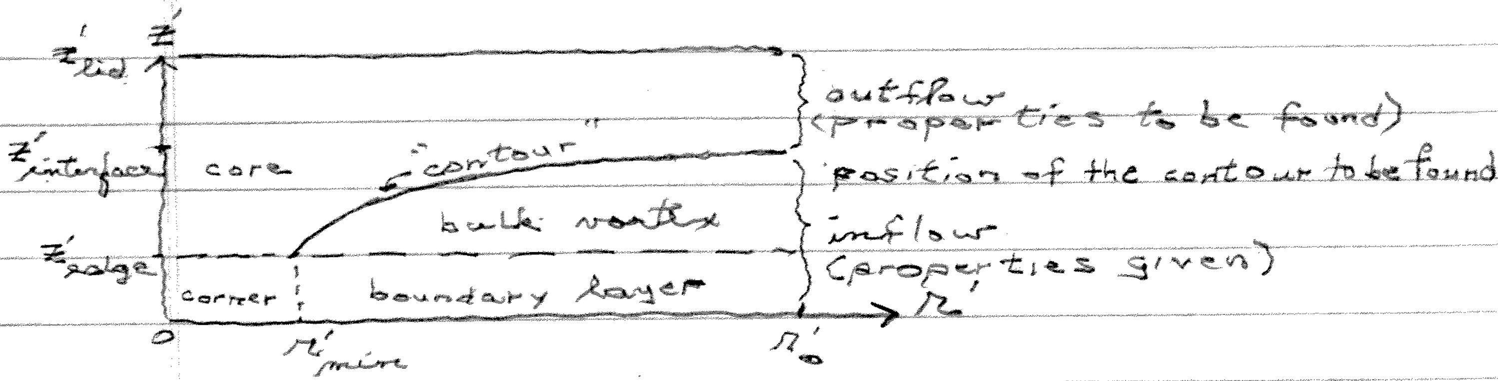
Diffusive Model of the Bulk-Vortex Module



Notes.

— Prime superscript denotes a dimensional quantity.

— Parameters z'_{edge}, $z'_{interface}$, r'_{min}, r'_0 given; the dimension z'_{lid} is computed in the (preliminary) tephigram method.

— The efflux (mass/time) at $r'=r'_0$, $z'_{interface} < z' < z'_{lid}$ equals the influx at $r'=r'_0$, $0 \leq z' < z'_{interface}$. For the tephigram method, measured ambient conditions hold for $r'=r'_0$, $0 < z' < z'_{lid}$. Here, known ambient conditions hold only for $r'=r'_0$, $0 < z' < z'_{interface}$, where convectively unstable stratification holds for circumstances of interest.

— In the Carrier/Hammond/George treatment, the angular momentum per unit mass, $r'v'(r',z') + \Omega' r'^2$, is constant on streamlines (inviscid treatment). Here, v' is (relative) swirl, i.e., swirl in noninertial coordinates rotating with the Earth at the locally pertinent angular speed Ω', termed the Coriolis parameter. We also adopt $r'v'(r') + \Omega' r'^2 = r'_0 v'_0 + \Omega' r_0'^2$, where $v'(r'_0) = v'_0$, and $0 < \epsilon \ll 1$ where $\epsilon \equiv v'_0/(\Omega' r'_0)$, given.

In the CHG treatment, in the bulk-vortex module:

$$Y(r', z') = Y_{amb}(z') \equiv Y(r_0', z'),$$

$$E'(r', z') = E'_{amb}(z') \equiv E'(r_0', z'),$$

where: $Y \equiv \rho_v'/\rho' = \sigma (RH) P'(T')/p' = \sigma\, p_v'/p'$

$$E = c_p' T' + L' Y + g' z' + q'^2/2$$
$$\simeq c_p' T' + L' Y + g' z' + v'^2/2.$$

{RH denotes relative humidity; σ given, dimensionless; $P'(T') \equiv$ saturation vapor pressure}

That is, the relative speed $q'^2 = u'^2 + v'^2 + w'^2 \simeq v'^2$, and sometimes we drop even the v'^2 for tractability, as in the ambient $r' \to r_0'$. We simply cite CHG for justification for this approximation. Physically, CHG are saying that, as air entering the bulk vortex at $r' = r_0'$ moves to smaller r', it is sinking slowly into the boundary-layer module, for compatibility with the boundary-layer solution. As the air descends to smaller z', it takes on the properties of air that formerly occupied that position at smaller z'. We recall from the tephigram method:

$$E'_{amb}(z') \equiv c_p' T'_{amb}(z') + L' Y_{amb}(z') + g' z' + \underbrace{\cancel{\frac{v'^2(z')}{2}}}_{\simeq 0}$$

$$Y_{amb}(z') \equiv \sigma\, RH_{amb}(z')\, P[T'_{amb}(z')] / p'_{amb}(z')$$

where for a given ambient we have the data

$$T'_{amb}(p'),\quad RH_{amb}(p') \equiv p_v'(p')/P[T'_{amb}(p')]$$

Also, we have [recall that $\rho'(r_0', z') \equiv \rho'_{amb}(z')$]

$$dp'_{amb}(z')/dz' = -\rho'_{amb}(z')\, g' = -\{\rho'_{amb}[p'(z')]\}\, g'.$$

In other words, we need to be able to switch between the inverse functions $p'_{amb}(z') \leftrightarrow z'_{amb}(p')$.
The reference state is taken to be $p'_{amb}(r'_o, 0) = p'_{ref}$, $T'_{amb}(r'_o, 0) = T'_{ref}$.

