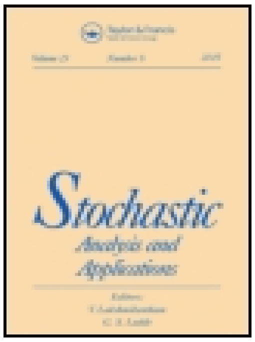

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ALMOST SURE CONVERGENCE OF STOCHASTIC INTEGRALS IN HILBERT SPACES*

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ABSTRACT

It is proved that, under the usual measurability and the square-integrability conditions, the stochastic integral of an operator-valued integrand with respect to a continuous local martingale in Hilbert spaces can be defined for almost every sample path via a sequence of simple progressively measurable approximations. The result generalizes McKean's construction of the Itô integral with respect to a one-dimensional Brownian motion.

1. INTRODUCTION

Let b_t be a one-dimensional Brownian motion and let $f(t,\omega)$ be a nonanticipating Brownian functional. As is well known, the stochastic integral $\int_0^t f(s)db_s$, $0 \le t \le T$, is usually defined in an $L^2(\Omega)$-sense under the condition

$$E\int_0^T f^2(s)ds < \infty . \tag{1.1}$$

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In McKean [8], following Itô [4], it was shown that, under a weaker condition

$$P\{\int_0^T f^2(s)ds < \infty\} = 1 , \tag{1.2}$$

the Itô integral $\int_0^t f(s)db_s$ can be defined for almost every Brownian path uniformly in $t \in [0,T]$. The extension of Itô integral to a stochastic integral with respect to a martingale was initiated by Doob [3] and later further developed by Kunita and Watanabe [6].

For a Brownian motion (or Wiener process) in an abstract Wiener space, the Itô integral with values in a Hilbert space was defined by Kuo [7]. Stochastic integrals based on martingales taking values in Hilbert spaces were considered by Kunita [5] and Metivier [9], among others. As far as we know, the stochastic integrals in Hilbert spaces have only been defined in the Itô sense by an L^2-isometry. However, in applications to stochastic control of distributed-parameter systems and nonlinear filtering, it is of practical interest to define stochastic integrals pathwise. This is the object of the present investigation. As the main result of the paper, it will be shown that, under an analogue of condition (1.2), the stochastic integral with respect to a Hilbert space-valued continuous local martingale can be defined in the sense of McKean through a sequence of simple function approximations. If the integrand is nonrandom, it is easy to define the resulting Wiener integral almost surely by applying an exponential martingale inequality of Chow and Menaldi [2]. The introduction of such inequality was motivated by a large deviation problem for stochastic partial differential equations studied in Chow [1]. For a general integrand, we will adopt the exponential martingale approach by McKean [8] with necessary technical innovation.

2. PRELIMINARIES AND TECHNICAL LEMMAS

Let H (or K) be a real separable Hilbert space with norm $|\cdot|$ and inner product $(\cdot,\cdot)$. Let $\mathcal{L}(K,H)$ denote the space of bounded linear operators from

K into H, of which $\mathcal{L}_2(K,H)$ is a subspace consisting of all Hilbert-Schmidt operators with norm $\|\cdot\|_2$. For a nuclear operator $Q \in \mathcal{L}_1(K,K)$, we denote by $\mathcal{L}_Q(K,H)$ the set of all linear operators $\Gamma : Q^{1/2}K \to H$ such that $\Gamma Q^{1/2} \in \mathcal{L}_2(K,H)$. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in T})$ be a complete right-continuous stochastic basis and let $\{M_t,\ t \in T\}$ be a K-valued cadlag martingale with respect to this basis, where the time interval $[0,T]$ is simply denoted by T. Then, by the Doob-Meyer decomposition, there exist unique processes, known as quadratic variations, $\langle M\rangle_t$ and $\langle\langle M\rangle\rangle_t$ such that $|M_t|^2 - \langle M\rangle_t$ and $M_t \otimes M_t - \langle\langle M\rangle\rangle_t$ are martigales, where $\otimes$ denotes the tensor product. The process $\langle\langle M\rangle\rangle_t$ is related to the increasing process $\langle M\rangle_t$ through the formula:

$$\langle\langle M\rangle\rangle_t = \int_0^t Q_M(s)\, d\langle M\rangle_s \,, \tag{2.1}$$

where Q_M is a positive symmetric operator with

$$\mathrm{Tr}.Q_M(s) = 1 \tag{2.2}$$

for each $s \in [0,T]$ a.s. in ω.

