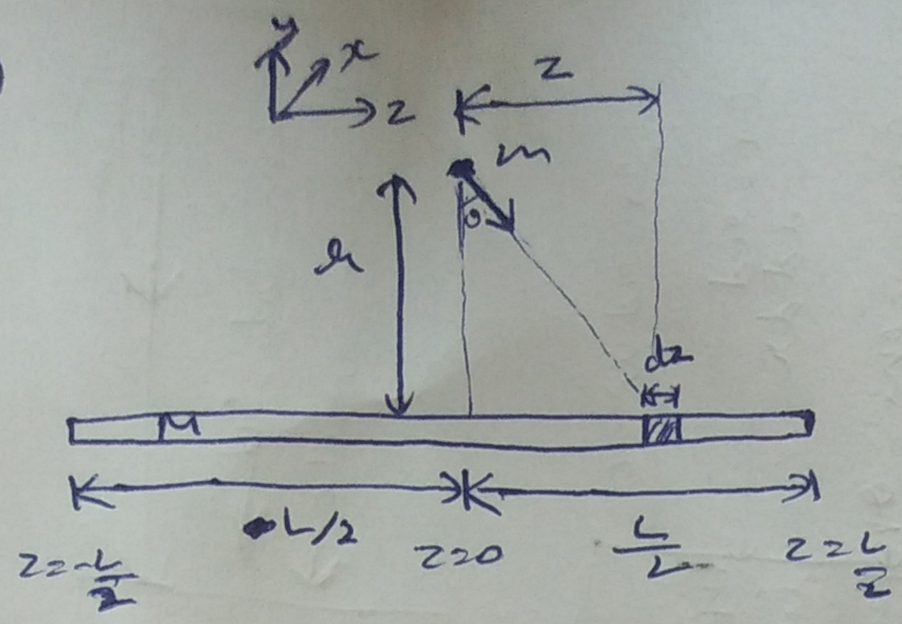
6)



Consider a small element of length dz on rod. Let $d\vec{F}$ be the force exerted by it on point mass m.

By symmetry, it can be seen that net force exerted by rod on point mass is perpendicular to the rod as ~~net~~ force along the rod is equal & opposite for both halves of the rod about point mass. so, net force has only components along x & y direction. so, $\Rightarrow \boxed{F_z = 0}$ only x & y components of $d\vec{F}$ are considered. Let mass of small element be dM.

$$\Rightarrow \boxed{dM = \frac{M}{L}\,dz}$$

$$d\vec{F}_x = -\frac{Gm\left(\frac{M}{L}dz\right)}{R^3}\vec{x}$$

$$d\vec{F}_y = -\frac{Gm\left(\frac{M}{L}dz\right)}{R^3}\vec{y}$$

where $R = \sqrt{r^2+z^2}$

& $r = \sqrt{x^2+y^2}$

because R is distance of element dz from point mass & r is distance of point mass from mid-point of rod.

$$\Rightarrow d\vec{F}_x + d\vec{F}_y = -\frac{GMm}{R^3L}(\vec{x}+\vec{y})\,dz$$

$$\Rightarrow \vec{F}_x + \vec{F}_y = -\int_{-L/2}^{L/2}\frac{GMm}{R^3L}(\vec{x}+\vec{y})\,dz$$

~~$\Rightarrow |d\vec{F}_x + d\vec{F}_y| = \frac{GMm}{}$~~

$$\Rightarrow \vec{F}_x + \vec{F}_y = -\frac{GMm}{L}(\vec{x}+\vec{y})\int_{-L/2}^{L/2}\frac{dz}{R^3}$$

; since $F_z = 0$, $\vec{F}_x + \vec{F}_y = \vec{F}$.

Also, $R = \sqrt{r^2+z^2}$

$$\Rightarrow \vec{F} = -\frac{GMm}{L}(\vec{x}+\vec{y})\int_{-L/2}^{L/2}\frac{dz}{(r^2+z^2)^{3/2}} = -\frac{GMm}{L}\sqrt{x^2+y^2}\,\hat{c}\int_{-L/2}^{L/2}\frac{dz}{(x^2+y^2+z^2)^{3/2}}$$

$\{r = \sqrt{x^2+y^2}\}$

where $\hat{c}$ is unit vector pointing ~~perpendicularly~~ outwards from mid-point of rod to point mass m.

