Introduction to Dependent Types

Eagan Technology Unconference

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Agenda

Section Outline

Quick Question

How many are familiar with this topic?

A Joke

This is not a `m-` tutorial, and nothing here will involve burritos.

Disclaimer

There will be many code examples with *very* loose translations to imperative/OOP as we go along. Though please keep in mind that these are merely made up syntactical translations, the actual concepts may differ vastly.

About This Talk

Example languages with dependent types:

- `Idris`
- `Epigram`
- `Agda`
- `Coq`

About This Talk

Example languages with dependent types:

- `Idris`
- `Epigram`
- `Agda`
- `Coq`

But we will be using `Haskell` though.

Honestly, it's because they're way over his head...

Dependent Types

Why do we want them?

- more expressive `type system`
- encode stronger invariants
- proving correctness of code

Teaser

Example please:

```
infixr 5 :>
data Vect (n :: Nat) a where
  VNil :: Vect 0 a
  (:>) :: a -> Vect n a -> Vect (n :+ 1) a

vs :: Vect 6 Int
vs = 4 :> 8 :> 15 :> 16 :> 23 :> 42 :> VNil
```

Translation* please:

```
enum Vect<Nat n, A> {
  Vect<0, A> VNil,
  Vect<n :+ 1, A> VCons<A>(A a, Vect<n, A> va)
}

Vect<6, Int> vs = VCons(4, VCons(8, VCons(15, VCons(16,
    VCons(23, VCons(42, VNil))))));
```

(*) supreme looseness and totally made-up syntax!!!

Section Outline

Values and Types

Values has Types, or Values are classified by Types.

```
..., -1, 0, 1, 2, 3, ... :: Int

True, False :: Bool

'a', 'b', 'c' :: Char

"abc" :: String ~ [Char]
```

About Data Types

How are data types defined?

- Some are built in magic: Int, Char, function arrow
- Some are built in sugar: list, tuples
  - We can define equivalent non-sugar version ourselves
- Rest can be user defined: Bool, String, Maybe

About Data Types

What are the data types like?

- Multiple Value constructors
- Paremetrize over another Type
- Recursive definition
- Synonyms of other Types
- Combination of the above

Defining Data Types

Define new data type with data.

- Left hand side (LHS) - Type constructor
- Right hand side (RHS) - Value constructor

Type and Value constructors are capticalized.

Our First Example!

Define a person:

```
-- | params for firstname, lastname, age respectively
data Person = Person String String Int

barbara :: Person
barbara = Person "Barbara" "Smith" 30
```

The Type of the Person Value constructor:

```
Person :: String -> String -> Int -> Person
```

A loose translation:

```
enum Person {
  Person(String firstname, String lastname, Int age)
}

Person barbara = new Person("Barbara", "Smith", 30)
```

Multiple Value Constructors

Data can have multiple Value constructors:

```
data Bool = False | True

data Weekdays = Sunday | Monday | Tuesday | Wednesday
              | Thursday | Friday | Saturday
```

A loose translation:

```
enum Bool { False, True }

enum Weekdays {
  Sunday, Monday, Tuesday, Wednesday, Thursday, Friday,
      Saturday
}
```

Multiple Value Constructor

You can do type aliasing with type:

```
type String = [Char]
type Side = Double
type Radius = Double
```

For example:

```
data Shape = Triangle Side Side Side
           | Rectangle Side Side
           | Circle Radius
```

A loose translation:

```
enum Shape {
  Triangle(Double side1, Double side2, Double side3),
  Rectangle(Double length, Double width),
  Circle(Double radius)
}
```

Multiple Value Constructor

Recall Side ∼ Radius ∼ Double:

```
data Shape = Triangle Side Side Side
           | Rectangle Side Side
           | Circle Radius
```

Types of the 3 Value constructors:

```
Triangle  :: Side -> Side -> Side -> Shape
Rectangle :: Side -> Side -> Shape
Circle    :: Radius -> Shape
```

Example Shapes:

```
myTri, myRect, myCir :: Shape
myTri  = Triangle 2.1 3.2 5
myRect = Rectangle 4 4
myCir  = Circle 7.2
```