$$p'_{r} = \rho'_{r} R' T'$$

$$p'(r', z') = \rho'(r', z') R' T'(r', z') \Rightarrow p_{amb}(z') = \rho_{amb}(z') R' T'_{amb}(z')$$

It will be convenient to approximate the equation of state for the gas as

$$\rho'(r', z') \approx \rho'_{amb}(z')\, T'(r', z')$$

in the bulk-gas module, for some purposes. This says merely that the density change in the bulk-vortex module is owing to hydrostatics mostly, because even the most intense hurricane is highly subsonic. We are also saying that we track water vapor only for its large condensational/evaporative heat; aside from that, water vapor is a trace species (< 3% by mass contribution to air).

Since the flow is quasisteady and axisymmetric, the secondary flow is treated by introduction of the streamfunction $\eta'(r', z')$; so the radial velocity component $u'(r', z')$ and the axial velocity component $w'(r', z')$ are given by

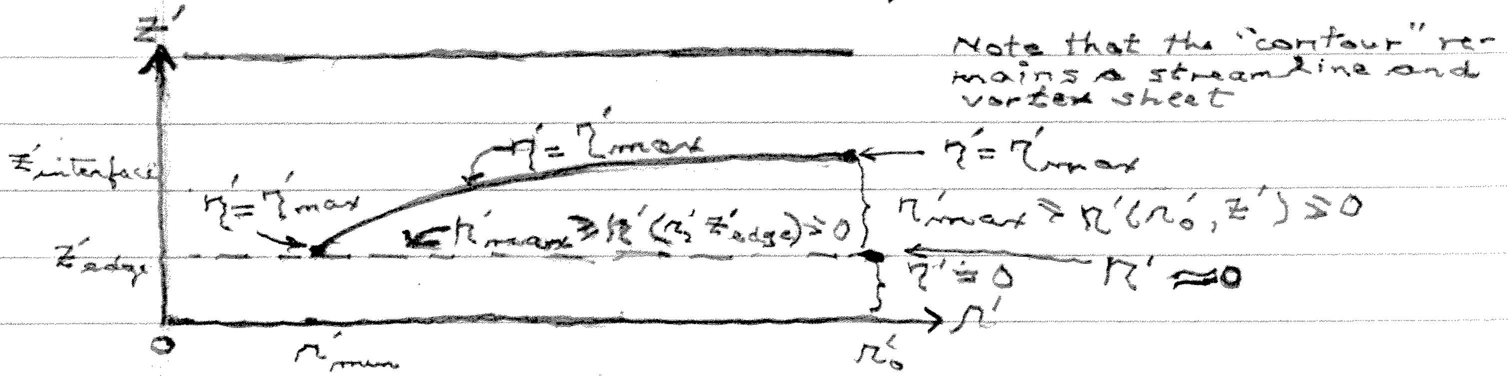
$$\rho'(r', z')\, u'(r', z')\, r' = -\frac{\partial \eta'}{\partial z'}, \quad \rho'(r', z')\, w'(r', z')\, r' = \frac{\partial \eta'}{\partial r'}$$

If the azimuthal component of vorticity $\omega'_\theta = 0$ within the bulk-vortex module, then

$$\omega'_\theta(r', z') = -\left[\frac{\partial w'}{\partial r'} - \frac{\partial u'}{\partial z'}\right] = 0 \Rightarrow \frac{\partial}{\partial z'}\left(\frac{1}{\rho' r'}\frac{\partial \eta'}{\partial z'}\right) + \frac{\partial}{\partial r'}\left(\frac{1}{\rho' r'}\frac{\partial \eta'}{\partial r'}\right) = 0$$

For a first cut, $\rho'(r', z') \rightarrow \rho'_{amb}(z')$ seems reasonable.

For Dirichlet boundary condition on $\Gamma'(r', z')$ in the bulk-vortex module,



At $r' = r'_0$:

$$\rho' u' r'_0 = -\frac{\partial \Gamma'}{\partial z'} \Rightarrow \Gamma'(r'_0, z') - 0 = r'_0 \int_{z'_{edge}}^{z'} \rho'(r'_0, z')[u(r'_0, z')]\,dz'$$

$$\Gamma'_{max} = r'_0 \int_{z'_{edge}}^{z'_{interface}} \rho'_{amb}(z')[-u_{in}(z')]\,dz'$$

If $[u_{in}(z')] = -u_{in}$, const, then

$$\Gamma'_{max} = \frac{r'_0}{g'}\left[p'_{amb}(z'_{interface}) - p'_{amb}(z'_{edge})\right](-u_{in})$$

The inflow $(-u'_{in})$ follows since Γ'_{max} is known.

