Multinomial and Gaussian Naïve Bayes

Slides from Tom Mitchell

Naïve Bayes in a Nutshell

Bayes rule:

$$P(Y = y_k | X_1 \ldots X_n) = \frac{P(Y = y_k) P(X_1 \ldots X_n | Y = y_k)}{\sum_j P(Y = y_j) P(X_1 \ldots X_n | Y = y_j)}$$

Assuming conditional independence among X_i's:

$$P(Y = y_k | X_1 \ldots X_n) = \frac{P(Y = y_k) \prod_i P(X_i | Y = y_k)}{\sum_j P(Y = y_j) \prod_i P(X_i | Y = y_j)}$$

So, classification rule for X^{new} = < X_1, …, X_n > is:

$$Y^{new} \leftarrow \arg\max_{y_k} \; P(Y = y_k) \prod_i P(X_i^{new} | Y = y_k)$$

Another way to view Naïve Bayes (Boolean Y):

Decision rule: is this quantity greater or less than 1?

$$\frac{P(Y=1|X_1 \ldots X_n)}{P(Y=0|X_1 \ldots X_n)} = \frac{P(Y=1)\prod_i P(X_i|Y=1)}{P(Y=0)\prod_i P(X_i|Y=0)}$$

Naïve Bayes: classifying text documents

- Classify which emails are spam?
- Classify which emails promise an attachment?

I am pleased to announce that Bob Frederking of the Language Technologies Institute is our new Associate Dean for Graduate Programs. In this role, he oversees the many issues that arise with our multiple masters and PhD programs. Bob brings to this position considerable experience with the masters and PhD programs in the LTI.

I would like to thank Frank Pfenning, who has served ably in this role for the past two years.

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Randal E. Bryant
Dean and University Professor

How shall we represent text documents for Naïve Bayes?

Learning to classify documents: P(Y|X)

- Y discrete valued.
  - e.g., Spam or not
- $X = \langle X_1, X_2, \ldots X_n \rangle$ = document

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- X_i is a random variable describing…

Learning to classify document: P(Y|X) the "Bag of Words" model

- Y discrete valued. e.g., Spam or not
- $X = <X_1, X_2, \ldots X_n>$ = document

- X_i are iid random variables. Each represents the word at its position i in the document
- Generating a document according to this distribution = rolling a 50,000 sided die, once for each word position in the document

- The observed counts for each word follow a ??? distribution

Multinomial Distribution

- $P(\theta)$ and $P(\theta|D)$ have the same form

Eg. 2 Dice roll problem (6 outcomes instead of 2)

Likelihood is ~ Multinomial($\theta = \{\theta_1, \theta_2, \dots, \theta_k\}$)

$$P(\mathcal{D} \mid \theta) = \theta_1^{\alpha_1}\theta_2^{\alpha_2}\dots\theta_k^{\alpha_k}$$

If prior is Dirichlet distribution,

$$P(\theta) = \frac{\prod_{i=1}^{k}\theta_i^{\beta_i-1}}{B(\beta_1,\dots,\beta_k)} \sim \text{Dirichlet}(\beta_1,\dots,\beta_k)$$

Then posterior is Dirichlet distribution

$$P(\theta|D) \sim \text{Dirichlet}(\beta_1+\alpha_1,\dots,\beta_k+\alpha_k)$$

For Multinomial, conjugate prior is Dirichlet distribution.

Bag of Words Approach

aardvark	0
about	2
all	2
Africa	1
apple	0
anxious	0
...	
gas	1
...	
oil	1
…	
Zaire	0

MAP estimates for bag of words

Map estimate for multinomial

$$\theta_i = \frac{\alpha_i + \beta_i - 1}{\sum_{m=1}^{k} \alpha_m + \sum_{m=1}^{k} (\beta_m - 1)}$$

$$\theta_{aardvark} = P(X_i = \text{aardvark}) = \frac{\#\text{ observed 'aardvark'} \;+ \#\text{ hallucinated 'aardvark'} - 1}{\#\text{ observed words} \;+ \#\text{ hallucinated words} - k}$$

What β's should we choose?