Let $F_\cdot : [0,T] \to \mathcal{L}_Q(K,H)$ be a progressively measurable process such that

$$E\int_0^T \|F_s Q_M^{1/2}(s)\|_2^2\, d\langle M\rangle_s < +\infty \,.$$

Then the stochastic integral

$$J_t = \int_0^t F_s dM_s \tag{2.3}$$

is well defined. As usual, the common procedure consists of finding a sequence of simple progressively measurable processes (s.p.m.p's) F_s^n's such that

$$\lim_{n\to\infty} E\int_0^T \|(F_s - F_s^n)Q_M^{1/2}\|_2^2\, d\langle M\rangle_s = 0 \tag{2.4}$$

and defining, for each n, the stochastic integral J_t^n as a finite sum in the obvious way. Then the stochastic integral (2.4) is defined as the $L^2(\Omega)$-limit of J_t^n. The material given above can be found in a book by Metivier and Pellaumail [10].

To define the stochastic integral as an almost sure limit, we need a few technical lemmas. First let us recall the Itô lemma in a Hilbert space:

Lemma 2.1. Let V_t be a H-valued process of bounded variation and let M_t be a H-valued continuous local martingale. Define $U_t = V_t + M_t$. If $\varphi : H \to I\!R$ is twice (Fréchet) differentiable in H such that the first and second derivatives $\varphi'(h) \in H$ and $\varphi''(h) \in \mathcal{L}_1(H, H)$ are locally bounded and continuous, then the following Itô formula holds:

$$\varphi(U_t) = \varphi(U_0) + \int_0^t \varphi'(U_s)dV_s + \frac{1}{2}\,\mathrm{Tr.} \int_0^t \varphi''(U_s)\, d\langle\langle M\rangle\rangle_s \,. \tag{2.5}$$

When $V_t \equiv 0$, this formula is given in Metivier and Pellaumail [10]. The formula in the present form is due to Pardoux [11]. Now let X_t be a H-valued continuous local martingale with $X_0 = 0$. We introduce two real-valued processes depending on a parameter $\lambda > 0$ as follows:

$$Y_t^\lambda = \lambda |X_t|^2 - \int_0^t \{\lambda + 2\lambda^2 |Q_X^{1/2}(s) X_s|^2\} d\langle X\rangle_s \tag{2.6}$$

and

$$Z_t^\lambda = \exp\{Y_t^\lambda\} \,. \tag{2.7}$$

Lemma 2.2. For each $\lambda > 0$, the exponential process Z_t^λ is a local martingale.
Proof. Define

$$\varphi(h) = |h|^2, \quad h \in H \,.$$

It is clear that

$$\varphi'(h) = 2h$$

and

$$\varphi''(h) = 2I \,,$$

where I denotes the identity operator on H. By the Itô formula (2.5), we have

$$|X_t|^2 = \varphi(X_t) = 2\int_0^t (X_s, dX_s) + \langle X\rangle_t \,, \tag{2.8}$$

where we made use of the fact that $\mathrm{Tr}.\langle\langle X\rangle\rangle_t = \langle X\rangle_t$. A substitution of (2.8) into (2.6) yields

$$Y_t^\lambda = V_t + M_t ,$$

where

$$V_t = -2\lambda^2 \int_0^t |Q_X^{1/2}(s)X_s|^2 \, d\langle X\rangle_s$$

is a real-valued process of bounded variation and

$$M_t = 2\lambda \int_0^t \langle X_s, dX_s\rangle$$

is a real-valued continuous local martingale. In view of the fact

$$\langle M\rangle_t = 4\lambda^2 \int_0^t |Q_X^{1/2}(s)X_s|^2 \, d\langle X\rangle_s = -2V_t ,$$

by applying the Itô formula (2.5) to Y_t^λ with $H = I\!R$, we get

$$\begin{aligned} Z^\lambda &= 1 + \int_0^t Z_s^\lambda dM_s + \int_0^t Z_s^\lambda dV_s + \frac{1}{2}\int_0^t Z_s^\lambda d\langle M\rangle_s \\ &= 1 + \int_0^t Z_s^\lambda dM_s . \end{aligned}$$

Therefore, Z_t^λ is a local martingale. □

Let us set $Y_t^\lambda = \lambda \eta_t^\lambda$ so that

$$\eta_t^\lambda = |X_t|^2 - \langle X\rangle_t - 2\lambda \int_0^t |Q_X^{1/2}(s)X_s|^2 \, d\langle X\rangle_s . \tag{2.9}$$

The following lemma provides the key estimate that is crucial in obtaining the main result.

