MATHEMATICS
CLASS XI

INDEX

CHAPTER-I

SETS

2 MARK QUESTIONS

1. Representation of Sets:
There are 2 methods to represent a set

a.Roster or Tabular Form: In this form, we list all the members of the set within braces { } and separate by commas.

b.Set-builder Form: In the set-builder form, we list the property or properties satisfied by all the elements of the sets.

2.How Many Types of Sets ?

- **Empty Sets:** A set which will not have any element is known as an empty set or the void set or null set. It is denoted by Φ or {}.
- **Singleton Set:** A set which has a single element, is known as a singleton set.
- **Finite and Infinite Set:** A set which has a finite number of elements, is called a finite set, if not the set is known as an infinite set.
- **Equal Sets:** Two sets A & B are said to be equal, if all elements of A is also an element of B, i.e. two equal sets will have the same elements.
- **Equivalent Sets:** Two finite sets A & B are said to be equal if number of elements in both sets are equal, i.e. $n(A) = n(B)$

3. Subsets: Set A will be a subset of Set B if every element of Set A is also present in Set B. In simple words, set A is contained inside Set B.

For Example: If set A has {A, B} and set B has {A, B, C}, then A will be the subset of B since elements of A are also present in set B

4. Venn-Diagrams: Venn diagrams are diagrams that show the relationship between two sets. The universal set U is represented by a point inside a rectangle, and its subsets are represented by points within closed curves within the rectangle in Venn diagrams.

5. Union of Sets: The union of two sets A & B is denoted by A ∪ B will be the set of all those elements which are either in set A or in set B or in both A and B.

6. Intersection of Sets: The intersection of two sets A & B is denoted by A ∩ B, and it is the set of all elements which are common in both set A and set B.

7. Let A = {1, 2, 3, 4, 5, 6}. Insert the appropriate symbol ∈or ∉ in the blank spaces:

(i) 5…A (ii) 8…A (iii) 0…A

(iv) 4…A (v) 2…A (vi) 10…A

Solution:

(i) $5 \in A$

(ii) $8 \notin A$

(iii) $0 \notin A$

(iv) $4 \in A$

(v) $2 \in A$

(vi) $10 \notin A$

8. How many elements has P (A), if A = Φ?

Solution:

If A is a set with m elements

n (A) = m then n [P (A)] = 2^m

If A = Φ we get n (A) = 0

n [P(A)] = $2^0 = 1$

Therefore, P (A) has one element

9. Write the following as intervals:

(i) {x: $x \in$ R, $-4 < x \leq 6$}

(ii) {x: $x \in$ R, $-12 < x < -10$}

(iii) {x: $x \in$ R, $0 \leq x < 7$}

(iv) {x: $x \in$ R, $3 \leq x \leq 4$}

Solution:

(i) {x: $x \in$ R, $-4 < x \leq 6$} = (–4, 6]

(ii) {x: $x \in$ R, $-12 < x < -10$} = (–12, –10)

(iii) {x: $x \in$ R, $0 \leq x < 7$} = [0, 7)

(iv) {x: $x \in$ R, $3 \leq x \leq 4$} = [3, 4]

10. Write the following intervals in set-builder form:

(i) (–3, 0)

(ii) [6, 12]

(iii) (6, 12]

(iv) [–23, 5)

Solution:

(i) $(-3, 0) = \{x: x \in R, -3 < x < 0\}$

(ii) $[6, 12] = \{x: x \in R, 6 \leq x \leq 12\}$

(iii) $(6, 12] = \{x: x \in R, 6 < x \leq 12\}$

(iv) $[-23, 5) = \{x: x \in R, -23 \leq x < 5\}$

4 MARK QUESTIONS

1. Write the following sets in the set-builder form:

(i) (3, 6, 9, 12)

(ii) {2, 4, 8, 16, 32}

(iii) {5, 25, 125, 625}

(iv) {2, 4, 6 …}

(v) {1, 4, 9 … 100}

Solution:

(i) {3, 6, 9, 12}

The given set can be written in the set-builder form as $\{x: x = 3n, n \in \mathbf{N} \text{ and } 1 \leq n \leq 4\}$

(ii) {2, 4, 8, 16, 32}

We know that $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, $16 = 2^4$, and $32 = 2^5$.

Therefore, the given set {2, 4, 8, 16, 32} can be written in the set-builder form as $\{x: x = 2^n, n \in \mathbf{N} \text{ and } 1 \leq n \leq 5\}$.

(iii) {5, 25, 125, 625}

We know that $5 = 5^1$, $25 = 5^2$, $125 = 5^3$, and $625 = 5^4$.

Therefore, the given set {5, 25, 125, 625} can be written in the set-builder form as $\{x: x = 5^n, n \in \mathbf{N} \text{ and } 1 \leq n \leq 4\}$.

(iv) {2, 4, 6 …}

{2, 4, 6 …} is a set of all even natural numbers

Therefore, the given set {2, 4, 6 …} can be written in the set-builder form as {*x*: *x* is an even natural number}.

(v) {1, 4, 9 … 100}

We know that $1 = 1^2$, $4 = 2^2$, $9 = 3^2$ …$100 = 10^2$.

Therefore, the given set {1, 4, 9… 100} can be written in the set-builder form as
$\{x: x = n^2, n \in N \text{ and } 1 \leq n \leq 10\}$.

2. Match each of the set on the left in the roster form with the same set on the right described in set-builder form:

(i) {1, 2, 3, 6} (a) {x: x is a prime number and a divisor of 6}

(ii) {2, 3} (b) {x: x is an odd natural number less than 10}

(iii) {M, A, T, H, E, I, C, S} (c) {x: x is a natural number and divisor of 6}

(iv) {1, 3, 5, 7, 9} (d) {x: x is a letter of the word MATHEMATICS}

Solution:

(i) Here the elements of this set are natural number as well as divisors of 6. Hence, (i) matches with (c).

(ii) 2 and 3 are prime numbers which are divisors of 6. Hence, (ii) matches with (a).

(iii) The elements are the letters of the word MATHEMATICS. Hence, (iii) matches with (d).

(iv) The elements are odd natural numbers which are less than 10. Hence, (v) matches with (b).

3. Which of the following sets are finite or infinite?

(i) The set of months of a year

(ii) {1, 2, 3 …}

(iii) {1, 2, 3 … 99, 100}

(iv) The set of positive integers greater than 100

(v) The set of prime numbers less than 99

Solution:

(i) The set of months of a year is a finite set as it contains 12 elements.

(ii) {1, 2, 3 …} is an infinite set because it has infinite number of natural numbers.

(iii) {1, 2, 3 …99, 100} is a finite set as the numbers from 1 to 100 are finite.

(iv) The set of positive integers greater than 100 is an infinite set as the positive integers which are greater than 100 are infinite.

(v) The set of prime numbers less than 99 is a finite set as the prime numbers which are less than 99 are finite.

4. State whether each of the following set is finite or infinite:

(i) The set of lines which are parallel to the *x*-axis

(ii) The set of letters in the English alphabet

(iii) The set of numbers which are multiple of 5

(iv) The set of animals living on the earth

(v) The set of circles passing through the origin (0, 0)

Solution:

(i) The set of lines which are parallel to the *x*-axis is an infinite set as the lines which are parallel to the *x*-axis are infinite.

(ii) The set of letters in the English alphabet is a finite set as it contains 26 elements.

(iii) The set of numbers which are multiple of 5 is an infinite set as the multiples of 5 are infinite.

(iv) The set of animals living on the earth is a finite set as the number of animals living on the earth is finite.

(v) The set of circles passing through the origin (0, 0) is an infinite set as infinite number of circles can pass through the origin.

5. In the following, state whether A = B or not:

(i) A = {a, b, c, d}; B = {d, c, b, a}

(ii) A = {4, 8, 12, 16}; B = {8, 4, 16, 18}

(iii) A = {2, 4, 6, 8, 10}; B = {x: x is positive even integer and $x \leq 10$}

(iv) A = {x: x is a multiple of 10}; B = {10, 15, 20, 25, 30 …}

Solution:

(i) A = {a, b, c, d}; B = {d, c, b, a}

Order in which the elements of a set are listed is not significant.

Therefore, A = B.

(ii) A = {4, 8, 12, 16}; B = {8, 4, 16, 18}

We know that $12 \in A$ but $12 \notin B$.

Therefore, $A \neq B$

(iii) A = {2, 4, 6, 8, 10};

B = {x: x is a positive even integer and $x \leq 10$} = {2, 4, 6, 8, 10}

Therefore, A = B

(iv) A = {x: x is a multiple of 10}

B = {10, 15, 20, 25, 30 …}

We know that $15 \in B$ but $15 \notin A$.

Therefore, $A \neq B$

6. Are the following pair of sets equal? Give reasons.

(i) A = {2, 3}; B = {x: x is solution of $x^2 + 5x + 6 = 0$}

(ii) A = {x: x is a letter in the word FOLLOW}; B = {y: y is a letter in the word WOLF}

Solution:

(i) A = {2, 3}; B = {x: x is solution of $x^2 + 5x + 6 = 0$}

$x^2 + 5x + 6 = 0$ can be written as

$x(x + 3) + 2(x + 3) = 0$

By further calculation

$(x + 2)(x + 3) = 0$

So we get

$x = -2$ or $x = -3$

Here

A = {2, 3}; B = {–2, –3}

Therefore, A ≠ B

(ii) A = {*x*: *x* is a letter in the word FOLLOW} = {F, O, L, W}

B = {*y*: *y* is a letter in the word WOLF} = {W, O, L, F}

Order in which the elements of a set which are listed is not significant.

Therefore, A = B.

7. Write down all the subsets of the following sets:

(i) {*a*}

(ii) {*a*, *b*}

(iii) {1, 2, 3}

(iv) Φ

Solution:

(i) Subsets of {*a*} are

Φ and {*a*}.

(ii) Subsets of {*a*, *b*} are

Φ, {*a*}, {*b*}, and {*a*, *b*}.

(iii) Subsets of {1, 2, 3} are

Φ, {1}, {2}, {3}, {1, 2}, {2, 3}, {1, 3}, and {1, 2, 3}.

(iv) Only subset of Φ is Φ.

8. If A = {1, 2, 3, 4}, B = {3, 4, 5, 6}, C = {5, 6, 7, 8} and D = {7, 8, 9, 10}; find

(i) A ∪ B

(ii) A ∪ C

(iii) B ∪ C

(iv) B ∪ D

(v) A ∪ B ∪ C

(vi) A ∪ B ∪ D

(vii) B ∪ C ∪ D

Solution:

It is given that

A = {1, 2, 3, 4], B = {3, 4, 5, 6}, C = {5, 6, 7, 8} and D = {7, 8, 9, 10}

(i) A ∪ B = {1, 2, 3, 4, 5, 6}

(ii) A ∪ C = {1, 2, 3, 4, 5, 6, 7, 8}

(iii) B ∪ C = {3, 4, 5, 6, 7, 8}

(iv) B ∪ D = {3, 4, 5, 6, 7, 8, 9, 10}

(v) A ∪ B ∪ C = {1, 2, 3, 4, 5, 6, 7, 8}

(vi) A ∪ B ∪ D = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

(vii) B ∪ C ∪ D = {3, 4, 5, 6, 7, 8, 9, 10}

7 MARK QUESTIONS

1. Write the following sets in roster form:

(i) A = {x: x is an integer and –3 < x < 7}.

(ii) B = {x: x is a natural number less than 6}.

(iii) C = {x: x is a two-digit natural number such that the sum of its digits is 8}

(iv) D = {x: x is a prime number which is divisor of 60}.

(v) E = The set of all letters in the word TRIGONOMETRY.

(vi) F = The set of all letters in the word BETTER.

Solution:

(i) A = {x: x is an integer and –3 < x < 7}

–2, –1, 0, 1, 2, 3, 4, 5, and 6 only are the elements of this set.

Hence, the given set can be written in roster form as

A = {–2, –1, 0, 1, 2, 3, 4, 5, 6}

(ii) B = {x: x is a natural number less than 6}

1, 2, 3, 4, and 5 only are the elements of this set

Hence, the given set can be written in roster form as

B = {1, 2, 3, 4, 5}

(iii) C = {x: x is a two-digit natural number such that the sum of its digits is 8}

17, 26, 35, 44, 53, 62, 71, and 80 only are the elements of this set

Hence, the given set can be written in roster form as

C = {17, 26, 35, 44, 53, 62, 71, 80}

(iv) D = {x: x is a prime number which is divisor of 60}

Here 60 = 2 × 2 × 3 × 5

2, 3 and 5 only are the elements of this set

Hence, the given set can be written in roster form as

D = {2, 3, 5}

(v) E = The set of all letters in the word TRIGONOMETRY

TRIGONOMETRY is a 12 letters word out of which T, R and O are repeated.

Hence, the given set can be written in roster form as

E = {T, R, I, G, O, N, M, E, Y}

(vi) F = The set of all letters in the word BETTER

BETTER is a 6 letters word out of which E and T are repeated.

Hence, the given set can be written in roster form as

F = {B, E, T, R}

2. List all the elements of the following sets:

(i) A = {*x*: *x* is an odd natural number}

(ii) B = {*x*: *x* is an integer, $-1/2 < x < 9/2$}

(iii) C = {*x*: *x* is an integer, $x^2 \leq 4$}

(iv) D = {*x*: *x* is a letter in the word "LOYAL"}

(v) E = {*x*: *x* is a month of a year not having 31 days}

(vi) F = {*x*: *x* is a consonant in the English alphabet which proceeds *k*}.

Solution:

(i) A = {*x*: *x* is an odd natural number}

So the elements are A = {1, 3, 5, 7, 9 …..}

(ii) B = {*x*: *x* is an integer, $-1/2 < x < 9/2$}

We know that $-1/2 = -0.5$ and $9/2 = 4.5$

So the elements are B = {0, 1, 2, 3, 4}.

(iii) C = {*x*: *x* is an integer, $x^2 \leq 4$}

We know that

$(-1)^2 = 1 \le 4$; $(-2)^2 = 4 \le 4$; $(-3)^2 = 9 > 4$

Here

$0^2 = 0 \le 4$, $1^2 = 1 \le 4$, $2^2 = 4 \le 4$, $3^2 = 9 > 4$

So we get

C = {–2, –1, 0, 1, 2}

(iv) D = {*x*: *x* is a letter in the word "LOYAL"}

So the elements are D = {L, O, Y, A}

(v) E = {*x*: *x* is a month of a year not having 31 days}

So the elements are E = {February, April, June, September, November}

(vi) F = {*x*: *x* is a consonant in the English alphabet which proceeds *k*}

So the elements are F = {b, c, d, f, g, h, j}

3. From the sets given below, select equal sets:

A = {2, 4, 8, 12}, B = {1, 2, 3, 4}, C = {4, 8, 12, 14}, D = {3, 1, 4, 2}

E = {–1, 1}, F = {0, *a*}, G = {1, –1}, H = {0, 1}

Solution:

A = {2, 4, 8, 12}; B = {1, 2, 3, 4}; C = {4, 8, 12, 14}

D = {3, 1, 4, 2}; E = {–1, 1}; F = {0, *a*}

G = {1, –1}; H = {0, 1}

We know that

$8 \in A$, $8 \notin B$, $8 \notin D$, $8 \notin E$, $8 \notin F$, $8 \notin G$, $8 \notin H$

$A \ne B$, $A \ne D$, $A \ne E$, $A \ne F$, $A \ne G$, $A \ne H$

It can be written as

$2 \in A$, $2 \notin C$

Therefore, A ≠ C

3 ∈ B, 3 ∉ C, 3 ∉ E, 3 ∉ F, 3 ∉ G, 3 ∉ H

B ≠ C, B ≠ E, B ≠ F, B ≠ G, B ≠ H

It can be written as

12 ∈ C, 12 ∉ D, 12 ∉ E, 12 ∉ F, 12 ∉ G, 12 ∉ H

Therefore, C ≠ D, C ≠ E, C ≠ F, C ≠ G, C ≠ H

4 ∈ D, 4 ∉ E, 4 ∉ F, 4 ∉ G, 4 ∉ H

Therefore, D ≠ E, D ≠ F, D ≠ G, D ≠ H

Here, E ≠ F, E ≠ G, E ≠ H

F ≠ G, F ≠ H, G ≠ H

Order in which the elements of a set are listed is not significant.

B = D and E = G

Therefore, among the given sets, B = D and E = G.

4. Make correct statements by filling in the symbols ⊂ or ⊄ in the blank spaces:

(i) {2, 3, 4} … {1, 2, 3, 4, 5}

(ii) {*a*, *b*, *c*} … {*b*, *c*, *d*}

(iii) {*x*: *x* is a student of Class XI of your school} … {*x*: *x* student of your school}

(iv) {*x*: *x* is a circle in the plane} … {*x*: *x* is a circle in the same plane with radius 1 unit}

(v) {*x*: *x* is a triangle in a plane}…{*x*: *x* is a rectangle in the plane}

(vi) {*x*: *x* is an equilateral triangle in a plane}… {*x*: *x* is a triangle in the same plane}

(vii) {*x*: *x* is an even natural number} … {*x*: *x* is an integer}

Solution:

(i) $\{2, 3, 4\} \subset \{1, 2, 3, 4, 5\}$

(ii) $\{a, b, c\} \not\subset \{b, c, d\}$

(iii) {*x*: *x* is a student of Class XI of your school} ⊂ {*x*: *x* student of your school}

(iv) {*x*: *x* is a circle in the plane} ⊄ {*x*: *x* is a circle in the same plane with radius 1 unit}

(v) {*x*: *x* is a triangle in a plane} ⊄ {*x*: *x* is a rectangle in the plane}

(vi) {*x*: *x* is an equilateral triangle in a plane} ⊂ {*x*: *x* is a triangle in the same plane}

(vii) {*x*: *x* is an even natural number} ⊂ {*x*: *x* is an integer}

5. Let A= {1, 2, {3, 4}, 5}. Which of the following statements are incorrect and why?

(i) $\{3, 4\} \subset A$

(ii) $\{3, 4\}\} \in A$

(iii) $\{\{3, 4\}\} \subset A$

(iv) $1 \in A$

(v) $1 \subset A$

(vi) $\{1, 2, 5\} \subset A$

(vii) $\{1, 2, 5\} \in A$

(viii) $\{1, 2, 3\} \subset A$

(ix) $\Phi \in A$

(x) $\Phi \subset A$

(xi) $\{\Phi\} \subset A$

Solution:

It is given that A= {1, 2, {3, 4}, 5}

(i) $\{3, 4\} \subset A$ is incorrect

Here $3 \in \{3, 4\}$; where, $3 \notin A$.

(ii) $\{3, 4\} \in A$ is correct

{3, 4} is an element of A.

(iii) {{3, 4}} ⊂ A is correct

{3, 4} ∈ {{3, 4}} and {3, 4} ∈ A.

(iv) 1∈A is correct

1 is an element of A.

(v) 1⊂ A is incorrect

An element of a set can never be a subset of itself.

(vi) {1, 2, 5} ⊂ A is correct

Each element of {1, 2, 5} is also an element of A.

(vii) {1, 2, 5} ∈ A is incorrect

{1, 2, 5} is not an element of A.

(viii) {1, 2, 3} ⊂ A is incorrect

3 ∈ {1, 2, 3}; where, 3 ∉ A.

(ix) Φ ∈ A is incorrect

Φ is not an element of A.

(x) Φ ⊂ A is correct

Φ is a subset of every set.

(xi) {Φ} ⊂ A is incorrect

Φ∈{Φ}; where, Φ ∈A.

6. Given the sets A = {1, 3, 5}, B = {2, 4, 6} and C = {0, 2, 4, 6, 8}, which of the following may be considered as universals set (s) for all the three sets A, B and C?

(i) {0, 1, 2, 3, 4, 5, 6}

(ii) Φ

(iii) {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

(iv) {1, 2, 3, 4, 5, 6, 7, 8}

Solution:

(i) We know that A ⊂{0, 1, 2, 3, 4, 5, 6}

B ⊂{0, 1, 2, 3, 4, 5, 6}

So C ⊄{0, 1, 2, 3, 4, 5, 6}

Hence, the set {0, 1, 2, 3, 4, 5, 6} cannot be the universal set for the sets A, B, and C.

(ii) A ⊄Φ, B ⊄Φ, C ⊄Φ

Hence, Φ cannot be the universal set for the sets A, B, and C.

(iii) A ⊂{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

B ⊂{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

C ⊂{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Hence, the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10} is the universal set for the sets A, B, and C.

(iv) A ⊂{1, 2, 3, 4, 5, 6, 7, 8}

B ⊂{1, 2, 3, 4, 5, 6, 7, 8}

So C ⊄{1, 2, 3, 4, 5, 6, 7, 8}

Hence, the set {1, 2, 3, 4, 5, 6, 7, 8} cannot be the universal set for the sets A, B, and C.

7. Find the intersection of each pair of sets:

(i) X = {1, 3, 5} Y = {1, 2, 3}

(ii) A = {*a*, *e*, *i*, *o*, *u*} B = {*a*, *b*, *c*}

(iii) A = {*x*: *x* is a natural number and multiple of 3}

B = {*x*: *x* is a natural number less than 6}

(iv) A = {*x*: *x* is a natural number and 1 < *x* ≤ 6}

B = {*x*: *x* is a natural number and 6 < *x* < 10}

(v) A = {1, 2, 3}, B = Φ

Solution:

(i) X = {1, 3, 5}, Y = {1, 2, 3}

So the intersection of the given set can be written as

X ∩ Y = {1, 3}

(ii) A = {*a*, *e*, *i*, *o*, *u*}, B = {*a*, *b*, *c*}

So the intersection of the given set can be written as

A ∩ B = {*a*}

(iii) A = {*x*: *x* is a natural number and multiple of 3} = (3, 6, 9 …}

B = {*x*: *x* is a natural number less than 6} = {1, 2, 3, 4, 5}

So the intersection of the given set can be written as

A ∩ B = {3}

(iv) A = {*x*: *x* is a natural number and $1 < x \leq 6$} = {2, 3, 4, 5, 6}

B = {*x*: *x* is a natural number and $6 < x < 10$} = {7, 8, 9}

So the intersection of the given set can be written as

A ∩ B = Φ

(v) A = {1, 2, 3}, B = Φ

So the intersection of the given set can be written as

A ∩ B = Φ

8. If A = {*x*: *x* is a natural number}, B ={*x*: *x* is an even natural number}

C = {*x*: *x* is an odd natural number} and D = {*x*: *x* is a prime number}, find

(i) A ∩ B

(ii) A ∩ C

(iii) A ∩ D

(iv) B ∩ C

(v) B ∩ D

(vi) C ∩ D

Solution:

It can be written as

A = {x: x is a natural number} = {1, 2, 3, 4, 5 …}

B ={x: x is an even natural number} = {2, 4, 6, 8 …}

C = {x: x is an odd natural number} = {1, 3, 5, 7, 9 …}

D = {x: x is a prime number} = {2, 3, 5, 7 …}

(i) A ∩B = {x: x is a even natural number} = B

(ii) A ∩ C = {x: x is an odd natural number} = C

(iii) A ∩ D = {x: x is a prime number} = D

(iv) B ∩ C = Φ

(v) B ∩ D = {2}

(vi) C ∩ D = {x: x is odd prime number}

SUMMARY

Sets can be manipulated using various operations, such as union, intersection, and complement, allowing for the analysis of relationships between different groups of objects. They are widely used in various branches of mathematics, including algebra, calculus, and probability theory, as well as in computer science and everyday problem-solving.

CHAPTER-II

RELATIONS AND FUNCTIONS

2 MARK QUESTIONS

1. If find the values of *x* and *y*.

Solution:

Given,

As the ordered pairs are equal, the corresponding elements should also be equal.

Thus,

x/3 + 1 = 5/3 and y – 2/3 = 1/3

Solving, we get

x + 3 = 5 and 3y – 2 = 1 [Taking L.C.M. and adding]

x = 2 and 3y = 3

Therefore,

x = 2 and y = 1

2. If set A has 3 elements and set B = {3, 4, 5}, then find the number of elements in (A × B).

Solution:

Given, set A has 3 elements, and the elements of set B are {3, 4, and 5}.

So, the number of elements in set B = 3

Then, the number of elements in (A × B) = (Number of elements in A) × (Number of elements in B)

= 3 × 3 = 9

Therefore, the number of elements in (A × B) will be 9.

3. If G = {7, 8} and H = {5, 4, 2}, find G × H and H × G.

Solution:

Given, G = {7, 8} and H = {5, 4, 2}

We know that,

The Cartesian product of two non-empty sets P and Q is given as

P × Q = {(p, q): $p \in$ P, $q \in$ Q}

So,

G × H = {(7, 5), (7, 4), (7, 2), (8, 5), (8, 4), (8, 2)}

H × G = {(5, 7), (5, 8), (4, 7), (4, 8), (2, 7), (2, 8)}

4. State whether each of the following statements is true or false. If the statement is false, rewrite the given statement correctly.

(i) If P = {m, n} and Q = {n, m}, then P × Q = {(m, n), (n, m)}

(ii) If A and B are non-empty sets, then A × B is a non-empty set of ordered pairs (x, y) such that $x \in$ A and $y \in$ B.

(iii) If A = {1, 2}, B = {3, 4}, then A × (B ∩ Φ) = Φ

Solution:

(i) The statement is false. The correct statement is

If P = {m, n} and Q = {n, m}, then

P × Q = {(m, m), (m, n), (n, m), (n, n)}

(ii) True

(iii) True

5. If A = {–1, 1}, find A × A × A.

Solution:

The A × A × A for a non-empty set A is given by

A × A × A = {(a, b, c): a, b, c ∈ A}

Here, it is given A = {–1, 1}

So,

A × A × A = {(–1, –1, –1), (–1, –1, 1), (–1, 1, –1), (–1, 1, 1), (1, –1, –1), (1, –1, 1), (1, 1, –1), (1, 1, 1)}

6. If A × B = {(a, x), (a, y), (b, x), (b, y)}. Find A and B.

Solution:

Given,

A × B = {(a, x), (a, y), (b, x), (b, y)}

We know that the Cartesian product of two non-empty sets, P and Q is given by:

P × Q = {(p, q): p ∈ P, q ∈ Q}

Hence, A is the set of all first elements, and B is the set of all second elements.

Therefore, A = {a, b} and B = {x, y}

4 MARK QUESTIONS

1. Let A = {1, 2}, B = {1, 2, 3, 4}, C = {5, 6} and D = {5, 6, 7, 8}. Verify that

(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(ii) $A \times C$ is a subset of $B \times D$

Solution:

Given,

$A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$, $C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$

(i) To verify: $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Now, $B \cap C = \{1, 2, 3, 4\} \cap \{5, 6\} = \Phi$

Thus,

L.H.S. = $A \times (B \cap C) = A \times \Phi = \Phi$

Next,

$A \times B = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4)\}$

$A \times C = \{(1, 5), (1, 6), (2, 5), (2, 6)\}$

Thus,

R.H.S. = $(A \times B) \cap (A \times C) = \Phi$

Therefore, L.H.S. = R.H.S.

Hence verified

(ii) To verify: $A \times C$ is a subset of $B \times D$

First,

$A \times C = \{(1, 5), (1, 6), (2, 5), (2, 6)\}$

And,

B × D = {(1, 5), (1, 6), (1, 7), (1, 8), (2, 5), (2, 6), (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8)}

Now, it's clearly seen that all the elements of set A × C are the elements of set B × D.

Thus, A × C is a subset of B × D.

Hence verified

2. Let A = {1, 2} and B = {3, 4}. Write A × B. How many subsets will A × B have? List them.

Solution:

Given,

A = {1, 2} and B = {3, 4}

So,

A × B = {(1, 3), (1, 4), (2, 3), (2, 4)}

Number of elements in A × B is n(A × B) = 4

We know that,

If C is a set with $n(C) = m$, then $n[P(C)] = 2^m$.

Thus, the set A × B has $2^4 = 16$ subsets.

And these subsets are as given below:

Φ, {(1, 3)}, {(1, 4)}, {(2, 3)}, {(2, 4)}, {(1, 3), (1, 4)}, {(1, 3), (2, 3)}, {(1, 3), (2, 4)}, {(1, 4), (2, 3)}, {(1, 4), (2, 4)}, {(2, 3), (2, 4)}, {(1, 3), (1, 4), (2, 3)}, {(1, 3), (1, 4), (2, 4)}, {(1, 3), (2, 3), (2, 4)}, {(1, 4), (2, 3), (2, 4)}, {(1, 3), (1, 4), (2, 3), (2, 4)}

3. Let A and B be two sets such that n(A) = 3 and n (B) = 2. If (x, 1), (y, 2), (z, 1) are in A × B, find A and B, where x, y and z are distinct elements.

Solution:

Given,

$n(A) = 3$ and $n(B) = 2$; and $(x, 1)$, $(y, 2)$, $(z, 1)$ are in A × B.

We know that,

A = Set of first elements of the ordered pair elements of A × B

B = Set of second elements of the ordered pair elements of A × B

So, clearly, x, y, and z are the elements of A; and

1 and 2 are the elements of B.

As $n(A) = 3$ and $n(B) = 2$, it is clear that set A = $\{x, y, z\}$ and set B = $\{1, 2\}$

4. Define a relation R on the set N of natural numbers by R = $\{(x, y): y = x + 5$, x is a natural number less than 4; $x, y \in N\}$. Depict this relationship using roster form. Write down the domain and the range.

Solution:

The relation R is given by:

R = $\{(x, y): y = x + 5$, x is a natural number less than 4, $x, y \in \mathbf{N}\}$

The natural numbers less than 4 are 1, 2, and 3.

So,

R = $\{(1, 6), (2, 7), (3, 8)\}$

Now,

The domain of R is the set of all first elements of the ordered pairs in the relation.

Hence, Domain of R = $\{1, 2, 3\}$

The range of R is the set of all second elements of the ordered pairs in the relation.

Hence, Range of R = $\{6, 7, 8\}$

5. Let f be the subset of Z × Z defined by $f = \{(ab, a + b): a, b \in Z\}$. Is f a function from Z to Z: justify your answer.

Solution:

Given relation, f is defined as

$f = \{(ab, a + b): a, b \in Z\}$

We know that a relation f from a set A to a set B is said to be a function if every element of set A has unique images in set B.

As $2, 6, -2, -6 \in Z$, $(2 \times 6, 2 + 6), (-2 \times -6, -2 + (-6)) \in f$

i.e., $(12, 8), (12, -8) \in f$

It's clearly seen that the same first element, 12, corresponds to two different images (8 and –8).

Therefore, the relation f is not a function.

6. Let A = {9, 10, 11, 12, 13} and let f: A → N be defined by $f(n)$ = the highest prime factor of n. Find the range of f.

Solution:

Given,

A = {9, 10, 11, 12, 13}

Now, f: A → **N** is defined as

$f(n)$ = The highest prime factor of n

So,

Prime factor of 9 = 3

Prime factors of 10 = 2, 5

Prime factor of 11 = 11

Prime factors of 12 = 2, 3

Prime factor of 13 = 13

Thus, it can be expressed as

$f(9)$ = The highest prime factor of 9 = 3

$f(10)$ = The highest prime factor of 10 = 5

$f(11)$ = The highest prime factor of 11 = 11

$f(12)$ = The highest prime factor of 12 = 3

$f(13)$ = The highest prime factor of 13 = 13

The range of f is the set of all $f(n)$, where $n \in$ A.

Therefore,

Range of f = {3, 5, 11, 13}

7 MARK QUESTIONS

1. The Cartesian product A × A has 9 elements among which are found (–1, 0) and (0, 1). Find the set A and the remaining elements of A × A.

Solution:

We know that,

If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.

Also, $n(A \times A) = n(A) \times n(A)$

Given,

$n(A \times A) = 9$

So, $n(A) \times n(A) = 9$

Thus, $n(A) = 3$

Also, given that the ordered pairs (–1, 0) and (0, 1) are two of the nine elements of A × A.

And, we know in $A \times A = \{(a, a): a \in A\}$

Thus, –1, 0, and 1 have to be the elements of A.

As $n(A) = 3$, clearly A = {–1, 0, 1}

Hence, the remaining elements of set A × A are as follows:

(–1, –1), (–1, 1), (0, –1), (0, 0), (1, –1), (1, 0), and (1, 1)

2. Let A = {1, 2, 3, … , 14}. Define a relation R from A to A by R = {(x, y): $3x – y = 0$, where $x, y \in A$}. Write down its domain, codomain and range.

Solution:

The relation R from A to A is given as:

R = {(x, y): $3x - y = 0$, where x, $y \in$ A}

= {(x, y): $3x = y$, where x, $y \in$ A}

So,

R = {(1, 3), (2, 6), (3, 9), (4, 12)}

Now,

The domain of R is the set of all first elements of the ordered pairs in the relation.

Hence, Domain of R = {1, 2, 3, 4}

The whole set A is the codomain of the relation R.

Hence, Codomain of R = A = {1, 2, 3, …, 14}

The range of R is the set of all second elements of the ordered pairs in the relation.

Hence, Range of R = {3, 6, 9, 12}

3. A = {1, 2, 3, 5} and B = {4, 6, 9}. Define a relation R from A to B by R = {(x, y): the difference between x and y is odd; $x \in$ A, $y \in$ B}. Write R in roster form.

Solution:

Given,

A = {1, 2, 3, 5} and B = {4, 6, 9}

The relation from A to B is given as

R = {(x, y): the difference between x and y is odd; $x \in$ A, $y \in$ B}

Thus,

R = {(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)}

4. The figure shows a relationship between the sets P and Q. Write this relation

(i) in set-builder form (ii) in roster form

What is its domain and range?

Solution:

From the given figure, it's seen that

P = {5, 6, 7}, Q = {3, 4, 5}

The relation between P and Q:

Set-builder form

(i) R = {(x, y): $y = x - 2$; $x \in$ P} or R = {(x, y): $y = x - 2$ for x = 5, 6, 7}

Roster form

(ii) R = {(5, 3), (6, 4), (7, 5)}

Domain of R = {5, 6, 7}

Range of R = {3, 4, 5}

5. Let A = {1, 2, 3, 4, 6}. Let R be the relation on A defined by

{(*a*, *b*): *a*, *b* ∈ A, *b* is exactly divisible by *a*}.

(i) Write R in roster form

(ii) Find the domain of R

(iii) Find the range of R

Solution:

Given,

A = {1, 2, 3, 4, 6} and relation R = {(a, b): a, $b \in$ A, b is exactly divisible by a}

Hence,

(i) R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)}

(ii) Domain of R = {1, 2, 3, 4, 6}

(iii) Range of R = {1, 2, 3, 4, 6}

6. Determine the domain and range of the relation R defined by R = {(x, x + 5): $x \in$ {0, 1, 2, 3, 4, 5}}.

Solution:

Given,

Relation R = {(x, x + 5): $x \in$ {0, 1, 2, 3, 4, 5}}

Thus,

R = {(0, 5), (1, 6), (2, 7), (3, 8), (4, 9), (5, 10)}

So,

Domain of R = {0, 1, 2, 3, 4, 5} and,

Range of R = {5, 6, 7, 8, 9, 10}

7. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.

(i) {(2, 1), (5, 1), (8, 1), (11, 1), (14, 1), (17, 1)}

(ii) {(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)}

(iii) {(1, 3), (1, 5), (2, 5)}

Solution:

(i) {(2, 1), (5, 1), (8, 1), (11, 1), (14, 1), (17, 1)}

As 2, 5, 8, 11, 14, and 17 are the elements of the domain of the given relation having their unique images, this relation can be called a function.

Here, domain = {2, 5, 8, 11, 14, 17} and range = {1}

(ii) {(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)}

As 2, 4, 6, 8, 10, 12, and 14 are the elements of the domain of the given relation having their unique images, this relation can be called a function.

Here, domain = {2, 4, 6, 8, 10, 12, 14} and range = {1, 2, 3, 4, 5, 6, 7}

(iii) {(1, 3), (1, 5), (2, 5)}

It's seen that the same first element, i.e., 1, corresponds to two different images, i.e., 3 and 5; this relation cannot be called a function.

8. Find the domain and range of the following real function:

(i) $f(x) = -|x|$ (ii) $f(x) = \sqrt{(9 - x^2)}$

Solution:

(i) Given,

$f(x) = -|x|, x \in R$

We know that,

As $f(x)$ is defined for $x \in R$, the domain of f is R.

It is also seen that the range of $f(x) = -|x|$ is all real numbers except positive real numbers.

Therefore, the range of f is given by $(-\infty, 0]$.

(ii) $f(x) = \sqrt{(9 - x^2)}$

As $\sqrt{(9 - x^2)}$ is defined for all real numbers that are greater than or equal to -3 and less than or equal to 3, for $9 - x^2 \geq 0$.

So, the domain of $f(x)$ is $\{x: -3 \leq x \leq 3\}$ or $[-3, 3]$.

Now,

For any value of x in the range [–3, 3], the value of $f(x)$ will lie between 0 and 3.

Therefore, the range of $f(x)$ is $\{x: 0 \le x \le 3\}$ or [0, 3].