Naïve Bayes Algorithm – discrete X_i

- Train Naïve Bayes (examples)

  for each value y_k

  estimate $\pi_k \equiv P(Y = y_k)$

  for each value x_{ij} of each attribute X_i

  estimate $\theta_{ijk} \equiv P(X_i = x_{ij} | Y = y_k)$

  prob that word x_{ij} appears in position i, given $Y=y_k$

- Classify (X^{new})

$$Y^{new} \leftarrow \arg\max_{y_k} \; P(Y = y_k) \prod_i P(X_i^{new} | Y = y_k)$$

$$Y^{new} \leftarrow \arg\max_{y_k} \; \pi_k \prod_i \theta_{ijk}$$

* Additional assumption: word probabilities are position independent

$$\theta_{ijk} = \theta_{mjk} \text{ for } i \neq m$$

Twenty NewsGroups

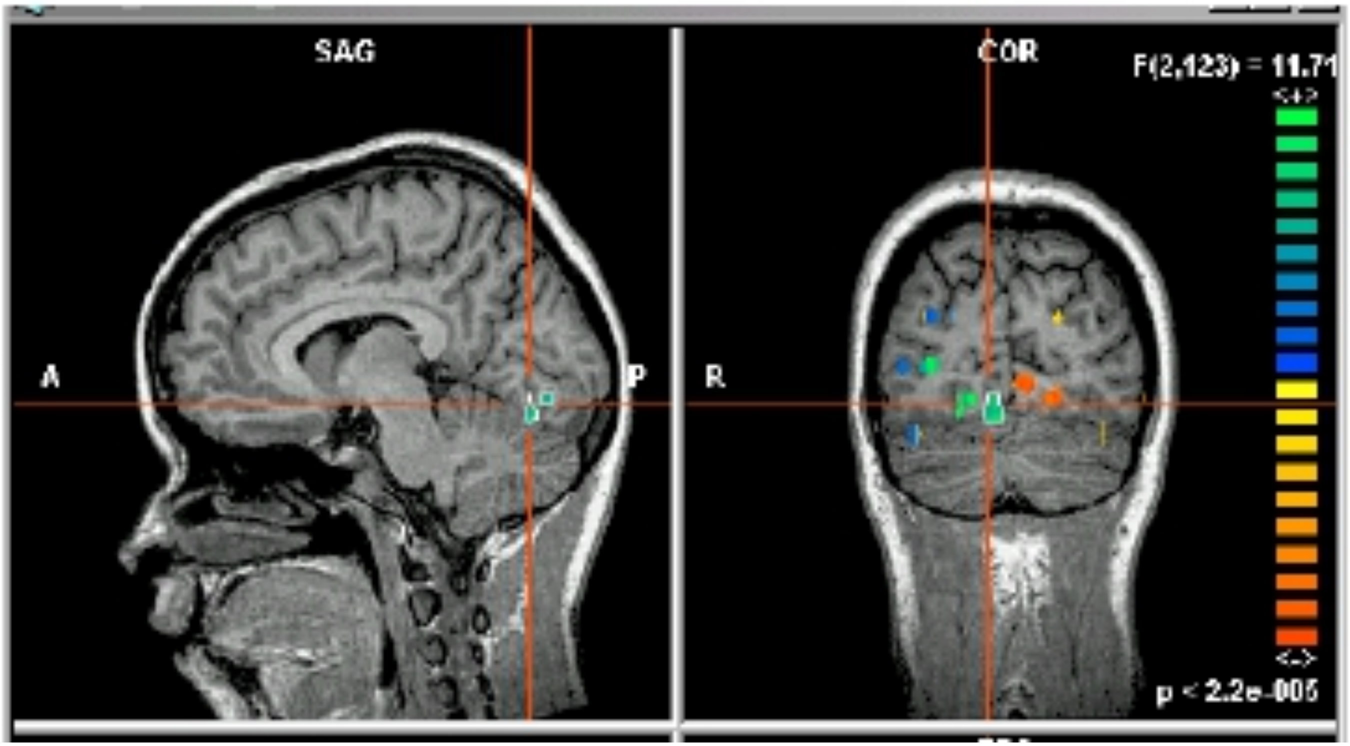
Given 1000 training documents from each group
Learn to classify new documents according to which newsgroup it came from

