Assignment 6: Rayleight Quotient Iteration; Splitting Methods

ACM 106a: Introductory Methods of Computational Mathematics (Fall 2013)

Due date: Thursday, November 21, 2013

1 The spectral radius

Recall that the spectral radius of the matrix A is defined as

$$\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

Show that:

(a) $\rho(A) = \sigma_1(A) = \|A\|_2$ if A is Hermitian.

(b) On the other hand, show that we may have $\rho(A) < \|A\|_2$ if A is not Hermitian. In particular, show that the matrix

$$A = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

satisfies $\|A\|_2 > m$, while $\rho(A) = 1$. In this case, $\rho(A) < \|A\|_2$ if $m \geq 1$.

(c) $\rho(A) \leq \|A^r\|^{1/r}$ for any operator norm $\|\cdot\|$ and positive r.

Hint: Try taking the "rth power" of either side of $Ax = \lambda x$.

(d) $\rho(A) = \lim_{r\to\infty} \|A^r\|^{1/r}$ for any operator norm if A is normal (this is actually true in general, but the general proof is significantly more complicated).

Hint: If we can show that $\|A^r\|/\rho(A)^r \to 1$, we are done. To do so, it may be helpful to use Part (c) and an expansion of the form $A = \sum \lambda_i q_i q_i^T$, where q_i is the eigenvector of A corresponding to λ_i. Using this expansion we can get a similar expansion of A^r.

2 Convergence of the Jacobi and Gauss-Seidel methods

In this problem, we develop convergence results for special classes of linear systems. Recall that a splitting method converges if and only if the update matrix R has spectral radius satisfying $\rho(R) < 1$.

(a) Suppose that we have the 2×2 linear system $Ax = b$, where $A \in \mathbf{R}^{2\times 2}$ is symmetric positive definite, i.e., A is symmetric and $x^T Ax > 0$ for all $x \in \mathbf{R}^2$, or equivalently, all eigenvalues of A are positive. Show that Jacobi and Gauss-Seidel iteration converge in this case.

Hint: It may be useful to derive analytic formulas for the eigenvalues of A and the update matrices R_J and R_{GS}, and then use positivity of the eigenvalues of A to bound $\rho(R_J)$ and $\rho(R_{GS})$. To compute R_{GS} it might be helpful to use the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(b) Next, consider the 2×2 matrix

$$A = \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix}.$$

Under what conditions on α do the Jacobi and Gauss-Seidel methods converge?

(c) Finally, consider the 2×2 block matrix

$$A = \begin{pmatrix} I & S \\ -S^T & I \end{pmatrix},$$

where $S \in \mathbf{R}^{n \times n}$. Using the singular value decomposition of S, establish conditions ensuring convergence of the Jacobi and Gauss-Seidel methods to the solution the system $Ax = b$.

Hint: It may be helpful to use one of the many identities giving formulas for the determinant of a 2×2 block matrix to compute the characteristic polynomial of the update matrices R_J and R_{GS}.

3 Rayleigh Quotient Iteration

In this problem, we investigate the convergence rate of Rayleigh quotient iteration for computing eigenvalues of real symmetric and unsymmetric matrices.

(a) Implement the Rayleigh Quotient Iteration in Matlab. You may use the built-in linear system solver "\" to solve any linear systems inside your loop.

(b) Generate a random 4×4 orthogonal matrix U and consider

$$A = UDU^T,$$

where $D = \mathrm{Diag}(2, 4, 13, 27)$.

Choose several initial iterates and test convergence of the algorithm towards the corresponding eigenvectors and eigenvalues. Your starting points should be chosen so that at least one converges to each eigenvector of A. Graph your error on a log scale as a function of the number of iterations. Roughly at rate is the algorithm converging?

Hint: To obtain your starting points, it is acceptable to choose a weighted sum of the columns of U, where enough weight is concentrated on one column of U to ensure convergence to the corresponding eigenvector of A. Error should be measured as the forward error between our current iterate and the desired eigenvector (or current Rayleigh quotient and desired eigenvalue).

(c) Repeat Part (b) with a random 4×4 nonsingular, well-conditioned matrix S. That is, you should generate a random matrix $S \in \mathbf{R}^{4 \times 4}$ with condition number $\kappa(S) = \sigma_1(S)/\sigma_4(S) \approx 10$, and take

$$A = SDS^{-1}.$$

Then test convergence to the eigenvectors and eigenvalues of A using a few different initial iterates as before.

Hint: To generate S it may be useful to generate each component of its singular value decomposition and then take their product.

Submission Instructions:

- Assignments are due at the **start** of class (1pm) on the due date.
- Write your name and ID# clearly on all pages, and underline your last name.
- **Matlab files:** please submit a single zip/rar/etc file with file name in the format **Lastname_Firstname_ID#_A6** to **homework.acm106a@gmail.com** that decompresses to a single folder of the same name containing the following:

- A single thoroughly commented Matlab script file containing all commands used for each assigned programming/simulation problem. File names should have format **Lastname_Firstname_ID#_A6_P#** :

  e.g. **Ames_Brendan_12345678_A6_P4** for problem 4.

- All Matlab functions used to perform any assigned programming/simulation problems with appropriate file names. Please add a comment to the beginning of each file with the format **Lastname_Firstname_ID#_A6_P#**

- A diary file of your session for each programming/simulation problem with file name in the format **Lastname_Firstname_ID#_A6_P#_DIARY.txt**. Please also submit a hard copy of the diary and any relevant derivations, pseudocode, etc. you may want considered for partial credit with your submitted solution sets at the beginning of class.