

Introduction to Dependent Types

Eagan Technology Unconference

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Section Outline

1 Preface

Quick Question

How many are familiar with this topic?

A Joke

This is not a $\mathsf{m-}$ tutorial.

A Joke

This is not a `m-` tutorial. Nor is it a `lens` tutorial

A Joke

This is not a `m-` tutorial. Nor is it a `lens` tutorial (aka the new new `m-` tutorial...)

A Joke

This is not a `m-` tutorial. Nor is it a `lens` tutorial (aka the new new `m-` tutorial.....because arrows were the new `m-` tutorials).

About This Talk

Agda, Idris, Coq and co^* have full support for dependent types.

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Because of that, it's harder to see the build up, so we won't be directly using them in this talk.

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Agda, Idris, Coq and co^* have full support for dependent types.

Because of that, it's harder to see the build up, so we won't be directly using them in this talk.

Honestly though, it's because they're way over my head :(

() There was another mini joke here. . .*

About This Talk

But we will be using Haskell though :)

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It's not truly dependent, but we can do more and more with each language extension that comes along.

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But we will be using Haskell though :)

It's not truly dependent, but we can do more and more with each language extension that comes along.

For the examples, there will also be *very* loose translation to imperative/OOP. Though please keep in mind that these are merely syntax translations, the actual concepts can differ vastly.

Section Outline

- 2 Review of Basics
 - Values and Types
 - Defining Data Types
 - Functions

Values and Types

Values has Types, or Values are classified by Types.

..., -1, 0, 1, 2, 3, ... :: Int

Values and Types

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```
..., -1, 0, 1, 2, 3, ... :: Int
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```
True, False :: Bool
```

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```

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True, False :: Bool
```

```
'a', 'b', 'c' :: Char
```

Values and Types

Values has Types, or Values are classified by Types.

```
..., -1, 0, 1, 2, 3, ... :: Int
```

```
True, False :: Bool
```

```
'a', 'b', 'c' :: Char
```

```
"abc" :: String ~ [Char]
```

Values are also called Terms

About Types

How are data types defined?

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- Some are built in magic: `Int`, `Char`, functions

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 - We can define equivalent non-sugar version ourselves

About Types

How are data types defined?

- Some are built in magic: `Int`, `Char`, functions
- Some are built in sugar: list, tuples
 - We can define equivalent non-sugar version ourselves
- Rest can be user defined: `Bool`, `String`, `Maybe`

About Types

What are the data types like?

About Types

What are the data types like?

- Multiple **Value** constructors

About Types

What are the data types like?

- Multiple **Value** constructors
- Parametrize over another type

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- Multiple **Value** constructors
- Parametrize over another type
- Recursive definition

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- Synonyms of other types

About Types

What are the data types like?

- Multiple **Value** constructors
- Parametrize over another type
- Recursive definition
- Synonyms of other types
- A combination of the above

Defining Data Types

Define new data type with `data`.

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- Left hand side (`LHS`) - `Type` constructor
- Right hand side (`RHS`) - `Value` constructor

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Define new data type with `data`.

- Left hand side (`LHS`) - `Type` constructor
- Right hand side (`RHS`) - `Value` constructor

`Type` and `Value` constructors are capticalized.

Our First Example!

Define a person:

```
-- | params for firstname , lastname , age respectively  
data Person = Person String String Int
```

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A loose translation:

```
enum Person {  
  Person(String firstname, String lastname, Int age)  
}
```

Our First Example!

Define a person:

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data Person = Person String String Int
```

A loose translation:

```
enum Person {  
  Person(String firstname, String lastname, Int age)  
}
```

In this example, the **Type** and **Value** constructor have the same name. The **Type** of the **Person** constructor:

```
Person :: String -> String -> Int -> Person
```

```
bobby :: Person
```

```
bobby = Person "Bobby" "Smith" 23
```

Our First Example!

Define a person:

```
-- | params for firstname, lastname, age respectively
data Person = Person String String Int
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A loose translation:

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enum Person {
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In this example, the **Type** and **Value** constructor have the same name. The **Type** of the **Person** constructor:

```
Person :: String -> String -> Int -> Person
```

```
bobby :: Person
```

```
bobby = Person "Bobby" "Smith" 23
```

```
-- a loose translation:
```

```
Person bobby = new Person("Bobby", "Smith", 23)
```


Multiple Value Constructors

Data can have multiple **Value** constructors:

```
data Bool = False | True
```

```
data Weekdays = Sunday | Monday | Tuesday | Wednesday  
               | Thursday | Friday | Saturday
```

Does this remind you of anything?