9. Let f, g: R → R be defined, respectively by $f(x) = x + 1$, $g(x) = 2x – 3$. Find $f + g$, $f – g$ and f/g.

Solution:

Given the functions f, g: R → R is defined as

$f(x) = x + 1$, $g(x) = 2x – 3$

Now,

$(f + g)(x) = f(x) + g(x) = (x + 1) + (2x – 3) = 3x – 2$

Thus, $(f + g)(x) = 3x – 2$

$(f – g)(x) = f(x) – g(x) = (x + 1) – (2x – 3) = x + 1 – 2x + 3 = – x + 4$

Thus, $(f – g)(x) = –x + 4$

$f/g(x) = f(x)/g(x)$, $g(x) \neq 0$, $x \in R$

$f/g(x) = x + 1/ 2x – 3$, $2x – 3 \neq 0$

Thus, $f/g(x) = x + 1/ 2x – 3$, $x \neq 3/2$

10. Let $f = \{(1, 1), (2, 3), (0, –1), (–1, –3)\}$ be a function from Z to Z defined by $f(x) = ax + b$, for some integers a, b. Determine a, b.

Solution:

Given, $f = \{(1, 1), (2, 3), (0, –1), (–1, –3)\}$

And the function defined as, $f(x) = ax + b$

For $(1, 1) \in f$

We have, $f(1) = 1$

So, $a \times 1 + b = 1$

$a + b = 1$ …. (i)

And for $(0, -1) \in f$

We have $f(0) = -1$

$a \times 0 + b = -1$

$b = -1$

On substituting $b = -1$ in (i), we get

$a + (-1) = 1 \Rightarrow a = 1 + 1 = 2$.

Therefore, the values of a and b are 2 and –1, respectively.

11. Let R be a relation from N to N defined by R = {(a, b): a, b ∈ N and $a = b^2$}. Are the following true?

(i) (a, a) ∈ R, for all a ∈ N
(ii) (a, b) ∈ R, implies (b, a) ∈ R

(iii) (a, b) ∈ R, (b, c) ∈ R implies (a, c) ∈ R

Justify your answer in each case.

Solution:

Given relation R = {(a, b): a, b ∈ N and $a = b^2$}

(i) It can be seen that 2 ∈ N; however, $2 \neq 2^2 = 4$.

Thus, the statement "(a, a) ∈ R, for all a ∈ N" is not true.

(ii) Its clearly seen that (9, 3) ∈ N because 9, 3 ∈ N and $9 = 3^2$.

Now, $3 \neq 9^2 = 81$; therefore, (3, 9) ∉ N

Thus, the statement "(a, b) ∈ R, implies (b, a) ∈ R" is not true.

(iii) It's clearly seen that (16, 4) ∈ R, (4, 2) ∈ R because 16, 4, 2 ∈ N and $16 = 4^2$ and $4 = 2^2$.

Now, $16 \neq 2^2 = 4$; therefore, (16, 2) ∉ N

Thus, the statement "(a, b) ∈ R, (b, c) ∈ R implies (a, c) ∈ R" is not true.

12. Let A = {1, 2, 3, 4}, B = {1, 5, 9, 11, 15, 16} and *f* = {(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)}. Are the following true?

(i) *f* is a relation from A to B (ii) *f* is a function from A to B

Justify your answer in each case.

Solution:

Given,

A = {1, 2, 3, 4} and B = {1, 5, 9, 11, 15, 16}

So,

A × B = {(1, 1), (1, 5), (1, 9), (1, 11), (1, 15), (1, 16), (2, 1), (2, 5), (2, 9), (2, 11), (2, 15), (2, 16), (3, 1), (3, 5), (3, 9), (3, 11), (3, 15), (3, 16), (4, 1), (4, 5), (4, 9), (4, 11), (4, 15), (4, 16)}

Also, given that,

f = {(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)}

(i) A relation from a non-empty set A to a non-empty set B is a subset of the Cartesian product A × B.

It's clearly seen that *f* is a subset of A × B.

Therefore, *f* is a relation from A to B.

(ii) As the same first element, i.e., 2 corresponds to two different images (9 and 11), relation *f* is not a function.

SUMMARY

Relation: A relation from set A to set B is the set of ordered pairs from A to B. Function: A function from set A to set B is a relation such that every element of A is mapped to exactly one element of B.

CHAPTER-III

TRIGONOMETRIC FUNCTIONS

2 MARK QUESTIONS

1. Find the degree measures corresponding to the following radian measures (Use π = 22/7)

(i) 11/16

(ii) -4

(iii) 5π/3

(iv) 7π/6

Solution:

(i) 11/16

Here π radian = 180°

(ii) -4

Here π radian = 180°

(iii) 5π/3

Here π radian = 180°

We get

= 300^o

(iv) $7\pi/6$

Here π radian = 180°

We get

= 210^o

2. A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?

Solution:

It is given that

No. of revolutions made by the wheel in

1 minute = 360

1 second = 360/60 = 6

We know that

The wheel turns an angle of 2π radian in one complete revolution.

In 6 complete revolutions, it will turn an angle of $6 \times 2\pi$ radian = 12π radian

Therefore, in one second, the wheel turns an angle of 12π radian.

3. sin x = 3/5, x lies in second quadrant.

Solution:

It is given that

sin x = 3/5

We can write it as

We know that

$\sin^2 x + \cos^2 x = 1$

We can write it as

$\cos^2 x = 1 - \sin^2 x$

4. sin 765°

Solution:

We know that values of sin x repeat after an interval of 2π or 360°

So we get

By further calculation

$= \sin 45^o$

$= 1/\sqrt{2}$

5. cosec (–1410°)

Solution:

We know that values of cosec x repeat after an interval of 2π or 360°

So we get

By further calculation

$= \text{cosec } 30^o = 2$

4 MARK QUESTIONS

1. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.

Solution:

The dimensions of the circle are

Diameter = 40 cm

Radius = 40/2 = 20 cm

Consider AB be as the chord of the circle i.e. length = 20 cm

In ΔOAB,

Radius of circle = OA = OB = 20 cm

Similarly AB = 20 cm

Hence, ΔOAB is an equilateral triangle.

$\theta = 60° = \pi/3$ radian

In a circle of radius *r* unit, if an arc of length *l* unit subtends an angle ϑ radian at the centre

We get $\theta = 1/r$

Therefore, the length of the minor arc of the chord is $20\pi/3$ cm.

2. Find the angle in radian though which a pendulum swings if its length is 75 cm and the tip describes an arc of length

(i) 10 cm (ii) 15 cm (iii) 21 cm

Solution:

In a circle of radius r unit, if an arc of length l unit subtends an angle θ radian at the centre, then θ = 1/r

We know that r = 75 cm

(i) l = 10 cm

So we get

θ = 10/75 radian

By further simplification

θ = 2/15 radian

(ii) l = 15 cm

So we get

θ = 15/75 radian

By further simplification

θ = 1/5 radian

(iii) l = 21 cm

So we get

θ = 21/75 radian

By further simplification

θ = 7/25 radian

3. cot x = 3/4, x lies in third quadrant.

Solution:

It is given that

cot x = 3/4

We can write it as

We know that

$1 + \tan^2 x = \sec^2 x$

We can write it as

$1 + (4/3)^2 = \sec^2 x$

Substituting the values

$1 + 16/9 = \sec^2 x$

$\cos^2 x = 25/9$

$\sec x = \pm 5/3$

Here x lies in the third quadrant so the value of sec x will be negative

$\sec x = -5/3$

We can write it as

4. sec x = 13/5, x lies in fourth quadrant.

Solution:

It is given that

$\sec x = 13/5$

We can write it as

We know that

$\sin^2 x + \cos^2 x = 1$

We can write it as

$\sin^2 x = 1 - \cos^2 x$

Substituting the values

$\sin^2 x = 1 - (5/13)^2$

$\sin^2 x = 1 - 25/169 = 144/169$

$\sin^2 x = \pm 12/13$

Here x lies in the fourth quadrant so the value of sin x will be negative

$\sin x = - 12/13$

We can write it as

5. tan x = -5/12, x lies in second quadrant.

Solution:

It is given that

$\tan x = - 5/12$

We can write it as

We know that

$1 + \tan^2 x = \sec^2 x$

We can write it as

$1 + (-5/12)^2 = \sec^2 x$

Substituting the values

$1 + 25/144 = \sec^2 x$

$\sec^2 x = 169/144$

$\sec x = \pm 13/12$

Here x lies in the second quadrant so the value of sec x will be negative sec x = – 13/12

We can write it as

6. $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$

Solution:

We get

= [2 cos 4x cos (-2x)] [-2 sin 4x sin (-2x)]

It can be written as

= [2 cos 4*x* cos 2*x*] [–2 sin 4*x* (–sin 2*x*)]

So we get

= (2 sin 4*x* cos 4*x*) (2 sin 2*x* cos 2*x*)

= sin 8x sin 4x

= RHS

7. $\cot 4x (\sin 5x + \sin 3x) = \cot x (\sin 5x - \sin 3x)$

Solution:

Consider

LHS = cot 4x (sin 5x + sin 3x)

It can be written as

Using the formula

$= 2 \cos 4x \cos x$

Hence, LHS = RHS.

7 MARK QUESTIONS

1. $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$

Solution:

By further simplification

$= 2 \sin 4x \cos (-2x) + 2 \sin 4x$

It can be written as

$= 2 \sin 4x \cos 2x + 2 \sin 4x$

Taking common terms

$= 2 \sin 4x (\cos 2x + 1)$

Using the formula

$= 2 \sin 4x (2 \cos^2 x - 1 + 1)$

We get

$= 2 \sin 4x (2 \cos^2 x)$

$= 4\cos^2 x \sin 4x$

= R.H.S.

2. $\cos 4x = 1 - 8\sin^2 x \cos^2 x$

Solution:

Consider

LHS = $\cos 4x$

We can write it as

$= \cos 2(2x)$

Using the formula $\cos 2A = 1 - 2\sin^2 A$

$= 1 - 2\sin^2 2x$

Again by using the formula $\sin 2A = 2\sin A \cos A$

$= 1 - 2(2\sin x \cos x)^2$

So we get

$= 1 - 8\sin^2 x \cos^2 x$

= R.H.S.

3. $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$

Solution:

Consider

L.H.S. $= \cos 6x$

It can be written as

$= \cos 3(2x)$

Using the formula $\cos 3A = 4\cos^3 A - 3\cos A$

$= 4\cos^3 2x - 3\cos 2x$

Again by using formula $\cos 2x = 2\cos^2 x - 1$

$= 4[(2\cos^2 x - 1)^3 - 3(2\cos^2 x - 1)$

By further simplification

$= 4[(2\cos^2 x)^3 - (1)^3 - 3(2\cos^2 x)^2 + 3(2\cos^2 x)] - 6\cos^2 x + 3$

We get

$= 4[8\cos^6 x - 1 - 12\cos^4 x + 6\cos^2 x] - 6\cos^2 x + 3$

By multiplication

$= 32 \cos^6 x - 4 - 48 \cos^4 x + 24 \cos^2 x - 6 \cos^2 x + 3$

On further calculation

$= 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1$

= R.H.S.

4. $\sec^2 2x = 1 - \tan 2x$

Solution:

It is given that

$\sec^2 2x = 1 - \tan 2x$

We can write it as

$1 + \tan^2 2x = 1 - \tan 2x$

$\tan^2 2x + \tan 2x = 0$

Taking common terms

$\tan 2x (\tan 2x + 1) = 0$

Here

$\tan 2x = 0$ or $\tan 2x + 1 = 0$

If $\tan 2x = 0$

$\tan 2x = \tan 0$

We get

$2x = n\pi + 0$, where $n \in Z$

$x = n\pi/2$, where $n \in Z$

$\tan 2x + 1 = 0$

We can write it as

$\tan 2x = -1$

So we get

Here

$2x = n\pi + 3\pi/4$, where $n \in Z$

$x = n\pi/2 + 3\pi/8$, where $n \in Z$

Hence, the general solution is $n\pi/2$ or $n\pi/2 + 3\pi/8$, $n \in Z$.

5. sin x + sin 3x + sin 5x = 0

Solution:

It is given that

$\sin x + \sin 3x + \sin 5x = 0$

We can write it as

$(\sin x + \sin 5x) + \sin 3x = 0$

Using the formula

By further calculation

$2 \sin 3x \cos (-2x) + \sin 3x = 0$

It can be written as

$2 \sin 3x \cos 2x + \sin 3x = 0$

By taking out the common terms

sin 3x (2 cos 2x + 1) = 0

Here

sin 3x = 0 or 2 cos 2x + 1 = 0

If sin 3x = 0

$3x = n\pi$, where $n \in Z$

We get

$x = n\pi/3$, where $n \in Z$

If 2 cos 2x + 1 = 0

cos 2x = – 1/2

By further simplification

$= -\cos \pi/3$

$= \cos (\pi - \pi/3)$

So we get

$\cos 2x = \cos 2\pi/3$

Here

6. (sin 3*x* + sin *x*) sin *x* + (cos 3*x* – cos *x*) cos *x* = 0

Solution:

Consider

LHS = (sin 3x + sin x) sin x + (cos 3x – cos x) cos x

By further calculation

$= \sin 3x \sin x + \sin^2 x + \cos 3x \cos x - \cos^2 x$

Taking out the common terms

$= \cos 3x \cos x + \sin 3x \sin x - (\cos^2 x - \sin^2 x)$

Using the formula

cos (A – B) = cos A cos B + sin A sin B

= cos (3x – x) – cos 2x

So we get

= cos 2x – cos 2x

= 0

= RHS

SUMMARY

The trigonometric functions are the basis of all trigonometry. They assign real Numbers to angle measures based on certain ratios. There are six trigonometric functions: sine, cosine, tangent, cosecant, secant, and cotangent.

CHAPTER-IV

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

2 MARK QUESTIONS

1. (5i) (-3/5i)

Solution:

(5i) (-3/5i) = 5 x (-3/5) x i^2

= -3 x -1 [i^2 = -1]

= 3

Hence,

(5i) (-3/5i) = 3 + i0

2. $i^9 + i^{19}$

Solution:

$i^9 + i^{19} = (i^2)^4 . i + (i^2)^9 . i$

$= (-1)^4 . i + (-1)^9 .i$

= 1 x i + -1 x i

= i – i

= 0

Hence,

$i^9 + i^{19}$ = 0 + i0

3. i^{-39}

Solution:

$i^{-39} = 1/ i^{39} = 1/ i^{4 \times 9 + 3} = 1/ (1^9 \times i^3) = 1/ i^3 = 1/ (-i)$ [$i^4 = 1$, $i^3 = -I$ and $i^2 = -1$]

Now, multiplying the numerator and denominator by i we get

$i^{-39} = 1 \times i / (-i \times i)$

$= i/ 1 = i$

Hence,

$i^{-39} = 0 + i$

4. 3(7 + *i*7) + *i*(7 + *i*7)

Solution:

$3(7 + i7) + i(7 + i7) = 21 + i21 + i7 + i^2 7$

$= 21 + i28 – 7$ [$i^2 = -1$]

$= 14 + i28$

Hence,

$3(7 + i7) + i(7 + i7) = 14 + i28$

5. (1 – *i*) – (–1 + *i*6)

Solution:

$(1 – i) – (–1 + i6) = 1 – i + 1 – i6$

$= 2 – i7$

Hence,

$(1 - i) - (-1 + i6) = 2 - i7$

6. $(1 - i)^4$

Solution:

$(1 - i)^4 = [(1 - i)^2]^2$

$= [1 + i^2 - 2i]^2$

$= [1 - 1 - 2i]^2$ $[i^2 = -1]$

$= (-2i)^2$

$= 4(-1)$

$= -4$

Hence, $(1 - i)^4 = -4 + 0i$

7. 4 – 3i

Solution:

Let's consider $z = 4 - 3i$

Then,

$= 4 + 3i$ and

$|z|^2 = 4^2 + (-3)^2 = 16 + 9 = 25$

Thus, the multiplicative inverse of $4 - 3i$ is given by z^{-1}

8. $\sqrt{5} + 3i$

Solution:

Let's consider $z = \sqrt{5} + 3i$

$|z|^2 = (\sqrt{5})^2 + 3^2 = 5 + 9 = 14$

Thus, the multiplicative inverse of $\sqrt{5} + 3i$ is given by z^{-1}

4 MARK QUESTIONS

1. $x^2 + 3 = 0$

Solution:

Given the quadratic equation,

$x^2 + 3 = 0$

On comparing it with $ax^2 + bx + c = 0$, we have

$a = 1$, $b = 0$, and $c = 3$

So, the discriminant of the given equation will be

$D = b^2 - 4ac = 0^2 - 4 \times 1 \times 3 = -12$

Hence, the required solutions are

2. $2x^2 + x + 1 = 0$

Solution:

Given the quadratic equation,

$2x^2 + x + 1 = 0$

On comparing it with $ax^2 + bx + c = 0$, we have

$a = 2$, $b = 1$, and $c = 1$

So, the discriminant of the given equation will be

$D = b^2 - 4ac = 1^2 - 4 \times 2 \times 1 = 1 - 8 = -7$

Hence, the required solutions are

3. $x^2 + 3x + 9 = 0$

Solution:

Given the quadratic equation,

$x^2 + 3x + 9 = 0$

On comparing it with $ax^2 + bx + c = 0$, we have

$a = 1$, $b = 3$, and $c = 9$

So, the discriminant of the given equation will be

$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 9 = 9 - 36 = -27$

Hence, the required solutions are

4. $-x^2 + x - 2 = 0$

Solution:

Given the quadratic equation,

$-x^2 + x - 2 = 0$

On comparing it with $ax^2 + bx + c = 0$, we have

$a = -1$, $b = 1$, and $c = -2$

So, the discriminant of the given equation will be

$D = b^2 - 4ac = 1^2 - 4 \times (-1) \times (-2) = 1 - 8 = -7$

Hence, the required solutions are

5. $x^2 + 3x + 5 = 0$

Solution:

Given the quadratic equation,

$x^2 + 3x + 5 = 0$

On comparing it with $ax^2 + bx + c = 0$, we have

$a = 1$, $b = 3$, and $c = 5$

So, the discriminant of the given equation will be

$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 5 = 9 - 20 = -11$

Hence, the required solutions are

6. $x^2 - x + 2 = 0$

Solution:

Given the quadratic equation,

$x^2 - x + 2 = 0$

On comparing it with $ax^2 + bx + c = 0$, we have

$a = 1$, $b = -1$, and $c = 2$

So, the discriminant of the given equation is

$D = b^2 - 4ac = (-1)^2 - 4 \times 1 \times 2 = 1 - 8 = -7$

Hence, the required solutions are

7. $\sqrt{2}x^2 + x + \sqrt{2} = 0$

Solution:

Given the quadratic equation,

$\sqrt{2}x^2 + x + \sqrt{2} = 0$

On comparing it with $ax^2 + bx + c = 0$, we have

$a = \sqrt{2}$, $b = 1$, and $c = \sqrt{2}$

So, the discriminant of the given equation is

$D = b^2 - 4ac = (1)^2 - 4 \times \sqrt{2} \times \sqrt{2} = 1 - 8 = -7$

Hence, the required solutions are

7 MARK QUESTIONS

1. $x^2 + x/\sqrt{2} + 1 = 0$

Solution:

Given the quadratic equation,

$x^2 + x/\sqrt{2} + 1 = 0$

It can be rewritten as,

$\sqrt{2}x^2 + x + \sqrt{2} = 0$

On comparing it with $ax^2 + bx + c = 0$, we have

$a = \sqrt{2}$, $b = 1$, and $c = \sqrt{2}$

So, the discriminant of the given equation is

$D = b^2 - 4ac = (1)^2 - 4 \times \sqrt{2} \times \sqrt{2} = 1 - 8 = -7$

Hence, the required solutions are

2. Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

Solution:

Let's assume $z = (x - iy)(3 + 5i)$

And,

$(3x + 5y) - i(5x - 3y) = -6 - 24i$

On equating real and imaginary parts, we have

$3x + 5y = -6$ …… (i)

$5x - 3y = 24$ …… (ii)

Performing (i) x 3 + (ii) x 5, we get

$(9x + 15y) + (25x - 15y) = -18 + 120$

$34x = 102$

$x = 102/34 = 3$

Putting the value of *x* in equation (i), we get

$3(3) + 5y = -6$

$5y = -6 - 9 = -15$

$y = -3$

Therefore, the values of *x* and *y* are 3 and –3, respectively.

3.Express the given complex number (-3) in the polar form.

Solution:

Given, complex number is -3.

Let $r \cos \theta = -3$ …(1)

and $r \sin \theta = 0$ …(2)

Squaring and adding (1) and (2), we get

$r^2\cos^2\theta + r^2\sin^2\theta = (-3)^2$

Take r^2 outside from L.H.S, we get

$r^2(\cos^2\theta + \sin^2\theta) = 9$

We know that, $\cos^2\theta + \sin^2\theta = 1$, then the above equation becomes,

$r^2 = 9$

$r = 3$ (Conventionally, $r > 0$)

Now, subsbtitute the value of r in (1) and (2)

3 cos θ = -3 and 3 sin θ = 0

cos θ = -1 and sin θ = 0

Therefore, θ = π

Hence, the polar representation is,

-3 = r cos θ + i r sin θ

3 cos π + 3 sin π = 3(cos π + i sin π)

Thus, the required polar form is 3 cos π+ 3i sin π = 3(cos π+i sin π)

4.Solve the given quadratic equation $2x^2 + x + 1 = 0$.

Solution:

Given quadratic equation: $2x^2 + x + 1 = 0$

Now, compare the given quadratic equation with the general form $ax^2 + bx + c = 0$

On comparing, we get

a = 2, b = 1 and c = 1

Therefore, the discriminant of the equation is:

$D = b^2 - 4ac$

Now, substitute the values in the above formula

$D = (1)^2 - 4(2)(1)$

D = 1- 8

D = -7

Therefore, the required solution for the given quadratic equation is

$x =[-b \pm \sqrt{D}]/2a$

$x = [-1 \pm \sqrt{-7}]/2(2)$

We know that, $\sqrt{-1} = i$

$x = [-1 \pm \sqrt{7}i] / 4$

Hence, the solution for the given quadratic equation is $(-1 \pm \sqrt{7}i) / 4$.

5.For any two complex numbers z_1 and z_2, show that $Re(z_1z_2) = Rez_1\ Rez_2 – Imz_1Imz_2$

Solution:

Given: z_1 and z_2 are the two complex numbers

To prove: $Re(z_1z_2) = Rez_1\ Rez_2 – Imz_1Imz_2$

Let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$

Now, $z_1\ z_2 =(x_1+iy_1)(x_2+iy_2)$

Now, split the real part and the imaginary part from the above equation:

$\Rightarrow x_1(x_2+iy_2) +iy_1(x_2+iy_2)$

Now, multiply the terms:

$= x_1x_2+ix_1y_2+ix_2y_1+i^2y_1y_2$

We know that, $i^2 = -1$, then we get

$= x_1x_2+ix_1y_2+ix_2y_1-y_1y_2$

Now, again seperate the real and the imaginary part:

$= (x_1x_2 -y_1y_2) +i (x_1y_2+x_2y_1)$

From the above equation, take only the real part:

$\Rightarrow Re (z_1z_2) =(x_1x_2 -y_1y_2)$

It means that,

$\Rightarrow Re(z_1z_2) = Rez_1\ Rez_2 - Imz_1Imz_2$

Hence, the given statement is proved.

6.Find the modulus of [(1+i)/(1-i)] – [(1-i)/(1+i)]

Solution:

Given: [(1+i)/(1-i)] – [(1-i)/(1+i)]

Simplify the given expression, we get:

$[(1+i)/(1-i)] - [(1-i)/(1+i)] = [(1+i)^2 - (1-i)^2]/ [(1+i)(1-i)]$
$= (1+i^2+2i-1-i^2+2i)) / (1^2+1^2)$

Now, cancel out the terms,

$= 4i/2$

$= 2i$

Now, take the modulus,

$| [(1+i)/(1-i)] - [(1-i)/(1+i)]| = |2i| = \sqrt{2^2} = 2$

Therefore, the modulus of [(1+i)/(1-i)] – [(1-i)/(1+i)] is 2.

SUMMARY

Complex numbers and quadratic equations are intimately connected, particularly when dealing with equations that lack real solutions. Here's a concise summary:

- **Complex Numbers**: These are numbers expressed in the form a+bia + bia+bi, where aaa and bbb are real numbers and iii is the imaginary unit (i2=−1i^2 = -1i2=−1).

- **Quadratic Equations**: These are second-degree polynomial equations in the form ax2+bx+c=0ax^2 + bx + c = 0ax2+bx+c=0, where aaa, bbb, and ccc are constants and xxx is the variable.

CHAPTER-V

LINEAR INEQUALITIES

2 MARK QUESTIONS

1. $x/3 > x/2 + 1$

Solution:

$-x/6 > 1$

$-x > 6$

$x < -6$

∴ The solutions of the given inequality are defined by all the real numbers less than – 6.

Hence, the required solution set is $(-\infty, -6)$

2. Ravi obtained 70 and 75 marks in the first two unit tests. Find the minimum marks he should get in the third test to have an average of at least 60 marks.

Solution:

Let us assume that x is the marks obtained by Ravi in his third unit test.

According to the question, all the students should have an average of at least 60 marks

$(70 + 75 + x)/3 \geq 60$

$= 145 + x \geq 180$

$= x \geq 180 - 145$

$= x \geq 35$

Hence, all the students must obtain 35 marks in order to have an average of at least 60 marks

3. To receive Grade 'A' in a course, one must obtain an average of 90 marks or more in five examinations (each of 100 marks). If Sunita's marks in the first four examinations are 87, 92, 94 and 95, find the minimum marks that Sunita must obtain in the fifth examination to get Grade 'A' in the course.

Solution:

Let us assume Sunita scored x marks in her fifth examination

Now, according to the question, in order to receive A grade in the course, she must obtain an average of 90 marks or more in her five examinations

$(87 + 92 + 94 + 95 + x)/5 \geq 90$

$= (368 + x)/5 \geq 90$

$= 368 + x \geq 450$

$= x \geq 450 - 368$

$= x \geq 82$

Hence, she must obtain 82 or more marks in her fifth examination

4. $2x + y \geq 6$

Solution:

Given $2x + y \geq 6$

Now, draw a solid line $2x + y = 6$ in the graph ($\because 2x + y = 6$ is included in the given question)

Now, consider $2x + y \geq 6$

Select a point (0, 0)

$\Rightarrow 2 \times (0) + 0 \geq 6$

⇒ $0 \geq 6$ (this is false)

∴ Solution region of the given inequality is above the line $2x + y = 6$. (away from the origin)

The graph is as follows:

5. $3x + 4y \leq 12$

Solution:

Given $3x + 4y \leq 12$

Now, draw a solid line $3x + 4y = 12$ in the graph (∵$3x + 4y = 12$ is included in the given question)

Now, consider $3x + 4y \leq 12$

Select a point (0, 0)

⇒ $3 \times (0) + 4 \times (0) \leq 12$

⇒ $0 \leq 12$ (this is true)

∴ Solution region of the given inequality is below the line $3x + 4y = 12$. (i.e., origin is included in the region)

The graph is as follows:

6. $y + 8 \geq 2x$

Solution:

Given $y + 8 \geq 2x$

Now, draw a solid line $y + 8 = 2x$ in the graph (∵$y + 8 = 2x$ is included in the given question)

Now, consider $y + 8 \geq 2x$

Select a point (0, 0)

$\Rightarrow (0) + 8 \geq 2 \times (0)$

$\Rightarrow 0 \leq 8$ (this is true)

$\therefore$ Solution region of the given inequality is above the line y + 8 = 2x. (i.e., origin is included in the region)

The graph is as follows:

7. x – y ≤ 2

Solution:

Given x – y ≤ 2

Now, draw a solid line x – y = 2 in the graph ($\because$ x – y = 2 is included in the given question).

Now, consider x – y ≤ 2

Select a point (0, 0)

$\Rightarrow (0) - (0) \leq 2$

$\Rightarrow 0 \leq 2$ (this is true)

$\therefore$ Solution region of the given inequality is above the line x – y = 2. (i.e., origin is included in the region)

The graph is as follows:

8. y < – 2

Solution:

Given y < – 2

Now, draw a dotted line y = – 2 in the graph ($\because$ y = – 2 is excluded in the given question)

Now, consider y < – 2

Select a point (0, 0)

$\Rightarrow 0 < -2$ (this is false)

$\therefore$ Solution region of the given inequality is below the line $y < -2$. (i.e., away from the origin)

The graph is as follows:

9. $x > -3$

Solution:

Given $x > -3$

Now, draw a dotted line $x = -3$ in the graph ($\because x = -3$ is excluded in the given question)

Now, consider $x > -3$

Select a point (0, 0)

$\Rightarrow 0 > -3$

$\Rightarrow 0 > -3$ (this is true)

$\therefore$ Solution region of the given inequality is right to the line $x > -3$. (i.e., origin is included in the region)

The graph is as follows:

4 MARK QUESTIONS

1. Solve 24x < 100, when

(i) x is a natural number.

(ii) x is an integer.

Solution:

(i) Given that $24x < 100$

Now we have to divide the inequality by 24 then we get $x < 25/6$

Now when x is a natural integer then

It is clear that the only natural number less than 25/6 are 1, 2, 3, 4.

Thus, 1, 2, 3, 4 will be the solution of the given inequality when x is a natural number.

Hence {1, 2, 3, 4} is the solution set.

(ii) Given that $24x < 100$

Now we have to divide the inequality by 24 then we get $x < 25/6$

now when x is an integer then

It is clear that the integer number less than 25/6 are…-1, 0, 1, 2, 3, 4.

Thus, solution of $24x < 100$ are…,-1, 0, 1, 2, 3, 4, when x is an integer.

Hence {…, -1, 0, 1, 2, 3, 4} is the solution set.

2. Solve – 12x > 30, when

(i) x is a natural number.

(ii) x is an integer.

Solution:

(i) Given that, $-12x > 30$

Now, by dividing the inequality by -12 on both sides we get, x < -5/2

When x is a natural integer then

It is clear that there is no natural number less than -2/5 because -5/2 is a negative number and natural numbers are positive numbers.

Therefore there would be no solution of the given inequality when x is a natural number.

(ii) Given that, – 12x > 30

Now by dividing the inequality by -12 on both sides we get, x < -5/2

When x is an integer then

It is clear that the integer number less than -5/2 are…, -5, -4, – 3

Thus, solution of – 12x > 30 is …,-5, -4, -3, when x is an integer.

Therefore the solution set is {…, -5, -4, -3}

3. 3(x – 1) ≤ 2 (x – 3)

Solution:

Given that, 3(x – 1) ≤ 2 (x – 3)

By multiplying, the above inequality can be written as

3x – 3 ≤ 2x – 6

Now, by adding 3 to both the sides, we get

3x – 3+ 3 ≤ 2x – 6+ 3

3x ≤ 2x – 3

Again, by subtracting 2x from both the sides,

$3x - 2x \leq 2x - 3 - 2x$

$x \leq -3$

Therefore, the solutions of the given inequality are defined by all the real numbers less than or equal to -3.

Hence, the required solution set is $(-\infty, -3]$

4. 3 (2 – x) ≥ 2 (1 – x)

Solution:

Given that, $3(2 - x) \geq 2(1 - x)$

By multiplying, we get

$6 - 3x \geq 2 - 2x$

Now, by adding 2x to both the sides,

$6 - 3x + 2x \geq 2 - 2x + 2x$

$6 - x \geq 2$

Again, by subtracting 6 from both the sides, we get

$6 - x - 6 \geq 2 - 6$

$-x \geq -4$

Multiplying throughout inequality by negative sign, we get

$x \leq 4$

∴ The solutions of the given inequality are defined by all the real numbers greater than or equal to 4.

Hence the required solution set is $(-\infty, 4]$

5. 3x – 2 < 2x + 1

Solution:

Given,

$3x - 2 < 2x + 1$

Solving the given inequality, we get

$3x - 2 < 2x + 1$

$= 3x - 2x < 1 + 2$

$= x < 3$

Now, the graphical representation of the solution is as follows:

6. $3(1 - x) < 2(x + 4)$

Solution:

Given,

$3(1 - x) < 2(x + 4)$

Solving the given inequality, we get

$3(1 - x) < 2(x + 4)$

Multiplying, we get

$= 3 - 3x < 2x + 8$

On rearranging, we get

$= 3 - 8 < 2x + 3x$

$= -5 < 5x$

Now by dividing 5 on both sides, we get

$-5/5 < 5x/5$

$= -1 < x$

7. The longest side of a triangle is 3 times the shortest side and the third side is 2 cm shorter than the longest side. If the perimeter of the triangle is at least 61 cm, find the minimum length of the shortest side.

Solution:

Let us assume the length of the shortest side of the triangle to be x cm.

∴ According to the question, the length of the longest side = 3x cm

And, length of third side = (3x – 2) cm

As, the least perimeter of the triangle = 61 cm

Thus, $x + 3x + (3x - 2)$ cm ≥ 61 cm

$= 7x - 2 \geq 61$

$= 7x \geq 63$

Now dividing by 7, we get

$= 7x/7 \geq 63/7$

$= x \geq 9$

Hence, the minimum length of the shortest side will be 9 cm.

8. $2x - 3y > 6$

Solution:

Given $2x - 3y > 6$

Now draw a dotted line $2x - 3y = 6$ in the graph ($\because 2x - 3y = 6$ is excluded in the given question)

Now Consider $2x - 3y > 6$

Select a point (0, 0)

$\Rightarrow 2 \times (0) - 3 \times (0) > 6$

$\Rightarrow 0 > 6$ (this is false)

$\therefore$ Solution region of the given inequality is below the line $2x - 3y > 6$. (Away from the origin)

The graph is as follows:

7 MARK QUESTIONS

1. Solve $5x – 3 < 7$, when

(i) x is an integer

(ii) x is a real number

Solution:

(i) Given that, $5x – 3 < 7$

Now by adding 3 on both sides, we get,

$5x – 3 + 3 < 7 + 3$

The above inequality becomes

$5x < 10$

Again, by dividing both sides by 5 we get,

$5x/5 < 10/5$

$x < 2$

When x is an integer, then

It is clear that that the integer number less than 2 are…, -2, -1, 0, 1.

Thus, solution of $5x – 3 < 7$ is …,-2, -1, 0, 1, when x is an integer.

Therefore the solution set is {…, -2, -1, 0, 1}

(ii) Given that, $5x – 3 < 7$

Now by adding 3 on both sides, we get,

$5x – 3 + 3 < 7 + 3$

Above inequality becomes

$5x < 10$

Again, by dividing both sides by 5, we get,

$5x/5 < 10/5$

$x < 2$

When x is a real number, then

It is clear that the solutions of $5x – 3 < 7$ will be given by $x < 2$ which states that all the real numbers that are less than 2.

Hence the solution set is $x \in (-\infty, 2)$

2. Solve 3x + 8 >2, when

(i) x is an integer.

(ii) x is a real number.

Solution:

(i) Given that, $3x + 8 > 2$

Now by subtracting 8 from both sides, we get,

$3x + 8 – 8 > 2 – 8$

The above inequality becomes,

$3x > – 6$

Again by dividing both sides by 3, we get,

$3x/3 > -6/3$

Hence $x > -2$

When x is an integer, then

It is clear that the integer numbers greater than -2 are -1, 0, 1, 2,…

Thus, solution of $3x + 8 > 2$ is -1, 0, 1, 2,… when x is an integer.

Hence the solution set is {-1, 0, 1, 2,…}

(ii) Given that, $3x + 8 > 2$

Now by subtracting 8 from both sides we get,

$3x + 8 – 8 > 2 – 8$

The above inequality becomes,

$3x > – 6$

Again, by dividing both sides by 3, we get,

$3x/3 > -6/3$

Hence $x > -2$

When x is a real number.

It is clear that the solutions of $3x + 8 >2$ will be given by $x > -2$ which means all the real numbers that are greater than -2.

Therefore the solution set is $x \in (-2, \infty)$

Solve the inequalities in Exercises 5 to 16 for real x.

3. 4x + 3 < 5x + 7

Solution:

Given that, $4x + 3 < 5x + 7$

Now by subtracting 7 from both the sides, we get

$4x + 3 – 7 < 5x + 7 – 7$

The above inequality becomes,

$4x – 4 < 5x$

Again, by subtracting 4x from both the sides,

$4x – 4 – 4x < 5x – 4x$

$x > – 4$

∴The solutions of the given inequality are defined by all the real numbers greater than -4.

The required solution set is $(-4, \infty)$

4. $3x – 7 > 5x – 1$

Solution:

Given that,

$3x – 7 > 5x – 1$

Now, by adding 7 to both the sides, we get

$3x – 7 +7 > 5x – 1 + 7$

$3x > 5x + 6$

Again, by subtracting 5x from both the sides,

$3x – 5x > 5x + 6 – 5x$

$-2x > 6$

Dividing both sides by -2 to simplify, we get

$-2x/-2 < 6/-2$

$x < -3$

∴ The solutions of the given inequality are defined by all the real numbers less than -3.

Hence the required solution set is $(-\infty, -3)$

5. $3(x – 2)/5 \leq 5 (2 – x)/3$

Solution:

Given that,

Now by cross-multiplying the denominators, we get

$9(x- 2) \leq 25 (2 – x)$

$9x – 18 \leq 50 – 25x$

Now adding 25x both the sides,

$9x – 18 + 25x \leq 50 – 25x + 25x$

$34x – 18 \leq 50$

Adding 25x both the sides,

$34x – 18 + 18 \leq 50 + 18$

$34x \leq 68$

Dividing both sides by 34,

$34x/34 \leq 68/34$

$x \leq 2$

6. 2 (2x + 3) – 10 < 6 (x – 2)

Solution:

Given that,

$2 (2x + 3) – 10 < 6 (x – 2)$

By multiplying, we get

$4x + 6 – 10 < 6x – 12$

On simplifying, we get

$4x – 4 < 6x – 12$

$4x – 6x < -12 + 4$

$-2x < -8$

Dividing by 2, we get;

$-x < -4$

Multiply by "-1" and change the sign.

