Part I

Itô's integration and applications of Itô's formula

The following books are useful for the course:

- Diffusions, Markov Processes, and Martingales: Volume 1 (CUP, Cambridge Mathematical Library) 2000, by L.C.G. Rogers and D. Williams.
- Diffusions, Markov Processes, and Martingales: Volume 2 (CUP, Cambridge Mathematical Library) 2000, by L.C.G. Rogers and D Williams.

We will follow especially the content and presentation in volume 2. Both these books are available online through the College library. Also useful will be

- Brownian Motion and Stochastic Calculus, Springer 1998, I. Karatzas, S. Shreve.

0 Prerequisites on stochastic processes

We will work with stochastic process $X = (X_t)_{t \geq 0}$ which will always (at least) be assumed jointly measurable with respect to $\mathcal{B}([0, \infty)) \times \mathcal{F}$ when considered as a map $(t, \omega) \to X_t(\omega)$. Throughout we work with processes on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}\right)$ which we assume satisfies the *usual conditions.* This means that $\mathcal{F}$ is $\mathbb{P}-$complete, the filtration is right-continuous in that $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{s>t} \mathcal{F}_s$, and that $\mathcal{F}_0$ contains all $\mathbb{P}$-null subsets of $\mathcal{F}$. Given a filtered probability space, it is always possible to define an augmentation which satisfies the usual condition. The real significance of this assumption is that it allows us to work with regularised modifications of semimartingales. To understand the type of regularity we need, we recall that a process is said to be *càdlàg* if its sample paths are right-continuous and have finite left-limits; that is, for every $\omega \in \Omega$ we have

$$\lim_{s>t, s \to t} X_s(\omega) = X_t(\omega) \text{ for all } t \geq 0$$

and

$$\lim_{s<t, s \to t} X_s(\omega) \text{ exists for all } t > 0.$$

The following result gives a necessary and sufficient condition for the existence of a càdlàg modification of a supermartingale Y on a setup satisfying the usual conditions. The resulting modification will again be a supermartingale with respect to the same setup.

Theorem 0.1 (Doob's regularisation) *Suppose* $Y = (Y_t)_{t\geq 0}$ *is a supermartingale relative to a setup* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *which satisfies the usual conditions. Then* Y *has a càdlàg modification* $X = (X_t)_{t\geq 0}$ *if and only if the function* $t \mapsto \mathbb{E}[Y_t]$ *is right-continuous from* $[0,\infty)$ *into* $\mathbb{R}$*. In this case,* X *is also a supermartingale on* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$.

Proof. See Theorem 67.7 of Rogers and Williams vol. 1. ■

We will need the following two important results.

Theorem 0.2 (Doob's L^2-inequality) *Suppose that* X *is a submartingale which is bounded in* L^2 *then* $X_t \to X_\infty$ *in* L^2 *as* $t \to \infty$ *and furthermore*

$$\mathbb{E}\left[\sup_{t\geq 0} |X_t|^2\right] \leq 4 \sup_{t\geq 0} \mathbb{E}\left[|X_t|^2\right] = 4\mathbb{E}\left[|X_\infty|^2\right].$$

Definition 0.3 *Let* T *be a stopping time, the* σ*-field* $\mathcal{F}_T$ *is defined as follows:* $A \in \mathcal{F}_T$ *iff* $A \cap \{T \leq t\} \in \mathcal{F}_T$ *for every* $t \geq 0$.

Theorem 0.4 (Optional-sampling theorem) *Let* X *be a UI martingale and suppose* $0 \leq S \leq T \leq \infty$ *are stopping times, then*

$$\mathbb{E}[X_T | \mathcal{F}_S] = X_S.$$

Moreover $\{X_T : T \text{ is a stopping time}\}$ *is UI.*

1 Spaces of martingales

The development of the stochastic integral has the main goal of arriving at integration-by-parts and change-of-variables formulas (the latter is known as Itô's formula). These describe the basic rules by which stochastic integrals can be manipulated. An important step in this is to prove the existence of the quadratic variation process $[M]$ of a local martingale M as a suitable limit of a sum of the squares of the increments of M. This is made easier by assuming that M has continuous sample paths.

1.1 Hilbert spaces of martingales

Let $\mathcal{M}^2$ denote the space of L^2-bounded martingales (not necessarily continuous, at this stage). Recall that by the martingale convergence theorem we may associate to any $M \in \mathcal{M}^2$ a unique element $M_\infty \in L^2(\mathcal{F}_\infty)$ such that $M_t \to M_\infty$ as $t \to \infty$. Conversely, and random variable M_∞ in $L^2(\mathcal{F}_\infty)$ itself characterises a unique martingale through $(\mathbb{E}[M_\infty | \mathcal{F}_t])_{t\geq 0}$ and because (Jensen's inequality)

$$\sup_{t\geq 0} \mathbb{E}\left[\mathbb{E}[M_\infty | \mathcal{F}_t]^2\right] = \mathbb{E}\left[M_\infty^2\right],$$

the martingale produced in this way belongs to $\mathcal{M}^2$. This defines the inverse of the map $M \mapsto M_\infty$, which is then describes a bijection from $\mathcal{M}^2$onto the Hilbert space $L^2(\mathcal{F}_\infty)$. This allows us to define a Hilbert space structure on $\mathcal{M}^2$ by setting

$$\langle M, N\rangle_2 = \mathbb{E}[M_\infty N_\infty].$$

The corresponding norm is denoted $||M||_2 := \langle M, M\rangle_2^{1/2}$. The main reason we study stochastic integration first for L^2-bounded martingales is the convenient use that can be made of Hilbert space theory.

Definition 1.1 *We let $c\mathcal{M}^2 \subset \mathcal{M}^2$ denote the subset of continuous L^2-bounded martingales.*

Lemma 1.2 *$c\mathcal{M}^2$ is a closed subspace of $\mathcal{M}^2$.*

Proof. That $c\mathcal{M}^2$ is a vector subspace of $\mathcal{M}^2$ is clear. Suppose $(M(n))_{n=1}^\infty \subset c\mathcal{M}^2$ is a sequence that converges to a martingale M in $\mathcal{M}^2$. Let $(M_\infty(n))_{n=1}^\infty$ and M_∞ denote the respective corresponding limiting random variables in $L^2(\mathcal{F}_\infty)$. Then, by Doob's L^2-inequality, we have

$$\mathbb{E}\left[\sup_{t\geq 0} |M_t(n) - M_t|^2\right] \leq 4\mathbb{E}\left[(M_\infty(n) - M_\infty)^2\right] \to 0 \text{ as } n \to \infty,$$

so that, along subsequence at least, we have a.s. $M(n_k) \to M$ uniformly on $[0, \infty]$. It follows that M is continuous. ∎

Remark 1.3 *The subspaces $\mathcal{M}_0^2$ consisting of L^2-bounded martingales starting from $M_0 = 0$ is a closed, as is $c\mathcal{M}_0^2$.*

2 Local martingales and quadratic variation

We say that a process M is a local martingale starting at 0 if $M_0 = 0$ and if there exists a *reducing sequence* of stopping times $0 \leq T_n \leq T_{n+1} \uparrow \infty$ a.s. such that $M^{T_n} := (M_{t \wedge T_n})_{t \geq 0}$ is a martingale for every n. Obviously, the class of local martingales depends on the filtration.

Definition 2.1 *M is a local martingale if: (i) M_0 is $\mathcal{F}_0$-measurable, and (ii) if $(M_t - M_0)_{t \geq 0}$ is a local martingale starting at 0.*

We will denote the class of local martingale and local martingales starting at 0 respectively by $\mathcal{M}_{\mathrm{loc}}$and $\mathcal{M}_{0,\mathrm{loc}}$ and likewise $\mathrm{c}\mathcal{M}_{\mathrm{loc}}$and $\mathrm{c}\mathcal{M}_{0,\mathrm{loc}}$ for their continuous counterparts.

Exercise 2.2 *Shows that the class of local martingales is stables under stopping; that is, if $M \in \mathcal{M}_{loc}$ and T is a stopping time then $M^T \in \mathcal{M}_{loc}$.*

When we use $\mathrm{c}\mathcal{M}_{\mathrm{loc}}$ we mean $(\mathrm{c}\mathcal{M})_{\mathrm{loc}}$ so that the localisation applies to property of being continuous as well as that of being a martingale. Of course, in this case it makes no difference:

$$(\mathrm{c}\mathcal{M})_{\mathrm{loc}} = \mathrm{c}(\mathcal{M}_{\mathrm{loc}})$$

– continuity itself being a local property– but sometimes care is needed deciding when given property will 'commute' with localisation in this way. An important lemma concerning $\mathrm{c}\mathcal{M}_{0,\mathrm{loc}}$ is the following.

Lemma 2.3 *If $M \in c\mathcal{M}_{0,loc}$ then there exists a reducing sequence $(T_n)_{n=1}^{\infty}$ for M such that M^{T_n} is a* ***bounded*** *martingale for every n.*

Proof. See the *Stochastic Processes* course notes. ■

This lemma is important for localising the stochastic integral as we will recall later. It will not be true in general without the continuity assumption. An immediate consequence is that

$$c\mathcal{M}_{0,\mathrm{loc}} = c\mathcal{M}^2_{0,\mathrm{loc}}.$$

The proof of the existence of the quadratic variation is also much easier for continuous local martingale than discontinuous ones. We recall the following result (again from *Stochastic Processes*).

Theorem 2.4 *If $M \in c\mathcal{M}_{0,loc}$ then there exists a unique adapted, continuous non-decreasing process $[M]$ starting at 0 such that $M^2 - [M] \in c\mathcal{M}_{0,loc}$.*

Remark 2.5 *A few comments are in order. Firstly, for $M \in c\mathcal{M}_{loc}$ we set $[M] := [M - M_0]$. Secondly, the connection between $[M]$ and the sample paths of M is that $[M]$ is the limit in probability, uniformly on compact subsets of $[0, \infty)$, of the sequence of processes*

$$(t, \omega) \mapsto \sum_{k=1}^{\infty} \left(M_{t_k^n}(\omega) - M_{t_{k-1}^n}(\omega) \right)^2$$

where $t_k^n := \min(t, k2^{-n})$ *.Hence, the reason why* $[M]$ *is called the quadratic variation of* M*.*

The quadratic covariation $[M,N]$ of two continuous local martingale M and N is defined by polarisation; that is

$$[M,N]_t = \frac{1}{4}\left([M+N]_t - [M-N]_t\right),$$

so that $[M,N]$ is bilinear in M and N.

Proposition 2.6 (Properties of quadratic covariation) *Let* $M, N \in c\mathcal{M}_{0,loc}$ *then*

1. *For any stopping time* T *we have* $[M,N]^T = \left[M^T, N^T\right]$
2. $[M,N]$ *is the unique finite variation process starting at*0 *such that* $MN - [M,N] \in c\mathcal{M}_{0,loc}$*.*

Proof. The uniqueness of $[M]$ gives immediately that $\left[M^T\right] = [M]^T$ a.s. The rest is an exercise in using the definition of $[M,N]$. ■

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Lemma 2.7 *If* $M \in c\mathcal{M}_0^2$ *then* $M^2 - [M]$ *is a UI (uniformly integrable) martingale.*

Proof. Note that because $[M]$ is non-decreasing we have $[M]_t \uparrow [M]_\infty$ as $t \to \infty$ and $[M]_\infty = \sup_{t \geq 0} [M]_t$. From the monotone convergence theorem we have

$$\mathbb{E}\left[\sup_{t\geq 0} [M]_t\right] = \lim_{t\to\infty} \mathbb{E}\left[[M]_t\right] = \lim_{t\to\infty} \mathbb{E}\left[M_t^2\right] = \mathbb{E}\left[M_\infty^2\right] < \infty.$$

It follows that

$$\sup_{t\geq 0}\left(M_t^2 - [M]_t\right) \in L^1 \tag{1}$$

which gives uniform integrability of $M^2 - [M]$. We know from Theorem 2.4 that $M^2 - [M]$ is a local martingale. Recall from *Stochastic Processes* that a local martingale X is a martingale whenever the random variables

$$\{X_T : T \text{ a stopping time, } T \leq t\}$$

are UI for all $t \geq 0$. In the case where $X = M^2 - [M]$ this can be verified at once using (1). ■

Let $M \in \mathcal{M}_{\mathrm{c,loc}}$ be a continuous local martingale. Our aim now is to understand the stochastic integral

$$\int H_s dM_s$$

for a wide class of stochastic processes H. As usual, we start by defining the stochastic integral for processes for which the integral is easy to interpret.

3 Itô's stochastic integral

We recall the L^2 construction of the Itô integral $H \cdot M = \int HdM$ in several stages.

3.1 Itô integration for elementary processes

We develop the basic assumptions needed on the integrand $H : (0, \infty) \times \Omega \to \mathbb{R}$. A process $H = (H_t)_{t>0}$ will be called **bounded elementary** if it has the form

$$H_t(\omega) = \sum_{k=1}^{n} Z_k(\omega) 1_{(S_k(\omega), T_k(\omega)]}(t), \tag{2}$$

where $0 \leq S_1 \leq T_1 \leq S_2 \leq T_2 \leq .... \leq S_n \leq T_n$ is a sequence of stopping times, and each Z_k is bounded $\mathcal{F}_{S_k}$-measurable random variable. We further let

$$||H||_\infty = \inf \left\{ M > 0 : \sup_{t \geq 0} |H_t| \leq M \text{ a.s.} \right\},$$

and denote the class of bounded elementary processes by bξ.

Exercise 3.1 *Prove that the class bξ is a vector space and is an algebra.*

Given H in bξ of the form (2), we define the integral $H \cdot M = (H \cdot M)_{t \geq 0}$ for any process M by setting

$$(H \cdot M)_t := \sum_{k=1}^{n} Z_k (M_{t \wedge T_k} - M_{t \wedge S_k}).$$

Lemma 3.2 *Let $H \in$bξ be of the form (2) and suppose $M \in c\mathcal{M}_0^2$, then $H \cdot M$ is in $c\mathcal{M}_0^2$ and*

$$\mathbb{E}\left[(H \cdot M)_\infty^2\right] = \mathbb{E}\left[\sum_{k=1}^{n} Z_k^2 \left([M]_{T_k} - [M]_{S_k}\right)\right] = \mathbb{E}\left[\int_0^\infty H_u^2 d[M]_u\right]$$

Proof. That $(H \cdot M)_0 = 0$ is immediate as is the continuity of $t \to (H \cdot M)_0$. It can be checked directly that $H \cdot M$ is a martingale. To finish let $t > 0$ then

$$\begin{aligned} & \mathbb{E}\left[(H \cdot M)_t^2\right] \\ = & \sum_{k=1}^{n} \mathbb{E}\left[Z_k^2 (M_{t \wedge T_k} - M_{t \wedge S_k})^2\right] \\ & +2 \sum_{j<k}^{n} \mathbb{E}\left[Z_j Z_k \left(M_{t \wedge T_j} - M_{t \wedge S_j}\right) Z_k (M_{t \wedge T_k} - M_{t \wedge S_k})\right]. \end{aligned}$$

For bounded stopping times $U \leq V$, the optional stopping theorem gives

$$\mathbb{E}\left[M_V^2 - [M]_V \,\middle|\, \mathcal{F}_U\right] = M_U^2 - [M]_U .$$

From this and standard properties of conditional expectation the last expression simplifies to

$$\mathbb{E}\left[(H \cdot M)_t^2\right] = \mathbb{E}\left[\sum_{k=1}^{n} Z_k^2 \left([M]_{t\wedge T_k} - [M]_{t\wedge S_k}\right)\right] = \mathbb{E}\left[\int_0^t H_u^2 d\,[M]_u\right].$$

Letting $t \to \infty$ and using the monotone convergence theorem we recover the result. ■

3.2 Previsible processes

We are interested in extending the definition of $H \cdot M$ beyond bounded elementary integrands. Noting that each process in bξ is automatically adapted and caglad (left-continuous with right-limits), we define the following.

