Introduction to Dependent Types

Eagan Technology Unconference

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Section Outline

Quick Question

How many are familiar with this topic?

A Joke

This is not a `m-` tutorial.

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Nor is it a `lens` tutorial

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Nor is it a `lens` tutorial (aka the new new `m-` tutorial...

A Joke

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Nor is it a `lens` tutorial (aka the new new `m-` tutorial...
...because `arrows` *were* the new `m-` tutorials).

About This Talk

`Agda`, `Idris`, `Coq` and co* have full support for dependent types.

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Because of that, it's harder to see the build up, so we won't be directly using them in this talk.

Honestly though, it's because they're way over my head :(

(*) There was another mini joke here...

About This Talk

But we will be using `Haskell` though :)

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It's not truely dependent, but we can do more and more with each `language extension` that comes along.

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But we will be using `Haskell` though :)

It's not truely dependent, but we can do more and more with each `language extension` that comes along.

For the examples, there will also be *very* loose translation to imperative/OOP. Though please keep in mind that these are merely syntax translations, the actual concepts can differ vastly.

Section Outline

Values and Types

Values has Types, or Values are classified by Types.

```
..., -1, 0, 1, 2, 3, ... :: Int
```

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True, False :: Bool
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Values and Types

Values has Types, or Values are classified by Types.

```
..., -1, 0, 1, 2, 3, ... :: Int

True, False :: Bool

'a', 'b', 'c' :: Char

"abc" :: String ~ [Char]
```

Values are also called Terms

About Types

How are data types defined?

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- Some are built in magic: Int, Char, functions

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- Some are built in magic: Int, Char, functions
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    - We can define equivalent non-sugar version ourselves

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How are data types defined?

- Some are built in magic: Int, Char, functions
- Some are built in sugar: list, tuples
    - We can define equivalent non-sugar version ourselves
- Rest can be user defined: Bool, String, Maybe

About Types

What are the data types like?

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- Multiple Value constructors

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- Multiple Value constructors
- Paremetrize over another type

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About Types

What are the data types like?

- Multiple Value constructors
- Paremetrize over another type
- Recursive definition
- Synonyms of other types
- A combination of the above

Defining Data Types

Define new data type with data.

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- Left hand side (LHS) - Type constructor
- Right hand side (RHS) - Value constructor

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- Left hand side (LHS) - Type constructor
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Type and Value constructors are capticalized.

Our First Example!

Define a person:

```
-- | params for firstname, lastname, age respectively
data Person = Person String String Int
```

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A loose translation:

```
enum Person {
  Person(String firstname, String lastname, Int age)
}
```

Our First Example!

Define a person:

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A loose translation:

```
enum Person {
  Person(String firstname, String lastname, Int age)
}
```

In this example, the Type and Value constructor have the same name. The Type of the Person constructor:

```
Person :: String -> String -> Int -> Person

bobby :: Person
bobby = Person "Bobby" "Smith" 23
```

Our First Example!

Define a person:

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-- | params for firstname, lastname, age respectively
data Person = Person String String Int
```

A loose translation:

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enum Person {
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}
```

In this example, the Type and Value constructor have the same name. The Type of the Person constructor:

```
Person :: String -> String -> Int -> Person

bobby :: Person
bobby = Person "Bobby" "Smith" 23

-- a loose translation:
Person bobby = new Person("Bobby", "Smith", 23)
```

Multiple Value Constructors

Data can have multiple Value constructors:

```
data Bool = False | True

data Weekdays = Sunday | Monday | Tuesday | Wednesday
              | Thursday | Friday | Saturday
```

Does this remind you of anything?

Multiple Value Constructors

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data Bool = False | True

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```

Does this remind you of anything?

