1 Introduction

When I was an undergraduate, I heard a confusing joke. The joke goes like this:

"What does a drowning number theorist say?"

"Log log log log log log log..."

Having only taken an elementary number theory course at the time I first heard the joke, I was stymied. What could logarithms possibly have to do with number theory? I had learned things about modular arithmetic, quadratic reciprocity, and prime numbers. Everything was algebraic and exact, not analytic and approximate.

Now, however, it seems only natural that logarithms play a key role in number theory. Indeed, the central theme of much of my research involves counting solutions to Diophantine problems[1] by analyzing the growth rates of those solutions. When those solutions grow exponentially, the number of solutions is often controlled by a logarithm.

My work can be subdivided into two broad categories:

1. I use Diophantine inequalities to solve number theoretic problems.

2. I prove new Diophantine inequalities which are interesting in their own right.

In this statement, I describe one project on which I work in each category. In the former category, I explain my work on Thue equations. In the latter category, I write about polynomial root separation.

2 Using Diophantine Inequalities: Thue Equations

2.1 Introduction

I primarily study certain Diophantine equations called Thue equations. A *Thue equation* is an equation of the form $F(x, y) = h$ where $F(x, y) \in \mathbb{Z}[x, y]$ is homogeneous[2] and irreducible (over $\mathbb{Q}$) of degree $n \geqslant 3$. The individual equations $F(x, y) = h$ are difficult to study, and it is easier to look at the collection of equations $F(x, y) = 0, \pm 1, \pm 2, \ldots, \pm h$. More concisely, we could write

$$|F(x, y)| \leqslant h, \tag{1}$$

and we call this inequality *Thue's inequality.*

The motivating question of my research is: how many integer-pair solutions $(x, y) \in \mathbb{Z}^2$ are there to Thue's inequality? It is not obvious that there are only finitely many, though Thue proved this in 1909 [22]. The main idea behind counting the number of integer-pair solutions to (1) is to observe that solutions (x, y) correspond to rational approximations x/y of certain algebraic numbers, and to show that the denominators of those rational approximations must grow quickly. The growth rates of the denominators produce bounds on the number of solutions to Thue's inequality.

[1] Diophantine problems are—very loosely—problems about integer solutions to polynomial equations with integer coefficients.

[2] This means that $F(x, y) = \sum_{j=0}^{n} a_j x^j y^{n-j}$.

In particular, Mahler [15], Siegel [20], and Mueller and Schmidt [16, 17, 18] have all found upper bounds on the number of solutions to (1) in terms of certain features of $F(x, y)$. The most important features of

$$F(x,y) = \sum_{j=0}^{n} a_j x^j y^{n-j}$$

are

1. the *degree*, $\deg(F) = n$,
2. the *height*, $H(F) = \max_{0 \leqslant j \leqslant n} |a_j|$, and
3. the number of nonzero summands, $s := \#\{j : a_j \neq 0\}$.

Letting $N(F, h)$ denote the number of integer-pair solutions to (1), Mueller and Schmidt were able to prove[3]

$$N(F,h) \ll s^2 \log(s) h^{2/n}(1 + \log(h^{1/n})),$$

and they made the following two conjectures.

Conjecture 2.1 (Mueller and Schmidt, [18]). $N(F, h) \ll sh^{2/n}$.

Conjecture 2.2 (Mueller and Schmidt, [18]). *Suppose that $H(F)$ is large[4] relative to h. Then $N(F, h)$ is bounded from above by a function that does not depend on h or $H(F)$.*

Mueller and Schmidt prove neither conjecture, though they provide compelling evidence, and they prove weaker versions of each conjecture. Their techniques involve classifying every solution to (1) as small, medium, or large, and they use different approaches to bound the number of solutions in each category. My results on Thue equations can be best understood in the context of these two conjectures.

2.2 Results

In my dissertation [10], I prove a result which makes progress towards Conjecture 2.1 and which fully supports Conjecture 2.2. My first result improves the bounds on the number of medium solutions to inequality (1).