comp.graphics	misc.forsale
comp.os.ms-windows.misc	rec.autos
comp.sys.ibm.pc.hardware	rec.motorcycles
comp.sys.mac.hardware	rec.sport.baseball
comp.windows.x	rec.sport.hockey
alt.atheism	sci.space
soc.religion.christian	sci.crypt
talk.religion.misc	sci.electronics
talk.politics.mideast	sci.med
talk.politics.misc	
talk.politics.guns	

Naive Bayes: 89% classification accuracy

What if we have continuous X_i?

Eg., image classification: X_i is real-valued i_{th} pixel



What if we have continuous X_i?

Eg., image classification: X_i is real-valued i^{th} pixel

Naïve Bayes requires $P(X_i \mid Y=y_k)$, but X_i is real (continuous)

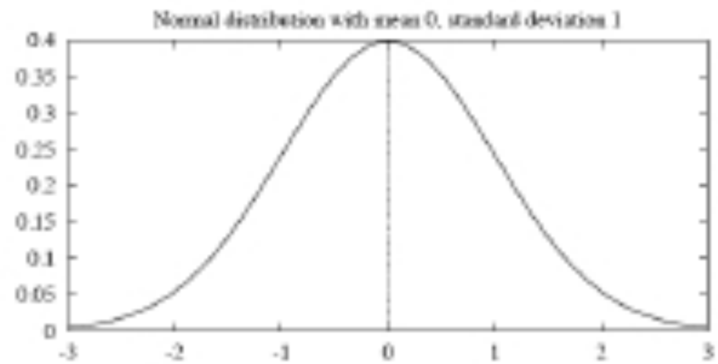
$$P(Y = y_k|X_1 \dots X_n) = \frac{P(Y = y_k) \prod_i P(X_i|Y = y_k)}{\sum_j P(Y = y_j) \prod_i P(X_i|Y = y_j)}$$

Common approach: assume $P(X_i \mid Y=y_k)$ follows a Normal (Gaussian) distribution

Gaussian Distribution

(also called "Normal")

p(x) is a *probability density function*, whose integral (not sum) is 1



$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The probability that X will fall into the interval (a, b) is given by

$$\int_a^b p(x)dx$$

- Expected, or mean value of X, $E[X]$, is

$$E[X] = \mu$$

- Variance of X is

$$Var(X) = \sigma^2$$

- Standard deviation of X, σ_X, is

$$\sigma_X = \sigma$$

What if we have continuous X_i ?

Gaussian Naïve Bayes (GNB): assume

$$p(X_i = x | Y = y_k) = \frac{1}{\sqrt{2\pi\sigma_{ik}^2}} \ e^{-\frac{1}{2}(\frac{x-\mu_{ik}}{\sigma_{ik}})^2}$$

Sometimes assume variance

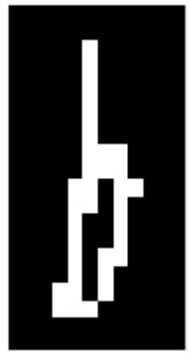
- is independent of Y (i.e., σ_i),
- or independent of X_i (i.e., σ_k)
- or both (i.e., σ)

Bayesian Learning

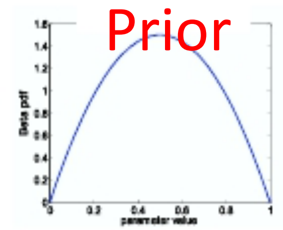
What if Features are Continuous?

