Assignment 3: The LU and Cholesky Decompositions

ACM 106a: Introductory Methods of Computational Mathematics (Fall 2013)

Due date: Thursday, October 24, 2013

1 Why not to explicitly evaluate of the inverse (unless absolutely necessary)

Suppose that $A \in \mathbf{R}^{n\times n}$ is nonsingular. Explain how to efficiently solve the matrix equation

$$AX = B,$$

where $B \in \mathbf{R}^{n\times m}$, using Gaussian elimination with partial pivoting. In particular, the inverse A^{-1} of A is the solution of the matrix equation

$$AX = I.$$

Give a formula for A^{-1} using the LU decomposition $A = PLU$. Estimate the number of flops required to solve the general matrix equation $AX = B$ and, in particular, find the inverse. How many flops are needed to solve the linear system $Ax = b$, $b \in \mathbf{R}^n$, by first computing A^{-1} and then taking $x = A^{-1}b$? How does this compare to the number of flops needed for Gaussian elimination with partial pivoting?

2 The LDL Decomposition

Suppose that $A \in \mathbf{R}^{n\times n}$ is nonsingular and there exists unique lower triangular matrix $L \in \mathbf{R}^{n\times n}$ with diagonal entries equal to 1 and unique upper triangular matrix $U \in \mathbf{R}^{n\times n}$ such that

$$A = LU.$$

Show that if A is also symmetric, i.e., $A^T = A$, then there exists a unique *diagonal* matrix $D \in \mathbf{R}^{n\times n}$ such that

$$A = LDL^T.$$

Hint: First show that if general A (not necessarily symmetric) has LU decomposition $A = LU$, then there exist unique upper triangular matrix $\tilde{U} \in \mathbf{R}^{n\times n}$ with all diagonal entries equal to 1 and diagonal matrix $D \in \mathbf{R}^{n\times n}$ such that $A = LD\tilde{U}$.

3 Fun with "\"

The Matlab command mldivide (backslash) does a reasonably good job of intuiting and exploiting structure of the matrix A when solving the system $A \setminus b$. If A is symmetric it will use a Cholesky decomposition, if A is upper triangular it will use back substitution, etc. In this exercise, we will test how robust this detection of matrix type is to error.

(a) Suppose that we would like to solve the linear system $Ax = b$, where $b \in \mathbf{R}^n$ and $A \in \mathbf{R}^{n\times n}$ is a nonsingular diagonal matrix. Design a linear time algorithm, i.e., needing $O(n)$ flops, for solving $Ax = b$ for diagonal A. Write a Matlab function implementing your algorithm.

Hint: You should be able to implement your algorithm using a single line of Matlab code by exploiting vectorized operations (i.e., you should avoid using loops).

(b) We now test your algorithm and Matlab's backslash command. Randomly generate 25 program instances for $n = 1000$ using the Matlab commands

$$A = \text{diag}(\text{rand}(n, 1)); \qquad b = \text{rand}(n, 1);$$

and solve using your algorithm from Part (a) and Matlab's backslash command $A \setminus b$.

Next, add noise by perturbing A by βE, $\beta = 10^{-15}$, to obtain the following matrices:

1. $A_1 = A + \beta E_1$, where $E_1 = e_1 e_2^T$, and e_1, e_2 are the first two standard basis vectors (first two columns of I);
2. $A_2 = A + \beta E_2$, where $E_2 = E_1 + E_1^T$;
3. $A_3 = A + \beta E_3$, where E_3 is a random upper triangular matrix generated using the Matlab command
$$E_3 = \text{triu}(\text{rand}(n));$$
4. $A_4 = A + \beta E_4$, where E_4 is a random symmetric matrix generated using the Matlab commands
$$E_4 = \text{rand}(n); \qquad E_4 = (E_4 + E_4')/2;$$
5. $A_5 = A + \beta E_5$, where E_5 is a random unsymmetric matrix generated using the Matlab command $E_5 = \text{rand}(n)$.

Report the average (and standard deviation) of time required to solve the perturbed problems using the backslash command and the unperturbed problem with backslash and your algorithm from Part (a). Explain any observed differences in computational complexity, citing the structure of the perturbed matrix A_i and the "default" algorithm for this type of matrix.

Submission Instructions:

- Assignments are due at the **start** of class (1pm) on the due date.
- Write your name and ID# clearly on all pages, and underline your last name.
- **Matlab files:** please submit a single zip/rar/etc file with file name in the format **Lastname_Firstname_ID#_A3** to **homework.acm106a@gmail.com** that decompresses to a single folder of the same name containing the following:
  - A single thoroughly commented Matlab script file containing all commands used for each assigned programming/simulation problem. File names should have format **Lastname_Firstname_ID#_A3_P#** :

    e.g. **Ames_Brendan_12345678_A3_P4** for problem 4.
  - All Matlab functions used to perform any assigned programming/simulation problems with appropriate file names. Please add a comment to the beginning of each file with the format **Lastname_Firstname_ID#_A3_P#**
  - A diary file of your session for each programming/simulation problem with file name in the format **Lastname_Firstname_ID#_A3_P#_DIARY.txt**. Please also submit a hard copy of the diary and any relevant derivations, pseudocode, etc. you may want considered for partial credit with your submitted solution sets at the beginning of class.