Parametrization

Types can parametrize over another type:

```
data Id a = Id a

intIdwrtSum :: Id Int
intIdwrtSum = Id 0
```

With:

```
Id :: a -> Id a
```

A loose translation:

```
enum Id<A> {
  Id(A a)
}
```

Tuple

Parametrize over 2 types - 2-tuple!

```
data Tuple a b = Tuple a b
```

With:

```
Tuple :: a -> b -> Tuple a b
```

A loose translation:

```
enum Tuple<A, B> {
  Tuple<A, B>(A a, B b)
}
```

Tuple

Actual built-in sugar:

```
   data Tuple a b = Tuple a b
=> data (,) a b = (,) a b
=> data (a, b) = (a, b)
```

An example:

```
type Employed = Bool

barbara, chet, luffy :: (Person, Employed)
barbara = (Person "Barbara" "Smith" 30, True)
chet    = (Person "Chet" "Awesome-Laser" 2, False)
luffy   = (Person "Luffy D." "Monkey" 19, False)
```

Maybe

Like Bool, but parametrizes a Type *a* over the True part:

```
data Maybe a = Nothing | Just a
```

With:

```
Nothing :: Maybe a
Just    :: a -> Maybe a
```

A loose translation:

```
enum Maybe<A> {
  Nothing,
  Just<A>(A a)
}
```

Maybe

From previous slide:

```
data Maybe a = Nothing | Just a
```

Say more with Occupation:

```
type Occupation = Maybe String

barbara2, chet2, luffy2 :: (Person, Occupation)
barbara2 = (Person "Barbara" "Smith" 30, Just "dancer")
chet2    = (Person "Chet" "Awesome-Laser" 2, Nothing)
luffy2   = (Person "Luffy D." "Monkey" 19, Just "pirate"
    )
```

Either

Like Bool, but parametrizes over both True and False:

```
data Either a b = Left a | Right b
```

With:

```
Left  :: a -> Either a b
Right :: b -> Either a b
```

A loose translation:

```
enum Either<A, B> {
  Left(A a),
  Right(B b)
}
```

Either

From previous slide:

```
data Either a b = Left a | Right b
```

Refine with Earning:

```
type Earning = Either String Int

barbara3, chet3, luffy3 :: (Person, Earning)
barbara3 = (Person "Barbara" "Smith" 30,
            Right 100000)
chet3    = (Person "Chet" "Awesome-Laser" 2,
            Left "Is a baby")
luffy3   = (Person "Luffy D." "Monkey" 19,
            Right 2000000)
```

Types with Recursion

Natural number:

```
data Nat = Z | S Nat
```

With:

```
Z :: Nat
S :: Nat -> Nat
```

A loose translation:

```
enum Nat {
  Z,
  S(Nat n)
}
```

Types with Recursion

Natural number:

```
data Nat = Z | S Nat

Z :: Nat
S :: Nat -> Nat

0 ∼ Z
1 ∼ S Z
2 ∼ S (S Z)
3 ∼ S (S (S Z))
```

Types with Recursion

List - recursive Type that parametrizes over another Type:

```
data List a = Nil | Cons a (List a)
```

With:

```
Nil  :: List a
Cons :: a -> List a -> List a
```

A loose translation:

```
enum List<A> {
  Nil,
  Cons<A>(A a, List<A> as)
}
```

Types with Recursion

Actual built-in sugar is something like:

```
   data List a = Nil | Cons a (List a)
=> data [] a = [] | (:) a ([] a)
=> data [a] = [] | (:) a [a]
```

Sugar that List:

```
ints :: List Int
ints = Cons 1 (Cons 2 (Cons 3 (Cons 4 Nil)))

-- built-in sugar
ints :: [] Int
ints = 1 : 2 : 3 : 4 : []

-- 2x the sugar!
ints :: [Int]
ints = [1, 2, 3, 4]
```

Functions

Maps Values of a Type to Values of another Type:

```
even :: Int -> Bool
even 0 = True
even n = if rem n 2 == 0
         then True
         else False
```

Not as loose translation:

```
Bool even(Int n) {
  switch n:
    case n == 0:
      return True;
    default:
      if rem(n, 2) == 0:
        return True;
      else
        return False;
}
```

