Naïve Bayes

Some slides courtesy of Vibhav Gogate, Carlos Guestrin, Chris Bishop, Dan Weld and Luke Zettlemoyer.

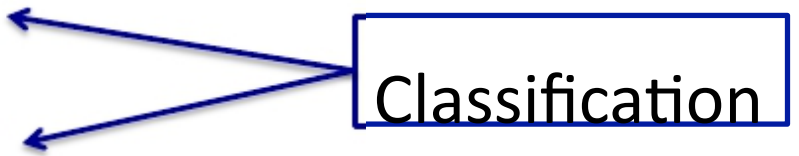
Supervised Learning of Classifiers

Find f

- Given: Training set $\{(x_i, y_i) \mid i = 1 \ldots n\}$
- Find: A good approximation to $f : X \rightarrow Y$

Examples: what are X and Y ?

- Spam Detection
  - Map email to {Spam,Ham}
- Digit recognition
  - Map pixels to {0,1,2,3,4,5,6,7,8,9}
- Stock Prediction
  - Map new, historic prices, etc. to $\Re$(the real numbers)



Bayesian Categorization/Classification

- Let the set of categories be $\{c_1, c_2, \ldots c_n\}$
- Let E be description of an instance.
- Determine category of E by determining for each c_i

$$P(c_i \mid E) = \frac{P(c_i)P(E \mid c_i)}{P(E)}$$

- P(E) can be ignored (normalization constant)

$$P(c_i \mid E) \sim P(c_i)P(E \mid c_i)$$

Text classification

- Classify e-mails
  - Y = {Spam,NotSpam}
- Classify news articles
  - Y = {what is the topic of the article?}
- Classify webpages
  - Y = {Student, professor, project, …}

- What to use for features, **X**?

Features **X** are word sequence in document
X_i for i^{th} word in article

Article from rec.sport.hockey

```
Path: cantaloupe.srv.cs.cmu.edu!das-news.harvard.e
From: xxx@yyy.zzz.edu (John Doe)
Subject: Re: This year's biggest and worst (opinic
Date: 5 Apr 93 09:53:39 GMT

I can only comment on the Kings, but the most
obvious candidate for pleasant surprise is Alex
Zhitnik. He came highly touted as a defensive
defenseman, but he's clearly much more than that.
Great skater and hard shot (though wish he were
more accurate). In fact, he pretty much allowed
the Kings to trade away that huge defensive
liability Paul Coffey. Kelly Hrudey is only the
biggest disappointment if you thought he was any
good to begin with. But, at best, he's only a
mediocre goaltender. A better choice would be
Tomas Sandstrom, though not through any fault of
his own, but because some thugs in Toronto decided
```

Features for Text Classification

- **X** is sequence of words in document
- **X** (and hence P(**X**|Y)) is huge!!!
  - Article at least 1000 words, $\mathbf{X}=\{X_1,\ldots,X_{1000}\}$
  - X_i represents i^{th}word in document, i.e., the domain of X_i is entire vocabulary, e.g., Webster Dictionary (or more), 10,000 words, etc.
- $10{,}000^{1000} = 10^{4000}$
- Atoms in Universe: 10^{80}
  - We may have a problem…

Bag of Words Model

Typical additional assumption –

- **Position in document doesn't matter**:
  - $P(X_i=x_i|Y=y) = P(X_k=x_i|Y=y)$
  - (all position have the same distribution)
- Ignore the order of words
- Sounds really silly, but often works very well!

- From now on:
  - X_i = Boolean: "word$_i$ is in document"
  - $\mathbf{X} = X_1 \wedge \ldots \wedge X_n$

Bag of Words Approach

aardvark	0
about	2
all	2
Africa	1
apple	0
anxious	0
...	
gas	1
...	
oil	1
…	
Zaire	0

Bayesian Categorization

$P(y_1 \mid \mathbf{X}) \sim P(y_i)P(\mathbf{X}|y_i)$

- Need to know:
  - Priors: $P(y_i)$
  - Conditionals: $P(\mathbf{X} \mid y_i)$

- $P(y_i)$ are easily estimated from data.
  - If n_i of the examples in D are in y_i, then $P(y_i) = n_i / |D|$
- Conditionals:
  - $\mathbf{X} = X_1 \wedge \ldots \wedge X_n$
  - Estimate $P(X_1 \wedge \ldots \wedge X_n \mid y_i)$
- Too many possible instances to estimate!
  - *(exponential in n)*
  - Even ***with*** bag of words assumption!

