Hyper-parameters Estimation in Bayesian Models

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Joint work with Feng Yu (UT El Paso) and Lixin Shen (Syracuse University)

Linear Inverse Problems

- Consider a linear inverse problem: $\boldsymbol{y} = \mathsf{F}\boldsymbol{x} + \boldsymbol{\epsilon}$ (observations: $\boldsymbol{y}$; unknown parameters: $\boldsymbol{x}$)
  - It is usually ill-posed.
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- [ ] Regularization approach: impose a regularization/penalty term $R(\boldsymbol{x})$

$$\arg\min_{\boldsymbol{x}} \|\boldsymbol{y} - \mathsf{F}\boldsymbol{x}\|^2 + \lambda R(\boldsymbol{x})$$

  - Ridge regression: $R(\boldsymbol{x}) = \|\boldsymbol{x}\|^2$
  - Lasso: $R(\boldsymbol{x}) = \|\boldsymbol{x}\|_1$
  - Group Lasso: $R(\boldsymbol{x}) = \sum_{g=1}^{G} \|\boldsymbol{x}_g\|_2$

Bayesian Models

- Bayesian approach: introduce a prior distribution on $\boldsymbol{x}$
  - likelihood: $p(\boldsymbol{y}|\boldsymbol{x}) = \mathcal{N}(\boldsymbol{y}|\mathsf{F}\boldsymbol{x}, \sigma^2\mathrm{I})$
  - prior: $p(\boldsymbol{x})$
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- [ ] Challenges
  - The computation of $p(\boldsymbol{y}) = \int p(\boldsymbol{y}|\boldsymbol{x})p(\boldsymbol{x})d\boldsymbol{x}$ is usually intractable.
  - The choice of the prior distribution is crucial.

Hierarchical Bayesian Learning

- Consider a hierarchical prior (scale mixtures of Gaussians)
  - $p(\boldsymbol{x}|\boldsymbol{\gamma}) = \mathcal{N}(\boldsymbol{x}|\mathbf{0}, \mathrm{diag}(\gamma_1, \ldots, \gamma_n))$ hyper-parameters
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☐ Benefits

- ○ Scale mixture of Gaussians contains all symmetric stable distributions, Laplace distributions, logistic distributions, and exponential power distributions.
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  - ⊘ Individualized hyper-parameters encode sparsity.
- ☐ Challenge: how to estimate the hyper-parameters $\boldsymbol{\gamma}$?
  - ○ The dimension of hyper-parameters $\boldsymbol{\gamma}$ scales with $\boldsymbol{x}$.
  - ○ It is not easy to choose the hyper-prior.

Hyper-parameters Estimation

- Learning from data (Evidence approach)

$$\arg\max_{\boldsymbol{\gamma}} p(\boldsymbol{y}|\boldsymbol{\gamma}) = \int p(\boldsymbol{y}|\boldsymbol{x})p(\boldsymbol{x}|\boldsymbol{\gamma})d\boldsymbol{x}$$

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  - Existing methods: EM [Dempster et.al. 1977], MacKay [MacKay, 1992], Convex Bounding [Wipf and Nagarajan, 2009], etc.

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- [ ] Goal:develop more efficient algorithms

Alternating Minimization

- The objective function:

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☐ The challenge comes from the log-determinant term.

- ○ Find a separable surrogate for it.
- ○ Use first-order Taylor expansion.

Alternating Minimization with Linearization

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  - We could update each γ_i separately.
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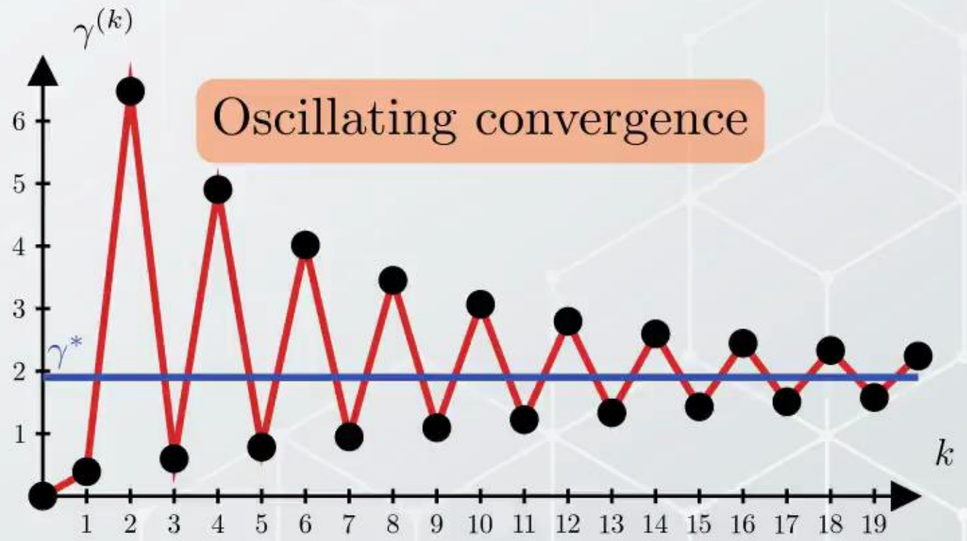