Functions with Recursion

Use recursion for recursive Types:

```
data Nat = Z | S Nat

toInt :: Nat -> Int
toInt Z     = 0
toInt (S n) = 1 + toInt n
```

Not as loose translation:

```
Int toInt(Nat n) {
  switch n:
    case Z:
      return 0;
    case (S m):   -- n ~ (S m)
      return 1 + toInt(m);
}
```

Functions with Recursion

Use recursion for recursive Types:

```
data Nat = Z | S Nat

toInt :: Nat -> Int
toInt Z     = 0
toInt (S n) = 1 + toInt n
```

Evaluation is a series of substitutions:

```
three = S (S (S Z)) :: Nat

  toInt three :: Int
= toInt (S (S (S Z)))
= 1 + toInt (S (S Z))
= 1 + 1 + toInt (S Z)
= 1 + 1 + 1 + toInt Z
= 1 + 1 + 1 + 0
= 1 + 1 + 1
= 1 + 2
= 3
```

Functions with Parametric Polymorphism

Functions can be parametric:

```
id :: a -> a
id x = x
```

Not as loose translation:

```
A id<A>(A a) {
  return a;
}
```

Functions with Parametric Polymorphism

Functions can be parametric:

```
data [a] = [] | (:) a [a]

-- actual name is (++)
append :: [a] -> [a] -> [a]
append []       ys = ys
append (x:xs) ys = x : append xs ys
```

A translation:

```
List<A> append(List<A> l1, List<A> l2) {
  switch l1:
    case Nil:
      return l2;
    case Cons(x, xs):
      List<A> rest = append(xs, l2);
      return Cons(x, rest);
}
```

Functions with Parametric Polymorphism

Functions can be parametric:

```
data [a] = [] | (:) a [a]

append :: [a] -> [a] -> [a]
append []     ys = ys
append (x:xs) ys = x : append xs ys
```

Evaluation is a series of substitutions:

```
xs = [4, 8] = 4 : 8 : [] :: [Int]
ys = [15, 16, 23, 42] = 15 : 16 : 23 : 42 : [] :: [Int]

  append xs ys :: [Int]
= append [4, 8] [15, 16, 23, 42]
= 4 : append [8] [15, 16, 23, 42]
= 4 : 8 : append [] [15, 16, 23, 42]
= 4 : 8 : [15, 16, 23, 42]
= 4 : [8, 15, 16, 23, 42]
= [4, 8, 15, 16, 23, 42]
```

Higher-order Functions

Functions that take functions as params:

```
-- actual name is ($)
apply :: (a -> b) -> a -> b
apply f x = f x

-- acutal name is (.)
compose :: (b -> c) -> (a -> b) -> (a -> c)
compose f g = \x -> f (g x)
```

Yay translations:

```
B apply(Func<A, B> f, A a) {
  return f(a);
}

Func<A, C> compose(Func<B, C> f, Func<A, B> g) {
  return x => f(g(x));
}
```

More Functions Examples

map:

```
map :: (a -> b) -> [a] -> [b]
map f []       = []
map f (x:xs) = f x : map f xs
```

A translation:

```
List<B> map(Func<A, B> f, List<A> la) {
  switch la:
    case Nil:
      return Nil;
    case Cons(a, as):
      B b = f(a)
      List<B> rest = map(f, as);
      return Cons(b, rest);
}
```

More Functions Examples

map:

```
map :: (a -> b) -> [a] -> [b]
map f []       = []
map f (x:xs) = f x : map f xs
```

Evaluation is a series of substitutions:

```
xs = [4, 8, 15, 16, 23, 42] :: [Int]
even :: Int -> Bool

  map even xs :: [Bool]
= map even [4, 8, 15, 16, 23, 42]
= even 4 : map even [8, 15, 16, 23, 42]
= True : even 8 : map even [15, 16, 23, 42]
= True : True : even 15 : map even [16, 23, 42]
= True : True : False : even 16 : map even [23, 42]
= True : True : False : True : even 23 : map even [42]
= True : True : False : True : False : even 24 : map even
    []
= True : True : False : True : False : True : []
= [True, True, False, True, False, True]
```

