Bounds on the Number of Solutions to Thue's Inequality

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7 February 2023

Setup

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  - $s = 3$
  - $H = 10$
- Let $h \in \mathbb{Z}_{>0}$

Foundational Result

Theorem (Thue, 1909)

$|F(x, y)| \leqslant h$ (known as Thue's Inequality) has finitely many integer pair solutions.

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Necessity of Hypotheses

- $\deg(F) \geqslant 3$ is necessary: if $d \in \mathbb{Z}_{>0}$ is not a square, then $F(x, y) = x^2 - dy^2$ is irreducible and homogeneous, and $|F(x, y)| \leqslant 1$ has infinitely many integer-pair solutions.

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- $F(x,y)$ being irreducible is also necessary: if $F(x,y)$ has a linear factor, say $mx - ny$, then any integer multiple of (n,m) is a solution to $|F(x,y)| \leqslant h$.

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- The homogeneity condition is also necessary: if $F(x,y) = x^6 + y^3$, then any integer pair of the form $(n, -n^2)$ will be a solution to $|F(x,y)| \leqslant h$.

Follow-Up Questions

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Approaches

1. Geometric
2. Algebraic

Changing Variables

Changing Variables

“Eliminating” h

Observe that $|x^5 + 3x^4y - y^5| \leqslant h$ if and only if

$$\left| \left(\frac{x}{h^{1/5}}\right)^5 + 3\left(\frac{x}{h^{1/5}}\right)^4 \left(\frac{y}{h^{1/5}}\right) - \left(\frac{y}{h^{1/5}}\right)^5 \right| \leqslant 1.$$

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Fact

$|F(x,y)| \leqslant h$ *if and only if*

$$\left|F\left(\frac{x}{h^{1/n}}, \frac{y}{h^{1/n}}\right)\right| \leqslant 1.$$

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Observe that $|x^5 + 3x^4y - y^5| \leqslant h$ with $y > 0$ if and only if

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Fact

$|F(x,y)| \leqslant h$ and $y > 0$ if and only if

$$\left|F\left(\frac{x}{y}, 1\right)\right| \leqslant \frac{h}{y^n}.$$

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$$\max(|x|, |y|) \leqslant C_{F,\kappa} e^{(\log h)^\kappa} = C_{F,\kappa} h^{(\log h)^{\kappa - 1}}$$

where $C_{F,\kappa}$ is an effectively computable constant depending only on $F(x,y)$ and κ.

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- Choose a $\kappa > n$.
- Compute $C_{F,\kappa}$.
- Test all pairs $(x,y) \in \mathbb{Z}^2$ satisfying $\max(|x|,|y|) \leqslant C_{F,\kappa} e^{(\log h)^\kappa}$ to see if $|F(x,y)| \leqslant h$.

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Question

Is this what the growth rate of $N(F,h)$ actually looks like?

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Note that $h^{0.32}$ grows much slower than $h^{2(\log h)^6}$.

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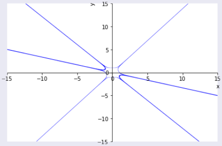
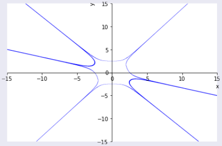


$$|x^5 + 3x^4y - y^5| = 1$$



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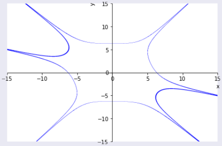

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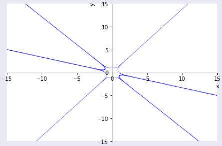

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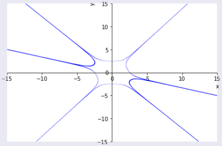

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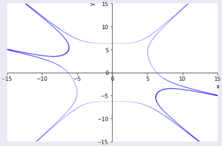

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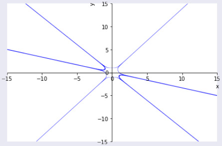

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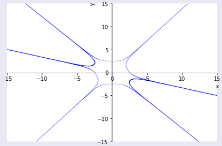