Eg., Character Recognition:

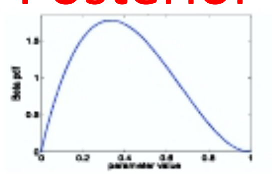
X_i is i_{th} pixel



Prior



Posterior



$$P(Y \mid \mathbf{X}) \propto P(\mathbf{X} \mid Y)\ P(Y)$$

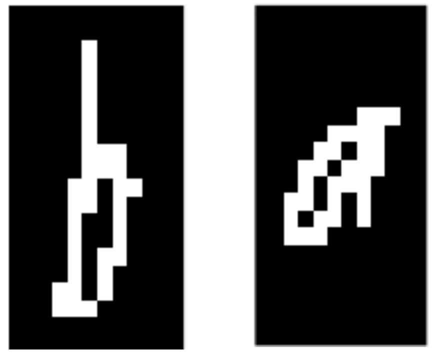
Data Likelihood

Bayesian Learning

What if Features are Continuous?

Eg., Character Recognition:

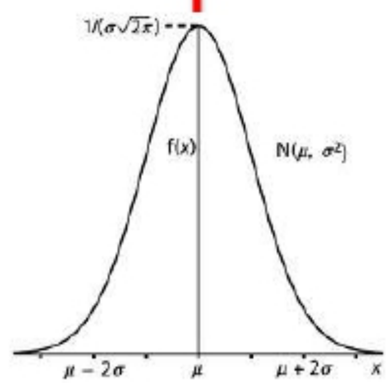
X_i is i_{th} pixel



$$P(Y \mid \mathbf{X}) \propto P(\mathbf{X} \mid Y)\, P(Y)$$

$$P(X_i = x \mid Y = y_k) = N(\mu_{ik}, \sigma_{ik})$$

$$N(\mu_{ik}, \sigma_{ik}) = \frac{1}{\sigma_{ik}\sqrt{2\pi}}\, e^{\frac{-(x-\mu_{ik})^2}{2\sigma_{ik}^2}}$$



Gaussian Naïve Bayes

Sometimes Assume Variance

- is independent of Y (i.e., σ_i),
- or independent of X_i (i.e., σ_k)
- or both (i.e., σ)

$$P(Y \mid \mathbf{X}) \propto P(\mathbf{X} \mid Y)\, P(Y)$$

$$P(X_i = x \mid Y = y_k) = N(\mu_{ik}, \sigma_{ik})$$

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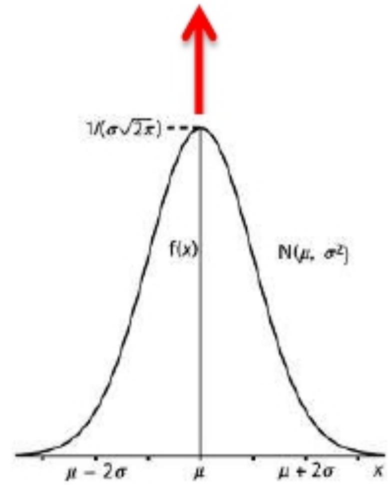


Learning Gaussian Parameters

Maximum Likelihood Estimates:

- Mean:

$$\widehat{\mu}_{MLE} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

- Variance:

$$\widehat{\sigma}^2_{MLE} = \frac{1}{N}\sum_{i=1}^{N} (x_i - \widehat{\mu})^2$$

Learning Gaussian Parameters

Maximum Likelihood Estimates:

- Mean:

$$\hat{\mu}_{ik} = \frac{1}{\sum_j \delta(Y^j = y_k)} \sum_j X_i^j \delta(Y^j = y_k)$$

jth training example

$\delta(x)$=1 if x true, else 0

- Variance:

$$\hat{\sigma}^2_{MLE} = \frac{1}{N} \sum_{i=1}^{N} (x_i - \hat{\mu})^2$$

Learning Gaussian Parameters

Maximum Likelihood Estimates:

- Mean:

$$\widehat{\mu}_{ik} = \frac{1}{\sum_j \delta(Y^j = y_k)} \sum_j X_i^j \delta(Y^j = y_k)$$

