DIC13

3. *Arithmetic coding / decoding*

10-04, 10-08, 10-11 SXD

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3.5. Arithmetic decoding

3.1. Segment coding

Consider a source alphabet $\mathcal{A} = \{a_1, \dots, a_n\}$, with probabilities $\{p_1, \dots, p_n\}$. Let $M = a_{j_1} \cdots a_{j_N}$ be a source message of length N. In *segment coding* (the first step in what will be *arithmetic coding*) a segment (or interval)

$$S_M = [l_M, h_M) \subset [0,1)$$

is assigned to any such M. Moreover, this assignment has the following properties:

1. $h_M - l_M = P(M)$
2. $S_M \cap S_{M'} = \emptyset$ for any pair two distinct messages M, M' of length N.
3. $\bigcup_M S_M = [0,1)$.
4. If M is a prefix of M', then $S_{M'} \subset S_M$.

So the length of S_M is $P(M)$; the different S_M, for fixed N, form a *partition* of $[0,1)$, and the partition for $N' > N$ is a refinement of that for N.

The case $N = 1$

For messages of length 1 (any one of the symbols a_j of $\mathcal{A}$), we set

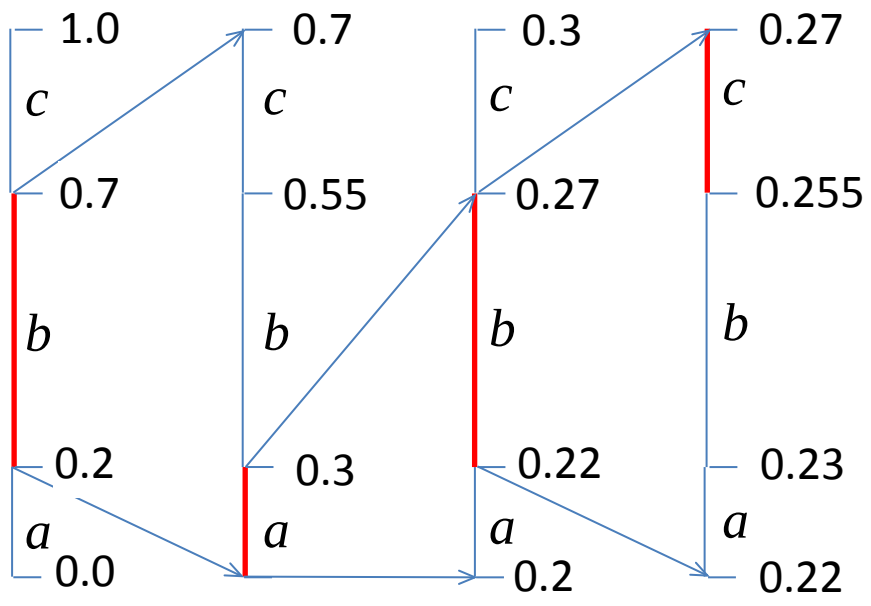
$$S_{a_j} = \left[p_1 + \cdots + p_{j-1}, p_1 + \cdots + p_{j-1} + p_j\right) = [\sigma_{j-1}, \sigma_j),$$

where we define $\sigma_j = p_1 + \cdots + p_{j-1} + p_j$, $j = 1, \dots, n$ (we say that $\sigma_1, \dots, \sigma_n$) is the *cumulative* probability distribution. Thus we have

$$l_{a_j} = \sigma_{j-1},\ h_{a_j} = \sigma_j,\ h_{a_j} - l_{a_j} = p_j$$

From the definitions it follows that $0 < p_1 = \sigma_1 < \cdots < \sigma_n = 1$ and hence that the segments S_{a_j} cover $[0,1)$ and are pairwise disjoint. See the left column of the illustration on next page for an example in which $n = 3$.

Example. Before considering the general construction, we first describe it in a simple example. We will see how to obtain the segment S_M for the message $babc$ produced by the source $\{'a': 0.2,\ 'b': 0.5,\ 'c': 0.3\}$.



Example

$\mathcal{A} = \{a, b, c\}$, $p = (0.2, 0.5, 0.3)$, $M = babc$.

