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22 September 2022

Homework 1

1.) a.) From page 38, we know that a survival function of the form $S(x) = \exp\{1 - \exp[(\lambda x)^{\alpha}]\}$ has a hazard rate $\alpha \lambda^{\alpha} x^{\alpha - 1} \exp\{(\lambda x)^{\alpha}\}$. From page 27, we know that the hazard function can be expressed as $h(x) = -d \ln[S(x)]/dx$ (2.3.2)

Plugging in $\alpha = 0.5$, we can derive our hazard function;

$$h(x) = -\frac{d}{dx} \ln(\exp\{1 - \exp[(\lambda x)^{0.5}]\}$$

$$= -\frac{d}{dx}(\cancel{1} - \exp[(\lambda x)^{1/2}]) \quad \text{(Swap out 0.5 for } 1/2)$$

$$= (-1) \cdot \exp[(\lambda x)^{1/2}] \cdot \frac{1}{2} \cdot (\lambda x)^{-1/2} \cdot \frac{d}{dx}(\lambda x) \quad \text{(chain rule)}$$

$$= \exp[(\lambda x)^{1/2}] \cdot \frac{1}{2} \cdot \lambda \cdot [(\lambda x)^{-1/2}]$$

$$= \exp[(\lambda x)^{1/2}] \cdot \frac{1}{2} \cdot \lambda \cdot \lambda^{-1/2} \cdot x^{-1/2}$$

$$= \frac{1}{2} \cdot \left(\frac{\lambda}{x}\right)^{1/2} \cdot \exp[(\lambda x)^{1/2}]$$

Which matches the above form. We calculate the derivative.

$$h'(x) = \frac{d}{dx}\left[\frac{1}{2} \cdot \left(\frac{\lambda}{x}\right)^{1/2} \cdot \exp[(\lambda x)^{1/2}]\right]$$

Product Rule

$$\frac{d}{dx}[(f(x))(g(x))] = f(x)(g'(x)) + (f'(x)g(x))$$

$$= \frac{1}{2}\left[\frac{1}{2} \cdot \left(\frac{\lambda}{x}\right)^{-1/2} \cdot \frac{d}{dx}\left(\frac{\lambda}{x}\right) \cdot \exp[(\lambda x)^{1/2}] \ldots\right.$$

$$\left.\ldots + \left(\frac{\lambda}{x}\right)^{1/2} \exp[(\lambda x)^{1/2}] \cdot \frac{d}{dx}(\lambda x)^{1/2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} \cdot \left(\frac{\lambda}{x}\right)^{-1/2} \cdot \lambda \cdot \left(\frac{-1}{x^2}\right) \exp[(\lambda x)^{1/2}] + \left(\frac{\lambda}{x}\right)^{1/2} \exp[(\lambda x)^{1/2}] \cdot \frac{1}{2}(\lambda x)^{-1/2} \ldots\right.$$

$$\left.\ldots \cdot \lambda\right]$$

We simplify...

$$= \frac{1}{2}\left[-\frac{1}{2} \cdot \lambda^{1/2} \cdot x^{-3/2} \cdot \exp[(\lambda x)^{1/2}] + \frac{1}{2} \cdot \lambda \cdot \frac{1}{x} \cdot \exp[(\lambda x)^{1/2}]\right]$$

$$h'(x) = -\frac{1}{4} \cdot \lambda^{1/2} \cdot x^{-3/2} \cdot \exp[(\lambda x)^{1/2}] + \frac{1}{4} \cdot \frac{\lambda}{x} \cdot \exp[(\lambda x)^{1/2}]$$

Set $h'(x) = 0$ to find change point

$$\cancel{\frac{1}{4}} \cdot \lambda^{1/2} \cdot x^{-3/2} \cdot \cancel{\exp[(\lambda x)^{1/2}]} = \cancel{\frac{1}{4}} \cdot \frac{\lambda}{x} \cdot \cancel{\exp[(\lambda x)^{1/2}]}$$

$$(x^{3/2})\frac{\lambda^{1/2}}{x^{3/2}} = \frac{\lambda}{\cancel{x}}\,(\cancel{x})$$

$$\frac{\lambda^{1/2}}{x^{1/2}} = \lambda$$

$$\lambda^{1/2} = \lambda x^{1/2}$$

$$(\lambda^{-1/2})^2 = (x^{1/2})^2$$

$$\boxed{\frac{1}{\lambda} = x}$$

When $x < 1/\lambda$, $h(x)$ is decreasing, and increasing when $x > 1/\lambda$. Thus the behavior of $h(x)$ changes where $x = 1/\lambda$. We know from 5. on page 44 that an exponential power function with $\alpha < 1$ is bath tub shaped, which we have above.

b.) We can use the form from page 38 to find our hazard function when $\alpha = 2$:

$$h(x) = 2\lambda^2 x \exp\{(\lambda x)^2\}$$

$$h'(x) = \frac{d}{dx}\, 2 \cdot \lambda^2 \cdot x \exp\{(\lambda x)^2\}$$

$$= 2\lambda^2 \exp\{(\lambda x)^2\} + 2\lambda^2 x \exp\{(\lambda x)^2\} \cdot 2(\lambda x) \cdot \lambda$$

$$= 2\lambda^2 \exp\{(\lambda x^2)\} + 4\lambda^4 x^2 \exp\{(\lambda x)^2\}$$

We can see that $\forall x < 0$ and $\forall x > 0$, the term $2\lambda^2 \exp\{(\lambda x^2)\} > 0$ and the term $4\lambda^4 x^2 \exp\{(\lambda x)^2\} > 0$, thus $h(x)$ is monotonically increasing.