A loose translation:

```
enum Bool { False, True }

enum Weekdays {
  Sunday, Monday, Tuesday, Wednesday, Thursday, Friday,
        Saturday
}
```

Multiple Value Constructor

You can do type aliasing with type:

```
type Side = Double
type Radius = Double
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For example:

```
data Shape = Triangle Side Side Side
           | Rectangle Side Side
           | Circle Radius
```

Multiple Value Constructor

You can do type aliasing with type:

```
type Side = Double
type Radius = Double
```

For example:

```
data Shape = Triangle Side Side Side
           | Rectangle Side Side
           | Circle Radius
```

A loose translation:

```
enum Shape {
  Triangle(Double side1, Double side2, Double side3),
  Rectangle(Double length, Double width),
  Circle(Double radius)
}
```

Multiple Value Constructor

Recall Side $\sim$ Radius $\sim$ Double:

```
data Shape = Triangle Side Side Side
           | Rectangle Side Side
           | Circle Radius
```

Multiple Value Constructor

Recall Side ~ Radius ~ Double:

```
data Shape = Triangle Side Side Side
           | Rectangle Side Side
           | Circle Radius
```

Types of the Value constructors:

```
Triangle  :: Side -> Side -> Side -> Shape
Rectangle :: Side -> Side -> Shape
Circle    :: Radius -> Shape
```

Multiple Value Constructor

Recall Side ∼ Radius ∼ Double:

```
data Shape = Triangle Side Side Side
           | Rectangle Side Side
           | Circle Radius
```

Types of the Value constructors:

```
Triangle  :: Side -> Side -> Side -> Shape
Rectangle :: Side -> Side -> Shape
Circle    :: Radius -> Shape
```

Example Shapes:

```
myTri, myRect, myCir :: Shape
myTri  = Triangle 2.1 3.2 5
myRect = Rectangle 4 4
myCir  = Circle 7.2
```

Parametrization

Types can parametrize over another type:

```
data Identity a = Identity a
```

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A loose translation:

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enum Identity<A> {
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```
data Identity a = Identity a
```

A loose translation:

```
enum Identity<A> {
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}
```

The Type of the Identity constructor:

```
Identity :: a -> Identity a

intIdwrtSum :: Indentity Int
intIdwrtSum = Identity 0
```

Tuple

Parametrize over 2 types - 2-tuple!

```
data Tuple a b = Tuple a b
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```

A loose translation:

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enum Tuple<A, B> {
  Tuple(A a, B b)
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data Tuple a b = Tuple a b
```

A loose translation:

```
enum Tuple<A, B> {
  Tuple(A a, B b)
}
```

With:

```
Tuple :: a -> b -> Tuple a b
```

Tuple

Actual built-in sugar:

```
   data Tuple a b = Tuple a b
=> data (,) a b = (,) a b
=> data (a, b) = (a, b)
```

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   data Tuple a b = Tuple a b
=> data (,) a b = (,) a b
=> data (a, b) = (a, b)
```

An example:

```
type Employed = Bool

barbara, chet, luffy :: (Person, Employed)
barbara = (Person "Barbara" "Sakura" 30, True)
chet    = (Person "Chet" "Awesome-Laser" 2, False)
luffy   = (Person "Luffy D." "Monkey" 19, False)
```

Maybe

Like Bool, but parametrize an *a* over the True part:

```
data Maybe a = Nothing | Just a
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A loose translation:

```
enum Maybe<A> {
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}
```

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Like Bool, but parametrize an *a* over the True part:

```
data Maybe a = Nothing | Just a
```

A loose translation:

```
enum Maybe<A> {
  Nothing,
  Just(A a)
}
```

The Types of the two Value constructors:

```
Nothing :: Maybe a
Just    :: a -> Maybe a
```

Maybe

From previous slide:

```
data Maybe a = Nothing | Just a
```

Maybe

From previous slide:

```
data Maybe a = Nothing | Just a
```

Say more with the occupation:

```
type Occupation = Maybe String

barbara, chet, luffy :: (Person, Occupation)
barbara = (Person "Barbara" "Sakura" 30, Just "dancer")
chet    = (Person "Chet" "Awesome-Laser" 2, Nothing)
luffy   = (Person "Luffy D." "Monkey" 19, Just "pirate")
```

