Modeling with Separable Differential Equations

KEY

Why does it matter?

How does it actually work?

What does it enable you to do?

Example

Recall: Newton's Law of Cooling states that the rate of cooling of an object is proportional to the difference between the temperature of the object and the temperature of the surrounding medium.

a) Write a differential that gives the temperature of an object at time t, given that the initial temperature is T_0, and the temperature of the surrounding medium is T_s.

b) Solve this initial value problem.

c) Use your results to assist with the following murder investigation.

The body of Victor "Vic" Timm was found in the library at 11:00 p.m. by the night watchman, Tom Peepers. The temperature of the body upon discovery was 70^0 F. Two hours later the body was 73^0 F. There are three suspects, Vic's sister Molly Gunn, her son Carey A. Gunn ("Sonny" to his friends) and Vic's fiancée Mary Ritch. Assume that Molly has an alibi from 2:00 p.m. to 7:00 p.m., Sonny had an alibi from 4:00 pm to 11:00 pm., and Mary has an alibi from noon to 4:00 pm and 7:00 pm to 11:00 pm., determine when Vic Timm died and who likely murdered him.

1.

The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t. After 3 hours, it is observed that 500 bacteria are present. After 10 hours, 5000 bacteria are present.

a) Write a differential that gives the number of bacteria after t hours.

t = time. P = population at time t, K = constant

$\frac{dP}{dt}$ = rate of change of P

$$\frac{dP}{dt} = KP$$

b) Solve the differential equation to find the number of bacteria after t hours, assuming that the initial number of bacteria is P_0.

$$\frac{dP}{P} = k\,dt \Rightarrow \ln|P| = kt + C \Rightarrow |P| = e^{kt+C}$$

$P = \pm e^{C} e^{kt}$. Set $A = \pm e^{C}$. Then $P = Ae^{kt}$

$$P(0) = P_0 \Rightarrow P_0 = Ae^{0} \Rightarrow A = P_0 \Rightarrow P = P_0 e^{kt}$$

c) What was the initial number of bacteria? (Round your answer to the nearest integer.)

$P(3) = 500 \Rightarrow 500 = P_0 e^{3K}$ * $\quad P(10) = 5000 \Rightarrow 5000 = P_0 e^{10K}$ **

Dividing * and **: $10 = e^{7K} \Rightarrow K = \frac{\ln 10}{7} \approx 0.328941$

Substituting in *: $500 = P_0 e^{3(0.328941)}$

so $P_0 \approx 186$ bacteria

d) How long will it take for the initial number of bacteria to double?

$$P = 186\, e^{0.328941t}$$

$$2(1\cancel{86}) = 1\cancel{86}\, e^{0.328941t} \Rightarrow 2 = e^{0.328941t}$$

$t = 2.11$ hours

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2.
A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate proportional to the glucose concentration at that time.

a) Write a differential equation that models the concentration, C, of the glucose solution in the bloodstream at time t.

t = time; C = concentration at time t

$\frac{dC}{dt}$ = rate of change of C; K = constant of prop.

r = Entering rate of solution

$$\frac{dC}{dt} = r - KC$$

rate in rate out

b) Solve the differential equation to find the concentration of the glucose solution in the bloodstream at time t, assuming that the initial concentration is C_0.

$$\frac{dC}{r-KC} = dt$$

$$-\frac{1}{K}\ln|r-KC| = t + c_1$$

$$\ln|r-KC| = -Kt + c_2$$

$$|r-KC| = e^{-Kt+c_2}$$

$$|r-KC| = e^{c_2}e^{-Kt}$$

$$r-KC = \pm e^{c_2}e^{-Kt}$$

Set $A = \pm e^{c_2}$

$$r-KC = Ae^{-Kt}$$

When $t=0$, $C=C_0$

$$r-KC_0 = Ae^0 \Rightarrow$$

$A = r-KC_0$. Thus,

$$r-KC = (r-KC_0)e^{-Kt}$$

Solving for C:

$$C = \frac{r}{K} + \left(C_0 - \frac{r}{K}\right)e^{-Kt}$$

c) In a complete English sentence, interpret the meaning of $\lim_{t\to\infty} C(t)$. Explain how you could have predicted the answer by considering the model found in part a).

$$\lim_{t\to\infty} e^{-Kt} = 0 \Rightarrow \lim_{t\to\infty} C(t) = \frac{r}{K}.$$

As time passes, the concentration will eventually be $\frac{r}{K}$.

