I b) $f(x) = x^{a-1} e^{-x}$ $\Rightarrow f'(x) = (a-1)x^{a-2}e^{-x} + x^{a-1}(-e^{-x})$

$\Rightarrow f'(x) = x^{a-2}e^{-x}(a-1+(-x))$

$\Rightarrow \boxed{f'(x) = -x^{a-2}e^{-x}(x-(a-1))}$

$\Rightarrow$ $f'(x) = 0$ at $x = a-1$ since $x-(a-1) = 0$

For ~~x~~ $0 < x < a-1$, $x-(a-1) < 0$

$\Rightarrow$ $f'(x) > 0$ for $0 < x < a-1$

similarly, $x-(a-1) > 0$ for $x > a-1$

and hence $f'(x) < 0$ for $x > a-1$

~~$\Rightarrow f(x)$~~ $\Rightarrow f(x)$ increases $f(0) = e^{-0}(0)^{a-1} = 0 \Rightarrow f(0) = 0$

from $f(0) = 0$ at $x = 0$ till $x = a-1$ to $f(a-1)$

and then decreases after $x = a-1$

$\Rightarrow$ $\boxed{f(x) \text{ has maximum at } x = a-1}$

c) $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\,dx$; Put $z = \frac{x}{c+x}$ $\Rightarrow \boxed{x = \frac{cz}{1-z}}$

$= \int_0^1 \left(\frac{cz}{1-z}\right)^{a-1} e^{\frac{cz}{z-1}} \frac{c}{(1-z)^2}\,dz$

$\Rightarrow dx = \frac{c(1-z)+cz}{(1-z)^2}\,dz$

$\Rightarrow \boxed{dx = \frac{c}{(1-z)^2}\,dz}$

$= \int_0^1 \frac{c^a z^{a-1}}{(1-z)^{a+1}} e^{\frac{cz}{z-1}}\,dz$

$= c^a \int_0^1 \frac{z^{a-1}}{(1-z)^{a+1}} e^{\frac{cz}{z-1}}\,dz$

Let $g(z) = \frac{z^{a-1}}{(1-z)^{a+1}} e^{\frac{cz}{z-1}}$

$\Rightarrow g'(z) = z^{a-1}e^{\frac{cz}{z-1}}(-(a+1))(1-z)^{-a-2} + (a-1)\frac{z^{a-2}}{(1-z)^{a+1}}e^{\frac{cz}{z-1}}$

$+ \frac{z^{a-1}}{(1-z)^{a+1}} e^{\frac{cz}{z-1}} \frac{-c}{(1-z)^2}$

$= \frac{z^{a-2}}{(1-z)^{a+2}} e^{\frac{cz}{z-1}} \left(z(a+1) + (a-1)(1-z) - \frac{cz}{1-z}\right)$

$\boxed{g'(z) = \frac{z^{a-2}}{(1-z)^{a+3}} e^{\frac{cz}{z-1}} \left(z(a+1)(1-z) + (a-1)(1-z)^2 - cz\right)}$

$$z(a+1)(1-z) + (a-1)(1-z)^2 - cz = (1-z)\left(z(a+1) + (a-1)(1-z)\right) - cz$$

$$= (1-z)(2z + a - 1) - cz$$

$$\Rightarrow \boxed{g'(z) = \frac{z^{a-2}}{(1-z)^{a+3}} e^{\frac{cz}{z-1}} \left[(1-z)(2z+a-1) - cz\right]}$$

~~$\Rightarrow g'(z) = 0$ if $(1-c$~~ ~~$g(z)$~~

~~For maximum~~

$g'(z) = 0$ at root of $(1-z)(2z+a-1) - cz = 0$ which lies between 0 and 1

$$2z^2 + (a+c-3)z + 1 - a = 0$$

$$\Rightarrow \boxed{z_0 = \frac{\sqrt{(a+c-3)^2 + 8(a-1)} - (a+c-3)}{4}}$$

We want $z_0 = \frac{1}{2}$ so that $g'(z_0) = g'(\frac{1}{2}) = 0$ and $g(z)$ is maximum at $z = \frac{1}{2}$

$$\Rightarrow \frac{1}{2} = \frac{\sqrt{(a+c-3)^2 + 8(a-1)} - (a+c-3)}{4}$$

$$\Rightarrow (2 + a + c - 3)^2 = (a+c-3)^2 + 8(a-1)$$

$$\Rightarrow (a+c-3)^2 + 4(a+c-3) + 4 = (a+c-3)^2 + 8(a-1)$$

~~$\Rightarrow 4(a+c) = 8a - 8 \Rightarrow 4(a-c) =$~~

$$\Rightarrow 4(a+c) - 8 = 8a - 8 \quad \Rightarrow 4(a+c) = 8a$$

$$\Rightarrow a + c = 2a$$

$$\Rightarrow \boxed{c = a}$$

$c = a$ and at $x = c$, $z = \frac{1}{2}$

$g(z)$ increases below $z = \frac{1}{2}$ and decreases above $z = \frac{1}{2}$

I (d) $x^{a-1} = e^{(a-1)\ln x}$

$$\Rightarrow x^{a-1} e^{-x} = e^{(a-1)\ln x} e^{-x}$$

$$\Rightarrow \boxed{x^{a-1} e^{-x} = e^{(a-1)\ln x - x}}$$

At large x, e^{-x} becomes very small & x^{a-1} becomes very large. These can underflow & overflow respectively. Hence, even though $x^{a-1}e^{-x}$ product is within numerical range, we can't calculate it.

But $\ln x$ & hence $(a-1)\ln x$, along with x can be precisely calculated at higher values of x & since we substract them before taking exponential, if $x^{a-1}e^{-x}$ is within numerical range $e^{(a-1)\ln x - x}$ is also withing numlerical range.

At very large x, however, $x^{a-1}e^{-x}$ becomes very small & hence not possible to calculate with computer.