Theorem 2.3 (K., [10]). *Let $N_M(F, h)$ denote the number of primitive,[5] medium solutions to* (1). *Then*

$$N_M(F,h) \ll s\left(1 + \log\left(s + \frac{\log h}{\max(1, \log H(f))}\right)\right).$$

[3]For any set X and functions $f, g : X \to \mathbb{R}_{\geqslant 0}$, we use the Vinogradov notation $f(x) \ll g(x)$ to mean that there is an absolute constant $C > 0$ so that for every $x \in X$, $f(x) \leqslant Cg(x)$.

[4]I omit precision for the sake of brevity here.

[5]For technical reasons, it is standard to first bound the number of solutions (x, y) with $\gcd(x, y) = 1$, and then later convert those into bounds on the total number of solutions.

This result makes progress towards Conjecture 2.1 in that it improves on Bengoechea's work in [1]. Additionally, it verifies Conjecture 2.2 for medium solutions.

In my paper [11], I focus on Conjecture 2.2 in the particular case when $s = 3$ and $h = 1$, meaning that we are looking at Thue equations of the form $ax^n + bx^k y^{n-k} + cy^n = \pm 1$. If $s = 3$, then $F(x, y)$ is known as a *trinomial*, since it has three terms. Thomas has already done some work on this case, e.g. in [21], in which he shows that if $F(x, y)$ is a trinomial, if $n \geqslant 6$ and if $h = 1$, then there are no more than $8w(n) + 8$ integer pair solutions to (1) where $w(n)$ is piecewise defined by the following table:

n	6	7	8	9	10–11	12–16	17–37	$\geqslant 38$
$w(n)$	16	13	11	9	8	7	6	5

I show that $w(n)$ can be improved as follows:

Theorem 2.4 (K., [11])**.** *The function $w(n)$ in Thomas' result can be replaced by $z(n)$, defined with the following table.*

n	6	7	8–218	$\geqslant 219$
$z(n)$	15	12	$w(n)$	4

It is worth noting that 4 is the best possible value that could be obtained for $z(n)$ using any approach analogous to Thomas', though it remains possible that $z(n)$ could be reduced to 4 for values of n less than 219.

2.3 Current and Future Work

Currently, I am working with Eva Goedhart and Sumin Leem to improve a result in [7]. Earp-Lynch et al. examine a broader class of Thue equations where the coefficients and solutions are allowed to come from a fixed quadratic imaginary number field. They prove that for a certain family of Thue equations (which depend on a parameter t), if $|t| \geqslant 100$, the equations have no solutions. Goedhart, Leem, and I would like to extend their results to include values of t with $|t| \leqslant 100$, and so far, we have improved the methods in [7] to account for values of t satisfying $78 \leqslant |t| \leqslant 100$. Once we account for the remaining values of t, it is likely that we will have some crude code for solving this broader family of Thue equations. Refining, improving, and generalizing this code could be an interesting project for an undergraduate who is interested in computational number theory.

In the future, I would like to focus on the unique insight which Thomas provides in [21] and which Grundman and Wisniewski extend in [9]. In particular, I would like to generalize Thomas' results from $h = 1$ to larger values of h.

Second, while both [21] and [9] establish a new approximation philosophy, they rely on results from older papers like [2] which use a different approximation philosophy. By re-aligning the methods of [2], I expect that Thomas' results could be further improved.

Finally, it is worth noting that foundational papers on Thue's Inequality did not always use s in their bounds on $N(F, h)$. For instance, Bombieri and Schmidt's paper [2] on the Thue-Siegel method uses $n + 1$ as an upper bound on s, and expresses all of its bounds in terms of n. I would like to update their results to account for s and see what consequences those changes would bring about. Updating Bombieri and Schmidt's results to account for s would make for a good undergraduate research project for a student interested in theoretical number theory.

3 New Diophantine Inequalities: Separation

While my work on Thue equations tends to involve using and improving inequalities that others have developed, I am also interested in proving new inequalities that others can use. My major project in this category is my work on polynomial root separation.