$x > 4$

∴ The solutions of the given inequality are defined by all the real numbers greater than 4.

Hence, the required solution set is $(4, \infty)$.

7. $37 - (3x + 5) \geq 9x - 8 (x - 3)$

Solution:

Given that, $37 - (3x + 5) \geq 9x - 8 (x - 3)$

On simplifying, we get

$= 37 - 3x - 5 \geq 9x - 8x + 24$

$= 32 - 3x \geq x + 24$

On rearranging,

$= 32 - 24 \geq x + 3x$

$= 8 \geq 4x$

$= 2 \geq x$

8. $5x - 3 \geq 3x - 5$

Solution:

We have,

$5x - 3 \geq 3x - 5$

Solving the given inequality, we get

$5x - 3 \geq 3x - 5$

On rearranging, we get

$= 5x - 3x \geq -5 + 3$

On simplifying,

$= 2x \geq -2$

Now, dividing by 2 on both sides, we get

$= x \geq -1$

The graphical representation of the solution is as follows:

9. Find all pairs of consecutive odd positive integers both of which are smaller than 10 such that their sum is more than 11.

Solution:

Let us assume x to be the smaller of the two consecutive odd positive integers.

$\therefore$ The other integer is $= x + 2$

It is also given in the question that both the integers are smaller than 10.

$\therefore x + 2 < 10$

$x < 8$ … (i)

Also, it is given in the question that the sum of two integers is more than 11.

$\therefore x + (x + 2) > 11$

$2x + 2 > 11$

$x > 9/2$

$x > 4.5$ … (ii)

Thus, from (i) and (ii), we have,

x is an odd integer and it can take values 5 and 7.

Hence, possible pairs are (5, 7) and (7, 9)

10. Find all pairs of consecutive even positive integers, both of which are larger than 5 such that their sum is less than 23.

Solution:

Let us assume x is the smaller of the two consecutive even positive integers.

∴ The other integer = x + 2

It is also given in the question that both the integers are larger than 5.

∴ $x > 5$ ….(i)

Also, it is given in the question that the sum of two integers is less than 23.

∴ $x + (x + 2) < 23$

$2x + 2 < 23$

$x < 21/2$

$x < 10.5$ … (ii)

Thus, from (i) and (ii) we have x is an even number and it can take values 6, 8 and 10.

Hence, possible pairs are (6, 8), (8, 10) and (10, 12).

SUMMARY

Linear inequalities are the expressions where any two values are compared by the inequality symbols such as, '<', '>', '≤' or '≥'. These values could be numerical or algebraic or a combination of both.

CHAPTER-VI

PERMUTATIONS AND COMBINATIONS

2 MARK QUESTIONS

1. A coin is tossed 3 times, and the outcomes are recorded. How many possible outcomes are there?

Solution:

Given A coin is tossed 3 times, and the outcomes are recorded.

The possible outcomes after a coin toss are head and tail.

The number of possible outcomes at each coin toss is 2.

∴ The total number of possible outcomes after 3 times = $2 \times 2 \times 2 = 8$

2. Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other?

Solution:

Given 5 flags of different colours.

We know the signal requires 2 flags.

The number of flags possible for the upper flag is 5.

Now, as one of the flags is taken, the number of flags remaining for the lower flag in the signal is 4.

The number of ways in which signal can be given = $5 \times 4 = 20$

3. Evaluate
(i) 8!

(ii) 4! – 3!

Solution:

(i) Consider 8!

We know that $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 40320$

(ii) Consider 4!-3!

$4!-3! = (4 \times 3!) - 3!$

The above equation can be written as

$= 3!\,(4-1)$

$= 3 \times 2 \times 1 \times 3$

$= 18$

4. Is 3! + 4! = 7!?

Solution:

Consider LHS 3! + 4!

Computing the left-hand side, we get

$3! + 4! = (3 \times 2 \times 1) + (4 \times 3 \times 2 \times 1)$

$= 6 + 24$

$= 30$

Again, considering RHS and computing, we get

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

Therefore, LHS ≠ RHS

Therefore, $3! + 4! \neq 7!$

5. From a committee of 8 persons, in how many ways can we choose a chairman and a vice chairman, assuming one person cannot hold more than one position?

Solution:

Total number of people in committee = 8

Number of positions to be filled = 2

⇒ Number of permutations =

6. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once?

Solution:

Total number of different letters in EQUATION = 8

Number of letters to be used to form a word = 8

⇒ Number of permutations =

7. How many chords can be drawn through 21 points on a circle?

Solution:

Given 21 points on a circle.

We know that we require two points on the circle to draw a chord.

∴ The number of chords is are

⇒ ${}^{21}C_2$=

∴ The total number of chords that can be drawn is 210

4 MARK QUESTIONS

1. How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5, assuming that

(i) Repetition of the digits is allowed?

(ii) Repetition of the digits is not allowed?

Solution:

(i) Let the 3-digit number be ABC, where C is at the units place, B at the tens place and A at the hundreds place.

Now, when repetition is allowed,

The number of digits possible at C is 5. As repetition is allowed, the number of digits possible at B and A is also 5 at each.

Hence, the total number possible 3-digit numbers $=5 \times 5 \times 5 = 125$

(ii) Let the 3-digit number be ABC, where C is at the units place, B at the tens place and A at the hundreds place.

Now, when repetition is not allowed,

The number of digits possible at C is 5. Suppose one of 5 digits occupies place C; now, as the repletion is not allowed, the possible digits for place B are 4, and similarly, there are only 3 possible digits for place A.

Therefore, the total number of possible 3-digit numbers$=5 \times 4 \times 3=60$

2. How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, and 6 if the digits can be repeated?

Solution:

Let the 3-digit number be ABC, where C is at the unit's place, B at the tens place and A at the hundreds place.

As the number has to be even, the digits possible at C are 2 or 4 or 6. That is, the number of possible digits at C is 3.

Now, as repetition is allowed, the digits possible at B is 6. Similarly, at A, also, the number of digits possible is 6.

Therefore, The total number of possible 3-digit numbers = $6 \times 6 \times 3 = 108$

3. How many 4-letter codes can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?

Solution:

Let the 4-letter code be 1234.

In the first place, the number of letters possible is 10.

Suppose any 1 of the ten occupies place 1.

Now, as repetition is not allowed, the number of letters possible at place 2 is 9. Now, at 1 and 2, any 2 of the 10 alphabets have been taken. The number of alphabets left for place 3 is 8, and similarly, the number of alphabets possible at 4 is 7.

Therefore, the total number of 4-letter codes=$10 \times 9 \times 8 \times 7$=5040

4. How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 and no digit appears more than once?

Solution:

Let the five-digit number be ABCDE. Given that the first 2 digits of each number are 67. Therefore, the number is 67CDE.

As repetition is not allowed and 6 and 7 are already taken, the digits available for place C are 0,1,2,3,4,5,8,9. The number of possible digits at place C is 8. Suppose one of them is taken at C; now the digits possible at place D is 7. And similarly, at E, the possible digits are 6.

∴ The total five-digit numbers with given conditions = $8 \times 7 \times 6 = 336$

5. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 6, 7, if no digit is repeated?

Solution:

An even number means that the last digit should be even.

The number of possible digits at one's place = 3 (2, 4 and 6)

⇒ Number of permutations=

One of the digits is taken at one's place; the number of possible digits available = 5

⇒ Number of permutations=

Therefore, the total number of permutations =3 × 20=60

6, if no digit is repeated. How many of these will be even?

Solution:

Total number of digits possible for choosing = 5

Number of places for which a digit has to be taken = 4

As there is no repetition allowed,

⇒ Number of permutations =

The number will be even when 2 and 4 are in one's place.

The possibility of (2, 4) at one's place = 2/5 = 0.4

The total number of even numbers = 120 × 0.4 = 48

7. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected.

Solution:

Given a bag contains 5 black and 6 red balls

The number of ways we can select 2 black balls from 5 black balls is $^5C_2$

The number of ways we can select 3 red balls from 6 red balls is $^6C_3$

The number of ways 2 black and 3 red balls can be selected is $^5C_2 \times {}^6C_3$

∴ The number of ways in which 2 black and 3 red balls can be selected from 5 black and 6 red balls is 200.

8. How many 6-digit numbers can be formed from the digits 0, 1, 3, 5, 7 and 9, which are divisible by 10 and no digit is repeated?

Solution:

The number is divisible by 10 if the unit place has 0 in it.

The 6-digit number is to be formed out of which unit place is fixed as 0.

The remaining 5 places can be filled by 1, 3, 5, 7 and 9.

Here, n = 5

And the numbers of choice available are 5.

So, the total ways in which the rest of the places can be filled are $^5P_5$

9. In how many ways can the letters of the word ASSASSINATION be arranged so that all the S's are together?

Solution:

In the given word ASSASSINATION, there are 4 'S'. Since all the 4 'S' have to be arranged together, let us take them as one unit.

The remaining letters are= 3 'A', 2 'I', 2 'N', T

The number of letters to be arranged is 9 (including 4 'S').

Using the formula

where n is the number of terms and p_1, p_2 p_3 are the number of times the repeating letters repeat themselves.

Here, p_1= 3, p_2= 2, p_3 = 2

Putting the values in formula we get

7 MARK QUESTIONS

1. How many 4-digit numbers are there with no digit repeated?

Solution:

To find the four-digit number (digits do not repeat),

We will have 4 places where 4-digits are to be put.

So, at the thousand's place = There are 9 ways as 0 cannot be at the thousand's place = 9 ways

At the hundredth's place = There are 9 digits to be filled as 1 digit is already taken = 9 ways

At the ten's place = There are now 8 digits to be filled as 2 digits are already taken = 8 ways

At unit's place = There are 7 digits that can be filled = 7 ways

The total number of ways to fill the four places = $9 \times 9 \times 8 \times 7 = 4536$ ways

So, a total of 4536 four-digit numbers can be there with no digits repeated.

2. How many words, with or without meaning, can be made from the letters of the word MONDAY, assuming that no letter is repeated, if.
(i) 4 letters are used at a time,

(ii) All letters are used at a time,
(iii) All letters are used, but the first letter is a vowel.

Solution:

(i) Number of letters to be used =4

⇒ Number of permutations =

(ii) Number of letters to be used = 6

⇒ Number of permutations =

(iii) Number of vowels in MONDAY = 2 (O and A)

⇒ Number of permutations in vowel =

Now, the remaining places = 5

Remaining letters to be used =5

⇒ Number of permutations =

Therefore, the total number of permutations = 2 × 120 =240

3. In how many ways can the letters of the word PERMUTATIONS be arranged if the

(i) Words start with P and end with S,

(ii) Vowels are all together,
(iii) There are always 4 letters between P and S?

Solution:

(i) Total number of letters in PERMUTATIONS =12

The only repeated letter is T; 2times

The first and last letters of the word are fixed as P and S, respectively.

Number of letters remaining =12 – 2 = 10

⇒ Number of permutations =

(ii) Number of vowels in PERMUTATIONS = 5 (E, U, A, I, O)

Now, we consider all the vowels together as one.

Number of permutations of vowels = 120

Now, the total number of letters = 12 – 5 + 1= 8

⇒ Number of permutations =

Therefore, the total number of permutations = 120 × 20160 = 2419200

(iii) The number of places is as 1 2 3 4 5 6 7 8 9 10 11 12

There should always be 4 letters between P and S.

Possible places of P and S are 1 and 6, 2 and 7, 3 and 8, 4 and 9, 5 and 10, 6 and 11, 7 and 12

Possible ways =7,

Also, P and S can be interchanged,

No. of permutations =2 × 7 =14

The remaining 10 places can be filled with 10 remaining letters,

∴ No. of permutations =

Therefore, the total number of permutations = 14 × 1814400 =25401600

4. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls?

Solution:

Given 5 boys and 4 girls in total.

We can select 3 boys from 5 boys in ${}^5C_3$ ways.

Similarly, we can select 3 boys from 54 girls in ${}^4C_3$ ways.

∴ The number of ways a team of 3 boys and 3 girls can be selected is ${}^5C_3 \times {}^4C_3$

$\Rightarrow {}^5C_3 \times {}^4C_3 =$

$\Rightarrow {}^5C_3 \times {}^4C_3 = 10 \times 4 = 40$

∴ The number of ways a team of 3 boys and 3 girls can be selected is ${}^5C_3 \times {}^4C_3 = 40$ ways

5. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.

Solution:

Given 6 red balls, 5 white balls and 5 blue balls.

We can select 3 red balls from 6 red balls in ${}^6C_3$ ways.

Similarly, we can select 3 white balls from 5 white balls in ${}^5C_3$ ways.

Similarly, we can select 3 blue balls from 5 blue balls in ${}^5C_3$ ways.

∴ The number of ways of selecting 9 balls is ${}^6C_3 \times {}^5C_3 \times {}^5C_3$

∴ The number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour is ${}^6C_3 \times {}^5C_3 \times {}^5C_3 = 2000$

6. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination.

Solution:

Given a deck of 52 cards.

There are 4 Ace cards in a deck of 52 cards.

According to the question, we need to select 1 Ace card out of the 4 Ace cards.

∴ The number of ways to select 1 Ace from 4 Ace cards is ${}^4C_1$

⇒ More 4 cards are to be selected now from 48 cards (52 cards – 4 Ace cards)

∴ The number of ways to select 4 cards from 48 cards is ${}^{48}C_4$

∴ The number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination is 778320.

7. In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers?

Solution:

Given 17 players, in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers.

There are 5 players that can bowl, and we can require 4 bowlers in a team of 11.

∴ The number of ways in which bowlers can be selected is: ${}^{5}C_4$

Now, other players left are = 17 – 5(bowlers) = 12

Since we need 11 players in a team and already 4 bowlers have been selected, we need to select 7 more players from 12.

∴ The number of ways we can select these players is: ${}^{12}C_7$

∴ The total number of combinations possible is: ${}^{5}C_4 \times {}^{12}C_7$

∴ The number of ways we can select a team of 11 players where 4 players are bowlers from 17 players is 3960.

8. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student?

Solution:

Given 9 courses are available and 2 specific courses are compulsory for every student.

Here, 2 courses are compulsory out of 9 courses, so a student needs to select 5 – 2 = 3 courses

∴ The number of ways in which 3 ways can be selected from 9 – 2(compulsory courses) = 7 are $^7C_3$

∴ The number of ways a student selects 5 courses from 9 courses where 2 specific courses are compulsory is 35.

9. How many words, with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word DAUGHTER?

Solution:

The word DAUGHTER has 3 vowels A, E, and U and 5 consonants D, G, H, T and R.

The three vowels can be chosen in $^3C_2$ as only two vowels are to be chosen.

Similarly, the five consonants can be chosen in $^5C_3$ ways.

∴ The number of choosing 2 vowels and 5 consonants would be $^3C_2 \times {}^5C_3$

= 30

∴ The total number of ways of is 30.

Each of these 5 letters can be arranged in 5 ways to form different words = $^5P_5$

Total number of words formed would be = 30 × 120 = 3600

10. How many words, with or without meaning, can be formed using all the letters of the word EQUATION at a time so that the vowels and consonants occur together?

Solution:

In the word EQUATION, there are 5 vowels (A, E, I, O, U) and 3 consonants (Q, T, N).

The numbers of ways in which 5 vowels can be arranged are $^5C_5$

…………… (i)

Similarly, the numbers of ways in which 3 consonants can be arranged are $^3P_3$

…………….. (ii)

There are two ways in which vowels and consonants can appear together.

(AEIOU) (QTN) or (QTN) (AEIOU)

∴ The total number of ways in which vowel and consonant can appear together are $2 \times {}^5C_5 \times {}^3C_3$

∴ $2 \times 120 \times 6 = 1440$

11. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of:
(i) Exactly 3 girls?

(ii) At least 3 girls?

(iii) At most 3 girls?

Solution:

(i) Given exactly 3 girls.

The total numbers of girls are 4.

Out of which, 3 are to be chosen.

∴ The number of ways in which choice would be made = $^4C_3$

Numbers of boys are 9 out of which 4 are to be chosen which is given by $^9C_4$

Total ways of forming the committee with exactly three girls.

= $^4C_3 \times {}^9C_4$

=

(ii) Given at least 3 girls.

There are two possibilities for making a committee choosing at least 3 girls.

There are 3 girls and 4 boys, or there are 4 girls and 3 boys.

Choosing three girls we have done in (i)

Choosing four girls and 3 boys would be done in $^4C_4$ ways.

And choosing 3 boys would be done in $^9C_3$

Total ways = $^4C_4 \times {}^9C_3$

The total number of ways of making the committee are

504 + 84 = 588

(iii) Given at most 3 girls

In this case, the numbers of possibilities are

0 girl and 7 boys

1 girl and 6 boys

2 girls and 5 boys

3 girls and 4 boys

Number of ways to choose 0 girl and 7 boys = $^4C_0 \times {}^9C_7$

The number of choosing 3 girls and 4 boys has been done in (1)

= 504

The total number of ways in which a committee can have at most 3 girls are = 36 + 336 + 756 + 504 = 1632

12. If the different permutations of all the letters of the word EXAMINATION are listed as in a dictionary, how many words are there in this list before the first word starts with E?

Solution:

In a dictionary, words are listed alphabetically, so to find the words

Listed before E should start with the letter either A, B, C or D.

But the word EXAMINATION doesn’t have B, C or D.

Hence, the words should start with the letter A

The remaining 10 places are to be filled in by the remaining letters of the word EXAMINATION which are E, X, A, M, 2N, T, 2I, 0

Since the letters are repeating, the formula used would be

Where n is the remaining number of letters, p_1 and p_2 are the number of times the repeated terms occurs.

The number of words in the list before the word starting with E

= words starting with letter A = 907200

13. The English alphabet has 5 vowels and 21 consonants. How many words with two different vowels and 2 different consonants can be formed from the alphabet?

Solution:

We know that there are 5 vowels and 21 consonants in the English alphabet.

Choosing two vowels out of 5 would be done in ${}^5C_2$ ways.

Choosing 2 consonants out of 21 can be done in ${}^{21}C_2$ ways.

The total number of ways to select 2 vowels and 2 consonants

$= {}^5C_2 \times {}^{21}C_2$

Each of these four letters can be arranged in four ways ${}^4P_4$

Total numbers of words that can be formed are

$24 \times 2100 = 50400$

14. It is required to seat 5 men and 4 women in a row so that the women occupy even places. How many such arrangements are possible?

Solution:

Given there is a total of 9 people.

Women occupy even places, which means they will be sitting in 2nd, 4th, 6th and 8th place where as men will be sitting in 1st, 3rd, 5th,7th and 9th place.

4 women can sit in four places and ways they can be seated= ${}^4P_4$

5 men can occupy 5 seats in 5 ways.

The number of ways in which these can be seated = $^5P_5$

The total numbers of sitting arrangements possible are

$24 \times 120 = 2880$

15. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can the excursion party be chosen?

Solution:

In this question, we get 2 options, which are

(i) Either all 3 will go

Then, the remaining students in the class are: 25 – 3 = 22

The number of students remained to be chosen for party = 7

Number of ways to choose the remaining 22 students = $^{22}C_7$

=

(ii) None of them will go

The students going will be 10.

Remaining students eligible for going = 22

The number of ways in which these 10 students can be selected are $^{22}C_{10}$

The total number of ways in which students can be chosen is

= 170544 + 646646 = 817190

SUMMARY

A permutation is an act of arranging objects or numbers in order. Combinations are the way of selecting objects or numbers from a group of objects or collections, in such a way that the order of the objects does not matter.

CHAPTER-VII

BINOMIAL THEOREM

2 MARK QUESTIONS

1 . Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

Solution:

By splitting the given 1.1 and then applying the binomial theorem, the first few terms of $(1.1)^{10000}$ can be obtained as

$(1.1)^{10000} = (1 + 0.1)^{10000}$

$= (1 + 0.1)^{10000}\ C_1\ (1.1)$ + other positive terms

$= 1 + 10000 \times 1.1$ + other positive terms

$= 1 + 11000$ + other positive terms

> 1000

$(1.1)^{10000} > 1000$

2. $(x^2 - y\ x)^{12}$, $x \neq 0$

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^{n}C_{r}\ a^{n-r}\ b^{r}$

Here $n = 12$, $a= x^2$ and $b = -y\ x$

Substituting the values, we get

$T_{n+1} = {}^{12}C_r \times x^{2(12-r)}\ (-1)^r\ y^r\ x^r$

$= -1^r\ {}^{12}c_r . x^{24-2r} .\ y^r$

3. Find the 4th term in the expansion of $(x – 2y)^{12}$.

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^{n}C_{r}\, a^{n-r}\, b^{r}$

Here, a= x, n =12, r= 3 and b = -2y

By substituting the values, we get

$T_4 = {}^{12}C_3\, x^9\, (-2y)^3$

$= -1760\, x^9\, y^3$

4 MARK QUESTIONS

1. $(1 - 2x)^5$

Solution:

From binomial theorem expansion, we can write as

$(1 - 2x)^5$

$= {}^5C_0 (1)^5 - {}^5C_1 (1)^4 (2x) + {}^5C_2 (1)^3 (2x)^2 - {}^5C_3 (1)^2 (2x)^3 + {}^5C_4 (1)^1 (2x)^4 - {}^5C_5 (2x)^5$

$= 1 - 5 (2x) + 10 (4x)^2 - 10 (8x^3) + 5 (16 x^4) - (32 x^5)$

$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

2. Using the binomial theorem, find $(96)^3$.

Solution:

Given $(96)^3$

96 can be expressed as the sum or difference of two numbers, and then the binomial theorem can be applied.

The given question can be written as $96 = 100 - 4$

$(96)^3 = (100 - 4)^3$

$= {}^3C_0 (100)^3 - {}^3C_1 (100)^2 (4) - {}^3C_2 (100) (4)^2 - {}^3C_3 (4)^3$

$= (100)^3 - 3 (100)^2 (4) + 3 (100) (4)^2 - (4)^3$

$= 1000000 - 120000 + 4800 - 64$

$= 884736$

3. Using the binomial theorem, find $(102)^5$.

Solution:

Given $(102)^5$

102 can be expressed as the sum or difference of two numbers, and then the binomial theorem can be applied.

The given question can be written as 102 = 100 + 2

$(102)^5 = (100 + 2)^5$

$= {}^5C_0 (100)^5 + {}^5C_1 (100)^4 (2) + {}^5C_2 (100)^3 (2)^2 + {}^5C_3 (100)^2 (2)^3 + {}^5C_4 (100) (2)^4 + {}^5C_5 (2)^5$

$= (100)^5 + 5 (100)^4 (2) + 10 (100)^3 (2)^2 + 5 (100) (2)^3 + 5 (100) (2)^4 + (2)^5$

= 1000000000 + 1000000000 + 40000000 + 80000 + 8000 + 32

= 11040808032

4. Using the binomial theorem, find $(101)^4$.

Solution:

Given $(101)^4$

101 can be expressed as the sum or difference of two numbers, and then the binomial theorem can be applied.

The given question can be written as 101 = 100 + 1

$(101)^4 = (100 + 1)^4$

$= {}^4C_0 (100)^4 + {}^4C_1 (100)^3 (1) + {}^4C_2 (100)^2 (1)^2 + {}^4C_3 (100) (1)^3 + {}^4C_4 (1)^4$

$= (100)^4 + 4 (100)^3 + 6 (100)^2 + 4 (100) + (1)^4$

= 100000000 + 4000000 + 60000 + 400 + 1

= 104060401

5. Using the binomial theorem, find $(99)^5$m.

Solution:

Given $(99)^5$

99 can be written as the sum or difference of two numbers then the binomial theorem can be applied.

The given question can be written as 99 = 100 -1

$(99)^5 = (100 – 1)^5$

$= {}^5C_0 (100)^5 – {}^5C_1 (100)^4 (1) + {}^5C_2 (100)^3 (1)^2 – {}^5C_3 (100)^2 (1)^3 + {}^5C_4 (100) (1)^4 – {}^5C_5 (1)^5$

$= (100)^5 – 5 (100)^4 + 10 (100)^3 – 10 (100)^2 + 5 (100) – 1$

= 1000000000 – 5000000000 + 10000000 – 100000 + 500 – 1

= 9509900499

6. $(x^2 – y)^6$

Solution:

The general term T_{r+1} in the binomial expansion is given by

$T_{r+1} = {}^n C_r a^{n-r} b^r$........ (i)

Here, $a = x^2$, $n = 6$ and $b = -y$

Putting values in (i)

$T_{r+1} = {}^6C_r x^{2(6-r)} (-1)^r y^r$

$= -1^r \, {}^6c_r .x^{12 – 2r}. y^r$

7 MARK QUESTIONS

1. Find $(a + b)^4 – (a – b)^4$. Hence, evaluate

Solution:

Using the binomial theorem, the expression $(a + b)^4$ and $(a – b)^4$ can be expanded

$(a + b)^4 = {}^4C_0\, a^4 + {}^4C_1\, a^3\, b + {}^4C_2\, a^2\, b^2 + {}^4C_3\, a\, b^3 + {}^4C_4\, b^4$

$(a – b)^4 = {}^4C_0\, a^4 – {}^4C_1\, a^3\, b + {}^4C_2\, a^2\, b^2 – {}^4C_3\, a\, b^3 + {}^4C_4\, b^4$

Now $(a + b)^4 – (a – b)^4 = {}^4C_0\, a^4 + {}^4C_1\, a^3\, b + {}^4C_2\, a^2\, b^2 + {}^4C_3\, a\, b^3 + {}^4C_4\, b^4 – [{}^4C_0\, a^4 – {}^4C_1\, a^3\, b + {}^4C_2\, a^2\, b^2 – {}^4C_3\, a\, b^3 + {}^4C_4\, b^4]$

$= 2\, ({}^4C_1\, a^3\, b + {}^4C_3\, a\, b^3)$

$= 2\, (4a^3\, b + 4ab^3)$

$= 8ab\, (a^2 + b^2)$

Now by substituting $a = \sqrt{3}$ and $b = \sqrt{2}$, we get

$(\sqrt{3} + \sqrt{2})^4 – (\sqrt{3} – \sqrt{2})^4 = 8\, (\sqrt{3})\, (\sqrt{2})\, \{(\sqrt{3})^2 + (\sqrt{2})^2\}$

$= 8\, (\sqrt{6})\, (3 + 2)$

$= 40\, \sqrt{6}$

2. Find $(x + 1)^6 + (x – 1)^6$. Hence or otherwise evaluate

Solution:

Using binomial theorem, the expressions $(x + 1)^6$ and $(x - 1)^6$ can be expressed as

$(x + 1)^6 = {}^6C_0\, x^6 + {}^6C_1\, x^5 + {}^6C_2\, x^4 + {}^6C_3\, x^3 + {}^6C_4\, x^2 + {}^6C_5\, x + {}^6C_6$

$(x - 1)^6 = {}^6C_0\, x^6 - {}^6C_1\, x^5 + {}^6C_2\, x^4 - {}^6C_3\, x^3 + {}^6C_4\, x^2 - {}^6C_5\, x + {}^6C_6$

Now, $(x + 1)^6 - (x - 1)^6 = {}^6C_0\, x^6 + {}^6C_1\, x^5 + {}^6C_2\, x^4 + {}^6C_3\, x^3 + {}^6C_4\, x^2 + {}^6C_5\, x + {}^6C_6 - [{}^6C_0\, x^6 - {}^6C_1\, x^5 + {}^6C_2\, x^4 - {}^6C_3\, x^3 + {}^6C_4\, x^2 - {}^6C_5\, x + {}^6C_6]$

$= 2\,[{}^6C_0\, x^6 + {}^6C_2\, x^4 + {}^6C_4\, x^2 + {}^6C_6]$

$= 2\,[x^6 + 15x^4 + 15x^2 + 1]$

Now by substituting $x = \sqrt{2}$, we get

$(\sqrt{2} + 1)^6 - (\sqrt{2} - 1)^6 = 2\,[(\sqrt{2})^6 + 15(\sqrt{2})^4 + 15(\sqrt{2})^2 + 1]$

$= 2\,(8 + 15 \times 4 + 15 \times 2 + 1)$

$= 2\,(8 + 60 + 30 + 1)$

$= 2\,(99)$

$= 198$

3. Show that $9^{n+1} - 8n - 9$ is divisible by 64 whenever n is a positive integer.

Solution:

In order to show that $9^{n+1} - 8n - 9$ is divisible by 64, it has to be shown that $9^{n+1} - 8n - 9 = 64\,k$, where k is some natural number.

Using the binomial theorem,

$(1 + a)^m = {}^mC_0 + {}^mC_1\, a + {}^mC_2\, a^2 + \dots + {}^mC_m\, a^m$

For a = 8 and m = n + 1 we get

$(1 + 8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1\,(8) + {}^{n+1}C_2\,(8)^2 + \dots + {}^{n+1}C_{n+1}\,(8)^{n+1}$

$9^{n+1} = 1 + (n + 1)\,8 + 8^2\,[{}^{n+1}C_2 + {}^{n+1}C_3\,(8) + \dots. + {}^{n+1}C_{n+1}\,(8)^{n-1}]$

$9^{n+1} = 9 + 8n + 64\,[{}^{n+1}C_2 + {}^{n+1}C_3\,(8) + \dots. + {}^{n+1}C_{n+1}\,(8)^{n-1}]$

$9^{n+1} – 8n – 9 = 64\,k$

Where $k = [{}^{n+1}C_2 + {}^{n+1}C_3\,(8) + \dots. + {}^{n+1}C_{n+1}\,(8)^{n-1}]$ is a natural number

Thus, $9^{n+1} – 8n – 9$ is divisible by 64 whenever n is a positive integer.

Hence proved.

4. a^5b^7 in $(a – 2b)^{12}$

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^{n}C_{r}\,a^{n-r}\,b^{r}$

Here a = a, b = -2b & n =12

Substituting the values, we get

$T_{r+1} = {}^{12}C_r\,a^{12-r}\,(-2b)^r$………. (i)

To find a^5

We equate $a^{12-r} = a^5$

r = 7

Putting r = 7 in (i)

$T_8 = {}^{12}C_7\,a^5\,(-2b)^7$

$= -101376\,a^5\,b^7$

Hence, the coefficient of $a^5b^7 = -101376$.

Write the general term in the expansion of

5. In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

Solution:

We know that the general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here n= m+n, a = 1 and b= a

Substituting the values in the general form

$T_{r+1} = {}^{m+n}C_r\, 1^{m+n-r}\, a^r$

$= {}^{m+n}C_r\, a^r$………… (i)

Now, we have that the general term for the expression is,

$T_{r+1} = {}^{m+n}C_r\, a^r$

Now, for coefficient of a^m

$T_{m+1} = {}^{m+n}C_m\, a^m$

Hence, for the coefficient of a^m, the value of r = m

So, the coefficient is ${}^{m+n}C_m$

Similarly, the coefficient of a^n is ${}^{m+n}C_n$

6. The coefficients of the $(r – 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ are in the ratio 1:3:5. Find n and r.

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here, the binomial is $(1+x)^n$ with a = 1 , b = x and n = n

The $(r+1)^{th}$ term is given by

$T_{(r+1)} = {}^{n}C_{r}\ 1^{n-r}\ x^{r}$

$T_{(r+1)} = {}^{n}C_{r}\ x^{r}$

The coefficient of $(r+1)^{th}$ term is ${}^{n}C_{r}$

The r^{th} term is given by $(r-1)^{th}$ term

$T_{(r+1-1)} = {}^{n}C_{r-1}\ x^{r-1}$

$T_{r} = {}^{n}C_{r-1}\ x^{r-1}$

∴ the coefficient of r^{th} term is ${}^{n}C_{r-1}$

For $(r-1)^{th}$ term, we will take $(r-2)^{th}$ term

$T_{r-2+1} = {}^{n}C_{r-2}\ x^{r-2}$

$T_{r-1} = {}^{n}C_{r-2}\ x^{r-2}$

∴ the coefficient of $(r-1)^{th}$ term is ${}^{n}C_{r-2}$

Given that the coefficient of $(r-1)^{th}$, r^{th} and $r+1^{th}$ term are in ratio 1:3:5

Therefore,

$\Rightarrow 5r = 3n - 3r + 3$

$\Rightarrow 8r - 3n - 3 = 0$…………2

We have 1 and 2 as

$n - 4r \pm 5 = 0$…………1

$8r - 3n - 3 = 0$…………….2

Multiplying equation 1 by number 2

$2n - 8r + 10 = 0$……………….3

Adding equations 2 and 3

2n -8r +10 =0

-3n – 8r – 3 =0

⇒ -n = -7

n =7 and r = 3

7. Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^{n}C_r\, a^{n-r}\, b^r$

The general term for binomial $(1+x)^{2n}$ is

$T_{r+1} = {}^{2n}C_r\, x^r$ …………………..1

To find the coefficient of x^n

r = n

$T_{n+1} = {}^{2n}C_n\, x^n$

The coefficient of $x^n = {}^{2n}C_n$

The general term for binomial $(1+x)^{2n-1}$ is

$T_{r+1} = {}^{2n-1}C_r\, x^r$

To find the coefficient of x^n

Putting n = r

$T_{r+1} = {}^{2n-1}C_r\, x^n$

The coefficient of $x^n = {}^{2n-1}C_n$

We have to prove

Coefficient of x^n in $(1+x)^{2n}$ = 2 coefficient of x^n in $(1+x)^{2n-1}$

Consider LHS = ${}^{2n}C_n$

8. Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here, a = 1, b = x and n = m

Putting the value

$T_{r+1} = {}^mC_r\, 1^{m-r}\, x^r$

$= {}^mC_r\, x^r$

We need the coefficient of x^2

∴ putting r = 2

$T_{2+1} = {}^mC_2\, x^2$

The coefficient of $x^2 = {}^mC_2$

Given that coefficient of $x^2 = {}^mC_2 = 6$

⇒ m (m – 1) = 12

⇒ $m^2 - m - 12 = 0$

⇒ $m^2 - 4m + 3m - 12 = 0$

⇒ m (m – 4) + 3 (m – 4) = 0

⇒ (m+3) (m – 4) = 0

⇒ m = – 3, 4

We need the positive value of m, so m = 4

9. Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 – x)^7$ using binomial theorem.

Solution:

$(1 + 2x)^6 = {}^6C_0 + {}^6C_1 (2x) + {}^6C_2 (2x)^2 + {}^6C_3 (2x)^3 + {}^6C_4 (2x)^4 + {}^6C_5 (2x)^5 + {}^6C_6 (2x)^6$

$= 1 + 6 (2x) + 15 (2x)^2 + 20 (2x)^3 + 15 (2x)^4 + 6 (2x)^5 + (2x)^6$

$= 1 + 12 x + 60x^2 + 160 x^3 + 240 x^4 + 192 x^5 + 64x^6$

$(1 – x)^7 = {}^7C_0 – {}^7C_1 (x) + {}^7C_2 (x)^2 – {}^7C_3 (x)^3 + {}^7C_4 (x)^4 – {}^7C_5 (x)^5 + {}^7C_6 (x)^6 – {}^7C_7 (x)^7$

$= 1 – 7x + 21x^2 – 35x^3 + 35x^4 – 21x^5 + 7x^6 – x^7$

$(1 + 2x)^6 (1 – x)^7 = (1 + 12 x + 60x^2 + 160 x^3 + 240 x^4 + 192 x^5 + 64x^6) (1 – 7x + 21x^2 – 35x^3 + 35x^4 – 21x^5 + 7x^6 – x^7)$

$192 – 21 = 171$

Thus, the coefficient of x^5 in the expression $(1+2x)^6(1-x)7$ is 171.

10. If a and b are distinct integers, prove that a – b is a factor of $a^n – b^n$, whenever n is a positive integer. [Hint write $a^n = (a – b + b)^n$ and expand]

Solution:

In order to prove that $(a – b)$ is a factor of $(a^n – b^n)$, it has to be proved that

$a^n – b^n = k (a – b)$ where k is some natural number.

a can be written as $a = a – b + b$

$a^n = (a – b + b)^n = [(a – b) + b]^n$

$= {}^nC_0 (a – b)^n + {}^nC_1 (a – b)^{n-1} b + …… + {}^n C_n b^n$

$a^n – b^n = (a – b) [(a –b)^{n-1} + {}^nC_1 (a – b)^{n-1} b + …… + {}^n C_n b^n]$

$a^n – b^n = (a – b) k$

Where $k = [(a –b)^{n-1} + {}^nC_1 (a – b)^{n-1} b + …… + {}^n C_n b^n]$ is a natural number

This shows that (a – b) is a factor of ($a^n - b^n$), where n is a positive integer.

