Point Estimation

Some slides courtesy of Vibhav Gogate, Carlos Guestrin, Chris Bishop,
Dan Weld and Luke Zettlemoyer.

Your first consulting job

Billionaire in Dallas asks:

- He says: I have thumbtack, if I flip it, what's the probability it will fall with the nail up?
- You say: Please flip it a few times:

Your first consulting job

Billionaire in Dallas asks:

– He says: I have thumbtack, if I flip it, what's the probability it will fall with the nail up?

– You say: Please flip it a few times:

– You say: The probability is:

  - P(H) = 3/5

– **He says: Why???**

– You say: Because…

Thumbtack – Binomial Distribution

- $P(\text{Heads}) = \theta$, $P(\text{Tails}) = 1-\theta$

x1 x2 x3 x4 … xN

$$\theta \; * \; (1-\theta) \; * \; \theta \; * \; (1-\theta) \; \ldots \; * \; \theta$$

- Flips are *i.i.d.*:
  - Independent events
  - Identically distributed according to Binomial distribution

Thumbtack – Binomial Distribution

- P(Heads) = θ, P(Tails) = 1-θ

x1 x2 x3 x4 xN

$\theta * (1-\theta) * \theta * (1-\theta) \ldots * \theta$

- Sequence *D* of α_H Heads and α_T Tails

$$P(\mathcal{D} \mid \theta) = \theta^{\alpha_H}(1-\theta)^{\alpha_T}$$

Maximum Likelihood Estimation

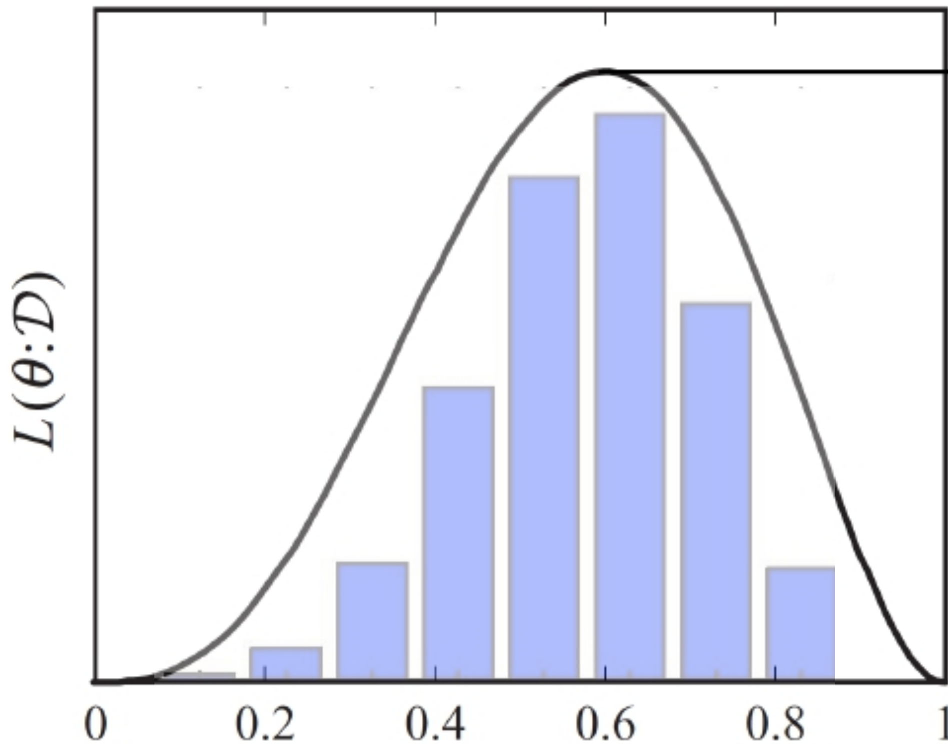
- **Data:** Observed set D of α_H Heads and α_T Tails
- **Hypothesis:** Binomial distribution
- **Learning:** finding θ is an optimization problem
  - What's the objective function?