Either

Like Bool, but parametrize over both True and False:

```
data Either a b = Left a | Right b
```

Either

Like Bool, but parametrize over both True and False:

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data Either a b = Left a | Right b
```

A loose translation:

```
enum Either<A, B> {
  Left(A a),
  Right(B b)
}
```

Either

Like Bool, but parametrize over both True and False:

```
data Either a b = Left a | Right b
```

A loose translation:

```
enum Either<A, B> {
  Left(A a),
  Right(B b)
}
```

The two Value constructors have Types:

```
Left  :: a -> Either a b
Right :: b -> Either a b
```

Either

From previous slide:

```
data Either a b = Left a | Right b
```

Either

From previous slide:

```
data Either a b = Left a | Right b
```

Refine with more details:

```
type Earning = Either String Int

barbara, chet, luffy :: (Person, Earning)
barbara = (Person "Barbara" "Sakura" 30,
          Right 100000)
chet    = (Person "Chet" "Awesome-Laser" 2,
          Left "Is a baby")
luffy   = (Person "Luffy D." "Monkey" 19,
          Right 2000000)
```

Types with Recursion

Natural number:

```
data Nat = Z | S Nat

Z :: Nat
S :: Nat -> Nat
```

Types with Recursion

Natural number:

```
data Nat = Z | S Nat

Z :: Nat
S :: Nat -> Nat
```

A loose translation:

```
enum Nat {
  Z,
  S(Nat n)
}
```

Types with Recursion

Natural number:

```
data Nat = Z | S Nat

Z :: Nat
S :: Nat -> Nat

0 ∼ Z
1 ∼ S Z
2 ∼ S (S Z)
3 ∼ S (S (S Z))
```

Types with Recursion

List - recursive type while parametrize over another type:

```
data List a = Nil | Cons a (List a)

Nil :: List a
Cons :: a -> List a -> List a
```

Types with Recursion

List - recursive type while parametrize over another type:

```
data List a = Nil | Cons a (List a)

Nil :: List a
Cons :: a -> List a -> List a
```

A loose translation:

```
enum List<A> {
  Nil,
  Cons(A a, List<A> as)
}
```

Types with Recursion

Actual built-in sugar is something like:

```
   data List a = Nil | Cons a (List a)
=> data [] a = [] | (:) a ([] a)
=> data [a] = [] | (:) a [a]
```

Types with Recursion

Actual built-in sugar is something like:

```
   data List a = Nil | Cons a (List a)
=> data [] a = [] | (:) a ([] a)
=> data [a] = [] | (:) a [a]
```

De-sugar that list:

```
ints :: List Int
ints = Cons 1 (Cons 2 (Cons 3 (Cons 4 Nil)))

-- built-in sugar
ints :: [] Int
ints = 1 : 2 : 3 : 4 : []

-- 2x the sugar!
ints :: [Int]
ints = [1, 2, 3, 4]
```

Functions

Maps Values of a Type to another Type:

```
even :: Int -> Bool
even 0 = True
even n = if rem n 2 == 0
         then True
         else False
```

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Maps Values of a Type to another Type:

```
even :: Int -> Bool
even 0 = True
even n = if rem n 2 == 0
         then True
         else False
```

Not as loose translation:

```
Bool even (Int n) {
  switch n:
    case n == 0:
      return True;
    default:
      if rem(n, 2) == 0
        return True;
      else
        return False;
}
```

Functions with Recursion

Use recursion for recursive types:

```
toInt :: Nat -> Int
toInt Z     = 0
toInt (S n) = 1 + toInt n
```

Functions with Recursion

Use recursion for recursive types:

```
toInt :: Nat -> Int
toInt Z     = 0
toInt (S n) = 1 + toInt n
```

Not as loose translation:

```
Int toInt (Nat n) {
  switch n:
    case Z:
      return 0;
    case (S m):   -- n ~ (S m)
      return 1 + toInt(m);
}
```

Functions with Recursion

Use recursion for recursive types:

```
toInt :: Nat -> Int
toInt Z     = 0
toInt (S n) = 1 + toInt n
```

Evaluation is a series of substitutions:

```
three = ∼ S (S (S Z)) :: Nat

   toInt three :: [Int]
=  toInt (S (S (S Z)))
= 1 + toInt (S (S Z))
= 1 + 1 + toInt (S Z)
= 1 + 1 + 1 + toInt Z
= 1 + 1 + 1 + 0
= 1 + 1 + 1
= 1 + 2
= 3
```

Functions with Parametric Polymorphism

Functions can be parametric:

```
id :: a -> a
id x = x
```

Functions with Parametric Polymorphism

Functions can be parametric:

```
id :: a -> a
id x = x
```

Not as loose translation:

```
A id<A>(A a) {
  return a;
}
```

Functions with Parametric Polymorphism

Functions can be parametric:

```
append :: [a] -> [a] -> [a]
append []     ys = ys
append (x:xs) ys = x : append xs ys
```

Functions with Parametric Polymorphism

Functions can be parametric:

```
append :: [a] -> [a] -> [a]
append []     ys = ys
append (x:xs) ys = x : append xs ys
```

A translation:

```
List<A> append(List<A> l1, List<A> l2) {
  switch l1:
    case Nil:
      return l2;
    case Cons(x, xs):
      List<A> rest = append(xs, l2);
      return Cons(x, rest);
}
```

Functions with Parametric Polymorphism

Functions can be parametric:

```
append :: [a] -> [a] -> [a]
append []     ys = ys
append (x:xs) ys = x : append xs ys
```

Evaluation is a series of substitutions:

```
xs = [4, 8] = 4 : 8 : [] :: [Int]
ys = [15, 16, 23, 42] = 15 : 16 : 23 : 42 : [] :: [Int]

  append xs ys :: [Int]
= append [4, 8] [15, 16, 23, 42]
= 4 : append [8] [15, 16, 23, 42]
= 4 : 8 : append [] [15, 16, 23, 42]
= 4 : 8 : [15, 16, 23, 42]
= 4 : [8, 15, 16, 23, 42]
= [4, 8, 15, 16, 23, 42]
```

Higher-order Functions

Functions that take functions as params:

```
-- actual name is ($)
apply :: (a -> b) -> a -> b
apply f x = f x

-- acutal name is (.)
compose :: (b -> c) -> (a -> b) -> (a -> c)
compose f g = \x -> f (g x)
```

Higher-order Functions

Functions that take functions as params:

```
-- actual name is ($)
apply :: (a -> b) -> a -> b
apply f x = f x

-- acutal name is (.)
compose :: (b -> c) -> (a -> b) -> (a -> c)
compose f g = \x -> f (g x)
```

Yay translations:

```
B apply(Func<A,B> f, A a) {
  return f(a);
}

Func<A,C> compose(Func<B,C> f, Func<A,B> g) {
  return x => f(g(x));
}
```

More Functions Examples

map:

```
map :: (a -> b) -> [a] -> [b]
map f []      = []
map f (x:xs) = f x : map f xs
```

More Functions Examples

map:

```
map :: (a -> b) -> [a] -> [b]
map f []       = []
map f (x:xs) = f x : map f xs
```

A translation:

```
List<B> map(Func<A,B> f, List<A> la) {
  switch la:
    case Nil:
      return Nil;
    case Cons(a, as):
      B b = f(a)
      List<B> rest = map(f, as);
      return Cons(b, rest);
}
```

More Functions Examples

map:

```
map :: (a -> b) -> [a] -> [b]
map f []     = []
map f (x:xs) = f x : map f xs
```

Evaluation is a series of substitutions:

```
xs = [4, 8, 15, 16, 23, 42] :: [Int]
even :: Int -> Bool

  map even xs :: [Bool]
= map even [4, 8, 15, 16, 23, 42]
= even 4 : map even [8, 15, 16, 23, 42]
= True : even 8 : map even [15, 16, 23, 42]
= True : True : even 15 : map even [16, 23, 42]
= True : True : False : even 16 : map even [23, 42]
= True : True : False : True : even 23 : map even [42]
= True : True : False : True : False : even 24 : map even
    []
= True : True : False : True : False : True : []
= [True, True, False, True, False, True]
```

More Functions Examples

zip:

```
zip :: [a] -> [b] -> [(a,b)]
zip []      ys      = []
zip xs      []      = []
zip (x:xs) (y:ys) = (x,y) : zip xs ys
```

More Functions Examples

zip:

```
zip :: [a] -> [b] -> [(a,b)]
zip []     ys     = []
zip xs     []     = []
zip (x:xs) (y:ys) = (x,y) : zip xs ys
```

A translation:

```
List<Tuple<A,B>> zip(List<A> l1, List<A> l2) {
  switch l1:
    case Nil:
      return Nil;
    case Cons(a, as):
      switch l2:
        case Nil:
          return Nil;
        case Cons(b, bs):
          Tuple<A,B> front = Tuple(a, b);
          List<Tuple<A,B>> rest = zip(as, bs);
          return Cons(front, rest);
}
```

More Functions Examples

zip:

```
zip :: [a] -> [b] -> [(a,b)]
zip []       ys      = []
zip xs       []      = []
zip (x:xs) (y:ys) = (x,y) : zip xs ys
```

Evaluation is a series of substitutions:

```
xs = ['a', 'b', 'c'] :: [Char]
ys = [1, 2, 3, 4] :: [Int]

  zip xs ys :: [(Char, Int)]
= zip ['a', 'b', 'c'] [1, 2, 3, 4]
= ('a', 1) : zip ['b', 'c'] [2, 3, 4]
= ('a', 1) : ('b', 2) : zip ['c'] [3, 4]
= ('a', 1) : ('b', 2) : ('c', 3) : zip [] [4]
= ('a', 1) : ('b', 2) : ('c', 3) : []
= [('a', 1), ('b', 2), ('c', 3)]
```

Section Outline

3 What is Dependent Type
- λ-Calculus
- Extensions on λ-calculus

λ-Calculus

So far, we have seen:

λ-Calculus

So far, we have seen:

- function application

λ-Calculus

So far, we have seen:

- function application
- function abstraction (aka higher-order functions)

λ-Calculus

So far, we have seen:

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So far, we have seen:

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- function abstraction (aka higher-order functions)
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- substitution

λ-Calculus

So far, we have seen:

- function application
- function abstraction (aka higher-order functions)
- variable binding
- substitution

=> basis for simply typed λ-calculus.

λ-Calculus

Q: Sure, but can we have more?

λ-Calculus

Q: Sure, but can we have more?
A: Yes, extend λ-calculus so we can have more forms of abstractions.

λ-Calculus

Q: Sure, but can we have more?
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Q: But how?

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Q: Sure, but can we have more?
A: Yes, extend λ-calculus so we can have more forms of abstractions.

Q: But how?
A: What if I told you. . .

λ-Calculus

Q: Sure, but can we have more?
A: Yes, extend λ-calculus so we can have more forms of abstractions.

Q: But how?
A: What if I told you...
...you already know at least 2 axes of extension :)

Subtype Polymorphism

Given data types T and P, if there is a relation between T and P by some notion of substitutability with T in place of P, then we say T is a `subtype` of the `supertype` P, denoted by $T <: P$.

Subtype Polymorphism

Given data types T and P, if there is a relation between T and P by some notion of substitutability with T in place of P, then we say T is a `subtype` of the `supertype` P, denoted by $T <: P$.

The is an extension on λ-calculus with `subtype polymorphism` and is denoted by $\lambda_{<:}$.

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=> Object Oriented Programming.

Subtype Polymorphism

Given data types T and P, if there is a relation between T and P by some notion of substitutability with T in place of P, then we say T is a `subtype` of the `supertype` P, denoted by $T <: P$.

The is an extension on λ-calculus with `subtype polymorphism` and is denoted by $\lambda_{<:}$.
=> Object Oriented Programming.

Though this is not an axis that we are interested in.

Parametric Polymorphism

Introduce a mechanism of universal quantification over Types: Types can abstract over Types, allows for `generic data types` and `generic functions`.

Parametric Polymorphism

Introduce a mechanism of universal quantification over Types:
Types can abstract over Types, allows for `generic data types` and `generic functions`.
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Parametric Polymorphism

Introduce a mechanism of universal quantification over Types:
Types can abstract over Types, allows for `generic data types` and `generic functions`.
=> Generic Programming.

Recall:

```
data Maybe a = Nothing | Just a
data List a = Nil | Cons a List a

id :: a -> a
map :: (a -> b) -> [a] -> [b]
```

Parametric Polymorphism

Introduce a mechanism of universal quantification over Types: Types can abstract over Types, allows for `generic data types` and `generic functions`.
=> Generic Programming.

Recall:

```
data Maybe a = Nothing | Just a
data List a = Nil | Cons a List a

id :: a -> a
map :: (a -> b) -> [a] -> [b]
```

The name for this extension is formally `second order` λ`-calculus`, aka `System F`, denoted by $\lambda 2$,

Value and Type Interdependency

Re-thinking functions:

```
even :: Int -> Bool
even 0 = True
even n = if rem n 2 == 0
         then True
         else False
```

f maps numbers to True and False.

Value and Type Interdependency

Re-thinking functions:

```
even :: Int -> Bool
even 0 = True
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=> Values on RHS depends on the Values on LHS

Value and Type Interdependency

Re-thinking functions:

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even :: Int -> Bool
even 0 = True
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```

f maps numbers to True and False.

=> Values on RHS depends on the Values on LHS

=> Values depending on Values

Value and Type Interdependency

Re-thinking parametrized data types:

```
data Maybe a = Nothing | Just a

data List a = Nil | Cons a (List a)
```

Value and Type Interdependency

Re-thinking parametrized data types:

```
data Maybe a = Nothing | Just a

data List a = Nil | Cons a (List a)
```

Maybe and List take a Type and return Value constructors

Value and Type Interdependency

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Are we seeing a pattern yet?

Value and Type Interdependency

Then what about the other cases of dependencies?

Value and Type Interdependency

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- Values depending on Values: λ-calculus

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Value and Type Interdependency

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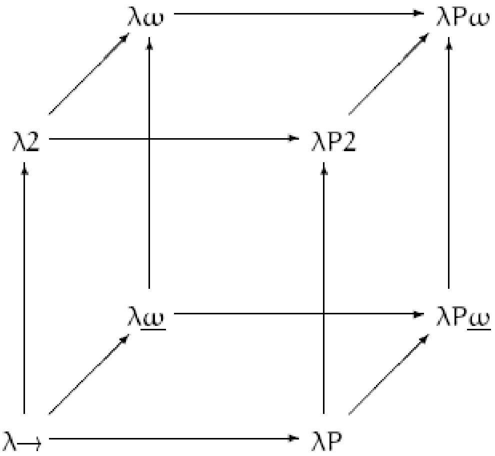
- Types depending on Types: $\lambda\underline{\omega}$
  => Type-level programming via type operators
- Types depending on Values: $\lambda\Pi$
  => Dependent types

Lambda Cube



System F_c

Currently, Haskell as of GHC 7.10.2 doesnot have true type operators. Achieves type-level programming through `type families` and equality on Types. This axis of extension on $\lambda 2$ is termed System F_c.

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Currently, Haskell as of GHC 7.10.2 doesnot have true type operators. Achieves type-level programming through `type families` and equality on Types. This axis of extension on $\lambda 2$ is termed System F_c.

Haskell is not truely dependent either because of the strong distinction between Values and Types. But with a handful of `language extensions` and the current Kind system, we are ready to fake dependent types in Haskell.