By our stipulation of the case $N = 1$, we know that

$$S_b = [0.2, 0.7).$$

Let us proceed now to assign an interval to ba (second colum of the illustration), then to bab (third column) and finally to $babc$ (fourth column). The interval S_{ba} is obtained by subdividing

$$S_b = [l_b = 0.2, h_b = 0.7)$$

into segments of relative length $p(a) = 0.2$, $p(b) = 0.5$ and $p(c) = 0.3$ and choosing the a-segment as S_{ba}. So the division points are

$$l_{ba} = 0.2, h_{ba} = l_{bb} = 0.2 + 0.5 \times \sigma(a) = 0.30,$$

$$h_{bb} = l_{bc} = 0.2 + 0.5 \times \sigma(b) = 0.55, h_{bc} = 0.2 + 0.5 \times \sigma(c) = 0.70 \text{ , so}$$

$$S_{ba} = [0.2,0.3), S_{bb} = [0.30, 0.55), S_{bc} = [0.55,0.70).$$

Now the interval S_{bab} is obtained in a similar way: subdivide S_{ba} into intervals that are proportional to $p(a), p(b)$ and $p(c)$ and choose as S_{bab} the segment corresponding to b. Actually we have

$$h_{baa} = l_{bab} = 0.2 + 0.1 \times \sigma(a) = 0.22,$$

$$h_{bab} = l_{bac} = 0.2 + 0.1 \times \sigma(b) = 0.27,$$

$$h_{bac} = 0.2 + 0.1 \times \sigma(c) = 0.30,$$

and hence

$$S_{bab} = [l_{bab}, h_{bab}) = [0.22,0.27).$$

Finally we get, following the same procedure with S_{bab},

$$S_{babc} = \big[0.22 + 0.05 \times \sigma(b), 0.22 + 0.05 \times \sigma(c)\big) = [0.255,0.270).$$

The general case

Suppose that we already know the interval $S_M = [l_M, h_M)$ of a message of length N and that $h_M - l_M = P(M)$. Then the interval of $M' = Ma_j$ is defined as follows:

$$S_{M'} = [l_{M'}, h_{M'}) = \left[l_M + P(M) \times \sigma_{j-1}, l_M + P(M) \times \sigma_j\right).$$

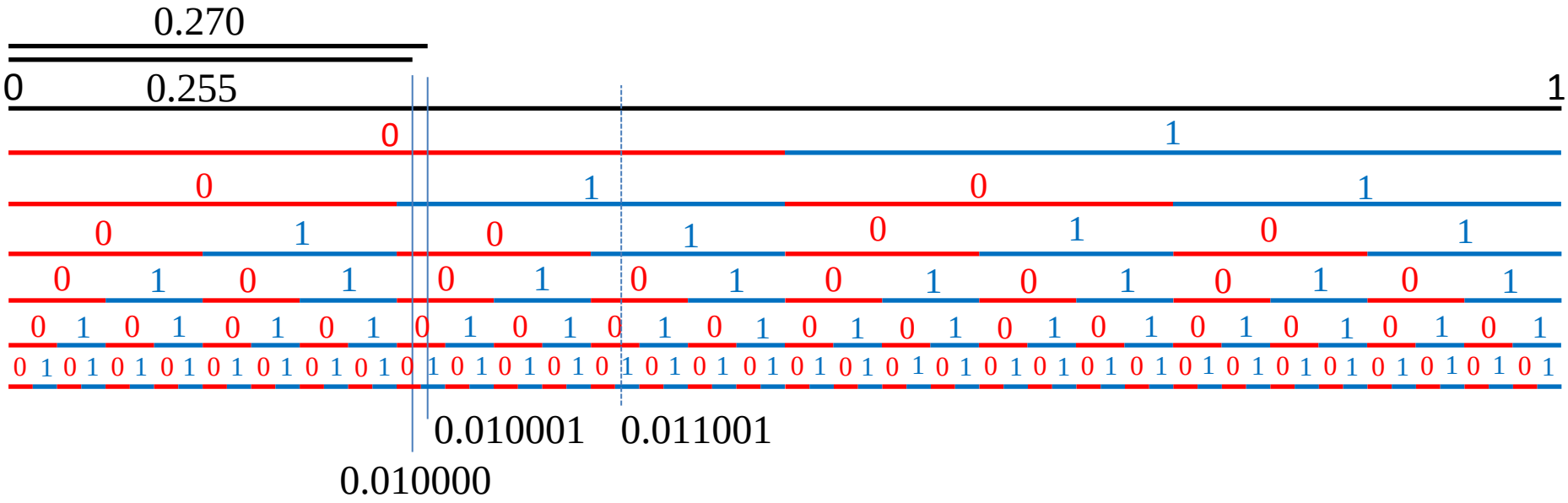
We note that

$$h_{M'} - l_{M'} = P(M) \times \left(\sigma_j - \sigma_{j-1}\right) = P(M)p_j = P(M').$$

Remark. The conditions 1-4 at the beginning are a direct consequence of the definitions.

3.2. Binary representation of segments

Binary representation of numbers in the unit segment



The binary unit segment

Examples. 0.5→0.1, 0.25→0.01, 0.75→0.11, 0.125→0.001;

0.255→0.01000, 0.270→0.01001;

0.011001→1/4+1/8+1/64=25/64=0.390625.