$\cancel{\vec{F} = F\hat{c}}$ ~~where~~ $\left(\hat{c} = \frac{x}{\sqrt{x^2+y^2}}\hat{x} + \frac{y}{\sqrt{x^2+y^2}}\hat{y}\right)$ — ①

②

$\Rightarrow \vec{F} = -F\hat{c}$ where F is given by ②.

$$\boxed{F = \frac{GMm}{L}\sqrt{x^2+y^2}\int_{-L/2}^{L/2}\frac{dz}{(x^2+y^2+z^2)^{3/2}}}$$

$\Rightarrow$ $\vec{F}$ is pointing towards centre of rod and its magnitude is given by F in ②.

$$\int_{-L/2}^{L/2}\frac{dz}{(x^2+y^2+z^2)^{3/2}} = \left[\frac{z}{(x^2+y^2)(x^2+y^2+z^2)^{1/2}}\right]_{-L/2}^{L/2} = \frac{1}{x^2+y^2}\left[\frac{z}{(x^2+y^2+z^2)^{1/2}}\right]_{-L/2}^{L/2}$$

$$= \frac{1}{x^2+y^2}\left(\frac{\frac{L}{2}}{\left(x^2+y^2+\left(\frac{L}{2}\right)^2\right)^{1/2}} - \frac{-\frac{L}{2}}{\left(x^2+y^2+\left(\frac{L}{2}\right)^2\right)^{1/2}}\right)$$

$$\Rightarrow \boxed{\frac{L}{(x^2+y^2)\sqrt{x^2+y^2+\left(\frac{L}{2}\right)^2}} = \int_{-L/2}^{L/2}\frac{dz}{(x^2+y^2+z^2)^{3/2}}} \quad — ③$$

Using ② & ③,

$$\boxed{F = \frac{GMm}{\sqrt{(x^2+y^2)\left(x^2+y^2+\frac{L^2}{4}\right)}}} \quad — ④$$

$\vec{F_x} = -F(\hat{c}\cdot\hat{x})\hat{x}$; $\vec{F_y} = -F(\hat{c}\cdot\hat{y})\hat{y}$

Using ①, $\boxed{\vec{F_x} = -F\frac{x}{\sqrt{x^2+y^2}}\hat{x} \;;\; \vec{F_y} = -F\frac{y}{\sqrt{x^2+y^2}}\hat{y}}$

$\boxed{\vec{F_x} = m\ddot{x}\hat{x} \;;\; \vec{F_y} = m\ddot{y}\hat{y}}$ $\Rightarrow m\ddot{x} = \frac{-Fx}{\sqrt{x^2+y^2}}$; $m\ddot{y} = \frac{-Fy}{\sqrt{x^2+y^2}}$

$$\Rightarrow \boxed{\ddot{x} = -\left(\frac{F}{m\sqrt{x^2+y^2}}\right)x \;;\; \ddot{y} = -\left(\frac{F}{m\sqrt{x^2+y^2}}\right)y} \quad — ⑤$$

Using ④, $\frac{F}{m\sqrt{x^2+y^2}} = \frac{GM}{(x^2+y^2)\sqrt{x^2+y^2+\frac{L^2}{4}}}$

Using (5),

$$\ddot{x} = -\frac{GM}{x^2+y^2}\,\frac{x}{\sqrt{x^2+y^2+\frac{L^2}{4}}}$$

$$\ddot{y} = -\frac{GM}{x^2+y^2}\,\frac{y}{\sqrt{x^2+y^2+\frac{L^2}{4}}}$$

$r=\sqrt{x^2+y^2} \Rightarrow r^2=x^2+y^2$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = -\frac{GM}{r^2}\,\frac{x}{\sqrt{r^2+\frac{L^2}{4}}}\,;\quad \frac{d^2y}{dt^2} = -\frac{GM}{r^2}\,\frac{y}{\sqrt{r^2+\frac{L^2}{4}}} \quad \text{where } r=\sqrt{x^2+y^2}}$$