11. Evaluate

Solution:

Using the binomial theorem, the expression $(a + b)^6$ and $(a - b)^6$ can be expanded

$(a + b)^6 = {}^6C_0\, a^6 + {}^6C_1\, a^5\, b + {}^6C_2\, a^4\, b^2 + {}^6C_3\, a^3\, b^3 + {}^6C_4\, a^2\, b^4 + {}^6C_5\, a\, b^5 + {}^6C_6\, b^6$

$(a - b)^6 = {}^6C_0\, a^6 - {}^6C_1\, a^5\, b + {}^6C_2\, a^4\, b^2 - {}^6C_3\, a^3\, b^3 + {}^6C_4\, a^2\, b^4 - {}^6C_5\, a\, b^5 + {}^6C_6\, b^6$

Now $(a + b)^6 - (a - b)^6 = {}^6C_0\, a^6 + {}^6C_1\, a^5\, b + {}^6C_2\, a^4\, b^2 + {}^6C_3\, a^3\, b^3 + {}^6C_4\, a^2\, b^4 + {}^6C_5\, a\, b^5 + {}^6C_6\, b^6 - [{}^6C_0\, a^6 - {}^6C_1\, a^5\, b + {}^6C_2\, a^4\, b^2 - {}^6C_3\, a^3\, b^3 + {}^6C_4\, a^2\, b^4 - {}^6C_5\, a\, b^5 + {}^6C_6\, b^6]$

Now by substituting a = √3 and b = √2, we get

$(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6 = 2\, [6\, (\sqrt{3})^5\, (\sqrt{2}) + 20\, (\sqrt{3})^3\, (\sqrt{2})^3 + 6\, (\sqrt{3})\, (\sqrt{2})^5]$

$= 2\, [54(\sqrt{6}) + 120\, (\sqrt{6}) + 24\, \sqrt{6}]$

$= 2\, (\sqrt{6})\, (198)$

$= 396\, \sqrt{6}$

SUMMARY

The binomial theorem states the principle for expanding the algebraic expression $(x + y)^n$ and expresses it as a sum of the terms involving individual exponents of variables x and y. Each term in a binomial expansion is associated with a numeric value which is called coefficient.

CHAPTER-VIII
SEQUENCES AND SERIES

2 MARK QUESTIONS

1. $a_n = n (n + 2)$

Solution:

Given,

n^{th} term of a sequence $a_n = n (n + 2)$

On substituting n = 1, 2, 3, 4, and 5, we get the first five terms

$a_1 = 1(1 + 2) = 3$

$a_2 = 2(2 + 2) = 8$

$a_3 = 3(3 + 2) = 15$

$a_4 = 4(4 + 2) = 24$

$a_5 = 5(5 + 2) = 35$

Hence, the required terms are 3, 8, 15, 24, and 35.

2. $a_n = n/n+1$

Solution:

Given the n^{th} term, $a_n = n/n+1$

On substituting n = 1, 2, 3, 4, 5, we get

Hence, the required terms are 1/2, 2/3, 3/4, 4/5 and 5/6.

3. $a_n = 2^n$

Solution:

Given the n^{th} term, $a_n = 2^n$

On substituting n = 1, 2, 3, 4, 5, we get

$a_1 = 2^1 = 2$

$a_2 = 2^2 = 4$

$a_3 = 2^3 = 8$

$a_4 = 2^4 = 16$

$a_5 = 2^5 = 32$

Hence, the required terms are 2, 4, 8, 16, and 32.

4. $\boldsymbol{a_n = (2n - 3)/6}$

Solution:

Given the n^{th} term, $a_n = (2n - 3)/6$

On substituting n = 1, 2, 3, 4, 5, we get

Hence, the required terms are -1/6, 1/6, 1/2, 5/6 and 7/6..

5. $\boldsymbol{a_n = (-1)^{n-1} 5^{n+1}}$

Solution:

Given the n^{th} term, $a_n = (-1)^{n-1} 5^{n+1}$

On substituting n = 1, 2, 3, 4, 5, we get

Hence, the required terms are 25, –125, 625, –3125, and 15625.

6. $a_n = 4n - 3$; a_{17}, a_{24}

Solution:

Given,

The n^{th} term of the sequence is $a_n = 4n - 3$

On substituting $n = 17$, we get

$a_{17} = 4(17) - 3 = 68 - 3 = 65$

Next, on substituting $n = 24$, we get

$a_{24} = 4(24) - 3 = 96 - 3 = 93$

7. $a_n = n^2/2^n$; a^7

Solution:

Given,

The n^{th} term of the sequence is $a_n = n^2/2^n$

Now, on substituting $n = 7$, we get

$a_7 = 7^2/2^7 = 49/\ 128$

8. $a_n = (-1)^{n-1}\ n^3$; a_9

Solution:

Given,

The n^{th} term of the sequence is $a_n = (-1)^{n-1}\ n^3$

On substituting $n = 9$, we get

$a_9 = (-1)^{9-1}\ (9)^3 = 1 \times 729 = 729$

9. The sums of *n* terms of two arithmetic progressions are in the ratio 5*n* + 4: 9*n* + 6. Find the ratio of their 18^{th} terms.

Solution:

Let a_1, a_2, and d_1, d_2 be the first terms and the common difference of the first and second arithmetic progression, respectively.

Then, from the question, we have

10. If the sum of the first *p* terms of an A.P. is equal to the sum of the first *q* terms, then find the sum of the first (*p* + *q*) terms.

Solution:

Let's take *a* and *d* to be the first term and the common difference of the A.P., respectively.

Then, it is given that

Therefore, the sum of (p + q) terms of the A.P. is 0.

11. Sum of the first *p, q* and *r* terms of an A.P. are *a, b* and *c*, respectively.

Prove that

Solution:

Let a_1 and *d* be the first term and the common difference of the A.P., respectively.

Then, according to the question, we have

Now, subtracting (2) from (1), we get

12. The ratio of the sums of *m* and *n* terms of an A.P. is m^2: n^2. Show that the ratio of the m^{th} and the n^{th} term is (2*m* – 1): (2*n* – 1).

Solution:

Let's consider that a and b are the first term and the common difference of the A.P., respectively.

Then, from the question, we have

Hence, the given result is proved.

13. If is the A.M. between *a* and *b*, then find the value of *n*.

Solution:

The A.M between a and b is given by (a + b)/2

Then, according to the question,

Thus, the value of n is 1.

14. The 5th, 8th and 11th terms of a G.P. are *p*, *q* and *s*, respectively. Show that $q^2 = ps$.

Solution:

Let's take a to be the first term and r to be the common ratio of the G.P.

Then, according to the question, we have

$a_5 = a\ r^{5-1} = a\ r^4 = p$ … (i)

$a_8 = a\ r^{8-1} = a\ r^7 = q$ … (ii)

$a_{11} = a\ r^{11-1} = a\ r^{10} = s$ … (iii)

Dividing equation (ii) by (i), we get

15. The 5th, 8th and 11th terms of a G.P. are *p*, *q* and *s*, respectively. Show that $q^2 = ps$.

Solution:

Let's take a to be the first term and r to be the common ratio of the G.P.

Then, according to the question, we have

$a_5 = a\,r^{5-1} = a\,r^4 = p$ … (i)

$a_8 = a\,r^{8-1} = a\,r^7 = q$ … (ii)

$a_{11} = a\,r^{11-1} = a\,r^{10} = s$ … (iii)

Dividing equation (ii) by (i), we get

16. The 4th term of a G.P. is the square of its second term, and the first term is –3. Determine its 7th term.

Solution:

Let's consider a to be the first term and r to be the common ratio of the G.P.

Given, $a = -3$

And we know that,

$a_n = ar^{n-1}$

So, $a_4 = ar^3 = (-3)\,r^3$

$a_2 = a\,r^1 = (-3)\,r$

Then, from the question, we have

$(-3)\,r^3 = [(-3)\,r]^2$

$\Rightarrow -3r^3 = 9\,r^2$

$\Rightarrow r = -3$

$a_7 = a\,r^{\,7-1} = a\,r^6 = (-3)\,(-3)^6 = -\,(3)^7 = -2187$

Therefore, the seventh term of the G.P. is –2187.

17. If the 4th, 10th and 16th terms of a G.P. are *x, y* and *z*, respectively. Prove that *x*, *y*, *and z* are in G.P.

Solution:

Let a be the first term and r be the common ratio of the G.P.

According to the given condition,

$a_4 = a\,r^3 = x$ … (1)

$a_{10} = a\,r^9 = y$ … (2)

$a_{16} = a\,r^{15} = z$ … (3)

On dividing (2) by (1), we get

4 MARK QUESTIONS

1. $a_1 = 3$, $a_n = 3a_{n-1} + 2$ for all $n > 1$

Solution:

Given, $a_n = 3a_{n-1} + 2$ and $a_1 = 3$

Then,

$a_2 = 3a_1 + 2 = 3(3) + 2 = 11$

$a_3 = 3a_2 + 2 = 3(11) + 2 = 35$

$a_4 = 3a_3 + 2 = 3(35) + 2 = 107$

$a_5 = 3a_4 + 2 = 3(107) + 2 = 323$

Thus, the first 5 terms of the sequence are 3, 11, 35, 107 and 323.

Hence, the corresponding series is

3 + 11 + 35 + 107 + 323 …….

2. $a_1 = -1$, $a_n = a_{n-1}/n$, $n \geq 2$

Solution:

Given,

$a_n = a_{n-1}/n$ and $a_1 = -1$

Then,

$a_2 = a_1/2 = -1/2$

$a_3 = a_2/3 = -1/6$

$a_4 = a_3/4 = -1/24$

$a_5 = a_4/5 = -1/120$

Thus, the first 5 terms of the sequence are -1, -1/2, -1/6, -1/24 and -1/120.

Hence, the corresponding series is

-1 + (-1/2) + (-1/6) + (-1/24) + (-1/120) + …….

3. $a_1 = a_2 = 2$, $a_n = a_{n-1} – 1$, $n > 2$

Solution:

Given,

$a_1 = a_2$, $a_n = a_{n-1} – 1$

Then,

$a_3 = a_2 – 1 = 2 – 1 = 1$

$a_4 = a_3 – 1 = 1 – 1 = 0$

$a_5 = a_4 – 1 = 0 – 1 = -1$

Thus, the first 5 terms of the sequence are 2, 2, 1, 0 and -1.

The corresponding series is

2 + 2 + 1 + 0 + (-1) + ……

4. The Fibonacci sequence is defined by

$1 = a_1 = a_2$ and $a_n = a_{n-1} + a_{n-2}$, $n > 2$

Find a_{n+1}/a_n, for n = 1, 2, 3, 4, 5

Solution:

Given,

$1 = a_1 = a_2$

$a_n = a_{n-1} + a_{n-2}$, $n > 2$

So,

$a_3 = a_2 + a_1 = 1 + 1 = 2$

$a_4 = a_3 + a_2 = 2 + 1 = 3$

$a_5 = a_4 + a_3 = 3 + 2 = 5$

$a_6 = a_5 + a_4 = 5 + 3 = 8$

Thus,

5. If the sum of *n* terms of an A.P. is $3n^2 + 5n$ and its m^{th} term is 164, find the value of *m*.

Solution:

Let's consider *a* and *b* to be the first term and the common difference of the A.P., respectively.

$a_m = a + (m - 1)d = 164$ … (1)

The sum of the terms is given by,

$S_n = n/2\ [2a + (n-1)d]$

6. A man starts repaying a loan as the first instalment of Rs. 100. If he increases the instalment by Rs 5 every month, what amount will he pay in the 30^{th} instalment?

Solution:

Given,

The first instalment of the loan is Rs 100.

The second instalment of the loan is Rs 105, and so on as the instalment increases by Rs 5 every month.

Thus, the amount that the man repays every month forms an A.P.

And then, A.P. is 100, 105, 110, …

Where the first term, $a = 100$

Common difference, $d = 5$

So, the 30th term in this A.P. will be

$A_{30} = a + (30 - 1)d$

$= 100 + (29)(5)$

$= 100 + 145$

$= 245$

Therefore, the amount to be paid in the 30th instalment will be Rs 245.

7. The difference between any two consecutive interior angles of a polygon is 5°. If the smallest angle is 120°, find the number of the sides of the polygon.

Solution:

It's understood from the question that the angles of the polygon will form an A.P. with a common difference $d = 5°$ and first term $a = 120°$.

And we know that the sum of all angles of a polygon with n sides is $180° (n - 2)$.

Thus, we can say

Thus, a polygon having 9 and 16 sides will satisfy the condition in the question.

8. The 4th term of a G.P. is the square of its second term, and the first term is –3. Determine its 7th term.

Solution:

Let's consider a to be the first term and r to be the common ratio of the G.P.

Given, $a = -3$

And we know that,

$a_n = ar^{n-1}$

So, $a_4 = ar^3 = (-3)\, r^3$

$a_2 = a\, r^1 = (-3)\, r$

Then, from the question, we have

$(-3)\, r^3 = [(-3)\, r]^2$

$\Rightarrow -3r^3 = 9\, r^2$

$\Rightarrow r = -3$

$a_7 = a\, r^{7-1} = a\, r^6 = (-3)\, (-3)^6 = -\, (3)^7 = -2187$

Therefore, the seventh term of the G.P. is –2187.

9. Which term of the following sequences:

(a) 2, 2√2, 4,… is 128 ? (b) √3, 3, 3√3,… is 729 ?

(c) 1/3, 1/9, 1/27, … is 1/19683 ?

Solution:

(a) The given sequence, 2, 2√2, 4,…

We have,

a = 2 and r = 2√2/2 = √2

Taking the n^{th} term of this sequence as 128, we have

Therefore, the 13^{th} term of the given sequence is 128.

(ii) Given the sequence, √3, 3, 3√3,…

We have,

a = √3 and r = 3/√3 = √3

Taking the n^{th} term of this sequence to be 729, we have

Therefore, the 12^{th} term of the given sequence is 729.

(iii) Given sequence, 1/3, 1/9, 1/27, …

a = 1/3 and r = (1/9)/(1/3) = 1/3

Taking the n^{th} term of this sequence to be 1/19683, we have

Therefore, the 9^{th} term of the given sequence is 1/19683.

10. For what values of *x,* the numbers -2/7, x, -7/2 are in G.P?

Solution:

The given numbers are -2/7, x, -7/2

Common ratio = x/(-2/7) = -7x/2

Also, common ratio = (-7/2)/x = -7/2x

Therefore, for *x* = ± 1, the given numbers will be in G.P.

11. A man starts repaying a loan as the first instalment of Rs. 100. If he increases the instalment by Rs 5 every month, what amount will he pay in the 30^{th} instalment?

Solution:

Given,

The first instalment of the loan is Rs 100.

The second instalment of the loan is Rs 105, and so on as the instalment increases by Rs 5 every month.

Thus, the amount that the man repays every month forms an A.P.

And then, A.P. is 100, 105, 110, …

Where the first term, $a = 100$

Common difference, $d = 5$

So, the 30th term in this A.P. will be

$A_{30} = a + (30 – 1)d$

$= 100 + (29)(5)$

$= 100 + 145$

$= 245$

Therefore, the amount to be paid in the 30th instalment will be Rs 245.

12. The difference between any two consecutive interior angles of a polygon is 5°. If the smallest angle is 120°, find the number of the sides of the polygon.

Solution:

It's understood from the question that the angles of the polygon will form an A.P. with a common difference $d = 5°$ and first term $a = 120°$.

And we know that the sum of all angles of a polygon with n sides is $180° (n – 2)$.

Thus, we can say

Thus, a polygon having 9 and 16 sides will satisfy the condition in the question.

13. Find the 20th and nth terms of the G.P. 5/2, 5/4, 5/8, ………

Solution:

Given G.P. is 5/2, 5/4, 5/8, ………

Here, a = First term = 5/2

r = Common ratio = (5/4)/(5/2) = ½

Thus, the 20^{th} term and n^{th} term

14. The sum of the first three terms of a G.P. is 39/10, and their product is 1. Find the common ratio and the terms.

Solution:

Let a/r, a, ar be the first three terms of the G.P.

$a/r + a + ar = 39/10$ …… (1)

$(a/r)(a)(ar) = 1$ …….. (2)

From (2), we have

$a^3 = 1$

Hence, a = 1 [Considering real roots only]

Substituting the value of a in (1), we get

$1/r + 1 + r = 39/10$

$(1 + r + r^2)/r = 39/10$

$10 + 10r + 10r^2 = 39r$

$10r^2 - 29r + 10 = 0$

$10r^2 - 25r - 4r + 10 = 0$

$5r(2r - 5) - 2(2r - 5) = 0$

$(5r - 2)(2r - 5) = 0$

Thus,

$r = 2/5$ or $5/2$

Therefore, the three terms of the G.P. are 5/2, 1 and 2/5.

15. How many terms of G.P. 3, 3^2, 3^3, … are needed to give the sum 120?

Solution:

Given G.P. is 3, 3^2, 3^3, …

Let's consider that *n* terms of this G.P. be required to obtain the sum 120.

We know that,

Here, $a = 3$ and $r = 3$

Equating the exponents, we get $n = 4$

Therefore, four terms of the given G.P. are required to obtain the sum 120.

7 MARK QUESTIONS

1. Find the sum of odd integers from 1 to 2001.

Solution:

The odd integers from 1 to 2001 are 1, 3, 5, …1999, 2001.

It clearly forms a sequence in A.P.

Where the first term, *a* = 1

The common difference, *d* = 2

Now,

$a + (n - 1)d = 2001$

$1 + (n-1)(2) = 2001$

$2n – 2 = 2000$

$2n = 2000 + 2 = 2002$

$n = 1001$

We know,

$S_n = n/2\ [2a + (n-1)d]$

Therefore, the sum of odd numbers from 1 to 2001 is 1002001.

2. Find the sum of all natural numbers lying between 100 and 1000, which are multiples of 5.

Solution:

The natural numbers lying between 100 and 1000, which are multiples of 5, are 105, 110, … 995.

It clearly forms a sequence in A.P.

Where the first term, a = 105

The common difference, d = 5

Now,

$a + (n - 1)d = 995$

$105 + (n – 1)(5) = 995$

$105 + 5n – 5 = 995$

$5n = 995 – 105 + 5 = 895$

$n = 895/5$

$n = 179$

We know,

$S_n = n/2\ [2a + (n-1)d]$

Therefore, the sum of all natural numbers lying between 100 and 1000, which are multiples of 5, is 98450.

3. In an A.P, the first term is 2, and the sum of the first five terms is one-fourth of the next five terms. Show that the 20th term is –112.

Solution:

Given,

The first term (a) of an A.P = 2

Let's assume *d* is the common difference of the A.P.

So, the A.P. will be 2, 2 + *d*, 2 + 2*d*, 2 + 3*d*, …

Then,

Sum of first five terms = 10 + 10*d*

Sum of next five terms = 10 + 35*d*

From the question, we have

$10 + 10d = \frac{1}{4}(10 + 35d)$

$40 + 40d = 10 + 35d$

$30 = -5d$

$d = -6$

$a_{20} = a + (20 – 1)d = 2 + (19)(-6) = 2 – 114 = -112$

Therefore, the 20^{th} term of the A.P. is –112.

4. How many terms of the A.P. -6, -11/2, -5, …. are needed to give the sum –25?

Solution:

Let's consider the sum of *n* terms of the given A.P. as –25.

We known that,

$S_n = n/2\ [2a + (n-1)d]$

where *n* = number of terms, *a* = first term, and *d* = common difference

So here, *a* = –6

$d = -11/2 + 6 = (-11 + 12)/ 2 = 1/2$

Thus, we have

5. If the sum of a certain number of terms of the A.P. 25, 22, 19, … is 116. Find the last term.

Solution:

Given A.P.,

25, 22, 19, …

Here,

First term, $a = 25$ and

Common difference, $d = 22 – 25 = -3$

Also given, the sum of a certain number of terms of the A.P. is 116.

The number of terms is n.

So, we have

$S_n = n/2 [2a + (n-1)d] = 116$

$116 = n/2 [2(25) + (n-1)(-3)]$

$116 \times 2 = n [50 – 3n + 3]$

$232 = n [53 – 3n]$

$232 = 53n – 3n^2$

$3n^2 – 53n + 232 = 0$

$3n^2 – 24n – 29n + 232 = 0$

$3n(n – 8) – 29(n – 8) = 0$

$(3n – 29) (n – 8) = 0$

Hence,

$n = 29/3$ or $n = 8$

As n can only be an integral value, $n = 8$

Thus, the 8th term is the last term of the A.P.

$a_8 = 25 + (8 – 1)(-3)$

$= 25 – 21$

$= 4$

6. Find the sum to *n* terms of the A.P., whose k^{th} term is $5k + 1$.

Solution:

Given, the k^{th} term of the A.P. is $5k + 1$.

k^{th} term $= a_k = a + (k – 1)d$

And,

$a + (k – 1)d = 5k + 1$

$a + kd – d = 5k + 1$

On comparing the coefficient of k, we get $d = 5$

$a – d = 1$

$a – 5 = 1$

$\Rightarrow a = 6$

7. If the sum of *n* terms of an A.P. is $(pn + qn^2)$, where *p* and *q* are constants, find the common difference.

Solution:

We know that,

$S_n = n/2\ [2a + (n-1)d]$

From the question, we have

On comparing the coefficients of n^2 on both sides, we get

$d/2 = q$

Hence, $d = 2q$

Therefore, the common difference of the A.P. is $2q$.

8. Insert five numbers between 8 and 26 such that the resulting sequence is an A.P.

Solution:

Let's assume A_1, A_2, A_3, A_4, and A_5 to be five numbers between 8 and 26 such that 8, A_1, A_2, A_3, A_4, A_5, 26 are in an A.P.

Here, we have,

$a = 8$, $b = 26$, $n = 7$

So,

$26 = 8 + (7 – 1)\ d$

$6d = 26 – 8 = 18$

$d = 3$

Now,

$A_1 = a + d = 8 + 3 = 11$

$A_2 = a + 2d = 8 + 2 \times 3 = 8 + 6 = 14$

$A_3 = a + 3d = 8 + 3 \times 3 = 8 + 9 = 17$

$A_4 = a + 4d = 8 + 4 \times 3 = 8 + 12 = 20$

$A_5 = a + 5d = 8 + 5 \times 3 = 8 + 15 = 23$

Therefore, the required five numbers between 8 and 26 are 11, 14, 17, 20, and 23.

9. Between 1 and 31, *m* numbers have been inserted in such a way that the resulting sequence is an A.P. and the ratio of 7th and (m – 1)th numbers is 5: 9. Find the value of *m*.

Solution:

Let's consider $a_1, a_2, \ldots a_m$ be m numbers such that 1, $a_1, a_2, \ldots a_m$, 31 is an A.P.

And here,

$a = 1$, $b = 31$, $n = m + 2$

So, $31 = 1 + (m + 2 - 1)(d)$

$30 = (m + 1) d$

$d = 30/(m + 1)$ ……. (1)

Now,

$a_1 = a + d$

$a_2 = a + 2d$

$a_3 = a + 3d$ …

Hence, $a_7 = a + 7d$

$a_{m-1} = a + (m - 1) d$

According to the question, we have

Therefore, the value of m is 14.

10. Find the 12th term of a G.P. whose 8th term is 192, and the common ratio is 2.

Solution:

Given,

The common ratio of the G.P., $r = 2$

And, let a be the first term of the G.P.

Now,

$a_8 = ar^{8-1} = ar^7$

$ar^7 = 192$

$a(2)^7 = 192$

$a(2)^7 = (2)^6\ (3)$

11. The sum of the first three terms of a G.P. is 16, and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to *n* terms of the G.P.

Solution:

Let's assume the G.P. to be $a, ar, ar^2, ar^3, \ldots$

Then, according to the question, we have

$a + ar + ar^2 = 16$ and $ar^3 + ar^4 + ar^5 = 128$

$a(1 + r + r^2) = 16 \ldots (1)$ and,

$ar^3(1 + r + r^2) = 128 \ldots (2)$

Dividing equation (2) by (1), we get

$r^3 = 8$

$r = 2$

Now, using $r = 2$ in (1), we get

$a (1 + 2 + 4) = 16$

$a (7) = 16$

$a = 16/7$

Now, the sum of terms is given as

12. Given a G.P. with *a* = 729 and 7th term 64, determine S_7.

Solution:

Given,

$a = 729$ and $a_7 = 64$

Let r be the common ratio of the G.P.

Then, we know that, $a_n = a\, r^{n-1}$

$a_7 = ar^{7-1} = (729)r^6$

$\Rightarrow 64 = 729\, r^6$

$r^6 = 64/729$

$r^6 = (2/3)^6$

$r = 2/3$

And we know that

13. Find a G.P. for which the sum of the first two terms is –4 and the fifth term is 4 times the third term.

Solution:

Consider a to be the first term and r to be the common ratio of the G.P.

Given, $S_2 = -4$

Then, from the question, we have

And,

$a_5 = 4 \times a_3$

$ar^4 = 4ar^2$

$r^2 = 4$

$r = \pm 2$

Using the value of r in (1), we have

Therefore, the required G.P is

-4/3, -8/3, -16/3, …. Or 4, -8, 16, -32, ……

SUMMARY

What does a Sequence and a Series Mean? A sequence is defined as an arrangement of numbers in a particular order. On the other hand, a series is defined as the sum of the elements of a sequence.

CHAPTER-IX

STRAIGHT LINES

2 MARK QUESTIONS

1. Find the slope of the line, which makes an angle of 30° with the positive direction of the y-axis measured anticlockwise.

Solution:

We know that if a line makes an angle of 30° with the positive direction of the y-axis measured anti-clock-wise, then the angle made by the line with the positive direction of the x-axis measured anti-clock-wise is 90° + 30° = 120°

∴ The slope of the given line is tan 120° = tan (180° – 60°)

= – tan 60°

= $-\sqrt{3}$

2. Find the value of x for which the points (x, – 1), (2, 1) and (4, 5) are collinear.

Solution:

If the points (x, – 1), (2, 1) and (4, 5) are collinear, then the Slope of AB = Slope of BC

Then, (1+1)/(2-x) = (5-1)/(4-2)

2/(2-x) = 4/2

2/(2-x) = 2

2 = 2(2-x)

2 = 4 – 2x

2x = 4 – 2

$2x = 2$

$x = 2/2$

$= 1$

∴ The required value of x is 1.

3. A line passes through (x_1, y_1) and (h, k). If the slope of the line is m, show that $k - y_1 = m (h - x_1)$.

Solution:

Given: the slope of the line is ‘m’.

The slope of the line passing through (x_1, y_1) and (h, k) is $(k - y_1)/(h - x_1)$

So,

$(k - y_1)/(h - x_1) = m$

$(k - y_1) = m (h - x_1)$

Hence, proved.

4. Consider the following population and year graph (Fig 10.10), find the slope of the line AB and using it, find what will be the population in the year 2010?

Solution:

We know that line AB passes through points A (1985, 92) and B (1995, 97).

Its slope will be $(97 - 92)/(1995 - 1985) = 5/10 = 1/2$

Let ‘y’ be the population in the year 2010. Then, according to the given graph, AB must pass through point C (2010, y)

So now, slope of AB = slope of BC

15/2 = y – 97

y = 7.5 + 97 = 104.5

∴ The slope of line AB is 1/2, while in the year 2010, the population will be 104.5 crores

5. Write the equations for the x-and y-axes.

Solution:

The y-coordinate of every point on the x-axis is 0.

∴ The equation of the x-axis is y = 0.

The x-coordinate of every point on the y-axis is 0.

∴ The equation of the y-axis is y = 0.

6. Passing through the point (– 4, 3) with slope 1/2

Solution:

Given:

Point (-4, 3) and slope, m = 1/2

We know that the point (x, y) lies on the line with slope m through the fixed point (x_0, y_0) only if its coordinates satisfy the equation $y – y_0 = m (x – x_0)$

So, y – 3 = 1/2 (x – (-4))

y – 3 = 1/2 (x + 4)

2(y – 3) = x + 4

2y – 6 = x + 4

x + 4 – (2y – 6) = 0

x + 4 – 2y + 6 = 0

$x - 2y + 10 = 0$

∴ The equation of the line is $x - 2y + 10 = 0$

7. Passing through (0, 0) with slope m.

Solution:

Given:

Point (0, 0) and slope, m = m

We know that the point (x, y) lies on the line with slope m through the fixed point (x_0, y_0) only if its coordinates satisfy the equation $y - y_0 = m (x - x_0)$

So, $y - 0 = m (x - 0)$

$y = mx$

$y - mx = 0$

∴ The equation of the line is $y - mx = 0$

8. The length L (in centimetres) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if L = 124.942 when C = 20 and L= 125.134 when C = 110, express L in terms of C.

Solution:

Let us assume 'L' along X-axis and 'C' along Y-axis; we have two points (124.942, 20) and (125.134, 110) in XY-plane.

We know that the equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

4 MARK QUESTIONS

1. Draw a quadrilateral in the Cartesian plane whose vertices are (– 4, 5), (0, 7), (5, – 5) and (– 4, –2). Also, find its area.

Solution:

Let ABCD be the given quadrilateral with vertices A (-4,5), B (0,7), C (5.-5) and D (-4,-2).

Now, let us plot the points on the Cartesian plane by joining the points AB, BC, CD, and AD, which give us the required quadrilateral.

To find the area, draw diagonal AC.

So, area (ABCD) = area (ΔABC) + area (ΔADC)

Then, area of triangle with vertices (x_1,y_1) , (x_2, y_2) and (x_3,y_3) is

Are of Δ ABC = $\frac{1}{2}[x_1 (y_2 – y_3) + x_2 (y_3 – y_1) + x_3 (y_1 – y_2)]$

= $\frac{1}{2}[-4 (7 + 5) + 0 (-5 – 5) + 5 (5 – 7)]$ unit^2

= $\frac{1}{2}[-4 (12) + 5 (-2)]$ unit^2

= $\frac{1}{2}(58)$ unit^2

= 29 unit^2

Are of Δ ACD = $\frac{1}{2}[x_1 (y_2 – y_3) + x_2 (y_3 – y_1) + x_3 (y_1 – y_2)]$

= $\frac{1}{2}[-4 (-5 + 2) + 5 (-2 – 5) + (-4) (5 – (-5))]$ unit^2

= $\frac{1}{2}[-4 (-3) + 5 (-7) – 4 (10)]$ unit^2

= $\frac{1}{2}(-63)$ unit^2

= -63/2 unit^2

Since area cannot be negative, area Δ ACD = 63/2 unit^2

Area (ABCD) = 29 + 63/2

$= 121/2$ unit^2

2. The base of an equilateral triangle with side 2a lies along the y-axis such that the mid-point of the base is at the origin. Find the vertices of the triangle.

Solution:

Let us consider ABC, the given equilateral triangle with side 2a.

Where, AB = BC = AC = 2a

In the above figure, assuming that the base BC lies on the x-axis such that the mid-point of BC is at the origin, i.e., BO = OC = a, where O is the origin.

The coordinates of point C are (0, a) and that of B are (0,-a).

The line joining a vertex of an equilateral Δ with the mid-point of its opposite side is perpendicular.

So, vertex A lies on the y –axis.

By applying Pythagoras' theorem,

$(AC)^2 = OA^2 + OC^2$

$(2a)^2 = a^2 + OC^2$

$4a^2 - a^2 = OC^2$

$3a^2 = OC^2$

$OC = \surd 3a$

Co-ordinates of point C = **±** √3a, 0

∴ The vertices of the given equilateral triangle are (0, a), (0, -a), (√3a, 0)

Or (0, a), (0, -a) and (-√3a, 0)

3. Find the distance between P (x_1, y_1) and Q (x_2, y_2) when: (i) PQ is parallel to the y-axis, (ii) PQ is parallel to the x-axis.

Solution:

Given:

Points P (x_1, y_1) and Q(x_2, y_2)

(i) When PQ is parallel to the y-axis, then $x_1 = x_2$

So, the distance between P and Q is given by

$= |y_2 – y_1|$

(ii) When PQ is parallel to the x-axis, then $y_1 = y_2$

So, the distance between P and Q is given by =

=

$= |x_2 – x_1|$

4. Find a point on the x-axis which is equidistant from points (7, 6) and (3, 4).

Solution:

Let us consider (a, 0) to be the point on the x-axis that is equidistant from the point (7, 6) and (3, 4).

So,

Now, let us square on both sides; we get,

$a^2 – 14a + 85 = a^2 – 6a + 25$

$-8a = -60$

$a = 60/8$

$= 15/2$

∴ The required point is (15/2, 0)

5. If three points (h, 0), (a, b) and (0, k) lie on a line, show that a/h + b/k = 1

Solution:

Let us consider if the given points A (h, 0), B (a, b) and C (0, k) lie on a line.

Then, the slope of AB = slope of BC

$(b - 0)/(a - h) = (k - b)/(0 - a)$

By simplifying, we get

$-ab = (k-b)(a-h)$

$-ab = ka - kh - ab + bh$

$ka + bh = kh$

Divide both sides by kh; we get

$ka/kh + bh/kh = kh/kh$

$a/h + b/k = 1$

Hence, proved.

6. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points P (0, – 4) and B (8, 0).

Solution:

The co-ordinates of the mid-point of the line segment joining the points P (0, – 4) and B (8, 0) are (0+8)/2, (-4+0)/2 = (4, -2)

The slope 'm' of the line non-vertical line passing through the point (x_1, y_1) and

(x_2, y_2) is given by $m = (y_2 – y_1)/(x_2 – x_1)$ where, $x \neq x_1$

The slope of the line passing through (0, 0) and (4, -2) is (-2-0)/(4-0) = -1/2

∴ The required slope is -1/2.

7. Find the angle between the x-axis and the line joining the points (3, –1) and (4, –2).

Solution:

The Slope of the line joining the points (3, -1) and (4, -2) is given by

$m = (y_2 – y_1)/(x_2 – x_1)$ where, $x \neq x_1$

m = (-2 –(-1))/(4-3)

= (-2+1)/(4-3)

= -1/1

= -1

The angle of inclination of the line joining the points (3, -1) and (4, -2) is given by

tan θ = -1

θ = (90° + 45°) = 135°

∴ The angle between the x-axis and the line joining the points (3, –1) and (4, –2) is 135°.

8. The owner of a milk store finds that he can sell 980 litres of milk each week at Rs. 14/litre and 1220 litres of milk each week at Rs. 16/litre. Assuming a linear relationship between the selling price and demand, how many litres could he sell weekly at Rs. 17/litre?

Solution:

Assuming the relationship between the selling price and demand is linear.

Let us assume the selling price per litre along X-axis and demand along Y-axis, we have two points (14, 980) and (16, 1220) in XY-plane.

We know that the equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$y - 980 = 120 (x - 14)$

$y = 120 (x - 14) + 980$

When x = Rs 17/litre,

$y = 120 (17 - 14) + 980$

$y = 120(3) + 980$

$y = 360 + 980 = 1340$

∴ The owner can sell 1340 litres weekly at Rs. 17/litre.

9. P (a, b) is the mid-point of a line segment between axes. Show that the equation of the line is x/a + y/b = 2

Solution:

Let AB be a line segment whose midpoint is P (a, b).

Let the coordinates of A and B be (0, y) and (x, 0), respectively.

$a (y - 2b) = -bx$

$ay - 2ab = -bx$

$bx + ay = 2ab$

Divide both sides with ab, then

Hence, proved.

7 MARK QUESTIONS

1. Without using Pythagoras' theorem, show that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right-angled triangle.

Solution:

The vertices of the given triangle are (4, 4), (3, 5) and (–1, –1).

The slope (m) of the line non-vertical line passing through the point (x_1, y_1) and

(x_2, y_2) is given by $m = (y_2 – y_1)/(x_2 – x_1)$ where, $x \neq x_1$

So, the slope of the line AB (m_1) = (5-4)/(3-4) = 1/-1 = -1

The slope of the line BC (m_2) = (-1-5)/(-1-3) = -6/-4 = 3/2

The slope of the line CA (m_3) = (4+1)/(4+1) = 5/5 = 1

It is observed that $m_1.m_3$ = -1.1 = -1

Hence, the lines AB and CA are perpendicular to each other.

∴ given triangle is right-angled at A (4, 4)

And the vertices of the right-angled Δ are (4, 4), (3, 5) and (-1, -1)

2. Without using the distance formula, show that points (– 2, – 1), (4, 0), (3, 3) and (–3, 2) are the vertices of a parallelogram.

Solution:

Let the given point be A (-2, -1) , B (4, 0) , C (3, 3) and D (-3, 2)

So now, the slope of AB = (0+1)/(4+2) = 1/6

The slope of CD = (3-2)/(3+3) = 1/6

Hence, the Slope of AB = Slope of CD

∴ AB || CD

Now,

The slope of BC = (3-0)/(3-4) = 3/-1 = -3

The slope of AD = (2+1)/(-3+2) = 3/-1 = -3

Hence, the Slope of BC = Slope of AD

∴ BC || AD

Thus, the pair of opposite sides are quadrilateral are parallel, so we can say that ABCD is a parallelogram.

Hence, the given vertices, A (-2, -1), B (4, 0), C(3, 3) and D(-3, 2) are vertices of a parallelogram.

3. The slope of a line is double the slope of another line. If the tangent of the angle between them is 1/3, find the slopes of the lines.