Definition 3.3 *The **previsible σ-field** $\mathcal{P}$ is the smallest σ-field on $(0,\infty)\times\Omega$ such that every adapted caglad process is $\mathcal{P}$-measurable. A process H is called previsible if it measurable, w.r.t. $\mathcal{P}$ and the Borel σ-field on $\mathbb{R}$, when viewed as a function $H : (0,\infty)\times\Omega \to \mathbb{R}$*

The next proposition clinches the previsible processes as the right class to work with to extend the Itô integral.

Proposition 3.4 $\mathcal{P} = \sigma\,(b\xi)$

Proof. Step1 :$\sigma\,(\mathrm{b}\xi) \subseteq \mathcal{P}$ We show that $H \in$bξ is previsible. To do so, we suppose $H = Z1_{(S,T]}$ leaving the general case as an exercise.Since caglad is obvious, we only need to show adaptedness of H. If $t > 0$ then we have

$$H_t = \lim_{n\to\infty} Z1_{[S+n^{-1},T+n^{-1})}\,(t) \text{ a.s.}$$

so it suffices to prove that $Z1_{[U,V)}$ is adapted, whenever $U \leq V$ are stopping times. This follows since for any Borel subset $B \subseteq \mathbb{R}$ with $0 \notin B$ we have

$$H_t^{-1}\,(B) = G_{B,t} := Z^{-1}\,(B) \cap \{U \leq t\} \cap \{V < t\} \in \mathcal{F}_t,$$

whereas if $0 \in B$ then $H_t^{-1}\,(B) = G_{B,t} \cup \{U < t\} \cup \{V \geq t\} \in \mathcal{F}_t$.

Step 2 :$\mathcal{P} \subseteq \sigma\,(\mathrm{b}\xi)$ Let H is any adapted caglad process and define for $n = 1, 2, ...$

$$H_t^n\,(\omega) = \sum_{k=1}^{n2^n} \max\left(\min\left(H_{\frac{k-1}{2^n}}\,(\omega)\,, n\right), -n\right) 1_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right]}\,(t) \in \mathrm{b}\xi.$$

then, using left-continuity, $H = \lim_{n\to\infty} H^n$ and it follows that H is $\sigma\,(\mathrm{b}\xi)$-measurable. ■

We can use this Proposition in concert with the following very useful result.

Theorem 3.5 (Monotone class theorem) *Let $\mathcal{H}$ be a (real) vector space consisting of bounded functions from a set S into $\mathbb{R}$. Assume that $\mathcal{H}$ contains constant functions, is closed under uniform convergence, and satisfies the following: for any uniformly bounded and non-negative sequence $(f_n)_{n=1}^{\infty}$ in $\mathcal{H}$ such that the limit $f(s) := \lim_{n\to\infty} f_n(s)$ exists for every $s \in S$, we have $f \in \mathcal{H}$. If $\mathcal{H}$ contains a class $\mathcal{C}$ which is closed under multiplication, then $\mathcal{H}$ contains every bounded $\sigma(\mathcal{C})$-measurable function from S into $\mathbb{R}$.*

We arrive at the following

Lemma 3.6 *Let $\mathcal{H}$ be a real vector space of processes on $(0,\infty)\times\Omega$. Assume also that $\mathcal{H}$ contains the constant functions, is closed under uniform convergence, and for any uniformly bounded non-negative sequence of (H_n) in $\mathcal{H}$ such that $H = \lim H_n$ we have $H \in \mathcal{H}$. If $\mathcal{H}$ contains $b\xi$ then it also contains every bounded previsible process on $(0,\infty)\times\Omega$.*

Why is this result so important? Suppose $M \in c\mathcal{M}_0^2$, then we define the space $L^2(M)$ as the class of previsible processes H for which

$$\mathbb{E}\left[\int_0^\infty H_s^2 d[M]_s\right] < \infty.$$

Viewing processes as measurable functions, this is the same as the space

$$L^2((0,\infty)\times\Omega,\mathcal{P},\mu)$$

where μ is the unique measure satisfying

$$\mu((s,t]\times A) = \mathbb{E}\left[1_A([M_t]-[M_s])\right], \tag{3}$$

for $0 < s \le t < \infty$ and $A \in \mathcal{F}_t$. As such, $L^2(M)$ is a Hilbert space with norm

$$||H||_{L^2(M)} = \mathbb{E}\left[\int_0^\infty H_s^2 d[M]_s\right]^{1/2}.$$

Exercise 3.7 *Verify the assertions above: show that (3) determines an unique extension of μ to a measure on $((0,\infty)\times\Omega,\mathcal{P})$ (hint: recall Caratheodory's extension theorem), then convince yourself that $L^2((0,\infty)\times\Omega,\mathcal{P},\mu) = L^2(M)$.*

Each element of $L^2(M)$ is, strictly speaking, an equivalence class of previsible processes but, as usual, we work with a representative process from the equivalence class..

Using Proposition 3.4 and Lemma 3.6 we obtain the following result.

Corollary 3.8 *Suppose $M \in c\mathcal{M}_0^2$, then $b\xi$ is a dense subspace of $L^2(M)$.*

Proof. It is easy to see that $b\xi \subset L^2(M)$. Let U denote the closure of $b\xi$ in $L^2(M)$, then U satisfies the conditions of Lemma 3.6 (exercise: check this) and, as such, U contains every bounded previsible process. In particular, U then contains the subspace simple functions $S = \{\sum \lambda_i 1_{A_i} : A_i \in \mathcal{P}\}$ and hence $L^2(M) = \bar{S} \subseteq U \subseteq L^2(M)$ so that $L^2(M) = U$, as required. ■

Definition 3.9 *A process V is called evanescent if* $\mathbb{P}(V_t = 0, \forall t) = 0$.

Remark 3.10 *Recall that all processes are assumed jointly measurable. Two processes are then indistinguishable if their difference is an evanescent process.*

Lemma 3.11 *An evanescent process on $(0, \infty) \times \Omega$ is previsible.*

Proof. See Rogers and Williams vol. 2 IV lemma 13.8

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4 Localisation and extension of the Itô integral

We continue to assume that $M \in \mathrm{c}\mathcal{M}_0^2$. With the results of the previous lecture in hand, we can now extend the definition of the $H \cdot M = \int H dM$ to any integrand $H \in L^2(M)$. This follows from elementary Hilbert space theory, since the map

$$I : \mathrm{b}\xi \to L^2(\mathcal{F}_\infty), \; I(H) = (H \cdot M)_\infty$$

is a linear isometry, which then extends uniquely to an isometry from $\overline{\mathrm{b}\xi} = L^2(M)$ onto a closed subspace of $L^2(\mathcal{F}_\infty)$. Recall from Subsection 1.1 that each $L^2(\mathcal{F}_\infty)$ is associated uniquely (up to equivalence) to a process in $\mathcal{M}^2$; using Lemma 1.2, the remark following it and the fact that $I(\mathrm{b}\xi) \subset \mathrm{c}\mathcal{M}_0^2$, the process associated to $(H \cdot M)_\infty$ will belong to $\mathrm{c}\mathcal{M}_0^2$. This is the Itô integral of $H \in L^2(M)$ with respect to $M \in \mathrm{c}\mathcal{M}_0^2$. From the isometry property we still have

$$\mathbb{E}\left[(H \cdot M)_\infty^2\right] = ||H \cdot M||_2^2 = ||H||_{L^2(M)}^2 = \mathbb{E}\left[\int_0^\infty H_s^2 d[M]_s\right]$$

Notation 4.1 *We denote the Itô integral defined in this way variously by* $(H \cdot M)_\cdot$, $\int_0^\cdot H_u dM_u$.

4.1 Localisation

In practice the assumption of L^2-boundedness on M and the consequent requirement that $H \in L^2(M)$ are too strong for applications. We can relax both by localising the definitions using stopping times. To see how the arguments goes, observe first that if H is a bounded previsible process and T a stopping time then $H1_{[0,T]}$ is bounded and previsible. We next recall the following result, a version of which has been proved in *Stochastic Processes.*

Proposition 4.2 *Let M be a continuous local martingale. Assume H and K are bounded previsible process and T is a stopping time. Then we have:*

1. $(H1_{[0,T]} \cdot M) = H \cdot M^T = (H \cdot M)^T$.
2. $[H \cdot M] = H^2 \cdot [M] := \int_0^\cdot H_u^2 d[M]_u$.
3. $H \cdot (K \cdot M) = (HK) \cdot M$.

We can localise the integrands using remark above: we call H a locally bounded previsible process (denoted: $\mathrm{lb}\mathcal{P}$) if there exists a sequence of stopping times $0 \le T_n \le T_{n+1} \uparrow \infty$ a.s. such that $H1_{[0,T_n]}$ is a bounded previsible process for every n. The first item in Proposition 4.2 allows us to extend the definition of $H \cdot M$ to the case where $H \in \mathrm{lb}\mathcal{P}$ and $M \in \mathrm{c}\mathcal{M}_{0,\mathrm{loc}}^2$. To do this we take a reducing sequence $(S_n)_{n=1}^\infty$ for M, and observe that $(R_n)_{n=1}^\infty = (S_n \wedge T_n)_{n=1}^\infty$

simultaneously reduces H to a bounded previsible process and M to an element of $c\mathcal{M}_0^2$ (we still, of course, have $R_n \uparrow \infty$ a.s). We then take

$$(H \cdot M)^{R_n} = \left(H1_{[0,R_n]} \cdot M\right) = \left(H1_{[0,R_n]} \cdot M^{R_n}\right)$$

which is easily seen to be well-defined. Putting things together we get:

Lemma 4.3 *For any $H \in lb\mathcal{P}$ and $M \in c\mathcal{M}_{0,loc}^2$ we have $H \cdot M \in c\mathcal{M}_{0,loc}^2$.*

The following result extends the stochastic integral to $c\mathcal{M}_{0,\mathrm{loc}}$, i.e. removing the assumption of L^2-boundedness on M.

Theorem 4.4 *Suppose M is a continuous local martingale with $M_0 = 0$ and H is a previsible process satisfying*

$$\mathbb{P}\left(\int_0^t H_s^2 d\left[M\right]_s < \infty, \forall t > 0\right) = 1.$$

Then the integral $H \cdot M$ exists and is a continuous local martingale starting at 0.

Proof. Let $(S_n)_{n=1}^{\infty}$ be a reducing sequence for M that makes M^{S_n} a bounded martingale (remember Lemma 2.3). Define T_n to be the stopping time

$$T_n := \inf\left\{t > 0 : \int_0^t H_s^2 d\left[M\right]_s > n\right\},$$

so that $0 \leq T_n \leq T_{n+1} \uparrow \infty$ a.s. Letting $R_n := S_n \wedge T_n$ we see that $M^{R_n} \in c\mathcal{M}_0^2$ and $H1_{[0,R_n]} \in L^2\left(M^{R_n}\right)$; it follows that $H^{R_n} \cdot M^{R_n}$ exists and is an L^2-bounded continuous martingale. Hence $(H \cdot M)$ is well-defined by

$$(H \cdot M)^{R_n} = H^{R_n} \cdot M^{R_n},$$

and is thus a continuous local martingale. ■

5 Itô's formula

5.1 Covariation of stochastic integrals

Theorem 5.1 (Kunita-Watanabe) *For any continuous local martingales M and N and any H which is locally bounded and previsible, we have*

$$[H \cdot M, N] = H \cdot [M, N] = [M, H \cdot N] \, .$$

Proof. We first show that $[H \cdot M, N] = [M, H \cdot N]$. By localising, we may assume if necessary that H is bounded previsible and that M and N are bounded martingales. It suffices then to show that

$$(H \cdot M) \, N - M \, (H \cdot N)$$

is a UI martingale. This will follow if we can show that

$$\mathbb{E} \left[(H \cdot M)_T \, N_T \right] = \mathbb{E} \left[(H \cdot N)_T \, M_T \right]$$

or, equivalently,

$$\mathbb{E} \left[\left(H \cdot M^T \right)_\infty N^T_\infty \right] = \mathbb{E} \left[\left(H \cdot N^T \right)_\infty M^T_\infty \right] .$$

We prove this now replacing M^T and N^T by M and N for simplicity and assuming $H = Z 1_{(U,V]} \in \mathrm{b}\xi$; we leave the case of general H as an exercise for the reader. This gives

$$\begin{aligned}
\mathbb{E} \left[(H \cdot M)_\infty N_\infty \right] &= \mathbb{E} \left[Z \left(M_V - M_U \right) N_\infty \right] \\
&= \mathbb{E} \left[Z \left(M_V - M_U \right) N_V \right] \\
&= \mathbb{E} \left[Z M_V N_V \right] - \mathbb{E} \left[Z M_V N_U \right] \\
&= \mathbb{E} \left[Z \left(N_V - N_U \right) M_V \right] \\
&= \mathbb{E} \left[(H \cdot N)_\infty M_\infty \right] .
\end{aligned}$$

To see that $[H \cdot M, N] = H \cdot [M, N]$ we equate

$$[(H+1) \cdot M] = H^2 \cdot [M] + 2H \cdot [M] + [M]$$

with

$$[H \cdot M + M] = H^2 \cdot [M] + 2 \, [H \cdot M, M] + [M]$$

to learn that $H \cdot [M] = [H \cdot M, M]$. Whereupon

$$\begin{aligned}
H \cdot [M, N] &= \frac{1}{4} \left(H \cdot [M+N] - H \cdot [M-N] \right) \\
&= \frac{1}{4} \left([H \cdot (M+N), M+N] - [H \cdot (M-N), M-N] \right) \\
&= \frac{1}{4} \left([H \cdot M, N] + [H \cdot N, M] + [H \cdot M, N] + [H \cdot N, M] \right) \\
&= [H \cdot M, N] \, .
\end{aligned}$$

■

5.2 Continuous semimartingales

A process X is called a continuous semimartingale if it has the decomposition

$$X = X_0 + M + A,$$

where X_0 is a $\mathcal{F}_0$-measurable random variable, $M \in \mathrm{c}\mathcal{M}_{0,\mathrm{loc}}$ and A is a continuous adapted process of finite variation with $A_0 = 0$. Quadratic variation is well-defined by

$$[X] = [M]$$

and since integration against A is understood using Lebesgue-Stieltjes integration (even for non-previsible integrands!) we may define the Itô integral $(H \cdot X)$ for any $H \in \mathrm{lb}\mathcal{P}$ in the obvious way by

$$H \cdot X = H \cdot M + H \cdot A.$$

Note that $H \cdot X$ is another continuous semimartingale and $[H \cdot X] = [H \cdot M] = H^2 \cdot [M]$.

Remark 5.2 *An $\mathbb{R}^d$-valued process $X = \left(X^1, ..., X^d\right)$ is a continuous semimartingale if each X^i is a continuous semimartingale as defined above. In this case, the quadratic variation is the matrix-valued process $[X] = \left(\left[X^i, X^j\right]\right)$ $_{1 \leq i,j, \leq d}$.*

5.3 Itô's formula

We have seen that Itô integration preserve the continuous semimartingale property. Itô's formula is essentially the statement that C^2 transformations preserve semimartingales. We recall the statement of Itô's formula here together with the integration-by-parts result on which it rests. A sketch proof was given in *Stochastic Processes*

Theorem 5.3 *We have:*

1. *(Integration-by-parts) If X and Y are continuous semimartingales then*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X, Y]_t .$$

2. *(Itô's formula) Let $X = \left(X^1, ..., X^d\right)$ be a $\mathbb{R}^d-$valued continuous semimartingale. For any $f \in C^2\left(\mathbb{R}^d\right)$, $f(X)$ is a continuous semimartingale and*

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f(X_s)}{\partial x_i} dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f(X_s)}{\partial x_i \partial x_j} d\left[X^i, X^j\right]_s . \quad (4)$$

Remark 5.4 *If any of the components of $X = \left(X^1, ..., X^d\right)$ has finite variation, then we need only assume f to be only C^1 in these components.*

6 Lévy's characterisation of Brownian motion

We recall the definition of **Brownian motion**.

Definition 6.1 *Given* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *an* $\mathbb{R}^d$*-valued stochastic process* $B = \left(B_t^1, ..., B_t^d\right)_{t\geq 0}$ *is an* $\{\mathcal{F}_t\}$*-Brownian motion if it is* $\{\mathcal{F}_t\}$*-adapted and*

1. $B_0 = 0$;
2. $t \mapsto B$ *is continuous on* $[0, \infty)$
3. *For every* $s < t$, *the increment* $B_t - B_s$ *is independent of* $\mathcal{F}_s$ *and has the multivariate normal distribution with zero mean and covariance matrix* $(t-s)\, I_d$.

