Assignment 5: Eigenproblems

ACM 106a: Introductory Methods of Computational Mathematics (Fall 2013)

Due date: Thursday, November 14, 2013

1 Normal matrices

Recall that a matrix $A \in \mathbf{C}^{n\times n}$ is *normal* if $A^*A = AA^*$.

(a) Prove that a normal triangular matrix must also be diagonal.

Hint: try using induction to show all off-diagonal entries must be zero.

(b) Use Part (a) to show that a matrix $A \in \mathbf{C}^{n\times n}$ is normal if and only if its Schur form (the triangular matrix T in the Schur Decomposition $A = QTQ^*$) is normal.

(c) Use the results of Parts (a) and (b) to show that a matrix A is normal if and only if it has n orthogonal eigenvectors.

Hint: consider the unitary matrices Q in the Schur Decomposition $A = QTQ^*$.

2 Inverse iteration and ill-conditioned systems

Recall that the *inverse iteration* algorithm requires the solution of a linear system of the form

$$(A - \sigma I)y_{i+1} = x_i$$

during each iteration i. If the shift σ is very close to an eigenvalue of A, the matrix $A - \sigma I$ is very close to being singular, i.e., $A - \sigma I$ is ill-conditioned. This would seem to be a problem; if $A - \sigma I$ is ill-conditioned we may not be able to obtain accurate approximations of our approximate eigenvectors x_{i+1} and y_{i+1}.

In this problem, we will verify that these apparent issues do not actually affect the accuracy of the algorithm. For the sake of simplicity, suppose that $A \in \mathbf{R}^{n\times n}$ is a real symmetric matrix. Recall that this implies that A has n real eigenvalues $\{\lambda_1, \dots, \lambda_n\}$, corresponding to the set of orthonormal eigenvectors $\{q_1, q_2, \dots, q_n\}$. Suppose further that A is well-conditioned so that we may solve linear systems of the form $Ay = x$ backward stably. Finally, suppose that the eigenvalue λ_1 is simple, with eigenvector q_1.

The next two problems show that the approximate eigenvectors x_k, y_k obtained during each step of inverse iteration are very close to being in the direction q_1 if the shift is close to λ_1, regardless of the condition number of $A - \sigma I$. That is, the large error in y_k caused by the ill-conditioning of $A - \sigma I$ is only in the magnitude of y_{k+1} and not in the direction; normalizing to get x_{k+1} should correct any of these errors.

(a) Suppose that the shift $\sigma = \lambda_1 + \mu$ for some very small $\mu \in \mathbf{R}$. Suppose that during the kth step of inverse iteration, we are required to solve the linear system

$$(A - \sigma I + \delta A)\tilde{y}_{k+1} = x_k, \tag{1}$$

for some $\|\delta A\|_2 = O(\epsilon)$. That is, we solve the system $(A - \sigma I)y_{k+1} = x_k$ backward stably, with error matrix δA, to obtain $\tilde{y}_{k+1}$. Show that

$$\|(A - \lambda_1 I)x_{k+1}\|_2 \le (|\mu| + \|\delta A\|_2) + \frac{1}{\|\tilde{y}_{k+1}\|_2},$$

where $x_{k+1} = \tilde{y}_{k+1}/\|\tilde{y}_{k+1}\|_2$. This implies that $(A - \lambda_1 I)x_{k+1}$ is very small, i.e., x_{k+1} and $\tilde{y}_{k+1}$ are close to being eigenvectors corresponding to λ_1, if $\tilde{y}_{k+1}$ is sufficiently large.

(b) We next verify that $\|\tilde{y}_{k+1}\|_2$ is large if $|y_0^T q_1|$ is large.

Recall that the fact that A is real symmetric implies that its set of eigenvectors $\{q_1, \dots, q_n\}$ forms an orthonormal basis for $\mathbf{R}^n$. This means that we may decompose x_k as $x_k = \sum \alpha_i q_i$ for some $\alpha \in \mathbf{R}^n$. Show that we have

$$\|\tilde{y}_{k+1}\|_2 \geq \frac{|\alpha_1|}{|\mu| + \|\delta A\|_2}$$

if $x_k = \sum \alpha_i q_i$.