Solution:

Let us consider 'm_1' and 'm' be the slope of the two given lines such that $m_1 = 2m$

We know that if θ is the angle between the lines l1 and l2 with slope m_1 and m_2, then

$1+2m^2 = -3m$

$2m^2 +1 +3m = 0$

2m (m+1) + 1(m+1) = 0

(2m+1) (m+1)= 0

m = -1 or -1/2

If m = -1, then the slope of the lines are -1 and -2

If m = -1/2, then the slope of the lines are -1/2 and -1

Case 2:

$2m^2 - 3m + 1 = 0$

$2m^2 - 2m - m + 1 = 0$

$2m (m - 1) - 1(m - 1) = 0$

$m = 1$ or $1/2$

If $m = 1$, then the slope of the lines are 1 and 2

If $m = 1/2$, then the slope of the lines are 1/2 and 1

∴ The slope of the lines are [-1 and -2] or [-1/2 and -1] or [1 and 2] or [1/2 and 1]

4. Passing through (2, 2√3) and inclined with the x-axis at an angle of 75°.

Solution:

Given: point $(2, 2\sqrt{3})$ and $\theta = 75°$

Equation of line: $(y - y_1) = m (x - x_1)$

where, m = slope of line = $\tan\theta$ and (x_1, y_1) are the points through which line passes

∴ $m = \tan 75°$

$75° = 45° + 30°$

Applying the formula:

We know that the point (x, y) lies on the line with slope m through the fixed point (x_1, y1), only if its coordinates satisfy the equation $y - y_1 = m (x - x_1)$

Then, $y - 2\sqrt{3} = (2 + \sqrt{3}) (x - 2)$

$y - 2\sqrt{3} = 2x - 4 + \sqrt{3}x - 2\sqrt{3}$

$y = 2x - 4 + \sqrt{3}x$

$(2 + \sqrt{3})x - y - 4 = 0$

∴ The equation of the line is $(2 + \sqrt{3})x - y - 4 = 0$

5. Intersecting the x-axis at a distance of 3 units to the left of origin with slope –2.

Solution:

Given:

Slope, $m = -2$

We know that if a line L with slope m makes x-intercept d, then the equation of L is

$y = m(x - d)$.

If the distance is 3 units to the left of the origin, then $d = -3$

So, $y = (-2)(x - (-3))$

$y = (-2)(x + 3)$

$y = -2x - 6$

$2x + y + 6 = 0$

∴ The equation of the line is $2x + y + 6 = 0$

6. Intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30^o with the positive direction of the x-axis.

Solution:

Given: $\theta = 30°$

We know that slope, $m = \tan\theta$

$m = \tan 30° = (1/\sqrt{3})$

We know that the point (x, y) on the line with slope m and y-intercept c lies on the line only if $y = mx + c$

If the distance is 2 units above the origin, $c = +2$

So, $y = (1/\sqrt{3})x + 2$

$y = (x + 2\sqrt{3}) / \sqrt{3}$

$\sqrt{3}\, y = x + 2\sqrt{3}$

$x - \sqrt{3}\, y + 2\sqrt{3} = 0$

∴ The equation of the line is $x - \sqrt{3}\, y + 2\sqrt{3} = 0$

7. Passing through the points (–1, 1) and (2, – 4).

Solution:

Given:

Points (-1, 1) and (2, -4)

We know that the equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$y - 1 = -5/3\ (x + 1)$

$3\ (y - 1) = (-5)\ (x + 1)$

$3y - 3 = -5x - 5$

$3y - 3 + 5x + 5 = 0$

$5x + 3y + 2 = 0$

∴ The equation of the line is $5x + 3y + 2 = 0$

8. Perpendicular distance from the origin is 5 units, and the angle made by the perpendicular with the positive x-axis is 30º.

Solution:

Given: $p = 5$ and $\omega = 30°$

We know that the equation of the line having normal distance p from the origin and angle ω, which the normal makes with the positive direction of the x-axis, is given by $x \cos \omega + y \sin \omega = p$.

Substituting the values in the equation, we get

$x \cos 30° + y \sin 30° = 5$

$x(\sqrt{3} / 2) + y(1/2) = 5$

$\sqrt{3}x + y = 5(2) = 10$

$\sqrt{3}x + y - 10 = 0$

∴ The equation of the line is $\sqrt{3}x + y - 10 = 0$

9. The vertices of ΔPQR are P (2, 1), Q (–2, 3) and R (4, 5). Find the equation of the median through the vertex R.

Solution:

Given:

Vertices of ΔPQR, i.e., P (2, 1), Q (-2, 3) and R (4, 5)

Let RL be the median of vertex R.

So, L is a midpoint of PQ.

We know that the midpoint formula is given by

.

∴ L = = (0, 2)

We know that the equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

y – 5 = -3/-4 (x-4)

(-4) (y – 5) = (-3) (x – 4)

-4y + 20 = -3x + 12

-4y + 20 + 3x – 12 = 0

3x – 4y + 8 = 0

∴ The equation of median through the vertex R is 3x – 4y + 8 = 0

10. Find the equation of the line passing through (–3, 5) and perpendicular to the line through the points (2, 5) and (–3, 6).

Solution:

Given:

Points are (2, 5) and (-3, 6).

We know that slope, m = (y_2 – y_1)/(x_2 – x_1)

= (6 – 5)/(-3 – 2)

= 1/-5 = -1/5

We know that two non-vertical lines are perpendicular to each other only if their slopes are negative reciprocals of each other.

Then, m = (-1/m)

= -1/(-1/5)

= 5

We know that the point (x, y) lies on the line with slope m through the fixed point (x_0, y_0), only if its coordinates satisfy the equation $y - y_0 = m(x - x_0)$

Then, $y - 5 = 5(x - (-3))$

$y - 5 = 5x + 15$

$5x + 15 - y + 5 = 0$

$5x - y + 20 = 0$

∴ The equation of the line is $5x - y + 20 = 0$

11. A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio 1: n. Find the equation of the line.

Solution:

We know that the coordinates of a point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio m: n are

We know that slope, $m = (y_2 - y_1)/(x_2 - x_1)$

$= (3 - 0)/(2 - 1)$

$= 3/1$

$= 3$

We know that two non-vertical lines are perpendicular to each other only if their slopes are negative reciprocals of each other.

Then, $m = (-1/m) = -1/3$

We know that the point (x, y) lies on the line with slope m through the fixed point (x_0, y_0), only if its coordinates satisfy the equation $y - y_0 = m(x - x_0)$

Here, the point is

$3((1 + n) y – 3) = (-(1 + n) x + 2 + n)$

$3(1 + n) y – 9 = – (1 + n) x + 2 + n$

$(1 + n) x + 3(1 + n) y – n – 9 – 2 = 0$

$(1 + n) x + 3(1 + n) y – n – 11 = 0$

∴ The equation of the line is $(1 + n) x + 3(1 + n) y – n – 11 = 0$

12. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point (2, 3).

Solution:

Given: the line cuts off equal intercepts on the coordinate axes, i.e., $a = b$

We know that equation of the line intercepts a and b on the x-and the y-axis, respectively, which is

$x/a + y/b = 1$

So, $x/a + y/a = 1$

$x + y = a$ … (1)

Given: point (2, 3)

$2 + 3 = a$

$a = 5$

Substitute value of 'a' in (1), we get

$x + y = 5$

$x + y – 5 = 0$

∴ The equation of the line is $x + y – 5 = 0$

13. Find the equation of the line passing through the point (2, 2) and cutting off intercepts on the axes whose sum is 9.

Solution:

We know that equation of the line-making intercepts a and b on the x-and the y-axis, respectively, is $x/a + y/b = 1$. … (1)

Given: sum of intercepts = 9

$a + b = 9$

$b = 9 – a$

Now, substitute the value of b in the above equation, and we get

$x/a + y/(9 – a) = 1$

Given: the line passes through point (2, 2)

So, $2/a + 2/(9 – a) = 1$

$[2(9 – a) + 2a] / a(9 – a) = 1$ $[18 – 2a + 2a] / a(9 – a) = 1$
$18/a(9 – a) = 1$

$18 = a (9 – a)$

$18 = 9a – a^2$

$a^2 – 9a + 18 = 0$

Upon factorising, we get

$a^2 – 3a – 6a + 18 = 0$

$a (a – 3) – 6 (a – 3) = 0$

$(a – 3) (a – 6) = 0$

$a = 3$ or $a = 6$

Let us substitute in (1)

Case 1 ($a = 3$):

Then $b = 9 – 3 = 6$

$x/3 + y/6 = 1$

$2x + y = 6$

$2x + y – 6 = 0$

Case 2 ($a = 6$):

Then $b = 9 – 6 = 3$

$x/6 + y/3 = 1$

$x + 2y = 6$

$x + 2y – 6 = 0$

∴ The equation of the line is $2x + y – 6 = 0$ or $x + 2y – 6 = 0$

14. Find the equation of the line through the point (0, 2), making an angle 2π/3 with the positive x-axis. Also, find the equation of the line parallel to it and crossing the y-axis at a distance of 2 units below the origin.

Solution:

Given:

Point (0, 2) and $\theta = 2\pi/3$

We know that $m = \tan \theta$

$m = \tan (2\pi/3) = -\sqrt{3}$

We know that the point (x, y) lies on the line with slope m through the fixed point (x_0, y_0), only if its coordinates satisfy the equation $y – y_0 = m (x – x_0)$

$y – 2 = -\sqrt{3} (x – 0)$

$y – 2 = -\sqrt{3} x$

$\sqrt{3}x + y – 2 = 0$

Given, the equation of the line parallel to the above-obtained equation crosses the y-axis at a distance of 2 units below the origin.

So, the point = (0, -2) and $m = -\sqrt{3}$

From point slope form equation,

$y – (-2) = -\sqrt{3}(x – 0)$

$y + 2 = -\sqrt{3}x$

$\sqrt{3}x + y + 2 = 0$

∴ The equation of the line is $\sqrt{3}x + y – 2 = 0$, and the line parallel to it is $\sqrt{3}x + y + 2 = 0$

15. The perpendicular from the origin to a line meets it at the point (–2, 9). Find the equation of the line.

Solution:

Given:

Points are origin (0, 0) and (-2, 9).

We know that slope, $m = (y_2 – y_1)/(x_2 – x_1)$

$= (9 – 0)/(-2-0)$

$= -9/2$

We know that two non-vertical lines are perpendicular to each other only if their slopes are negative reciprocals of each other.

$m = (-1/m) = -1/(-9/2) = 2/9$

We know that the point (x, y) lies on the line with slope m through the fixed point (x_0, y_0) only if its coordinates satisfy the equation $y – y_0 = m (x – x_0)$

$y – 9 = (2/9) (x – (-2))$

$9(y – 9) = 2(x + 2)$

$9y - 81 = 2x + 4$

$2x + 4 - 9y + 81 = 0$

$2x - 9y + 85 = 0$

∴ The equation of the line is $2x - 9y + 85 = 0$

16. Point R (h, k) divides a line segment between the axes in the ratio 1: 2. Find the equation of the line.

Solution:

Let us consider AB to be the line segment, such that r (h, k) divides it in the ratio 1: 2.

So, the coordinates of A and B be (0, y) and (x, 0), respectively.

We know that the coordinates of a point dividing the line segment join the points (x_1, y_1) and (x_2, y_2) internally in the ratio m: n is

$h = 2x/3$ and $k = y/3$

$x = 3h/2$ and $y = 3k$

∴ A = (0, 3k) and B = (3h/2, 0)

We know that the equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$3h(y - 3k) = -6kx$

$3hy - 9hk = -6kx$

$6kx + 3hy = 9hk$

Let us divide both sides by 9hk, and we get,

$2x/3h + y/3k = 1$

∴ The equation of the line is given by $2x/3h + y/3k = 1$

17. By using the concept of the equation of a line, prove that the three points (3, 0), (– 2, – 2) and (8, 2) are collinear.

Solution:

According to the question,

If we have to prove that the given three points (3, 0), (– 2, – 2) and (8, 2) are collinear, then we have to also prove that the line passing through the points (3, 0) and (– 2, – 2) also passes through the point (8, 2).

By using the formula,

The equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$-5y = -2 (x – 3)$

$-5y = -2x + 6$

$2x – 5y = 6$

If $2x – 5y = 6$ passes through (8, 2),

$2x – 5y = 2(8) – 5(2)$

$= 16 – 10$

$= 6$

= RHS

The line passing through points (3, 0) and (– 2, – 2) also passes through the point (8, 2).

Hence, proved. The given three points are collinear.

18. Reduce the following equations into slope-intercept form and find their slopes and the y-intercepts.
(i) x + 7y = 0

(ii) 6x + 3y – 5 = 0
(iii) y = 0

Solution:

(i) x + 7y = 0

Given:

The equation is x + 7y = 0

The slope-intercept form is represented in the form 'y = mx + c', where m is the slope and c is the y-intercept.

So, the above equation can be expressed as

y = -1/7x + 0

∴ The above equation is of the form y = mx + c, where m = -1/7 and c = 0

(ii) 6x + 3y – 5 = 0

Given:

The equation is 6x + 3y – 5 = 0

The slope-intercept form is represented in the form 'y = mx + c', where m is the slope and c is the y-intercept.

So, the above equation can be expressed as

3y = -6x + 5

y = -6/3x + 5/3

= -2x + 5/3

∴ The above equation is of the form $y = mx + c$, where $m = -2$ and $c = 5/3$

(iii) $y = 0$

Given:

The equation is $y = 0$

The slope-intercept form is given by '$y = mx + c$', where m is the slope and c is the y-intercept.

$y = 0 \times x + 0$

∴ The above equation is of the form $y = mx + c$, where $m = 0$ and $c = 0$

19. Reduce the following equations into intercept form and find their intercepts on the axes.

(i) $3x + 2y – 12 = 0$

(ii) $4x – 3y = 6$

(iii) $3y + 2 = 0$

Solution:

(i) $3x + 2y – 12 = 0$

Given:

The equation is $3x + 2y – 12 = 0$

The equation of the line in intercept form is given by $x/a + y/b = 1$, where 'a' and 'b' are intercepted on the x-axis and the y-axis, respectively.

So, $3x + 2y = 12$

Now, let us divide both sides by 12; we get

$3x/12 + 2y/12 = 12/12$

$x/4 + y/6 = 1$

∴ The above equation is of the form $x/a + y/b = 1$, where $a = 4$, $b = 6$

The intercept on the x-axis is 4.

The intercept on the y-axis is 6.

(ii) $4x - 3y = 6$

Given:

The equation is $4x - 3y = 6$

The equation of the line in intercept form is given by $x/a + y/b = 1$, where 'a' and 'b' are intercepted on the x-axis and the y-axis, respectively.

So, $4x - 3y = 6$

Now, let us divide both sides by 6; we get

$4x/6 - 3y/6 = 6/6$

$2x/3 - y/2 = 1$

$x/(3/2) + y/(-2) = 1$

∴ The above equation is of the form $x/a + y/b = 1$, where $a = 3/2$, $b = -2$

The intercept on the x-axis is 3/2.

The intercept on the y-axis is -2.

(iii) $3y + 2 = 0$

Given:

The equation is $3y + 2 = 0$

The equation of the line in intercept form is given by $x/a + y/b = 1$, where 'a' and 'b' are intercepted on the x-axis and the y-axis, respectively.

So, $3y = -2$

Now, let us divide both sides by -2; we get

$3y/-2 = -2/-2$

$3y/-2 = 1$

$y/(-2/3) = 1$

∴ The above equation is of the form $x/a + y/b = 1$, where $a = 0$, $b = -2/3$

The intercept on the x-axis is 0.

The intercept on the y-axis is $-2/3$.

SUMMARY

Slope of line passing through the points (x_1, y_1) and (x_2,y_2) is $m = (y_2 - y_1) /(x_2 - x_1)$. Equation of a line when it makes intercepts 'a' and 'b' on the x-and y-axis: $x/a + y/b = 1$. Equation of any line with slope 'm' that has x-intercept 'd' is $y=m (x – d)$.

.

CHAPTER-X

CONIC SECTIONS

2 MARK QUESTIONS

1. $y^2 = 10x$

Solution:

Given:

The equation is $y^2 = 10x$

Here, we know that the coefficient of x is positive.

So, the parabola open towards the right.

On comparing this equation with $y^2 = 4ax$, we get,

$4a = 10$

$a = 10/4 = 5/2$

Thus, co-ordinates of the focus = (a,0) = (5/2, 0)

Since the given equation involves y^2, the axis of the parabola is the x-axis.

∴ The equation of directrix, x =-a, then,

$x = -5/2$

Length of latus rectum = 4a = 4(5/2) = 10

2. Vertex (0, 0); focus (3, 0)

Solution:

Given:

Vertex (0, 0) and focus (3, 0)

We know that the vertex of the parabola is (0, 0) and the focus lies on the positive x-axis. [x-axis is the axis of the parabola.]

The equation of the parabola is of the form $y^2 = 4ax$.

Since, the focus is (3, 0), a = 3

∴ The equation of the parabola is $y^2 = 4 \times 3 \times x$,

$y^2 = 12x$

3. Vertex (0, 0); focus (–2, 0)

Solution:

Given:

Vertex (0, 0) and focus (-2, 0)

We know that the vertex of the parabola is (0, 0) and the focus lies on the positive x-axis. [x-axis is the axis of the parabola.]

The equation of the parabola is of the form $y^2=-4ax$.

Since, the focus is (-2, 0), a = 2

∴ The equation of the parabola is $y^2 = -4 \times 2 \times x$,

$y^2 = -8x$

4. Vertex (0, 0) passing through (2, 3) and axis is along x-axis.

Solution:

We know that the vertex is (0, 0) and the axis of the parabola is the x-axis

The equation of the parabola is either of the form $y^2 = 4ax$ or $y^2 = -4ax$.

Given that the parabola passes through point (2, 3), which lies in the first quadrant.

So, the equation of the parabola is of the form $y^2 = 4ax$, while point (2, 3) must satisfy the equation $y^2 = 4ax$.

Then,

$3^2 = 4a(2)$

$3^2 = 8a$

$9 = 8a$

$a = 9/8$

Thus, the equation of the parabola is

$y^2 = 4\ (9/8)x$

$= 9x/2$

$2y^2 = 9x$

∴ The equation of the parabola is $2y^2 = 9x$

5. Vertex (0, 0), passing through (5, 2) and symmetric with respect to y-axis.

Solution:

We know that the vertex is (0, 0) and the parabola is symmetric about the y-axis.

The equation of the parabola is either of the form $x^2 = 4ay$ or $x^2 = -4ay$.

Given that the parabola passes through point (5, 2), which lies in the first quadrant.

So, the equation of the parabola is of the form $x^2 = 4ay$, while point (5, 2) must satisfy the equation $x^2 = 4ay$.

Then,

$5^2 = 4a(2)$

$25 = 8a$

$a = 25/8$

Thus, the equation of the parabola is

$x^2 = 4\ (25/8)y$

$x^2 = 25y/2$

$2x^2 = 25y$

$\therefore$ The equation of the parabola is $2x^2 = 25y$

4 MARK QUESTIONS

1. Centre (0, 2) and radius 2

Solution:

Given:

Centre (0, 2) and radius 2

Let us consider the equation of a circle with centre (h, k) and

Radius r is given as $(x - h)^2 + (y - k)^2 = r^2$

So, centre (h, k) = (0, 2) and radius (r) = 2

The equation of the circle is

$(x - 0)^2 + (y - 2)^2 = 2^2$

$x^2 + y^2 + 4 - 4y = 4$

$x^2 + y^2 - 4y = 0$

∴ The equation of the circle is $x^2 + y^2 - 4y = 0$

2. Centre (–2, 3) and radius 4

Solution:

Given:

Centre (-2, 3) and radius 4

Let us consider the equation of a circle with centre (h, k).

Radius r is given as $(x - h)^2 + (y - k)^2 = r^2$

So, centre (h, k) = (-2, 3) and radius (r) = 4

The equation of the circle is

$(x + 2)^2 + (y - 3)^2 = (4)^2$

$x^2 + 4x + 4 + y^2 – 6y + 9 = 16$

$x^2 + y^2 + 4x – 6y – 3 = 0$

∴ The equation of the circle is $x^2 + y^2 + 4x – 6y – 3 = 0$

3. Centre (1/2, 1/4) and radius (1/12)

Solution:

Given:

Centre (1/2, 1/4) and radius 1/12

Let us consider the equation of a circle with centre (h, k).

Radius r is given as $(x – h)^2 + (y – k)^2 = r^2$

So, centre (h, k) = (1/2, 1/4) and radius (r) = 1/12

The equation of the circle is

$(x – 1/2)^2 + (y – 1/4)^2 = (1/12)^2$

$x^2 – x + ¼ + y^2 – y/2 + 1/16 = 1/144$

$x^2 – x + ¼ + y^2 – y/2 + 1/16 = 1/144$

$144x^2 – 144x + 36 + 144y^2 – 72y + 9 – 1 = 0$

$144x^2 – 144x + 144y^2 – 72y + 44 = 0$

$36x^2 + 36x + 36y^2 – 18y + 11 = 0$

$36x^2 + 36y^2 – 36x – 18y + 11= 0$

∴ The equation of the circle is $36x^2 + 36y^2 – 36x – 18y + 11= 0$

4. Centre (1, 1) and radius √2

Solution:

Given:

Centre (1, 1) and radius $\sqrt{2}$

Let us consider the equation of a circle with centre (h, k).

Radius r is given as $(x - h)^2 + (y - k)^2 = r^2$

So, centre (h, k) = (1, 1) and radius (r) = $\sqrt{2}$

The equation of the circle is

$(x-1)^2 + (y-1)^2 = (\sqrt{2})^2$

$x^2 - 2x + 1 + y^2 - 2y + 1 = 2$

$x^2 + y^2 - 2x - 2y = 0$

∴ The equation of the circle is $x^2 + y^2 - 2x - 2y = 0$

5. Centre (–a, –b) and radius $\sqrt{(a^2 - b^2)}$

Solution:

Given:

Centre (-a, -b) and radius $\sqrt{(a^2 - b^2)}$

Let us consider the equation of a circle with centre (h, k) and

Radius r is given as $(x - h)^2 + (y - k)^2 = r^2$

So, centre (h, k) = (-a, -b) and radius (r) = $\sqrt{(a^2 - b^2)}$

The equation of the circle is

$(x + a)^2 + (y + b)^2 = (\sqrt{(a^2 - b^2)^2})$

$x^2 + 2ax + a^2 + y^2 + 2by + b^2 = a^2 - b^2$

$x^2 + y^2 + 2ax + 2by + 2b^2 = 0$

∴ The equation of the circle is $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$

In each of the following Exercise 6 to 9, find the centre and radius of the circles.

6. $(x + 5)^2 + (y - 3)^2 = 36$

Solution:

Given:

The equation of the given circle is $(x + 5)^2 + (y – 3)^2 = 36$

$(x – (-5))^2 + (y – 3)^2 = 6^2$ [which is of the form $(x – h)^2 + (y – k)^2 = r^2$]

Where, $h = -5$, $k = 3$ and $r = 6$

∴ The centre of the given circle is (-5, 3) and its radius is 6.

7. $x^2 + y^2 – 4x – 8y – 45 = 0$

Solution:

Given:

The equation of the given circle is $x^2 + y^2 – 4x – 8y – 45 = 0$.

$x^2 + y^2 – 4x – 8y – 45 = 0$

$(x^2 – 4x) + (y^2 - 8y) = 45$

$(x^2 – 2(x)(2) + 2^2) + (y^2 – 2(y)(4) + 4^2) – 4 – 16 = 45$

$(x – 2)^2 + (y – 4)^2 = 65$

$(x – 2)^2 + (y – 4)^2 = (\sqrt{65})^2$ [which is form $(x-h)^2 + (y-k)^2 = r^2$]

Where $h = 2$, $K = 4$ and $r = \sqrt{65}$

∴ The centre of the given circle is (2, 4) and its radius is $\sqrt{65}$.

8. $x^2 + y^2 – 8x + 10y – 12 = 0$

Solution:

Given:

The equation of the given circle is $x^2 + y^2 - 8x + 10y - 12 = 0$.

$x^2 + y^2 – 8x + 10y – 12 = 0$

$(x^2 – 8x) + (y^2 + 10y) = 12$

$(x^2 - 2(x)(4) + 4^2) + (y^2 - 2(y)(5) + 5^2) - 16 - 25 = 12$

$(x - 4)^2 + (y + 5)^2 = 53$

$(x - 4)^2 + (y - (-5))^2 = (\sqrt{53})^2$ [which is form $(x-h)^2 + (y-k)^2 = r^2$]

Where h = 4, K= -5 and r = $\sqrt{53}$

∴ The centre of the given circle is (4, -5) and its radius is $\sqrt{53}$.

9. $2x^2 + 2y^2 - x = 0$

Solution:

The equation of the given circle is $2x^2 + 2y^2 - x = 0$.

$2x^2 + 2y^2 - x = 0$

$(2x^2 + x) + 2y^2 = 0$

$(x^2 - 2(x)(1/4) + (1/4)^2) + y^2 - (1/4)^2 = 0$

$(x - 1/4)^2 + (y - 0)^2 = (1/4)^2$ [which is form $(x-h)^2 + (y-k)^2 = r^2$]

Where, h = ¼, K = 0, and r = ¼

∴ The center of the given circle is (1/4, 0) and its radius is 1/4.

10. Find the equation of the circle passing through the points (4,1) and (6,5) and whose centre is on the line 4x + y = 16.

Solution:

Let us consider the equation of the required circle to be $(x - h)^2 + (y - k)^2 = r^2$

We know that the circle passes through points (4,1) and (6,5)

So,

$(4 - h)^2 + (1 - k)^2 = r^2$ ……………..(1)

$(6 - h)^2 + (5 - k)^2 = r^2$ ………………(2)

Since, the centre (h, k) of the circle lies on line 4x + y = 16,

$4h + k = 16$………………… (3)

From the equation (1) and (2), we obtain

$(4 – h)^2 + (1 – k)^2 = (6 – h)^2 + (5 – k)^2$

$16 – 8h + h^2 + 1 - 2k + k^2 = 36 - 12h + h^2 + 15 – 10k + k^2$

$16 – 8h + 1 - 2k + 12h - 25 - 10k$

$4h + 8k = 44$

$h + 2k = 11$……………. (4)

On solving equations (3) and (4), we obtain h=3 and k= 4.

On substituting the values of h and k in equation (1), we obtain

$(4 – 3)^2 + (1 – 4)^2 = r^2$

$(1)^2 + (-3)^2 = r^2$

$1+9 = r^2$

$r = \sqrt{10}$

so now, $(x – 3)^2 + (y – 4)^2 = (\sqrt{10})^2$

$x^2 – 6x + 9 + y^2 – 8y + 16 = 10$

$x^2 + y^2 – 6x – 8y + 15 = 0$

∴ The equation of the required circle is $x^2 + y^2 – 6x – 8y + 15 = 0$

11. $y^2 = 12x$

Solution:

Given:

The equation is $y^2 = 12x$

Here, we know that the coefficient of x is positive.

So, the parabola opens towards the right.

On comparing this equation with $y^2 = 4ax$, we get,

$4a = 12$

$a = 3$

Thus, the co-ordinates of the focus = (a, 0) = (3, 0)

Since the given equation involves y^2, the axis of the parabola is the x-axis.

∴ The equation of directrix, $x = -a$, then,

$x + 3 = 0$

Length of latus rectum = $4a = 4 \times 3 = 12$

12. $x^2 = 6y$

Solution:

Given:

The equation is $x^2 = 6y$

Here, we know that the coefficient of y is positive.

So, the parabola opens upwards.

On comparing this equation with $x^2 = 4ay$, we get,

$4a = 6$

$a = 6/4$

$= 3/2$

Thus, the co-ordinates of the focus = (0,a) = (0, 3/2)

Since the given equation involves x^2, the axis of the parabola is the y-axis.

∴ The equation of directrix, $y =-a$, then, y = -3/2

Length of latus rectum = 4a = 4(3/2) = 6

13. $y^2 = -8x$

Solution:

Given:

The equation is $y^2 = -8x$

Here, we know that the coefficient of x is negative.

So, the parabola open towards the left.

On comparing this equation with $y^2 = -4ax$, we get,

-4a = -8

a = -8/-4 = 2

Thus, co-ordinates of the focus = (-a,0) = (-2, 0)

Since the given equation involves y^2, the axis of the parabola is the x-axis.

∴ Equation of directrix, x =a, then,

x = 2

Length of latus rectum = 4a = 4 (2) = 8

14. $x^2 = -16y$

Solution:

Given:

The equation is $x^2 = -16y$

Here, we know that the coefficient of y is negative.

So, the parabola opens downwards.

On comparing this equation with $x^2 = -4ay$, we get,

$-4a = -16$

$a = -16/-4$

$= 4$

Thus, co-ordinates of the focus = $(0,-a) = (0,-4)$

Since the given equation involves x^2, the axis of the parabola is the y-axis.

∴ The equation of directrix, $y = a$, then,

$y = 4$

Length of latus rectum = $4a = 4(4) = 16$

15. Vertices (±2, 0), foci (±3, 0)

Solution:

Given:

Vertices (±2, 0) and foci (±3, 0)

Here, the vertices are on the x-axis.

So, the equation of the hyperbola is of the form $x^2/a^2 – y^2/b^2 = 1$

Since the vertices are (±2, 0), so, $a = 2$

Since the foci are (±3, 0), so, $c = 3$

It is known that, $a^2 + b^2 = c^2$

So, $2^2 + b^2 = 3^2$

$b^2 = 9 – 4 = 5$

∴ The equation of the hyperbola is $x^2/4 – y^2/5 = 1$

16. Vertices (0, ± 5), foci (0, ± 8)

Solution:

Given:

Vertices (0, ± 5) and foci (0, ± 8)

Here, the vertices are on the y-axis.

So, the equation of the hyperbola is of the form $y^2/a^2 – x^2/b^2 = 1$

Since the vertices are (0, ±5), so, a = 5

Since the foci are (0, ±8), so, c = 8

It is known that, $a^2 + b^2 = c^2$

So, $5^2 + b^2 = 8^2$

$b^2 = 64 – 25 = 39$

∴ The equation of the hyperbola is $y^2/25 – x^2/39 = 1$

17. Vertices (0, ± 3), foci (0, ± 5)

Solution:

Given:

Vertices (0, ± 3) and foci (0, ± 5)

Here, the vertices are on the y-axis.

So, the equation of the hyperbola is of the form $y^2/a^2 – x^2/b^2 = 1$

Since the vertices are (0, ±3), so, a = 3

Since the foci are (0, ±5), so, c = 5

It is known that $a^2 + b^2 = c^2$

So, $3^2 + b^2 = 5^2$

$b^2 = 25 – 9 = 16$

∴ The equation of the hyperbola is $y^2/9 – x^2/16 = 1$

18. Foci (±5, 0), the transverse axis is of length 8.

Solution:

Given:

Foci (±5, 0) and the transverse axis is of length 8.

Here, the foci are on x-axis.

The equation of the hyperbola is of the form $x^2/a^2 – y^2/b^2 = 1$

Since the foci are (±5, 0), so, $c = 5$

Since the length of the transverse axis is 8,

$2a = 8$

$a = 8/2$

$= 4$

It is known that $a^2 + b^2 = c^2$

$4^2 + b^2 = 5^2$

$b^2 = 25 – 16$

$= 9$

∴ The equation of the hyperbola is $x^2/16 – y^2/9 = 1$

19. Foci (0, ±13), the conjugate axis is of length 24.

Solution:

Given:

Foci (0, ±13) and the conjugate axis is of length 24.

Here, the foci are on y-axis.

The equation of the hyperbola is of the form $y^2/a^2 – x^2/b^2 = 1$

Since the foci are $(0, \pm 13)$, so, $c = 13$

Since the length of the conjugate axis is 24,

$2b = 24$

$b = 24/2$

$= 12$

It is known that $a^2 + b^2 = c^2$

$a^2 + 12^2 = 13^2$

$a^2 = 169 - 144$

$= 25$

∴ The equation of the hyperbola is $y^2/25 - x^2/144 = 1$

20. Foci (± 3√5, 0), the latus rectum is of length 8.

Solution:

Given:

Foci $(\pm 3\sqrt{5}, 0)$ and the latus rectum is of length 8.

Here, the foci are on x-axis.

The equation of the hyperbola is of the form $x^2/a^2 - y^2/b^2 = 1$

Since the foci are $(\pm 3\sqrt{5}, 0)$, so, $c = \pm 3\sqrt{5}$

Length of latus rectum is 8

$2b^2/a = 8$

$2b^2 = 8a$

$b^2 = 8a/2$

$= 4a$

It is known that $a^2 + b^2 = c^2$

$a^2 + 4a = 45$

$a^2 + 4a – 45 = 0$

$a^2 + 9a – 5a – 45 = 0$

$(a + 9) (a -5) = 0$

$a = -9$ or 5

Since a is non – negative, $a = 5$

So, $b^2 = 4a$

$= 4 \times 5$

$= 20$

∴ The equation of the hyperbola is $x^2/25 – y^2/20 = 1$

21. Foci (± 4, 0), the latus rectum is of length 12

Solution:

Given:

Foci (± 4, 0) and the latus rectum is of length 12

Here, the foci are on x-axis.

The equation of the hyperbola is of the form $x^2/a^2 – y^2/b^2 = 1$

Since the foci are (± 4, 0), so, $c = 4$

Length of latus rectum is 12

$2b^2/a = 12$

$2b^2 = 12a$

$b^2 = 12a/2$

$= 6a$

It is known that $a^2 + b^2 = c^2$

$a^2 + 6a = 16$

$a^2 + 6a – 16 = 0$

$a^2 + 8a – 2a – 16 = 0$

$(a + 8) (a – 2) = 0$

$a = -8$ or 2

Since a is non – negative, $a = 2$

So, $b^2 = 6a$

$= 6 \times 2$

$= 12$

∴ The equation of the hyperbola is $x^2/4 – y^2/12 = 1$

22. Vertices (±7, 0), e = 4/3

Solution:

Given:

Vertices $(\pm7, 0)$ and $e = 4/3$

Here, the vertices are on the x- axis

The equation of the hyperbola is of the form $x^2/a^2 – y^2/b^2 = 1$

Since the vertices are $(\pm 7, 0)$, so, $a = 7$

It is given that $e = 4/3$

$c/a = 4/3$

$3c = 4a$

Substituting the value of a, we get

$3c = 4(7)$

$c = 28/3$

It is known that, $a^2 + b^2 = c^2$

$7^2 + b^2 = (28/3)^2$

$b^2 = 784/9 – 49$

$= (784 – 441)/9$

$= 343/9$

∴ The equation of the hyperbola is $x^2/49 – 9y^2/343 = 1$

23. Foci (0, ±√10), passing through (2, 3)

Solution:

Given:

Foci (0, ±√10) and passing through (2, 3)

Here, the foci are on y-axis.

The equation of the hyperbola is of the form $y^2/a^2 – x^2/b^2 = 1$

Since the foci are (±√10, 0), so, $c = \sqrt{10}$

It is known that $a^2 + b^2 = c^2$

$b^2 = 10 – a^2$ ………….. (1)

It is given that the hyperbola passes through point (2, 3)

So, $9/a^2 – 4/b^2 = 1$ … (2)

From equations (1) and (2), we get,

$9/a^2 – 4/(10-a^2) = 1$

$9(10 – a^2) – 4a^2 = a^2(10 –a^2)$

$90 – 9a^2 – 4a^2 = 10a^2 – a^4$

$a^4 – 23a^2 + 90 = 0$

$a^4 – 18a^2 – 5a^2 + 90 = 0$

$a^2(a^2 -18) -5(a^2 -18) = 0$

$(a^2 – 18) (a^2 -5) = 0$

$a^2 = 18$ or 5

In hyperbola, $c > a$ i.e., $c^2 > a^2$

So, $a^2 = 5$

$b^2 = 10 – a^2$

$= 10 – 5$

$= 5$

∴ The equation of the hyperbola is $y^2/5 – x^2/5 = 1$

7 MARK QUESTIONS

1. Find the equation of the circle passing through the points (2, 3) and (–1, 1) and whose centre is on the line x – 3y – 11 = 0.

Solution:

Let us consider the equation of the required circle to be $(x - h)^2 + (y - k)^2 = r^2$

We know that the circle passes through points (2,3) and (-1,1).

$(2 - h)^2 + (3 - k)^2 = r^2$ ……………..(1)

$(-1 - h)^2 + (1 - k)^2 = r^2$ ………………(2)

Since, the centre (h, k) of the circle lies on line x – 3y – 11= 0,

$h - 3k = 11$………………… (3)

From the equation (1) and (2), we obtain

$(2 - h)^2 + (3 - k)^2 = (-1 - h)^2 + (1 - k)^2$

$4 - 4h + h^2 + 9 - 6k + k^2 = 1 + 2h + h^2 + 1 - 2k + k^2$

$4 - 4h + 9 - 6k = 1 + 2h + 1 - 2k$

$6h + 4k = 11$……………. (4)

Now let us multiply equation (3) by 6 and subtract it from equation (4) to get,

$6h + 4k - 6(h - 3k) = 11 - 66$

$6h + 4k - 6h + 18k = 11 - 66$

$22k = -55$

$K = -5/2$

Substitute this value of K in equation (4) to get,

$6h + 4(-5/2) = 11$

$6h – 10 = 11$

$6h = 21$

$h = 21/6$

$h = 7/2$

We obtain $h = 7/2$and $k = -5/2$

On substituting the values of h and k in equation (1), we get

$(2 – 7/2)^2 + (3 + 5/2)^2 = r^2$

$[(4-7)/2]^2 + [(6+5)/2]^2 = r^2$
$(-3/2)^2 + (11/2)^2 = r^2$

$9/4 + 121/4 = r^2$

$130/4 = r^2$

The equation of the required circle is

$(x – 7/2)^2 + (y + 5/2)^2 = 130/4$

$[(2x-7)/2]^2 + [(2y+5)/2]^2 = 130/4$
$4x^2 -28x + 49 +4y^2 + 20y + 25 =130$

$4x^2 +4y^2 -28x + 20y – 56 = 0$

$4(x^2 +y^2 -7x + 5y – 14) = 0$

$x^2 + y^2 – 7x + 5y – 14 = 0$

∴ The equation of the required circle is $x^2 + y^2 – 7x + 5y – 14 = 0$

2. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point (2, 3).