$$P(\mathcal{D} \mid \theta) = \theta^{\alpha_H}(1-\theta)^{\alpha_T}$$

- **MLE:** Choose θ to maximize probability of *D*

$$\begin{aligned}\widehat{\theta} &= \arg\max_{\theta} \; P(\mathcal{D} \mid \theta) \\ &= \arg\max_{\theta} \; \ln P(\mathcal{D} \mid \theta)\end{aligned}$$

Data



Set the derivative to 0 and solve

Your first parameter learning algorithm

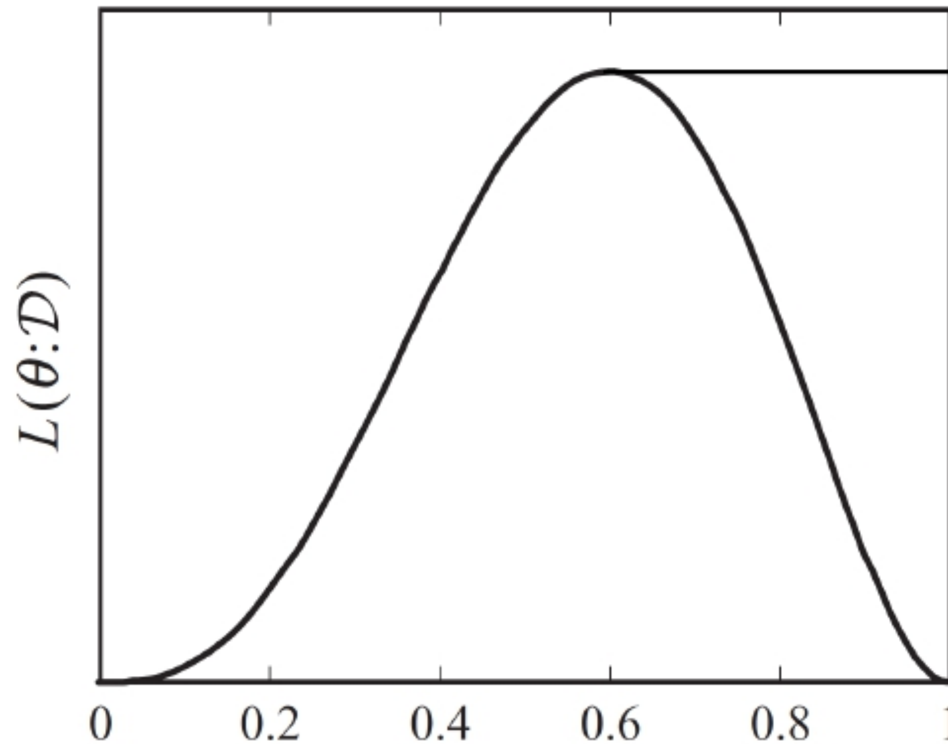
$$\begin{aligned}\widehat{\theta} &= \arg\max_{\theta} \quad \ln P(\mathcal{D} \mid \theta) \\ &= \arg\max_{\theta} \quad \ln \theta^{\alpha_H}(1-\theta)^{\alpha_T}\end{aligned}$$

- Set derivative to zero, and solve!

$$\begin{aligned}\frac{d}{d\theta} \ln P(\mathcal{D} \mid \theta) &= \frac{d}{d\theta}\left[\ln \theta^{\alpha_H}(1-\theta)^{\alpha_T}\right] \\ &= \frac{d}{d\theta}\left[\alpha_H \ln \theta + \alpha_T \ln(1-\theta)\right] \\ &= \alpha_H \frac{d}{d\theta} \ln \theta + \alpha_T \frac{d}{d\theta} \ln(1-\theta) \\ &= \frac{\alpha_H}{\theta} - \frac{\alpha_T}{1-\theta} \quad = 0\end{aligned}$$

$$\boxed{\widehat{\theta}_{MLE} \quad = \quad \frac{\alpha_H}{\alpha_H + \alpha_T}}$$

Data



$$\widehat{\theta}_{MLE} = \frac{\alpha_H}{\alpha_H + \alpha_T}$$

But, how many flips do I need?