Multiple Value Constructors

Data can have multiple **Value** constructors:

```
data Bool = False | True
```

```
data Weekdays = Sunday | Monday | Tuesday | Wednesday  
               | Thursday | Friday | Saturday
```

Does this remind you of anything?

A loose translation:

```
enum Bool { False, True }
```

```
enum Weekdays {  
    Sunday, Monday, Tuesday, Wednesday, Thursday, Friday,  
    Saturday  
}
```

Multiple Value Constructor

You can do type aliasing with `type`:

```
type Side = Double  
type Radius = Double
```

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For example:

```
data Shape = Triangle Side Side Side  
           | Rectangle Side Side  
           | Circle Radius
```

Multiple Value Constructor

You can do type aliasing with `type`:

```
type Side = Double  
type Radius = Double
```

For example:

```
data Shape = Triangle Side Side Side  
           | Rectangle Side Side  
           | Circle Radius
```

A loose translation:

```
enum Shape {  
  Triangle(Double side1, Double side2, Double side3),  
  Rectangle(Double length, Double width),  
  Circle(Double radius)  
}
```

Multiple Value Constructor

Recall $\text{Side} \sim \text{Radius} \sim \text{Double}$:

```
data Shape = Triangle Side Side Side
           | Rectangle Side Side
           | Circle Radius
```

Multiple Value Constructor

Recall `Side ~ Radius ~ Double`:

```
data Shape = Triangle Side Side Side
           | Rectangle Side Side
           | Circle Radius
```

Types of the `Value` constructors:

```
Triangle  :: Side -> Side -> Side -> Shape
Rectangle :: Side -> Side -> Shape
Circle    :: Radius -> Shape
```

Multiple Value Constructor

Recall `Side ~ Radius ~ Double`:

```
data Shape = Triangle Side Side Side
           | Rectangle Side Side
           | Circle Radius
```

Types of the `Value` constructors:

```
Triangle  :: Side -> Side -> Side -> Shape
Rectangle :: Side -> Side -> Shape
Circle    :: Radius -> Shape
```

Example `Shapes`:

```
myTri, myRect, myCir :: Shape
myTri  = Triangle 2.1 3.2 5
myRect = Rectangle 4 4
myCir  = Circle 7.2
```


Parametrization

Types can parametrize over another type:

```
data Identity a = Identity a
```

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A loose translation:

```
enum Identity<A> {  
  Identity(A a)  
}
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data Identity a = Identity a
```

A loose translation:

```
enum Identity<A> {  
  Identity(A a)  
}
```

The `Type` of the `Identity` constructor:

```
Identity :: a -> Identity a
```

```
intIdwrtSum :: Identity Int  
intIdwrtSum = Identity 0
```

Tuple

Parametrize over 2 types - 2-tuple!

```
data Tuple a b = Tuple a b
```

Tuple

Parametrize over 2 types - 2-tuple!

```
data Tuple a b = Tuple a b
```

A loose translation:

```
enum Tuple<A, B> {  
  Tuple(A a, B b)  
}
```

Tuple

Parametrize over 2 types - 2-tuple!

```
data Tuple a b = Tuple a b
```

A loose translation:

```
enum Tuple<A, B> {  
  Tuple(A a, B b)  
}
```

With:

```
Tuple :: a -> b -> Tuple a b
```

Tuple

Actual built-in sugar:

```
data Tuple a b = Tuple a b
=> data (,) a b = (,) a b
=> data (a, b) = (a, b)
```

Tuple

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```
data Tuple a b = Tuple a b
=> data (,) a b = (,) a b
=> data (a, b) = (a, b)
```

An example:

```
type Employed = Bool

barbara, chet, luffy :: (Person, Employed)
barbara = (Person "Barbara" "Sakura" 30, True)
chet    = (Person "Chet" "Awesome-Laser" 2, False)
luffy   = (Person "Luff D." "Monkey" 19, False)
```


Maybe

Like `Bool`, but parametrize an `a` over the `True` part:

```
data Maybe a = Nothing | Just a
```

Maybe

Like `Bool`, but parametrize an `a` over the `True` part:

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A loose translation:

```
enum Maybe<A> {  
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  Just(A a)  
}
```

Maybe

Like `Bool`, but parametrize an `a` over the `True` part:

```
data Maybe a = Nothing | Just a
```

A loose translation:

```
enum Maybe<A> {  
  Nothing,  
  Just(A a)  
}
```

The `Types` of the two `Value` constructors:

```
Nothing :: Maybe a  
Just    :: a -> Maybe a
```

Maybe

From previous slide:

```
data Maybe a = Nothing | Just a
```

Maybe

From previous slide:

```
data Maybe a = Nothing | Just a
```

Say more with the occupation:

```
type Occupation = Maybe String
```

```
barbara, chet, luffy :: (Person, Occupation)  
barbara = (Person "Barbara" "Sakura" 30, Just "dancer")  
chet    = (Person "Chet" "Awesome-Laser" 2, Nothing)  
luffy   = (Person "Luff D." "Monkey" 19, Just "pirate")
```

Either

Like `Bool`, but parametrize over both `True` and `False`:

```
data Either a b = Left a | Right b
```

Either

Like `Bool`, but parametrize over both `True` and `False`:

```
data Either a b = Left a | Right b
```

A loose translation:

```
enum Either<A, B> {  
  Left(A a),  
  Right(B b)  
}
```

Either

Like `Bool`, but parametrize over both `True` and `False`:

```
data Either a b = Left a | Right b
```

A loose translation:

```
enum Either<A, B> {  
  Left(A a),  
  Right(B b)  
}
```

The two `Value` constructors have `Types`:

```
Left  :: a -> Either a b  
Right :: b -> Either a b
```


Either

From previous slide:

```
data Either a b = Left a | Right b
```

Either

From previous slide:

```
data Either a b = Left a | Right b
```

Refine with more details:

```
type Earning = Either String Int
```

```
barbara, chet, luffy :: (Person, Earning)
barbara = (Person "Barbara" "Sakura" 30,
           Right 100000)
chet    = (Person "Chet" "Awesome-Laser" 2,
           Left "Is a baby")
luffy   = (Person "Luff D." "Monkey" 19,
           Right 2000000)
```

Types with Recursion

Natural number:

```
data Nat = Z | S Nat
```

```
Z :: Nat
```

```
S :: Nat -> Nat
```

Types with Recursion

Natural number:

```
data Nat = Z | S Nat
```

```
Z :: Nat
```

```
S :: Nat -> Nat
```

A loose translation:

```
enum Nat {  
  Z,  
  S(Nat n)  
}
```

Types with Recursion

Natural number:

```
data Nat = Z | S Nat
```

```
Z :: Nat
```

```
S :: Nat -> Nat
```

```
0 ~ Z
```

```
1 ~ S Z
```

```
2 ~ S (S Z)
```

```
3 ~ S (S (S Z))
```

Types with Recursion

List - recursive type while parametrize over another type:

```
data List a = Nil | Cons a (List a)
```

```
Nil :: List a
```

```
Cons :: a -> List a -> List a
```

Types with Recursion

List - recursive type while parametrize over another type:

```
data List a = Nil | Cons a (List a)
```

```
Nil :: List a
```

```
Cons :: a -> List a -> List a
```

A loose translation:

```
enum List<A> {  
  Nil,  
  Cons(A a, List<A> as)  
}
```

Types with Recursion

Actual built-in sugar is something like:

```
data List a = Nil | Cons a (List a)
=> data [] a = [] | (:) a ([] a)
=> data [a] = [] | (:) a [a]
```


Types with Recursion

Actual built-in sugar is something like:

```
data List a = Nil | Cons a (List a)
=> data [] a = [] | (:) a ([] a)
=> data [a] = [] | (:) a [a]
```

De-sugar that list:

```
ints :: List Int
ints = Cons 1 (Cons 2 (Cons 3 (Cons 4 Nil)))

-- built-in sugar
ints :: [] Int
ints = 1 : 2 : 3 : 4 : []

-- 2x the sugar!
ints :: [Int]
ints = [1, 2, 3, 4]
```

Functions

Maps **Values** of a **Type** to another **Type**:

```
even :: Int -> Bool
even 0 = True
even n = if rem n 2 == 0
         then True
         else False
```

Functions

Maps **Values** of a **Type** to another **Type**:

```
even :: Int -> Bool
even 0 = True
even n = if rem n 2 == 0
         then True
         else False
```

Not as loose translation:

```
Bool even (Int n) {
  switch n:
    case n == 0:
      return True;
  default:
    if rem(n, 2) == 0
      return True;
    else
      return False;
}
```

Functions with Recursion

Use recursion for recursive types:

```
toInt :: Nat -> Int
toInt Z      = 0
toInt (S n) = 1 + toInt n
```

Functions with Recursion

Use recursion for recursive types:

```
toInt :: Nat -> Int
toInt Z    = 0
toInt (S n) = 1 + toInt n
```

Not as loose translation:

```
Int toInt (Nat n) {
  switch n:
  case Z:
    return 0;
  case (S m): -- n ~ (S m)
    return 1 + toInt(m);
}
```

Functions with Recursion

Use recursion for recursive types:

```
toInt :: Nat -> Int
toInt Z      = 0
toInt (S n) = 1 + toInt n
```

Evaluation is a series of substitutions:

```
three = ~ S (S (S Z)) :: Nat
```

```
    toInt three :: [Int]
=   toInt (S (S (S Z)))
=  1 + toInt (S (S Z))
=  1 + 1 + toInt (S Z)
=  1 + 1 + 1 + toInt Z
=  1 + 1 + 1 + 0
=  1 + 1 + 1
=  1 + 2
=  3
```

Functions with Parametric Polymorphism

Functions can be parametric:

```
id :: a -> a  
id x = x
```

Functions with Parametric Polymorphism

Functions can be parametric:

```
id :: a -> a  
id x = x
```

Not as loose translation:

```
A id<A>(A a) {  
  return a;  
}
```


Functions with Parametric Polymorphism

Functions can be parametric:

```
append :: [a] -> [a] -> [a]
append []      ys = ys
append (x:xs) ys = x : append xs ys
```

Functions with Parametric Polymorphism

Functions can be parametric:

```
append :: [a] -> [a] -> [a]
append []      ys = ys
append (x:xs) ys = x : append xs ys
```

A translation:

```
List<A> append(List<A> l1, List<A> l2) {
  switch l1:
  case Nil:
    return l2;
  case Cons(x, xs):
    List<A> rest = append(xs, l2);
    return Cons(x, rest);
}
```

Functions with Parametric Polymorphism

Functions can be parametric:

```
append :: [a] -> [a] -> [a]
append []      ys = ys
append (x:xs) ys = x : append xs ys
```

Evaluation is a series of substitutions:

```
xs = [4, 8] = 4 : 8 : [] :: [Int]
ys = [15, 16, 23, 42] = 15 : 16 : 23 : 42 : [] :: [Int]

append xs ys :: [Int]
= append [4, 8] [15, 16, 23, 42]
= 4 : append [8] [15, 16, 23, 42]
= 4 : 8 : append [] [15, 16, 23, 42]
= 4 : 8 : [15, 16, 23, 42]
= 4 : [8, 15, 16, 23, 42]
= [4, 8, 15, 16, 23, 42]
```

Higher-order Functions

Functions that take functions as params:

```
-- actual name is ($)
apply :: (a -> b) -> a -> b
apply f x = f x

-- actual name is (.)
compose :: (b -> c) -> (a -> b) -> (a -> c)
compose f g = \x -> f (g x)
```

Higher-order Functions

Functions that take functions as params:

```
-- actual name is ($)
apply :: (a -> b) -> a -> b
apply f x = f x

-- actual name is (.)
compose :: (b -> c) -> (a -> b) -> (a -> c)
compose f g = \x -> f (g x)
```

Yay translations:

```
B apply(Func<A,B> f, A a) {
  return f(a);
}

Func<A,C> compose(Func<B,C> f, Func<A,B> g) {
  return x => f(g(x));
}
```

More Functions Examples

map:

```
map :: (a -> b) -> [a] -> [b]
map f []          = []
map f (x:xs)      = f x : map f xs
```

More Functions Examples

map:

```
map :: (a -> b) -> [a] -> [b]
map f []      = []
map f (x:xs) = f x : map f xs
```

A translation:

```
List<B> map(Func<A,B> f, List<A> la) {
  switch la:
  case Nil:
    return Nil;
  case Cons(a, as):
    B b = f(a)
    List<B> rest = map(f, as);
    return Cons(b, rest);
}
```

More Functions Examples

map:

```
map :: (a -> b) -> [a] -> [b]
map f []      = []
map f (x:xs) = f x : map f xs
```

Evaluation is a series of substitutions:

```
xs = [4, 8, 15, 16, 23, 42] :: [Int]
even :: Int -> Bool
```

```
map even xs :: [Bool]
= map even [4, 8, 15, 16, 23, 42]
= even 4 : map even [8, 15, 16, 23, 42]
= True : even 8 : map even [15, 16, 23, 42]
= True : True : even 15 : map even [16, 23, 42]
= True : True : False : even 16 : map even [23, 42]
= True : True : False : True : even 23 : map even [42]
= True : True : False : True : False : even 24 : map even
  []
= True : True : False : True : False : True : []
= [True, True, False, True, False, True]
```


More Functions Examples

zip:

```
zip :: [a] -> [b] -> [(a,b)]
zip []      ys      = []
zip xs      []      = []
zip (x:xs) (y:ys) = (x,y) : zip xs ys
```

More Functions Examples

zip:

```
zip :: [a] -> [b] -> [(a,b)]
zip []      ys      = []
zip xs      []      = []
zip (x:xs) (y:ys) = (x,y) : zip xs ys
```

A translation:

```
List<Tuple<A,B>> zip(List<A> l1, List<A> l2) {
  switch l1:
    case Nil:
      return Nil;
    case Cons(a, as):
      switch l2:
        case Nil:
          return Nil;
        case Cons(b, bs):
          Tuple<A,B> front = Tuple(a, b);
          List<Tuple<A,B>> rest = zip(as, bs);
          return Cons(front, rest);
}
```

More Functions Examples

zip:

```
zip :: [a] -> [b] -> [(a,b)]
zip []      ys      = []
zip xs      []      = []
zip (x:xs) (y:ys) = (x,y) : zip xs ys
```

Evaluation is a series of substitutions:

```
xs = ['a', 'b', 'c'] :: [Char]
ys = [1, 2, 3, 4] :: [Int]

zip xs ys :: [(Char, Int)]
= ('a', 1) : zip ['b', 'c'] [2, 3, 4]
= ('a', 1) : ('b', 2) : zip ['c'] [3, 4]
= ('a', 1) : ('b', 2) : ('c', 3) : zip [] [4]
= ('a', 1) : ('b', 2) : ('c', 3) : []
= [('a', 1), ('b', 2), ('c', 3)]
```

Section Outline

- 3 What is Dependent Type
 - λ -Calculus
 - Extensions on λ -calculus

λ -Calculus

So far, we have seen:

λ -Calculus

So far, we have seen:

- function application
- function abstraction (aka higher-order functions)

λ -Calculus

So far, we have seen:

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- function abstraction (aka higher-order functions)
- variable binding

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So far, we have seen:

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- function abstraction (aka higher-order functions)
- variable binding
- substitution

λ -Calculus

So far, we have seen:

- function application
- function abstraction (aka higher-order functions)
- variable binding
- substitution

\Rightarrow basis for simply typed λ -calculus.

λ -Calculus

Q: Sure, but can we have more?

λ -Calculus

Q: Sure, but can we have more?

A: Yes, extend λ -calculus so we can have more forms of abstractions.

λ -Calculus

Q: Sure, but can we have more?

A: Yes, extend λ -calculus so we can have more forms of abstractions.

Q: But how?

λ -Calculus

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A: What if I told you...

λ -Calculus

Q: Sure, but can we have more?

A: Yes, extend λ -calculus so we can have more forms of abstractions.

Q: But how?

A: What if I told you...

...you already know at least 2 axes of extension :)

Subtype Polymorphism

Given data types T and P , if there is a relation between T and P by some notion of substitutability with T in place of P , then we say T is a *subtype* of the *supertype* P , denoted by $T <: P$.

Subtype Polymorphism

Given data types T and P , if there is a relation between T and P by some notion of substitutability with T in place of P , then we say T is a *subtype* of the *supertype* P , denoted by $T <: P$.

There is an extension on λ -calculus with *subtype polymorphism* and is denoted by $\lambda_{<:}$.

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=> Object Oriented Programming.

Subtype Polymorphism

Given data types T and P , if there is a relation between T and P by some notion of substitutability with T in place of P , then we say T is a *subtype* of the *supertype* P , denoted by $T <: P$.

There is an extension on λ -calculus with *subtype polymorphism* and is denoted by $\lambda_{<:}$.

\Rightarrow Object Oriented Programming.

Though this is not an axis that we are interested in.

Parametric Polymorphism

Introduce a mechanism of universal quantification over **Types**:

Types can abstract over **Types**, allows for *generic data types* and *generic functions*.

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Introduce a mechanism of universal quantification over **Types**:

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=> Generic Programming.

Parametric Polymorphism

Introduce a mechanism of universal quantification over **Types**:

Types can abstract over **Types**, allows for *generic data types* and *generic functions*.

=> Generic Programming.

Recall:

```
data Maybe a = Nothing | Just a
data List a = Nil | Cons a List a
```

```
id :: a -> a
map :: (a -> b) -> [a] -> [b]
```

Parametric Polymorphism

Introduce a mechanism of universal quantification over **Types**:

Types can abstract over **Types**, allows for *generic data types* and *generic functions*.

=> Generic Programming.

Recall:

```
data Maybe a = Nothing | Just a
data List a = Nil | Cons a List a
```

```
id :: a -> a
map :: (a -> b) -> [a] -> [b]
```

The name for this extension is formally *second order λ -calculus*, aka *System F*, denoted by $\lambda 2$,

Value and Type Interdependency

Re-thinking functions:

```
even :: Int -> Bool
even 0 = True
even n = if rem n 2 == 0
         then True
         else False
```

f maps **numbers** to **True** and **False**.

Value and Type Interdependency

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=> **Values** on RHS depends on the **Values** on LHS

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=> **Values** depending on **Values**

Value and Type Interdependency

Re-thinking parametrized data types:

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Value and Type Interdependency

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Maybe and List take a Type and return Value constructors

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=> Values depending on Types

=> Parametric polymorphism of $\lambda 2$ again

Are we seeing a pattern yet?

Value and Type Interdependency

Then what about the other cases of dependencies?

Value and Type Interdependency

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Value and Type Interdependency

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- **Values** depending on **Values**: λ -calculus
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- **Types** depending on **Types**: λ_{ω}
=> Type-level programming via type operators

Value and Type Interdependency

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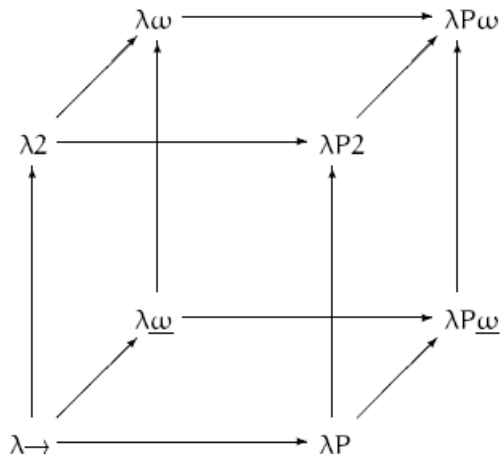
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Value and Type Interdependency

Then what about the other cases of dependencies?

- **Values** depending on **Values**: λ -calculus
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=> Type-level programming via type operators
- **Types** depending on **Values**: $\lambda\Pi$
=> Dependent types

Lambda Cube



System F_c

Currently, Haskell as of GHC 7.10.2 doesnot have true type operators. Achieves type-level programming through *type families* and equality on [Types](#). This axis of extension on $\lambda 2$ is termed System F_c .

System F_c

Currently, Haskell as of GHC 7.10.2 doesnot have true type operators. Achieves type-level programming through *type families* and equality on **Types**. This axis of extension on $\lambda 2$ is termed System F_c .

Haskell is not truely dependent either because of the strong distinction between **Values** and **Types**. But with a handful of *language extensions* and the current **Kind** system, we are ready to fake dependent types in Haskell.

Teaser

Example please:

```
data Vec (n :: Nat) a where
  VNil  :: Vec 0 a
  (:>) :: a -> Vec n a -> Vec (n + 1) a

vs :: Vec 6 Int
vs = 4 :> 8 :> 15 :> 16 :> 23 :> 42 :> VNil
```

Teaser

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  (:>) :: a -> Vec n a -> Vec (n + 1) a

vs :: Vec 6 Int
vs = 4 :> 8 :> 15 :> 16 :> 23 :> 42 :> VNil
```

Translation* please:

```
enum Vec<Nat n, A> {
  Vec<0, n> VNil,
  Vec<n + 1, n> VCons(A a, Vec<n, A> va)
}

Vec<6, Int> vs = VCons(4, VCons(8, VCons(15, VCons(16,
  VCons(23, VCons(42, VNil))))));
```

(*) *supreme looseness and totally made-up syntax*

Section Outline

- 4 Steps toward Dependent Types
 - Kinds
 - Language Extensions

Kinds

Q: **Types** classify **Values**, but what classifies **Types**?