Alternating Minimization with Quadratic Surrogates

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Theorem [Feng, Shen, and S., 2024]

If τ_k is properly chosen, the proposed algorithm converges to a stationary point $\boldsymbol{\gamma}^*$ of the objective function $L(\boldsymbol{\gamma})$. That is, for each i, either $\gamma_i^*=0$ or $\frac{\partial L(\boldsymbol{\gamma}^*)}{\partial\gamma_i}=0$.

Numerical Results: Denoising

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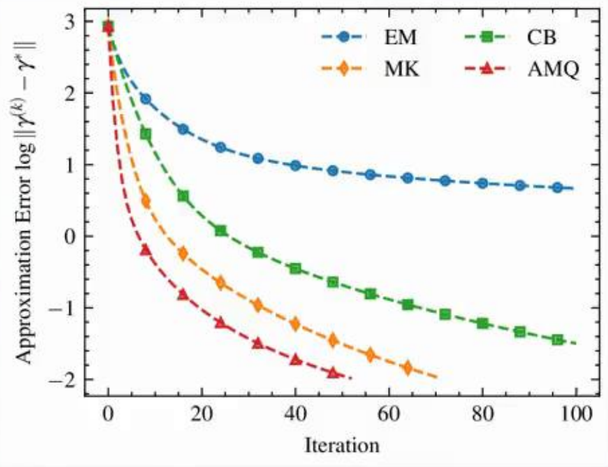
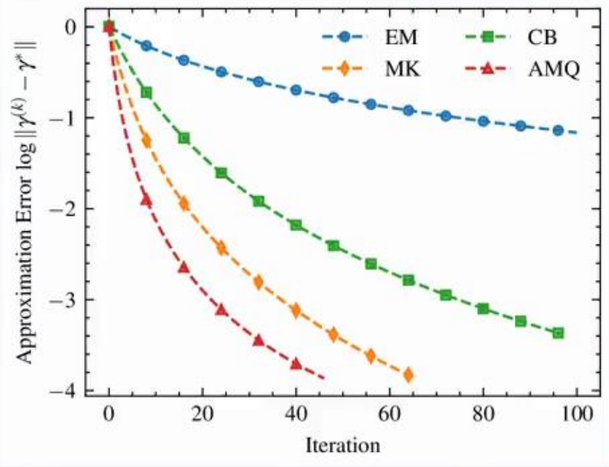
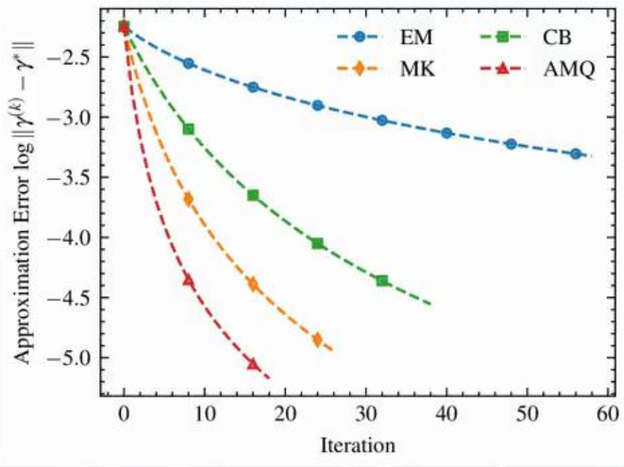
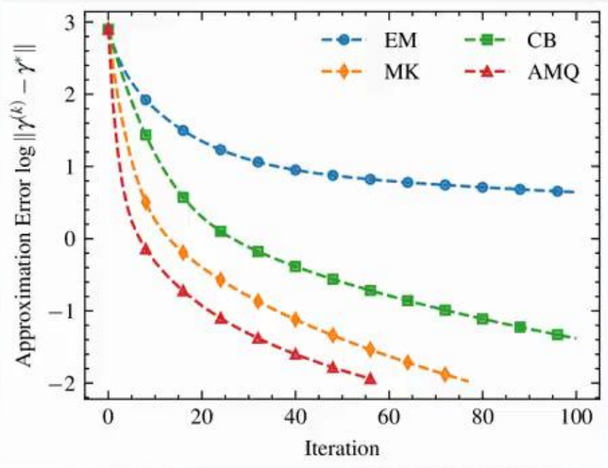
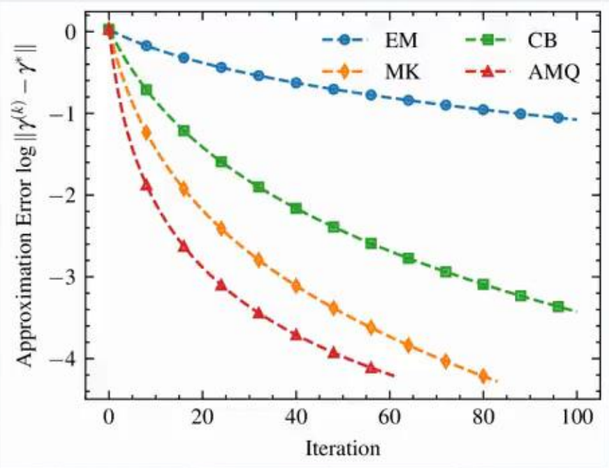
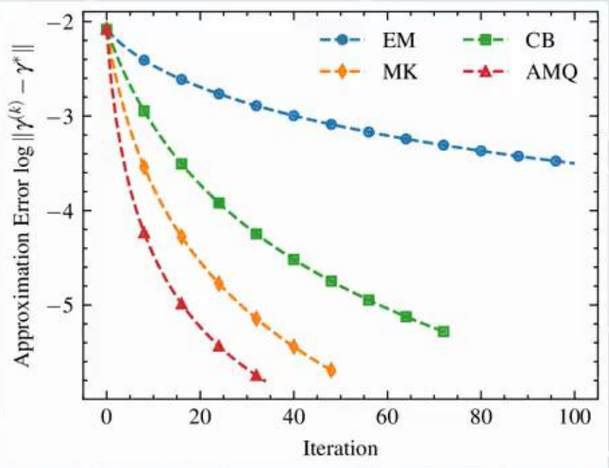
	$\sigma^2 = 10$	$\sigma^2 = 1$	$\sigma^2 = 0.1$
$s = 20\%$			
$s = 90\%$			