Remark 6.2 *Brownian motion (BM) is also called Wiener's process; the law of* B *induced on the space of continuous paths is called Wiener's measure.*

Exercise 6.3 *Let* $B = \left(B^1, ..., B^d\right)$ *be d-dimensional* $\{\mathcal{F}_t\}$*-Brownian motion. Prove that each* B^i *is a martingale and that*

$$\left[B^i\right]_t = t, \text{for } i = 1, ..., d, \text{ while } \left[B^i, B^j\right] = 0 \text{ for } i \neq j. \tag{5}$$

The following result extends the previous exercise: any continuous local martingale $\left(B^1, ..., B^d\right)$ satisfying (5) must be Brownian motion.

Theorem 6.4 (Lévy's characterisation of BM) *Let* $B = \left(B^1, ..., B^d\right)$ *be a d-dimensional continuous local martingale starting at zero defined on* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$. *Assume that*

$$\left(\left[B^i, B^j\right]_t\right)_{i,j\in\{1,...,d\}} = tI_d \in \mathbb{R}^{d\times d},$$

then B *is d-dimensional* $\{\mathcal{F}_t\}$*-Brownian motion.*

Proof. See Stochastic Processes. ■

The following result characterises all continuous local martingales as being Brownian motion up to a time-change by the quadratic variation.

Theorem 6.5 (Dubins-Schwarz characterisaton of $c\mathcal{M}_{0,\mathbf{loc}}$) *Let* M *be a continuous local martingale starting at zero, defined on* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *and satisfying*

$$[M]_t \uparrow \infty \text{ as } t \to \infty \text{ a.s.}$$

For $t \geq 0$ *define the stopping time*

$$\tau_t = \inf\{s > 0 : [M]_s > t\}$$

and let $\mathcal{G}_t := \mathcal{F}_{\tau_t}$. *Then the process* $(B_t) = (M_{\tau_t})$ *is Brownian motion on* $\left(\Omega, \mathcal{F}, \{\mathcal{G}_t\}_{t\geq 0}, \mathbb{P}\right)$ *and moreover* $M = B_{[M]_t}$.

Proof. To see that B is adapted to $\{\mathcal{G}_t\}_{t\geq 0}$ it is sufficient to prove that $B_t^{-1}((\infty, x]) \in \mathcal{G}_t$ for every $t \geq 0$ and $x \in \mathbb{R}$. From the definitions of B and $\mathcal{G}_t$, this amounts to checking that

$$A = \left\{\omega : M_{\tau_t(\omega)}(\omega) \leq x\right\} \cap \{\omega : \tau_t \leq s\} \in \mathcal{F}_s$$

for all $s \geq 0$. Using the uniform continuity of M on $[0, s]$ we have

$$A = \cap_{m=1}^{\infty} \cup_{n=1}^{\infty} \cap_{N=n}^{\infty} \cup_{j=1}^{N} \left\{M_{js/N} \leq x + \frac{1}{m}\right\} \cap \left\{\frac{(j-1)\,s}{N} < \tau_t \leq \frac{js}{N}\right\} \in \mathcal{F}_s$$

(exercise: prove this fact).

We now prove continuity of B is continuous. Since B is the composition $M \circ \tau$, it will be continuous on any interval where $[M]$ is *strictly* increasing because the inverse function τ will also be continuous and strictly increasing on this interval. It thus suffices to show that M is constant on any interval on which $[M]$ is constant. For each $q \in \mathbb{Q} \cap \mathbb{R}_+$ we introduce the stopping time

$$S_q = \inf\left\{t > q : [M]_t > [M]_q\right\}$$

and prove that M is constant on $[q, S_q]$. By localisation we may assume M is a bounded continuous martingale. Then since $M^2 - [M]$ is a UI martingale, we can use Theorem 0.4 to see that

$$\mathbb{E}\left[\left(M_{S_q} - M_q\right)^2 \Big| \mathcal{F}_q\right] = \mathbb{E}\left[M_{S_q}^2 - M_q^2 \Big| \mathcal{F}_q\right] = \mathbb{E}\left[[M]_{S_q} - [M]_q \Big| \mathcal{F}_q\right] = 0$$

using $[M]_{S_q} = [M]_q$ (continuity of $[M]$). It follows that $M_{S_q} = M_q$ a.s. from which it is easy to prove that M is constant on $[q, S_q]$.

We finish the proof by showing that B and $B^2 - t$ are $\{\mathcal{G}_t\}$-local martingales; Levy's characterisation will then show that B is Brownian motion. We define the stopping times

$$T_n = \inf\{t > 0 : |M_t| > n\}$$

and let $U_n = [M]_{T_n}$. It can be shown that $\tau_{U_n} = T_n$ (exercise: Do this. It is a little subtle, you will need to use the definition of T_n and the argument in the last paragraph). We then have that

$$B_{t\wedge U_n} = M_{\tau_t}^{T_n}$$

and since $\{U_n \leq t\} = \{T_n \leq \tau_t\} \in \mathcal{F}_{T_n} \cap \mathcal{F}_{\tau_t} \subset \mathcal{F}_{\tau_t} = \mathcal{G}_t$ we know that U_n is a $\{\mathcal{G}_t\}$-stopping time. To finish, we apply Theorem 0.4 to the bounded (hence UI) martingale M^{T_n} to obtain

$$\mathbb{E}\left[B_{t\wedge U_n} \middle| \mathcal{G}_s\right] = \mathbb{E}\left[M_{\tau_t}^{T_n} \middle| \mathcal{F}_{\tau_s}\right] = M_{\tau_s}^{T_n} = B_{s\wedge U_n}$$

and likewise

$$\begin{aligned}
\mathbb{E}\left[B_{t\wedge U_n}^2 - t \wedge U_n \middle| \mathcal{G}_s\right] &= \mathbb{E}\left[\left(M_{\tau_t}^{T_n}\right)^2 - [M]_{\tau_t} \wedge [M]_{T_n} \Big| \mathcal{F}_{\tau_s}\right] \\
&= \mathbb{E}\left[\left(M_{\tau_t}^{T_n}\right)^2 - [M]_{\tau_t}^{T_n} \Big| \mathcal{F}_{\tau_s}\right] \\
&= \left(M_{\tau_s}^{T_n}\right)^2 - [M]_{\tau_s}^{T_n} \\
&= B_{s\wedge U_n}^2 - s \wedge U_n.
\end{aligned}$$

■

7 Exponential martingales

Suppose that X is a continuous semimartingale starting at zero, then the there exists a unique semimartingale $Z = \mathcal{E}(X)$ which satisfies

$$Z_t = 1 + \int_0^t Z_s dX_s. \tag{6}$$

This SDE has an explicit solution given by $Z_t = \exp\left(X_t - \frac{1}{2}[X]_t\right)$; the process $\mathcal{E}(X)$ is called the *stochastic exponential of* X. It is easy to see that process is a solution to (6) by using Itô's formula. To see uniqueness we notice, again by using Itô's formula, that $Y_t = Z_t^{-1} = \exp\left(-X_t + \frac{1}{2}[X]_t\right)$ satisfies

$$dY_t = -Y_t dX_t + Y_t d[X]_t .$$

It follows that for any other solution $\tilde{Z}$ to (6) we have

$$d\left(Y_t\tilde{Z}_t\right) = Y_t d\tilde{Z}_t + \tilde{Z}_t dY_t + d\left[Y_t, \tilde{Z}_t\right] = 0,$$

i.e. $Y_t\tilde{Z}_t$ is constant and hence $Y_t\tilde{Z}_t = 1$ for all $t \geq 0$. This gives the uniqueness $\tilde{Z}_t = Z_t$. More generally if Z_0 is an $\mathcal{F}_0$-measurable random variable, then $Z_t = Z_0\mathcal{E}(X)_t$ solves the SDE

$$Z_t = Z_0 + \int_0^t Z_s dX_s.$$

Suppose now that M is a continuous local martingale with $M_0 = 0$ and let $\theta \in \mathbb{R}$. It follows from the definition (6) that $\mathcal{E}(\theta M)$ is again a continuous local martingale, and since $\mathcal{E}(\theta M)_t = \exp\left(\theta M_t - \frac{1}{2}\theta^2[M]_t\right)$ is non-negative it follows that $\mathcal{E}(\theta M)$ is a supermartingale[1]. In particular, we have that

$$\mathbb{E}\left[\exp\left(\theta M_t - \frac{1}{2}\theta^2[M]_t\right)\right] \leq 1 \tag{7}$$

for all $t \geq 0$. It is of interest to know when we have equality in (7). This is especially important when we later use $\mathcal{E}(\theta M)$ to give Radon-Nikodym derivatives describing a change-of-measure from one probability measure to an equivalent one. A sufficient condition to have $\mathbb{E}[\mathcal{E}(\theta M)_t] = 1$ is that $\mathcal{E}(\theta M)$ be a martingale; we can typically force this to be the case by demanding some integrability of the quadratic variation. A strong condition – where $[M]$ is bounded – is considered in the next theorem.

Theorem 7.1 *Suppose for every $t \geq 0$ there exists a (non-random) $K_t < \infty$ such that $[M]_t \leq K_t$ a.s., then*

$$\mathbb{P}\left(\max_{s\leq t} M_s > y\right) \leq \exp\left(\frac{-y^2}{2K_t}\right). \tag{8}$$

[1] We have used here the fact that a non-negative local martingale is a supermartingale: to see this take a reducing sequence and apply the conditional version of Fatou's lemma.

Proof. Let $M_t^* = \max_{s \le t} M_s$. For any $\theta, y > 0$ we have

$$\begin{aligned}\mathbb{P}\left(M_t^* > y\right) &\le \mathbb{P}\left(\max_{s \le t} \mathcal{E}\left(\theta M\right)_s > \exp\left(\theta y - \frac{1}{2}\theta^2 K_t\right)\right) \\ &\le \exp\left(-\theta y + \frac{1}{2}\theta^2 K_t\right);\end{aligned}$$

where the last inequality comes from Doob's maximal inequality and a localisation argument. By minimising the right-hand side with respect to $\theta > 0$ we obtain the stated estimate. ■

Under these conditions, we can use the conclusion (8) to prove that $\mathcal{E}\left(\theta M\right)$ is a martingale. Indeed, for $\theta > 0$ we have that

$$\begin{aligned}\mathbb{E}\left[\exp\left(\theta M_t^*\right)\right] &= \mathbb{E}\left[\int_0^\infty 1_{\{\exp(\theta M_t^*) > x\}} dx\right] \\ &= \int_0^\infty \mathbb{P}\left(\exp\left(\theta M_t^*\right) > x\right) dx \\ &= 1 + \int_{e^\theta}^\infty \mathbb{P}\left(M_t^* > y\right) \theta e^{\theta y} dy \\ &\le 1 + \theta \int_{e^\theta}^\infty \exp\left(\frac{-y^2}{2K_t} + \theta y\right) y \\ &< \infty. \end{aligned} \tag{9}$$

We already know that $\mathcal{E}\left(\theta M\right)$ is a local martingale so that, along a reducing sequence (T_n), we have

$$\mathbb{E}\left[\mathcal{E}\left(\theta M\right)_t^{T_n} \middle| \mathcal{F}_s\right] = \mathcal{E}\left(\theta M\right)_s^{T_n} \tag{10}$$

for $s < t$. From (9) we have that $\sup_n \mathcal{E}\left(\theta M\right)_t^{T_n} \le \exp\left(\theta M_t^*\right) \in L^1$ and the conditional dominated convergence theorem allows us to take the limit as $n \to \infty$ in (10).

8 Girsanov's theorem

We continue to work on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$, and we consider how the class of local martingales is affected by changing $\mathbb{P}$ to an equivalent probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$. We recall that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent (written as: $\mathbb{P} \simeq \mathbb{Q}$) if for every $A \in \mathcal{F}$ we have $\mathbb{P}(A) = 0$ if and only if $\mathbb{Q}(A)$; that is, the set of $\mathbb{P}$-null sets is the same as the set of $\mathbb{Q}$-null sets. An important result concerning equivalent probability measures is the Radon-Nikodym theorem: $\mathbb{P} \simeq \mathbb{Q}$ iff there exists a random variable $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ with $Z > 0$ a.s. such that

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}[Z 1_A].$$

It is then easy to show that Z is unique a.s. and that $\mathbb{E}^{\mathbb{P}}[Z] = 1$. The random variable Z is usually denoted $\frac{d\mathbb{Q}}{d\mathbb{P}}$ and is referred to as the Radon-Nikodym derivative of $\mathbb{Q}$ w.r.t. $\mathbb{P}$. The inverse $Z^{-1} \in L^1(\Omega, \mathcal{F}, \mathbb{Q})$ and

$$\mathbb{P}(A) = \mathbb{E}^{\mathbb{Q}}\left[Z^{-1} 1_A\right]$$

so that, for equivalent measures, we have

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^{-1} \quad \text{a.s.}$$

Given a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ we let $(Z_t)_{t\geq 0}$ denote the $\left(\{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$-martingale defined by

$$Z_t = \mathbb{E}^{\mathbb{P}}\left[\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right| \mathcal{F}_t\right]$$

That (Z_t) is a UI martingale follows from $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

Exercise 8.1 *Show that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent on $(\Omega, \mathcal{F}_t)$ for any $t \geq 0$. From the Radon-Nikodym theorem there then exists a Radon-Nikodym derivative of $\mathbb{Q}$ w.r.t. $\mathbb{P}$. This is a random variable on $(\Omega, \mathcal{F}_t)$ which we denote by*

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}.$$

Prove that

$$Z_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}.$$

Lemma 8.2 *$(M_t)_{t\geq 0}$ is a $(\{\mathcal{F}_t\}, \mathbb{Q})$-local martingale iff $(Z_t M_t)_{t\geq 0}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$-local martingale*

Proof. Let $0 \leq T_n \leq T_{n+1} \uparrow \infty$ $\mathbb{Q}$-a.s. be a reducing sequence of stopping times for M under $\mathbb{Q}$. Note that $\mathbb{Q}(\{T_n \uparrow \infty\}) = 1 = \mathbb{P}(\{T_n \uparrow \infty\})$ because $\mathbb{P} \simeq \mathbb{Q}$. We prove that ZM^{T_n} is a $\mathbb{P}$-martingale for every n; it will then follow that $\left(ZM^{T_n}\right)^{T_n} = Z^{T_n}M^{T_n} = (ZM)^{T_n}$ is a martingale for every n. To see this let $s \le t$ and take any $A \in \mathcal{F}_s$ then if $N := M^{T_n}$ we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[N_t 1_A\right] &= \mathbb{E}^{\mathbb{P}}\left[Z N_t 1_A\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[Z_t N_t 1_A\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[Z_s Z_s^{-1} \mathbb{E}^{\mathbb{P}}\left[\left. Z_t N_t \right| \mathcal{F}_s\right] 1_A\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[Z Z_s^{-1} \mathbb{E}^{\mathbb{P}}\left[\left. Z_t N_t \right| \mathcal{F}_s\right] 1_A\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[Z_s^{-1} \mathbb{E}^{\mathbb{P}}\left[\left. Z_t N_t \right| \mathcal{F}_s\right] 1_A\right],
\end{aligned}$$

from which we obtain

$$\mathbb{E}^{\mathbb{Q}}\left[\left. N_t \right| \mathcal{F}_s\right] = Z_s^{-1} \mathbb{E}^{\mathbb{P}}\left[\left. Z_t N_t \right| \mathcal{F}_s\right]$$

so that N is a $\mathbb{Q}$-martingale iff NZ is a $\mathbb{P}$-martingale. ■

We now make the assumption that Z is a continuous martingale. In this case, we can make a link with exponential martingales since $\left(Z_t^{-1}\right)_{t \ge 0}$ is locally bounded and previsible and the stochastic integral $X_t := \int_0^t Z_s^{-1} dZ_s$ is well-defined and yields a process in $c\mathcal{M}_{0,\mathrm{loc}}$. By definition we then have

$$Z_t = Z_0 + \int_0^t Z_s dX_s,$$

so that $Z = Z_0 \mathcal{E}(X)$.