A: **Kinds**

Introducing ★

```
-- built-in magic: infinitely many value constructors
data Int = ... | -1 | 0 | 1 | 2 | ...
data Bool = False | True
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```
Int :: *
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```
Int :: *
```

```
Bool ::
```


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Introducing Constraint

Haskell has *typeclasses* that very loosely resemble interfaces in OOP. A basic **Typeclass** consists of a collection of function signatures for a **Type** to implement. Afterward, this **Typeclass** instance can be used to provide *contexts* for functions.

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```
Show -- types that can be serialized to String
Eq   -- types that can be compared for equality
Ord  -- types that can be ordered
Num  -- types that are like numbers: +, -, *, ...
      -- and many others
```

Introducing Constraint

An example:

```
data Ordering = LT | EQ | GT

show      :: Show a => a -> String    -- toString()
(==)     :: Eq a  => a -> a -> Bool
compare  :: Ord a => a -> a -> Ordering
(+)      :: Num a => a -> a -> a
sequenceA :: (Applicative f, Traversable t) => t (f a)
          -> f (t a)
```

Introducing Constraint

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          -> f (t a)
```

A loose translation with `:` for implements:

```
enum Ordering { LT, EQ, GT }

String show<A>(A a) where A : Show
Bool equals<A>(A a, A a) where A : Eq
Ordering compare<A>(A a, A a) where A : Ord
A plus<A>(A a, A a) where A : Num
F<T<_>> sequenceA<F,T>(T<F<_>> tfa) where F :
    Applicative, T : Traversable
```

Introducing Constraint

These **Typeclass** contexts have **Kind Constraint**.

```
Show :: * -> Constraint
```

```
Eq    :: * -> Constraint
```

```
Ord   :: * -> Constraint
```

```
Num   :: * -> Constraint
```

Introducing Constraint

These **Typeclass** contexts have **Kind Constraint**.

```
Show  :: * -> Constraint
Eq    :: * -> Constraint
Ord   :: * -> Constraint
Num   :: * -> Constraint
```

```
{-# LANGUAGE ConstraintKinds #-}
type ShowCxt a b = (Show a, Show b)
```

```
sameSerialization :: ShowCtxt a b => a -> b -> Bool
sameSerialization a b = show a == show b
```

```
ShowCxt :: * -> * -> Constraint
```


Other Kinds

There are other **Kinds** aside from ***** and **Constraint**

```
import GHC.Prim

(*)          -- kind of fully realized type
(#)         -- kind of unboxed stuff used internally
Constraint -- kind of constraints and type equality
OpenKind  -- superkind of (*) and (#)
AnyK      -- polymorphic kind for flexible arity
```

Other Kinds

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(#)         -- kind of unboxed stuff used internally
Constraint  -- kind of constraints and type equality
OpenKind    -- superkind of (*) and (#)
AnyK        -- polymorphic kind for flexible arity

-- the only sort, sorts classify kinds
(*), (#), Constraint, OpenKind, AnyK :: BOX
BOX :: BOX
```

All these **Kinds** are built-in and inferred as of GHC 7.10.2.

Language Extensions

Compiler extensions that enable a variety of new functionalities:

Language Extensions

Compiler extensions that enable a variety of new functionalities:

- Syntax extension
- Type-level programming
- Generic programming
- FFI
- Type disambiguation
- Typeclass extension

Language Extensions

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- Syntax extension
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Each extension has a name, and is enabled with the `LANGUAGE` pragma.

GADTs

Define data and explicit give type signatures to the **Value** constructors

```
data Bool = False | True
data Maybe a = Nothing | Just a
data List a = Nil | Cons a (List a)
```

Becomes:

```
data Bool where
  False :: Bool
  True  :: Bool

data Maybe a where
  Nothing :: Maybe a
  Just    :: a -> Maybe a

data List a where
  Nil    :: List a
  Cons   :: a -> List a -> List a
```

KindSignatures

Type Operators

DataKinds

Type Families

Section Outline

5 Questions

Questions?