More Functions Examples

zip:

```
zip :: [a] -> [b] -> [(a,b)]
zip []       ys      = []
zip xs       []      = []
zip (x:xs) (y:ys) = (x,y) : zip xs ys
```

A translation:

```
List<Tuple<A, B>> zip(List<A> l1, List<B> l2) {
  switch l1:
    case Nil:
      return Nil;
    case Cons(a, as):
      switch l2:
        case Nil:
          return Nil;
        case Cons(b, bs):
          Tuple<A, B> front = Tuple(a, b);
          List<Tuple<A, B>> rest = zip(as, bs);
          return Cons(front, rest);
}
```

More Functions Examples

zip:

```
zip :: [a] -> [b] -> [(a,b)]
zip []     ys     = []
zip xs     []     = []
zip (x:xs) (y:ys) = (x,y) : zip xs ys
```

Evaluation is a series of substitutions:

```
xs = ['a', 'b', 'c'] :: [Char]
ys = [1, 2, 3, 4] :: [Int]

  zip xs ys :: [(Char, Int)]
= zip ['a', 'b', 'c'] [1, 2, 3, 4]
= ('a', 1) : zip ['b', 'c'] [2, 3, 4]
= ('a', 1) : ('b', 2) : zip ['c'] [3, 4]
= ('a', 1) : ('b', 2) : ('c', 3) : zip [] [4]
= ('a', 1) : ('b', 2) : ('c', 3) : []
= [('a', 1), ('b', 2), ('c', 3)]
```

Section Outline

3 What is Dependent Type
- λ-Calculus
- Extensions on λ-calculus

λ-Calculus

So far, we have seen:

- function application
- function abstraction (aka higher-order functions)
- variable binding
- substitution

=> basis for `simply typed` λ`-calculus`.

λ-Calculus

You: Sure...
Me: Ah, yes, we want to extend λ-`calculus` so we can have more forms of abstractions!

λ-Calculus

You: Sure...
Me: Ah, yes, we want to extend λ-`calculus` so we can have more forms of abstractions!

Q: But how?
A: What if I tell you...
...you should already be familiar with 2 axes of extension :)

Subtype Polymorphism

Given data types T and P, if there is a relation between T and P by some notion of substitutability with T in place of P, then we say T is a `subtype` of the `supertype` P, denoted by $T <: P$.

The is an extension on λ-`calculus` with `subtype polymorphism` and is denoted by $\lambda_{<:}$.
=> `Object Oriented Programming`

Though this is not an axis that we are interested in.

Parametric Polymorphism

Introduce a mechanism of universal quantification over Types: Types can abstract over Types, allows for generic data types and generic functions.
=> Generic Programming

Recall:

```
data Maybe a = Nothing | Just a
data List a = Nil | Cons a (List a)

(.) :: (b -> c) -> (a -> b) -> (a -> c)
map :: (a -> b) -> [a] -> [b]
```

The name for this extension is formally second order λ-calculus, aka System F, denoted by $\lambda 2$.

Value and Type Interdependency

Re-thinking functions:

```
even :: Int -> Bool
even 0 = True
even n = if rem n 2 == 0
         then True
         else False
```

even maps Ints to True and False.

=> Values on RHS depends on the Values on LHS

=> Values depending on Values

=> Ordinary λ-calculus

Value and Type Interdependency

Re-thinking parametrized data types:

```
data Maybe a = Nothing | Just a

data List a = Nil | Cons a (List a)
```

Maybe and List take a Type and return Value constructors
=> Values on RHS depends on the Type on LHS
=> Values depending on Types
=> `Parametric polymorphism` of $\lambda 2$

Value and Type Interdependency

Then what about the other cases of dependencies?

- Values depending on Values: λ-`calculus`
- Values depending on Types: $\lambda 2$, `System F`

Value and Type Interdependency

Then what about the other cases of dependencies?