- Variance:

$$\widehat{\sigma}_{ik}^2 = \frac{1}{\sum_j \delta(Y^j = y_k) - 1} \sum_j (X_i^j - \widehat{\mu}_{ik})^2 \delta(Y^j = y_k)$$

Gaussian Naïve Bayes Algorithm – continuous X_i (but still discrete Y)

- Train Naïve Bayes (examples)

  for each value y_k

  estimate* $\pi_k \equiv P(Y = y_k)$

  for each attribute X_i estimate $P(X_i|Y = y_k)$

  - class conditional mean μ_{ik}, variance σ_{ik}

- Classify (X^{new})

$$Y^{new} \leftarrow \arg\max_{y_k} \; P(Y = y_k) \prod_i P(X_i^{new}|Y = y_k)$$

$$Y^{new} \leftarrow \arg\max_{y_k} \; \pi_k \prod_i \mathcal{N}(X_i^{new}; \mu_{ik}, \sigma_{ik})$$

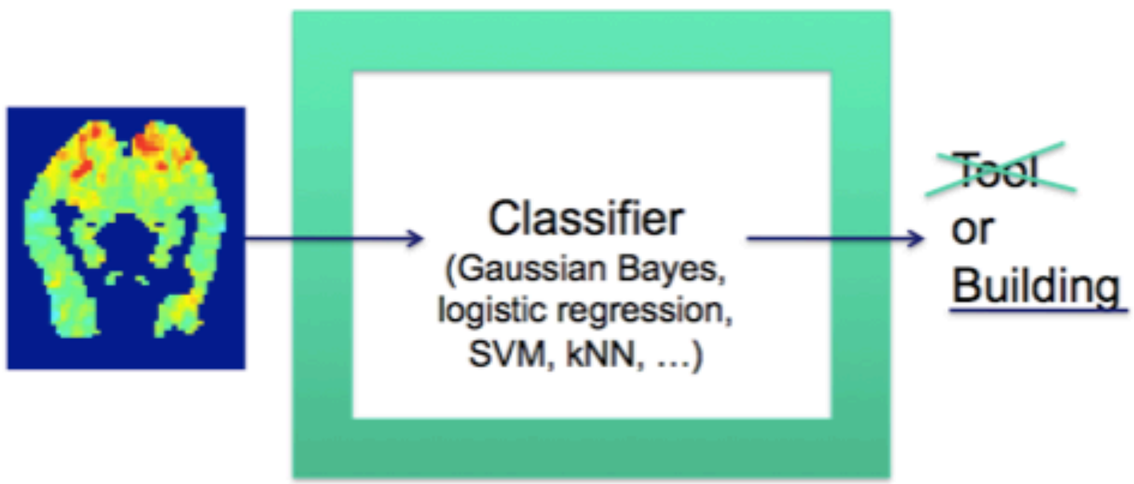
* probabilities must sum to 1, so need estimate only n-1 parameters...

How many parameters must we estimate for Gaussian Naïve Bayes if Y has k possible values, X=<X1, … Xn>?

