Notes on the Boundary-Layer Dynamics and Energetics

- Dynamics (unchanged from JFM work for a constant-density model):

$$\phi = r'u'/(\Omega' r_0'^2),\ \psi = r'v'/(\Omega' r_0'^2),\ w = w'/(2\Omega'\nu')^{1/2}$$
$$\zeta = z'/(\nu'/2\Omega')^{1/2},\ x = r'^2/r_0'^2$$

$$\phi_x + w_\zeta = 0$$
$$\phi\phi_x + w\phi_\zeta + (\Psi^2 - \psi^2 - \phi^2)/2x - (\psi - \Psi) - \phi_{\zeta\zeta} = 0$$
$$\phi\psi_x + w\psi_\zeta + \phi - \psi_{\zeta\zeta} = 0$$

$$\zeta \to \zeta_{edge}:\ \phi \to 0,\ \psi \to \Psi(x)$$
$$\zeta = 0\ :\ \phi = \psi = w = 0$$

near $x \to 1$:
$$\phi \approx -[\Psi(x)]\sin(2^{-1/2}\zeta)\exp(-2^{-1/2}\zeta)$$
$$\psi \approx [\Psi(x)]\left[1 - \cos(2^{-1/2}\zeta)\exp(-2^{-1/2}\zeta)\right]$$
$$w = 2^{-1/2}\left[\Psi_x(x)\right]\left\{1 - \left[\sin(2^{-1/2}\zeta) + \cos(2^{-1/2}\zeta)\right]\left[\exp(-2^{-1/2}\zeta)\right]\right\}$$

Seek, in particular, $w(x, \zeta \to \zeta_{edge}) \equiv W(x)$ for $\Psi(x)$ of interest by integrating to smaller x from x almost as large as unity. Note that
$$w(x \to 1, \zeta \to \zeta_{edge}) \equiv W(x) \to 2^{-1/2}\Psi_x(x)$$

Adopt $(0 < \epsilon \ll 1)$
$$\Psi(x) = rV(r) = 1 + \epsilon - x,\quad \epsilon \equiv v_0'/(\Omega_0' r_0'),\ \text{given}$$

Note that the integral over ζ from $\zeta = 0$ to $\zeta \to \zeta_{edge}$ of $\phi(x \to 1, \zeta)$ is small ($\approx 2^{-1/2}\epsilon$) since ϕ changes sign repeatedly.

- Introduce a "volumetric" streamfunction η^*, distinct from (but related to) the mass/time streamfunction η. The quantities $\eta^{*\prime}$ and η' are the dimensional counterparts. In prior notes "Diffusive Model of the Bulk-Vortex Module",

$$\eta'(r', z'_{edge}) - 0 = \int_{r'_o}^{r'} \underline{r}' \rho'(\underline{r}', z'_{edge})\, w'(\underline{r}', z'_{edge})\, d\underline{r}';$$

$$\eta'_{max} = \eta'(r'_{min}, z'_{edge}) = \int_{r'_{min}}^{r'_o} \underline{r}' \rho'(\underline{r}', z'_{edge}) \left[-W'(\underline{r}')\right] d\underline{r}';$$

where $W'(r') = W(r)\,(2\nu'\Omega')^{1/2}$, $x = r^2 = (r'/r'_o)^2$,

so $\quad dx = 2r\,dr = 2r'\,dr'/r_o'^2$.

~~Here,~~

$$\eta^*(x_{min}, \zeta) = -\int_0^{\zeta} \phi(x_{min}, \zeta)\, d\zeta, \text{ so } \eta^*_{max} = \eta^*(x_{min}, \zeta_{edge});$$

also,

$$\eta^*_{max} = \eta^*(x_{min}, \zeta_{edge}) = \int_1^{x_{min}} w(x, \zeta_{edge})\, dx = \int_{x_{min}}^1 \left[-W(x)\right] dx$$

$$= \int_{r'_{min}}^{r'_o} \frac{r'\left[-W'(r')\right] dr'}{\left[\frac{\nu'}{2r_o'^2\Omega'}\right]^{1/2} \Omega' r_o'^3} = \int_{r'_{min}}^{r'_o} \frac{r'\left[-W(r')\right]}{\left(E/2\right)^{1/2} \Omega' r_o'^3} dr'$$

$$\eta^*_{max}(\Omega' r_o'^3)(E/2)^{1/2} = \int_{r'_{min}}^{r'_o} r'\left[-W'(r')\right] dr'.$$

~~$$\eta'_{max} = \rho'_c\, \eta^*_{max}\, \{\Omega' r_o'^3\}\{E/2\}^{3/2} = \rho'_c \int_{r'_{min}}^{r'_o} r'\left[-W(r')\right] dr'.$$~~

$$= \int_{r'_{min}}^{r'_o} r' \rho'(r', z'_{edge}) \left[-W'(r')\right] dr'$$

$$= \rho'_{ref}\, \Omega' r_o'^3 (E/2)^{1/2} \int_{x_{min}}^{1} \rho(x, \zeta_{edge}) \left[-W(x)\right] dx$$

where ρ'_c is a quantity with the dimensions of a density that links η'_{max} and η^*_{max}.