Need to Simplify Somehow

- Too many probabilities
  - $P(x_1 \wedge x_2 \wedge x_3 \mid y_i)$

$P(x_1 \wedge x_2 \wedge x_3 \mid \text{spam})$
$P(x_1 \wedge x_2 \wedge \neg x_3 \mid \text{spam})$
$P(x_1 \wedge \neg x_2 \wedge x_3 \mid \text{spam})$
….
$P(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \mid \neg \text{spam})$

- Can we assume some are the same?
  - $P(x_1 \wedge x_2 \mid y_i) = P(x_1 \mid y_i)\, P(x_2 \mid y_i)$

Conditional Independence

- X is **conditionally independent** of Y given Z, if the probability distribution for X is independent of the value of Y, given the value of Z

$$(\forall i, j, k) P(X = i | Y = j, Z = k) = P(X = i | Z = k)$$

- e.g.,

$$P(Thunder | Rain, Lightning) = P(Thunder | Lightning)$$

- Equivalent to:

$$P(X, Y \mid Z) = P(X \mid Z) P(Y \mid Z)$$

Naïve Bayes

- Naïve Bayes assumption:
  - Features are independent given class:

$$P(X_1, X_2|Y) = P(X_1|X_2, Y)P(X_2|Y)$$
$$= P(X_1|Y)P(X_2|Y)$$

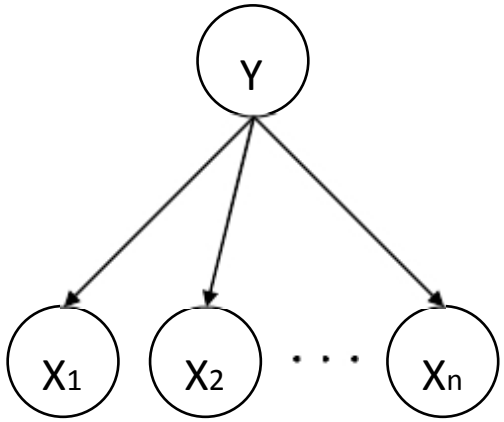
  - More generally:

$$P(X_1 \ldots X_n|Y) = \prod_i P(X_i|Y)$$

- How many parameters now?
  - Suppose **X** is composed of *n* binary features

The Naïve Bayes Classifier

- **Given:**
  - Prior P(Y)
  - *n* conditionally independent features **X** given the class Y
  - For each X_i, we have likelihood $P(X_i|Y)$



- **Decision rule:**

$$y^* = h_{NB}(\mathbf{x}) = \arg\max_y P(y) P(x_1, \ldots, x_n \mid y)$$
$$= \arg\max_y P(y) \prod_i P(x_i|y)$$

MLE for the parameters of NB

- Given dataset, count occurrences for all pairs
  - Count($X_i=x_i$,$Y=y$)----- How many pairs?

- MLE for discrete NB, simply:
  - Prior:

$$P(Y = y) = \frac{Count(Y = y)}{\sum_{y'} Count(Y = y')}$$

  - Likelihood:

$$P(X_i = x|Y = y) = \frac{Count(X_i = x, Y = y)}{\sum_{x'} Count(X_i = x', Y = y)}$$

NAÏVE BAYES CALCULATIONS

Day	Outlook	Temperature	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Subtleties of NB Classifier: #1 Violating the NB Assumption

- Usually, features are not conditionally independent:

$$P(X_1 \ldots X_n | Y) \neq \prod_i P(X_i | Y)$$

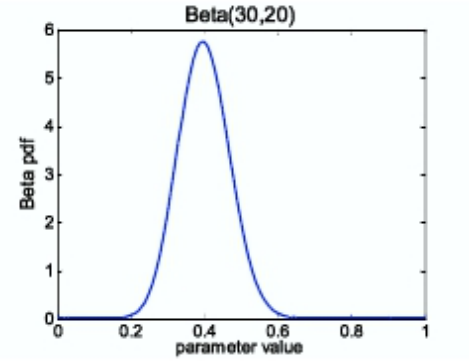
- Actual probabilities P(Y|**X**) often biased towards 0 or 1
- Nonetheless, NB is the single most used classifier out there
  - NB often performs well, even when assumption is violated
  - [Domingos & Pazzani '96] discuss some conditions for good performance