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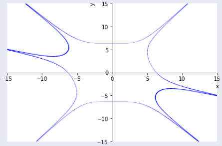

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$$N(F,h) = \text{number of lattice points “inside” } |F(x,y)| \leqslant h$$
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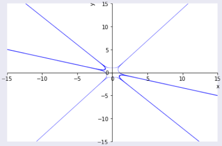

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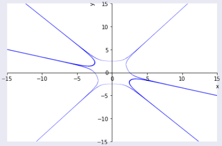

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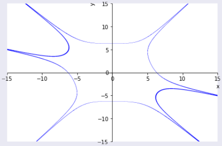

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&=: V(F,h)
\end{aligned}
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Exploring dependence on h

Volume

In general

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Implication

Since $N(F,h)\approx V(F,h)=h^{2/n}V(F,1)$, it would make sense if $N(F,h)\approx h^{2/n}\cdot C_F$ where C_F is a constant depending only on F.

Exploring dependence on h

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Theorem (Mahler, 1934)

For any $F \in \mathbb{Z}[x, y]$ irreducible and homogeneous of degree $n \geqslant 3$, there exists a constant $C(F)$ so that

$$|N(F, h) - V(F, h)| \leqslant C(F) \cdot h^{\frac{1}{n-1}}$$

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Theorem (Mahler, 1934)

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$$|N(F, h) - V(F, h)| \leqslant C(F) \cdot h^{\frac{1}{n-1}}$$

Corollary

We have

$$h^{-2/n} N(F, h) = V(F, 1) + O_F\left(h^{-\frac{1}{n+3}}\right),$$

i.e. $N(F, h) = O_F(h^{2/n})$.

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From Mahler's Theorem

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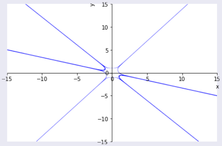
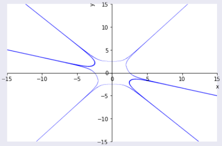
The “Long Tendrils”

Some Pictures

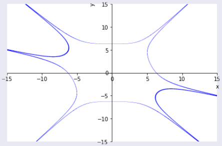
Recall that we had the previous pictures:



$$|x^5 + 3x^4y - y^5| = 1$$



$$|x^5 + 3x^4y - y^5| = 100$$



$$|x^5 + 3x^4y - y^5| = 10^4$$

Question

What’s the deal with the linear parts?

A Connection to $\mathbb{Q}$

Translating to One Variable

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- By symmetry, we could also count rational approximations to roots of $g(Y) = F(1, Y)$.

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- We can connect primitive solution counts to total solution counts using partial summation methods.

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- Hence, the line $Y = \frac{X}{\alpha}$ is a subset of $\{(x, y) \in \mathbb{R}^2 : |F(x, y)| \leqslant h\}$.
- Therefore, real roots α correspond with tendrils of slope $\frac{1}{\alpha}$.

Example

Let $F(x, y) = x^5 + 3x^4y - y^5$

Bounds on the Number of Solutions to Thue's Inequality

Greg Knapp

Thue's Inequality
Introduction
Geometry
Algebra
Results

Counting Techniques
Different Solution Types
Large Solutions
Medium Solutions
Small Solutions
In Total

Binomials and Trinomials
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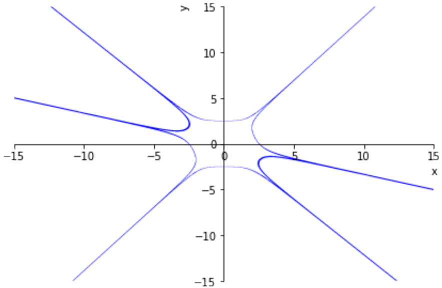

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If $g(x) \in \mathbb{R}[x]$ has degree n, then g has no more than n real roots.

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Lemma (Schmidt, 1987)

Suppose $g(x) \in \mathbb{R}[x]$ has $s+1$ nonzero terms and $g(0) \neq 0$. Then g has no more than $2s$ real roots.

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Answer

Maybe we need to consider some complex roots, but we only need to consider good approximations to $\ll s$ roots.