2.18.) a) $f(x) = \begin{cases} 1/100 & 0 < x < 100 \\ 0 & \text{elsewhere} \end{cases}$

From page 22, we know the Survival function $S(x)$ can be defined as:

$$S(x) = Pr(X > x) \quad (2.2.1)$$

We can also rewrite this as

$$S(x) = 1 - F(x)$$

where $F(x)$ is the cumulative distribution function. We know the C.D.F of the uniform distribution to be

$$F(x) \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & x \in [a, b] \\ 1 & x > b \end{cases}$$

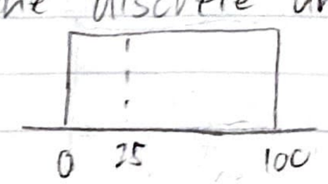
We solve:

$$\begin{aligned} S(x) &= 1 - F(x) \quad\quad a < x < b \\ &= 1 - \frac{(x-a)}{(b-a)} \\ &= \frac{b-a}{b-a} - \frac{x-a}{b-a} \\ &= \frac{b-x}{b-a} \quad\quad a < x < b \end{aligned}$$

$S(x)$ at $x = 25$ days $= \frac{100-25}{100-0} = \frac{3}{4}$ | $S(x)$ at $x = 50$ days $= \frac{100-50}{100-0} = \frac{1}{2}$ | $S(x)$ at $x = 75$ days $= \frac{100-75}{100-0} = \frac{1}{4}$

b.) For the discrete uniform

$x = 25$



$$mrl(25) = \frac{1}{2}(25 + 100) - x$$

$$= 62.5 - 25 = \boxed{37.5}$$

b.) $x = 25$ days , $S(x) = 3/4$

$$mrl(25) = \int_{25}^{100} (u-25)\, \tfrac{1}{100}\, du \Big/ \tfrac{3}{4}$$

$$= \frac{4}{3} \cdot \frac{1}{100} \int_{25}^{100} u - 25$$

$$= \frac{4}{3} \frac{1}{100} \left[\frac{1}{2}u^2 - 25u \right]_{25}^{100}$$

$$= \frac{4}{3} \cdot \frac{1}{100} \left[\left(\frac{10{,}000}{2} - 2{,}500 \right) - \left(\frac{625}{2} - 625 \right) \right]$$

$$= \frac{4}{300} \left[\frac{9375}{2} - 1875 \right] = \frac{18750}{300} - \frac{7500}{300} = \boxed{37.5}$$

For quicker calculations, we will use discrete method.

$x = 50$, $S(x) = 1/2$

$$mrl(50) = \frac{1}{2}(50 + 100) - 50$$

$$= \frac{1}{2}(150) - 50 = \boxed{25}$$

$$75 - 50 = \boxed{25}$$

$x = 75$ $S(x) = 1/4$

$$mrl(75) = \frac{1}{2}(75 + 100) - 75$$

$$= 87.5 - 75 = \boxed{12.5}$$

c.) Since our distribution is uniform, the mean and median for each value of x will be the same. Respectively, 37.5, 25, and 12.5 for x = 25, 50, and 75.

3.) a.) For λ_1: (Relapse) The log-likelihood:

$$\ell\ell = \sum_{i=1}^{m} \left[\ln(\lambda_1) - (\lambda_1 t_i)\right] + \sum_{j=1}^{k} (-\lambda_2 u_j)$$

$$\ell\ell = m \ln(\lambda_1) - \lambda_1(T+U)$$

Likelihood: $6 \ln(\hat{\lambda}_1) + \lambda_1(180)$

$$\ell\ell' = \frac{m}{\hat{\lambda}_1} - (T+U)$$

$T = \sum_{i=1}^{T} t_i = (5+8+12+24+32+\ldots) = 98$

$$0 = \frac{6}{\hat{\lambda}_1} - (98+82)$$

$m = 6$

$U = \sum_{j=1}^{k} u_j = 16+17+19+30 = 82$

$$180 = \frac{6}{\hat{\lambda}_1}$$

$$\boxed{\hat{\lambda}_1 = 6/180 = 1/30}$$ MLE

For λ_2: (Death)

$$\ell\ell' = \frac{m'}{\hat{\lambda}_2} - (T'+U')$$

$T' = \sum_{i=1}^{T} t_i' = 11+12+15+45 = 83$

$$0 = \frac{4}{\hat{\lambda}_2} - (83+143)$$

$U' = \sum u_j' = 33+28+16+17+19+30 = 143$

$$226 = \frac{4}{\hat{\lambda}_2}$$

$m' = 4$

$$\boxed{\hat{\lambda}_2 = \frac{4}{226} = 2/113}$$

Likelihood: $4 \ln(\hat{\lambda}_2) + \hat{\lambda}_2(226)$

b.) λ_2 w/ non-observed relapse

$$\ell\ell = \frac{m}{\hat{\lambda}_2} - (T+U)$$

$T = 83 \quad U = 33+28 = 61$

$$0 = \frac{4}{\hat{\lambda}_2} - (83+61)$$

$$144 = \frac{4}{\hat{\lambda}_2}$$

$$\hat{\lambda}_2 = 4/144 = 1/36$$

We observe that when we truncate our data, our value for $\hat{\lambda}_2$ becomes over 3 times larger.

4.) See R html