Theorem 8.3 (Girsanov's theorem) *Suppose that M is in $c\mathcal{M}_{0,loc}$ w.r.t. $\mathbb{P}$ then*

$$N_t := M_t - [M, X]_t = M_t - \int_0^t Z_s^{-1} d[M, Z]_s$$

is in $c\mathcal{M}_{0,loc}$ w.r.t. $\mathbb{Q}$ and $[N] = [M]$.

Proof. The identity $[N] = [M]$ follows from the definition of quadratic variation for continuous semimartingales. From the previous lemma, we need to show that NZ is a $\mathbb{P}$ local martingale. Thus using Itô's formula for the product we have

$$d(N_t Z_t) = N_t dZ_t + Z_t dM_t - Z_t d[M, X]_t + d[N, Z]_t$$

and since $[N, Z]_t = [M, Z]_t = \int_0^t Z_s d[M, Z]_s$ it follows that

$$N_t Z_t = \int_0^t N_s dZ_s + \int_0^t Z_s dM_s \in c\mathcal{M}_{0,\mathrm{loc}}.$$

■

9 The Cameron-Martin theorem

As an example of Girsanov's theorem we work on the particular probability space $\Omega = C\left([0,\infty),\mathbb{R}^d\right)$ of continuous paths with the topology of uniform convergence. We can define the canonical process by $(B_t)_{t\geq 0}$ considering the evaluation mapping $B_t(\omega) = \omega(t)$. We let $\mathcal{F} = \sigma(B_u : u \geq 0)$ and take as the measure on $(\Omega,\mathcal{F})$ the unique probability measure that gives the canonical process $(B_t)_{t\geq 0}$ the distribution of Brownian motion. We can define a filtration by setting

$$\mathcal{F}_t = \sigma(B_u : u \leq t)$$

to be the natural filtration of B and set $\mathcal{F} = \sigma(B_u : u \geq 0)$. Note that this filtration need not satisfy the usual conditions, however we can always find a minimal augmentation which does have this property.

Definition 9.1 (usual augmentation) *By the usual augmentation of* $\left(\Omega,\mathcal{G},\{\mathcal{G}_t\}_{t\geq 0},\mathbb{P}\right)$ *we mean the minimal (w.r.t. inclusion)* $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}\right)$ *satisfying the usual conditions such that* $\mathcal{G}_t \subseteq \mathcal{F}_t$ *for all* t*, and* $\mathcal{G} \subseteq \mathcal{F}$.

Lemma 9.2 (Blumenthal's zero-one law) $\mathcal{G}_0$ *is trivial, i.e. if* $A \in \mathcal{G}_0$ *then* $\mathbb{P}(A) \in \{0,1\}$.

Theorem 9.3 (The martingale representation theorem) *Suppose that* $Y \in L^2(\mathcal{G}_\infty)$. *Then there exists a previsible process* H *with* $\mathbb{E}\left[\int_0^\infty |H_u|^2\,du\right] < \infty$ *such that*

$$\mathbb{E}[Y|\,\mathcal{G}_t] = \mathbb{E}[Y] + \int_0^t \langle H_u, dB_u\rangle .$$

Furthermore every $\{\mathcal{G}_t\}$*-local martingale is continuous and can be represented as a stochastic integral w.r.t.* B *of a previsible process* H *with*

$$\mathbb{P}\left(\int_0^t |H_u|^2\,du < \infty,\ \forall t\right) = 1.$$

We do not give proofs of these results here. We are now in a position to prove the following.

Theorem 9.4 (Cameron-Martin theorem) *Let* $\mathbb{Q}$ *be a probability measure on* $(\Omega,\mathcal{G})$ *such that* $\mathbb{Q} \simeq \mathbb{P}$ *then there exists a previsible process* H *such that*

$$Z_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{G}_t} = \exp\left(\int_0^t \langle H_u, dB_u\rangle - \frac{1}{2}\int_0^t |H_u|^2\,du\right).$$

The process $W_t = B_t - \int_0^t H_u du$ *is a Brownian motion under* $\mathbb{Q}$.

Proof. From Blumenthal's zero-one law we have $Z_0 = 1$. Since $Z = (Z_t)_{t\geq 0}$ is a martingale relative to $\left(\{\mathcal{G}_t\}_{t\geq 0}, \mathbb{P}\right)$ the martingale representation theorem shows that Z is continuous and hence $X_t = \int_0^t Z_s^{-1} dZ_s$ is a continuous local martingale w.r.t. $\left(\{\mathcal{G}_t\}_{t\geq 0}, \mathbb{P}\right)$. The martingale representation theorem gives the existence of previsible process H such that

$$X_t = \int_0^t \langle H_s, dB_s \rangle ,$$

and since $Z = \mathcal{E}(X)$ we have

$$Z_t = \exp\left(X_t - \frac{1}{2}[X]_t\right) = \exp\left(\int_0^t \langle H_s, dB_s \rangle - \frac{1}{2}\int_0^t |H_s|^2 \, ds\right).$$

For the final assertion, we use Girsanov's theorem to see that

$$W_t^i = B_t^i - \left[B^i, X\right]_t = B_t^i - \int_0^t H_s^i ds$$

is a $\mathbb{Q}$-local martingale. The $\mathbb{Q}$-covariation equals $\left[W^i, W^j\right]_t = \left[B^i, B^j\right]_t = \delta_{i,j} t$, and so by Lèvy's characterisaton W is a Brownian motion under $\mathbb{Q}$. ■

10 The Burkholder-Davis-Gundy inequalities

We now prove a fundamental estimate for continuous local martingales. Given $M \in c\mathcal{M}_{0,\text{loc}}$ we define the maximal function by

$$M_t^* = \sup_{s \le t} |M_s| .$$

The Burkholder-Davis-Gundy (BDG) inequalities relate the moments of the maximal function to the moments of the quadratic variation of M. To give the statement, we must first make the following definition.

Definition 10.1 *Let $F : [0, \infty) \to [0, \infty)$ be a continuous increasing function such that $F(0) = 0$. We say F is moderate if there exists $\alpha > 1$ such that*

$$\sup_{x>0} \frac{F(\alpha x)}{F(x)} < \infty. \tag{11}$$

Exercise 10.2 *If F is moderate show that for any $\beta > 1$*

$$\sup_{x>0} \frac{F(\beta x)}{F(x)} < \infty.$$

Hint: Assume (11) for some $\alpha > 1$ and use the identity

$$\frac{F(\alpha^n x)}{F(x)} = \prod_{k=1}^{n} \frac{F\left(\alpha^{n-k+1} x\right)}{F(\alpha^{n-k} x)}.$$

Remark 10.3 *$F(x) = x^p$ is moderate for any $p > 0$. The function $F(x) = e^x$ is not moderate.*

Theorem 10.4 (Burkholder-Davis-Gundy) *Assume $F : [0, \infty) \to [0, \infty)$ is a moderate function. There exist constant $c < C$ such that for any $M \in c\mathcal{M}_{0,loc}$*

$$c\mathbb{E}\left[F\left([M]_\infty^{1/2}\right)\right] \le \mathbb{E}\left[F(M_\infty^*)\right] \le C\mathbb{E}\left[F\left([M]_\infty^{1/2}\right)\right]$$

The proof of this result turns on the following general lemma.

Lemma 10.5 (Good λ-inequality) *Suppose $X, Y \ge 0$ are random variables. Assume that there exists $\beta > 1$ such that for all $\lambda > 0, \delta > 0$,*

$$\mathbb{P}(X > \beta\lambda, Y \le \delta\lambda) \le \psi(\delta)\, \mathbb{P}(X > \lambda), \tag{12}$$

where $\psi : (0, \infty) \to (0, \infty)$ is such that $\lim_{\delta \downarrow 0} \psi(\delta) = 0$. Then for any moderate function $F : [0, \infty) \to [0, \infty)$ we have

$$\mathbb{E}[F(X)] \le C\mathbb{E}[F(Y)], \tag{13}$$

where C depends only on F, β and ψ.

Proof. If $\mathbb{E}[F(Y)] = \infty$ there is nothing to prove, so we may assume $F(Y)$ is integrable. If (12) holds for X then it holds for $X \wedge n$; we may thus assume that $\mathbb{E}[F(X)] < \infty$ and prove (13). Since F is moderate, choose $\gamma > 0$ such that $F(x/\beta) \geq \gamma F(x)$. Integrating (12) w.r.t. the Lebesgue-Stieltjes measure associated with F gives on the RHS

$$\psi(\delta)\int_0^\infty \mathbb{E}\left[1_{\{X>\lambda\}}\right]F(d\lambda) = \psi(\delta)\,\mathbb{E}\left[\int_0^\infty 1_{\{X>\lambda\}}F(d\lambda)\right] = \psi(\delta)\,\mathbb{E}[F(X)]$$

and for the LHS

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty 1_{\{Y/\delta\leq\lambda<X/\beta\}}F(d\lambda)\right] &= \mathbb{E}\left[\left(\int_0^{X/\beta}F(d\lambda) - \int_0^{Y/\delta}F(d\lambda)\right)^+\right] \\ &\geq \mathbb{E}[F(X/\beta)] - \mathbb{E}[F(Y/\delta)] \\ &\geq \gamma\mathbb{E}[F(X)] - \mathbb{E}[F(Y/\delta)],\end{aligned}$$

so that $\gamma\mathbb{E}[F(X)] - \mathbb{E}[F(Y/\delta)] \leq \psi(\delta)\,\mathbb{E}[F(X)]$ and hence for some $d < \infty$

$$(\gamma - \psi(\delta))\,\mathbb{E}[F(X)] \leq \mathbb{E}[F(Y/\delta)] \leq d\mathbb{E}[F(Y)]$$

Choose δ small enough that $\gamma - \psi(\delta) > \gamma/2$ to give

$$\mathbb{E}[F(X)] \leq \frac{2d}{\gamma}\mathbb{E}[F(Y)].$$

■

Using this result we can prove Theorem 10.4.

Proof. (BDG inequalities) We verify the BDG inequalities for the pair of random variables M_∞^* and $[M]_\infty^{1/2}$. To do so we first make a general observation: for any $X \in \mathrm{c}\mathcal{M}_{0,\mathrm{loc}}$, with T_a the stopping time

$$T_a^X = \inf\{t > 0 : X_t = a\}, \tag{14}$$

then for $a < 0 < b$ we have

$$\mathbb{P}(T_b < T_a | \mathcal{F}_0) = \frac{-a}{b-a}.$$

This is self-evident if X is a martingale by optional-stopping. The case where $X \in \mathrm{c}\mathcal{M}_{0,\mathrm{loc}}$ follows by considering the martingale X^{τ_K} for $K > \max(-a, b)$, wherein $\tau_K = \inf\{t > 0 : |X_t| > K\}$, and noticing that $T_a^X = T_b^{X^{\tau_K}}$ and $T_b^X = T_b^{X^{\tau_K}}$.

Suppose then that $\beta > 1$ and $\lambda > 0$ and assume that $0 < \delta < \beta - 1$. Introduce the stopping time

$$\tau = \inf\{t > 0 : |M_t| > \lambda\}$$

and the process $X_t := (M_{t+\tau} - M_\tau)^2 - \left([M]_{t+\tau} - [M]_\tau\right)$, which is an element of $\mathrm{c}\mathcal{M}_{0,\mathrm{loc}}$ defined relative to the filtration $\{\mathcal{G}_t\} := \{\mathcal{F}_{\tau+t}\}$. The crucial observation is then that

$$\left\{M_\infty^* > \beta\lambda, [M]_\infty^{1/2} \leq \delta\lambda\right\} \subseteq \left\{T_b^X < T_a^X\right\}$$

where $a = -\delta^2\lambda^2$, $b := (\beta-1)^2\lambda^2 - \delta^2\lambda^2$ and where T^X is as in (14). Hence we have

$$\mathbb{P}\left(M_\infty^* > \beta\lambda, [M]_\infty^{1/2} \le \delta\lambda \middle| \mathcal{F}_\tau \right) \le \frac{\delta^2}{\beta^2 - 1},$$

from which we deduce that that

$$\begin{aligned}
\mathbb{P}\left(M_\infty^* > \beta\lambda, [M]_\infty^{1/2} \le \delta\lambda\right) &= \mathbb{P}\left(M_\infty^* > \beta\lambda, [M]_\infty^{1/2} \le \delta\lambda, \tau < \infty\right) \\
&= \mathbb{E}\left[\mathbb{P}\left(M_\infty^* > \beta\lambda, [M]_\infty^{1/2} \le \delta\lambda \middle| \mathcal{F}_\tau \right) 1_{\{\tau<\infty\}}\right] \\
&\le \frac{\delta^2}{\beta^2 - 1}\mathbb{P}(\tau < \infty) \\
&= \frac{\delta^2}{\beta^2 - 1}\mathbb{P}(M_\infty^* > \lambda)
\end{aligned}$$

as required. We deduce from Lemma 10.5 that $\mathbb{E}\left[F\left(M_\infty^*\right)\right] \le C\mathbb{E}\left[F\left([M]_\infty^{1/2}\right)\right]$. The other inequality can be seen by reworking the above proof with M_∞^* and $[M]_\infty^{1/2}$ interchanged. ■

Example 10.6 *The following two examples are typical of how this result is used in SDE theory: suppose that $M = B$ is a Brownian motion and H, G are bounded previsible processes*

1. *For any $p > 0$,*

$$\mathbb{E}\left[\left|\int_0^T H_u dB_u\right|^p\right] \le C_p \mathbb{E}\left[\left|\int_0^T H_u^2 du\right|^{p/2}\right].$$

2. *Suppose $X_t = X_0 + \int_0^t G_u du + \int_0^t H_u dB_u$, then if $X_0 \in L^p$ and $p \ge 2$ we can bound $\mathbb{E}\left[|X_T|^p\right]$ by*

$$\begin{aligned}
& 3^{p-1}\left(\mathbb{E}\left[|X_0|^p\right] + \mathbb{E}\left[\left|\int_0^T G_u du\right|^p\right] + \mathbb{E}\left[\left|\int_0^T H_u^2 du\right|^{p/2}\right].\right) \\
\le\ & 3^{p-1}\left(\mathbb{E}\left[|X_0|^p\right] + T^{p-1}\mathbb{E}\left[\int_0^T |G_u|^p du\right] + T^{p/2-1}\mathbb{E}\left[\left|\int_0^T |H_u|^p du\right|\right]\right).
\end{aligned}$$

11 Tanaka's formula and semimartingale local time

Given a continuous semimartingale X Itô's formula allows us to compute the semimartingale decomposition of $f(X_t)$ whenever $f \in C^2$. Sometimes we would like to relax the C^2 assumption.

Theorem 11.1 (Tanaka's formula) *Suppose X is a continuous semimartingale. There exists a continuous, increasing adapted process $L = (L_t)_{t \geq 0}$ such that*

$$|X_t| = |X_0| + \int_0^t sgn(X_s)\, dX_s + L_t, \tag{15}$$

wherein the function

$$sgn(x) := \begin{cases} 1 & x > 0 \\ -1 & x \leq 0 \end{cases}.$$

Furthermore we have that

$$\int_0^t 1_{\{X_s \neq 0\}}(s)\, dL_s = 0,$$

so that L increases only when X is zero.

Remark 11.2 *The formula (15) is called Tanaka's formula. The process L is called the local time of X at* 0.