Subtleties of NB Classifier: #2 Insufficient Training Data

- What if you never see a training instance where X1=a and Y=b
  - You never saw, Y=Spam, X1={Enlargement}
  - P(X1=Enlargement|Y=Spam)=0
- Thus no matter what values X2, X3,….,Xn take:
  - P(X1=Enlargement, X2=a2,…,Xn=an|Y=Spam)=0
  - Why?

$$y^* = h_{NB}(\mathbf{x}) = \arg\max_y P(y) P(x_1, \ldots, x_n \mid y)$$
$$= \arg\max_y P(y) \prod_i P(x_i|y)$$

For Binary Features: We already know the answer!



$$P(\theta \mid \mathcal{D}) = \frac{\theta^{\beta_H+\alpha_H-1}(1-\theta)^{\beta_T+\alpha_T-1}}{B(\beta_H+\alpha_H, \beta_T+\alpha_T)} \sim Beta(\beta_H+\alpha_H, \beta_T+\alpha_T)$$

- MAP: use most likely parameter

$$\widehat{\theta} = \arg\max_{\theta} P(\theta \mid \mathcal{D}) = \frac{\alpha_H + \beta_H - 1}{\alpha_H + \beta_H + \alpha_T + \beta_T - 2}$$

- Beta prior equivalent to extra observations for each feature
- As $N \rightarrow 1$, prior is "forgotten"
- **But, for small sample size, prior is important!**

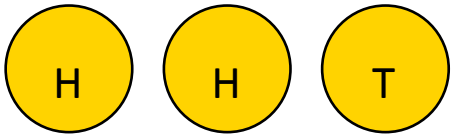
That's Great for Binomial

- Works for Spam / Ham
- What about multiple classes
  - Eg, given a wikipedia page, predicting type

Multinomials: Laplace Smoothing

- Laplace's estimate:
  - Pretend you saw every outcome k extra times

$$P_{LAP,k}(x) = \frac{c(x) + k}{N + k|X|}$$



$$P_{LAP,0}(X) = \left\langle \frac{2}{3}, \frac{1}{3} \right\rangle$$

$$P_{LAP,1}(X) = \left\langle \frac{3}{5}, \frac{2}{5} \right\rangle$$

$$P_{LAP,100}(X) = \left\langle \frac{102}{203}, \frac{101}{203} \right\rangle$$

  - What's Laplace with k = 0?
  - k is the strength of the prior
  - Can derive this as a MAP estimate for multinomial with *Dirichlet priors*

- Laplace for conditionals:
  - Smooth each condition independently:

$$P_{LAP,k}(x|y) = \frac{c(x,y) + k}{c(y) + k|X|}$$

Probabilities: Important Detail!

- $P(\text{spam} \mid X_1 \ldots X_n) = \prod_i P(\text{spam} \mid X_i)$

Any more potential problems here?

- We are multiplying lots of small numbers
  Danger of underflow!
  - $0.5^{57} = 7 \text{ E } -18$
- Solution? Use logs and add!
  - $p_1 * p_2 = e^{\log(p1)+\log(p2)}$
  - Always keep in log form

Naïve Bayes Posterior Probabilities

- Classification results of naïve Bayes
  - I.e. the class with maximum posterior probability…
  - Usually fairly accurate (?!?!?)
- However, due to the inadequacy of the conditional independence assumption…
  - Actual posterior-***probability*** estimates ***not*** accurate.
  - Output probabilities generally very close to 0 or 1.

NB with Bag of Words for Text Classification

- Learning phase:
  - Prior $P(Y_m)$
    - Count how many documents from each topic (prior)
  - $P(X_i|Y_m)$
    - Let B be the bag of words created from the union of all docs
    - Let B_m be a bag of words formed from all the docs in topic m
    - Let #(i, B) be the number of times word i is in bag B
    - $P(X_i \mid Y_m)$ is proportional to (#(i, B_m)+1)
- Test phase:
  - For each document
    - Use naïve Bayes decision rule

