Volterra process manipulation

As in Abi Jaber, Larsson, & Pulido (2017, Lemma 2.1), let $K \in L^2_{\text{loc}}(\mathbb{R}_+)$, $F \in L^1_{\text{loc}}(\mathbb{R}_+)$ and M be a continuous local martingale. Define

$$(F * K)_t := \int_0^t F(t-s)K(s)ds, \quad (K * dM)_t := \int_0^t K(t-s)dM_s,$$

then we wish to show that

$$(F * K) * dM = F * (K * dM).$$

Proof.

$$\begin{aligned}
((F * K) * dM)_t &= \int_0^t \int_0^{t-s} F(t-s-u)K(u)du dM_s \\
&= \int_0^t \int_s^t F(t-v)K(v-s)dv dM_s \quad (\text{sub. } u = v - s) \\
&= \int_0^t \int_0^v F(t-v)K(v-s)dM_s dv \quad (\text{stoch. Fubini}) \\
&= \int_0^t F(t-v) \int_0^v K(v-s)dM_s dv \\
&= (F * (K * dM))_t.
\end{aligned}$$

□

References

Abi Jaber, E., Larsson, M., & Pulido, S. (2017). Affine Volterra processes. *e-print arXiv:1708.08796v2*.