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Heuristic

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Conjecture (Mueller and Schmidt, 1987)

s^2 can be replaced by s and $(1 + \log h^{1/n})$ is unnecessary.

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Theorem (Bennett, 2001)

$ax^n - by^n = 1$ has at most one solution in positive integers x and y.

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Theorem (Thomas, 2000)

For $n \geqslant 39$ and F a trinomial, the number of solutions to $|F(x, y)| = 1$ is less than or equal to 48.

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Theorem (Akhtari and Bengoechea, 2020)

The number of positive, primitive solutions of $|F(x, y)| \leqslant h$ when h is small relative to the discriminant of F is $\ll s \log s$. If $n \geqslant s^2$, then the number of positive, primitive solutions is $\ll s$.

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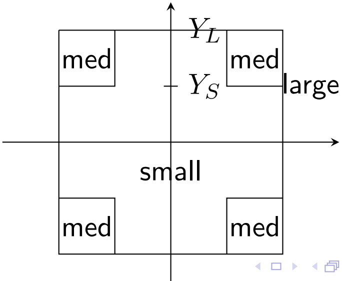

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- Technique: archimedean Newton polygons

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Medium Solution Setup

Lemma (Mueller and Schmidt, 1987)

There is a set S of roots of $f(x) = F(x,1)$ and a set S^* of roots of $g(y) = F(1,y)$ both with size $\ll s$ so that for any solution to $|F(x,y)| \leqslant h$ with $|x|, |y| > Y_S$, there exists $\alpha \in S$ or $\alpha^* \in S^*$ so that

$$\left|\alpha - \frac{x}{y}\right| \leqslant \frac{K}{y^{n/s}} \quad \text{or} \quad \left|\alpha^* - \frac{y}{x}\right| < \frac{K}{x^{n/s}}$$

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where K depends on F and h.

Moral

There's a set of $\ll s$ algebraic numbers so that any primitive solution to $|F(x,y)| \leqslant h$ with $x, y > Y_S$ gives a rational number $\frac{x}{y}$ or $\frac{y}{x}$ which is close to one of those algebraic numbers.

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Goal

Fix $\alpha \in S$ and count the number of rationals which satisfy

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Setup

- Recall that a (positive) medium solution has $Y_S < x, y < Y_L$.
- Fix α, enumerate the medium solutions which satisfy the above inequality, and order them so that

$$Y_S < y_0 \leqslant y_1 \leqslant \cdots \leqslant y_t < Y_L.$$

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- Use the fact that if $\frac{x_i}{y_i}$ and $\frac{x_{i+1}}{y_{i+1}}$ are close to α, they are close to each other:

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- This is known as The Gap Principle

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Counting with Gaps

Using $y_{i+1} > \frac{y_i^{\frac{n}{s}-1}}{K}$ together with $Y_S < y_0 \leqslant y_1 \leqslant \cdots \leqslant y_t < Y_L$, we can find bounds on t.

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$$Y_L \geqslant y_t \geqslant \frac{y_{t-1}^{\frac{n}{s}-1}}{K}$$

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Multiply both sides of

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$$Y_L K^{\frac{-1}{\frac{n}{s}-2}} \geqslant \left(Y_S K^{\frac{-1}{\frac{n}{s}-2}}\right)^{(\frac{n}{s}-1)^t}$$

and solve the inequality for t to find...

Counting

Lemma (K., 2021)

If $n \geqslant 3s$ and there are $t+1$ medium solutions associated to α, then

$$t \leqslant \frac{\log\left[\frac{\log Y_L K^{-1/(\frac{n}{s}-2)}}{\log Y_S K^{-1/(\frac{n}{s}-2)}}\right]}{\log\left(\frac{n}{s}-1\right)}$$

Moreover, this bound is sharp.

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Reducing the above constants into terms of n, s, h, H, using $n \geqslant 3s$, and applying the fact that there are $\ll s$ roots α that we need to care about, we find...