$$\widehat{\theta}_{MLE} = \frac{\alpha_H}{\alpha_H + \alpha_T}$$

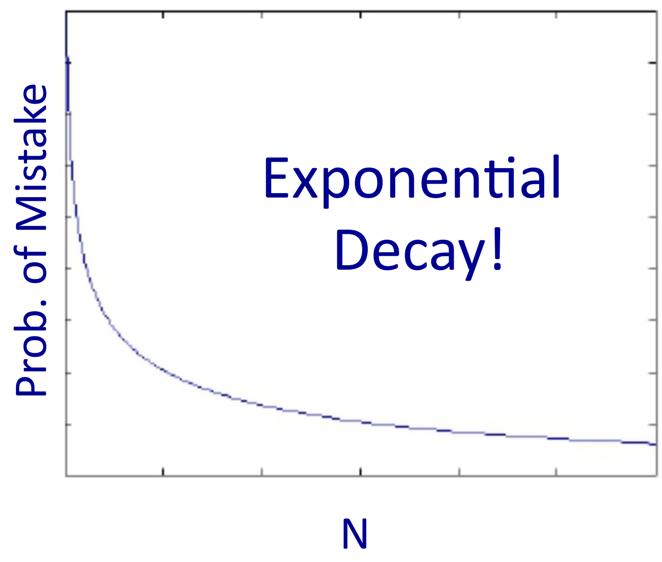
- Billionaire says: I flipped 3 heads and 2 tails.
- You say: θ = 3/5, I can prove it!
- He says: What if I flipped 30 heads and 20 tails?
- You say: Same answer, I can prove it!
- **He says: What's better?**
- You say: Umm… The more the merrier???
- He says: Is this why I am paying you the big bucks???
- You say: I will give you a theoretical bound.

A bound (from Hoeffding's inequality)

For $N = \alpha_H + \alpha_T$, and $\widehat{\theta}_{MLE} = \frac{\alpha_H}{\alpha_H + \alpha_T}$

Let θ^* be the true parameter, for any $\varepsilon > 0$:

$$P(|\,\widehat{\theta} - \theta^*\,| \geq \epsilon) \leq 2e^{-2N\epsilon^2}$$



PAC Learning

- PAC: Probably Approximately Correct
- Billionaire says: I want to know the thumbtack θ, within ε = 0.1, with probability at least 1-δ = 0.95.
- How many flips? Or, how big do I set N?

$$P(|\,\widehat{\theta} - \theta^*\,| \geq \epsilon) \;\leq\; 2e^{-2N\epsilon^2}$$

$$\delta \geq 2e^{-2N\epsilon^2} \geq P(\text{mistake})$$

$$\ln \delta \geq \ln 2 - 2N\epsilon^2$$

$$N \geq \frac{\ln(2/\delta)}{2\epsilon^2}$$

Interesting! Lets look at some numbers!

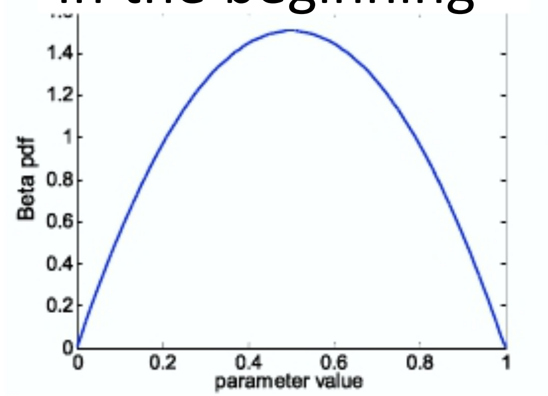
ε = 0.1, δ=0.05

$$N \geq \frac{\ln(2/0.05)}{2 \times 0.1^2} \approx \frac{3.8}{0.02} = 190$$

What if I have prior beliefs?

