Core Module: Derivation of the Streamfunction Equation (for a Tropical-Storm Scenario: no eye)

· The core module is steady, axisymmetric, inviscid, and saturated.

— So the vapor mass fraction $Y' \to Y_s'(T', p')$, where $Y_s'(T', p') = \sigma \frac{P'(T')}{p'} = 0.622 \frac{P'(T')}{p'}$, $P'\{T'\}$ given

note: sometimes I write $P'\{T'\}$ as $p_s'\{T'\}$, an old habit. Y' is a contradiction because prime denotes dimensional, and the vapor mass fraction is dimensionless, but I reserve the symbol Y for a normalized vapor mass fraction.

— Y' is not conserved on streamlines $\eta'(r', z')$ owing to condensation.

— So there are just three integrals that are constant on streamlines in the core:

$E'\{\eta'\} = c_p' T' + L' Y_s'(T', p') + g' z' + q'^2/2$

where we often approximate $q'^2/2 \to v'^2/2$

but more generally $q'^2 = u'^2 + v'^2 + w'^2$.

$S'(\eta') = c_p \ln(T'/T_r') - R_a' \ln(p'/p_r') + \underline{L' Y_s'(T', p')/T'}$

by choice: gas const for dry air

$T_r' = T_{ref} = T'(r_o', 0) = T_{amb}'(0)$

$p_r' = p_{ref} = p'(r_o', 0) = p_{amb}'(0)$

$\Gamma'(\eta') = r' v' + \Omega' r'^2$

— $p'/p_r' = (\rho'/\rho_r')(T'/T_r')$

Solving the S' equation for p'/p_r', and substituting for h'/h_r' from the E' equation gives

$p'\{r', z', \eta'(r', z'), T'[r', z', \eta'(r', z')]\}/p_r'$

$$\left(\frac{T'}{T_r'}\right)^{\gamma/(\gamma-1)} \exp\left[-\frac{S'(\eta')}{R_a'} + \frac{E'(\eta') - g'z' - q'^2/2}{R_a' T'} - \frac{\gamma}{\gamma-1}\right]$$

$\rho'\{r', z', \eta'(r', z'), T'[r', z', \eta'(r', z')]\}/\rho_r' =$

$$\left(\frac{T'}{T_r'}\right)^{1/(\gamma-1)} \exp\left[-\frac{S'(\eta')}{R_a'} + \frac{E'(\eta') - g'z' - q'^2/2}{R_a' T'} - \frac{\gamma}{\gamma-1}\right]$$

~~note~~: $E'(\eta')$ is an implicit algebraic equation for $T'[r', z', \eta'(r', z')]$, once the streamfunction $\eta'(r', z')$ is found.

Since $p' = p'\{r', z', \eta'(r', z'), T'[r', z', \eta'(r', z')]\}$, by the chain rule:

$$\left\{\frac{\partial p'}{\partial r'}\left[r', z', \eta'(r', z'), T'[r', z', \eta'(r', z')]\right]\right\}_{z'} = \left(\frac{\partial p'}{\partial \eta'}\right)_{T', r', z'} \left(\frac{\partial \eta'}{\partial r'}\right)_{z'}$$

$$+ \left(\frac{\partial p'}{\partial r'}\right)_{T', \eta', z'} + \left(\frac{\partial p'}{\partial T'}\right)_{\eta', r', z'} \left(\frac{\partial T'}{\partial \eta'}\right)_{r', z'} \left(\frac{\partial \eta'}{\partial r'}\right)_{z'} + \left(\frac{\partial p'}{\partial T'}\right)_{\eta', r', z'} \left(\frac{\partial T'}{\partial r'}\right)_{\eta', z'}$$

We need this expression because we seek the streamfunction equation from the conservation of radial momentum:

$$\rho' u' \frac{\partial u'}{\partial r'} + \rho' w' \frac{\partial u'}{\partial z'} - 2\Omega' \rho' v' - \rho' \frac{v'^2}{r'} = -\left(\frac{\partial p'}{\partial r'}\right)_{z'}$$

There are three partial derivatives of p' to be found.

$$\left(\frac{\partial \Gamma'}{\partial r'}\right)_{z'} = \rho' w' r' \quad \text{from continuity}$$

$$c_p' T' + \frac{L'\sigma\, P_s'(T')/P_r'}{\left(\frac{T'}{T_r'}\right)^{\gamma/(\gamma-1)} \exp\left\{-\frac{S'(\eta')}{R_a'} + \frac{E'(\eta') - g'z' - \underset{\sim}{q}'^2/2}{R_a' T'} - \frac{\gamma}{\gamma-1}\right\}}$$