Proof. We approximate the function $f(x) = |x|$ by a C^2 function and use Itô's formula. To do so, we let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth non-decreasing function such that $\phi(x) = -1$ for $x \leq 0$ and $\phi(x) = 1$ for $x \geq 1$ (exercise: convince yourself that there exists such a function). We then define

$$f_n(x) = \begin{cases} \int_0^x \phi(ny)\, dy & x \geq 0 \\ \int_x^0 \phi(ny)\, dy & x < 0 \end{cases},$$

an easy exercise shows that

$$\sup_x |f_n(x) - |x|| \leq \frac{2}{n}, \tag{16}$$

and furthermore $f_n'(x) \leq f_{n+1}'(x) \uparrow \text{sgn}(x)$ as $n \to \infty$. Applying Itô's formula to $f_n(X_t)$ gives

$$f_n(X_t) = f_n(X_0) + \int_0^t f_n'(X_s)\, dX_s + \frac{1}{2}\int_0^t f_n''(X_s)\, d[X]_s. \tag{17}$$

The second-order derivative of f_n is given by $f_n''(x) = n\phi'(nx)$ on $(0, 1/n)$ and is zero otherwise. It follows that $f_n''(X_s) \geq 0$ a.s. and hence for each n the process

$$C_t^n := \frac{1}{2}\int_0^t f_n''(X_s)\, d[X]_s$$

is non-decreasing in t. Write $X = X_0 + M + A$ its decomposition as a semi-martingale, where, by localisation, we may assume that M is a bounded and continuous martingale and A is a process of finite variation. Using Itô's isometry, the uniform convergence (16), and two applications of the dominated convergence theorem (w.r.t. $\int ..d[M]$ and w.r.t. $\mathbb{E}$) we have that

$$\begin{aligned}\mathbb{E}\left[\left\{\int_0^\infty \left(f_n'\left(X_s\right) - \operatorname{sgn}\left(X_s\right)\right) dM_s\right\}^2\right] &= \mathbb{E}\left[\int_0^\infty \left(f_n'\left(X_s\right) - \operatorname{sgn}\left(X_s\right)\right)^2 d\left[M\right]_s\right] \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

From Doob's L^2 inequality we obtain uniform convergence as $n \to \infty$

$$\sup_{t>0}\left|\int_0^t f_n'\left(X_s\right) dM_s - \int_0^t \operatorname{sgn}\left(X_s\right) dM_s\right| \to 0$$

in L^2 and (along a subsequence) almost surely. We also have, as $n \to \infty$,

$$\begin{aligned} & \sup_{t>0}\left|\int_0^t f_n'\left(X_s\right) dA_s - \int_0^t \operatorname{sgn}\left(X_s\right) dA_s\right| \\ \leq & \sup_{t>0}\int_0^t \left|f_n'\left(X_s\right) - \operatorname{sgn}\left(X_s\right)\right| \left|dA_s\right| \\ = & \int_0^\infty \left|f_n'\left(X_s\right) - \operatorname{sgn}\left(X_s\right)\right| \left|dA_s\right| \\ \to & \; 0 \end{aligned}$$

again from the dominated convergence theorem.

Rewriting (17) as

$$f_n\left(X_t\right) - f_n\left(X_0\right) - \int_0^t f_n'\left(X_s\right) dX_s = C_t^n$$

we have shown that the left-hand side converges uniformly to $|X_t| - |X_0| - \int_0^t \operatorname{sgn}(X_s)\, dX_s$; it follows that the continuous process C^n converges uniformly to a (necessarily continuous) process L. Since C^n was non-decreasing for every n, L must also be a non-decreasing processes. Finally since

$$\int_0^t 1_{\{|X_s|>1/n\}} dC_s^m = 0 \text{ for all } m \geq n$$

we have $\int_0^t 1_{\{|X_s|>1/n\}} dL_s = 0$ for every n. Letting $n \to \infty$ we learn that $\int_0^t 1_{\{X_s \neq 0\}}(s)\, dL_s = 0$. ■

Part II

Stochastic differential equations

12 Pathwise uniqueness and strong solutions of stochastic differential equations

We will work on a setup consisting of $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ which satisfies the usual conditions on which there is defined an $\mathbb{R}^d$-valued Brownian motion $(B_t)_{t\geq 0}$ and a $\mathcal{F}_0$-measurable random variable $\xi \in \mathbb{R}^n$.Our aim in this section is to understand stochastic differential equations of the form

$$X_t = \xi + \int_0^t \sigma\left(s, X.\right) dB_s + \int_0^t b\left(s, X.\right) ds, \tag{18}$$

where $\sigma : \mathbb{R}_+ \times C\left(\mathbb{R}_+, \mathbb{R}^n\right) \to \mathrm{L}\left(\mathbb{R}^d, \mathbb{R}^n\right)$ and $b : \mathbb{R}_+ \times C\left(\mathbb{R}_+, \mathbb{R}^n\right) \to \mathbb{R}^n$ satisfy the following condition.

Condition 12.1 *For every continuous adapted $\mathbb{R}^n$-valued process X, $\left(\sigma\left(t, X.\right)\right)_{t\geq 0}$ and $\left(b\left(t, X.\right)\right)_{t\geq 0}$ are, respectively, a previsible process and an adapted process and they are such that*

$$\mathbb{P}\left(\int_0^t \left|\sigma\left(s, X.\right)\right|^2 ds + \int_0^t \left|b\left(s, X.\right)\right| ds < \infty\right) = 1. \tag{19}$$

General conditions on σ and b for this to hold can be found in volume 2 of Rogers and Williams.

Example 12.2 *If $\sigma\left(t, x.\right) = \sigma\left(x_t\right)$ and $b\left(t, x.\right) = b\left(x_t\right)$ and if σ and b are Borel measurable then the condition is satisfied and we say the SDE (18) is a diffusion SDE.*

Definition 12.3 *We say that* ***pathwise uniqueness*** *holds for (18) if for any setup $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ ξ and B, where B is an $\{\mathcal{F}_t\}$-Brownian motion, and any two continuous semimartingales X and Y relative to $\{\mathcal{F}_t\}_{t\geq 0}$ for which (19) holds and for which*

$$\begin{aligned} X_t &= \xi + \int_0^t \sigma\left(s, X.\right) dB_s + \int_0^t b\left(s, X.\right) ds \text{ and} \\ Y_t &= \xi + \int_0^t \sigma\left(s, Y.\right) dB_s + \int_0^t b\left(s, Y.\right) ds, \end{aligned}$$

we have $\mathbb{P}\left(X_t = Y_t, \forall t\right) = 1$; that is, X and Y are indistinguishable.

Pathwise uniqueness guarantees that there is never more than one solution for any given setup. If there is one and only one solution for every setup, the SDE is called an **exact SDE**

Definition 12.4 *The SDE (18) is exact if on any setup,* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right), \xi$ *and* B, *there exists a semimartingale satisfying (18) and (19), and this semimartingale is unique up to indistinguishability.*

Our immediate goal is to find conditions under which the SDE (18) is exact. As in the theory of ODEs, Lipschitz bounds on σ and b lead to useful estimates.

Condition 12.5 *We say* σ *and* b *are Lipschitz continuous if there exists* $C < \infty$ *such that for all* $t \geq 0$ *and all* $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, *we have*

$$
\begin{aligned}
|\sigma(t, x.) - \sigma(t, y.)| &\leq C \sup_{0\leq s\leq t} |x_s - y_s| \quad \text{and} \\
|b(t, x.) - b(t, y.)| &\leq C \sup_{0\leq s\leq t} |x_s - y_s|.
\end{aligned}
$$

Example 12.6 *In the diffusion setting where* $\sigma(t, x.) = \sigma(x_t)$ *and* $b(t, x.) = b(x_t)$ *this condition will be satisfied if* σ *and* b *are Lipschitz continuous functions in the usual sense.*

Lemma 12.7 *Let* $T > 0$ *and* $p \geq 2$ *and suppose that* $\xi \in L^p(\mathcal{F}_0)$. *Let* σ *be a previsible process with values in* $L\left(\mathbb{R}^d, \mathbb{R}^n\right)$ *and* b *an adapted* $\mathbb{R}^n$*-valued process and let* X *be the semimartingale given by*

$$
X_t = \xi + \int_0^t \sigma_s dB_s + \int_0^t b_s ds.
$$

Then for some constant $C = C(p, n, T) < \infty$ *we have for all* $0 \leq t \leq T$

$$
\mathbb{E}\left[\sup_{s\leq t} |X_s|^p\right] \leq C\mathbb{E}\left[|\xi|^p + \int_0^t (|\sigma_s|^p + |b_s|^p)\, ds\right]. \tag{20}
$$

Proof. We first have that

$$
\sup_{s\leq t} |X_s|^p \leq 3^{p-1}\left(|\xi|^p + \sup_{s\leq t}\left|\int_0^s \sigma_s dB_u\right|^p + \sup_{s\leq t}\left|\int_0^s b_u du\right|^p\right).
$$

Then, from the BDG inequalities we have that

$$
\mathbb{E}\left[\sup_{s\leq t}\left|\int_0^s \sigma_s dB_u\right|^p\right] \leq C_p \mathbb{E}\left[\left(\int_0^t |\sigma_u|^2 du\right)^{p/2}\right] \leq C_p t^{p/2-1}\mathbb{E}\left[\int_0^t |\sigma_u|^p du\right].
$$

We also have

$$
\mathbb{E}\left[\sup_{s\leq t}\left|\int_0^s b_u du\right|^p\right] \leq t^{p-1}\mathbb{E}\left[\int_0^t |b_u|^p du\right].
$$

Putting things together leads to the estimate (20). ■

Suppose that σ and b now are Lipschitz continuous in the sense of Condition 12.5. If X and Y satisfy

$$\begin{aligned} X_t &= \xi + \int_0^t \sigma(s, X.)\, dB_s + \int_0^t b(s, X.)\, ds \text{ and} \\ Y_t &= \eta + \int_0^t \sigma(s, Y.)\, dB_s + \int_0^t b(s, Y.)\, ds, \end{aligned}$$

then we can use the last lemma to obtain the estimate

$$\mathbb{E}\left[\sup_{s\le t} |X_s - Y_s|^p\right] \le C\left\{\mathbb{E}\left[|\xi-\eta|^p\right] + \mathbb{E}\left[\int_0^t \sup_{u\le s} |X_u - Y_u|^p\, ds\right]\right\}. \qquad (21)$$

13 Existence and uniqueness

The following lemma is often useful in estimating the growth of a solution to a differential equation.

Lemma 13.1 (Gronwall's lemma) *Suppose that $h : [0, T] \to \mathbb{R}$ is a continuous function and that there exists $c > 0$ and $d > 0$ such that*

$$h_t \leq c + d \int_0^t h_s ds \text{ for all } t \in [0, T].$$

Then $h_t \leq ce^{dt}$ for all $t \in [0, T]$.

Proof. Let $x_t := c + d \int_0^t h_s ds$ and notice that x is C^1 and x also satisfies

$$x_t \leq c + d \int_0^t x_s ds.$$

We have $h_t \leq x_t$ and it will suffice to prove that $x_t \leq ce^{dt}$ for all $t \in [0, T]$. This follows from $\frac{d}{dt}\left(x_t e^{-dt}\right) = d\left(h_t - x_t\right) e^{dt} \leq 0$ since we then have $x_t e^{-dt} \leq x_0 = d$. ■

Theorem 13.2 *Assume that σ and b satisfy Conditions 12.1 and 12.5 and further that there exists C_T with*

$$\sup_{0 \leq t \leq T} \left(|\sigma(t, 0)| + |b(t, 0)|\right) = C_T < \infty \ . \tag{22}$$

Let $\xi \in L^2(\mathcal{F}_0)$ then the SDE (18) is exact.

Proof. We will construct a unique solution on the usual augmentation of the natural filtration generated by B (recall Definition 9.1). Let L_T^2 denote the space of continuous adapted processes on this setup which satisfy

$$||X||_T := \mathbb{E}\left[\sup_{s \leq T} |X_s|^2\right]^{1/2} < \infty.$$

The conditions ensure that the map $X \to I(X)$ is well-defined on L_T^2 where

$$I(X)_t = \xi + \int_0^t \sigma(s, X.)\, dB_s + \int_0^t b(s, X.)\, ds,$$

furthermore we can use (21) with $p = 2$ and (22) to see that $I(X) \in L_T^2$:

$$\mathbb{E}\left[\sup_{s \leq T} |X_s|^2\right] \leq c_1(T) \left\{\mathbb{E}\left[|\xi|^2\right] + \mathbb{E}\left[\int_0^T \left(|\sigma(s, X.)|^2 + |b(s, X.)|^2\right) ds\right]\right\},$$

the right hand side is finite since

$$\begin{aligned}\int_0^T |\sigma(s,X.)|^2\,ds &\leq 2\left(\int_0^T |\sigma(s,0)|^2\,ds + \int_0^T |\sigma(s,X.)-\sigma(s,0)|^2\,ds\right)\\ &\leq 2C_T^2 T + 2C^2\int_0^T \sup_{0\leq s\leq t} |X_s|^2\,ds \in L^1\end{aligned}$$

and likewise for $\int_0^T |b(s,X.)|^2\,ds$.

A process $X \in L_T^2$ is a solution to the SDE (18) if and only if it is a fixed point of I; we thus prove that I has a unique fixed point. We define a sequence in L_T^2 by taking

$$X^0 \equiv \xi \text{ and then } X^{n+1} = I(X^n) \text{ for } n = 0,1,2,...$$

We let $\Delta_t^{n+1} = \mathbb{E}\left[\sup_{s\leq t}\left|X_s^{n+1} - X_s^n\right|^2\right]$ and use (22) to derive the estimate

$$\begin{aligned}\Delta_t^{n+1} &= \mathbb{E}\left[\sup_{s\leq t}\left|I(X^n)_s - I\left(X^{n-1}\right)_s\right|^2\right]\\ &\leq c_2(T)\int_0^t \mathbb{E}\left[\sup_{0\leq u\leq s}\left|X_s^n - X_s^{n-1}\right|^2\right]ds\\ &= c_2(T)\int_0^t \Delta_s^n ds\end{aligned}$$

for all $t \leq T$. A simple induction gives that

$$\begin{aligned}\Delta_t^n &\leq c_2(T)^n \int_0^t\int_0^{t_n}...\int_0^{t_2} \Delta_{t_1}^0 dt_1...dt_{n-1}dt_n\\ &\leq \frac{c_2(T)^n t^n \Delta_T^0}{n!},\end{aligned}$$

for all $t \leq T$. From which it follows that $(X^n)_{n=0}^\infty$ is Cauchy and hence there exists a limit X such that $X = \lim_{n\to\infty} X^n$ uniformly in L^2.

The process X is adapted and continuous (as the uniform limit of such processes). To see that X is a fixed point of I we again apply (22) to give

$$\mathbb{E}\left[\sup_{s\leq T}\left|X_s^{n+1} - I(X)_s\right|^2\right] \leq c_3(T)\mathbb{E}\left[\sup_{s\leq T}|X_s^n - X_s|^2\right] \to 0,$$

as $n \to \infty$, i.e. $I(X) = X$. Finally we prove uniqueness: if X' is another solution then for all $0 \leq t \leq T$

$$\mathbb{E}\left[\sup_{s\leq t}|X_s - X_s'|^2\right] \leq c_4(T)\int_0^t \mathbb{E}\left[\sup_{s\leq t}|X_u - X_u'|^2\right]ds,$$

whereupon Gronwall's lemma gives that

$$\mathbb{E}\left[\sup_{s\leq t}|X_s - X_s'|^2\right] = 0,$$

for all $0 \leq t \leq T$ and thus X and X' are indistinguishable. ■

Remark 13.3 *The assumption that ξ is square-integrable can be relaxed by a localisation. The Lipschitz condition can be replaced by a local Lipschitz one: for every $N \in \mathbb{N}$, there exists $C = C_N < \infty$ such that we have*

$$\begin{aligned} |\sigma(t, x.) - \sigma(t, y.)| &\leq C \sup_{0 \leq s \leq t} |x_s - y_s| \text{ and} \\ |b(t, x.) - b(t, y.)| &\leq C \sup_{0 \leq s \leq t} |x_s - y_s| . \end{aligned}$$

for all $t \geq 0$ and all $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$ satisfying $\sup_{0 \leq s \leq t} |x_s| \vee \sup_{0 \leq s \leq t} |y_s| \leq N$.

13.1 Diffusions

We turn attention now to the case where $\sigma(t, x.) = \sigma(x_t)$ and $b(t, x.) = b(x_t)$. One of the advantages of SDEs is that they provide a direct method for constructing a diffusion process.