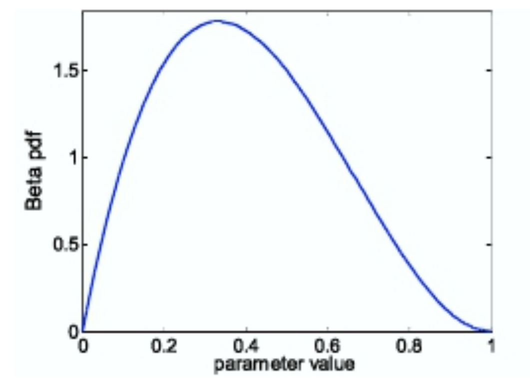
- Billionaire says: Wait, I know that the thumbtack is “close” to 50-50. What can you do for me now?
- **You say: I can learn it the Bayesian way…**
- Rather than estimating a single θ, we obtain a distribution over possible values of θ

In the beginning



Observe flips
e.g.: {tails, tails}

After observations

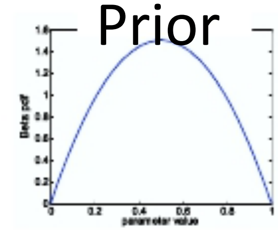


Bayesian Learning

Use Bayes rule:

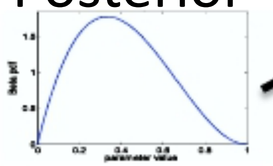
Data Likelihood

Prior



$$P(\theta \mid \mathcal{D}) = \frac{P(\mathcal{D} \mid \theta) P(\theta)}{P(\mathcal{D})}$$

Posterior



Normalization

Or equivalently: $P(\theta \mid \mathcal{D}) \propto P(\mathcal{D} \mid \theta) P(\theta)$

Bayesian Learning for Thumbtacks

$$P(\theta \mid \mathcal{D}) \;\propto\; P(\mathcal{D} \mid \theta)P(\theta)$$

Likelihood function is Binomial:

$$P(\mathcal{D} \mid \theta) = \theta^{\alpha_H}(1-\theta)^{\alpha_T}$$

- What about prior?
  - Represent expert knowledge
  - Simple posterior form
- Conjugate priors:
  - Closed-form representation of posterior
  - **For Binomial, conjugate prior is Beta distribution**

