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Sta 206
11 November 2022

Homework 6

2.) a.) $\hat{\beta}_k^* = \sqrt{n-1}\,(s_y) \times (r_k)$ for standardized model

$$r_k = \frac{\frac{1}{n-1}\sum_{i=1}^{n}(X_{ik}-\bar{X}_k)(Y_i-\bar{Y})}{s_k s_y}$$

In the standardized regression model, we can see that the value of $\hat{\beta}_{k_i}^*$ has no dependence on the other values of $\hat{\beta}_{k_j}^*$ where $i \neq j$. So regardless of how many terms our in model, the regression coefficients do not change.

For the original model, we have: $\hat{\beta}^* = \begin{pmatrix} \bar{Y} \\ \sqrt{n-1}\, s_y r_1 \\ \vdots \\ \sqrt{n-1}\, s_y r_{p-1} \end{pmatrix}$

$$\hat{\beta}_k = \frac{1}{\sqrt{n-1}\, s_{x_k}} \hat{\beta}_k^*$$

$$\hat{\beta}_k = \frac{1}{\sqrt{n-1}\, s_{x_k}} \cdot \sqrt{n-1}\, s_y r_k$$

$$\hat{\beta}_k = \frac{s_y}{s_{x_k}} \cdot r_{yk}$$

$\hat{\beta}_k = \hat{\beta}_k$ ∎ Thus, the coefficient remains uninfluenced for each $\hat{\beta}_{k_i}$ by any other $\hat{\beta}_{k_j}$ $i \neq j$.

2b.) $SSR(X_j \mid X_{(-j)}) = SSR(X_j)$

$SSR(X_j \mid X_{(-j)}) = SSE(X_{(-j)}) - SSE(X_j, X_{(-j)})$

$$SSE(X_I) = SSR(X_j \mid X_I) + SSE(X_1, X_2)$$

$X_j = X_2$

$X_I = X_1$

$$\boxed{X_I = X_{-j}}$$

b. $SSR(X_j \mid X_I) = SSE(X_I) - SSE(X_I, X_j)$

$= SSE(X_I^*) - SSE(X_I^*, X_j^*)$

$= Y'(I - H(X_I^*))Y - Y'(I - H(X_j^*, X_I))Y$

$H(x) = X(X^T X)^{-1} X^T$

$x = 1 \; X_j \; X_I$

$= (1 \; X_j^* X_I^*) \begin{pmatrix} n & 0 \\ 0 & I_{p-1} \end{pmatrix} \begin{pmatrix} 1 \\ X_j^* \\ X_I^* \end{pmatrix}$

$= \frac{1}{n} 1 1^T + X_I^* X_I^{*T} + X_j^* X_j^{*T}$

$= \begin{pmatrix} 1 \\ X_I^* \end{pmatrix} (1 \; X_I^{*T})$

$= \begin{pmatrix} n & 0 \\ 0 & I_{p-1} \end{pmatrix}$

$= (1 \; X_I^*) \begin{pmatrix} n & 0 \\ 0 & I_{p-1} \end{pmatrix} \begin{pmatrix} 1 \\ X_I^* \end{pmatrix}$

$= \frac{1}{n} 1 1^T + X_I^* X_I^{*T}$

$= Y'(\cancel{I} - \cancel{\tfrac{1}{n} 1 1^T} - \cancel{X_I^* X_I^{*T}})Y - Y^T(\cancel{I} - \cancel{\tfrac{1}{n} 1 1^T} - \cancel{X_I^* X_I^{*T}} - X_j^* X_j^{*T})Y$

$= Y^T (X_j^* X_j^{*T}) Y$

$SSR(X_j^*) = Y'(H(X_j^*) - \frac{1}{n} J_n)Y$

$= Y'(\frac{1}{n} 1 1^T + X_j^* X_j^{*T} - \frac{1}{n} J_n)Y$

$= Y'(X_j^* X_j^{*T})Y \; \blacksquare$

3.) $VIF_1 = VIF_2 = \frac{1}{1-R_1^2} = \frac{1}{1-R_2^2}$

The kth diagonal element of the inverse correlation matrix r_{XX}^{-1} is the VIF

$$r_{XX} = \begin{bmatrix} 1 & r_{12} \\ r_{21} & 1 \end{bmatrix} \qquad r_{XX}^{-1} = \frac{1}{1-(r_{12}\cdot r_{21})} \begin{bmatrix} 1 & -r_{12} \\ -r_{21} & 1 \end{bmatrix}$$

$$r_{XX}^{-1} = \begin{bmatrix} \frac{1}{(1-r_{12}\cdot r_{21})} & -r_{12} \\ -r_{21} & \frac{1}{1-(r_{12}\cdot r_{21})} \end{bmatrix}$$

$$r_{XX}^{-1} = \begin{bmatrix} \frac{1}{(1-r_{12}^2)} & -r_{12} \\ -r_{21} & \frac{1}{(1-r_{12}^2)} \end{bmatrix}$$

$$r_{XX}^{-1} = \begin{bmatrix} \frac{1}{1-R_1^2} & -r_{12} \\ -r_{21} & \frac{1}{1-R_2^2} \end{bmatrix}$$ By hint...

We know that the VIF is the kth diagonal element, thus the proof is complete.