3.3. Arithmetic coding

The basic idea of *arithmetic coding* is to select an element in the ment $S_M = [l_M, h_M)$ that requires the minimal number of bits. Then we encode $\boldsymbol{M}$ using the binary word formed with those bits.

This can be accomplished as follows. Suppose that the first bit that is different in the binary representations of $\boldsymbol{l_M}$ and $\boldsymbol{h_M}$ is the $\boldsymbol{r}$-th, so that we will have

$$l_M = 0.b_1 b_2 \cdots b_{r-1} 0 \cdots, \; h_M = 0.b_1 b_2 \cdots b_{r-1} 1 \cdots$$

If $0.b_1 b_2 \cdots b_{r-1} 1 < h_M$, then the number $0.b_1 b_2 \cdots b_{r-1} 1$, or the word $b_1 b_2 \cdots b_{r-1} 1$, satisfies the requirements, for $0.b_1 b_2 \cdots b_{r-1} 1 \in S_M$ and any other number in the interval S_M will require more bits. Thus we encode M as $b_1 b_2 \cdots b_{r-1} 1$. If $0.b_1 b_2 \cdots b_{r-1} 1 = h_M$, then the shortest binary word will be $b_1 b_2 \cdots b_{r-1}$ if $l_M = 0.b_1 b_2 \cdots b_{r-1} 0$. Otherwise $l_M = 0.b_1 b_2 \cdots b_{r-1} 0x$, with $x \neq 0$. If $x = 0 \cdots$, we can take $0.b_1 b_2 \cdots b_{r-1} 01$, else if $x = 1 \cdots 10$ (s ones), we may take $0.b_1 b_2 \cdots b_{r-1} 01 \cdots 11$ ($s + 1$ ones).

Remark. For the decoding, it will be convenient to include in the encoding the length of the original message. Thus it can be represented as a pair (L, c), where L is the length of the of the message and c is the binary string given by the arithmetic encoding.

For example, if S = [‘a’ : 0.2, ‘b’ : 0.5: ‘c’ : 0.3] and M=”babc”, then the arithmetic encoding is [4,”010001”], for the dimals of 0.255 and 0.270 are, respectively:

01000001010 ⋯

01000101000 ⋯

3.4. Effective computations

```
def accumulate(P):
    S=[]        # for the accumulated distribution
    a=0.0       # the current accumulated value
    for x in P:
        a += x[1]   # x[1]: probability of s=x[0]
        S.append((x[0],a))   # add pair (s,a) to S.
    return(S)

def IE(M, P):
    S=accumulate(P)
    P=dict(P); S=dict(S)
    l=0.0; h=1.0; u=h-l
    for x in M:        # see formulas on page 6
        h=l+S[x]*u; u=P[x]*u; l=h-u
    return [l,h]
```

```
def BE(a,b,nb=58):
    l=dec2bin(a,nb)
    h=dec2bin(b,nb)
    r=0
    while l[r]==h[r]: # see discussion of page 8
        r=r+1
    h=h[:r]+'1'
    if bin2dec(h)<b: return h
    else:
        if bin2dec(l[:r])==a: return l[:r]
        x=l[(r+1):]
        if x[0]=='0': return l[:r]+'01'
        j=x.index('0')  # it will fail if no 0 in x
        return l[:r+1+j]+'1'
def AE(M, P, nb=58):
    l, h = IE(M, P)
    return([len(M),BE(l,h,nb)])
```

3.5. Arithmetic decoding

Suppose we have the arithmetically encoded message

$C = [N, x]$ (N the number of symbols, x the binary string encoding).

Let P be a representation of the source as a list of pairs (symbol, probability). Let A denote the list of pairs (symbol, cumulative-probability). The decoding algorithm can be described as follows:

```
def AD(C,P):
      N = C[0]; x=C[1]; x=bin2dec(x)
      A=accumulate(P);   P=dict(P)
      l=0; h=1; M=""
      for j in range(N):
         u = h-l
         for (k,q) in A:
               if (l+q*u <= x): continue
               else: break
         M += k; h = l + q*u; l = h-P[k]*u
   return M
```

Example. If P=[('a', 0.25), ('b', 0.4}, ('c', 0.15), ('d', 0.1), ('e', 0.1)] and

M= "badbbdcbabea",

then the arithmetic encoding AE(M,P) is

[12,"01010101101110110111100101"]

(see the examples in AE-AD.py).

Remark. This description of arithmetic encoding and decoding would work if floats had unlimited precision, but in pyzo it only works for short messages (of the order of 54 encoded bits, due to the fact of that floats have 16 significant digits, which amounts to 16*log(10)/log(2)≃ 53.15 bits.