Ψ ← dimensionless, so it has no prime superscript

$$= E'(\eta') - g'z' - \underset{\sim}{q}'^2/2$$

This is a useful form from which to obtain two needed partial derivatives of T'.

$$\left(\frac{\partial p'}{\partial r'}\right)_{T', \eta', z'} = \rho'\left[-u'\frac{\partial u'}{\partial r'} - v'\frac{\partial v'}{\partial r'} - w'\frac{\partial w'}{\partial r'}\right]$$

$$\left(\frac{\partial p'}{\partial \eta'}\right)_{T', r', z'} = \rho'\left[-T'\frac{dS'(\eta')}{d\eta'} + \frac{dE'(\eta')}{d\eta'} - v'\frac{\partial v'}{\partial \eta'}\right]$$

$$\Gamma'(\eta') = r'v' + \Omega' r'^2 \Rightarrow v' = \frac{\Gamma'(\eta') - \Omega' r'^2}{r'}$$

$$\left(\frac{\partial v'}{\partial \eta'}\right)_{r', z'} = \frac{d\Gamma'(\eta')}{d\eta'}\frac{1}{r'}$$

$$= \rho'\left[-T'\frac{dS'(\eta')}{d\eta'} + \frac{dE'(\eta')}{d\eta'} - \frac{v'}{r'}\frac{d\Gamma'(\eta')}{d\eta'}\right]$$

$$\left(\frac{\partial p'}{\partial T'}\right)_{\eta', r', z'} = \rho'\left[c_p' - \frac{E'(\eta') - g'z' - \underset{\sim}{q}'^2/2}{T'}\right]$$

$$\left(\frac{\partial T'}{\partial \eta'}\right)_{r', z'} = \frac{\frac{dE'(\eta')}{d\eta'}}{c_p' + R_a' T'[\chi]\left\{\frac{dP_s'(T')/dT'}{P_s'(T')} - \frac{\gamma}{\gamma-1}\frac{1}{T'} + \frac{E'(\eta') - g'z' - \underset{\sim}{q}'^2/2}{R_a' T'^2}\right\}}$$

$$\chi \equiv \frac{L'\sigma\, P_s'(T')/P_r'}{\Psi\,(R_a' T')}$$

χ is dimensionless so it has no prime

$$\left(\frac{\partial T'}{\partial r'}\right)_{\eta', z'} = \frac{-\left(u'\frac{\partial u'}{\partial r'} + v'\frac{\partial v'}{\partial r'} + w'\frac{\partial w'}{\partial r'}\right)\left[1+\chi\right]}{c_p + R_a' T'\left[\chi\right]\left\{\frac{dp_s'(T')/dT'}{p_s'(T')} - \frac{\gamma}{\gamma-1}\frac{1}{T'} + \frac{E'(\eta') - g'z' - q'^2/2}{R_a' T'^2}\right\}}$$

Substituting for the pressure gradient in the equation for conservation of radial momentum:

$$\rho' u'\frac{\partial u'}{\partial r'} + \rho' w'\frac{\partial u'}{\partial z'} - 2\rho'\Omega' v' - \rho' v'^2/r' =$$

$$-\rho'\left[-T'\frac{dS'(\eta')}{d\eta'} + \frac{dE'(\eta')}{d\eta'} - v'\frac{\partial v'}{\partial \eta'}\right]\rho' w' r'$$

$$-\rho'\left[-u'\frac{\partial u'}{\partial r'} - v'\frac{\partial v'}{\partial r'} - w'\frac{\partial w'}{\partial r'}\right]_{T', \eta', z'}$$

$$-\rho'\left\{c_p - \frac{E'(\eta') - g'z' - q'^2/2}{T'}\right\}\rho' w' r'.$$

$$\left\{\frac{\frac{dE'}{d\eta'}(\eta')\left[1+\chi\right] - v'\left(\frac{\partial v'}{\partial r'}\right)_{\eta', z'}\left[1+\chi\right] - T'\frac{dS'(\eta')}{d\eta'}\left[\chi\right]}{c_p' + R_a' T'\left[\chi\right]\left\{\frac{dp_s'(T')/dT'}{p_s'(T')} - \frac{\gamma}{\gamma-1}\frac{1}{T'} + \frac{E'(\eta') - g'z' - q'^2/2}{R_a' T'^2}\right\}}\right\}$$