Solution:

Let us consider the equation of the required circle to be $(x – h)^2+ (y – k)^2 = r^2$

We know that the radius of the circle is 5 and its centre lies on the x-axis, k = 0 and r = 5.

So now, the equation of the circle is $(x – h)^2 + y^2 = 25$.

It is given that the circle passes through the point (2, 3) so the point will satisfy the equation of the circle.

$(2 – h)^2 + 3^2 = 25$

$(2 – h)^2 = 25-9$

$(2 – h)^2 = 16$

$2 – h = \pm \sqrt{16} = \pm 4$

If $2-h = 4$, then $h = -2$

If $2-h = -4$, then $h = 6$

Then, when $h = -2$, the equation of the circle becomes

$(x + 2)^2 + y^2 = 25$

$x^2 + 12x + 36 + y^2 = 25$

$x^2 + y^2 + 4x – 21 = 0$

When $h = 6$, the equation of the circle becomes

$(x – 6)^2 + y^2 = 25$

$x^2 -12x + 36 + y^2 = 25$

$x^2 + y^2 -12x + 11 = 0$

∴ The equation of the required circle is $x^2 + y^2 + 4x – 21 = 0$ and $x^2 + y^2 -12x + 11 = 0$

3. Find the equation of the circle passing through (0,0) and making intercepts a and b on the coordinate axes.

Solution:

Let us consider the equation of the required circle to be $(x – h)^2 + (y – k)^2 = r^2$

We know that the circle passes through (0, 0),

So, $(0 – h)^2 + (0 – k)^2 = r^2$

$h^2 + k^2 = r^2$

Now, The equation of the circle is $(x - h)^2 + (y - k)^2 = h^2 + k^2$.

It is given that the circle intercepts a and b on the coordinate axes.

i.e., the circle passes through points (a, 0) and (0, b).

So, $(a - h)^2 + (0 - k)^2 = h^2 + k^2$……………..(1)

$(0 - h)^2 + (b - k)^2 = h^2 + k^2$………………(2)

From equation (1), we obtain

$a^2 - 2ah + h^2 + k^2 = h^2 + k^2$

$a^2 - 2ah = 0$

$a(a - 2h) = 0$

$a = 0$ or $(a - 2h) = 0$

However, $a \neq 0$; hence, $(a - 2h) = 0$

$h = a/2$

From equation (2), we obtain

$h^2 - 2bk + k^2 + b^2 = h^2 + k^2$

$b^2 - 2bk = 0$

$b(b - 2k) = 0$

$b = 0$ or $(b - 2k) = 0$

However, $a \neq 0$; hence, $(b - 2k) = 0$

$k = b/2$

So, the equation is

$(x - a/2)^2 + (y - b/2)^2 = (a/2)^2 + (b/2)^2$

$[(2x-a)/2]^2 + [(2y-b)/2]^2 = (a^2 + b^2)/4$
$4x^2 - 4ax + a^2 + 4y^2 - 4by + b^2 = a^2 + b^2$

$4x^2 + 4y^2 - 4ax – 4by = 0$

$4(x^2 + y^2 - 7x + 5y – 14) = 0$

$x^2 + y^2 – ax – by = 0$

∴ The equation of the required circle is $x^2 + y^2 – ax – by = 0$

4. Find the equation of a circle with centre (2,2) and passes through the point (4,5).

Solution:

Given:

The centre of the circle is given as (h, k) = (2,2)

We know that the circle passes through point (4,5), the radius (r) of the circle is the distance between the points (2,2) and (4,5).

$r = \sqrt{[(2-4)^2 + (2-5)^2]}$

$= \sqrt{[(-2)^2 + (-3)^2]}$

$= \sqrt{[4+9]}$

$= \sqrt{13}$

The equation of the circle is given as

$(x– h)^2+ (y – k)^2 = r^2$

$(x –h)^2 + (y – k)^2 = (\sqrt{13})^2$

$(x –2)^2 + (y – 2)^2 = (\sqrt{13})^2$

$x^2 – 4x + 4 + y^2 – 4y + 4 = 13$

$x^2 + y^2 – 4x – 4y = 5$

∴ The equation of the required circle is $x^2 + y^2 – 4x – 4y = 5$

5. Does the point (–2.5, 3.5) lie inside, outside or on the circle $x^2 + y^2 = 25$?

Solution:

Given:

The equation of the given circle is $x^2 + y^2 = 25$.

$x^2 + y^2 = 25$

$(x - 0)^2 + (y - 0)^2 = 5^2$ [which is of the form $(x - h)^2 + (y - k)^2 = r^2$]

Where, h = 0, k = 0 and r = 5.

So the distance between point (-2.5, 3.5) and the centre (0,0) is

$= \sqrt{[(-2.5 - 0)^2 + (-3.5 - 0)^2]}$

$= \sqrt{(6.25 + 12.25)}$

$= \sqrt{18.5}$

= 4.3 [which is < 5]

Since, the distance between point (-2.5, -3.5) and the centre (0, 0) of the circle is less than the radius of the circle, point (-2.5, -3.5) lies inside the circle.

6. $x^2/36 + y^2/16 = 1$

Solution:

Given:

The equation is $x^2/36 + y^2/16 = 1$

Here, the denominator of $x^2/36$ is greater than the denominator of $y^2/16$.

So, the major axis is along the x-axis, while the minor axis is along the y-axis.

On comparing the given equation with $x^2/a^2 + y^2/b^2 = 1$, we get

a = 6 and b = 4.

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(36-16)}$

$= \sqrt{20}$

$= 2\sqrt{5}$

Then,

The coordinates of the foci are $(2\sqrt{5}, 0)$ and $(-2\sqrt{5}, 0)$.

The coordinates of the vertices are $(6, 0)$ and $(-6, 0)$

Length of major axis = $2a = 2 (6) = 12$

Length of minor axis = $2b = 2 (4) = 8$

Eccentricity, $e = c/a = 2\sqrt{5}/6 = \sqrt{5}/3$

Length of latus rectum = $2b^2/a = (2\times16)/6 = 16/3$

7. $x^2/4 + y^2/25 = 1$

Solution:

Given:

The equation is $x^2/4 + y^2/25 = 1$

Here, the denominator of $y^2/25$ is greater than the denominator of $x^2/4$.

So, the major axis is along the x-axis, while the minor axis is along the y-axis.

On comparing the given equation with $x^2/a^2 + y^2/b^2 = 1$, we get

$a = 5$ and $b = 2$.

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(25-4)}$

$= \sqrt{21}$

Then,

The coordinates of the foci are (0, √21) and (0, -√21).

The coordinates of the vertices are (0, 5) and (0, -5)

Length of the major axis = 2a = 2 (5) = 10

Length of the minor axis = 2b = 2 (2) = 4

Eccentricity, e = c/a = √21/5

Length of latus rectum = $2b^2/a = (2\times2^2)/5 = (2\times4)/5 = 8/5$

8. $x^2/16 + y^2/9 = 1$

Solution:

Given:

The equation is $x^2/16 + y^2/9 = 1$ or $x^2/4^2 + y^2/3^2 = 1$

Here, the denominator of $x^2/16$ is greater than the denominator of $y^2/9$.

So, the major axis is along the x-axis, while the minor axis is along the y-axis.

On comparing the given equation with $x^2/a^2 + y^2/b^2 = 1$, we get

a = 4 and b = 3.

$c = \sqrt{(a^2 – b^2)}$

$= \sqrt{(16-9)}$

$= \sqrt{7}$

Then,

The coordinates of the foci are (√7, 0) and (-√7, 0).

The coordinates of the vertices are (4, 0) and (-4, 0)

Length of the major axis = 2a = 2 (4) = 8

Length of the minor axis = 2b = 2 (3) = 6

Eccentricity, $e = c/a = \sqrt{7}/4$

Length of latus rectum $= 2b^2/a = (2\times3^2)/4 = (2\times9)/4 = 18/4 = 9/2$

9. $x^2/25 + y^2/100 = 1$

Solution:

Given:

The equation is $x^2/25 + y^2/100 = 1$

Here, the denominator of $y^2/100$ is greater than the denominator of $x^2/25$.

So, the major axis is along the y-axis, while the minor axis is along the x-axis.

On comparing the given equation with $x^2/b^2 + y^2/a^2 = 1$, we get

$b = 5$ and $a =10$.

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(100-25)}$

$= \sqrt{75}$

$= 5\sqrt{3}$

Then,

The coordinates of the foci are $(0, 5\sqrt{3})$ and $(0, -5\sqrt{3})$.

The coordinates of the vertices are $(0, \sqrt{10})$ and $(0, -\sqrt{10})$

Length of the major axis $= 2a = 2\ (10) = 20$

Length of the minor axis $= 2b = 2\ (5) = 10$

Eccentricity, $e = c/a = 5\sqrt{3}/10 = \sqrt{3}/2$

Length of latus rectum $= 2b^2/a = (2\times5^2)/10 = (2\times25)/10 = 5$

10. $x^2/49 + y^2/36 = 1$

Solution:

Given:

The equation is $x^2/49 + y^2/36 = 1$

Here, the denominator of $x^2/49$ is greater than the denominator of $y^2/36$.

So, the major axis is along the x-axis, while the minor axis is along the y-axis.

On comparing the given equation with $x^2/a^2 + y^2/b^2 = 1$, we get

b = 6 and a =7

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(49-36)}$

$= \sqrt{13}$

Then,

The coordinates of the foci are $(\sqrt{13}, 0)$ and $(-\sqrt{3}, 0)$.

The coordinates of the vertices are (7, 0) and (-7, 0)

Length of the major axis = 2a = 2 (7) = 14

Length of the minor axis = 2b = 2 (6) = 12

Eccentricity, $e = c/a = \sqrt{13}/7$

Length of latus rectum $= 2b^2/a = (2\times6^2)/7 = (2\times36)/7 = 72/7$

11. $x^2/100 + y^2/400 = 1$

Solution:

Given:

The equation is $x^2/100 + y^2/400 = 1$

Here, the denominator of $y^2/400$ is greater than the denominator of $x^2/100$.

So, the major axis is along the y-axis, while the minor axis is along the x-axis.

On comparing the given equation with $x^2/b^2 + y^2/a^2 = 1$, we get

$b = 10$ and $a = 20$.

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(400-100)}$

$= \sqrt{300}$

$= 10\sqrt{3}$

Then,

The coordinates of the foci are $(0, 10\sqrt{3})$ and $(0, -10\sqrt{3})$.

The coordinates of the vertices are $(0, 20)$ and $(0, -20)$

Length of the major axis $= 2a = 2\ (20) = 40$

Length of the minor axis $= 2b = 2\ (10) = 20$

Eccentricity, $e = c/a = 10\sqrt{3}/20 = \sqrt{3}/2$

Length of latus rectum $= 2b^2/a = (2\times10^2)/20 = (2\times100)/20 = 10$

12. $36x^2 + 4y^2 = 144$

Solution:

Given:

The equation is $36x^2 + 4y^2 = 144$ or $x^2/4 + y^2/36 = 1$ or $x^2/2^2 + y^2/6^2 = 1$

Here, the denominator of $y^2/6^2$ is greater than the denominator of $x^2/2^2$.

So, the major axis is along the y-axis, while the minor axis is along the x-axis.

On comparing the given equation with $x^2/b^2 + y^2/a^2 = 1$, we get

$b = 2$ and $a = 6$.

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(36-4)}$

$= \sqrt{32}$

$= 4\sqrt{2}$

Then,

The coordinates of the foci are $(0, 4\sqrt{2})$ and $(0, -4\sqrt{2})$.

The coordinates of the vertices are $(0, 6)$ and $(0, -6)$

Length of the major axis $= 2a = 2 (6) = 12$

Length of the minor axis $= 2b = 2 (2) = 4$

Eccentricity, $e = c/a = 4\sqrt{2}/6 = 2\sqrt{2}/3$

Length of latus rectum $= 2b^2/a = (2\times2^2)/6 = (2\times4)/6 = 4/3$

13. $16x^2 + y^2 = 16$

Solution:

Given:

The equation is $16x^2 + y^2 = 16$ or $x^2/1 + y^2/16 = 1$ or $x^2/1^2 + y^2/4^2 = 1$

Here, the denominator of $y^2/4^2$ is greater than the denominator of $x^2/1^2$.

So, the major axis is along the y-axis, while the minor axis is along the x-axis.

On comparing the given equation with $x^2/b^2 + y^2/a^2 = 1$, we get

$b = 1$ and $a = 4$.

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(16-1)}$

$= \sqrt{15}$

Then,

The coordinates of the foci are $(0, \sqrt{15})$ and $(0, -\sqrt{15})$.

The coordinates of the vertices are $(0, 4)$ and $(0, -4)$

Length of the major axis $= 2a = 2 (4) = 8$

Length of the minor axis $= 2b = 2 (1) = 2$

Eccentricity, $e = c/a = \sqrt{15}/4$

Length of latus rectum $= 2b^2/a = (2\times 1^2)/4 = 2/4 = ½$

14. $4x^2 + 9y^2 = 36$

Solution:

Given:

The equation is $4x^2 + 9y^2 = 36$ or $x^2/9 + y^2/4 = 1$ or $x^2/3^2 + y^2/2^2 = 1$

Here, the denominator of $x^2/3^2$ is greater than the denominator of $y^2/2^2$.

So, the major axis is along the x-axis, while the minor axis is along the y-axis.

On comparing the given equation with $x^2/a^2 + y^2/b^2 = 1$, we get

$a = 3$ and $b = 2$.

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(9-4)}$

$= \sqrt{5}$

Then,

The coordinates of the foci are $(\sqrt{5}, 0)$ and $(-\sqrt{5}, 0)$.

The coordinates of the vertices are $(3, 0)$ and $(-3, 0)$

Length of the major axis = 2a = 2 (3) = 6

Length of the minor axis = 2b = 2 (2) = 4

Eccentricity, e = c/a = $\sqrt{5}/3$

Length of latus rectum = $2b^2/a = (2\times2^2)/3 = (2\times4)/3 = 8/3$

In each of the following Exercises 10 to 20, find the equation for the ellipse that satisfies the given conditions:

15. Vertices (± 5, 0), foci (± 4, 0)

Solution:

Given:

Vertices (± 5, 0) and foci (± 4, 0)

Here, the vertices are on the x-axis.

So, the equation of the ellipse will be of the form $x^2/a^2 + y^2/b^2 = 1$, where 'a' is the semi-major axis.

Then, a = 5 and c = 4.

It is known that $a^2 = b^2 + c^2$.

So, $5^2 = b^2 + 4^2$

$25 = b^2 + 16$

$b^2 = 25 - 16$

$b = \sqrt{9}$

= 3

∴ The equation of the ellipse is $x^2/5^2 + y^2/3^2 = 1$ or $x^2/25 + y^2/9 = 1$

16. Vertices (0, ± 13), foci (0, ± 5)

Solution:

Given:

Vertices (0, ± 13) and foci (0, ± 5)

Here, the vertices are on the y-axis.

So, the equation of the ellipse will be of the form $x^2/b^2 + y^2/a^2 = 1$, where 'a' is the semi-major axis.

Then, a =13 and c = 5.

It is known that $a^2 = b^2 + c^2$.

$13^2 = b^2+5^2$

$169 = b^2 + 15$

$b^2 = 169 – 125$

$b = \sqrt{144}$

= 12

∴ The equation of the ellipse is $x^2/12^2 + y^2/13^2 = 1$ or $x^2/144 + y^2/169 = 1$

17. Vertices (± 6, 0), foci (± 4, 0)

Solution:

Given:

Vertices (± 6, 0) and foci (± 4, 0)

Here, the vertices are on the x-axis.

So, the equation of the ellipse will be of the form $x^2/a^2 + y^2/b^2 = 1$, where 'a' is the semi-major axis.

Then, a = 6 and c = 4.

It is known that $a^2 = b^2 + c^2$.

$6^2 = b^2+4^2$

$36 = b^2 + 16$

$b^2 = 36 - 16$

$b = \sqrt{20}$

∴ The equation of the ellipse is $x^2/6^2 + y^2/(\sqrt{20})^2 = 1$ or $x^2/36 + y^2/20 = 1$

18. Ends of major axis (± 3, 0), ends of minor axis (0, ±2)

Solution:

Given:

Ends of major axis (± 3, 0) and ends of minor axis (0, ±2)

Here, the major axis is along the x-axis.

So, the equation of the ellipse will be of the form $x^2/a^2 + y^2/b^2 = 1$, where 'a' is the semi-major axis.

Then, a = 3 and b = 2.

∴ The equation for the ellipse $x^2/3^2 + y^2/2^2 = 1$ or $x^2/9 + y^2/4 = 1$

19. Ends of major axis (0, ±√5), ends of minor axis (±1, 0)

Solution:

Given:

Ends of major axis (0, ±√5) and ends of minor axis (±1, 0)

Here, the major axis is along the y-axis.

So, the equation of the ellipse will be of the form $x^2/b^2 + y^2/a^2 = 1$, where 'a' is the semi-major axis.

Then, $a = \sqrt{5}$ and b = 1.

∴ The equation for the ellipse $x^2/1^2 + y^2/(\sqrt{5})^2 = 1$ or $x^2/1 + y^2/5 = 1$

20. Length of major axis 26, foci (±5, 0)

Solution:

Given:

Length of major axis is 26 and foci (±5, 0)

Since the foci are on the x-axis, the major axis is along the x-axis.

So, the equation of the ellipse will be of the form $x^2/a^2 + y^2/b^2 = 1$, where 'a' is the semi-major axis.

Then, 2a = 26

a = 13 and c = 5.

It is known that $a^2 = b^2 + c^2$.

$13^2 = b^2+5^2$

$169 = b^2 + 25$

$b^2 = 169 – 25$

b = √144

= 12

∴ The equation of the ellipse is $x^2/13^2 + y^2/12^2 = 1$ or $x^2/169 + y^2/144 = 1$

21. Length of minor axis 16, foci (0, ±6).

Solution:

Given:

Length of minor axis is 16 and foci (0, ±6).

Since the foci are on the y-axis, the major axis is along the y-axis.

So, the equation of the ellipse will be of the form $x^2/b^2 + y^2/a^2 = 1$, where 'a' is the semi-major axis.

Then, 2b =16

b = 8 and c = 6.

It is known that $a^2 = b^2 + c^2$.

$a^2 = 8^2 + 6^2$

= 64 + 36

=100

$a = \sqrt{100}$

= 10

∴ The equation of the ellipse is $x^2/8^2 + y^2/10^2 = 1$ or $x^2/64 + y^2/100 = 1$

22. Foci (±3, 0), a = 4

Solution:

Given:

Foci (±3, 0) and a = 4

Since the foci are on the x-axis, the major axis is along the x-axis.

So, the equation of the ellipse will be of the form $x^2/a^2 + y^2/b^2 = 1$, where 'a' is the semi-major axis.

Then, c = 3 and a = 4.

It is known that $a^2 = b^2 + c^2$.

$a^2 = 8^2 + 6^2$

= 64 + 36

= 100

$16 = b^2 + 9$

$b^2 = 16 – 9$

= 7

∴ The equation of the ellipse is $x^2/16 + y^2/7 = 1$

23. b = 3, c = 4, centre at the origin; foci on the x axis.

Solution:

Given:

b = 3, c = 4, centre at the origin and foci on the x axis.

Since the foci are on the x-axis, the major axis is along the x-axis.

So, the equation of the ellipse will be of the form $x^2/a^2 + y^2/b^2 = 1$, where 'a' is the semi-major axis.

Then, b = 3 and c = 4.

It is known that $a^2 = b^2 + c^2$.

$a^2 = 3^2 + 4^2$

= 9 + 16

=25

a = √25

= 5

∴ The equation of the ellipse is $x^2/5^2 + y^2/3^2$ or $x^2/25 + y^2/9 = 1$

24. Centre at (0, 0), major axis on the y-axis and passes through the points (3, 2) and (1, 6).

Solution:

Given:

Centre at (0, 0), major axis on the y-axis and passes through the points (3, 2) and (1, 6).

Since the centre is at (0, 0) and the major axis is on the y- axis, the equation of the ellipse will be of the form $x^2/b^2 + y^2/a^2 = 1$, where 'a' is the semi-major axis.

The ellipse passes through points (3, 2) and (1, 6).

So, by putting the values x = 3 and y = 2, we get,

$3^2/b^2 + 2^2/a^2 = 1$

$9/b^2 + 4/a^2$…. (1)

And by putting the values x = 1 and y = 6, we get,

$1^1/b^2 + 6^2/a^2 = 1$

$1/b^2 + 36/a^2 = 1$ …. (2)

On solving equation (1) and (2), we get

$b^2 = 10$ and $a^2 = 40$.

∴ The equation of the ellipse is $x^2/10 + y^2/40 = 1$ or $4x^2 + y^2 = 40$

25. Major axis on the x-axis and passes through the points (4,3) and (6,2).

Solution:

Given:

Major axis on the x-axis and passes through the points (4, 3) and (6, 2).

Since the major axis is on the x-axis, the equation of the ellipse will be the form

$x^2/a^2 + y^2/b^2 = 1$…. (1) [Where 'a' is the semi-major axis.]

The ellipse passes through points (4, 3) and (6, 2).

So by putting the values x = 4 and y = 3 in equation (1), we get,

$16/a^2 + 9/b^2 = 1$ …. (2)

Putting, x = 6 and y = 2 in equation (1), we get,

$36/a^2 + 4/b^2 = 1$ …. (3)

From equation (2)

$16/a^2 = 1 – 9/b^2$

$1/a^2 = (1/16\ (1 – 9/b^2))$ …. (4)

Substituting the value of $1/a^2$ in equation (3) we get,

$36/a^2 + 4/b^2 = 1$

$36(1/a^2) + 4/b^2 = 1$

$36[1/16\ (1 - 9/b^2)] + 4/b^2 = 1$

$36/16\ (1 - 9/b^2) + 4/b^2 = 1$

$9/4\ (1 - 9/b^2) + 4/b^2 = 1$

$9/4 - 81/4b^2 + 4/b^2 = 1$

$-81/4b^2 + 4/b^2 = 1 - 9/4$

$(-81+16)/4b^2 = (4-9)/4$

$-65/4b^2 = -5/4$

$-5/4(13/b^2) = -5/4$

$13/b^2 = 1$

$1/b^2 = 1/13$

$b^2 = 13$

Now substituting the value of b^2 in equation (4) we get,

$1/a^2 = 1/16(1 - 9/b^2)$

$= 1/16(1 - 9/13)$

$= 1/16((13-9)/13)$

$= 1/16(4/13)$

$= 1/52$

$a^2 = 52$

Equation of ellipse is $x^2/a^2 + y^2/b^2 = 1$

By substituting the values of a^2 and b^2 in above equation we get,

$x^2/52 + y^2/13 = 1$

26. $16x^2 – 9y^2 = 576$

Solution:

Given:

The equation is $16x^2 – 9y^2 = 576$

Let us divide the whole equation by 576.We get

$16x^2/576 – 9y^2/576 = 576/576$

$x^2/36 – y^2/64 = 1$

On comparing this equation with the standard equation of hyperbola $x^2/a^2 – y^2/b^2 = 1$,

We get a = 6 and b = 8,

It is known that $a^2 + b^2 = c^2$

So,

$c^2 = 36 + 64$

$c^2 = \sqrt{100}$

c = 10

Then,

The coordinates of the foci are (10, 0) and (-10, 0).

The coordinates of the vertices are (6, 0) and (-6, 0).

Eccentricity, e = c/a = 10/6 = 5/3

Length of latus rectum = $2b^2/a = (2 \times 8^2)/6 = (2\times64)/6 = 64/3$

27. $5y^2 – 9x^2 = 36$

Solution:

Given:

The equation is $5y^2 – 9x^2 = 36$

Let us divide the whole equation by 36. We get

$5y^2/36 – 9x^2/36 = 36/36$

$y^2/(36/5) – x^2/4 = 1$

On comparing this equation with the standard equation of hyperbola $y^2/a^2 – x^2/b^2 = 1$,

We get $a = 6/\sqrt{5}$ and $b = 2$,

It is known that $a^2 + b^2 = c^2$

So,

$c^2 = 36/5 + 4$

$c^2 = 56/5$

$c = \sqrt{(56/5)}$

$= 2\sqrt{14}/\sqrt{5}$

Then,

The coordinates of the foci are $(0, 2\sqrt{14}/\sqrt{5})$ and $(0, – 2\sqrt{14}/\sqrt{5})$.

The coordinates of the vertices are $(0, 6/\sqrt{5})$ and $(0, -6/\sqrt{5})$.

Eccentricity, $e = c/a = (2\sqrt{14}/\sqrt{5}) / (6/\sqrt{5}) = \sqrt{14}/3$

Length of latus rectum $= 2b^2/a = (2 \times 2^2)/6/\sqrt{5} = (2\times4)/6/\sqrt{5} = 4\sqrt{5}/3$

28. $49y^2 – 16x^2 = 784$.

Solution:

Given:

The equation is $49y^2 – 16x^2 = 784$.

Let us divide the whole equation by 784, we get

$49y^2/784 – 16x^2/784 = 784/784$

$y^2/16 – x^2/49 = 1$

On comparing this equation with the standard equation of hyperbola $y^2/a^2 – x^2/b^2 = 1$,

We get $a = 4$ and $b = 7$,

It is known that $a^2 + b^2 = c^2$

So,

$c^2 = 16 + 49$

$c^2 = 65$

$c = \sqrt{65}$

Then,

The coordinates of the foci are $(0, \sqrt{65})$ and $(0, –\sqrt{65})$.

The coordinates of the vertices are $(0, 4)$ and $(0, -4)$.

Eccentricity, $e = c/a = \sqrt{65}/4$

Length of latus rectum $= 2b^2/a = (2 \times 7^2)/4 = (2\times49)/4 = 49/2$

In each Exercises 7 to 15, find the equations of the hyperbola satisfying the given conditions

SUMMARY

A conic section, conic or a quadratic curve is a curve obtained from a cone's surface intersecting a plane. The three types of conic section are the hyperbola, the parabola, and the ellipse; the circle is a special case of the ellipse, though it was sometimes called as a fourth type.

CHAPTER-XI

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

2 MARK QUESTIONS

1. A point is on the x-axis. What are its y-coordinate and z-coordinates?

Solution:

If a point is on the x-axis, then the coordinates of y and z are 0.

So the point is (x, 0, 0).

2. A point is in the XZ-plane. What can you say about its *y*-coordinate?

Solution:

If a point is in the XZ plane, then its y-co-ordinate is 0.

3. Fill in the blanks:
(i) The x-axis and y-axis, taken together, determine a plane known as _______.
(ii) The coordinates of points in the XY-plane are of the form _______.
(iii) Coordinate planes divide the space into ______ octants.

Solution:

(i) The x-axis and y-axis, taken together, determine a plane known as **XY Plane**.

(ii) The coordinates of points in the XY-plane are of the form **(x, y, 0)**.

(iii) Coordinate planes divide the space into **eight** octants.

4 MARK QUESTIONS

1. Find the coordinates of the point which divides the line segment joining the points (–2, 3, 5) and (1, –4, 6) in the ratio (i) 2:3 internally, (ii) 2:3 externally.

Solution:

Let the line segment joining the points P (-2, 3, 5) and Q (1, -4, 6) be PQ.

(i) 2:3 internally

By using the section formula,

We know that the coordinates of the point R, which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m:n, is given by:

Upon comparing, we have

$x_1 = -2$, $y_1 = 3$, $z_1 = 5$;

$x_2 = 1$, $y_2 = -4$, $z_2 = 6$ and

m = 2, n = 3

So, the coordinates of the point which divide the line segment joining the points P (– 2, 3, 5) and Q (1, – 4, 6) in the ratio 2:3 internally is given by:

Hence, the coordinates of the point which divides the line segment joining the points (-2, 3, 5) and (1, -4, 6) is (-4/5, 1/5, 27/5).

(ii) 2:3 externally

By using the section formula,

We know that the coordinates of the point R, which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) externally in the ratio m:n, is given by:

Upon comparing, we have

$x_1 = -2$, $y_1 = 3$, $z_1 = 5$;

$x_2 = 1$, $y_2 = -4$, $z_2 = 6$ and

$m = 2$, $n = 3$

So, the coordinates of the point which divide the line segment joining the points P (–2, 3, 5) and Q (1, –4, 6) in the ratio 2:3 externally is given by:

∴ The coordinates of the point which divides the line segment joining the points (-2, 3, 5) and (1, -4, 6) is (-8, 17, 3).

2. Given that P (3, 2, –4), Q (5, 4, –6) and R (9, 8, –10) are collinear. Find the ratio in which Q divides PR.

Solution:

Let us consider Q divides PR in the ratio k:1.

By using the section formula,

We know that the coordinates of the point R, which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m:n, is given by:

Upon comparing, we have,

$x_1 = 3$, $y_1 = 2$, $z_1 = -4$;

$x_2 = 9$, $y_2 = 8$, $z_2 = -10$ and

$m = k$, $n = 1$

So, we have

$9k + 3 = 5 (k+1)$

$9k + 3 = 5k + 5$

$9k – 5k = 5 – 3$

$4k = 2$

$k = 2/4$

$= ½$

Hence, the ratio in which Q divides PR is 1:2.

3. Find the ratio in which the YZ-plane divides the line segment formed by joining the points (–2, 4, 7) and (3, –5, 8).

Solution:

Let the line segment formed by joining the points P (-2, 4, 7) and Q (3, -5, 8) be PQ.

We know that any point on the YZ-plane is of the form (0, y, z).

So, let R (0, y, z) divides the line segment PQ in the ratio k:1.

Then,

Upon comparing, we have,

$x_1 = -2$, $y_1 = 4$, $z_1 = 7$;

$x_2 = 3$, $y_2 = -5$, $z_2 = 8$ and

$m = k$, $n = 1$

By using the section formula,

We know that the coordinates of the point R, which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m:n, is given by:

So we have,

$3k – 2 = 0$

$3k = 2$

$k = 2/3$

Hence, the ratio in which the YZ-plane divides the line segment formed by joining the points (–2, 4, 7) and (3, –5, 8) is 2:3.

4. Using the section formula, show that the points A (2, –3, 4), B (–1, 2, 1) and C (0, 1/3, 2) are collinear.

Solution:

Let point P divides AB in the ratio k:1.

Upon comparing, we have,

$x_1 = 2$, $y_1 = -3$, $z_1 = 4$;

$x_2 = -1$, $y_2 = 2$, $z_2 = 1$ and

$m = k$, $n = 1$

By using the section formula,

We know that the coordinates of the point R, which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m:n, is given by:

So we have,

Now, we check if, for some value of k, the point coincides with point C.

Put $(-k+2)/(k+1) = 0$

$-k + 2 = 0$

$k = 2$

When k = 2, then (2k-3)/(k+1) = (2(2)-3)/(2+1)

= (4-3)/3

= 1/3

And, (k+4)/(k+1) = (2+4)/(2+1)

= 6/3

= 2

∴ C (0, 1/3, 2) is a point which divides AB in the ratio 2:1 and is the same as P.

Hence, A, B, and C are collinear.

5. Find the coordinates of the points which trisect the line segment joining the points P (4, 2, –6) and Q (10, –16, 6).

Solution:

Let A (x_1, y_1, z_1) and B (x_2, y_2, z_2) trisect the line segment joining the points P (4, 2, -6) and Q (10, -16, 6).

A divides the line segment PQ in the ratio 1:2.

Upon comparing, we have,

x_1 = 4, y_1 = 2, z_1 = -6;

x_2 = 10, y_2 = -16, z_2 = 6 and

m = 1, n = 2

By using the section formula,

We know that the coordinates of the point R, which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m:n, is given by:

So, we have

Similarly, we know that B divides the line segment PQ in the ratio 2:1.

Upon comparing, we have,

$x_1 = 4$, $y_1 = 2$, $z_1 = -6$;

$x_2 = 10$, $y_2 = -16$, $z_2 = 6$ and

m = 2, n = 1

By using the section formula,

We know that the coordinates of the point R, which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m:n, is given by:

So, we have

∴ The coordinates of the points which trisect the line segment joining the points P (4, 2, – 6) and Q (10, –16, 6) are (6, -4, -2) and (8, -10, 2).

7 MARK QUESTIONS

1. Find the distance between the following pairs of points:

(i) (2, 3, 5) and (4, 3, 1)

(ii) (–3, 7, 2) and (2, 4, –1)

(iii) (–1, 3, –4) and (1, –3, 4)

(iv) (2, –1, 3) and (–2, 1, 3)

Solution:

(i) (2, 3, 5) and (4, 3, 1)

Let P be (2, 3, 5) and Q be (4, 3, 1)

By using the formula,

Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So, here

$x_1 = 2, y_1 = 3, z_1 = 5$

$x_2 = 4, y_2 = 3, z_2 = 1$

Distance PQ = $\sqrt{[(4 - 2)^2 + (3 - 3)^2 + (1 - 5)^2]}$

$= \sqrt{[(2)^2 + 0^2 + (-4)^2]}$

$= \sqrt{[4 + 0 + 16]}$

= $\sqrt{20}$

$= 2\sqrt{5}$

∴ The required distance is $2\sqrt{5}$ units.

(ii) (–3, 7, 2) and (2, 4, –1)

Let P be (– 3, 7, 2) and Q be (2, 4, –1)

By using the formula,

Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So, here

$x_1 = -3$, $y_1 = 7$, $z_1 = 2$

$x_2 = 2$, $y_2 = 4$, $z_2 = -1$

Distance PQ = $\sqrt{[(2 - (-3))^2 + (4 - 7)^2 + (-1 - 2)^2]}$

$= \sqrt{[(5)^2 + (-3)^2 + (-3)^2]}$

$= \sqrt{[25 + 9 + 9]}$

= $\sqrt{43}$

∴ The required distance is $\sqrt{43}$ units.

(iii) (–1, 3, –4) and (1, –3, 4)

Let P be (– 1, 3, – 4) and Q be (1, – 3, 4)

By using the formula,

Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So here,

$x_1 = -1$, $y_1 = 3$, $z_1 = -4$

$x_2 = 1$, $y_2 = -3$, $z_2 = 4$

Distance PQ = $\sqrt{[(1 - (-1))^2 + (-3 - 3)^2 + (4 - (-4))^2]}$

$= \sqrt{[(2)^2 + (-6)^2 + (8)^2]}$

$= \sqrt{[4 + 36 + 64]}$

= $\sqrt{104}$

$= 2\sqrt{26}$

∴ The required distance is $2\sqrt{26}$ units.

(iv) (2, –1, 3) and (–2, 1, 3)

Let P be (2, – 1, 3) and Q be (– 2, 1, 3)

By using the formula,

Distance PQ = $\sqrt{[(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2]}$

So here,

$x_1 = 2, y_1 = – 1, z_1 = 3$

$x_2 = – 2, y_2 = 1, z_2 = 3$

Distance PQ = $\sqrt{[(-2 – 2)^2 + (1 – (-1))^2 + (3 – 3)^2]}$

= $\sqrt{[(-4)^2 + (2)^2 + (0)^2]}$

= $\sqrt{[16 + 4 + 0]}$

= $\sqrt{20}$

= $2\sqrt{5}$

∴ The required distance is $2\sqrt{5}$ units.

2. Show that the points (–2, 3, 5), (1, 2, 3) and (7, 0, –1) are collinear.

Solution:

If three points are collinear, then they lie on the same line.

First, let us calculate the distance between the 3 points

i.e., PQ, QR and PR

Calculating PQ

P ≡ (–2, 3, 5) and Q ≡ (1, 2, 3)

By using the formula,

Distance PQ = $\sqrt{[(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2]}$

So, here

$x_1 = -2$, $y_1 = 3$, $z_1 = 5$

$x_2 = 1$, $y_2 = 2$, $z_2 = 3$

Distance PQ = $\sqrt{[(1 - (-2))^2 + (2 - 3)^2 + (3 - 5)^2]}$

= $\sqrt{[(3)^2 + (-1)^2 + (-2)^2]}$

= $\sqrt{[9 + 1 + 4]}$

= $\sqrt{14}$

Calculating QR

Q ≡ (1, 2, 3) and R ≡ (7, 0, – 1)

By using the formula,

Distance QR = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So, here

$x_1 = 1$, $y_1 = 2$, $z_1 = 3$

$x_2 = 7$, $y_2 = 0$, $z_2 = -1$

Distance QR = $\sqrt{[(7 - 1)^2 + (0 - 2)^2 + (-1 - 3)^2]}$

= $\sqrt{[(6)^2 + (-2)^2 + (-4)^2]}$

= $\sqrt{[36 + 4 + 16]}$

= $\sqrt{56}$

= $2\sqrt{14}$

Calculating PR

P ≡ (–2, 3, 5) and R ≡ (7, 0, – 1)

By using the formula,

Distance PR = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So here,

$x_1 = -2$, $y_1 = 3$, $z_1 = 5$

$x_2 = 7$, $y_2 = 0$, $z_2 = -1$

Distance PR = $\sqrt{[(7 - (-2))^2 + (0 - 3)^2 + (-1 - 5)^2]}$

$= \sqrt{[(9)^2 + (-3)^2 + (-6)^2]}$

$= \sqrt{[81 + 9 + 36]}$

$= \sqrt{126}$

$= 3\sqrt{14}$

Thus, PQ = $\sqrt{14}$, QR = $2\sqrt{14}$ and PR = $3\sqrt{14}$

So, PQ + QR = $\sqrt{14} + 2\sqrt{14}$

$= 3\sqrt{14}$

= PR

∴ The points P, Q and R are collinear.