Definition 13.4 *Suppose $a : \mathbb{R}^n \to \Lambda^n$ (Λ^n = the space of $n \times n$ non-negative definite matrices) and $b : \mathbb{R}^n \to \mathbb{R}^n$ are measurable. An (a, b)-diffusion is a continuous semimartingale $X = \left(\left(X_t^1, ..., X_t^n\right)\right)_{t \geq 0}$ on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ such that*

$$M_t^i = X_t^i - X_0^i - \int_0^t b^i(X_s)\,ds \in c\mathcal{M}_{0,loc} \ \forall i = 1, ..., n$$

with

$$\left[M^i, M^j\right]_t = \int_0^t a^{ij}(X_s)\,ds.$$

Suppose that $f \in C^\infty(\mathbb{R}^n)$ then for any (a, b)-diffusion X, then from Itô's formula we have that

$$\begin{aligned} & f(X_t) - f(X_0) \\ = & \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x^i}(X_s)\,dX_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\,d\left[X^i, X^j\right]_s \\ = & \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x^i}(X_s)\,dM_s^i + \int_0^t \mathcal{A}f(X_s)\,ds, \end{aligned}$$

where $\mathcal{A}$ is the second-order differential operator

$$\mathcal{A}f(x) = \sum_{i=1}^n b^i(x)\frac{\partial f}{\partial x^i}(x) + \frac{1}{2}\sum_{i,j=1}^n a^{ij}(x)\frac{\partial^2 f}{\partial x^i \partial x^j}(x).$$

It follows that

$$C_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\,ds$$

belongs to $\mathrm{c}\mathcal{M}_{0,\mathrm{loc}}$ for all $f \in C^\infty(\mathbb{R}^n)$. This gives an alternative characterisation of diffusions.

Lemma 13.5 *X is an (a, b)-diffusion if and only if $C^f \in c\mathcal{M}_{0,loc}$ for all $f \in C^\infty(\mathbb{R}^n)$.*

Proof. The "only if" part is clear from the above calculation. For the converse, we apply the condition to the functions $f(x) = x^i$ and $f(x) = x^i x^j$. ■

Exercise 13.6 *Prove that the condition in the previous lemma can be weakened to $C^f \in c\mathcal{M}_{0,loc}$ for all f smooth and compactly supported.*

The analytical approach to the construction of diffusions proceeds through the theory of Markov process; under some conditions on (a, b), the operator $\mathcal{A}$ becomes the infinitesimal generator of a Markov process. The probabilistic approach takes the view that a diffusion can be constructed directly by solving an SDE. For instance, if X solves

$$X_t = X_0 + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dB_s$$

then we have

$$\begin{aligned} X_t^i - X_0^i &= \int \mu^i(X_s)\, ds + \sum_{k=1}^d \int_0^t \sigma^{ik}(X_s)\, dB_s^k \\ \left[X^i, X^j\right]_t &= \int_0^t \sigma^{ik}(X_s)\, \sigma^{jk}(X_s)\, ds, \end{aligned}$$

hence X is a $\left(\sigma\sigma^T, \mu\right)$-diffusion. By choosing $\mu = b$ and σ such that $\sigma\sigma^T = a$ we see that X realises an (a, b)-diffusion.

Remark 13.7 *If $n = d$, then by taking $\sigma(x) := a(x)^{1/2}$, the unique non-negative definite square root of a, we can ensure that $\sigma\sigma^T = a$. Conditions on a that make σ locally Lipschitz are important.*

14 Weak solutions

Sometimes it is too strong a requirement to demand the existence of a solution for every $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$, ξ, and every $\{\mathcal{F}_t\}$-Brownian motion B. It is useful to have a less stringent notion of solution which demands that (18) holds for *some* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ and *some* $\{\mathcal{F}_t\}$-Brownian motion B.

Definition 14.1 *Let μ be a probability measure on $\mathbb{R}^n$. We say that (X, B) is a **weak solution** to (18) with initial distribution μ if there exists some $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$, which satisfies the usual conditions, such that B is an $\{\mathcal{F}_t\}$-Brownian motion and X is an $\{\mathcal{F}_t\}$-semimartingale for which:*

1. *X_0 has distribution μ;*
2. *$\forall t$, $\int_0^t \left[|\sigma(s, X)|^2 + |b(s, X)|\right] ds < \infty$ a.s.*
3. *$\forall t$ (18) holds a.s.*

Remark 14.2 *Notice that the Brownian motion is now part of the definition of a solution.*

There is an accompanying notion of uniqueness, uniqueness in law, which allows for the comparison of solutions possibly defined on different probability spaces.

Definition 14.3 *We say that the SDE (18) is **unique in law** (with initial distribution μ) if for any two weak solutions (X, B) and (X', B') with initial distribution μ, X and X' have the same law.*

The following example clarifies the relationship between the different notions of solution. Let $\mu = \delta_0$, and consider the following SDE in $\mathbb{R}$:

$$X_t = \int_0^t \operatorname{sgn}(X_s)\, dB.$$

Any solution (on any filtered probability space) must be a Brownian motion by Lévy's characterisation. Uniqueness in law is therefore immediate. On the other hand, a weak solution exists: given any Brownian motion W, take $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ to be the usual augmentation of the natural filtration of W and define B to be the Brownian motion (Lévy again!)

$$B_t = \int_0^t \operatorname{sgn}(W_s)\, dW_s.$$

The pair (W, B) are then related via

$$\int_0^t \operatorname{sgn}(W_s)\, dB_s = \int_0^t dW_t = W_t$$

and so forms a weak solution. Pathwise uniqueness cannot hold in this example, however: on the same setup $-W$ also satisfies

$$\int_0^t \operatorname{sgn}(-W_s)\, dB_s = -\int_0^t dW_t = -W_t$$

but W and $-W$ are self-evidently not indistinguishable. Worse still, W is not a solution on every setup since from Tanaka's theorem

$$B_t = |W_t| - L_t^0(W),$$

and the augmentation of $\{\sigma(B_s : s \leq t)\}$ is therefore contained in the augmentation of $\{\sigma(|W_s| : s \leq t)\}$, which is strictly smaller than $\{\mathcal{F}_t\}$. On the other hand for W to be a solution on this setup it must be $\{\sigma(B_s : s \leq t)\}$ adapted.

The following theorem completely resolves the relationship between weak solutions, pathwise uniqueness and exactness of SDEs.

Theorem 14.4 (Yamada-Watanabe) *The SDE (18) is exact if and only if there exists a weak solution* ***and*** *pathwise uniqueness holds for (18). In this case, uniqueness in law holds too.*

The proof is technical and long; see Rogers and Williams vol. 2 Theorem 17.1 for the full details.

15 The martingale problem formulation

Here we describe an equivalent way to formulate the notion of a weak solution. Suppose that there exists a weak solution (with initial distribution δ_y) to the SDE

$$X_t = y + \int_0^t \sigma(X_s)\,dB_s + \int_0^t b(X_s)\,ds. \tag{23}$$

We define $\Omega := W^n = C([0,\infty),\mathbb{R}^n)$, the space of continuous $\mathbb{R}^n$-valued paths. Let $Z_t : W^n \to \mathbb{R}^n$ for $t \geq 0$ be the canonical evaluations $Z_t(\omega) = \omega(t)$ and take $\mathcal{G}_t = \sigma(Z_s : s \leq t)$ and $\mathcal{G} := \sigma(\mathcal{G}_t : t \geq 0)$. Define a probability measure on $(\Omega, \mathcal{G})$ by taking $\mathbb{P}^y$ to be the law of the solution X to (23), so that $\left(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t\geq 0}, \mathbb{P}^y\right)$ is a filtered probability space (which does not, in general, satisfy the usual conditions). The following properties hold:

1. $\mathbb{P}^y(\{\omega : \omega_0 = y\}) = 1$;
2. For every $f \in C_c^\infty(\mathbb{R}^n)$ (smooth, compactly supported) the following process is a $\{\mathcal{G}_t\}_{t\geq 0}$-martingale under $\mathbb{P}^y$

$$C_t^f := f(\omega_t) - f(\omega_0) - \int_0^t \mathcal{A}f(\omega_s)\,ds, \tag{24}$$

wherein

$$\mathcal{A}f(x) = \sum_{i=1}^n b^i(x)\frac{\partial f}{\partial x^i}(x) + \frac{1}{2}\sum_{i,j=1}^n a^{i,j}(x)\frac{\partial^2 f}{\partial x^i \partial x^j}(x)$$

and, as usual, $a(x) := \sigma(x)\sigma(x)^T$.

Definition 15.1 *If for any $y \in \mathbb{R}^n$ a probability measure $\mathbb{P}^y$ on $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t\geq 0})$ satisfies the two conditions above, we say $\mathbb{P}^y$ solves the martingale problem for (a,b) starting at y.*

Remark 15.2 *If the process C^f in (24) is a martingale relative to $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t\geq 0})$ then it is also a martingale relative to the usual $\mathbb{P}^y$-augmentation of $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t\geq 0})$. See Rogers and Williams for the details.*

The above steps show how a weak solution to (37) gives rise to a solution to the martingale problem. The converse is also true: we can build a weak solution from a probability measure $\mathbb{P}^y$ satisfying the two stated conditions. We prove this now in a special case.

Theorem 15.3 *Let $\mathbb{P}^y$ solve the martingale problem for (a,b) starting at y. Then there exists a weak solution with initial distribution δ_y to the SDE*

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt. \tag{25}$$

Proof. We prove this assuming that $n = d$ (recall that B is a d-dimensional Brownian motion) and assuming that $\sigma(x)$ is invertible for every $x \in \mathbb{R}^n$. For the general proof, see Problems sheet 5.

In this case we let $X = (X_t)_{t\geq 0}$ be a process $X_t(\omega) = \omega(t)$, defined on the canonical space, and such that X has law $\mathbb{P}^y$. Define the sequence of $\{\mathcal{G}_t\}$-stopping times $(T_n)_{n=1}^{\infty}$ by

$$T_n = \inf\{t > 0 : |X_t| \geq n\}$$

then for any function $f \in C_c^{\infty}(\mathbb{R}^n)$ which is such that $f(x) = x^i$ on $B(0,n)$ (here $x = (x^1, ..., x^n) \in \mathbb{R}^n$) we have that $f(X_t) = X_t^i$ for $t \leq T_n$ and hence if we define

$$M_t := X_t - X_0 - \int_0^t b(X_s)\, ds,$$

then $\left(M^i_{t\wedge T_n}\right) = \left(C^f_{t\wedge T_n}\right)$ is a martingale. It follows that $M \in \mathrm{c}\mathcal{M}_{0,\mathrm{loc}}$. By applying the same argument to a function $f \in C_c^{\infty}(\mathbb{R}^n)$ such that $f(x) = x^i x^j$ on $B(0,n)$ we can deduce that

$$M_t^i M_t^j - \int_0^t a^{ij}(X_s)\, ds$$

is a local martingale so that

$$\left[M^i, M^j\right]_t = \int_0^t a^{ij}(X_s)\, ds.$$

Since X is continuous, adapted and $\sigma(x)$ is invertible for every x, the process $\sigma(X_s)^{-1} \in L(\mathbb{R}^n, \mathbb{R}^n)$ is locally bounded and previsible, hence

$$B_t = \int_0^t \sigma(X_s)^{-1}\, dM_s \in c\mathcal{M}_{0,\mathrm{loc}}.$$

We have

$$\begin{aligned}
\left[B^i, B^j\right]_t &= \sum_{k,l=1}^{n} \int_0^t \left[\sigma(X_s)^{-1}\right]^{i,k} \left[\left[\sigma(X_s)^{-1}\right]^{j,l}\right] d\left[M^k, M^l\right]_s \\
&= \sum_{k,l=1}^{n} \int_0^t \left[\sigma(X_s)^{-1}\right]^{i,k} \left[\left[\sigma(X_s)^{-1}\right]^{j,l}\right] a^{kl}(X_s)\, ds \\
&= \sum_{k,l=1}^{n} \sum_{m=1}^{n} \int_0^t \left[\sigma(X_s)^{-1}\right]^{i,k} \left[\sigma(X_s)^{-1}\right]^{j,l} \sigma(X_s)^{k,m} \sigma(X_s)^{l,m}\, ds \\
&= \sum_{k,l=1}^{n} \sum_{m=1}^{n} \int_0^t \delta_{i,m}\delta_{j,m} ds \\
&= t\delta_{i,j},
\end{aligned}$$

where δ is the Kroenecker delta. By Lévy's characterisation, B is a Brownian motion and we have

$$\int_0^t \sigma\left(X_s\right) dB_s = \int_0^t dM_t = M_t = X_t - X_0 - \int_0^t b\left(X_s\right) ds,$$

so that X and B form a weak solution. ■

16 Connections with partial differential equations

An important aspect of diffusions is their connection with PDE. We will assume that for every pair of initial conditions (t,x) there exists a weak solution (X,B) on a filtered space $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}\right)$ such that

$$X_s = x + \int_t^s \sigma(u,X_u)\,dB_u + \int_t^s b(u,X_u)\,du. \tag{26}$$

Note that $X\in\mathbb{R}^n$ depends on (t,x); we use $\mathbb{E}^{t,x}$ to denote expectation w.r.t. the initial condition (t,x). A simple use of Ito's formula shows in this case that

$$f(s,X_s) - f(t,x) - \int_t^s \left(\frac{\partial f}{\partial u}(u,X_s) + \mathcal{A}_u f(u,X_u)\right) du = \text{loc. mart.}$$

where

$$\mathcal{A}_u f(u,x) = \sum_{i=1}^n b^i(u,x)\frac{\partial f}{\partial x^i}(u,x) + \frac{1}{2}\sum_{i,j=1}^n a^{ij}(u,x)\frac{\partial^2 f}{\partial x^i \partial x^j}(u,x)$$

Lemma 16.1 *If σ and b are continuous and satisfy*

$$|\sigma(u,x)| \leq C(1+|x|) \;\; \text{and} \;\; |b(u,x)| \leq C(1+|x|) \tag{27}$$

for some $C>0$ then for every (t,x) it holds that

$$\mathbb{E}^{t,x}\left[\sup_{t\leq s\leq T}|X_s|^p\right] < \infty$$

for every $p>0$.

Proof. An exercise using the BDG inequalities. ■

For the following result we follow Karatzas and Shreve.

Theorem 16.2 (Feynman-Kac representation) *Let $f:\mathbb{R}^n\to\mathbb{R}$, $k:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ and $g:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ be continuous functions which satisfy*

$$\begin{aligned} |f(x)| &\leq D\left(1+|x|^r\right) \\ \max_{t\in[0,T]}|k(t,x)| &\leq D\left(1+|x|^r\right) \end{aligned}$$

for some $D>0, r>0$. Suppose there exists a function $u\in C^{1,2}([0,T]\times\mathbb{R}^n)$ which solves the Cauchy problem

$$\frac{\partial u}{\partial t} + \mathcal{A}_t u + g = ku, \;\; \text{in } [0,T)\times\mathbb{R}^n$$

with boundary condition $u(T,\cdot) = f(\cdot)$ *such that*

$$\max_{t\in[0,T]} |u(t,x)| \le C\left(1+|x|^{2k}\right)$$

for some $k \ge 1$ *and some* $C > 0$. *Then the solution* u *can be represented as*

$$u(t,x) = \mathbb{E}^{t,x}\left[f(X_T)\, e^{-\int_t^T k(s,X_s)ds} + \int_t^T g(s,X_s)\, e^{-\int_t^s k(u,X_u)du} ds\right].$$

Proof. Fix $t \in [0,T)$, and let

$$\tau_n = \inf\{s \ge t : |X_s| \ge n\}.$$

By applying Itô's formula on $[t, T\wedge\tau_n]$ to the process

$$Z_s := u(s,X_s)\exp\left(-\int_t^s k(u,X_u)\,du\right)$$

we obtain

$$u(t,X_t) = u(T\wedge\tau_n, X_{T\wedge\tau_n})\, e^{-\int_t^{T\wedge\tau_n} k(u,X_u)du} + \int_t^{T\wedge\tau_n} g(s,X_s)\, e^{-\int_t^s k(u,X_u)du} ds.$$

Taking expectations w.r.t. $\mathbb{P}^{t,x}$ shows that $u(t,x)$ equals

$$\mathbb{E}^{t,x}\left[f(X_T)\, e^{-\int_t^T k(s,X_s)ds} 1_{\{\tau_n > T\}} + u(\tau_n, X_{\tau_n})\, e^{-\int_t^T k(s,X_s)ds} 1_{\{\tau_n \le T\}}\right]$$
$$+\mathbb{E}^{t,x}\left[\int_t^{T\wedge\tau_n} g(s,X_s)\, e^{-\int_t^s k(u,X_u)du} ds\right]. \tag{28}$$

As $n \to \infty$ we have a.s.

$$\int_t^{T\wedge\tau_n} g(s,X_s)\, e^{-\int_t^s k(u,X_u)du} ds \to \int_t^T g(s,X_s)\, e^{-\int_t^s k(u,X_u)du} ds, \tag{29}$$

and because

$$\sup_{t\le s\le T} |g(s,X_s)| \le C\left(1 + \sup_{t\le s\le T} |X_s|^{2k}\right) \in L^1,$$

we can use the dominated convergence to deduce L^1-convergence in (29).