Teaser

Example please:

```
data Vec (n :: Nat) a where
  VNil :: Vec 0 a
  (:>) :: a -> Vec n a -> Vec (n + 1) a

vs :: Vec 6 Int
vs = 4 :> 8 :> 15 :> 16 :> 23 :> 42 :> VNil
```

Teaser

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vs :: Vec 6 Int
vs = 4 :> 8 :> 15 :> 16 :> 23 :> 42 :> VNil
```

Translation* please:

```
enum Vec<Nat n, A> {
  Vec<0, n> VNil,
  Vec<n + 1, n> VCons(A a, Vec<n, A> va)
}

Vec<6, Int> vs = VCons(4, VCons(8, VCons(15, VCons(16,
    VCons(23, VCons(42, VNil))))));
```

(*) supreme looseness and totally made-up syntax

Section Outline

Kinds

Q: Types classify Values, but what classifies Types?

Kinds

Q: Types classify Values, but what classifies Types?
A: Kinds

Introducing ⋆

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-- built-in magic: infinitely many value constructors
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Introducing Constraint

Haskell has *typeclasses* that very loosely resemble interfaces in OOP. A basic Typeclass consists of a collection of function signatures for a Type to implement. Afterward, this Typeclass instance can be used to provide `contexts` for functions.

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Eq    -- types that can be compared for equality
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```
Show  -- types that can be serialized to String
Eq    -- types that can be compared for equality
Ord   -- types that can be ordered
Num   -- types that are like numbers: +, -, *, ...
      -- and many others
```

Introducing Constraint

An example:

```
data Ordering = LT | EQ | GT

show      :: Show a => a -> String   -- toString()
(==)      :: Eq a => a -> a -> Bool
compare   :: Ord a => a -> a -> Ordering
(+)       :: Num a => a -> a -> a
sequenceA :: (Applicative f, Traversable t) => t (f a)
    -> f (t a)
```

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A loose translation with : for implements:

```
enum Ordering { LT, EQ, GT }

String show<A>(A a) where A : Show
Bool equals<A>(A a, A a) where A : Eq
Ordering compare<A>(A a, A a) where A : Ord
A plus<A>(A a, A a) where A : Num
F<T<_>> sequenceA<F,T>(T<F<_>> tfa) where F :
    Applicative, T : Traversable
```

Introducing Constraint

These Typeclass contexts have Kind Constraint.

```
Show :: * -> Constraint
Eq   :: * -> Constraint
Ord  :: * -> Constraint
Num  :: * -> Constraint
```

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```
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```
{-# LANGUAGE ConstraintKinds #-}
type ShowCxt a b = (Show a, Show b)

sameSerialization :: ShowContxt a b => a -> b -> Bool
sameSerialization a b = show a == show b

ShowCxt ::
```

Introducing Constraint

These Typeclass contexts have Kind Constraint.

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ShowCxt :: * -> * -> Constraint
```

Other Kinds

There are other Kinds aside from * and Constraint

```
import GHC.Prim

(*)            -- kind of fully realized type
(#)            -- kind of unboxed stuff used internally
Constraint     -- kind of constraints and type equality
OpenKind       -- superkind of (*) and (#)
AnyK           -- polymorphic kind for flexible arity
```

Other Kinds

There are other Kinds aside from * and Constraint

```
import GHC.Prim

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OpenKind      -- superkind of (*) and (#)
AnyK          -- polymorphic kind for flexible arity

-- the only sort, sorts classify kinds
(*), (#), Constraint, OpenKind, AnyK :: BOX
BOX :: BOX
```

All these Kinds are built-in and inferred as of GHC 7.10.2.

Language Extensions

Compiler extensions that enable a variety of new functionalities:

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- Syntax extension
- Type-level programming
- Generic programming
- FFI
- Type disambiguation
- Typeclass extension

Language Extensions

Compiler extensions that enable a variety of new functionalities:

- Syntax extension
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- Typeclass extension

Each extension has a name, and is enabled with the LANGUAGE pragma.

