The error-state Kalman filter

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This document hosted on Github accompanies the paper *Robust phased-array radio system aided inertial navigation using factor graph optimisation* and outlines in more detail the implementation of the error-state Kalman filter (ESKF) used as a benchmark estimator.

A. Error-state model

The baseline used for comparison is a standard ESKF with error-state

$$\delta\boldsymbol{x} \triangleq [\delta\boldsymbol{p}\ \delta\boldsymbol{v}\ \delta\boldsymbol{\theta}\ \delta\boldsymbol{b}_{\text{acc}}\ \delta\boldsymbol{b}_{\text{gyro}}]^\top \in \mathbb{R}^{15} \tag{1}$$

with linearised kinematics obtained from [1]

$$\delta\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})\delta\boldsymbol{x} + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{w} \tag{2}$$

where

$$\boldsymbol{F} = \begin{bmatrix} \mathbf{0}_3 & \boldsymbol{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & -\hat{\boldsymbol{R}}[\hat{\boldsymbol{f}}^b_{nb}]_\times & -\hat{\boldsymbol{R}} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & -[\hat{\boldsymbol{\omega}}^b_{nb}]_\times & \mathbf{0}_3 & -\boldsymbol{I}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -\frac{1}{\tau_{\text{acc}}}\boldsymbol{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -\frac{1}{\tau_{\text{gyro}}}\boldsymbol{I}_3 \end{bmatrix} \tag{3}$$

and

$$\boldsymbol{G} = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\hat{\boldsymbol{R}} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & -\boldsymbol{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \boldsymbol{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \boldsymbol{I}_3 \end{bmatrix} \tag{4}$$

are functions of the nominal state $\boldsymbol{x}$ and debiased IMU measurements, $\hat{\boldsymbol{f}}^b_{nb} \triangleq \boldsymbol{f}^b_{imu} - \hat{\boldsymbol{b}}^b_{\text{acc}}$ and $\hat{\boldsymbol{\omega}}^b_{nb} \triangleq \boldsymbol{\omega}^b_{imu} - \hat{\boldsymbol{b}}^b_{\text{gyro}}$. Furthermore, $\hat{\boldsymbol{R}} \triangleq \boldsymbol{R}^n_b(\hat{\boldsymbol{q}}^n_b)$ where the sub- and superscript of $\hat{\boldsymbol{R}}$ are omitted for notational simplicity. Assuming zero-order hold, the discrete-time transition matrix $\boldsymbol{F}_d$ and process noise covariance matrix $\boldsymbol{\mathcal{Q}}_d$ can then be obtained from Van Loan's formula [2] with (3) and (4) as input.

B. Filter prediction

The ESKF is driven by an INS with frequency $f_{\text{ins}} = \frac{1}{\tau_{\text{ins}}}$. For each iteration k, the INS will update its state $\boldsymbol{x}_{\text{ins}}$ using the discretised equations of motion given in (5)

$$\begin{aligned} \boldsymbol{q}^n_b[k+1] &= \boldsymbol{q}^n_b[k] \otimes \delta\boldsymbol{q}[k] \\ \boldsymbol{v}^n_{nb}[k+1] &= \boldsymbol{v}^n_{nb}[k] + \hat{\boldsymbol{R}}[k]\hat{\boldsymbol{f}}^b_{nb}[k] + \boldsymbol{g}^n \\ \boldsymbol{p}^n_{nb}[k+1] &= \boldsymbol{p}^n_{nb}[k] + \frac{2}{\tau_{\text{ins}}}(\boldsymbol{v}^n_{nb}[k] + \boldsymbol{v}^n_{nb}[k+1]) \\ \boldsymbol{b}_{\text{acc}}[k+1] &= \exp\left(\frac{\tau_{\text{ins}}}{\tau_{\text{acc}}}\right)\boldsymbol{b}_{\text{acc}}[k] \\ \boldsymbol{b}_{\text{gyro}}[k+1] &= \exp\left(\frac{\tau_{\text{ins}}}{\tau_{\text{gyro}}}\right)\boldsymbol{b}_{\text{gyro}}[k] \end{aligned} \tag{5}$$

with attitude increment defined as in Appx. E in [3].

$$\delta\boldsymbol{q}[k] \triangleq \begin{bmatrix} \cos\left(\frac{\|\boldsymbol{\alpha}[k]\|_2}{2}\right) \\ \sin\left(\frac{\|\boldsymbol{\alpha}[k]\|_2}{2}\right)\frac{\boldsymbol{\alpha}[k]}{\|\boldsymbol{\alpha}[k]\|_2} \end{bmatrix} \tag{6}$$

where $\boldsymbol{\alpha}$ is given by

$$\boldsymbol{\alpha}[k] \triangleq \hat{\boldsymbol{\omega}}^b_{nb}[k]\tau_{\text{ins}} \tag{7}$$

which is obtained from Appx. J in [3].