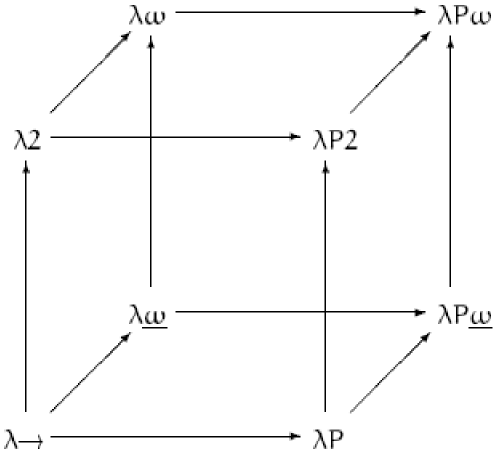
- Values depending on Values: λ-`calculus`
- Values depending on Types: $\lambda 2$, `System F`
- Types depending on Types: $\lambda \underline{\omega}$
  => `Type-level programming` via `type operators`

Value and Type Interdependency

Then what about the other cases of dependencies?

- Values depending on Values: λ-`calculus`
- Values depending on Types: $\lambda 2$, `System F`
- Types depending on Types: $\lambda \underline{\omega}$
  => `Type-level programming` via `type operators`
- Types depending on Values: $\lambda \Pi$
  => `Dependent types`

Lambda Cube



System F_c

Currently, `Haskell` as of GHC 7.10.2

- no true `type operators`
- `type-level programming` through:
  - `type families`
  - `equalities` and `coercions` on Types

This axis of extension on $\lambda 2$ is termed `System` $\mathtt{F}_c$.

System F_c

Currently, `Haskell` as of GHC 7.10.2

- not fully dependent either:
  - strong distinction between Values and Types
- emulate `dependent types` with:
  - handful of `language extensions`
  - Kind system

Section Outline

Kinds

Q: Types classify Values, but what classifies Types?
A: Kinds

Introducing ⋆

```
-- built-in magic: infinitely many value constructors
data Int = ... | -1 | 0 | 1 | 2 | ...
data Bool = False | True
data [a] = Nil | (:) a [a]
data Maybe a = Nothing | Just a
data (a, b) = (a, b)
data Either a b = Left a | Right b

Int :: *
Bool :: *
[Int] :: *
[] :: * -> *
Maybe Person :: *
Maybe :: * -> *
```

Introducing ⋆

```
-- built-in magic: infinitely many value constructors
data Int = ... | -1 | 0 | 1 | 2 | ...
data Bool = False | True
data [a] = Nil | (:) a [a]
data Maybe a = Nothing | Just a
data (a, b) = (a, b)
data Either a b = Left a | Right b

(Person, Bool) :: *
(,) Person :: * -> *
(,) :: * -> * -> *
Either String Earning :: *
Either String :: * -> *
Either :: * -> * -> *
```

Introducing Constraint

`Haskell` has `typeclasses` that very loosely resemble interfaces in OOP. A basic Typeclass consists of a collection of function signatures for a Type to implement. Afterward, this Typeclass `instance` can be used to provide `contexts` for functions.

```
Show  -- types that can be serialized to String
Eq    -- types that can be compared for equality
Ord   -- types that can be ordered
Num   -- types that are like numbers: +, -, *, ...
      -- and many others
```

Introducing Constraint

An example:

```
data Ordering = LT | EQ | GT

show      :: Show a => a -> String  -- toString()
(==)      :: Eq a => a -> a -> Bool
compare   :: Ord a => a -> a -> Ordering
(+)       :: Num a => a -> a -> a
sequenceA :: (Applicative f, Traversable t) => t (f a)
    -> f (t a)
```

A loose translation with `:` for `implements`:

```
enum Ordering { LT, EQ, GT }

String show<A>(A a) where A : Show
Bool equal<A>(A a, A a) where A : Eq
Ordering compare<A>(A a, A a) where A : Ord
A plus<A>(A a, A a) where A : Num
F<T<_>> sequenceA<F,T>(T<F<_>> tfa) where F :
    Applicative, T : Traversable
```

Introducing Constraint

These Typeclass contexts have Kind Constraint.

```
Show :: * -> Constraint
Eq   :: * -> Constraint
Ord  :: * -> Constraint
Num  :: * -> Constraint

{-# LANGUAGE ConstraintKinds #-}

type ShowContext a b = (Show a, Show b)

sameSerialization :: ShowContext a b => a -> b -> Bool
sameSerialization a b = show a == show b

ShowContext :: * -> * -> Constraint
```