Counting Medium Solutions

Theorem (K., 2021)

The number of primitive medium solutions to $|F(x,y)| \leqslant h$ *when* $n \geqslant 3s$ *is*

$$\begin{aligned} &\ll s\left(1 + \log\left(s + \frac{\log h}{\max(1, \log H)}\right)\right) \\ &\ll s\left(1 + \log s + \log^{+}\left(\frac{\log h}{\max(1, \log H)}\right)\right) \end{aligned}$$

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Recall:

Conjecture

If $h \leqslant H^{1-\frac{s}{n}-\rho}$, then the number of primitive solutions to $|F(x,y)| \leqslant h$ is bounded by a function only of s and ρ.

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Small solutions make up the bulk of the solutions and are tough to count.

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When $n > 4se^{2\Phi}$, the number of primitive small solutions to $|F(x,y)| \leqslant h$ is

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"Definition"

Here, Φ measures the "sparsity" of F and satisfies $\log^3 s \leqslant e^{\Phi} \ll s$.

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Recap

Bounds for Different Types of Solutions

Recall:

- The number of large primitive solutions is $\ll s$.
- The number of medium primitive solutions is $\ll s\left(1+\log s+\log^{+}\left(\frac{\log h}{\max(1,\log H)}\right)\right)$ when $n \geqslant 3s$.
- The number of small primitive solutions is $\ll se^{\Phi}h^{2/n}$ when $n > 4se^{2\Phi}$.

Summing Up

As a consequence:

Theorem (K., 2022)

When $n > 4se^{2\Phi}$, the number of primitive solutions to $|F(x,y)| \leqslant h$ is

$$\ll se^{\Phi}\left(1 + \log^{+}\left(\frac{\log h^{1/\log^3 s}}{\max(1, \log H)}\right)\right) h^{2/n}.$$

Compare to:

Theorem (Mueller and Schmidt, 1987)

The number of integer pair solutions to $|F(x,y)| \leqslant h$ is

$$\ll s^2 h^{2/n}(1 + \log h^{1/n}).$$

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Theorem (Bennett, 2001)

$ax^n - by^n = 1$ *has at most one solution in positive integers* x *and* y.

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Recall that we showed the Gap Principle previously: when $\frac{x}{y}$ and $\frac{x'}{y'}$ both approximate the same root of f and $y' \geqslant y > Y_S$, we had

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In general, K is very large.

Theorem (Thomas, 2000)

When $s = 2$, K can be improved substantially and Y_S can be taken to be less than 1 (eliminating any small solutions). As a consequence, there are explicit bounds on the number of solutions to $|F(x, y)| = 1$. If $n \geqslant 39$, then there are no more than 48 solutions to $|F(x, y)| = 1$.

Trinomials

Theorem (K., 2021)

When $s = 2$, there are no more than $C(n)$ solutions to $|F(x, y)| = 1$ where $C(n)$ is defined by

n	6 - 7	8	9 - 11	12 - 16	17 - 38	39 - 218	$\geqslant 219$
$C(n)$	128	96	72	64	56	48	40

See `https://arxiv.org/abs/2210.09631` for more details.

Trinomial Computations

Computations indicate that for the following degrees (vertical axis) and heights (horizontal axis), the maximum number of solutions to $|F(x,y)| = 1$ is given in the following table:

H	1	2	3	4	5	6	7	8	9	10	11	12	13
$n=6$	8	6	8	8	6	6	6	6	8	6	6	6	6
$n=7$	8	6	8	8	6	6	6	6	8	6	6	6	6
$n=8$	8	6	8	8	6	6	6	6	8	6	6	6	6
$n=9$	8	6	8	8	6	6	6	6	8	6	6	-	-
$n=10$	8	6	8	8	6	6	6	6	8	-	-	-	-
$n=11$	8	6	8	8	6	6	6	6	8	-	-	-	-
$n=12$	8	6	8	8	6	6	6	-	-	-	-	-	-
$n=13$	8	6	8	8	6	6	-	-	-	-	-	-	-
$n=14$	8	6	8	8	6	6	-	-	-	-	-	-	-
$n=15$	8	6	8	-	-	-	-	-	-	-	-	-	-

Thank you!

Questions?