GADTs

Define data and explicit give type signatures to the Value constructors.

```
data Bool = False | True
data Maybe a = Nothing | Just a
data List a = Nil | Cons a (List a)
```

GADTs

Define data and explicit give type signatures to the Value constructors.

```
data Bool = False | True
data Maybe a = Nothing | Just a
data List a = Nil | Cons a (List a)
```

Becomes:

```
{-# LANGUAGE GADTs #-}
data Bool where
  False :: Bool
  True  :: Bool

data Maybe a where
  Nothing :: Maybe a
  Just    :: a -> Maybe a

data List a where
  Nil  :: List a
  Cons :: a -> List a -> List a
```

GADTs

Define data and explicit give type signatures to the Value constructors.

```
data Bool = False | True
data Maybe a = Nothing | Just a
data List a = Nil | Cons a (List a)
```

Loose translations:

```
enum Bool {
  Bool False,
  Bool True
}

emum Maybe<A> {
  Maybe<A> Nothing,
  Maybe<A> Just(A a)
}

enum List<A> {
  List<A> Nil,
  List<A> Cons(A a, List<A> as)
}
```

KindSignatures

Specify the Kind of the Type variables:

```
{-# LANGUAGE GADTs #-}
{-# LANGUAGE KindSignatures #-}
data Bool :: * where
  False :: Bool
  True  :: Bool

data Maybe :: * -> * where
  Nothing :: Maybe a
  Just    :: a -> Maybe a

data List :: * -> * where
  Nil  :: List a
  Cons :: a -> List a -> List a
```

DataKinds

Kinds are built-in; no user defined Kinds.

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Want Values at the Type level though!

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Want Values at the Type level though!

=> Data kind promotion :)

DataKinds

Example:

```
data Bool = False | True
```

With DataKinds, we get something like:

```
{-# LANGUAGE DataKinds #-}
```

Kind		Bool
Type	Bool	'True | 'False
Value	True | False	

DataKinds

Example:

```haskell
data Nat = Z | S Nat
```

With DataKinds, we get something like:

```haskell
{-# LANGUAGE DataKinds #-}
```

Kind		Nat
Type	Nat	'Z | 'S Nat
Value	Z | S Nat	

Example

Example with GADTs:

```
{-# LANGUAGE DataKinds #-}
{-# LANGUAGE GADTs #-}
data Bool = False | True

data TextInput a where
  RawText  :: String -> TextInput 'False
  SafeText :: String -> TextInput 'True

sanitize :: TextInput a -> TextInput 'True
sanitize (RawText str)  = SafeText (htmlEncode str)
sanitize (SafeText str) = SafeText str
```

Example

```
{-# LANGUAGE DataKinds #-}
{-# LANGUAGE GADTs #-}
data Bool = False | True

data TextInput a where
  RawText  :: String -> TextInput 'False
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sanitize :: TextInput a -> TextInput 'True
sanitize (RawText str)  = SafeText (htmlEncode str)
sanitize x              = x
```

Example

Translation*:

```
enum Bool {
  Bool False,
  Bool True
}

enum TextInput<Bool A> {
  TextInput<'False> RawText(String str),
  TextInput<'True> SafeText(String str)
}

TextInput<'True> sanitize(TextInput<A> input) {
  switch input:
    case RawText(str):
      return SafeText(htmlEncode(str));
    default:
      return input;
}
```

Type Families

Type Operators

Head of the List

`head` returns the first element of the list:

```
-- from standard library
head :: [a] -> a
head []     = error "empty list"
head (x:xs) = x
```

`Elm` updated to:

```
head :: [a] -> Maybe a
head []     = Nothing
head (x:xs) = Just x
```

Alternatively:

```
head :: [a] -> Either String a
head []     = Left "empty list"
head (x:xs) = Right x
```

Section Outline

5 Closing
- Beyond
- Questions

Beyond Dependent Types

- Total functional languages
  - termination and totality check
  - disallow partial functions
  - distinguish between data and codata
- Proof assistant languages

Questions?