$$h_{NB}(\mathbf{x}) = \arg\max_y P(y) \prod_{i=1}^{LengthDoc} P(x_i|y)$$

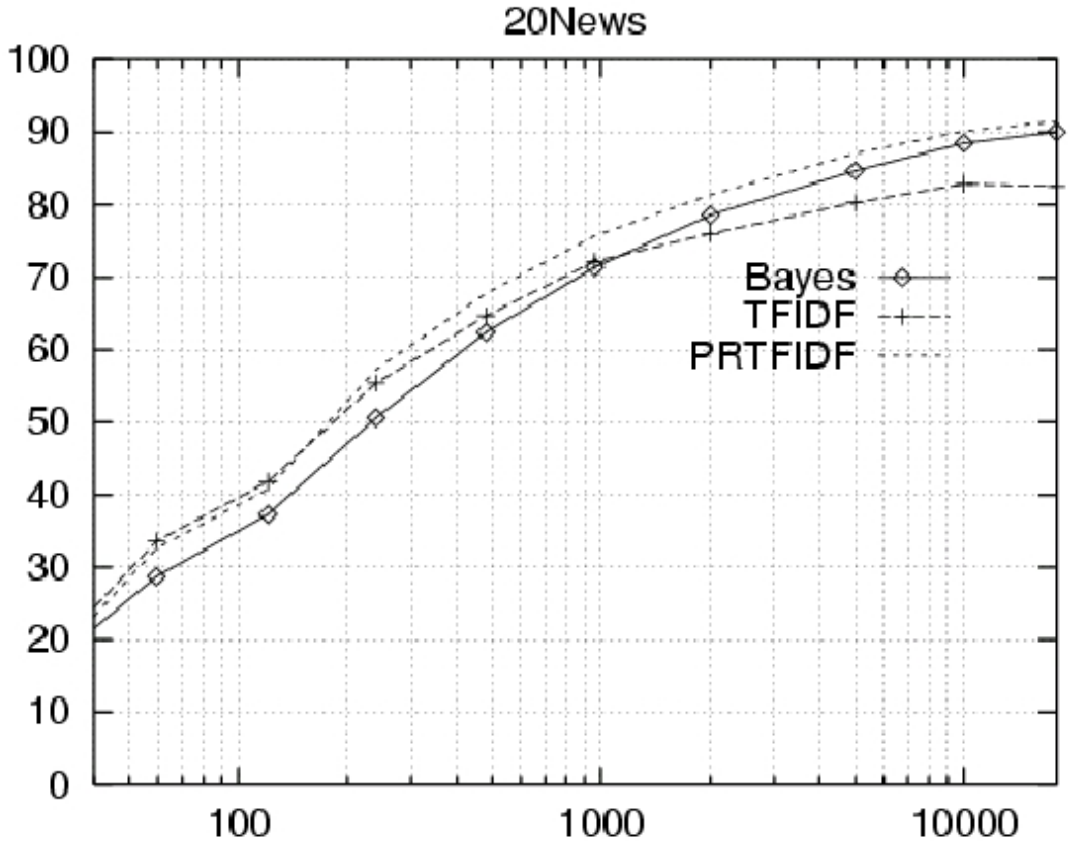
Twenty News Groups results

Given 1000 training documents from each group
Learn to classify new documents according to
which newsgroup it came from

comp.graphics	misc.forsale
comp.os.ms-windows.misc	rec.autos
comp.sys.ibm.pc.hardware	rec.motorcycles
comp.sys.mac.hardware	rec.sport.baseball
comp.windows.x	rec.sport.hockey
alt.atheism	sci.space
soc.religion.christian	sci.crypt
talk.religion.misc	sci.electronics
talk.politics.mideast	sci.med
talk.politics.misc	
talk.politics.guns	

Naive Bayes: 89% classification accuracy

Learning curve for Twenty News Groups



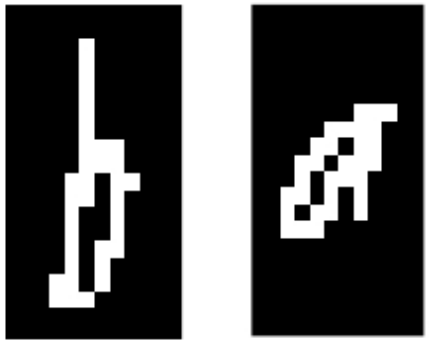
Accuracy vs. Training set size (1/3 withheld for test)

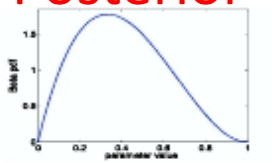
Bayesian Learning

What if Features are Continuous?