Aside: $\frac{\partial p'}{\partial r'}(r'_0, z'_{edge}) \approx \rho'(r', z'_{edge})\left[2\Omega' v'(r') + \frac{v'^2(r')}{r'}\right]$;

$p'(r'_0, z_{edge}) = p'_{amb}(z'_{edge})$; $v'(r')$ known:

$r' v'(r') + \Omega' r'^2 = r'_0 v'(r'_0) + \Omega' r_0'^2$;

$\rho'(r', z'_{edge}) = p'(r', z'_{edge}) / [R' T'(r', z'_{edge})]$,

$c_p T'(r', z'_{edge}) + L' Y'(z'_{edge}) + g' z'_{edge} + \frac{v'^2(r')}{2} = E'_{amb}(z'_{edge})$

This yields a good approximation to $p'(r'_{min}, z'_{edge})$.

Returning to estimation of $\Gamma'_{max} = \Gamma'(r'_{min}, z'_{edge})$:

$$\Gamma'(r', z'_{edge}) - 0 = \int_{r'_0}^{r'} \underline{r}'\, \rho'(\underline{r}', z'_{edge})\, w'(\underline{r}', z'_{edge})\, d\underline{r}'$$

from boundary-layer dynamics

$$\eta'_{max} = \int_{r'_{min}}^{r'_o} r' \rho'(r', z'_{edge}) \left[-w'_{\text{boundary layer, edge}}(r') \right] dr'$$

The entrainment into the boundary layer is estimated on an incompressible basis (constant-density treatment of the dynamics), but we accept this approximation. The density $\rho'(r', z'_{edge})$ is discussed just above. The quantity η'_{max} is an enormous number in SI units, and we use it for normalization of $\eta'(r', z')$ for computational convenience! So we deal with

$$\eta(r, z) = \eta'(r', z')/\eta'_{max},$$

where $r = r'/r'_o$, $z = z'/z'_{bl}$. We cannot readily nondimensionalize z' against r'_o because $r'_o \gg z'_{bl}$. It is true that r'_{min} is closer to z'_{bl} in size, but I prefer not to use it for nondimensionalization. We could use $[\nu'/(2\Omega')]^{1/2}$ for nondimensionalizing z' in the bulk-vortex module, since $z'_{interface} \simeq 3\,(z'_{edge})$, and $z'_{edge} \sim 5\left(\frac{\nu'}{2\Omega'}\right)^{1/2}$. I did not want to do this when the bulk-vortex module was being treated as inviscid, but now only the dynamics (not the energetics) is inviscid in the bulk-vortex module.

— The pressure field in the bulk-vortex module is given by inviscid dynamics:

$$\frac{\partial p'}{\partial z'} = -\rho' g', \quad \frac{\partial p'}{\partial r'} = \rho'\left[2\Omega' v' + \frac{v'^2}{r'}\right], \quad p'(r'_o, z') = p'_{amb}(z')$$

$$p'(r'_o, 0) = p'_{ref}$$

Conveniently,

$$p'(r',z') - p'(r'_0, z'_{edge}) \doteq -g' \int_{z'_{edge}}^{z'} \rho'_{amb}(z')\, dz' - \int_{r'}^{r'_0} \rho'(r', z'_{edge}) \left\{ 2\Omega' v'(r') + \frac{v'^2(r')}{r'} \right\} dr'$$

This approximation gives the pressure field to better accuracy along the top edge of the boundary layer and at the periphery, but less accurately elsewhere. Thus the contour is given relatively accurately near its end "points" (r'_{max}, z'_{edge}) and $(r'_0, z'_{interface})$.

In the above expression for $p'(r', z')$, $p'(r'_0, z'_{edge}) = p'_{ref}$. Also, integration yields $p'(r', z'_{edge})$ (see page 4), and $T'(r', z'_{edge})$ is available from the expression $E'(r', z'_{edge}) = E'_{amb}(z'_{edge})$ (see page 4). Thus $\rho'(r', z'_{edge}) = p'(r', z'_{edge}) / [R' T'(r', z'_{edge})]$.

Also, $\frac{s'(r', z')}{c_p'} = \ln\left\{ [T'(r', z') / T'_{ref}] \Big/ [p'(r', z') / p'_{ref}]^{(\gamma-1)/\gamma} \right\}$,

or $\frac{s'(r', z')}{R'} = \ln\left\{ [T'(r', z') / T'_{ref}]^{\gamma/(\gamma-1)} \Big/ [p'(r', z') / p'_{ref}] \right\}$.

To within $2\frac{1}{2}\%$, with $L' = 2.500 \times 10^6$ J/kg, with T' in K, vapor over water, $\ln P'(T') = \{\log_{10} e\}^{-1} [11.40 - 2353/T']$, P' in Pa; with $L' = 2.834 \times 10^6$ J/kg and T' in K, vapor over ice, $\ln P'(T') = \{\log_{10} e\}^{-1} [12.545 - 2665.8/T']$, P' in Pa.

I give more accurate expressions in the notebook. Nominally, freezing is at 273 K, but in fact in the atmosphere, freezing occurs closer to 263 or 253 K.

Notice that a reconnaissance flight at 10 k ft flies in the core module.