$$+\rho'\left\{c_p' - \frac{E'(\eta') - g'z' - q'^2/2}{T'}\right\}.$$

$$\left\{\frac{\left(u'\frac{\partial u'}{\partial r'} + v'\frac{\partial v'}{\partial r'} + w'\frac{\partial w'}{\partial r'}\right)_{\eta', z'}\left[1+\chi\right]}{c_p' + R_a' T'\left[\chi\right]\left\{\frac{dp_s'(T')/dT'}{p_s^2(T')} - \frac{\gamma}{\gamma-1}\frac{1}{T'} + \frac{E'(\eta') - g'z' - q'^2/2}{}\right\}}\right\}$$

We take: $v'\frac{\partial v'}{\partial r'} \gg \left(u'\frac{\partial u'}{\partial r'} + w'\frac{\partial w'}{\partial r'}\right)$ in the last term

We note: $-2\Omega v' - \frac{v'^2}{r'} = v'\left(\frac{\partial v'}{\partial r}\right)_\eta$;

$$c_p' - \frac{E'(\eta') - g'z' - q'^2/2}{T'} = -L' \gamma_s'(T')(T'$$

$$= -\frac{L'}{T'}\frac{\sigma p_s'(T')}{p'}$$

Cancel ρ' in all terms, and divide each term by w'.

$$\frac{\partial u'}{\partial z'} - \frac{\partial w'}{\partial r'} = -\rho' r' \left[-T' \frac{dS(\eta')}{d\eta'} + \frac{dE(\eta')}{d\eta'} - v' \frac{\partial v'}{\partial \eta'} \right]$$

$$-\rho' r' \left[\frac{L'}{T'} \frac{\sigma p_s'(T')}{p'} \right] \left\{ \frac{\frac{dE(\eta')}{d\eta'}[1+\chi] - v'\left(\frac{\partial v'}{\partial \eta'}\right)_{r',z'}[1+\chi] - T'\frac{dS(\eta')}{d\eta'}[\chi]}{c_p' + R_a' T'[\chi] \left\{ \frac{dp_s'(T')/dT'}{p_s'(T')} - \frac{\gamma}{\gamma-1}\frac{1}{T'} + \frac{E(\eta') - gz' - q'^2/2}{R_a' T'^2} \right\}} \right\}$$

$$+ \frac{1}{\mu'} \left[\frac{L'}{T'} \frac{\sigma p_s'(T')}{p'} \right] \left\{ \frac{v'\left(\frac{\partial v'}{\partial r'}\right)_{\eta',z'}[1+\chi]}{c_p' + R_a' T'[\chi] \left\{ \frac{dp_s'(T')/dT'}{p_s'(T')} - \frac{\gamma}{\gamma-1}\frac{1}{T'} + \frac{E(\eta') - gz' - q'^2/2}{R_a' T'^2} \right\}} \right\}$$

Since $v' = v'(\eta', r')$, a well-known identity is

$$\left(\frac{\partial \eta'}{\partial r'}\right)_{v'} \left(\frac{\partial v'}{\partial \eta'}\right)_{r'} \left(\frac{\partial r'}{\partial v'}\right)_{\eta'} = -1$$

For terms in the last two lines to cancel,

$$\rho' w' r' \left(v' \frac{\partial v'}{\partial \eta'}\right)_{r',z'} = -v'\left(\frac{\partial v'}{\partial r'}\right)_{\eta',z'}$$

But $\rho' w' r' = \left(\frac{\partial \eta'}{\partial r'}\right)_{z'}$, so

$$\left(\frac{\partial \eta'}{\partial r'}\right)_{z'} \left(\frac{\partial v'}{\partial \eta'}\right)_{r',z'} \left(\frac{\partial r'}{\partial v'}\right)_{\eta',z'} = -1$$

We take this to hold to good approximation so terms involving v' in the last two lines cancel.