To show the second term in (28) tends to zero as $n \to \infty$, we first note that

$$\left|\mathbb{E}^{t,x}\left[u(\tau_n, X_{\tau_n})\, e^{-\int_t^T k(s,X_s)ds} 1_{\{\tau_n \le T\}}\right]\right| \le C\left(1+n^{2k}\right)\mathbb{P}^{t,x}(\tau_n \le T),$$

and then

$$\mathbb{P}^{t,x}(\tau_n \le T) = \mathbb{P}^{t,x}\left(\sup_{t\le s\le T} |X_s| \ge n\right) \le n^{-m}\mathbb{E}^{t,x}\left[\sup_{t\le s\le T} |X_s|^m\right].$$

Choosing $m = m_0 > 2k$ the second term in(28) is bounded in absolute value by

$$C\left(1+n^{2k}\right)n^{-m_0}\mathbb{E}^{t,x}\left[\sup_{t\leq s\leq T}|X_s|^{m_0}\right]\to 0 \text{ as } n\to\infty.$$

Finally, the first term in (28) we have a.s.

$$f\left(X_T\right)e^{-\int_t^T k(s,X_s)ds}1_{\{\tau_n>T\}}\to f\left(X_T\right)e^{-\int_t^T k(s,X_s)ds},$$

and

$$\left|f\left(X_T\right)\exp\left(-\int_t^T k\left(s,X_s\right)ds\right)1_{\{\tau_n>T\}}\right|\leq C\left(1+\sup_{t\leq s\leq T}|X_s|^{2k}\right)\in L^1,$$

so the DCT can be used once more. ■

17 An application of Feynman-Kac

We consider application of the representation formula derived in the previous lecture to demonstrate the power of this result. We restrict attention to the time-homogeneous diffusions so that $\mathcal{A}_t = \mathcal{A}$. For now we also take $g \equiv 0$. Along the same lines as the proof of Theorem 16.2, we consider the initial-value Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} + ku &= \mathcal{A}u, \text{ in } \mathbb{R}^n \\ u(0, \cdot) &= f(\cdot), \end{aligned} \tag{30}$$

which we can represent stochastically as

$$u(t, x) = \mathbb{E}^x \left[f(X_t) \exp\left(-\int_0^t k(X_s)\, ds \right) \right]. \tag{31}$$

We let $\alpha > 0$, and define $w = w_\alpha$ to be the Laplace transform

$$w(x) = \int_0^\infty e^{-\alpha t} u(t, x).$$

A formal calculation using (30) and integration-by-parts then shows that

$$\begin{aligned} \mathcal{A}w(x) &= \int_0^\infty e^{-\alpha t} \partial_t u(t, x)\, dt + k(x) w(x) \\ &= -f(x) + \alpha w(x) + k(x) w(x); \end{aligned}$$

i.e., suppressing the x-dependence, we have shown that w solves

$$[\mathcal{A} - (\alpha + k)] w = -f.$$

From now on we assume that $n = 1$ and take

$$\mathcal{A}h = \frac{1}{2}\Delta h = \frac{1}{2}h''.$$

In this case the diffusion X under $\mathbb{P}^x$ is a Brownian motion started at x. Using the representation (31) and Fubini's theorem we learn that

$$w(x) = \mathbb{E}^x \left[\int_0^\infty f(X_t) \exp\left(-\alpha t - \int_0^t k(X_s)\, ds \right) dt \right]. \tag{32}$$

The following lemma is useful in cases where k is not everywhere continuous.

Lemma 17.1 *Let $n = 1$ and assume $k : \mathbb{R} \to [0, \infty)$ is piecewise continuous. Then for any $\alpha > 0$ the function $w = w_\alpha$ defined in (32) is C^2 and satisfies*

$$\frac{1}{2}w'' - (\alpha + k) w = -f \text{ on } \mathbb{R} \setminus D_k$$

where D_k are the discontinuity points of k.

Proof. See Karatzas and Shreve, Theorem 4.9 for an even more general result. ∎

17.1 The arcsine law for Brownian motion

We show how to use the results above to derive the *arcsine law,* a famous result of Lévy about Brownian motion.

Proposition 17.2 *Let B be a standard Brownian motion in $\mathbb{R}$ and let $T_+(t) := \int_0^t 1_{(0,\infty)}(B_s)\,ds = Leb\{s : s \in [0,t], B_s > 0\}$. Then we have*

$$\mathbb{P}(T_+(t) \le s) := F_t(s) = \frac{2}{\pi}\arcsin\sqrt{\frac{s}{t}} \text{ for } 0 \le s \le t.$$

Proof. By applying Lemma 17.1 with the choices $f \equiv 1$ and $k = \beta 1_{(0,\infty)}$ for $\beta > 0$, we see that for any $\alpha > 0$, w satisfies

$$\begin{aligned} \frac{1}{2}w'' - (\alpha+\beta)\,w &= -1 \text{ if } x > 0 \\ \frac{1}{2}w'' - \alpha w &= -1 \text{ if } x < 0. \end{aligned} \tag{33}$$

To ensure that w is C^2 we need to impose to conditions

$$w(0+) = w(0-) \text{ and } w'(0+) = w'(0-). \tag{34a}$$

Solving the differential equation (33) and discarding unbounded solutions gives the general solution

$$w(x) = \begin{cases} Ae^{-x\sqrt{2(\alpha+\beta)}} + (\alpha+\beta)^{-1} & x > 0 \\ Be^{-x\sqrt{2\alpha}} + \alpha^{-1} & x < 0 \end{cases}.$$

The boundary conditions (34a) then enforce that

$$\begin{aligned} A + (\alpha+\beta)^{-1} &= B + \alpha^{-1} \\ A\sqrt{\alpha+\beta} &= B\sqrt{\alpha}, \end{aligned}$$

and hence $A = \left(\sqrt{1+\beta/\alpha} - 1\right)(\alpha+\beta)^{-1}$. We have therefore computed the Laplace transform (w.r.t. t) of

$$u(t,x) = \mathbb{E}^x\left[\exp\left(-\beta T_+(t)\right)\right],$$

explicitly, at $x = 0$ we have

$$\int_0^\infty e^{-\alpha t}\mathbb{E}^0\left[\exp\left(-\beta T_+(t)\right)\right]dt = w(0) = \frac{1}{\sqrt{\alpha(\alpha+\beta)}}.$$

If we can show that

$$\int_0^\infty e^{-\alpha t}\int_0^\infty e^{-\beta s}\,dF_t(s)\,dt = \frac{1}{\sqrt{\alpha(\alpha+\beta)}} \tag{35}$$

then it will follow from the injectivity of the Laplace transform (twice!), that F_t is the distribution function of $T_+(t)$. But (35) follows since the left-hand side can be computed easily

$$\begin{aligned}
\frac{1}{\pi}\int_0^\infty e^{-\alpha t}\int_0^t \frac{e^{-\beta s}}{\sqrt{s\left(t-s\right)}}dsdt &= \frac{1}{\pi}\int_0^\infty \frac{e^{-\alpha(t-s)}}{\sqrt{t-s}}\int_0^t \frac{e^{-(\alpha+\beta)s}}{\sqrt{s}}dsdt \\
&= \frac{1}{\pi}\int_0^\infty \frac{e^{-\alpha u}}{\sqrt{u}}du\int_0^\infty \frac{e^{-(\alpha+\beta)s}}{\sqrt{s}}ds \\
&= \frac{1}{\pi}\sqrt{\frac{\pi}{\alpha}}\sqrt{\frac{\pi}{\alpha+\beta}} = \frac{1}{\sqrt{\alpha\left(\alpha+\beta\right)}}
\end{aligned}$$

■

Part III
One-dimensional SDEs

18 Pathwise uniqueness criteria

The analysis on SDEs in one-dimensional simplifies considerably. Pathwise uniqueness can be shown under weaker assumptions than (local) Lipschitz conditions which are essentially the best possible in dimensions greater than one. The analysis uses properties of Tanaka's theorem; we recall that for continuous semimartingales X we have

$$|X_t| = |X_0| + \int_0^t \operatorname{sgn}(X_s)\, dX_s + L_t^0(X),$$

where $L_t^0(X)$ is the local time of X at 0. Using this together with $X_t^+ = \frac{1}{2}(|X_t| + X_t)$ we obtain the useful fact that

$$\begin{aligned} X_t^+ &= X_0^+ + \frac{1}{2}\int_0^t (\operatorname{sgn}(X_s) + 1)\, dX_s + \frac{1}{2} L_t^0(X) \\ &= X_0^+ + \int_0^t 1_{\{s:X_s>0\}} dX_s + \frac{1}{2} L_t^0(X). \end{aligned} \tag{36}$$

We are interested in the pathwise uniqueness of the SDE

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dB_s + \int_0^t b(X_s)\, ds \tag{37}$$

where, for the moment, we assume that σ and b are Borel-measurable functions on $\mathbb{R}$. If we suppose that X and Y are two solutions to (37) on the same filtered space, then writing

$$X_t \vee Y_t := \max(X_t, Y_t)$$

(36) gives that

$$\begin{aligned} X_t \vee Y_t &= X_t + (Y_t - X_t)^+ \\ &= x_0 + \int_0^t \left\{\sigma(X_s) + [\sigma(Y_s) - \sigma(X_s)]\, 1_{\{Y_s > X_s\}}\right\} dB_s \\ &\quad + \int_0^t \left\{b(X_s) + [b(Y_s) - b(X_s)]\, 1_{\{Y_s > X_s\}}\right\} ds \\ &\quad + \frac{1}{2} L_t^0(Y - X) \\ &= x_0 + \int_0^t \sigma(X_s \vee Y_s)\, dB_s + \int_0^t b(X_s \vee Y_s)\, ds + \frac{1}{2} L_t^0(Y - X). \end{aligned}$$

Hence if the local times process $L^0(Y-X)$ vanishes identically then $X \vee Y$ is also a solution to (37). In this case, uniqueness in law will immediately give pathwise uniqueness because $X \vee Y, X$ and Y will all have the same law and hence for every $t \geq 0$

$$\mathbb{P}(X_t \vee Y_t = X_t) = 1 \text{ and } \mathbb{P}(X_t \vee Y_t = Y_t) = 1,$$

i.e. both $X_t \geq Y_t$ and $X_t \leq Y_t$ hold a.s. By continuity we have that X and Y are indistinguishable.

In light of this observation, it is useful to have a sufficient condition to ensure that $L_t^0(Y-X) = 0$; we will derive such a condition now. The proof will rely on the occupation density formula for local time which states that for any bounded measurable ϕ and continuous semimartingale Z

$$\int_0^t \phi(Z_u)\, d[Z]_u = \int_{\mathbb{R}} \phi(a)\, L_t^a(Z)\, da, \tag{38}$$

where $L_t^a(Z)$ denotes the local time of Z at a defined by

$$|Z_t - a| = |Z_0 - a| + \int_0^t \operatorname{sgn}(Z_s - a)\, dZ_s + L_t^a(Z).$$

For a proof of this see Rogers and Williams vol 2. It can also be shown that there exists a version of the two-parameter process $\{L_t^a(Z)\}_{a\in\mathbb{R}, t\geq 0}$ such that $(t,a) \to L_t^a(Z)$ is right-continuous in a and continuous in t, a fact that we will use. Again, Rogers and Williams supply the full details.

Lemma 18.1 *Let Z be a continuous semimartingale. Suppose there exists a continuous increasing function $\rho : (0,\infty) \to (0,\infty)$ such that*

$$\int_\epsilon^1 \rho(u)^{-1}\, du \uparrow \infty \text{ as } \epsilon \downarrow 0 \tag{39}$$

for which

$$\int_0^t \rho(Z_u)^{-1}\, d[Z]_u < \infty \tag{40}$$

then $L_t^0(Z) = 0$.

Proof. Apply (38) to $\phi_m(u) = \rho(u)^{-1} 1_{[m^{-1},\infty)}(u)$, let $m \to \infty$ and use the monotone convergence theorem to give

$$\int_0^t \rho(Z_u)^{-1}\, d[Z]_u = \int_0^\infty \rho(u)^{-1} L_t^a(Z)\, da.$$

By the right continuity of $a \to L_t^a(Z)$, the integral on the right hand side diverges unless $L_t^0(Z) = 0$. ■

The following result is a classical result of Yamada and Watanabe.

Theorem 18.2 (Yamada-Watanabe criteria for pathwise uniqueness) *Suppose there exists a a continuous increasing function $\rho : (0, \infty) \to (0, \infty)$ satisfying (39) such that for all $x \neq y$*

$$|\sigma(x) - \sigma(y)| \leq \rho(|x - y|)^{1/2}. \tag{41}$$

If b is Lipschitz continuous then pathwise uniqueness holds for the SDE

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dB_s + \int_0^t b(X_s)\, ds.$$

Proof. Let X and Y be two solutions and let $Z := Y - X$. We can verify condition (40) by applying (41) to give

$$\int_0^t \rho(Z_u)^{-1} d[Z]_u = \int_0^t \rho(Z_u)^{-1} (\sigma(Y_s) - \sigma(X_s))^2 \leq t,$$

and then by $L^0(Z) \equiv 0$ by the previous lemma. Applying (36) to Z gives

$$Z_t^+ = \int_0^t 1_{\{Z_s > 0\}} (\sigma(Y_s) - \sigma(X_s))\, dB_s + \int_0^t 1_{\{Z_s > 0\}} (b(Y_s) - b(X_s))\, ds.$$

Upon taking expectations and using the Lipschitzness of b we obtain the bound

$$\begin{aligned} \mathbb{E}\left[Z_t^+\right] &\leq |b|_{\mathrm{Lip}}\, \mathbb{E}\left[\int_0^t 1_{\{Z_s > 0\}} |Y_s - X_s|\, ds\right] \\ &= |b|_{\mathrm{Lip}} \int_0^t \mathbb{E}\left[Z_s^+\right] ds. \end{aligned}$$

Gronwall's lemma gives that $\mathbb{E}\left[Z_t^+\right] = 0$ i.e. $X_t \geq Y_t$ a.s. Interchanging the rôles of X and Y reverses the inequality. ■

19 A comparison theorem for one-dimensional SDEs

One important use of the one-dimensional SDEs is to provide sufficient conditions for non-explosion of multidimensional SDEs. At the heart of these arguments is the following comparison results of Ikeda and Watanabe.

Theorem 19.1 (Comparison theorem) *For $i = 1, 2$ let*

$$X_t^i = x_0^i + \int_0^t \sigma\left(X_s^i\right) dB_s + \int_0^t \beta_s^i ds.$$

Where σ satisfies condition (41), $X_0^1 \geq X_0^2$ a.s. Suppose further that there exist functions $b_i : \mathbb{R} \to \mathbb{R}$, at least one of which is Lipschitz, such that $b_1 \geq b_2$ pointwise and which satisfy for all $t \geq 0$

$$\beta_t^1 \geq b_1\left(X_t^1\right) \text{ and } b_2\left(X_t^2\right) \geq \beta_t^2 a.s.$$

Then, for all $t \geq 0$, $X_t^1 \geq X_t^2$ a.s.