Eg., Character Recognition:

X_i is ith pixel





$$P(Y \mid \mathbf{X}) \propto P(\mathbf{X} \mid Y)\, P(Y)$$

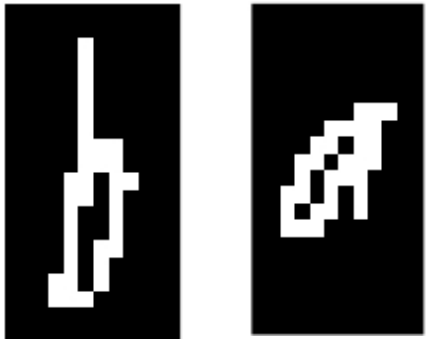
Data Likelihood

Bayesian Learning

What if Features are Continuous?

Eg., Character Recognition:

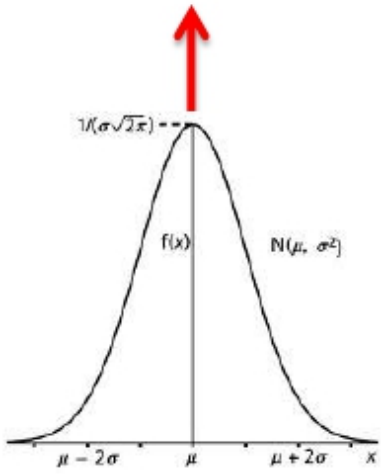
X_i is i^{th} pixel



$$P(Y \mid \mathbf{X}) \propto P(\mathbf{X} \mid Y)\, P(Y)$$

$$P(X_i = x \mid Y = y_k) = N(\mu_{ik}, \sigma_{ik})$$

$$N(\mu_{ik}, \sigma_{ik}) = \frac{1}{\sigma_{ik}\sqrt{2\pi}}\, e^{\frac{-(x-\mu_{ik})^2}{2\sigma_{ik}^2}}$$



Gaussian Naïve Bayes

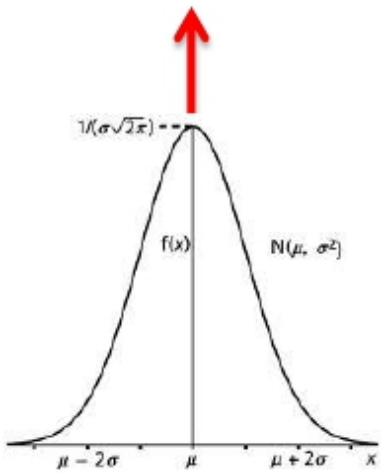
Sometimes Assume Variance

- is independent of Y (i.e., σ_i),
- or independent of X_i (i.e., σ_k)
- or both (i.e., σ)

$$P(Y \mid \mathbf{X}) \propto P(\mathbf{X} \mid Y)\, P(Y)$$

$$P(X_i = x \mid Y = y_k) = N(\mu_{ik}, \sigma_{ik})$$

$$N(\mu_{ik}, \sigma_{ik}) = \frac{1}{\sigma_{ik}\sqrt{2\pi}}\, e^{\frac{-(x-\mu_{ik})^2}{2\sigma_{ik}^2}}$$



Learning Gaussian Parameters

Maximum Likelihood Estimates:

- Mean:

$$\widehat{\mu}_{MLE} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

- Variance:

$$\widehat{\sigma}^2_{MLE} = \frac{1}{N}\sum_{i=1}^{N} (x_i - \widehat{\mu})^2$$

Learning Gaussian Parameters

Maximum Likelihood Estimates:

- Mean:

$$\hat{\mu}_{ik} = \frac{1}{\sum_j \delta(Y^j = y_k)} \sum_j X_i^j \delta(Y^j = y_k)$$

jth training example

$\delta(x)$=1 if x true, else 0

- Variance:

$$\hat{\sigma}^2_{MLE} = \frac{1}{N} \sum_{i=1}^{N} (x_i - \hat{\mu})^2$$

Learning Gaussian Parameters

Maximum Likelihood Estimates:

- Mean:

$$\hat{\mu}_{ik} = \frac{1}{\sum_j \delta(Y^j = y_k)} \sum_j X_i^j \delta(Y^j = y_k)$$

- Variance:

$$\hat{\sigma}_{ik}^2 = \frac{1}{\sum_j \delta(Y^j = y_k) - 1} \sum_j (X_i^j - \hat{\mu}_{ik})^2 \delta(Y^j = y_k)$$

What you need to know about Naïve Bayes

- Naïve Bayes classifier
  - What's the assumption
  - Why we use it
  - How do we learn it
  - Why is Bayesian estimation important
- Text classification
  - Bag of words model
- Gaussian NB
  - Features are still conditionally independent
  - Each feature has a Gaussian distribution given class