Other Kinds

There are other Kinds aside from * and Constraint

```
import GHC.Prim

(*)              -- kind of fully realized type
(#)              -- kind of unboxed stuff used internally
Constraint       -- kind of constraints and type equality
OpenKind         -- superkind of (*) and (#)
AnyK             -- polymorphic kind for flexible arity
```

Other Kinds

There are other Kinds aside from * and Constraint

```
import GHC.Prim

(*)           -- kind of fully realized type
(#)           -- kind of unboxed stuff used internally
Constraint    -- kind of constraints and type equality
OpenKind      -- superkind of (*) and (#)
AnyK          -- polymorphic kind for flexible arity

-- the only sort, sorts classify kinds
(*), (#), Constraint, OpenKind, AnyK :: BOX
BOX :: BOX
```

All these Kinds are built-in and inferred as of GHC 7.10.2.

All of this will be changed with the next GHC 8.0.1 release.

Language Extensions

Compiler extensions that enable a variety of new functionalities:

- Syntax extension
- Type-level programming
- Generic deriving
- FFI
- Type disambiguation
- Typeclass extension

Each extension has a name, and is enabled with the LANGUAGE pragma.

GADTs

Define data and explicit give type signatures to the Value constructors.

```
data Bool = False | True
data Maybe a = Nothing | Just a
data List a = Nil | Cons a (List a)
```

Becomes:

```
{-# LANGUAGE GADTs #-}
data Bool where
  False :: Bool
  True  :: Bool

data Maybe a where
  Nothing :: Maybe a
  Just    :: a -> Maybe a

data List a where
  Nil  :: List a
  Cons :: a -> List a -> List a
```

GADTs

Define data and explicit give type signatures to the Value constructors.

```
data Bool = False | True
data Maybe a = Nothing | Just a
data List a = Nil | Cons a (List a)
```

Loose translations:

```
enum Bool {
  Bool False,
  Bool True
}

enum Maybe<A> {
  Maybe<A> Nothing,
  Maybe<A> Just(A a)
}

enum List<A> {
  List<A> Nil,
  List<A> Cons(A a, List<A> as)
}
```

KindSignatures

Specify the Kind of the Type variables:

```
{-# LANGUAGE GADTs #-}
{-# LANGUAGE KindSignatures #-}
data Bool :: * where
  False :: Bool
  True  :: Bool

data Maybe :: * -> * where
  Nothing :: Maybe a
  Just    :: a -> Maybe a

data List :: * -> * where
  Nil  :: List a
  Cons :: a -> List a -> List a
```

DataKinds

Kinds are built-in; no user defined Kinds.

Want Values at the Type level though!

=> `Data kind promotion` :)

DataKinds

Example:

```
data Bool = False | True
```

With DataKinds, we get something like:

```
{-# LANGUAGE DataKinds #-}
```

Kind		Bool
Type	Bool	'True | 'False
Value	True | False	

DataKinds

Example:

```haskell
data Nat = Z | S Nat
```

With DataKinds, we get something like:

```haskell
{-# LANGUAGE DataKinds #-}
```

Kind		Nat
Type	Nat	'Z | 'S Nat
Value	Z | S Nat	

Example

Example with GADTs:

```
{-# LANGUAGE DataKinds #-}
{-# LANGUAGE GADTs #-}
{-# LANGUAGE KindSignatures #-}

data Bool = False | True

data TextInput (a :: Bool) where
  RawText  :: String -> TextInput 'False
  SafeText :: String -> TextInput 'True

sanitize :: TextInput a -> TextInput 'True
sanitize (RawText str)  = SafeText (htmlEncode str)
sanitize x              = x
```

Notice that the *a* here is phantom.

Example

Translation*:

```
enum Bool {
  Bool False,
  Bool True
}

enum TextInput<Bool b> {
  TextInput<'False> RawText(String str),
  TextInput<'True> SafeText(String str)
}

TextInput<'True> sanitize(TextInput<B> input) {
  switch input:
    case RawText(input):
      return SafeText(htmlEncode(input));
    default:
      return input;
}
```

(*) supreme looseness and totally made-up syntax!!!