$$\eta_{max} = \frac{\eta'_{max}}{\rho'_{ref}\,\Omega' r_o'^{3} (E/2)^{1/2}} = \int_{x_{min}}^{1} \rho(x,\zeta_{edge})\,[-W(x)]\,dx$$

$$= \frac{\rho'_c}{\rho'_{ref}}\,\eta^{*}_{max} = \frac{\rho'_c}{\rho'_{ref}} \int_{x_{min}}^{1} [-W(x)]\,dx$$

where $\rho'_{ref} = p'_{ref}/(R' T'_{ref}) = p'(r'_o, 0)/(R' T'(r'_o, 0))$,

$$\rho_c \equiv \rho'_c/\rho'_{ref}: \quad \rho_c = \frac{\int_{x_{min}}^{1} \rho(x,\zeta_{edge})\,[-W(x)]\,dx}{\int_{x_{min}}^{1} [-W(x)]\,dx}$$

Dimensionally,

$$\eta'_{max} = \rho'_c (\Omega' r_o'^{3})(E/2)^{1/2} \int_{x_{min}}^{1} [-W(x)]\,dx$$

$$= \rho'_{ref} (\Omega' r_o'^{3})(E/2)^{1/2} \int_{x_{min}}^{1} \rho(x,\zeta_{edge})\,[-W(x)]\,dx$$

We need this to conserve total mass/s throughput across the corner flow, which connects the boundary-layer and core modules.

- Energetics

  If we adopt the Reynolds analogy (equidiffusion),

  $$\phi E_x + w E_\zeta = E_{\zeta\zeta}, \quad \phi Y_x + w Y_\zeta = Y_{\zeta\zeta}$$

  where $\left[T'_{amb,o} \equiv T'_{ref} = T'(r'_o, 0); \rho'_{amb,o} \equiv \rho'_{ref} = \rho'(r'_o, 0); \text{etc.} \right]$

  $$E(x, \zeta) = \frac{E'(r', z')}{c'_p T'_{amb,o}} = \frac{c'_p T'(r', z') + L' Y'(r', z') + g' z' + q'^2/2}{c'_p T'_{amb,o}}$$

  $$Y(x, z) = \frac{Y'(r', z')}{Y'_{amb,o}}, \quad Y'_{amb,o} = \frac{(RH)_{amb,o}\, \delta P[T_{amb,o}]}{\rho'_{amb,o}}$$

  $$\text{so } E(x, \zeta) = T(x, z) + \left(\frac{L' Y'_{amb,o}}{c'_p T'_{amb,o}}\right) Y(x, z) + \left(\frac{g' (\nu'/2\Omega')^{1/2}}{c'_p T'_{amb,o}}\right)\zeta + \left(\frac{\Omega'^2 r_o'^2}{c'_p T'_{ref}}\right)\frac{v^2}{2}$$

  — We have let the mass fraction for water vapor, previously denoted $Y(r', z')$, be here denoted $Y'(r', z')$. We reserve Y for ~~th~~ normalized mass fraction (explicitly, so Y is unity at sea-level ambient).

  — $q'^2/2 \longrightarrow v'^2/2$ for our purposes in the def$^{\underline{n}}$ of $E'(r', z')$.

  — At $x \to 1$, $\zeta_{edge} \geq \zeta \geq 0$:

  $$E(x \to 1, \zeta) = E'_{amb}(z') / (c'_p T'_{amb,o}), \quad \zeta = \frac{z'}{(\nu'/2\Omega')^{1/2}}$$

  $$Y(x \to 1, \zeta) = Y'_{amb}(z') / Y'_{amb,o}, \quad \zeta = \frac{z'}{(\nu'/2\Omega')^{1/2}}$$

  — At $\zeta \to \zeta_{edge} \left(\equiv z'_{edge}/(\nu'/2\Omega')^{1/2}\right)$, $1 \geq x \geq x_{min}$:

  $$E(x, \zeta_{edge}) \longrightarrow E'_{amb}(z'_{edge}) / (c'_p T'_{amb,o})$$

  $$Y(x, \zeta_{edge}) \longrightarrow Y'_{amb}(z'_{edge}) / Y'_{amb,o}$$

  Note compatibility with conditions as $x \to 1$

At $\zeta = 0$, $1 \geq x \geq x_{min}$:

$$E(x,0) = T(x,0) + \frac{L' Y'_{amb,s}}{c_p' T'_{amb}} Y(x,0)$$

$Y(x,0)$ assigned; $T(x,0)$ assigned

$$T(x,0) = \frac{T'(r',0)}{T'_{ref}}$$

The temperature tends to fall 2-3 K from T'_{ref} as r' decreases for large, slowly translating vortices owing to ocean churning. Otherwise, $T'(r',0)$ remains at T'_{ref}.

$$Y(x,0) = \frac{RH(r')\, P'[T'(r',0)]/p'(r',0)}{RH(r_o')\, P'[T'_{ref}]/p'_{ref}}$$

As a best guess, $RH(r') \approx RH(r_o') \approx 0.85$ or so. Since we approximate the boundary layer as having constant density,

$$p'(r',0) = p'_{ref} - \left[\int_{r'}^{r_o'} \rho'(r', z'_{edge}) \left\{ 2\Omega' v'(r') + \frac{v'^2(r')}{r'} \right\} dr' \right]$$

$$= p'_{amb,s} + \left[p'(r', z'_{edge}) - p'_{amb}(z'_{edge}) \right]$$

$$= p'_{amb,s} + \left[p'_{edge}(r') - p'_{edge}(r_o') \right]$$