Proof. As in the Theorem 18.2, the conditions ensure that $L_t^0(Z) \equiv 0$ where $Z := X^2 - X^1$. It follows that

$$Z_t^+ = Z_0^+ + \int_0^t 1_{\{Z_s>0\}}\left(\sigma\left(X_s^2\right) - \sigma\left(X_s^2\right)\right) dB_s + \int_0^t 1_{\{Z_s>0\}}\left(\beta_s^2 - \beta_s^1\right) ds.$$

Noting that $Z_0^+ = 0$, by taking expectations and by assuming, b_1 is Lipschitz we obtain

$$\begin{aligned} 0 \leq \mathbb{E}\left[Z_t^+\right] &= \mathbb{E}\left[\int_0^t 1_{\{Z_s>0\}}\left(\beta_s^2 - \beta_s^1\right) ds\right] \\ &\leq \mathbb{E}\left[\int_0^t 1_{\{Z_s>0\}}\left(b_2\left(X_s^2\right) - b_1\left(X_s^1\right)\right) ds\right] \\ &\leq \mathbb{E}\left[\int_0^t 1_{\{Z_s>0\}}\left(b_1\left(X_s^2\right) - b_1\left(X_s^1\right)\right) ds\right] \\ &\leq |b_1|_{\text{Lip}}\, \mathbb{E}\left[\int_0^t 1_{\{Z_s>0\}}\left|X_s^2 - X_s^1\right| ds\right] \\ &= |b_1|_{\text{Lip}}\, \mathbb{E}\left[\int_0^t Z_s^+ ds\right]. \end{aligned}$$

From Gronwall's inequality we see that $\mathbb{E}\left[Z_t^+\right] = 0$ for all t, so that $Z^+ \equiv 0$ a.s. ■

Example 19.2 *Consider the squared Bessel process; that is the SDE*

$$dX_t = 2\sqrt{X_t^+} dB_t + \alpha dt,\ X_0 = 1 \tag{42}$$

for $\alpha \geq 0$. *The conditions of the Yamada-Watanabe theorem hold here with* $\rho(u) = u$ *on* $(0, \infty)$; *hence pathwise uniqueness holds as does uniqueness in law. To prove the existence of a weak solution, we construct a family of weak solutions* $\{X^n\}$ *on the same setup to the SDE*

$$dX_t^n = 2\sigma_n(X_t^n)\,dB_t + \alpha dt,\ X_0^n = 1,$$

wherein σ_n *is the bounded continuous function* $\sigma_n(x) = \sqrt{(x \wedge n)^+}$. *This can be done by solving the corresponding martingale problems (see Theorem 23.5 of Rogers and Williams for details). It follows that the SDE (42) is exact.*

Using Theorem 19.1 we can say more by comparing solution to SDE (42) with SDE

$$dY_t = 2\sqrt{Y_t^+}dB_t,\ Y_0 = 0,$$

which has the trivial solution $Y \equiv 0$. *An application of Theorem 19.1 gives that* $\mathbb{P}(X_t \geq 0\ \forall t) = 1$.

20 Transformation of one-dimensional SDEs

One-dimensional SDEs can often be neatly transformed such that the properties of its solutions become more immediately apparent. This technique gives rise to a different method for constructing one-dimensional diffusions which is more general than SDEs. We will not consider the full methodology here, but instead restrict attention to the implications it has for SDEs.

To start, consider the solution to the one-dimensional SDE

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt,\ X_0 = x$$

for $\sigma \in C(\mathbb{R})$ and b measurable. One idea might be to transform X to a local martingale: by defining $Y_t = s(X_t)$ and assuming that $s \in C^2(\mathbb{R})$ we can use Itô's formula to give

$$dY_t = s'(X_t)\,\sigma(X_t)\,dB_t + \left[s'(X_t)\,b(X_t) + \frac{1}{2}s''(X_t)\,\sigma(X_t)^2\right]dt. \tag{43}$$

We then look for a function s which satisfies

$$s'(x)\,b(x) = -\frac{1}{2}s''(x)\,\sigma(x)^2\,, \tag{44}$$

e.g., if we assume that $b\sigma^{-2}$ is in $L^1(\mathbb{R})$, then we can take s such that

$$s'(x) = \exp\left(-\int_{-\infty}^{x} 2b(y)\,\sigma(y)^{-2}\,dy\right).$$

Such a function is called a **scale function** of X. It is in $C^2(\mathbb{R})$ and strictly increasing and therefore has a continuous inverse. By defining

$$g(y) := s'\left(s^{-1}(y)\right)\sigma\left(s^{-1}(y)\right) = (s'\sigma)\circ s^{-1}(y)$$

and using (44) we see that

$$dY_t = g(Y_t)\,dB_t,\ Y_0 = y_0 := s(x)\,, \tag{45}$$

a local martingale.

We can try to take this further. Suppose Y is a solution to (45), then the quadratic variation of Y is given by

$$[Y]_u = \int_0^u g(Y_s)^2\,ds.$$

If g is bounded from below by $c > 0$ therefore we have $[Y]_t \uparrow \infty$ as $t \to \infty$, and the Dubins-Schwarz theorem yields a Brownian motion W such that $W_t = Y_{\tau(t)} - s(x)$ with

$$\tau(t) = \inf\{u > 0 : [Y]_u > t\}$$

so that

$$\int_0^{\tau(t)} g(Y_s)^2\,ds = t.$$

From differentiating this identity in t we learn that $\tau'(t)\, g\left(Y_{\tau(t)}\right)^2 = \tau'(t)\, g(W_t)^2 = 1$ hence

$$\tau(t) = \int_0^t g(W_s)^{-2}\, ds.$$

The point is now that the solution Y is determined by the Brownian motion W since

$$[Y]_t = A_t := \tau^{-1}(t) = \inf\left\{u > 0 : \int_0^u g(W_s)^{-2}\, ds > t\right\}, \tag{46}$$

and $Y_t = W_{A_t}$. Hence, uniqueness in law holds. Existence of a weak solution follows too by reversing this procedure: taking any probability space supporting a Brownian motion W; define A as in (46), and let $Y_t = W_{A_t}$; using $A'_t = g(W_{A_t})^2 = g(Y_t)^2$ gives that

$$[Y]_t = \int_0^t g(Y_s)^2\, ds.$$

Finally, by defining

$$B_t = \int_0^t g(Y_s)^{-1}\, dY_s,$$

we can learn from Lévy's charcaterisation that B is a Brownian motion with

$$\int_0^t g(Y_s)\, dB_s = Y_t - Y_0.$$

The pair (Y, B) is then a weak solution.

Remark 20.1 *From the occupation density formula we have*

$$\tau(t) = \int_0^t g(W_s)^{-2}\, ds = \int_{\mathbb{R}} g(u)^{-2} L_t^u(W)\, du =: \int_{\mathbb{R}} L_t^u(W)\, m(du) \tag{47}$$

*where $m(du) := g(u)^{-2}\, du$ is called the **speed measure** of the diffusion X. The above analysis suggest a general two-step method for constructing a diffusion in $\mathbb{R}$, namely for given s and m to take a Brownian motion B and let $X_t = s^{-1}\left(B_{\tau(t)}\right)$ for and τ computed from m as in (47). This method allows for the construction of diffusions more general than those specified by an SDE (the reasoning above relies on s being C^2 and m having a density).*

21 Non-explosion criteria for SDEs

We explore how the ideas of the last section give sufficient conditions for the non-explosion of solutions to the SDE

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt,\ X_0 = x \in \mathbb{R}^n. \tag{48}$$

Definition 21.1 (non-explosion) *Let X be a solution to (48), define*

$$T_n = \inf\{t > 0 : |X_t| > n\} \in [0, \infty]$$

and let $T^* = \sup_n T_n$. We say the X does not explode if $\mathbb{P}(T^* = \infty) = 1$.

An explosive solution is then one for which $\mathbb{P}(T^* < \infty) > 0$; that is, it has a non-zero probability of leaving every compact set in finite time. Khasminskii's criterion describes conditions which guarantee non-explosion. We first consider the one-dimesional case, deriving a one-sided criterion which prevents explosion to $+\infty$. To do this, we need to impose conditions on σ and b on some interval of the form $[1, \infty)$.

Condition 21.2 *$\sigma : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ are continuous on $[1, \infty)$ and $\sigma > 0$ on $[1, \infty)$.*

We define the scale functions $s : [1, \infty) \to [0, \infty)$ for the diffusion X on $[1, \infty]$ by

$$s'(x) = \exp\left(-2 \int_1^x b(u)\, \sigma(u)^{-2}\, du\right),\ s(1) = 0. \tag{49}$$

We recall the speed measure m is defined by

$$m'(y) = \sigma\left(s^{-1} y\right)^{-2} s'\left(s^{-1} y\right)^{-2},$$

Changing variables to $x = s^{-1}(x)$ this becomes

$$m'(x) = m'(y)\, s'(x) = \sigma(x)^{-2}\, s'(x)^{-1}. \tag{50}$$

on $[1, \infty)$. We let $m(1) = 0$.

It follows that

$$\frac{d}{ds} = \frac{dm}{ds} \frac{d}{dm} = \sigma(x)^{-2}\, s'(x)^{-2} \frac{d}{dm}. \tag{51}$$

Note that the generator associated with the diffusion X is $\mathcal{A} = \frac{1}{2} \sigma(x)^2 \frac{d^2}{dx^2} + b(x) \frac{d}{dx}$. We can re-write this on $[1, \infty)$ using

$$\begin{aligned} \frac{d}{dx} &= s' \frac{d}{ds} \text{ and} \\ \frac{d^2}{dx^2} &= -2b(x)\, \sigma(x)^{-2}\, s' \frac{d}{ds} + \sigma(x)^{-2} \frac{d^2}{dmds} \end{aligned}$$

together with (51) gives

$$\mathcal{A} = \frac{1}{2}\frac{d^2}{dmds}.$$

Now define inductively a sequence of functions $\{\psi_n\}_{n=0}^{\infty}$ with $\psi_n : [1,\infty) \to [0,\infty)$ by $\psi_0 \equiv 1$ and

$$\psi_{n+1}(x) = 2\int_1^x s'(y)\int_1^y m'(z)\,\psi_n(z)\,dzdy.$$

It's clear that each $\psi_n \geq 0$ is increasing and C^2 on $[1,\infty)$. Note that

$$\mathcal{A}\psi_{n+1} = \frac{1}{2}\frac{d^2\psi_{n+1}}{dmds} = \psi_n$$

furthermore by induction we have $\psi_n(x) \leq \psi_1(x)^n/n!$; the induction step here follows from

$$\begin{aligned}
\psi_{n+1}(x) &\leq 2\int_1^x s'(y)\,\psi_n(y)\,m(y)\,dy \\
&\leq \frac{1}{n!}\int_1^x \psi_1(y)^n\,d\psi_1(y) \\
&= \frac{\psi_1(x)^{n+1}}{(n+1)!}.
\end{aligned}$$

We can then define a function $\psi : \mathbb{R} \to \mathbb{R}$ by setting $\psi(x) = 1$ for $x < 1$ and for $x \geq 1$

$$\psi(x) = \sum_{n=0}^{\infty} \psi_n(x). \tag{52}$$

The series and its derivatives up to order two converge locally uniformly, hence ψ is also C^2 on $[1,\infty)$ and we have $\mathcal{A}\psi = \psi$ on $[1,\infty)$ and $\mathcal{A}\psi = 0 \leq \psi$ on $(-\infty, 1)$.

The function we have just constructed is used in a fundamental way in the following result.

Theorem 21.3 *Let* $S_n = \inf\{t > 0 : X_t > n\}$. *If*

$$\int_1^{\infty} s'(y)\int_1^y m'(z)\,dzdy = \infty \tag{53}$$

then $\mathbb{P}\left(\sup_{n\geq 1} S_n = \infty\right) = 1$.

Proof. We take the function ψ constructed above. By applying Itô's formula to the non-negative process $Y_t = e^{-t}\psi(X_t)$ we have

$$dY_t = \left[-e^{-t}\psi(X_t) + e^{-t}\mathcal{A}\psi(X_t)\right]dt + e^{-t}\psi'(X_t)\,\sigma(X_t)\,dB_t$$

so that Y is a supermartingale (recall that non-negative local martingales are supermartingales). It follows that

$$\psi\left(n\right)\mathbb{E}\left[e^{-S_n}\right]=\mathbb{E}\left[Y_{S_n}\right]\leq\mathbb{E}\left[Y_0\right]=\psi\left(x\right),$$

and with $S^*=\sup_{n\geq 1}S_n$ the dominated convergence theorem gives

$$\mathbb{E}\left[e^{-S^*}\right]=\lim_{n\to\infty}\mathbb{E}\left[e^{-S_n}\right]=\lim_{n\to\infty}\frac{\psi\left(x\right)}{\psi\left(n\right)}. \tag{54}$$

The condition (53) shows that $\psi_1\left(y\right)\uparrow\infty$ as $y\to\infty$. By the construction of ψ and the fact that $\psi_n\left(y\right)\leq\psi_1\left(y\right)^n/n!$ we have

$$1+\psi_1\left(y\right)\leq\psi\left(y\right)\leq\exp\left(\psi_1\left(y\right)\right)$$

giving that $\psi\left(y\right)\uparrow\infty$ as $y\to\infty$ and hence $\lim_{n\to\infty}\frac{\psi(x)}{\psi(n)}=0$. It follows from (54) that $\mathbb{P}\left(S^*=\infty\right)=1$, as required. ■

22 Khasminskii's explosion criterion for multi-dimensional diffusions

Using the ideas in the last lecture we can derive a non-explosion criterion for the multidimensional diffusion

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt,\ X_0 = x \in \mathbb{R}^n, \tag{55}$$

assuming that σ and b are continuous. Let $R_t = |X_t|^2 = \sum_{i=1}^n \left(X_t^i\right)^2$ then Itô's formula gives that

$$\begin{aligned}
dR_t &= 2\sum_{i=1}^n\sum_{j=1}^d X_t^i \sigma_{i,j}(X_t)\,dB_t^j + 2\sum_{i=1}^n\sum_{j=1}^d X_t^i b_i(X_t)\,dt + \sum_{i=1}^n d\left[X^i\right]_t \\
&= 2\sum_{i=1}^n\sum_{j=1}^d X_t^i \sigma_{i,j}(X_t)\,dB_t^j + 2\sum_{i=1}^n\sum_{j=1}^d X_t^i b_i(X_t)\,dt + \sum_{i,k=1}^n \sigma_{i,k}(X_t)\,\sigma_{i,k}(X_t)\,dt \\
&= 2X_t^T\sigma(X_t)\,dB_t + \left\{2X_t^T b(X_t) + Tr(a(X_t))\right\}dt,
\end{aligned}$$

wherein $a(x) = \sigma(x)\,\sigma(x)^T$ and we recall is non-negative definite. We thus have that

$$[R]_t = 4\left[\int X_s^T\sigma(X_s)\,dB_s\right]_t = \int_0^t 4X_s^T a(X_s)\,X_s ds.$$

This suggest defining the continuous functions θ and μ by

$$\begin{aligned}
\theta(r) &= \sup\left\{2\sqrt{x^T a(x)\,x} : x \in \mathbb{R}^n, |x|^2 = r\right\} \text{ and} \\
\mu(r) &= \sup\left\{2x^T b(x) + Tr(a(x)) : x \in \mathbb{R}^n, |x|^2 = r\right\}.
\end{aligned}$$

We can then formulate conditions on θ and μ to ensure non-explosion of X.

Theorem 22.1 (Khasminskii, 1960) *Suppose that θ is strictly positive on $[1.\infty)$, then the diffusion X in (55) is non-explosive if*

$$\int_1^\infty s'(y)\int_1^y m'(z)\,dzdy = \infty,$$

where s and m are as defined in (49) and (47) when σ and b are replaced respectively by θ and μ.

Proof. Let ψ be the constructed in (52) with σ replaced by θ and b replaced by μ. Applying Itô's formula again to $Y_t = e^{-t}\psi(R_t)$ gives we obtain

$$\begin{aligned}
dY_t &= e^{-t}\left\{-\psi(R_t) + \psi'(R_t)\left[2X_t^T b(X_t) + Tr(a(X_t))\right] + 2\psi''(R_t)\,X_s^T a(X_s)\,X_s\right\} \\
&\quad + d\,(\text{loc. mart}).
\end{aligned}$$

We have $\psi' > 0$ and $\psi'' > 0$ (ψ is C^2 increasing and convex) and thus Y is a supermartingale. The argument can be concluded as in the steps leading to (54) replacing S_n with the stopping times

$$T_n = \inf \{t > 0 : |X_t| > n\}.$$

■