Type Families

`Type families` - type level functions, computed and checked at compile time.

Comes in 2 flavors:

- type synonym families
- data families

and have a few options:

- associated vs. standalone
- open vs. closed[1]
- injectivity[2]

Type Families

At Value level:

```
data Nat = Z | S Nat

add :: Nat -> Nat -> Nat
add Z     m = m
add (S n) m = S (add n m)

  add (S (S Z)) (S Z)
= S (add (S Z) (S Z))
= S (S (add Z (S Z)))
= S (S (S Z))
```

Type Families

At Type level:

```
{-# LANGUAGE DataKinds #-}
{-# LANGUAGE TypeFamilies #-}

data Nat = Z | S Nat

type family Add (n :: Nat) (m :: Nat) :: Nat where
  Add 'Z      m = m
  Add ('S n) m = 'S (Add n m)

  Add ('S ('S 'Z)) ('S 'Z)
= 'S (Add ('S 'Z) ('S 'Z))
= 'S ('S (Add 'Z ('S 'Z)))
= 'S ('S ('S 'Z))
```

Type Operators

Allows usage of symbols in place of Type constructors and Type families.

```
{-# LANGUAGE DataKinds #-}
{-# LANGUAGE TypeFamilies #-}
{-# LANGUAGE TypeOperators #-}

data Nat = Z | S Nat

type family (:+) n m where
    'Z      :+ m = m
    ('S n)  :+ m = 'S (n :+ m)

    ('S ('S 'Z)) :+ ('S 'Z)
=   'S (('S 'Z) :+ ('S 'Z))
=   'S ('S ('Z :+ ('S 'Z)))
=   'S ('S ('S 'Z))
```

Extended Haskell

Assume `LANGUAGE extensions` are turned on from now on.

Bad news, no more translations :(

Vectors

Like List, but also indexed by Nat to indicate length.

List:

```
data List a where
  Nil  :: List a
  Cons :: a -> List a -> List a
```

Vector:

```
-- 'Z ~ 0
-- 'S n ~ n :+ 1
data Vect (n :: Nat) a where
  VNil :: Vect Z a
  (:>) :: a -> Vect n a -> Vect (S n) a

type Six = S (S (S (S (S (S Z)))))
vs :: Vect Six Int
vs = 4 :> 8 :> 15 :> 16 :> 23 :> 42 :> VNil
```

Vectors

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List:

```
data List a where
  Nil  :: List a
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```

Module GHC.TypeLits provide `type-level literals`:

```
-- 'Z ~ 0
-- 'S n ~ n :+ 1
data Vect (n :: Nat) a where
  VNil :: Vect 0 a
  (:>) :: a -> Vect n a -> Vect (n :+ 1) a

vs :: Vect 6 Int
vs = 4 :> 8 :> 15 :> 16 :> 23 :> 42 :> VNil
```

Head

head returns the first element of the List:

```
-- from standard library
-- useless unless we know the list is non-empty
head :: [a] -> a
head []      = error "empty list"
head (x:xs) = x
```

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Elm now uses Maybe:

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mhead :: [a] -> Maybe a
mhead []      = Nothing
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Head

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mhead []     = Nothing
mhead (x:xs) = Just x
```

With Vector:

```
vhead :: Vect (S n) a -> a
vhead (x:>xs) = x
```

Append

append concatenates 2 Lists:

```
append :: [a] -> [a] -> [a]
append []     ys = ys
append (x:xs) ys = x : append xs ys
```

With Vector:

```
vappend :: Vect n a -> Vect m a -> Vect (n :+ m) a
vappend VNil    ys = ys
vappend (x:>xs) ys = x :> vappend xs ys
```

Map

`map` maps a function over a List:

```
map :: (a -> b) -> [a] -> [b]
map f []     = []
map f (x:xs) = f x : map f xs
```

With Vector:

```
vmap :: (a -> b) -> Vect n a -> Vect n b
vmap f VNil     = VNil
vmap f (x:>xs) = f x :> vmap f xs
```