The error-state is not updated in between aiding measurements, but the predicted state covariance of the filter is updated using the discretised transition- and process noise covariance matrices obtained in Section -A using the standard state covariance filter prediction

$$\tilde{\boldsymbol{\mathcal{P}}}[k+1] = \boldsymbol{F}_d[k]\hat{\boldsymbol{\mathcal{P}}}[k]\boldsymbol{F}_d^\top[k] + \boldsymbol{\mathcal{Q}}_d[k] \tag{8}$$

where an initial estimate of the state covariance $\hat{\boldsymbol{\mathcal{P}}}[0]$ must be set during initialisation. $\tilde{\boldsymbol{\mathcal{P}}}$ is used directly as the state covariance estimate $\hat{\boldsymbol{\mathcal{P}}}$ between update steps.

C. Filter update

The system receives state observations $\boldsymbol{z}$ at rate $\tau_{\text{aid}} < \tau_{\text{ins}}$. In this paper, it is assumed that observations received at iteration k are synchronised. I.e., observations from multiple locators can be used directly with the IMU measurements at the same iteration without performing any validity checks.

For an iteration k, the innovation is computed as

$$\boldsymbol{\nu}[k] = \boldsymbol{z}[k] - \boldsymbol{h}(\boldsymbol{x}_{\text{ins}}[k]) \tag{9}$$

where $\boldsymbol{h}$ is the observation model of $\boldsymbol{z}$ with corresponding innovation covariance given by

$$\boldsymbol{\mathcal{S}}[k] = \boldsymbol{H}[k]\tilde{\boldsymbol{\mathcal{P}}}[k]\boldsymbol{H}[k]^\top + \boldsymbol{\mathcal{R}}[k] \tag{10}$$

where

$$\boldsymbol{H} \triangleq \left.\frac{\partial\boldsymbol{h}}{\partial\delta\boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_{\text{ins}}} \tag{11}$$

With this, the estimates of the error-state and state covariance are then computed using

$$\delta\hat{\boldsymbol{x}}[k] = \boldsymbol{W}[k]\boldsymbol{\nu}[k] \tag{12}$$

and

$$\hat{\boldsymbol{\mathcal{P}}}[k] = (\boldsymbol{I} - \boldsymbol{W}[k]\boldsymbol{H}[k])\tilde{\boldsymbol{\mathcal{P}}}[k] \tag{13}$$

where the Kalman gain $\boldsymbol{W}$ is given by

$$\boldsymbol{W}[k] = \tilde{\boldsymbol{\mathcal{P}}}[k]\boldsymbol{H}[k]^\top\boldsymbol{\mathcal{S}}[k]^{-1} \tag{14}$$

The derivations above follow the definitions from [1], with the additional step of defining the innovation $\boldsymbol{\nu}$ and $\boldsymbol{\mathcal{S}}$ as separate variables, which will be used when setting up the outlier rejection scheme and when evaluating filter consistency.

D. Injection of the error-state and filter reset

With the error-state estimate, the next step is to inject the error-state into the nominal state $\boldsymbol{x}_{\text{ins}}$

$$\boldsymbol{x}_{\text{ins}}[k] = \boldsymbol{x}_{\text{ins}}[k] \oplus \delta\hat{\boldsymbol{x}}[k] \tag{15}$$

which involves simple summation of all states except the attitude, where a quaternion product is computed, as noted in [1]. Finally, the ESKF must be reset, which involves setting $\delta\hat{\boldsymbol{x}} = 0$ and updating $\hat{\boldsymbol{\mathcal{P}}}$ to account for change in frame of the orientation error. As Solà explains in [1], this last step may be neglected in many cases, which results in a so-called trivial error reset for the filter, which is utilised in this project.

References

[1] J. Solà, "Quaternion kinematics for the error-state Kalman filter," Nov. 2017, arXiv:1711.02508 [cs]. [Online]. Available: http://arxiv.org/abs/1711.02508

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[3] P. D. Groves, *Principles of GNSS, inertial, and multisensor integrated navigation systems*, 2nd ed., ser. GNSS technology and application series. Boston: Artech House, 2013, oCLC: ocn820530994.