3. Verify the following:
(i) (0, 7, –10), (1, 6, –6), and (4, 9, –6) are the vertices of an isosceles triangle.

(ii) (0, 7, 10), (–1, 6, 6), and (–4, 9, 6) are the vertices of a right-angled triangle.

(iii) (–1, 2, 1), (1, –2, 5), (4, –7, 8), and (2, –3, 4) are the vertices of a parallelogram.

Solution:

(i) (0, 7, –10), (1, 6, –6), and (4, 9, – 6) are the vertices of an isosceles triangle.

Let us consider the points,

P(0, 7, –10), Q(1, 6, –6) and R(4, 9, –6)

If any 2 sides are equal, it will be an isosceles triangle

So, first, let us calculate the distance of PQ, QR

Calculating PQ

P ≡ (0, 7, – 10) and Q ≡ (1, 6, – 6)

By using the formula,

Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So, here,

$x_1 = 0, y_1 = 7, z_1 = -10$

$x_2 = 1, y_2 = 6, z_2 = -6$

Distance PQ = $\sqrt{[(1 - 0)^2 + (6 - 7)^2 + (-6 - (-10))^2]}$

$= \sqrt{[(1)^2 + (-1)^2 + (4)^2]}$

$= \sqrt{[1 + 1 + 16]}$

$= \sqrt{18}$

Calculating QR

Q ≡ (1, 6, – 6) and R ≡ (4, 9, – 6)

By using the formula,

Distance QR = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So, here

$x_1 = 1, y_1 = 6, z_1 = -6$

$x_2 = 4, y_2 = 9, z_2 = -6$

Distance QR = $\sqrt{[(4 - 1)^2 + (9 - 6)^2 + (-6 - (-6))^2]}$

$= \sqrt{[(3)^2 + (3)^2 + (-6+6)^2]}$

$= \sqrt{[9 + 9 + 0]}$

$= \sqrt{18}$

Hence, PQ = QR

18 = 18

2 sides are equal

∴ PQR is an isosceles triangle.

(ii) (0, 7, 10), (–1, 6, 6), and (–4, 9, 6) are the vertices of a right-angled triangle.

Let the points be

P(0, 7, 10), Q(– 1, 6, 6) & R(– 4, 9, 6)

First, let us calculate the distance of PQ, OR and PR

Calculating PQ

P ≡ (0, 7, 10) and Q ≡ (– 1, 6, 6)

By using the formula,

Distance PQ = $\sqrt{[(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2]}$

So here,

$x_1 = 0, y_1 = 7, z_1 = 10$

$x_2 = –1, y_2 = 6, z_2 = 6$

Distance PQ = $\sqrt{[(-1 – 0)^2 + (6 – 7)^2 + (6 – 10)^2]}$

= $\sqrt{[(-1)^2 + (-1)^2 + (-4)^2]}$

= $\sqrt{[1 + 1 + 16]}$

= $\sqrt{18}$

Calculating QR

Q ≡ (1, 6, –6) and R ≡ (4, 9, –6)

By using the formula,

Distance QR = $\sqrt{[(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2]}$

So, here

$x_1 = 1$, $y_1 = 6$, $z_1 = –6$

$x_2 = 4$, $y_2 = 9$, $z_2 = –6$

Distance QR = $\sqrt{[(4 – 1)^2 + (9 – 6)^2 + (-6 – (-6))^2]}$

= $\sqrt{[(3)^2 + (3)^2 + (-6+6)^2]}$

= $\sqrt{[9 + 9 + 0]}$

= $\sqrt{18}$

Calculating PR

P ≡ (0, 7, 10) and R ≡ (– 4, 9, 6)

By using the formula,

Distance PR = $\sqrt{[(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2]}$

So, here

$x_1 = 0$, $y_1 = 7$, $z_1 = 10$

$x_2 = – 4$, $y_2 = 9$, $z_2 = 6$

Distance PR = $\sqrt{[(-4 – 0)^2 + (9 – 7)^2 + (6 – 10)^2]}$

= $\sqrt{[(-4)^2 + (2)^2 + (-4)^2]}$

= $\sqrt{[16 + 4 + 16]}$

= $\sqrt{36}$

Now,

$PQ^2 + QR^2 = 18 + 18$

= 36

= PR^2

By using the converse of Pythagoras theorem,

∴ The given vertices P, Q & R are the vertices of a right–angled triangle at Q.

(iii) (–1, 2, 1), (1, –2, 5), (4, –7, 8), and (2, –3, 4) are the vertices of a parallelogram.

Let the points: A(–1, 2, 1), B(1, –2, 5), C(4, –7, 8) & D(2, –3, 4)

ABCD can be vertices of parallelogram only if opposite sides are equal.

i.e., AB = CD and BC = AD

First, let us calculate the distance

Calculating AB

A ≡ (–1, 2, 1) and B ≡ (1, –2, 5)

By using the formula,

Distance AB = $\sqrt{[(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2]}$

So, here

$x_1 = – 1, y_1 = 2, z_1 = 1$

$x_2 = 1, y_2 = – 2, z_2 = 5$

Distance AB = $\sqrt{[(1 – (-1))^2 + (-2 – 2)^2 + (5 – 1)^2]}$

= $\sqrt{[(2)^2 + (-4)^2 + (4)^2]}$

= $\sqrt{[4 + 16 + 16]}$

= $\sqrt{36}$

= 6

Calculating BC

B ≡ (1, –2, 5) and C ≡ (4, –7, 8)

By using the formula,

Distance BC = $\sqrt{[(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2]}$

So, here

$x_1 = 1, y_1 = – 2, z_1 = 5$

$x_2 = 4$, $y_2 = -7$, $z_2 = 8$

Distance BC = $\sqrt{[(4 - 1)^2 + (-7 - (-2))^2 + (8 - 5)^2]}$

= $\sqrt{[(3)^2 + (-5)^2 + (3)^2]}$

= $\sqrt{[9 + 25 + 9]}$

= $\sqrt{43}$

Calculating CD

C ≡ (4, –7, 8) and D ≡ (2, –3, 4)

By using the formula,

Distance CD = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So here,

$x_1 = 4$, $y_1 = -7$, $z_1 = 8$

$x_2 = 2$, $y_2 = -3$, $z_2 = 4$

Distance CD = $\sqrt{[(2 - 4)^2 + (-3 - (-7))^2 + (4 - 8)^2]}$

= $\sqrt{[(-2)^2 + (4)^2 + (-4)^2]}$

= $\sqrt{[4 + 16 + 16]}$

= $\sqrt{36}$

= 6

Calculating DA

D ≡ (2, –3, 4) and A ≡ (–1, 2, 1)

By using the formula,

Distance DA = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So, here

$x_1 = 2$, $y_1 = -3$, $z_1 = 4$

$x_2 = –1, y_2 = 2, z_2 = 1$

Distance DA = $\sqrt{[(-1 – 2)^2 + (2 – (-3))^2 + (1 – 4)^2]}$

= $\sqrt{[(-3)^2 + (5)^2 + (-3)^2]}$

= $\sqrt{[9 + 25 + 9]}$

= $\sqrt{43}$

Since AB = CD and BC = DA (given),

In ABCD, both pairs of opposite sides are equal.

∴ ABCD is a parallelogram.

4. Find the equation of the set of points which are equidistant from the points (1, 2, 3) and (3, 2, –1).

Solution:

Let A (1, 2, 3) & B (3, 2, –1)

Let point P be (x, y, z)

Since it is given that point P(x, y, z) is equal distance from point A(1, 2, 3) & B(3, 2, –1)

i.e. PA = PB

First, let us calculate

Calculating PA

P ≡ (x, y, z) and A ≡ (1, 2, 3)

By using the formula,

Distance PA = $\sqrt{[(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2]}$

So, here

$x_1 = x, y_1 = y, z_1 = z$

$x_2 = 1, y_2 = 2, z_2 = 3$

Distance PA = $\sqrt{[(1 - x)^2 + (2 - y)^2 + (3 - z)^2]}$

Calculating PB

P ≡ (x, y, z) and B ≡ (3, 2, –1)

By using the formula,

Distance PB = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So, here

$x_1 = x, y_1 = y, z_1 = z$

$x_2 = 3, y_2 = 2, z_2 = -1$

Distance PB = $\sqrt{[(3 - x)^2 + (2 - y)^2 + (-1 - z)^2]}$

Since PA = PB

Square on both sides, we get

$PA^2 = PB^2$

$(1 - x)^2 + (2 - y)^2 + (3 - z)^2 = (3 - x)^2 + (2 - y)^2 + (-1 - z)^2$

$(1 + x^2 - 2x) + (4 + y^2 - 4y) + (9 + z^2 - 6z)$

$(9 + x^2 - 6x) + (4 + y^2 - 4y) + (1 + z^2 + 2z)$

$-2x - 4y - 6z + 14 = -6x - 4y + 2z + 14$

$4x - 8z = 0$

$x - 2z = 0$

∴ The required equation is $x - 2z = 0$.

5. Find the equation of the set of points P, the sum of whose distances from A(4, 0, 0) and B(–4, 0, 0) is equal to 10.

Solution:

Let A(4, 0, 0) & B(– 4, 0, 0)

Let the coordinates of point P be (x, y, z)

Calculating PA

P ≡ (x, y, z) and A ≡ (4, 0, 0)

By using the formula,

Distance PA = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So, here

$x_1 = x$, $y_1 = y$, $z_1 = z$

$x_2 = 4$, $y_2 = 0$, $z_2 = 0$

Distance PA = $\sqrt{[(4 - x)^2 + (0 - y)^2 + (0 - z)^2]}$

Calculating PB,

P ≡ (x, y, z) and B ≡ (– 4, 0, 0)

By using the formula,

Distance PB = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So here,

$x_1 = x$, $y_1 = y$, $z_1 = z$

$x_2 = -4$, $y_2 = 0$, $z_2 = 0$

Distance PB = $\sqrt{[(-4 - x)^2 + (0 - y)^2 + (0 - z)^2]}$

It is given that,

PA + PB = 10

PA = 10 – PB

Square on both sides, we get

$PA^2 = (10 – PB)^2$

$PA^2 = 100 + PB^2 – 20\ PB$

$(4 – x)^2 + (0 – y)^2 + (0 – z)^2$

$100 + (– 4 – x)^2 + (0 – y)^2 + (0 – z)^2 – 20\ PB$

$(16 + x^2 – 8x) + (y^2) + (z^2)$

$100 + (16 + x^2 + 8x) + (y^2) + (z^2) – 20\ PB$

$20\ PB = 16x + 100$

$5\ PB = (4x + 25)$

Square on both sides again, we get

$25\ PB^2 = 16x^2 + 200x + 625$

$25\ [(– 4 – x)^2 + (0 – y)^2 + (0 – z)^2] = 16x^2 + 200x + 625$

$25\ [x^2 + y^2 + z^2 + 8x + 16] = 16x^2 + 200x + 625$

$25x^2 + 25y^2 + 25z^2 + 200x + 400 = 16x^2 + 200x + 625$

$9x^2 + 25y^2 + 25z^2 – 225 = 0$

∴ The required equation is $9x^2 + 25y^2 + 25z^2 – 225 = 0$.

6. Three vertices of a parallelogram ABCD are A(3, – 1, 2), B(1, 2, – 4) and C(– 1, 1, 2). Find the coordinates of the fourth vertex.

Solution:

Given:

ABCD is a parallelogram with vertices A (3, -1, 2), B (1, 2, -4), and C (-1, 1, 2).

Where, $x_1 = 3$, $y_1 = -1$, $z_1 = 2$;

$x_2 = 1, y_2 = 2, z_2 = -4;$

$x_3 = -1, y_3 = 1, z_3 = 2$

Let the coordinates of the fourth vertex be D (x, y, z).

We know that the diagonals of a parallelogram bisect each other, so the midpoints of AC and BD are equal, i.e., Midpoint of AC = Midpoint of BD ……….(1)

Now, by the midpoint formula, we know that the coordinates of the mid-point of the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are $[(x_1+x_2)/2, (y_1+y_2)/2, (z_1+z_2)/2]$

So we have,

= (2/2, 0/2, 4/2)

= (1, 0, 2)

$1 + x = 2, 2 + y = 0, -4 + z = 4$

$x = 1, y = -2, z = 8$

Hence, the coordinates of the fourth vertex are D (1, -2, 8).

7. Find the lengths of the medians of the triangle with vertices A (0, 0, 6), B (0, 4, 0) and C (6, 0, 0).

Solution:

Given:

The vertices of the triangle are A (0, 0, 6), B (0, 4, 0) and C (6, 0, 0).

$x_1 = 0, y_1 = 0, z_1 = 6;$

$x_2 = 0, y_2 = 4, z_2 = 0;$

$x_3 = 6, y_3 = 0, z_3 = 0$

So, let the medians of this triangle be AD, BE and CF, corresponding to the vertices A, B and C, respectively.

D, E and F are the midpoints of the sides BC, AC and AB, respectively.

By the midpoint formula, the coordinates of the midpoint of the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are [$(x_1+x_2)/2$, $(y_1+y_2)/2$, $(z_1+z_2)/2$]

So, we have

By the distance formula, the distance between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by

∴ The lengths of the medians of the given triangle are 7, $\sqrt{34}$ and 7.

8. If the origin is the centroid of the triangle PQR with vertices P (2a, 2, 6), Q (–4, 3b, –10) and R (8, 14, 2c), then find the values of a, b and c.

Solution:

Given:

The vertices of the triangle are P (2a, 2, 6), Q (-4, 3b, -10) and R (8, 14, 2c).

Where,

$x_1 = 2a$, $y_1 = 2$, $z_1 = 6$;

$x_2 = -4$, $y_2 = 3b$, $z_2 = -10$;

$x_3 = 8$, $y_3 = 14$, $z_3 = 2c$

We know that the coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), are $[(x_1+x_2+x_3)/3, (y_1+y_2+y_3)/3, (z_1+z_2+z_3)/3]$

So, the coordinates of the centroid of the triangle PQR are

$2a + 4 = 0$, $3b + 16 = 0$, $2c – 4 = 0$

$a = -2$, $b = -16/3$, $c = 2$

∴ The values of a, b and c are $a = -2$, $b = -16/3$, and $c = 2$.

9. Find the coordinates of a point on the y-axis, which are at a distance of 5√2 from the point P (3, –2, 5).

Solution:

Let the point on the y-axis be A (0, y, 0).

Then, it is given that the distance between the points A (0, y, 0) and P (3, -2, 5) is 5√2.

Now, by using the distance formula,

We know that the distance between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by

Distance of PQ = $\sqrt{[(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2]}$

So, the distance between the points A (0, y, 0) and P (3, -2, 5) is given by

Distance of AP = $\sqrt{[(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2]}$

$= \sqrt{[(3-0)^2 + (-2-y)^2 + (5-0)^2]}$

$= \sqrt{[3^2 + (-2-y)^2 + 5^2]}$

$= \sqrt{[(-2-y)^2 + 9 + 25]}$

$5\sqrt{2} = \sqrt{[(-2-y)^2 + 34]}$

Squaring on both sides, we get

$(-2 -y)^2 + 34 = 25 \times 2$

$(-2 -y)^2 = 50 – 34$

$4 + y^2 + (2 \times -2 \times -y) = 16$

$y^2 + 4y + 4 - 16 = 0$

$y^2 + 4y - 12 = 0$

$y^2 + 6y - 2y - 12 = 0$

$y (y + 6) - 2 (y + 6) = 0$

$(y + 6) (y - 2) = 0$

$y = -6, y = 2$

∴ The points (0, 2, 0) and (0, -6, 0) are the required points on the y-axis.

10. A point R with x-coordinate 4 lies on the line segment joining the points P (2, –3, 4) and Q (8, 0, 10). Find the coordinates of the point R.
[Hint: Suppose R divides PQ in the ratio k:1. The coordinates of the point R are given by

Solution:

Given:

The coordinates of the points are P (2, -3, 4) and Q (8, 0, 10).

$x_1 = 2, y_1 = -3, z_1 = 4$;

$x_2 = 8, y_2 = 0, z_2 = 10$

Let the coordinates of the required point be (4, y, z).

So, let the point R (4, y, z) divides the line segment joining the points P (2, -3, 4) and Q (8, 0, 10) in the ratio k:1.

By using the section formula,

We know that the coordinates of the point R, which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m:n, is given by:

So, the coordinates of the point R are given by

So, we have

8k + 2 = 4 (k + 1)

8k + 2 = 4k + 4

8k – 4k = 4 – 2

4k = 2

k = 2/4

= 1/2

Now, let us substitute the values, and we get

= 6

∴ The coordinates of the required point are (4, -2, 6).

11. If A and B be the points (3, 4, 5) and (–1, 3, –7), respectively, find the equation of the set of points P such that $PA^2 + PB^2 = k^2$, where k is a constant.

Solution:

Given:

Points A (3, 4, 5) and B (-1, 3, -7)

$x_1 = 3$, $y_1 = 4$, $z_1 = 5$;

$x_2 = -1$, $y_2 = 3$, $z_2 = -7$;

$PA^2 + PB^2 = k^2$ ……….(1)

Let the point be P (x, y, z).

Now, by using the distance formula,

We know that the distance between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by

So,

And

Now, substituting these values in (1), we have

$[(3 – x)^2 + (4 – y)^2 + (5 – z)^2] + [(-1 – x)^2 + (3 – y)^2 + (-7 – z)^2] = k^2$ $[(9 + x^2 – 6x) + (16 + y^2 – 8y) + (25 + z^2 – 10z)] + [(1 + x^2 + 2x) + (9 + y^2 – 6y) + (49 + z^2 + 14z)] = k^2$

$9 + x^2 – 6x + 16 + y^2 – 8y + 25 + z^2 – 10z + 1 + x^2 + 2x + 9 + y^2 – 6y + 49 + z^2 + 14z = k^2$

$2x^2 + 2y^2 + 2z^2 – 4x – 14y + 4z + 109 = k^2$

$2x^2 + 2y^2 + 2z^2 – 4x – 14y + 4z = k^2 – 109$

$2 (x^2 + y^2 + z^2 – 2x – 7y + 2z) = k^2 – 109$

$(x^2 + y^2 + z^2 – 2x – 7y + 2z) = (k^2 – 109)/2$

Hence, the required equation is $(x^2 + y^2 + z^2 – 2x – 7y + 2z) = (k^2 – 109)/2$.

SUMMARY

In three-dimensional geometry, the x-axis, y-axis and z-axis are the three coordinate axes of a rectangular Cartesian coordinate system. These lines are three mutually perpendicular lines. The values of the coordinate axes determine the location of the point in the coordinate plane.

CHAPTER-XII

LIMITS AND DERIVATIVES

2 MARK QUESTIONS

1. Evaluate the given limit:

Solution:

Given,

Substituting x = 3, we get

= 3 + 3

= 6

2. Evaluate the given limit:

Solution:

Given limit,

Substituting x = π, we get

= (π – 22 / 7)

3. Evaluate the given limit:

Solution:

Given limit,

Substituting r = 1, we get

= $\pi(1)^2$

$= \pi$

4. Evaluate the given limit:

Solution:

Given limit,

Substituting x = 4, we get

= [4(4) + 3] / (4 – 2)

= (16 + 3) / 2

= 19 / 2

5. Evaluate the given limit:

Solution:

Given limit,

Substituting x = -1, we get

= $[(-1)^{10} + (-1)^5 + 1] / (-1 - 1)$

= (1 – 1 + 1) / – 2

= – 1 / 2

6. Evaluate the given limit:

Solution:

Given limit,

= $[(0 + 1)^5 – 1] / 0$

=0

Since this limit is undefined,

Substitute x + 1 = y, then x = y – 1

7. Evaluate the given limit:

Solution:

= [a (0) + b] / c (0) + 1

= b / 1

= b

8. Evaluate the given limit:

Solution:

Given limit,

Substituting x = 1,

= [a $(1)^2$ + b (1) + c] / [c $(1)^2$ + b (1) + a]

= (a + b + c) / (a + b + c)

Given,

= 1

9. Evaluate the given limit:

Solution:

By substituting x = – 2, we get

10. Find the derivative of 99x at x = l00.

Solution:

Let f (x) = 99x,

From the first principle,

= 99

4 MARK QUESTIONS

1. Find the derivative of the following functions from the first principle.

(i) $x^3 – 27$

(ii) $(x – 1) (x – 2)$

(iii) $1 / x^2$

(iv) $x + 1 / x – 1$

Solution:

(i) Let $f(x) = x^3 – 27$

From the first principle,

(ii) Let $f(x) = (x – 1) (x – 2)$

From the first principle,

(iii) Let $f(x) = 1 / x^2$

From the first principle, we get

(iv) Let $f(x) = x + 1 / x – 1$

From the first principle, we get

2:Find the derivative of the function $x^2\cos x$.

Solution:

Given function is $x^2\cos x$

Let $y = x^2\cos x$

Differentiate with respect to x on both sides.

Then, we get:

$dy/dx = (d/dx)x^2\cos x$

Now, using the formula, we can write the above form as:

$dy/dx = x^2 (d/dx) \cos x + \cos x (d/dx)x^2$

Now, differentiate the function:

$dy/dx = x^2 (-\sin x) + \cos x (2x)$

Now, rearrange the terms, we will get:

$dy/dx = 2x \cos x – x^2 \sin x$

3:Find the positive integer "n" so that $\lim_{x\to 3}[(x^n– 3^n)/(x – 3)] = 108$.

Solution:

Given limit: $\lim_{x\to 3}[(x^n– 3^n)/(x – 3)] = 108$

Now, we have:

$\lim_{x\to 3}[(x^n– 3^n)/(x-3)] = n(3)^{n-1}$

$n(3)^{n-1} = 108$

Now, this can be written as:

$n(3)^{n-1} = 4 (27) = 4(3)^{4-1}$

Therefore, by comparing the exponents in the above equation, we get:

$n = 4$

Therefore, the value of positive integer "n" is 4.

4:Find the derivative of $f(x) = x^3$ using the first principle.

Solution:

By definition,

$f'(x) = \lim_{h\to 0} [f(x+h)-f(x)]/h$

Now, substitute $f(x)=x^3$ in the above equation:

$f'(x) = \lim_{h\to 0} [(x+h)^3-x^3]/h$

$f'(x) = \lim_{h\to 0} (x^3+h^3+3xh(x+h)-x^3)/h$

$f'(x) = \lim_{h\to 0} (h^2+3x(x+h))$

Substitute h = 0, we get:

$f'(x) = 3x^2$

Therefore, the derivative of the function $f'(x) = x^3$ is $3x^2$.

5:Determine the derivative of cosx/(1+sin x).

Solution:

Given function: cosx/(1+sin x)

Let y = cosx/(1+sin x)

Now, differentiate the function with respect to "x", we get

dy/dx = (d/dx) (cos x/(1+sin x))

Now, use the u/v formula in the above form, we get

$dy/dx = [(1+\sin x)(-\sin x) – (\cos x)(\cos x)]/(1+\sin x)^2$

$dy/dx = (-\sin x – \sin^2 x-\cos^2 x)/(1+\sin x)^2$

Now, take (-) outside from the numerator, we get:

$dy/dx = -(\sin x + \sin^2 x + \cos^2 x)/(1+\sin x)^2$

We know that $\sin^2 x + \cos^2 x = 1$

By substituting this, we can get:

$dy/dx = -(1+\sin x)/(1+\sin x)^2$

Cancel out (1+sin x) from both numerator and denominator, we get:

$dy/dx = -1/(1+\sin x)$

Therefore, the derivative of cosx/(1+sin x) is -1/(1+sin x).

6:$\lim_{x\to 0}$ |x|/x is equal to:

(a)1 (b)-1 (c)0 (d)does not exists

Solution:

A correct answer is an **option (d)**

Explanation:

The limit mentioned here is $x\to 0$

It has two possibilities:

Case 1: $x\to 0^+$

Now, substitute the limit in the given function:

$\lim_{x\to 0+} |x|/x = x/x = 1$

Case 2: $x\to0^-$

Now, substitute the limit in the given function:

$\lim_{x\to 0-} |x|/x = -x/x = -1$

Hence, the result for both cases varies, the solution is an option (D)

7:Evaluate the derivative of $f(x) = \sin^2x$ using Leibnitz product rule.

Solution:

Given function: $f(x) = \sin^2x$

Let $y= \sin^2x$

Now, by using Leibnitz product rule, we can write it as:

$dy/dx = (d/dx) \sin^2x$

Sin^2x can be written as (sin x)(sin x)

Now, it becomes:

$dy/dx = (d/dx) (\sin x)(\sin x)$

$dy/dx = (\sin x)'(\sin x) + (\sin x)(\sin x)'$

dy/dx = cos x sin x + sin x cos x

dy/dx = 2 sin x cos x

dy/dx = sin 2x

Therefore, the derivative of the function $\sin^2 x$ is sin 2x.

8. Evaluate the expression [cosec x-cot x] .

Ans. By rewriting cosec x and cot x in terms of cos x and sin x we get,

$$[cosec\ x - \cot cot\ \ x] = \left[\frac{1}{\sin sin\ x} - \frac{\cos cos\ x}{\sin sin\ x}\right]$$

By using the half angle formula of $cos\ x$ and $sin\ x$ we get,

$$\frac{2sin^2\frac{x}{2}}{2\sin sin\frac{x}{2}\cos cos\frac{x}{2}}$$

$$\frac{sin\frac{x}{2}}{\cos cos\frac{x}{2}}$$

$$= \tan tan\frac{x}{2}$$

=0

7 MARK QUESTIONS

1. Differentiate x **using first principle. [IMP-2012]**

Ans. The given function is f(x) = x

We can write it as

f(x+h) = cosec (x+h)

By definition of first principle,

$$f'(x) = \frac{[f(x+h)-f(x)]}{h}$$

By substituting the value of the functions

$$f'(x) = \frac{(x+h)-x}{h}$$

$$\frac{\frac{1}{\sin\sin(x+h)} - \frac{1}{\sin\sin x}}{h}$$

$$\frac{\sin\sin x - \sin\sin(x+h)}{h \sin\sin(x+h)\sin\sin x}$$

On simplifying we get,

$$\frac{2\cos\cos\left[\frac{2x+h}{2}\right]\sin\sin\left(\frac{-h}{2}\right)}{h\sin\sin(x+h)\sin\sin x}$$

$$\frac{2\cos\cos\left[x+\frac{h}{2}\right]\sin\sin\left(\frac{-h}{2}\right)}{h\sin\sin(x+h)\sin\sin x}$$

$$\frac{\sin\sin\frac{h}{2}}{\frac{h}{2}} \times \frac{\cos\cos\left(x+\frac{h}{2}\right)}{\cos\cos x \lim_{h\to 0}\sin\sin(x+h)}$$

Applying the limit h=0

$$f'(x) = 1\times\left(-\frac{\cos\cos x}{\sin\sin x\sin\sin x}\right)$$

$$= -\frac{\cos\cos x}{\sin\sin x\sin\sin x}$$

$$= -\operatorname{cosec} x \cot\cot x$$

2. Evaluate (a+h)2sin(a+h)−a2sinah$(a+h)2sin(a+h)-a2sinah$

Ans.

Given function is $\frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

Using the formula for derivative we get,

$$\frac{(a+h)^2 \sin\sin(a+h) - a^2 \sin\sin a}{h} = \frac{(a^2+2ah+h^2)\sin\sin(a+h) - a^2\sin\sin a}{h}$$

$$\frac{a^2\sin\sin(a+h) + 2ah\sin(a+h) + h^2\sin\sin(a+h) - a^2\sin\sin a}{h}$$

On simplifying we gwt,

$$\frac{a^2 2\cos\cos\frac{2a+h}{2}\sin\sin\frac{h}{2}}{2\frac{h}{2}} + 2a\sin\sin(a+h) + h\sin\sin(a+h)$$

$$= a^2 2\cos\cos\frac{2a+0}{2}\times 1 + 2a\sin\sin(a+0) + 0\times\sin\sin a$$

$$= a^2 2\cos\cos a + 2a\sin\sin a$$

3. Find the derivative of f(x) = secsec x by the first principle. [IMP-2010]

Ans. The given function is f(x) = secsec x

We can write it as

f(x+h) =secsec(x+h)

By definition of first principle, we have,

$$f'(x) = \frac{\sec sec\,(x+h) - \sec sec\,x}{h}$$

$$\frac{\frac{1}{\cos cos\,(x+h)} - \frac{1}{\cos cos\,x}}{h}$$

$$\frac{\cos cos\,x - \cos cos\,(x+h)}{\cos cos\,(x+h) \cos cos\,xh}$$

On simplifying we get,

$$\frac{-2 \sin sin\left[\frac{2x+h}{2}\right] \sin sin\left(\frac{-h}{2}\right)}{\cos cos\,(x+h) \cos cos\,xh}$$

$$\frac{2 \sin sin\left[\frac{2x+h}{2}\right] \sin sin\left(\frac{h}{2}\right)}{\cos cos\,(x+h) \cos cos\,xh}$$

$$\frac{2 \sin sin\,\frac{h}{2}}{2\frac{h}{2}} \times \frac{\sin sin\,\frac{2x+h}{2}}{\lim\limits_{h \to 0} \cos cos\,(x+h) \cos cos\,x}$$

Applying the limit h=0

$$= 1 \times \frac{\sin sin\,\frac{2x+0}{2}}{\cos cos\,(x+0) \cos cos\,x}$$

$$= \frac{\sin sin\,x}{\cos cos\,x \cos cos\,x}$$

$$= \tan tan\,x \sec sec\,x$$

4. Find the derivative of xn−anx−a$xn-anx-a$

Ans. By differentiating using the quotient rule

$$\frac{d}{dx}\frac{\left(x^{n}-a^{n}\right)}{x-a}=\frac{(x-a)\frac{d}{dx}\left(x^{n}-a^{n}\right)-\left(x^{n}-a^{n}\right)\frac{d}{dx}(x-a)}{(x-a)^{2}}$$

$$=\frac{(x-a)\left(nx^{n-1}-0\right)-\left(x^{n}-a^{n}\right)(1-0)}{(x-a)^{2}}$$

By simplifying the above equation, we get,

$$=\frac{nx^{n-1}(x-a)-x^{n}+a^{n}}{(x-a)^{2}}$$

$$=\frac{nx^{n}-nax^{n-1}-x^{n}+a^{n}}{(x-a)^{2}}$$

$$=\frac{x^{n}(n-1)-nax^{n-1}+a^{n}}{(x-a)^{2}}$$

SUMMARY

Limits and Derivatives are two major parts of differentiation and calculus. A limit can be defined as a value that a function is seen to approach as the input, yielding some value in return. The rate at which a function or quantity changes in relation to others can be termed as its derivative.

CHAPTER-XIII

STATISTICS

2 MARK QUESTIONS

Find the mean deviation about the mean for the data in Exercises 1 and 2.

1. 4, 7, 8, 9, 10, 12, 13, 17

Solution:-

First, we have to find ($\bar{x}$) of the given data.

So, the respective values of the deviations from mean,

i.e., $x_i - \bar{x}$ are, 10 – 4 = 6, 10 – 7 = 3, 10 – 8 = 2, 10 – 9 = 1, 10 – 10 = 0,

10 – 12 = – 2, 10 – 13 = – 3, 10 – 17 = – 7

6, 3, 2, 1, 0, -2, -3, -7

Now, absolute values of the deviations,

6, 3, 2, 1, 0, 2, 3, 7

MD = sum of deviations/ number of observations

= 24/8

= 3

So, the mean deviation for the given data is 3.

2. 38, 70, 48, 40, 42, 55, 63, 46, 54, 44

Solution:-

First, we have to find ($\overline{x}$) of the given data.

So, the respective values of the deviations from mean,

i.e., $x_i - \overline{x}$ are, 50 – 38 = -12, 50 -70 = -20, 50 – 48 = 2, 50 – 40 = 10, 50 – 42 = 8,

50 – 55 = – 5, 50 – 63 = – 13, 50 – 46 = 4, 50 – 54 = -4, 50 – 44 = 6

-12, 20, -2, -10, -8, 5, 13, -4, 4, -6

Now, absolute values of the deviations,

12, 20, 2, 10, 8, 5, 13, 4, 4, 6

MD = sum of deviations/ number of observations

= 84/10

= 8.4

So, the mean deviation for the given data is 8.4.

Find the mean deviation about the median for the data in Exercises 3 and 4.

3. 13, 17, 16, 14, 11, 13, 10, 16, 11, 18, 12, 17

Solution:-

First, we have to arrange the given observations into ascending order.

10, 11, 11, 12, 13, 13, 14, 16, 16, 17, 17, 18.

The number of observations is 12.

Then,

Median = ((12/2)th observation + ((12/2)+ 1)th observation)/2

(12/2)th observation = 6th = 13

(12/2)+ 1)th observation = 6 + 1

$= 7^{th} = 14$

Median = (13 + 14)/2

= 27/2

= 13.5

So, the absolute values of the respective deviations from the median, i.e., $|x_i - M|$ are

3.5, 2.5, 2.5, 1.5, 0.5, 0.5, 0.5, 2.5, 2.5, 3.5, 3.5, 4.5

Mean Deviation

= (1/12) × 28

= 2.33

So, the mean deviation about the median for the given data is 2.33.

4. 36, 72, 46, 42, 60, 45, 53, 46, 51, 49

Solution:-

First, we have to arrange the given observations into ascending order.

36, 42, 45, 46, 46, 49, 51, 53, 60, 72.

The number of observations is 10.

Then,

Median = $((10/2)^{th}$ observation + $((10/2)+ 1)^{th}$ observation)/2

$(10/2)^{th}$ observation = 5^{th} = 46

$(10/2)+ 1)^{th}$ observation = 5 + 1

$= 6^{th} = 49$

Median = (46 + 49)/2

= 95

= 47.5

So, the absolute values of the respective deviations from the median, i.e., $|x_i - M|$ are

11.5, 5.5, 2.5, 1.5, 1.5, 1.5, 3.5, 5.5, 12.5, 24.5

Mean Deviation

= (1/10) × 70

= 7

So, the mean deviation about the median for the given data is 7.

4 MARK QUESTIONS

Find the mean and variance for each of the data in Exercise 1 to 5.

1. 6, 7, 10, 12, 13, 4, 8, 12

Solution:-

So, $\bar{x}$ = (6 + 7 + 10 + 12 + 13 + 4 + 8 + 12)/8

= 72/8

= 9

Let us make the table of the given data and append other columns after calculations.

X_i	**Deviations from mean** ($x_i - \bar{x}$)	$(x_i - \bar{x})^2$
6	6 – 9 = -3	9
7	7 – 9 = -2	4
10	10 – 9 = 1	1
12	12 – 9 = 3	9
13	13 – 9 = 4	16
4	4 – 9 = – 5	25
8	8 – 9 = – 1	1
12	12 – 9 = 3	9
		74

We know that the Variance,

$\sigma^2 = (1/8) \times 74$

= 9.2

∴Mean = 9 and Variance = 9.25

Find the mean deviation about the mean for the data in Exercises 5 and 6.

2.

x_i	**5**	**10**	**15**	**20**	**25**
f_i	**7**	**4**	**6**	**3**	**5**

Solution:-

Let us make the table of the given data and append other columns after calculations.

X_i	**f_i**	**$f_i x_i$**	**$\lvert x_i - \bar{x} \rvert$**	**$f_i \lvert x_i - \bar{x} \rvert$**
5	7	35	9	63
10	4	40	4	16
15	6	90	1	6
20	3	60	6	18
25	5	125	11	55
	25	350		158

The sum of calculated data,

The absolute values of the deviations from the mean, i.e., $|x_i - \bar{x}|$, as shown in the table.

3.

x_i	**10**	**30**	**50**	**70**	**90**
f_i	**4**	**24**	**28**	**16**	**8**

Solution:-

Let us make the table of the given data and append other columns after calculations.

X_i	**f_i**	**$f_i x_i$**	**$\lvert x_i - \bar{x} \rvert$**	**$f_i \lvert x_i - \bar{x} \rvert$**
10	4	40	40	160
30	24	720	20	480
50	28	1400	0	0
70	16	1120	20	320
90	8	720	40	320
	80	4000		1280

4.

x_i	**5**	**7**	**9**	**10**	**12**	**15**
f_i	**8**	**6**	**2**	**2**	**2**	**6**

Solution:-

Let us make the table of the given data and append other columns after calculations.

X_i	**f_i**	**c.f.**	**$\lvert x_i - M \rvert$**	**$f_i \lvert x_i - M \rvert$**

5	8	8	2	16
7	6	14	0	0
9	2	16	2	4
10	2	18	3	6
12	2	20	5	10
15	6	26	8	48

Now, N = 26, which is even.

Median is the mean of the 13^{th} and 14^{th} observations. Both of these observations lie in the cumulative frequency of 14, for which the corresponding observation is 7.