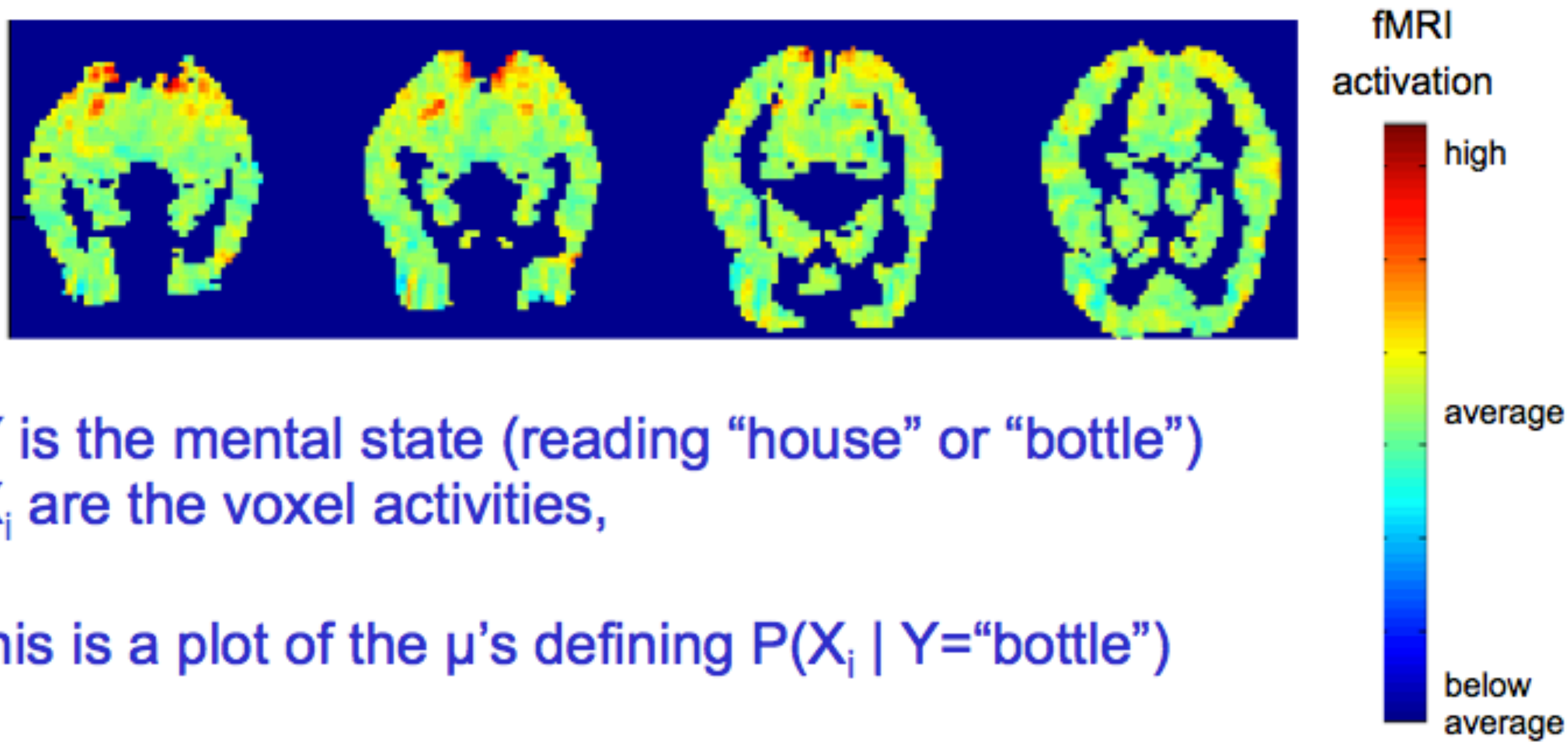
$$p(X_i = x|Y = y_k) = \frac{1}{\sqrt{2\pi\sigma_{ik}^2}} \; e^{-\frac{1}{2}(\frac{x-\mu_{ik}}{\sigma_{ik}})^2}$$

GNB Example: Classify a person's cognitive state, based on brain image

- reading a sentence or viewing a picture?
- reading the word describing a "Tool" or "Building"?
- answering the question, or getting confused?