Lemma 2.3. For each $\beta > 0$, we have

$$P\{\sup_{0\le t\le T} \eta_t^\lambda > \beta\} \le e^{-\lambda\beta} , \quad \forall\, \lambda > 0 . \tag{2.10}$$

Proof. Clearly the above inequality is equivalent to

$$P\{\sup_{0\le t\le T} Z_t^\lambda > e^{\lambda\beta}\} \le e^{-\lambda\beta} . \tag{2.11}$$

To verify (2.11), let $\{\tau_k\}$ be a localization sequence for Z_t^λ, that is, for each k, τ_k is a stopping time such that $Z_{t\wedge\tau_k}^\lambda$ is a martingale and $\tau_k \uparrow T$ as $k \to \infty$. Consequently, for each k fixed, we have

$$E(Z_{T\wedge\tau_k}^\lambda) = E(Z_0^\lambda) = 1 ,$$

and, by invoking Doob's martingale inequality,

$$P\{\sup_{0\le t\le T} Z_{t\wedge\tau_k}^\lambda > e^{\lambda\beta}\} \le e^{-\lambda\beta} E(Z_{T\wedge\tau_k}^\lambda) = e^{-\lambda\beta} .$$

Therefore, by the Monotone Convergence theorem, we obtain

$$P\{\sup_{0\le t\le T} Z_t^\lambda > e^{\lambda\beta}\} = \lim_{k\to\infty} P\{\sup_{0\le t\le T} Z_{t\wedge\tau_k}^\lambda > e^{\lambda\beta}\} \le e^{-\lambda\beta} ,$$

as to be shown. □

3. MAIN RESULT

We will show that the stochastic integral (2.4) can be defined pathwise under an analogue of condition (1.1), namely,

$$\int_0^T \|F_s Q_M^{1/2}\|_2^2 \, d\langle M\rangle_s < \infty \quad \text{a.s..} \tag{3.1}$$

As mentioned before, we first consider a s.p.m.p. of the form:

$$F_t^n(\omega) = \sum_{i=1}^n A_i(\omega) I_{[t_{i-1},t_i)} , \tag{3.2}$$

where $A_i : \Omega \to \mathcal{L}_Q(K,H)$ is $\mathcal{F}_{t_{i-1}}$-measurable for each i and I_B denotes the indicator function of set B. In this case, one simply define

$$J_t^n(\omega) = \int_0^t F_s^n \, dM_s \cong \sum_{i=1}^n A_i(\omega)(M_{t_i} - M_{t_{i-1}}) . \tag{3.3}$$

The following theorem constitutes the main result of this paper, which generalizes the McKean's result, in which $M_t = b_t$ is a one-dimensional Brownian motion.

Theorem 3.1. *Under condition* (3.1), there exists a sequence of s.p.m.p.'s $\{F^n_\cdot\}$ of the form (3.2) such that the corresponding stochastic integrals J^n_t's converge almost surely to a unique limit $J_t \cong \int_0^t F_s dM_s$, uniformly in t over $[0,T]$, that is,

$$\lim_{n\to\infty} \sup_{0\le t\le T} |J^n_t - J_t| = 0 \quad \text{a.s. in } \omega \; .$$

Proof. The proof will proceed along the line as in McKean [8] with some necessary modifications.

