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Homework 1

$\boxed{1.)}$ A3 - 1.) a Let $Y = g(x)$ s.t. $P(Y=y) = \sum_{g(x)=y} P(X=x)$

$$E(Y) = \sum_{y} y\, P(Y=y)$$
$$= \sum_{y} y \sum_{g(x)} P(X=x)$$
$$= \sum_{y}\sum_{g(x)} y\, P(X=x)$$
$$= \sum_{x} g(x)\cdot P(X=x) \quad \leftarrow \text{LOTUS}$$

2.) $$E(aX+b) = \sum_{x} (aX+b)\, P(X=x)$$
$$= \sum_{x} aX\cdot P(X=x) + bP(X=x)$$
$$= a\sum_{x} x\cdot P(X=x) + b\sum_{x}(X=x)$$
$$= a\cdot E(X) + b$$

3.) $$E(XY) = \sum_{x}\sum_{y} xy\, \underbrace{P(X=x, Y=y)}_{\text{joint distribution}}$$
$$= \sum_{x}\sum_{y} xy\, P(X=x)\cdot P(Y=y) \quad \leftarrow \text{by indep.}$$
$$= \sum_{x} x\, P(X=x) \sum_{y} y\, P(Y=y)$$
$$= E(X)\cdot E(Y)$$

4.) $$Cov(X,Y) = E([X-E(X)][Y-E(Y)])$$
$$= E\{XY - X\cdot E(Y) - Y\cdot E(X) + E(X)\cdot E(Y)\}$$

$$= E(XY) - E(X)E(Y) - E(X)\cdot E(Y) + E(X)E(Y)$$
$$= E(XY) - E(X)E(Y) \blacksquare$$

5.) $Cov(\sum_i a_i X_i + b_i, \sum_j c_j Y_j + d_j) = E[(\sum a_i X_i + b_i)(\sum c_j Y_j + d_j)] - E(\sum a_i X_i + b_i) \cdot E(\sum c_j Y_j + d_j)$

$$= E\left(\sum_i \sum_j a_i X_i c_j Y_j + a_i X_i d_j + b_i c_j Y_j + b_i d_j\right) - \ldots$$
$$\ldots - E(\sum a_i X_i + b_i) \cdot E(\sum c_j Y_j + d_j)$$
$$= \sum_i \sum_j (a_i c_j E(X_i Y_j) + a_i d_j E(X_i) + b_i c_j E(Y_j) + b_i d_j)$$
$$- \sum\sum [a_i E(X_i) + b_i][c_j E(Y_j) + d_j]$$
$$= \sum_i \sum_j a_i c_j E(X_i Y_j) + \cancel{a_i d_j E(X_i)} + \cancel{b_i c_j E(Y_j)} + \cancel{b_i d_j} - a_i c_j E(X_i) E(Y_j)$$
$$- \cancel{a_i d_j E(X_i)} - \cancel{b_i c_j E(Y_j)} - \cancel{b_i d_j}$$
$$= \sum_i \sum_j a_i c_j E(X_i, Y_j) - a_i c_j E(X_i) \cdot E(Y_j)$$
$$= \sum_i \sum_j a_i c_j Cov(X_i, Y_j) \blacksquare$$

6.) $Var(X) = E[(X - E(X))^2]$
$$= E[(X^2 - X \cdot E(X) - X \cdot E(X) + E(X)^2]$$
$$= E(X^2 - 2 \cdot X \cdot E(X) + E(X)^2]$$
$$= E(X^2) - 2E(X)^2 + E(X)^2$$
$$= E(X^2) - E(X)^2 \blacksquare$$

7.) $Var(X+Y) = E((X+Y)^2) - [E(X+Y)]^2$
$$= E(X^2 + 2XY + Y^2) - [E(X) + E(Y)]^2$$
$$= E(X^2) + 2 \cdot E(XY) + E(Y^2) - [E(X)^2 + 2E(X)E(Y) + E(Y)^2]$$
$$= [E(X^2) - E(X)^2] + [E(Y^2) - E(Y)^2] + 2E(XY) - 2E(X)E(Y)$$
$$= Var(X) + Var(Y) - 2Cov(X, Y) \blacksquare$$

8.) From Sheldon Ross "First Course in Probability"
$$0 \le Var\left(\frac{X}{\sigma_X} \pm \frac{Y}{\sigma_Y}\right) = 2 \pm 2 Corr(X, Y)$$

$2 + 2 Corr(X, Y) \ge 0$	$2 - 2 Corr(X, Y) \ge 0$
$2 Corr(X, Y) \ge -2$	$-2 Corr(X, Y) \ge -2$
$Corr(X, Y) \ge -1$	$Corr(X, Y) \le 1$ $\blacksquare$