Zip

zip creates pair-wise tuples:

```
zip :: [a] -> [b] -> [(a,b)]
zip (x:xs) (y:ys) = (x,y) : zip xs ys
zip xs     ys     = []
```

With Vector:

```
vzip :: Vect n a -> Vect n b -> Vect n (a, b)
vzip (x:>xs) (y:>ys) = (x,y) :> vzip xs ys
vzip VNil    VNil    = VNil
```

Zip

zip2 with Min type family:

```
type family Min n m where
  Min Z     m     = Z
  Min n     Z     = Z
  Min (S n) (S m) = S (Min n m)

vzip2 :: Vect n a -> Vect m b -> Vect (Min n m) (a, b)
vzip2 (x:>xs) (y:>ys) = (x,y) :> vzip2 xs ys
vzip2 xs      VNil    = VNil
vzip2 VNil    ys      = VNil
```

Heterogeneous List

Heterogeneous List indexed by List of Types:

```
infixr 5 :::>
data HList (t :: [*]) where
  HNil  :: HList '[]
  (:::>) :: t -> HList ts -> HList (t ': ts)

defaults :: HList '[Int, Bool, Maybe a]
defaults = 0 :::> False :::> Nothing :::> HNil
```

Heterogeneous Vector

Heterogeneous Vector indexed by a List of Types:

```
infixr 5 :>>
data HVect (n :: Nat) (t :: [*]) where
  HVNil  :: HVect Z '[]
  (:>>) :: t -> HVect n ts -> HVect (S n) (t ': ts)

vdefaults :: HVect 3 '[Int, Bool, Maybe a]
vdefaults = 0 :>> False :>> Nothing :>> HVNil
```

Replicate and Filter

`replicate` repeats an element n times:

```
replicate :: Int -> a -> [a]
replicate 0 x = []
replicate n x = x : replicate (n - 1) x
```

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```
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```

`filter` selects elements from a list for given predicate:

```
filter :: (a -> Bool) -> [a] -> [a]
filter f []       = []
filter f (x:xs) = if f x
                  then x : filter f xs
                  else filter f xs
```

Pi Types

Π-types - Values in Type signatures. Fake by deriving `singleton` instances of Sing data family to reflect values to the type level.

```
-- fake with singleton types
data instance Sing (n :: Nat) where
  SZ :: Sing Z
  SS :: Sing n -> Sing (S n)

-- so we have
SZ :: Sing Z
SS SZ :: Sing (S Z)
SS (SS SZ) :: Sing (S (S Z))
```

Pi Types

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```
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  SZ :: Sing Z
  SS :: Sing n -> Sing (S n)

-- so we have
SZ :: Sing Z
SS SZ :: Sing (S Z)
SS (SS SZ) :: Sing (S (S Z))

-- finally:
vreplicate :: Sing (n :: Nat) -> a -> Vect n a
vreplicate SZ      a = VNil
vreplicate (SS n) a = a :> vreplicate n a
```

Sigma Types

Σ-types - tuple where 2nd value depends on 1st:

```
-- borrowing Idris's ** dependent pair syntax
-- n ~ S (S (S Z)) ~ 3, singleton version
dpair :: (n :: Nat ** Vect n Char)
dpair = (3 ** 'a' :> 'b' :> 'c' :> VNil)

vfilter :: (a -> Bool) -> Vect n a
        -> (p :: Nat ** Vect p a)
```

Sigma Types

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-- borrowing Idris's ** dependent pair syntax
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vfilter :: (a -> Bool) -> Vect n a
        -> (p :: Nat ** Vect p a)
```

Credit to Ertugrul Söylemez:

```
-- Sugar ** for Sigma type constructor and Exists value
   constructor
data Sigma :: KProxy a -> (a -> *) -> * where
  (Exists :: Sing (x :: a) -> b x
       -> Sigma ('KProxy :: KProxy a) b
```

Section Outline

5 Closing
- Beyond
- Questions

Beyond Dependent Types

- Total functional languages
  - termination and totality check
  - disallow partial functions
  - distinction between data and codata
- Proof assistant languages
  - Ph.D. first please

Questions?