$$-\frac{\partial}{\partial z'}\left(\frac{1}{\rho' r'}\frac{\partial \eta'}{\partial z'}\right) - \frac{\partial}{\partial r'}\left(\frac{1}{\rho' r'}\frac{\partial \eta'}{\partial r'}\right) = -\rho' r' \left[-T'\frac{dS(\eta')}{d\eta'} + \frac{dE(\eta')}{d\eta} - \frac{v'}{r}\frac{d\Gamma(\eta')}{d\eta'} \right]$$

$$+\rho' r' \left[\frac{L' Y_s'(T',p')}{T'} \right] \left[\frac{(dE(\eta')/d\eta')[1+\chi] - T'(dS(\eta')/d\eta')[\chi]}{c_p' + R_a' T'[\chi] \left\{ \frac{dp_s'(T')/dT'}{p_s'(T')} + \frac{L' Y_s'(T',p')/T'}{R_a T'} \right\}} \right]$$

$$Y_s'(T',p') \equiv \sigma p_s(T')/p'$$

Recall:

$$\chi \equiv \frac{L'\sigma p_s'(T')/p_r'}{[\Upsilon] R_a' T'}, \quad \Upsilon = \left(\frac{T'}{T_r'}\right)^{\frac{\gamma}{\gamma-1}} \exp\left[-\frac{S'(\eta')}{R_a'} + \frac{L' Y_s'(T',p')}{R_a' T'}\right]$$

$$-\frac{\partial}{\partial z'}\left(\frac{1}{\rho' r'}\frac{\partial \eta'}{\partial r'}\right) - \frac{\partial}{\partial r'}\left(\frac{1}{\rho' r'}\frac{\partial \eta'}{\partial r'}\right) =$$

$$-\rho' r'\left\{ -T'\frac{dS'(\eta')}{d\eta'}\left[1 - \frac{[\chi] L' Y_s'(T',p')/T'}{c_p' + R_a' T'[\chi]\left\{\frac{dp_s'(T')/dT'}{p_s'(T')} + \frac{L' Y_s'(T',p')/T'}{R_a' T'}\right\}}\right]\right.$$

$$+ \frac{dE'(\eta')}{d\eta'}\left[1 - \frac{[1+\chi] L' Y_s'(T',p')/T'}{c_p' + R_a' T'[\chi]\left\{\frac{dp_s'(T')/dT'}{p_s'(T')} + \frac{L' Y_s'(T',p')/T'}{R_a' T'}\right\}}\right]$$

$$\left. - \frac{v'}{r'}\frac{d\Gamma'(\eta')}{d\eta'}\right\}$$

• Nondimensionalization

$$\tilde{Y}_s(T,p) \equiv Y_s'(T',p')/Y_{ref}', \quad Y_{ref}' \equiv Y'_{amb,surface}$$

$$T \equiv T'/T_{ref}, \quad p \equiv p'/p_{ref}, \quad p_s(T) = p_s'(T')/p_{ref}$$

$$\tilde{Y}_s(T,p) \equiv \frac{\sigma[p_s'(T')/p_{amb,s}']/(p'/p_{amb,s}')}{\sigma[(RH)_{amb,s}\, p_s'(T_{amb,s}')/p_{amb,s}']/(p_{amb,s}'/p_{amb,s}')}$$

$$= \frac{p_s(T)/p}{(RH)_{amb,s}\, p_s(1)}, \quad (RH)_{amb,s} \equiv \frac{(p_v')_{ref}}{p_s'(T_{ref}')}$$

$p_v' \equiv$ vapor pressure; $p_v = p_v'/p_{ref}$

$$S(\eta) = S'(\eta')/R_a', \quad \Pi_1 \equiv \frac{L' Y_{ref}'}{c_p' T_{ref}'}$$

[$\Pi_i \equiv$ dimensionless group i]

$$\Upsilon = T^{\gamma/(\gamma-1)} \exp\left[-S(\eta) + \Pi_1 \frac{\gamma}{\gamma-1}\frac{\tilde{Y}_s(T,p)}{T}\right]$$

$$X = \pi_1 \left\{\frac{\gamma}{\gamma-1}\right\} \frac{\tilde{Y}_s(T,p)\; p}{T\,[\,\Upsilon\,]}$$

$$\pi_2 \equiv (\Omega'^2 r_o'^2)/(c_p' T_{ref}')$$

$\pi_4 \equiv z_t'^2 / r_o'^2$ { z_t' is an alternate for z_{grid}' },
$S = S'/R_a'$, $(R_a'/c_p') = (\gamma-1)/\gamma$,
$\eta = \dfrac{\eta'}{\eta_{ref}'}$, $\eta_{ref}' \equiv \rho_{ref}' \, \Omega' r_o'^3 (E/2)^{1/2}$, $E \equiv \left(\dfrac{\nu'}{\Omega' r_o'^2}\right)$,