3.1 Introduction

For a polynomial

$$f(x) = b\prod_{i=1}^{n}(x - \alpha_i) \in \mathbb{C}[x],$$

define the *separation* of $f(x)$ to be

$$\operatorname{sep}(f) := \min_{\alpha_i \neq \alpha_j} |\alpha_i - \alpha_j|,$$

and define the *Mahler measure* of $f(x)$ to be

$$M(f) := |b|\prod_{i=1}^{n}\max(1, |\alpha_i|).$$

Separation is an important tool both in theoretical and in computational number theory. In the study of Thue equations, for example, Grundman and Wisniewski [9] use a result of Rump [19] which gives a lower bound on separation. In computational number theory, separation is used in root-finding algorithms [13].

Mahler measure is also a key tool because it contains information about both the roots and the coefficients of the polynomial. For example, it is the subject of Lehmer's conjecture.

I study the relationship between separation and Mahler measure, and specifically, I am interested in finding bounds on separation in terms of Mahler measure.

Mahler began the study of relating $\operatorname{sep}(f)$ to $M(f)$ in [14], where he showed that for any monic[6], separable[7] polynomial $f(x) \in \mathbb{Z}[x]$ with degree n,

$$\operatorname{sep}(f) > \frac{\sqrt{3}}{n^{(n+2)/2}M(f)^{n-1}}.$$

Mahler's result spurred many other authors to investigate questions about whether or not his bounds are optimal, and whether they can be improved in special cases. For examples in addition to the already cited works, see [3, 4, 5, 6, 8].

Each of the above papers is concerned with bounding $\operatorname{sep}(f)$ from below in terms of $M(f)$. However, I became curious about finding upper bounds on $\operatorname{sep}(f)$ in terms of $M(f)$, especially when I was unable to find any other work on this topic.

[6] A polynomial is monic if its leading coefficient is 1.

[7] A polynomial is separable if it has no repeated roots.

3.2 Results

In my dissertation [10], I conjectured that $\operatorname{sep}(f) \ll M(f)^{1/(n-1)}$, and I proved this conjecture in a few cases. Recently, however, Chi Hoi Yip and I proved something stronger. This result is nearly optimal in the sense that the only part of the upper bound that could be reduced is the constant 34.

Theorem 3.1 (K., Yip, [12]). *Let $f(x) \in \mathbb{C}[x]$ be a monic, separable polynomial of degree $n \geqslant 2$. Then*

$$\operatorname{sep}(f) \leqslant \min\left(2, \frac{34}{\sqrt{n}}\right) M(f)^{1/(n-1)}. \tag{2}$$

In addition to this general result, we are able to improve the exponent $1/(n-1)$ when $f(x) \in \mathbb{R}[x]$ has no real roots, and we are able to improve the constant $34/\sqrt{n}$ when $f(x) \in \mathbb{R}[x]$ has only real roots. We also find optimal constants C for which $\operatorname{sep}(f) \leqslant CM(f)^{1/(n-1)}$ for $f(x) \in \mathbb{R}[x]$ with small degrees.

3.3 Future Work

There are still some small degree cases where we have not determined the optimal constant C for an inequality of the form $\operatorname{sep}(f) \leqslant CM(f)^{1/(n-1)}$, and I would like to continue to search for these constants. Determining these constants could make for a good undergraduate research project since the techniques largely involve calculus and elementary geometry.

I would be interested in examining separation for special families of polynomials. For example, Chris Sinclair (University of Oregon) has suggested that studying the separation of Salem polynomials would be useful for his research.

More generally, I would like to see if results giving either upper or lower bounds on $\operatorname{sep}(f)$ can be improved by accounting for the number of nonzero summands of $f(x)$. The number of nonzero summands of $f(x)$ should play a role in the separation since, as Mueller and Schmidt observe in [18], the number of nonzero summands controls the distribution of roots.

4 Conclusion

Having worked in Diophantine approximation for several years now, I am no longer surprised when logarithms appear in my work. The Diophantine problems themselves remain as alluring and difficult as ever, but the sources of modern tools and techniques are within my grasp. I'm looking forward to introducing undergraduates to these (and related) problems!

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