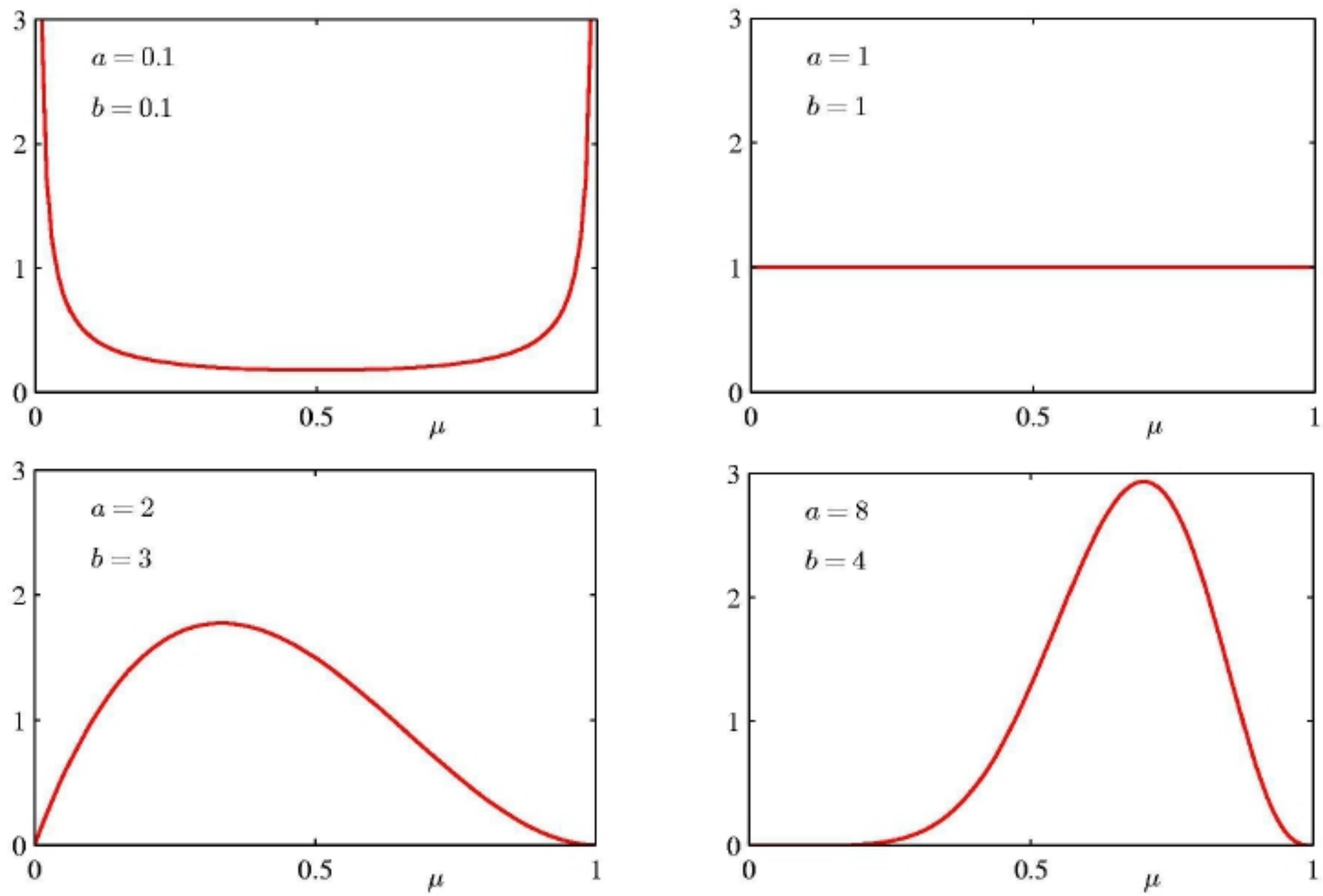
Beta Distribution

- Distribution over $\mu \in [0, 1]$.

$$B(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$$\begin{aligned} \mathrm{Beta}(\mu|a,b) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{a-1}(1-\mu)^{b-1} \\ \mathbb{E}[\mu] &= \frac{a}{a+b} \\ \mathrm{var}[\mu] &= \frac{ab}{(a+b)^2(a+b+1)} \end{aligned}$$

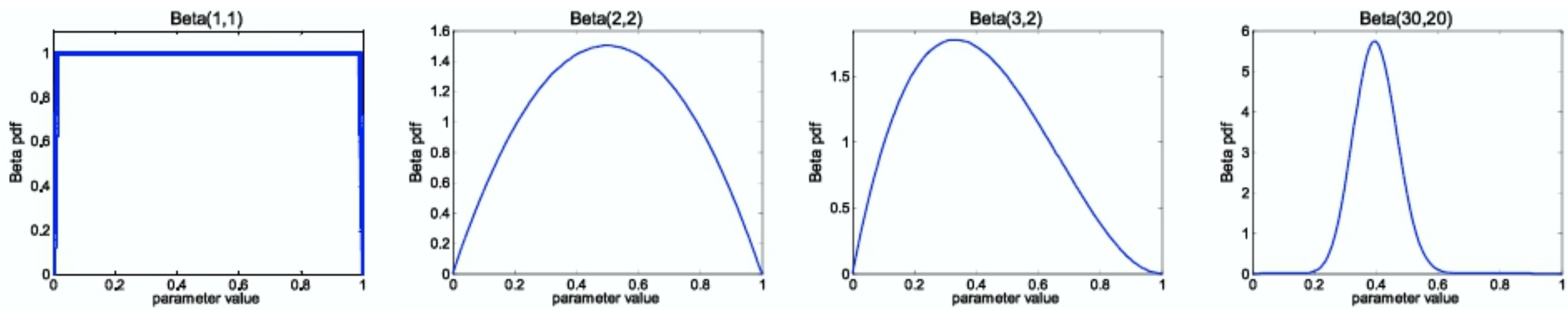
Beta Distribution



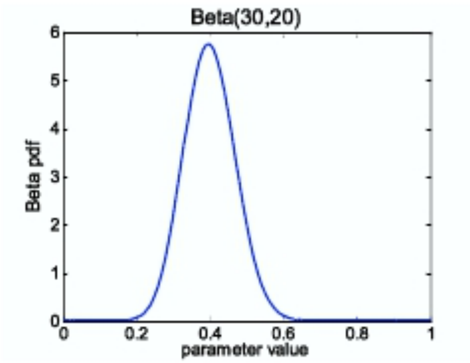
Posterior Distribution

- Prior: $Beta(\beta_H, \beta_T)$
- Data: α_H heads and α_T tails
- Posterior distribution:

$$P(\theta \mid \mathcal{D}) \sim Beta(\beta_H + \alpha_H, \beta_T + \alpha_T)$$



Bayesian Posterior Inference



- Posterior distribution:

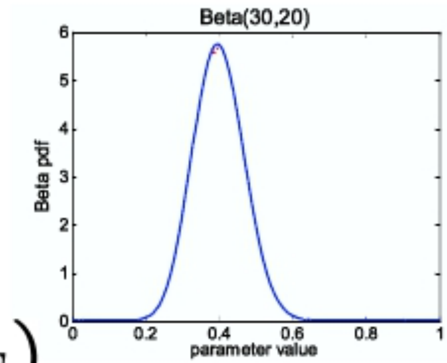
$$P(\theta \mid \mathcal{D}) \sim Beta(\beta_H + \alpha_H, \beta_T + \alpha_T)$$

- Bayesian inference:
  - No longer single parameter
  - For any specific f, the function of interest
  - Compute the expected value of f

$$E[f(\theta)] = \int_0^1 f(\theta) P(\theta \mid \mathcal{D}) d\theta$$

  - Integral is often hard to compute

MAP: Maximum a Posteriori Approximation



$$P(\theta \mid \mathcal{D}) \sim Beta(\beta_H + \alpha_H, \beta_T + \alpha_T)$$

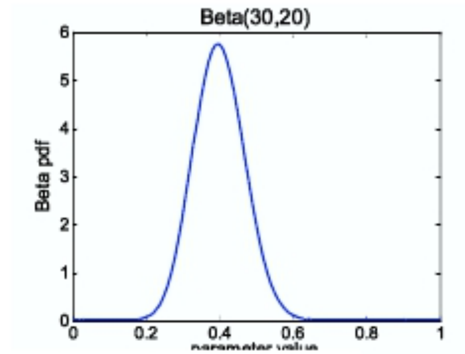
$$E[f(\theta)] = \int_0^1 f(\theta) P(\theta \mid \mathcal{D}) d\theta$$

- As more data is observed, Beta is more certain
- MAP: use most likely parameter to approximate the expectation

$$\widehat{\theta} = \arg\max_{\theta} P(\theta \mid \mathcal{D})$$

$$E[f(\theta)] \approx f(\widehat{\theta})$$

MAP for Beta distribution



$$P(\theta \mid \mathcal{D}) = \frac{\theta^{\beta_H+\alpha_H-1}(1-\theta)^{\beta_T+\alpha_T-1}}{B(\beta_H+\alpha_H, \beta_T+\alpha_T)} \sim Beta(\beta_H+\alpha_H, \beta_T+\alpha_T)$$

MAP: use most likely parameter:

$$\widehat{\theta} = \arg\max_{\theta} P(\theta \mid \mathcal{D}) = \frac{\alpha_H+\beta_H-1}{\alpha_H+\beta_H+\alpha_T+\beta_T-2}$$

Beta prior equivalent to extra thumbtack flips

As $N \rightarrow \infty$, prior is “forgotten”

But, for small sample size, prior is important!

Estimating Parameters

- Maximum Likelihood Estimate (MLE): choose θ that maximizes probability of observed data $\mathcal{D}$

$$\widehat{\theta} = \arg\max_{\theta} P(\mathcal{D} \mid \theta)$$

- Maximum a Posteriori (MAP) estimate: choose θ that is most probable given prior probability and the data

$$\widehat{\theta} = \arg\max_{\theta} P(\theta \mid \mathcal{D})$$

$$= \arg\max_{\theta} = \frac{P(\mathcal{D} \mid \theta)P(\theta)}{P(\mathcal{D})}$$