$r = r'/r_o'$, $z = z'/z_t'$, $p = p'/p_{ref}'$, $T = T'/T_{ref}'$, $\rho = \rho'/\rho_{ref}'$,

$p_v = p_v'/p_{ref}'$, $p_s = p_s'/p_{ref}'$, $p_{ref}' = \rho_{ref}' R_a' T_{ref}'$,
$\Gamma = \Gamma'/(\Omega' r_o'^2) = r v + r^2$, $v = v'/(\Omega' r_o')$.

$$-\frac{\partial}{\partial z}\left(\frac{1}{\rho r}\frac{\partial \eta}{\partial z}\right) - \pi_4 \frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial \eta}{\partial r}\right) = -\frac{\pi_4}{\pi_2 (E/2)}\, \rho r \left\{ -\pi_2 \frac{v}{r}\frac{d\Gamma(\eta)}{d\eta} \right. -$$

Interesting groupings: $E<<1$, $\pi_2 <<1$, $\pi_4^2 << 1$; we anticipate that $\pi_4/[\pi_2(E/2)] = O(1)$ for cases of physical interest: both sides enter.

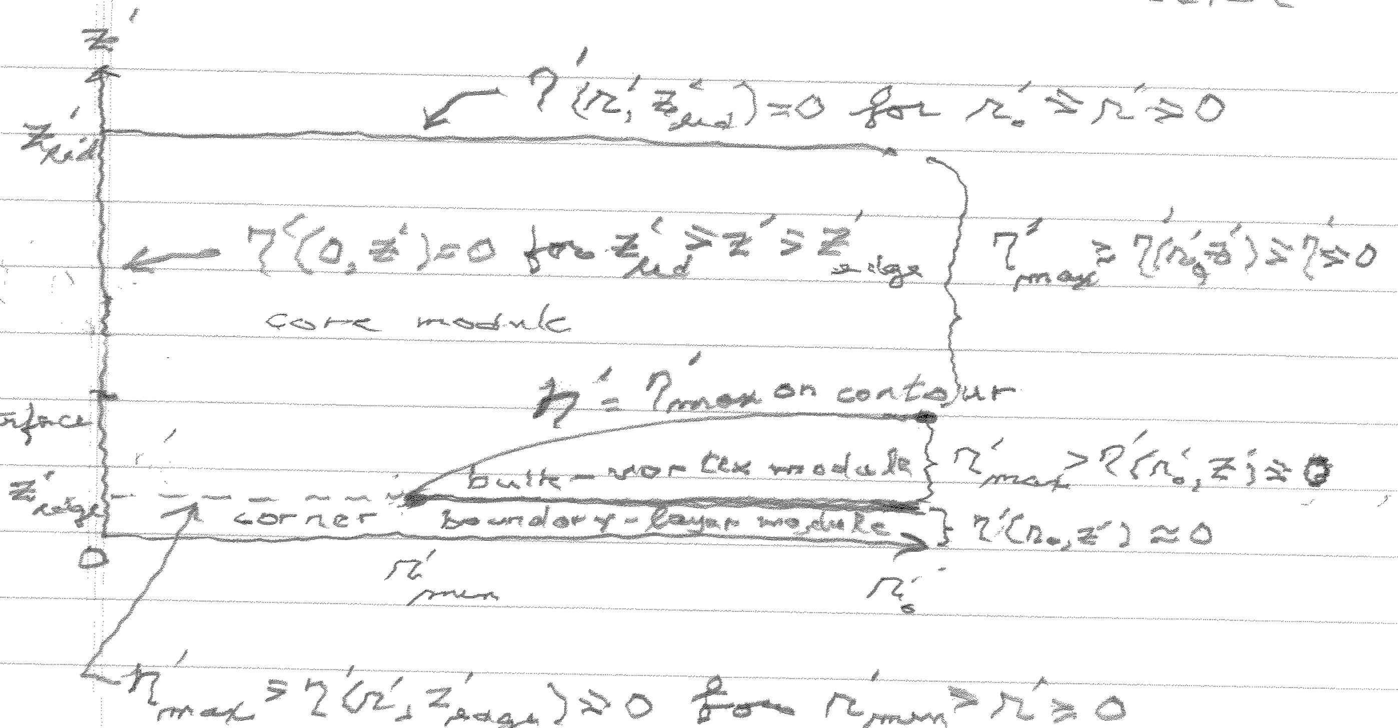
$$- \frac{\gamma-1}{\gamma} T \frac{dS}{d\eta}\left[1 - \frac{[X]\,\pi_1 \tilde{Y}_s(T,p)/T}{1+\pi_1\left[\tilde{Y}_s(T,p)/T\right]\left\{\pi_1 \dfrac{\tilde{Y}_s(T,p)}{T}\dfrac{\gamma}{\gamma-1} + \dfrac{T}{p_s(T)}\dfrac{dp_s(T)}{dT}\right\}}\right]$$