As the first step, we need to construct a sequence of s.p.m.p.'s $\{F^n_\cdot\}$ such that

$$\lim_{n\to\infty} \int_0^T \|(F_s - F^n_s)Q_M^{1/2}\|_2^2 \, d\langle M\rangle_s = 0 \quad \text{a.s..} \tag{3.4}$$

This will be done by a combination of truncation, smoothing and discretization of F. Note that the condition (3.1) does not guarantee that F_t be a.s. bounded and measurable in t. Therefore, in contrast with the classical case, we introduce the following truncation:

$$\tilde{F}_k(t,\omega) = F_t(\omega) I_{B^k_t} \; ,$$

where $B^k_t = \{\|F(t,\omega)Q_M^{1/2}(t,\omega)\|_2 \le k\}$. Then we define

$$\tilde{F}_{k\ell}(t,\omega) = \ell \int_{t-1/\ell}^{t} \tilde{F}_k(s,\omega)ds \; ,$$

and

$$\tilde{F}_{k\ell m}(t,\omega) = \tilde{F}_{k\ell}(\tfrac{[mt]}{m},\omega) \; .$$

Clearly, for each (t,ω), $\tilde{F}_k(t,\omega)$ converges to $F(t,\omega)$ as $k\to\infty$. Thus, by the Monotone Convergence theorem, we have

$$\lim_{k\to\infty} \int_0^T \|(F - \tilde{F}_k)Q_M^{1/2}\|_2^2 \, d\langle M\rangle_s = 0 \quad \text{a.s..} \tag{3.5}$$

Since, for each k, $\tilde{F}_k Q_M^{1/2} \in \mathcal{L}_2(K,H)$ is uniformly bounded, it is also clear that

$$\lim_{\ell\to\infty} \lim_{m\to\infty} \int_0^T \|(\tilde{F}_k - \tilde{F}_{k\ell m})Q_M^{1/2}\|_2^2 \, d\langle M\rangle_s = 0 \quad \text{a.s..} \tag{3.6}$$

Now, by combining (3.5) and(3.6), we get

$$\lim_{k\to\infty}\lim_{\ell\to\infty}\lim_{m\to\infty}\int_0^T \|(F-\tilde{F}_{k\ell m})Q_M^{1/2}\|_2^2\,d\langle M\rangle_s = 0 \quad \text{a.s..}$$

Therefore, for each $n \geq 1$, it is possible to take an array of subsequences $\{k_n, \ell_n, m_n\}$ such that

$$P\{\int_0^T \|(F_s - F_s^n)Q_M^{1/2}\|_2^2\,d\langle M\rangle_s > \frac{1}{2^n}\} \leq \frac{1}{2^n}\,, \tag{3.7}$$

where $F_s^n(\omega) = \tilde{F}_{k_n\ell_n m_n}(s,\omega)$ is a s.p.m.p. that has the desired property (3.4). Moreover, we claim that the corresponding stochastic integrals J_t^n's defined by (3.3) converge a.s. to a limit J_t uniformly in t over $[0,T]$. This can be proved as follows.

Let $C(T,H)$ denote the Banach space of H-valued continuous functions on $[0,T]$ with the sup-norm. It suffices to show that $\{J^n_\cdot\}$ is a Cauchy sequence in $C(T,H)$ a.s. in ω. To this end, let us set

$$X_t^n = J_t^n - J_t^{n-1} = \int_0^t (F_s^n - F_s^{n-1})\,dM_s$$

so that

$$\langle X^n\rangle_t = \int_0^t \|(F^n - F^{n-1})Q_M^{1/2}\|_2^2\,d\langle M\rangle_s\,.$$

Note that, for any $Y_t \in C(T,H)$,

$$|Q_{X^n}^{1/2}Y_s| \leq \|Q_{X^n}^{1/2}\|_2\,|Y_s| = \sqrt{\mathrm{Tr}.Q_{X^n}}\,|Y_s| \leq \sup_{0\leq t\leq T}|Y_t|\,,$$

since $\mathrm{Tr}.Q_{X^n} = 1$. Now we invoke Lemma 2.3 to get

$$\begin{aligned} &P\{\sup_{0\leq t\leq T}|X_t^n|(1-2\lambda_n\langle X^n\rangle_T) - \langle X^n\rangle_T > \beta_n\} \\ &\leq P\{\sup_{0\leq t\leq T}(|X_t^n|^2 - \langle X\rangle_t - 2\lambda_n\int_0^t |Q_{X^n}^{1/2}(s)X_s^n|^2\,d\langle X^n\rangle_s) > \beta_n\} \\ &\leq e^{-\lambda_n\beta_n} \leq \frac{1}{n^2}\,, \end{aligned} \tag{3.8}$$

if we choose $\lambda_n = (2^n\log n)^{1/2}$ and $\beta_n = 2(2^{-n}\log n)^{1/2}$. In view of (3.7) and the inequality

