphi}_t$. *Thus, if $a(t, x)$ is continuous, the equation (5′) has the unique solution almost surely, for a given Brownian motion $b(t)$.*

Example 3. (Nisio [2], Ex. 3).

$$\alpha(t, w) = a(\xi+ \int_0^t f(w(s))ds)$$

where $a(x)$ is a bounded, Borel measurable function on $\mathbb{R}$ such that $c \leqq a(x)$, $f(x)$ is a locally bounded Borel measurable function on $\mathbb{R}$ and $\xi \in \mathbb{R}$. The corresponding stochastic differential equations is now

$$\begin{cases} dX(t) = a(\xi+ \int_0^t f(X(s))ds)dB_t \\ X(0) = x \end{cases} \tag{7}$$

which, in a special case of $f(x) = x$, is essentially an equation for motion with random acceleration:

$$\begin{cases} dX(t) = \dot{X}(t)dt \\ d\dot{X}(t) = a(X(t))dB(t) \\ X(0) = \xi \\ \dot{X}(0) = x \end{cases} \tag{8}$$

Now, the equation (2) is given, in this case, as

$$\varphi_t = \int_0^t \frac{ds}{a^2(\xi + \int_0^{\varphi_s} f[x+(T^{\varphi}b)(u)]du)} \tag{9}$$

and

$$\begin{aligned} \int_0^{\varphi_s} f[x+(T^{\varphi}b)(u)]du &= \int_0^{\varphi_s} f[x+b(\varphi_u^{-1})]du \\ &= \int_0^s f[x+b(u)]d\varphi_u \\ &= \int_0^s f[x+b(u)]\dot{\varphi}_u du \end{aligned}$$

Thus, (9) is equivalent to

$$\begin{cases} \dot{\varphi}_t = \dfrac{1}{a^2(\xi + \int_0^t f[x+b(u)]\dot{\varphi}_u du)} \\ \varphi_0 = 0 \end{cases} \tag{10}$$

(10) is solved, for a given Brownian motion $b(t)$, uniquely in the following way: set

$$Z(t) = \int_0^t f[x+b(u)]\dot{\varphi}_u du. \quad \text{Then}$$

$$\dot{Z}(t) = f(x+b(t)) \cdot \dot{\varphi}_t = \frac{f(x+b(t))}{a^2(\xi + Z(t))}$$

and hence,

$$a^2(\xi + Z(t))\dot{Z}(t) = f(x+b(t)).$$

Therefore,

$$\int_0^t a^2(\xi + Z(s))\dot{Z}(s)ds = \int_0^t f[x+b(s)]ds$$

and hence,

$$A(Z(t)) = \int_0^t f[x+b(s)]ds$$

where $A(x) = \int_0^x a^2(\xi + y)dy$. Let $A^{-1}(x)$ be the inverse function of $x \mapsto A(x)$. Then

$$Z(t) = A^{-1}\Big(\int_0^t f[x+b(s)]ds\Big)$$

and thus, φ_t is solved as

$$\varphi_t = \int_0^t \frac{ds}{a^2(\xi + Z(s))} = \int_0^t \frac{ds}{a^2\left(\xi + A^{-1}\left(\int_0^s f[x+b(u)]du\right)\right)}. \tag{11}$$

Solution $X(t)$ of (7) is given as $X(t) = x + b(\varphi_t^{-1})$.

Example 4.

$$\alpha(t, w) = a(w(t), \xi + \int_0^t f(w(s))ds)$$

where $a(x, y)$ is a bounded continuous function on $\mathbb{R} \times \mathbb{R}$ such that $c \leq a(x, y)$. $f(x)$ is a bounded continuous function such that $c' \leq f(x)$ for some positive constant c' and $\xi \in \mathbb{R}$. Then, the equation (2) is given, in this case, as

$$\varphi_t = \int_0^t \frac{ds}{a^2[x + (T^{\varphi}b)(\varphi_s), \xi + \int_0^{\varphi_t} f[x + (T^{\varphi}b)(u)]du]} \tag{12}$$

for a given Brownian motion $b = (b(t))$.
Let

$$Z(t) = \int_0^{\varphi_t} f[x + (T^{\varphi}b)(u)]du = \int_0^t f[x + b(u)]\dot{\varphi}_u du.$$

Then, (12) is equivalent to

$$\begin{cases} \dot{Z}(t) = f(x + b(t))\dot{\varphi}_t \\ \qquad = \dfrac{f(x+b(t))}{a^2(x+b(t), \xi + Z(t))} \\ \qquad = \Phi(b(t), Z(t)) \\ Z(0) = 0 \end{cases} \tag{13}$$

where

$$\Phi(\eta, \zeta) = \frac{f(x+\eta)}{a^2(x+\eta, \xi+\zeta)}$$

As we remarked in Example 2 the equation (13) has the unique solution $Z(t)$ almost surely for a given Brownian motion $b(t)$. Then, $\varphi(t)$ is solved uniquely as

$$\varphi_t = \int_0^t \frac{ds}{a^2(x+b(s), \xi + Z(s))} \tag{14}$$

and solution $X(t)$ of

$$\begin{cases} dX(t) = a(X(t), \xi + \int_0^t f[X(s)]ds)dB(t) \\ X(0) = x \end{cases} \tag{15}$$

is given by $X(t) = x + b(\varphi_t^{-1})$.

Example 5.

$$\alpha(t,w) = a(w(t), \xi + \int_0^t f_1[w(t_1)]dt_1 \int_0^{t_1} f_2[w(t_2)] \int_0^{t_2} \cdots$$
$$\cdots \int_0^{t_{n-1}} f_n[w(t_n)]dt_n)$$

where $a(x, y)$ is a bounded Borel measurable function on $\mathbb{R}\times\mathbb{R}$ such that $c \leqq a(x, y)$, $f_1, f_2, \ldots, f_n$ are bounded Borel measurable functions on $\mathbb{R}$ and $\xi \in \mathbb{R}$. Then, the equation (2) is equivalent to

$$\varphi_t = \int_0^t \frac{1}{a^2(x+b(s), \xi + \int_0^s f_1(x+b(s_1))\dot{\varphi}_{s_1}ds_1 \int_0^{s_2} f_2(x+b(s_2))\dot{\varphi}_{s_2}ds_2 \cdots \int_0^{s_{n-1}} f_n(x+b(s_n))\dot{\varphi}_{s_n}ds_n)}. \tag{16}$$

From this, we see, for example, that if $a(x, y)$ satisfies $|a(z, y_1)-a(z, y_2)| \leq K|y_1-y_2|$ for all $z, y_1, y_2 \in \mathbb{R}$ (K is a constant) then φ_t is solved uniquely and hence, the corresponding stochastic differential equation is solved uniquely. This condition of the function a is much weaker than the Lipschitz condition on $\alpha(t, w)$ since we do not need any regularity condition on $f_1, f_2, \ldots f_n$.

References

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