Hint: try writing $\tilde{y}_{k+1}$ in terms of the eigenvectors $q_1, \dots, q_n$ and substituting into (1). Then consider the projection of either side of (1) onto the line in direction q_1 (the scalar given by multiplying either side by q_1^T).

3 QR iteration with shifts

Suppose that we perform QR iteration with shifts: starting with $A = A_1$, we update A_i by first factorizing $(A_i - \sigma_i I) = Q_i R_i$, and then setting $A_{i+1} = R_i Q_i + \sigma_i I$.

(a) Show that each step of this algorithm is equivalent to multiplying A on the left and the right by the unitary matrix $\bar{Q}_i = Q_1 Q_2 \cdots Q_i$:

$$A_{i+1} = \bar{Q}_i^* A \bar{Q}_i.$$

(b) Recall that a matrix H is in upper Hessenberg form if all entries below the first subdiagonal of H are equal to zero, i.e., $H_{ij} = 0$ for all $i \geq j + 2$. Show that the product of an upper Hessenberg matrix and a triangular matrix is in Hessenberg form. That is, prove that HT and TH are in upper Hessenberg form if H is upper Hessenberg and T is triangular.

(c) Use the result of Part (b) to show that QR iteration preserves upper Hessenberg form: if A_i is upper Hessenberg, then so is A_{i+1}.

Hint: First show that Q_i in the QR factorization $A_i - \sigma_i I = Q_i R_i$ must be upper Hessenberg.

4 QR iteration with bad shifts

In this problem, we consider applying QR iteration with and without shifts to the matrix

$$A = \begin{pmatrix} 5 & -1 & 0 & 0 \\ -1 & 5 & -1 & 0 \\ 0 & -1 & 5 & -1 \\ 0 & 0 & -1 & 5 \end{pmatrix}$$

(a) Write Matlab programs that perform explicit QR iteration with or without shifts (without reduction to Hessenberg form, or use of the Implicit Q theorem). Your code may use Matlab's built-in "qr" function wherever necessary.

When using shifts, you should take $\sigma_i = A_i(4,4)$. Declare your algorithm to be converged when the infinity norm of two subsequent diagonals $\|\text{diag}(A_{i+1} - A_i)\|_\infty$ is less than the stopping tolerance of 10^{-6}.

(b) Use your programs to attempt to compute the eigenvalues of the matrix A above. Compare those found to those given by the command "eig" in Matlab. Explain the observed phenomena.

(c) Revise your QR iteration with shifts to correct any observed inaccuracy in Part (b).

Hint: it may be illuminating to compare the sequence of QR factorizations obtained with shifting for A to those obtained without shifting for the matrix

$$B = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Submission Instructions:

- Assignments are due at the **start** of class (1pm) on the due date.
- Write your name and ID# clearly on all pages, and underline your last name.
- **Matlab files:** please submit a single zip/rar/etc file with file name in the format **Lastname_Firstname_ID#_A5** to **homework.acm106a@gmail.com** that decompresses to a single folder of the same name containing the following:
  - A single thoroughly commented Matlab script file containing all commands used for each assigned programming/simulation problem. File names should have format **Lastname_Firstname_ID#_A5_P#** :

    e.g. **Ames_Brendan_12345678_A5_P4** for problem 4.
  - All Matlab functions used to perform any assigned programming/simulation problems with appropriate file names. Please add a comment to the beginning of each file with the format **Lastname_Firstname_ID#_A5_P#**
  - A diary file of your session for each programming/simulation problem with file name in the format **Lastname_Firstname_ID#_A5_P#_DIARY.txt**. Please also submit a hard copy of the diary and any relevant derivations, pseudocode, etc. you may want considered for partial credit with your submitted solution sets at the beginning of class.