This is when $\frac{dC}{dt} = 0$, the equilibrium solution

3.

Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt have been dissolved. Another brine solution is pumped into the tank at a rate of 3 gal/min, and when the solution is well stirred, it is then pumped out at the same rate. The concentration of the solution entering is 4 lb/gal. Let $A(t)$ be the amount of salt in the tank at time $t > 0$.

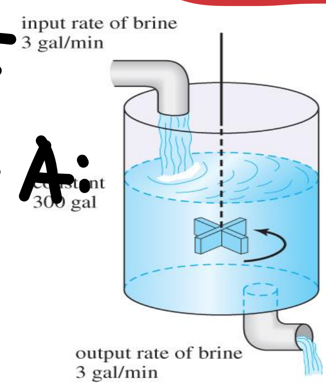
a) Write a differential equation that models the amount of salt, A, at time t.

t = time, A = Amount of salt at time t

$\frac{dA}{dt}$ = rate of change of A = rate in − rate out

$$\frac{dA}{dt} = 4\frac{lb}{gal} \cdot 3\frac{gal}{min} - \frac{A\,lb}{300\,gal}\, 3\frac{gal}{min} \Rightarrow \frac{dA}{dt} = 12 - \frac{A}{100}$$

b) Solve the differential equation to find the amount of salt, A, in the tank at time t.



$$\frac{dA}{dt} = \frac{1200 - A}{100} \Rightarrow \frac{dA}{1200 - A} = \frac{1}{100}dt$$

Integrating and solving for A:

$$1200 - A = Ce^{-\frac{1}{100}t}$$

when $t = 0$, $A = 50$

$$1200 - 50 = C \Rightarrow C = 1150$$

So $A = 1200 - 1150\, e^{-\frac{1}{100}t}$

c) How much salt is the tank after 60 minutes?

$A(60) = 568.87$ lbs

d) How long will it take for the tank to have 1060 lbs of salt?

$$1060 = 1200 - 1150\, e^{-\frac{1}{100}t} \Rightarrow t = 210.59 \text{ mins}$$

e) In a complete English sentence, interpret the meaning of $\lim_{t \to \infty} A(t)$.

$$\lim_{t \to \infty} e^{-\frac{1}{100}t} = 0 \Rightarrow \lim_{t \to \infty} A(t) = 1200.$$

Eventually, the tank will have 1200 lbs of salt.

4.

Help solve the following murder mystery!

The great detective Sherlock Holmes and his assistant Dr. Watson are discussing the murder of actor Cornelius McHam.

McHam was shot in the head, and his understudy, Barry Moore, was found standing over the body with the murder weapon in hand. Let's listen in.

Watson: Open-and-shut case Holmes! Moore is the murderer.

Holmes: Not so fast Watson! You are forgetting Newton's Law of Cooling!

Watson: Huh?

Holmes: Elementary my dear Watson! Moore was found standing over McHam at 10:06 p.m., at which time the coroner recorded a body temperature of 77.9°F and noted that the room thermostat was set to 72°F. At 11:06 p.m., the coroner took another reading and recorded a body temperature of 75.6°F. Since McHam's normal temperature is 98.6°F, and since Moore was on stage between 6:00 p.m. and 8:00 p.m., Moore is obviously innocent. Time of death is 7:03 pm

Watson: Huh?

Holmes: Sometimes you are so dull Watson. Ask any calculus student to figure it out for you.

Watson: Hrrumph….

Let T be the temperature at time t, where t is the number of hours after (or before) 10:06 pm.

$\frac{dT}{dt} = k(T - T_s) \Rightarrow \frac{dT}{T - T_s} = k\,dt$

$\ln|T - T_s| = kt + c \Rightarrow T - T_s = \pm e^{c+kt}$

$T - T_s = Ae^{kt}$

$T_s = 72 \Rightarrow T = 72 + Ae^{kt}$

$T(0) = 77.9 \Rightarrow 77.9 = 72 + Ae^{0} \Rightarrow A = 5.9$

So $T = 72 + 5.9e^{kt}$

$T(1) = 75.6 \Rightarrow 75.6 = 72 + 5.9e^{k} \Rightarrow k = -0.494$

Thus $T = 72 + 5.9e^{-0.494t}$

$T = 98.6 \Rightarrow 98.6 = 72 + 5.9e^{-0.494t} \Rightarrow t = -3.05$

or 3h 3mins before 10:06pm

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