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HW1 By Hand

2.11 Determinant definition for $A \in \mathbb{R}^{k \times k}$:

$$|A| = a_{11} \quad \text{if } k=1$$
$$|A| = \sum_{j=1}^{k} a_{1j} |A_{1j}| (-1)^{1+j} \quad \text{if } k>1$$

Proof:

$A \in \mathbb{R}^{p \times p}$ such that $A_{ij} \begin{cases} 0 & \text{if } i \neq j \\ a_{ij} & \text{if } i = j \end{cases}$

$$\det(A) = \sum_{j=1}^{p} a_{1j} |A_{1j}| (-1)^{1+j}$$

$j=1$

$$\det(A) = (-1)^{1+1} a_{11} \det(A_{11}) + 0 + \dots + 0$$
$$= a_{11} \det(A_{11})$$

$j=2$

$$\det(A) = a_{11}[a_{22} \det(A_{22}) + 0 + \dots + 0]$$
$$= a_{11} a_{22} \det(A_{22})$$

$\vdots$

$j = p-1$

$$\det(A) = a_{11} \cdot a_{22} \cdot \dots \cdot a_{p-1} \cdot \det(A_{p-1p-1})$$

$- A_{p-1p-1} \in \mathbb{R}^{1 \times 1}$ so we use the determinant definition for $k=1$, such that

$$\det(A) = a_{11} \cdot a_{22} \cdot \dots \cdot a_{p-1p-1} \cdot a_{pp} \quad \blacksquare$$

$\det(I) = 1$

P. 2.12

$$A = P \Lambda P'$$
$$\det(A) = \det(P) \cdot \det(\Lambda) \cdot \det(P')$$
$$\det(A) = \det(P) \cdot \det(P)' \cdot \det(\Lambda)$$
$$= \det(PP') \cdot \det(\Lambda)$$
$$= \det(I) \cdot \det(\Lambda)$$
$$= \det(\Lambda)$$

By 2.11 proof, determinant of diagonal is product of entries

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$$

Problem 9 P. 3.16

$$\Sigma_v = E[(V - \mu_v)(V - \mu_v)']$$
$$= E[VV' - V\mu_v' - \mu_v V' + \mu_v \mu_v']$$
$$= E(VV') - E(V\mu_v') - E(\mu_v V') + E(\mu_v \mu_v')$$
$$= E(VV') - \mu_v' E(V) - \mu_v E(V') + \mu_v \mu_v'$$
$$= E(V, V') - \mu_v' \mu_v - \mu_v \mu_v' + \mu_v \mu_v'$$
$$\Sigma_v = E(V, V') - \mu_v \mu_v'$$
$$\Sigma_v + \mu_v \mu_v' = E(V, V') \quad \blacksquare$$

Problem 11 P 4.2

a.) $f(x) = \dfrac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left\{-\dfrac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right\}$

$\mu = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 2 & \sqrt{2}/2 \\ \sqrt{2}/2 & 1 \end{bmatrix}$

$$\sigma_{12} = \rho_{12} \sqrt{\sigma_{11}} \sqrt{\sigma_{22}}$$
$$= 0.5 \cdot \sqrt{2} \cdot \sqrt{1}$$
$$= \sqrt{2}/2$$

$|\Sigma| = 2 \cdot 1 - \left(\dfrac{\sqrt{2}}{2}\right)^2 = 3/2$

$p = 2$

$|\Sigma|^{1/2} = \sqrt{3/2} \qquad \Sigma^{-1} = \dfrac{2}{3} \begin{bmatrix} 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & 2 \end{bmatrix}$

$$f(x) = \frac{1}{(2\pi)^{2/2} \sqrt{3/2}} \exp\left\{-\frac{1}{2} (x_1 - 0, x_2 - 2) \frac{2}{3} \begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 2 \end{pmatrix} \begin{pmatrix} x_1 - 0 \\ x_2 - 2 \end{pmatrix}\right\}$$

$$= \frac{1}{\sqrt{6\pi}} \exp\left\{-\frac{1}{2}\cdot\frac{1}{3}\left(x_1^2 - \sqrt{2}\,x_1(x_2-2) + 2(x_2-2)^2\right)\right\}$$

b.) $$(x-\mu)^T \Sigma^{-1}(x-\mu) = (x_1 \;\; \cancel{x_2} - 2)\, \tfrac{2}{3} \begin{pmatrix} 1 & -\sqrt{2}/2 \\ -\frac{\sqrt{2}}{2} & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 - 2 \end{pmatrix}$$

$$= \frac{2}{3}\left(x_1^2 - \sqrt{2}\,x_1(x_2-2) + 2(x_2-2)^2\right)$$

P 12 4.13

a.) $$|\Sigma| = \left| \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \right|$$

From 4.11 hint

$$= \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ \Sigma_{21} - \Sigma_{22}\Sigma_{22}^{-1}\Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}$$

Now determinant

$$= \det \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}$$

By 4.10

$$\det|\Sigma| = \det \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{bmatrix} = \det \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

$$= \det[\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}] \cdot \det[\Sigma_{22}]$$

b) From hint

$$(x-\mu)'\Sigma^{-1}(x-\mu) = \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}' \begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix}' \begin{bmatrix} (\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{bmatrix}$$

$$\times \begin{bmatrix} x_1-\mu_1-\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2) \\ x_2-\mu_2 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1-\mu_1)'I-(x_2-\mu_2)'\Sigma_{22}^{-1}\Sigma_{21} \\ (x_2-\mu_2)' \end{bmatrix} \times \begin{bmatrix} (\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{bmatrix}$$

$$\times \begin{bmatrix} x_1-\mu_1-\Sigma_{12}\Sigma_{22}(x_2-\mu_2) \\ (x_2-\mu_2) \end{bmatrix}$$

$$= \begin{bmatrix} (x_1-\mu_1)'I-(x_2-\mu_2)'\Sigma_{22}^{-1}\Sigma_{21}\cdot(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} \\ (x_2-\mu_2)'\Sigma_{22}^{-1} \end{bmatrix} \times$$

$$= [(x_1-\mu_1)-\Sigma_{22}^{-1}\Sigma_{21}(x_2-\mu_2)]' \times (\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}[x_1-\mu_1-\ldots$$
$$\ldots-\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)] + (x_2-\mu_2)'\Sigma_{22}^{-1}(x_2-\mu_2)$$

c) $E(X_1|X_2=x_2) = \mu_1+(x_2-\mu_2)\Sigma_{12}\Sigma_{22}^{-1}$

$Var(X_1|X_2=x_2) = \Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

$E(X_2) = \mu_2$

$Var(X_2) = \Sigma_{22}$

$X_2 \sim N(\mu_2, \Sigma_{22})$

$(X_1|X_2=x_2) \sim N(\mu_1+(x_2-\mu_2)\Sigma_{12}\Sigma_{22}^{-1}, \Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$.