3)

$$\frac{df}{dx} = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$$

Using Taylor's theorem,

$$f(x+\delta) = f(x) + \delta \frac{d}{dx} f(x) + \frac{\delta^2}{2} \frac{d^2}{dx^2} f(x+\zeta)$$

where ζ belongs to $(0, \delta)$

$$\implies f(x+\delta) - f(x) - \frac{\delta^2}{2} \frac{d^2}{dx^2} f(\zeta) = \delta \frac{d}{dx} f(x)$$

$$\implies \boxed{\frac{df}{dx} = \frac{f(x+\delta) - f(x)}{\delta} - \frac{\delta}{2} \frac{d^2}{dx^2} f(\zeta)}$$

where ζ belongs to $(0, \delta)$

Let

$$g(\delta) = \frac{f(x+\delta) - f(x)}{\delta}$$

$$E(\delta) = \frac{\delta}{2} \mid \frac{d^2}{dx^2} f(\zeta)|_{max}$$

where ζ belongs to $(0, \delta)$

If we approximate $\frac{df}{dx}$ by $g(\delta)$, error is less than or equal to $E(\delta)$. It can be easily seen that by reducing δ, we reduce $E(\delta)$ because $\frac{\delta}{2}$ decreases and $\frac{d^2}{dx^2} f(\zeta)|_{max}$ decreases. So, upper bound on error decreases with reducing δ. So, accuracy, in general, increases with reducing δ.

Due to finite numerical precision, representation of f is precise only upto finite numer of bits. For large $f(x+\delta)$, this error is very small in comparision to $f(x+\delta)$. So, it can be ignored. But for very small $f(x+\delta)$, this error is comparable to $f(x+\delta)$. So, this numerical precision error bounds the minimum error possible.

$f(x) = x(x-1)$

At $x = 1, f(x) = 0$. So, error in f(x) at x=1 is 0 since 0 can be precisely stored.

Let c be the number of precision bits. Then precision error in x is about $dx = 2^{-c-1}|x|$.

Consider product pq. Then maximum precision error in pq is ($|p\ dq| + |q\ dp|$) where dq and dp are precision errors.

So, error in $x(x-1)$ is $|x(dx)| + |(x-1)dx| = dx(|x| + |x-1|) = 2^{-c-1}|x|(|x| + |x-1|)$

For $x = 1$, precision error in x is 2^{-c}. So, error in $x(x-1)$ is 2^{-c}

Then precision error in δ is about $2^{-c-1}\delta$.

Consider a fraction $\frac{p}{q}$.

$$d(\frac{p}{q}) = \frac{dp - \frac{p}{q} dq}{q}$$

This equation represents the sensitivity of $\frac{p}{q}$ small errors in numerator and denominator.

So, maximum precision error in $\frac{p}{q}$ is $\frac{|dp| + \frac{p}{q}|dq|}{q}$.

So, maximum precision error in $\frac{f(x+\delta)}{\delta}$ is $\frac{2^{-c} + \$\frac{f(x+\delta)}{\delta}\$2^{-c}\delta}{\delta}$.

Let ϵ be upper bound on error.

$$\epsilon = 2^{-c-1}\frac{1+\delta}{\delta} + \frac{\delta}{2} \mid \frac{d^2}{dx^2} f(\zeta)|_{max}$$

where ζ belongs to $(0, \delta)$

$f(x) = x(x-1)$

$$\implies \frac{df}{dx} = 2x - 1$$

$$\implies \frac{d^2 f}{dx^2} = 2$$

$$\implies \epsilon = 2^{-c-1}\frac{1+\delta}{\delta} + \delta$$

$$\implies \epsilon = 2^{-c-1} + 2^{-c-1}\frac{1}{\delta} + \delta$$

ϵ is minimised when

$$\frac{2^{-c-1}}{\delta} = \delta$$

by AM-GM inequality. Also, qualitatively, one should decrease δ as long as numerical precision error is lower than taylor-series truncation error.

ϵ is minimised when

$$\delta = 2^{-(c+1)/2}$$

Putting $c = 52$, $\delta = 2^{-26.5} \approx 10^{-8}$

So, $\delta = 10^{-8}$ gives most accurate numerical derivative in the performed calculations.