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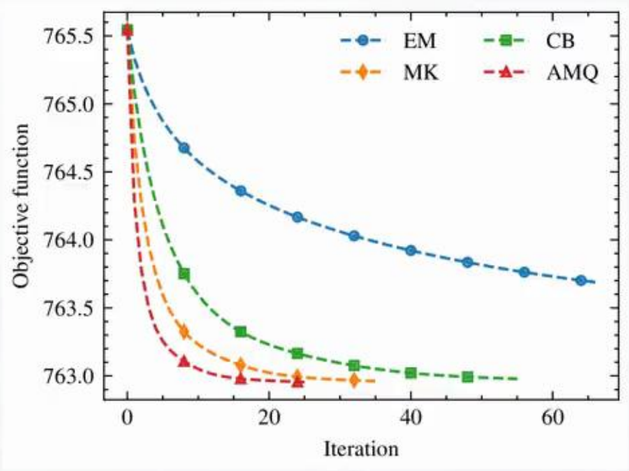
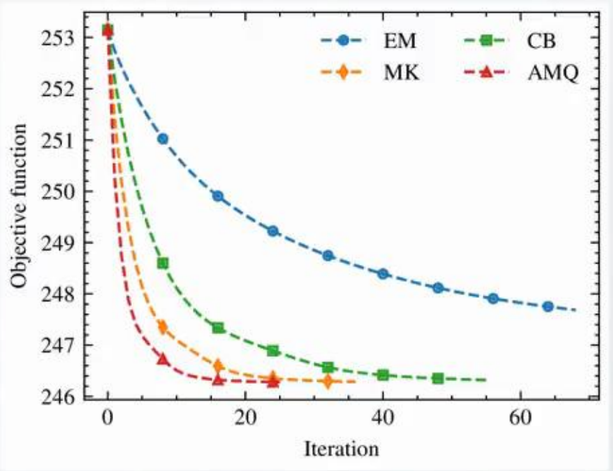
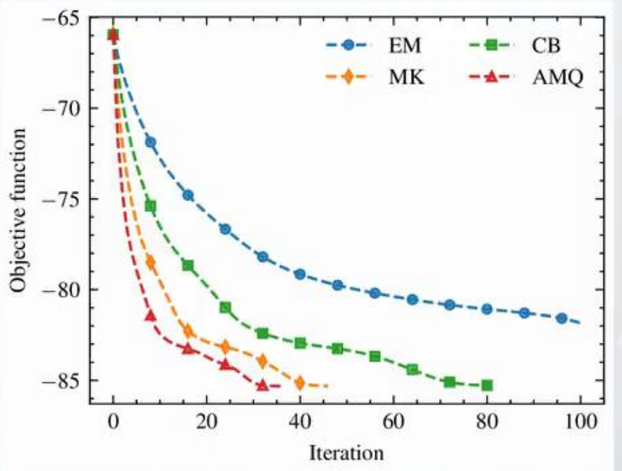
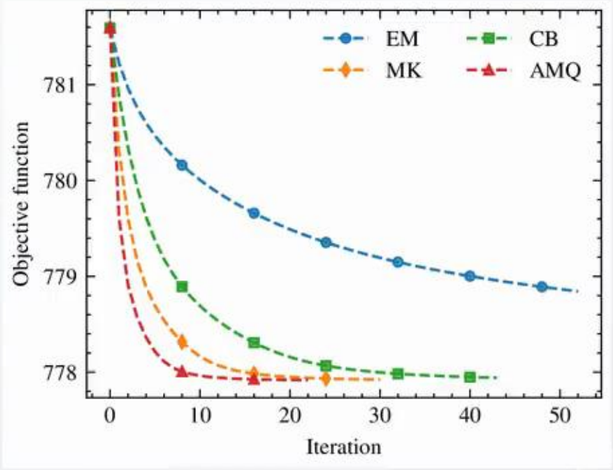
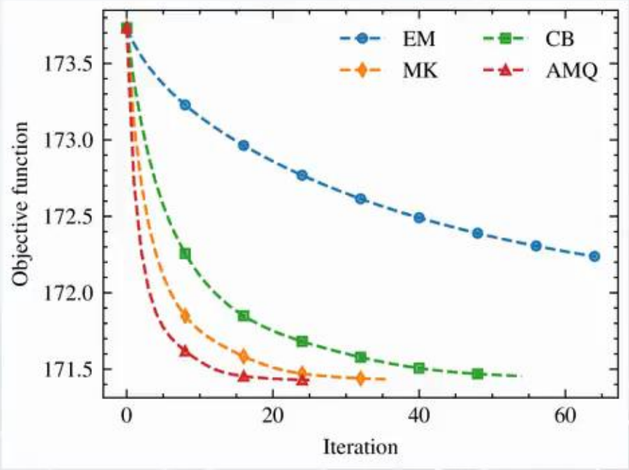
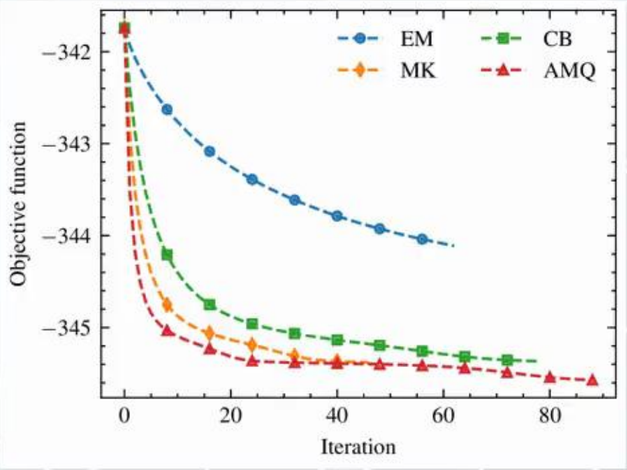
$s = 20\%$			
$s = 90\%$			