$$\begin{aligned}\langle X^n\rangle_T &= \int_0^T \|(F-F^n)Q_M^{1/2}-(F-F^{n-1})Q_M^{1/2}\|_2^2\,d\langle M\rangle_s\\ &\le 2\Big(\int_0^T \|(F-F^n)Q_M^{1/2}\|_2^2\,d\langle M\rangle_s+\int_0^T \|(F-F^{n-1})Q_M^{1/2}\|_2^2\,d\langle M\rangle_s\Big)\,,\end{aligned}$$

it follows that

$$\begin{aligned}P\{\langle X^n\rangle_T>\frac{6}{2^n}\} &\le P\{\int_0^T \|(F-F^n)Q_M^{1/2}\|_2^2\,d\langle M\rangle_s>\frac{1}{2^n}\}\\ &\qquad +P\{\int_0^T \|(F-F^{n-1})Q_M^{1/2}\|_2^2\,d\langle M\rangle_s>\frac{1}{2^{n-1}}\}\\ &\le \frac{1}{2^n}+\frac{1}{2^{n-1}}=\frac{3}{2^n}\,. \qquad (3.9)\end{aligned}$$

For any $\alpha_n>0$ and $\nu_n>0$, it is clear that

$$\begin{aligned}P\{\sup_{0\le t\le T}|X_t^n|>\alpha_n\} &\le P\{\sup_{0\le t\le T}|X_t^n|>\alpha_n,\,\langle X^n\rangle_T\le\nu_n\}+P\{\langle X^n\rangle_T>\nu_n\}\\ &\le P\{\sup_{0\le t\le T}|X_t^n|(1-2\lambda_n\langle X^n\rangle_T)-\langle X^n\rangle_T>\alpha_n^2(1-2\lambda_n\nu_n)-\nu_n\}\\ &\qquad +P\{\langle X^n\rangle_T>\nu_n\}\,. \qquad (3.10)\end{aligned}$$

If we set $\nu_n=6/2^n$ and choose $\alpha_n=\sqrt{(\beta_n+\nu_n)/(1-2\lambda_n\nu_n)}>0$ so that $\alpha_n^2(1-2\lambda_n\nu_n)-\nu_n=\beta_n$, the inequalities (3.8) – (3.10) imply that

$$P\{\sup_{0\le t\le T}|X_t^n|>\alpha_n\}\le\frac{1}{n^2}+\frac{3}{2^n}\,.$$

Thus we conclude from the Borel-Cantelli lemma that

$$P\{\sup_{0\le t\le T}|X_t^n|>\alpha_n\ \text{ i.o.}\}=0\,.$$

It is easy to check that $\sum_{n=n_0}^{\infty}\alpha_n<\infty$ for some $n_0>1$. Therefore, for almost every ω, $J_t^n=\int_0^t F_s^n dM_s$ forms a Cauchy sequence in $C(T,H)$. Denote the limit by $\int_0^t F_s dM_s$ so that

$$\lim_{n\to\infty}\sup_{0\le t\le T}|\int_0^t F_s dM_s-\int_0^t F_s^n dM_s|=0 \quad \text{a.s. in }\omega\,.$$

This completes the proof of our theorem. ◻

We remark that the stochastic integral defined above does not depend on the particular choice of simple approximations F^n and it is a H-valued continuous local martingale. It is easy to show that the basic properties of a stochastic integral in finite dimension also hold in the present case.

As a simple example, let $M_t = W_t$ be a Wiener process in K with covariance operator R so that

$$E(W_t, k) = 0 ,$$

and

$$E(W_t, k)(W_t, h) = (Rk, h) \qquad \forall k, h \in K .$$

Then we have $\langle\langle M \rangle\rangle_t = tR$, $\langle M \rangle_t = t(\text{Tr}.R)$ and $Q_M = (\text{Tr}.R)^{-1}R$. Therefore, by Theorem 3.1, for a progressively measurable process $F. : [0,T] \to \mathcal{L}_Q(K,H)$ satisfying

$$\int_0^T \|F_s R^{1/2}\|_2^2 \, ds < \infty \quad \text{a.s. in } \omega ,$$

there exists a sequence of s.p.m.p.'s $\{F_t^n\}$ such that $\int_0^t F_s^n dW_s$ converges pathwise to the corresponding Itô integral $\int_0^t F_s dW_s$ in $C(T, H)$.

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