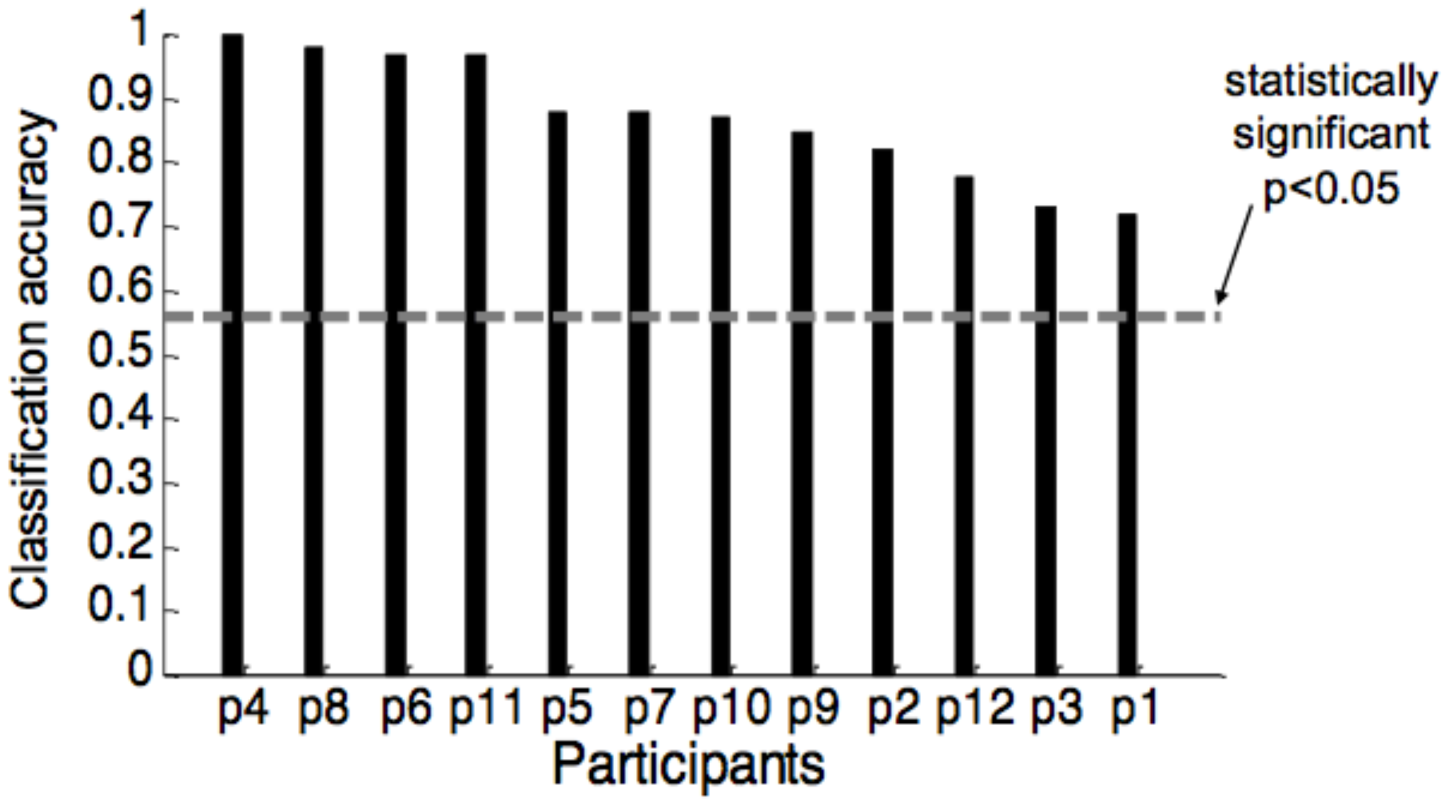
Mean activations over all training examples for Y="bottle"



Y is the mental state (reading "house" or "bottle")
X_i are the voxel activities,