Numerical Results: SAR and EEG

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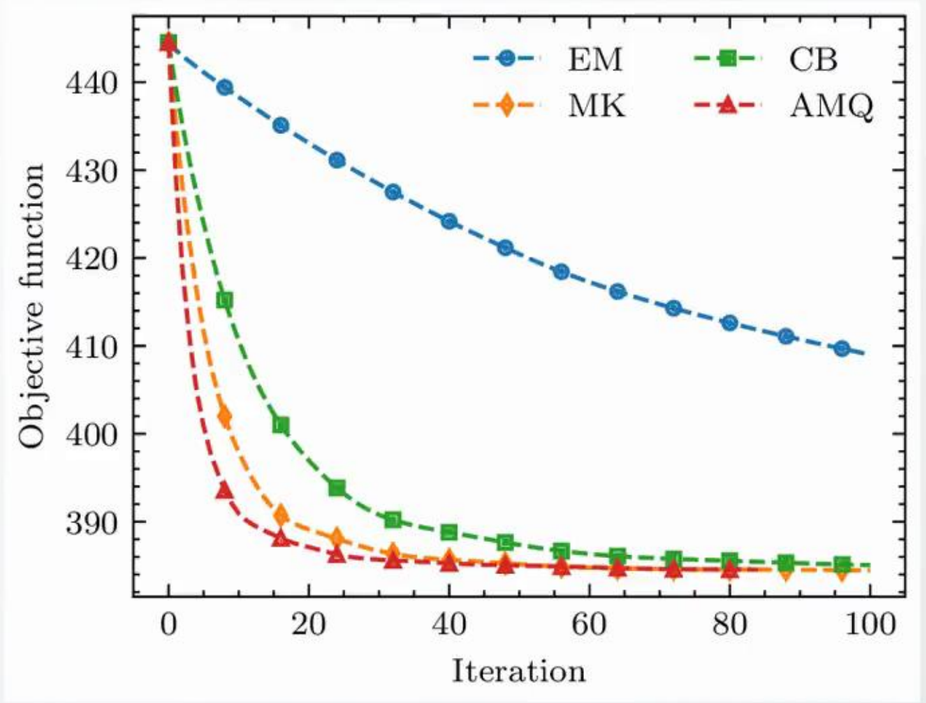
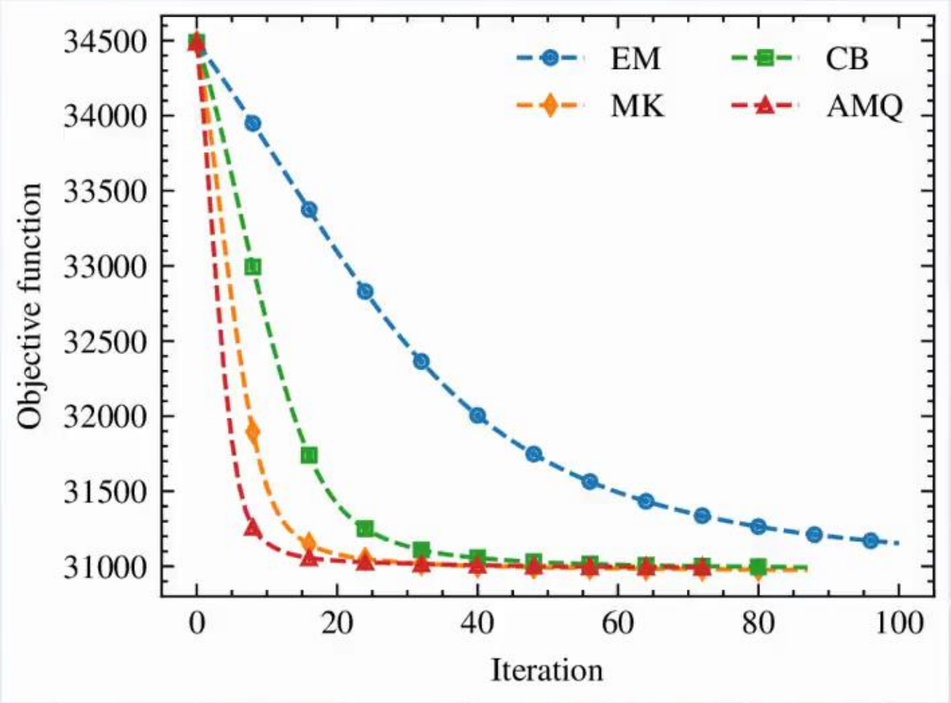
EEG



SAR



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  - We propose a new algorithm for hyper-parameters estimation in hierarchical (linear) Bayesian models.

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☐ Issues

- Non-Gaussian distributions
- Non-linear models
- Challenge: more efficient sampling (Monte Carlo) algorithms

Thank you for your attention!

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