$$+ \frac{dE}{d\eta}\left[1 - \frac{[1+X]\,\pi_1 \tilde{Y}_s(T,p)/T}{1+\pi_1\left[\tilde{Y}_s(T,p)/T\right]\left\{\pi_1 \dfrac{\tilde{Y}_s(T,p)}{T}\dfrac{\gamma}{\gamma-1} + \dfrac{T}{p_s(T)}\dfrac{dp_s(T)}{dT}\right\}}\right]$$

$\pi_1 \approx O(0.2)$ or so, typically

$\pi_4 = 10^{-3}$, $E = 10^{-1}$, $\pi_2 = 10^{-2}$, typically

Core Module: Boundary Conditions on Streamfunction



The boundary condition holding on $r' = r'_0$, $z'_{lid} \geq z' \geq z'_{interface}$ deserves discussion.

$\psi'(r'_0, z')$ should decrease monitonically from $\psi' = \psi'_{max}$ at $z' = z'_{interface}$ to $\psi' = 0$ at $z' = z'_{lid}$, but we really can't prescribe how.

$$-\rho' u'(r'_0, z') r'_0 = \frac{\partial \psi'(r'_0, z')}{\partial z'}, \quad \rho' w'(r'_0, z') r'_0 = \frac{\partial \psi'(r'_0, z')}{\partial r'}$$

$\frac{\partial \psi'(r'_0, z')}{\partial z'} < 0 \Rightarrow u'(r'_0, z') \geq 0$ (outflow). We expect $\frac{\partial \psi'}{\partial r'}(r'_0, z') = 0$ at $z' = z'_{interface}$ and $z' = z'_{lid}$ because these are streamlines (no cross flow), but $\frac{\partial \psi'(r_0, z')}{\partial r'_0}$ might be discontinuous [vortex sheet: inflow below $z' = z'_{interface}$, outflow above $z' = z'_{interface}$ ... though $\frac{\partial \psi'(r'_0, z'_{interface})}{\partial z'} = 0$ is not excluded (smooth transition)]. We might see whether it suffices to take

$$\frac{\partial \psi'}{\partial r'_0}(r_0, z') = 0, \quad z'_{lid} \geq z' \geq z'_{interface}$$

Core Module: Finding the Dependent Variables after the Streamfunction $\psi'(r', z')$ Is in Hand

- $v'(r', z') = \left[\Gamma'(\psi'(r', z')) - \Omega' r'^2\right] / r'$

  $v'(0, z') = 0$

- Simultaneous nonlinear algebraic equations for $p'(r', z')$ and $T'(r', z')$:

$$Y'(r', z') = \varepsilon P'[T'(r', z')] / p'(r', z')$$

$$c_p' T'(r', z') + L' Y'(r', z') + g' z' + \frac{v'^2(r', z')}{2} = E'[\psi'(r', z')]$$

$$\ln \left[\frac{T'(r', z')}{T_{ref}'}\right]^{\gamma/(\gamma-1)} - \ln \left[\frac{p'(r', z')}{p_{ref}'}\right] + \frac{L' Y'(r', z')}{R' T'(r', z')} = \frac{s'[\psi'(r', z')]}{R'}$$

- $\rho'(r', z') = p'(r', z') / [R' T'(r', z')]$

- $u'(r', z') = -\dfrac{1}{\rho'(r', z')} \dfrac{1}{r'} \dfrac{\partial \psi'(r', z')}{\partial z'}$

- $w'(r', z') = \dfrac{1}{\rho'(r', z')} \dfrac{1}{r'} \dfrac{\partial \psi'(r', z')}{\partial r'}$