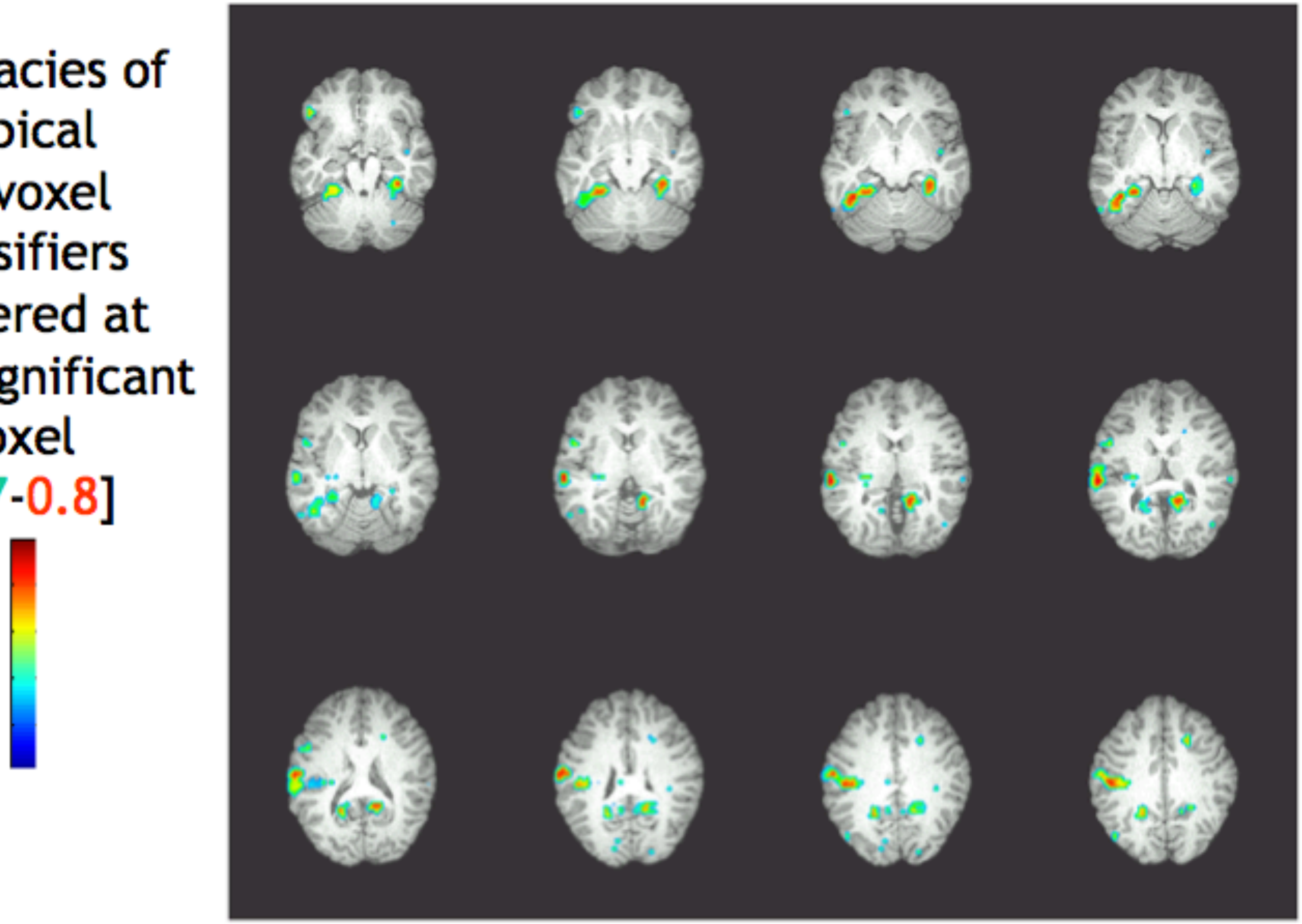
this is a plot of the μ's defining $P(X_i \mid Y=\text{"bottle"})$

Classification task: is person viewing a "tool" or "building"?



Where is information encoded in the brain?

Accuracies of cubical 27-voxel classifiers centered at each significant voxel [0.7-0.8]



Naïve Bayes: What you should know

- Designing classifiers based on Bayes rule
- Conditional independence
  - What it is
  - Why it's important
- Naïve Bayes assumption and its consequences
  - Which (and how many) parameters must be estimated under different generative models (different forms for $P(X|Y)$)
    - and why this matters
- How to train Naïve Bayes classifiers
  - MLE and MAP estimates
  - with discrete and/or continuous inputs X_i