Then,

Median = (13^{th} observation + 14^{th} observation)/2

= (7 + 7)/2

= 14/2

= 7

So, the absolute values of the respective deviations from the median, i.e., $|x_i – M|$ are shown in the table.

5.

x_i	15	21	27	30	35
f_i	3	5	6	7	8

Solution:-

Let us make the table of the given data and append other columns after calculations.

X_i	f_i	c.f.	$\|x_i - M\|$	$f_i \|x_i - M\|$
15	3	3	15	45
21	5	8	9	45
27	6	14	3	18
30	7	21	0	0
35	8	29	5	40

Now, N = 29, which is odd.

So, 29/2 = 14.5

The cumulative frequency greater than 14.5 is 21, for which the corresponding observation is 30.

Then,

Median = (15th observation + 16th observation)/2

= (30 + 30)/2

= 60/2

= 30

So, the absolute values of the respective deviations from the median, i.e., $|x_i - M|$ are shown in the table.

7 MARK QUESTIONS

1.

Income per day in ₹	0 – 100	100 – 200	200 – 300	300 – 400	400 – 500	500 – 600	600 – 700	700 – 800
Number of persons	4	8	9	10	7	5	4	3

Solution:-

Let us make the table of the given data and append other columns after calculations.

Income per day in ₹	Number of persons f_i	Midpoints x_i	$f_i x_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
0 – 100	4	50	200	308	1232
100 – 200	8	150	1200	208	1664
200 – 300	9	250	2250	108	972
300 – 400	10	350	3500	8	80
400 – 500	7	450	3150	92	644
500 – 600	5	550	2750	192	960
600 – 700	4	650	2600	292	1160
700 – 800	3	750	2250	392	1176
	50		17900		7896

2. Find the mean deviation about median for the following data.

Marks	0 -10	10 -20	20 – 30	30 – 40	40 – 50	50 – 60
Number of girls	6	8	14	16	4	2

Solution:-

Let us make the table of the given data and append other columns after calculations.

Marks	Number of girls f_i	Cumulative frequency (c.f.)	Mid - points x_i	$\|x_i - Med\|$	$f_i\|x_i - Med\|$
0 – 10	6	6	5	22.85	137.1
10 – 20	8	14	15	12.85	102.8
20 – 30	14	28	25	2.85	39.9
30 – 40	16	44	35	7.15	114.4
40 – 50	4	48	45	17.15	68.6
50 – 60	2	50	55	27.15	54.3
	50				517.1

The class interval containing $N^{th}/2$ or 25^{th} item is 20-30.

So, 20-30 is the median class.

Then,

Median = l + (((N/2) – c)/f) × h

Where, l = 20, c = 14, f = 14, h = 10 and n = 50

Median = 20 + (((25 – 14))/14) × 10

= 20 + 7.85

= 27.85

3.

Height in cms	**95 – 105**	**105 – 115**	**115 – 125**	**125 – 135**	**135 – 145**	**145 – 155**
Number of boys	**9**	**13**	**26**	**30**	**12**	**10**

Solution:-

Let us make the table of the given data and append other columns after calculations.

Height in cms	**Number of boys f_i**	**Midpoints x_i**	**f_ix_i**	**$\|x_i - \bar{x}\|$**	**$f_i\|x_i - \bar{x}\|$**
95 – 105	9	100	900	25.3	227.7
105 – 115	13	110	1430	15.3	198.9
115 – 125	26	120	3120	5.3	137.8
125 – 135	30	130	3900	4.7	141
135 – 145	12	140	1680	14.7	176.4
145 – 155	10	150	1500	24.7	247
	100		12530		1128.8

Find the mean and variance for each of the data in Exercise 1 to 5.

4. 6, 7, 10, 12, 13, 4, 8, 12

Solution:-

So, $\bar{x}$ = (6 + 7 + 10 + 12 + 13 + 4 + 8 + 12)/8

= 72/8

= 9

Let us make the table of the given data and append other columns after calculations.

X_i	**Deviations from mean $(x_i - \bar{x})$**	**$(x_i - \bar{x})^2$**
6	6 – 9 = -3	9
7	7 – 9 = -2	4
10	10 – 9 = 1	1

12	12 – 9 = 3	9
13	13 – 9 = 4	16
4	4 – 9 = – 5	25
8	8 – 9 = – 1	1
12	12 – 9 = 3	9
		74

We know that the Variance,

$\sigma^2 = (1/8) \times 74$

$= 9.2$

$\therefore$Mean = 9 and Variance = 9.25

5. First n natural numbers

Solution:-

We know that Mean = Sum of all observations/Number of observations

$\therefore$Mean, $\overline{x} = ((n(n + 1))2)/n$

$= (n + 1)/2$

and also, WKT Variance,

By substituting the value of $\overline{x}$, we get

WKT, $(a + b)(a – b) = a^2 – b^2$

$\sigma^2 = (n^2 – 1)/12$

∴Mean = $(n + 1)/2$ and Variance = $(n^2 – 1)/12$

6. First 10 multiples of 3

Solution:-

First, we have to write the first 10 multiples of 3.

3, 6, 9, 12, 15, 18, 21, 24, 27, 30

So, $\bar{x}$ = (3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 + 27 + 30)/10

= 165/10

= 16.5

Let us make the table of the data and append other columns after calculations.

X_i	**Deviations from mean** $(x_i – \bar{x})$	$(x_i – \bar{x})^2$
3	3 – 16.5 = -13.5	182.25
6	6 – 16.5 = -10.5	110.25
9	9 – 16.5 = -7.5	56.25
12	12 – 16.5 = -4.5	20.25
15	15 – 16.5 = -1.5	2.25
18	18 – 16.5 = 1.5	2.25
21	21 – 16.5 = – 4.5	20.25
24	24 – 16.5 = 7.5	56.25
27	27 – 16.5 = 10.5	110.25
30	30 – 16.5 = 13.5	182.25
		742.5

Then, the Variance

= (1/10) × 742.5

= 74.25

∴Mean = 16.5 and Variance = 74.25

7. An analysis of monthly wages paid to workers in two firms, A and B, belonging to the same industry, gives the following results:

	Firm A	Firm B
No. of wages earners	**586**	**648**
Mean of monthly wages	**Rs 5253**	**Rs 5253**
Variance of the distribution of wages	**100**	**121**

(i) Which firm, A or B, pays a larger amount as monthly wages?

(ii) Which firm, A or B, shows greater variability in individual wages?

Solution:-

(i) From the given table,

Mean monthly wages of firm A = Rs 5253

and Number of wage earners = 586

Then,

Total amount paid = 586 × 5253

= Rs 3078258

Mean monthly wages of firm B = Rs 5253

Number of wage earners = 648

Then,

Total amount paid = 648×5253

= Rs 34,03,944

So, firm B pays larger amount as monthly wages.

(ii) Variance of firm A = 100

We know that, standard deviation (σ)= $\sqrt{100}$

=10

Variance of firm B = 121

Then,

Standard deviation (σ)=$\sqrt{121}$

=11

Hence, the standard deviation is more in case of Firm B. That means, in firm B, there is greater variability in individual wages.

SUMMARY

Summary statistics provide a quick summary of data and are particularly useful for comparing one project to another, or before and after. There are two main types of summary statistics used in evaluation: measures of central tendency and measures of dispersion.

CHAPTER-XIV

PROBABILITY

2 MARK QUESTIONS

1. A coin is tossed three times.

Solution:-

Since either coin can turn up Head (H) or Tail (T), the possible outcomes may be

When 1 coin is tossed once the sample space = 2

Then,

The coin is tossed 3 times the sample space = $2^3 = 8$

Thus, the sample space is S = {HHH, THH, HTH, HHT, TTT, HTT, THT, TTH}

2. A die is thrown two times.

Solution:-

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible outcomes when the die is thrown.

Then, the total number of the sample space = (6 × 6)

= 36

Thus, the sample space is

S={(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,1),(3,2),(3,3)(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

3. A coin is tossed four times.

Solution:-

Since either coin can turn up Head (H) or Tail (T), the possible outcomes are

When 1 coin is tossed once the sample space = 2

Then,

The coin is tossed 3 times the sample space = 2^4 = 16

Thus, the sample space is S = {HHHH, THHH, HTHH, HHTH, HHHT, TTTT, HTTT, THTT, TTHT, TTTH, TTHH, HHTT, THTH, HTHT, THHT, HTTH}

4. A coin is tossed, and a die is thrown.

Solution:-

A coin can turn up either Head (H) or Tail (T), and this is the possible outcome.

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible numbers that come when the die is thrown.

Then, total number of space = (2 × 6) = 12

Thus, the sample space is,

S={(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)}

5. A coin is tossed, and then a die is rolled only in case a head is shown on the coin.

Solution:-

Since either coin can turn up Head (H) or Tail (T), these are the possible outcomes.

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible numbers that come when the die is thrown.

When the head is encountered,

Then, number of space = (1 × 6) = 6

Sample Space S_H= {H1, H2, H3, H4, H5, H6}

Now, the tail is encountered, Sample space S_T= {T}

Therefore the total sample space S = {H1, H2, H3, H4, H5, H6, T}

6. 2 boys and 2 girls are in Room X, and 1 boy and 3 girls are in Room Y. Specify the sample space for the experiment in which a room is selected and then a person.

Solution:-

From the question, it is given that

2 boys and 2 girls are in Room X

1 boy and 3 girls in Room Y

Let us assume b1, b2 and g1, g2 be 2 boys and 2 girls in Room X.

And also, assume b3 and g3, g4, g5 be 1 boy and 3 girls in Room Y.

The problem is solved by dividing it into two cases.

Case 1: Room X is selected

Sample Space S_x = {(X,b1),(X,b2),(X,g1),(X,g2)}

Case 2: Room Y is selected

Sample Space S_y ={(Y,b3),(Y,g3),(Y,g4),(Y,g5)}

The overall sample space

S={(X,b1),(X,b2),(X,g1),(X,g2),(Y,b3),(Y,g3),(Y,g4),(Y,g5)}

7. One die of red colour, one of white colour and one of blue colour are placed in a bag. One die is selected at random and rolled, its colour and the number on its uppermost face is noted. Describe the sample space.

Solution:-

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible numbers that come when the die is thrown.

And also, assume die of red colour be 'R', die of white colour be 'W', die of blue colour be 'B'.

So, the total number of sample space = (6 × 3) = 18

The sample space of the event is

S={(R,1),(R,2),(R,3),(R,4),(R,5),(R,6),(W,1),(W,2),(W,3),(W,4),(W,5),(W,6)
(B,1),(B,2),(B,3),(B,4),(B,5),(B,6)}

8. An experiment consists of recording boy–girl composition of families with 2 children.

(i) What is the sample space if we are interested in knowing whether it is a boy or girl in the order of their births?

(ii) What is the sample space if we are interested in the number of girls in the family?

Solution:-

Let us assume the boy be 'B' and the girl be 'G'.

(i) The sample space if we are interested in knowing whether it is a boy or girl in the order of their births, S = {GG, BB, GB, BG}

(ii) The sample space if we are interested in the number of girls in the family when there are two children in the family then

Sample space S = {2, 1, 0}

9. A box contains 1 red and 3 identical white balls. Two balls are drawn at random in succession without replacement. Write the sample space for this experiment.

Solution:-

From the question, it is given that a box contains 1 red and 3 identical white balls.

Let us assume 'R' be the event of the red ball being drawn, and 'W' be the event of the white ball being drawn.

Given in the question that white balls are identical; therefore, the event of drawing any one of the three white balls is the same.

Then, total number of sample space = $(2^2 - 1) = 3$

∴Sample space S = {WW, WR, RW}

10. An experiment consists of tossing a coin and then throwing it the second time if a head occurs. If a tail occurs on the first toss, then a die is rolled once. Find the sample space.

Solution:-

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible outcomes when the die is thrown.

Since either coin can turn up Head (H) or Tail (T), these are the possible outcomes.

Let us take,

Case 1: Head is encountered

Sample space S_1 = {HT, HH}

Case 2: Tail is encountered

Sample Space S_2 = {(T,1), (T,2), (T,3), (T,4), (T,5), (T,6)}

Then,

The Overall Sample space

S = {(HT), (HH), (T1), (T2), (T3), (T4), (T5), (T6)}

4 MARK QUESTIONS

1. Suppose 3 bulbs are selected at random from a lot. Each bulb is tested and classified as defective (D) or non-defective (N). Write the sample space of this experiment.

Solution:-

From the question,

'D' denotes the event that the bulb is defective, and 'N' denotes the event of non-defective bulbs.

Then,

Total number of Sample space = $2 \times 2 \times 2 = 8$

Thus, Sample space S = {DDD, DDN, DND, NDD, DNN, NDN, NND, NNN}

2. A coin is tossed. If the outcome is a head, a die is thrown. If the die shows up an even number, the die is thrown again. What is the sample space for the experiment?

Solution:-

Since either coin can turn up Head (H) or Tail (T), these are the possible outcomes.

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible outcomes when the die is thrown.

The problem can be solved by dividing it into 3 cases.

Case 1: The outcome is Head, and the corresponding number on the die shows an Odd number.

Total number of sample space = $(1 \times 3) = 3$

Sample space S_{HO} = {(H,1), (H,3), (H,5)}

Case 2: The outcome is Head, and the corresponding number on the die shows an Even number.

Total number of sample space = $(1 \times 3 \times 6) = 18$

S_{HE}={(H,2,1),(H,2,2),(H,2,3),(H,2,4),(H,2,5),(H,2,6),(H,4,1),(H,4,2),(H,4,3),(H,2,4),(H, 4,5),(H,4,6), (H,6,1),(H,6,2),(H,6,3),(H,6,4),(H,6,5),(H,6,6)}

Case 3: The outcome is Tail

Total number of sample space=1

Sample space S_T = {(T)}

The overall sample spaces

S={(H,1),(H,3),(H,5), (H,2,1),(H,2,2),(H,2,3),(H,2,4),(H,2,5),(H,2,6),(H,4,1),(H,4,2),(H,4,3),(H,2,4),(H,4,5),

(H,4,6), (H,6,1),(H,6,2),(H,6,3),(H,6,4),(H,6,5),(H,6,6),(T)}

3. The numbers 1, 2, 3 and 4 are written separately on four slips of paper. The slips are put in a box and mixed thoroughly. A person draws two slips from the box, one after the other, without replacement. Describe the sample space for the experiment.

Solution:-

From the question, it is given that 1, 2, 3, and 4 are the numbers written on the four slips.

When two slips are drawn without replacement, the first event has 4 possible outcomes, and the second event has 3 possible outcomes because 1 slip is already picked.

Therefore, the total number of possible outcomes = (4 × 3) = 12

Thus, the sample space is,

S={(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3)}

4. An experiment consists of rolling a die and then tossing a coin once if the number on the die is even. If the number on the die is odd, the coin is tossed twice. Write the sample space for this experiment.

Solution:-

Since either coin can turn up Head (H) or Tail (T), these are the possible outcomes.

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible outcomes when the die is thrown.

The following problem can be divided into two cases.

(i) The number on the die is even.

The sample space S_E={(2,H),(4,H),(6,H),(2,T),(4,T),(6,T)}

(ii) The number on the die is odd, and the coin is tossed twice.

The sample space

S_o = {(1,H,H), (3,H,H), (5,H,H), (1,H,T), (3,H,T), (5,H,T), (1,T,H), (3,T,H), (5,T,H), (1,T,T), (3,T,T), (5,T,T)}

Hence, the overall sample space for the problem= $S_E + S_o$

S = {(2,H), (4,H), (6,H), (2,T), (4,T), (6,T), (1,H,H), (3,H,H), (5,H,H), (1,H,T), (3,H,T), (5,H,T), (1,T,H), (3,T,H), (5,T,H), (1,T,T), (3,T,T), (5,T,T)}

5. A coin is tossed. If it shows a tail, we draw a ball from a box which contains 2 red and 3 black balls. If it shows head, we throw a die. Find the sample space for this experiment.

Solution:-

Since either coin can turn up Head (H) or Tail (T), these are the possible outcomes.

Let us assume R_1, R_2 denote the event the red balls are drawn, and B_1, B_2, B_3 denote the events black balls are drawn.

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible outcomes when the die is thrown.

(i) Coin shows the Tail.

So, the sample space S_T = {(TR_1), (TR_2), (TB_1), (TB_2), (TB_3)}

(ii) Coin shows the head.

So, the sample space S_H = {(H,1), (H,2), (H,3), (H,4), (H,5), (H,6)}

Hence, the overall sample space for the problem= $S_T + S_H$

S = {(T,R_1), (T,R_2), (T,B_1), (T,B_2), (T,B_3), (H,1), (H,2), (H,3), (H,4), (H,5), (H,6)}

6. A die is thrown repeatedly until a six comes up. What is the sample space for this experiment?

Solution:-

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible outcomes when the die is thrown.

As per the condition given in the question, a die is thrown repeatedly until a six comes up.

If six may come up for the first throw or six may come up on the second throw, this process will go continuously until the six comes.

The sample space when 6 comes on the very first throw S_1 = {6}

The sample space when 6 comes on the second throw S_2 = {(1,6), (2,6), (3,6), (4,6), (5,6)}

This event can go on for infinite times.

So, the sample space is infinitely defined

S = {(6), (1,6), (2,6), (3,6), (4,6), (5,6), (1,1,6), (1,2,6)……}

7 MARK QUESTIONS

1. A die is rolled. Let E be the event "die shows 4" and F be the event "die shows even number". Are E and F mutually exclusive?

Solution:-

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible outcomes when the die is thrown.

So, S = (1, 2, 3, 4, 5, 6)

As per the conditions given in the question,

E be the event "die shows 4"

E = (4)

F be the event "die shows even number".

F = (2, 4, 6)

$E \cap F = (4) \cap (2, 4, 6)$

= 4

$4 \neq \varphi$ … [because there is a common element in E and F]

Therefore, E and F are not mutually exclusive events.

2. A die is thrown. Describe the following events:

(i) A: a number less than 7 (ii) B: a number greater than 7

(iii) C: a multiple of 3 (iv) D: a number less than 4

(v) E: an even number greater than 4 (vi) F: a number not less than 3

Also, find $A \cup B$, $A \cap B$, $B \cup C$, $E \cap F$, $D \cap E$, $A - C$, $D - E$, $E \cap F'$, F'

Solution:-

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible outcomes when the die is thrown.

So, S = (1, 2, 3, 4, 5, 6)

As per the conditions given in the question,

(i) A: a number less than 7

All the numbers in the die are less than 7,

A = (1, 2, 3, 4, 5, 6)

(ii) B: a number greater than 7

There is no number greater than 7 on the die.

Then,

B= (φ)

(iii) C: a multiple of 3

There are only two numbers which are multiple of 3.

Then,

C= (3, 6)

(iv) D: a number less than 4

D= (1, 2, 3)

(v) E: an even number greater than 4

E = (6)

(vi) F: a number not less than 3

F= (3, 4, 5, 6)

Also, we have to find, A U B, A ∩ B, B U C, E ∩ F, D ∩ E, D – E, A – C, E ∩ F', F'

So,

A ∩ B = (1, 2, 3, 4, 5, 6) ∩ (φ)

= (φ)

B U C = (φ) U (3, 6)

= (3, 6)

E ∩ F = (6) ∩ (3, 4, 5, 6)

= (6)

D ∩ E = (1, 2, 3) ∩ (6)

= (φ)

D – E = (1, 2, 3) – (6)

= (1, 2, 3)

A – C = (1, 2, 3, 4, 5, 6) – (3, 6)

= (1, 2, 4, 5)

F' = S – F

= (1, 2, 3, 4, 5, 6) – (3, 4, 5, 6)

= (1, 2)

E ∩ F' = (6) ∩ (1, 2)

= (φ)

3. An experiment involves rolling a pair of dice and recording the numbers that come up. Describe the following events: A: the sum is greater than 8, B: 2 occurs on either die C: the sum is at least 7 and a multiple of 3. Which pairs of these events are mutually exclusive?

Solution:-

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible outcomes when the die is thrown.

In the question, it is given that pair of die is thrown, so the sample space will be

Now, we shall find whether pairs of these events are mutually exclusive or not.

(i) $A \cap B = \varphi$

There is no common element in A and B.

Therefore, A & B are mutually exclusive.

(ii) $B \cap C = \varphi$

There is no common element between.

Therefore, B and C are mutually exclusive.

(iii) $A \cap C$ {(3,6), (4,5), (5,4), (6,3), (6,6)}

$\Rightarrow$ {(3,6), (4,5), (5,4), (6,3), (6,6)} $\neq \varphi$

A and C have common elements.

Therefore, A and C are mutually exclusive.

4. Three coins are tossed once. Let A denotes the event 'three heads show", B denotes the event "two heads and one tail show", C denotes the event" three tails show and D denote the event 'a head shows on the first coin". Which events are

(i) Mutually exclusive? (ii) Simple? (iii) Compound?

Solution:-

Since either coin can turn up Head (H) or Tail (T), these are the possible outcomes.

But, now three coins are tossed once, so the possible sample space contains

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTH}

Now,

A: 'three heads'

A= (HHH)

B: “two heads and one tail”

B= (HHT, THH, HTH)

C: ‘three tails’

C= (TTT)

D: a head shows on the first coin

D= (HHH, HHT, HTH, HTT)

(i) Mutually exclusive

$A \cap B$ = (HHH) $\cap$ (HHT, THH, HTH)

= φ

Therefore, A and C are mutually exclusive.

$A \cap C$ = (HHH) $\cap$ (TTT)

= φ

There, A and C are mutually exclusive.

$A \cap D$ = (HHH) $\cap$ (HHH, HHT, HTH, HTT)

= (HHH)

$A \cap D \neq \varphi$

So they are not mutually exclusive

$B \cap C$ = (HHT, HTH, THH) $\cap$ (TTT)

= φ

Since there is no common element in B & C, so they are mutually exclusive.

$B \cap D$ = (HHT, THH, HTH) $\cap$ (HHH, HHT, HTH, HTT)

= (HHT, HTH)

$B \cap D \neq \varphi$

There are common elements in B & D.

So, they are not mutually exclusive.

C ∩ D = (TTT) ∩ (HHH, HHT, HTH, HTT)

= φ

There is no common element in C & D.

So they are not mutually exclusive.

(ii) Simple event

If an event has only one sample point of a sample space, it is called a simple (or elementary) event.

A = (HHH)

C = (TTT)

Both A & C have only one element.

So they are simple events.

(iii) Compound events

If an event has more than one sample point, it is called a Compound event.

B= (HHT, HTH, THH)

D= (HHH, HHT, HTH, HTT)

Both B & D have more than one element.

So, they are compound events.

5. Three coins are tossed. Describe

(i) Two events which are mutually exclusive.

(ii) Three events which are mutually exclusive and exhaustive.

(iii) Two events which are not mutually exclusive.

(iv) Two events which are mutually exclusive but not exhaustive.

(v) Three events which are mutually exclusive but not exhaustive.

Solution:-

Since either coin can turn up Head (H) or Tail (T), these are the possible outcomes.

But, now three coins are tossed once, so the possible sample space contains

S= (HHH, HHT, HTH, HTT, THH, THT, TTH, TTT)

(i) Two events which are mutually exclusive.

Let us assume A be the event of getting only head,

A = (HHH)

And also, let us assume B be the event of getting only Tail,

B = (TTT)

So, $A \cap B = \varphi$

Since there is no common element in A& B, these two are mutually exclusive.

(ii) Three events which are mutually exclusive and exhaustive.

Now,

Let us assume P be the event of getting exactly two tails,

P = (HTT, TTH, THT)

Let us assume Q be the event of getting at least two heads,

Q = (HHT, HTH, THH, HHH)

Let us assume R be the event of getting only one tail,

C= (TTT)

P ∩ Q = (HTT, TTH, THT) ∩ (HHT, HTH, THH, HHH)

= φ

There is no common element in P and Q.

Therefore, they are mutually exclusive.

$Q \cap R$ = (HHT, HTH, THH, HHH) $\cap$ (TTT)

= φ

There is no common element in Q and R.

Hence, they are mutually exclusive.

$P \cap R$ = (HTT, TTH, THT) $\cap$ (TTT)

= φ

There is no common element in P and R.

So they are mutually exclusive.

Now, P and Q, Q and R, and P and R are mutually exclusive

$\therefore$ P, Q, and R are mutually exclusive.

And also,

$P \cup Q \cup R$ = (HTT, TTH, THT, HHT, HTH, THH, HHH, TTT) = S

Hence P, Q and R are exhaustive events.

(iii) Two events which are not mutually exclusive.

Let us assume ‘A’ be the event of getting at least two heads,

A = (HHH, HHT, THH, HTH)

Let us assume ‘B’ be the event of getting only head,

B= (HHH)

Now, $A \cap B$ = (HHH, HHT, THH, HTH) $\cap$ (HHH)

= (HHH)

$A \cap B \neq \varphi$

There is a common element in A and B.

So they are not mutually exclusive.

(iv) Two events which are mutually exclusive but not exhaustive.

Let us assume ‘P’ be the event of getting only Head,

P = (HHH)

Let us assume ‘Q’ be the event of getting only tail,

Q = (TTT)

$P \cap Q = (HHH) \cap (TTT)$

$= \varphi$

Since there is no common element in P and Q,

These are mutually exclusive events.

But,

$P \cup Q = (HHH) \cup (TTT)$

= {HHH, TTT}

$P \cup Q \neq S$

Since $P \cup Q \neq S$, these are not exhaustive events.

(v) Three events which are mutually exclusive but not exhaustive.

Let us assume ‘X’ be the event of getting only head,

X = (HHH)

Let us assume ‘Y’ be the event of getting only tail,

Y = (TTT)

Let us assume ‘Z’ be the event of getting exactly two heads,

Z= (HHT, THH, HTH)

Now,

$X \cap Y = (HHH) \cap (TTT)$

$= \varphi$

$X \cap Z = (HHH) \cap (HHT, THH, HTH)$

$= \varphi$

$Y \cap Z = (TTT) \cap (HHT, THH, HTH)$

$= \varphi$

Therefore, they are mutually exclusive.

Also,

$X \cup Y \cup Z = (HHH\ TTT, HHT, THH, HTH)$

$X \cup Y \cup Z \neq S$

So, X, Y and Z are not exhaustive.

Hence, it is proved that X, Y and X are mutually exclusive but not exhaustive.

6. Refer to question 6 above and state true or false. (Give reasons for your answer)

(i) A and B are mutually exclusive

(ii) A and B are mutually exclusive and exhaustive

(iii) A = B^{I}

(iv) A and C are mutually exclusive

(v) A and B^{I} are mutually exclusive.

(vi) A^{I}, B^{I}, C are mutually exclusive and exhaustive.

Solution:-

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible outcomes when the die is thrown.

In the question, it is given that pair of die is thrown, so the sample space will be, By referring the question 6 above,

(i) A and B are mutually exclusive.

So, $(A \cap B) = \varphi$

So, A & B are mutually exclusive.

Hence, the given statement is true.

(ii) A and B are mutually exclusive and exhaustive.

$\Rightarrow A \cup B = S$

From statement (I), we have A and B are mutually exclusive.

∴ A and B are mutually exclusive and exhaustive.

Hence, the statement is true.

(iii) A = B

Therefore, the statement is true.

(iv) A and C are mutually exclusive.

We have,

$A \cap C = \{(2, 1), (2, 2), (2, 3), (4, 1)\}$

$A \cap C \neq \varphi$

A and C are not mutually exclusive.

Hence, the given statement is false.

(v) A and B' are mutually exclusive.

We have,

$A \cap B^{I} = A \cap A = A$

$\therefore A \cap B^{I} \neq \varphi$

So, A and B^{I} are not mutually exclusive.

Therefore, the given statement is false.

They are not mutually exclusive.

B^{I} and C are not mutually exclusive.

Therefore, A', B' and C are not mutually exclusive and exhaustive.

So, the given statement is false.

7. Which of the following cannot be a valid assignment of probabilities for outcomes of sample Space S = {ω_1, ω_2, ω_3, ω_4, ω_5, ω_6, ω_7}

Assignment

Assignment	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
(a)	0.1	0.01	0.05	0.03	0.01	0.2	0.6
(b)	1/7	1/7	1/7	1/7	1/7	1/7	1/7
(c)	0.1	0.2	0.3	0.4	0.5	0.6	0.7
(d)	-0.1	0.2	0.3	0.4	-0.2	0.1	0.3
(e)	1/14	2/14	3/14	4/14	5/14	6/14	15/14

Solution:-

(a) Condition (i): Each of the numbers $p(\omega_i)$ is positive and less than zero. Condition (ii): Sum of probabilities

0.01 + 0.05 + 0.03 + 0.01 + 0.2 + 0.6 = 1

Therefore, the given assignment is valid.

b) Condition (i): Each of the numbers $p(\omega_i)$ is positive and less than zero. Condition (ii): Sum of probabilities

= (1/7) + (1/7) + (1/7) + (1/7) + (1/7) + (1/7) + (1/7)

= 7/7

= 1

Therefore, the given assignment is valid.

c) Condition (i): Each of the numbers $p(\omega_i)$ is positive and less than zero. Condition (ii): Sum of probabilities

= 0.1 + 0.2 + 0.3 + 0.4 + 0.5 + 0.6 + 0.7

= 2.8 > 1

Therefore, the 2^{nd} condition is not satisfied

Which states that $p(w_i) \leq 1$

So, the given assignment is not valid.

d) The conditions of the axiomatic approach don't hold true in the given assignment, that is

1) Each of the numbers $p(w_i)$ is less than zero but also negative.

To be true, each of the numbers $p(w_i)$ should be less than zero and positive.

So, the assignment is not valid.

e) Condition (i): Each of the numbers $p(\omega_i)$ is positive and less than zero. Condition (ii): Sum of probabilities

= (1/14) + (2/14) + (3/14) + (4/14) + (5/14) + (6/14) + (7/14)

= (28/14) ≥ 1

The second condition doesn't hold true, so the assignment is not valid.

8. A coin is tossed twice, what is the probability that at least one tail occurs?

Solution:-

Since either coin can turn up Head (H) or Tail (T), these are the possible outcomes.

Here coin is tossed twice, then the sample space is S = (TT, HH, TH, HT)

∴ Number of possible outcomes n (S) = 4

Let A be the event of getting at least one tail.

∴ n (A) = 3

P(Event) = Number of outcomes favourable to the event/ Total number of possible outcomes

P(A) = n(A)/n(S)

= ¾

9. A die is thrown, find the probability of the following events.

(i) A prime number will appear.

(ii) A number greater than or equal to 3 will appear.

(iii) A number less than or equal to one will appear.

(iv) A number more than 6 will appear.

(v) A number less than 6 will appear.

Solution:-

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible outcomes when the die is thrown.

Here, S = {1, 2, 3, 4, 5, 6}

∴n(S) = 6

(i) A prime number will appear.

Let us assume 'A' be the event of getting a prime number, A = {2, 3, 5}

Then, n(A) = 3

P(Event) = Number of outcomes favourable to the event/ Total number of possible outcomes

∴P(A) = n(A)/n(S)

= 3/6

= ½

(ii) A number greater than or equal to 3 will appear.

Let us assume ‘B’ be the event of getting a number greater than or equal to 3,

B = {3, 4, 5, 6}

Then, n(B) = 4

P(Event) = Number of outcomes favourable to the event/ Total number of possible outcomes

∴P(B) = n(B)/n(S)

= 4/6

= 2/3

(iii) A number less than or equal to one will appear.

Let us assume ‘C’ be the event of getting a number less than or equal to 1,

C = {1}

Then, n (C) = 1

P(Event) = Number of outcomes favourable to the event/ Total number of possible outcomes

∴P(C) = n(C)/n(S)

= 1/6

(iv) A number more than 6 will appear.

Let us assume ‘D’ be the event of getting a number more than 6, then

D = {0)}

Then, n (D) = 0

P(Event) = Number of outcomes favourable to the event/ Total number of possible outcomes

∴P(D) = n(D)/n(S)

= 0/6

= 0

(v) A number less than 6 will appear.

Let us assume ‘E’ be the event of getting a number less than 6, then

E= (1, 2, 3, 4, 5)

Then, n (E) = 5

P(Event) = Number of outcomes favourable to the event/ Total number of possible outcomes

∴P(E) = n(E)/n(S)

= 5/6

10. A card is selected from a pack of 52 cards.

(a) How many points are there in the sample space?

(b) Calculate the probability that the card is an ace of spades.

(c) Calculate the probability that the card is (i) an ace (ii) black card

Solution:-

From the question, it is given that there are 52 cards in the deck.

(a) Number of points in the sample space = 52 (given)

∴n(S) = 52

(b) Let us assume ‘A’ be the event of drawing an ace of spades.

A= 1

Then, n (A) = 1

P(Event) = Number of outcomes favourable to the event/ Total number of possible outcomes

∴P(A) = n(A)/n(S)

= 1/52

(c) Let us assume 'B' be the event of drawing an ace. There are four aces.

Then, n (B)= 4

P(Event) = Number of outcomes favourable to the event/ Total number of possible outcomes

∴P(B) = n(B)/n(S)

= 4/52

= 1/13

(d) Let us assume 'C' be the event of drawing a black card. There are 26 black cards.

Then, n (C) = 26

P(Event) = Number of outcomes favourable to the event/ Total number of possible outcomes

∴P(C) = n(C)/n(S)

= 26/52

= ½

11. A fair coin with 1 marked on one face and 6 on the other and a fair die are both tossed. Find the probability that the sum of numbers that turn up is (i) 3 (ii) 12

Solution:-

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible outcomes when the die is thrown.

So, the sample space S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5),(6, 6)}

Then, n(S) = 12

(i) Let us assume 'P' be the event having the sum of numbers as 3.

P = {(1, 2)},

Then, n (P) = 1

P(Event) = Number of outcomes favourable to the event/ Total number of possible outcomes

∴P(P) = n(P)/n(S)

= 1/12

(ii) Let us assume ‘Q’ be the event having the sum of the number as 12.

Then Q = {(6, 6)}, n (Q) = 1

P(Event) = Number of outcomes favourable to the event/ Total number of possible outcomes

∴P(Q) = n(Q)/n(S)

= 1/12

12. There are four men and six women on the city council. If one council member is selected for a committee at random, how likely is it that it is a woman?

Solution:-

From the question, it is given that there are four men and six women on the city council.

Here total members in the council = 4 + 6 = 10,

Hence, the sample space has 10 points.

∴ n (S) = 10

The number of women is 6 … [given]

Let us assume ‘A’ be the event of selecting a woman.

Then n (A) = 6

P(Event) = Number of outcomes favourable to the event/Total number of possible outcomes

∴P(A) = n(A)/n(S)

= 6/10 … [divide both numerator and denominators by 2]

= 3/5

13. A fair coin is tossed four times, and a person wins Rs 1 for each head and loses Rs 1.50 for each tail that turns up.

From the sample space, calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.

Solution:-

Since either coin can turn up Head (H) or Tail (T), these are the possible outcomes.

But, now coin is tossed four times, so the possible sample space contains,

S = (HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, THHT, HTTH, THTH, TTHH,

TTTH, TTHT, THTT, HTTT, TTTT)

As per the condition given the question, a person will win or lose money depending upon the face of the coin, so

(i) For 4 heads = 1 + 1 + 1 + 1 = ₹ 4

So, he wins ₹ 4.

(ii) For 3 heads and 1 tail = 1 + 1 + 1 – 1.50

= 3 – 1.50

= ₹ 1.50

So, he will be winning ₹ 1.50.

(iii) For 2 heads and 2 tails = 1 + 1 – 1.50 – 1.50

= 2 – 3

= – ₹ 1

So, he will be losing ₹ 1.

(iv)For 1 head and 3 tails = 1 – 1.50 – 1.50 – 1.50

= 1 – 4.50

= – ₹ 3.50

So, he will be losing Rs. 3.50.

(v) For 4 tails = – 1.50 – 1.50 – 1.50 – 1.50

= – ₹ 6

So, he will be losing Rs. 6.

Now, the sample space of amounts is

S= {4, 1.50, 1.50, 1.50, 1.50, – 1, – 1, – 1, – 1, – 1, – 1, – 3.50, – 3.50, – 3.50, – 3.50, – 6}

Then, n (S) = 16

P (winning ₹ 4) = 1/16

P (winning ₹ 1.50) = 4/16 … [divide both numerator and denominator by 4]

= ¼

P (winning ₹ 1) = 6/16 … [divide both numerator and denominator by 2]

= 3/8

P (winning ₹ 3.50) = 4/16 … [divide both numerator and denominator by 4]

= ¼

P (winning ₹ 6) = 1/16

= 3/8

14. Three coins are tossed once. Find the probability of getting

(i) 3 heads (ii) 2 heads (iii) at least 2 heads

(iv) at most 2 heads (v) no head (vi) 3 tails

(vii) Exactly two tails (viii) no tail (ix) at most two tails

Solution:-

Since either coin can turn up Head (H) or Tail (T), these are the possible outcomes.

But, now three coins are tossed, so the possible sample space contains

S = {HHH, HHT, HTH, THH, TTH, HTT, TTT, THT}

Where s is sample space and here n(S) = 8

(i) 3 heads

Let us assume ‘A’ be the event of getting 3 heads.

n(A)= 1

∴P(A) = n(A)/n(S)

= 1/8

(ii) 2 heads

Let us assume ‘B’ be the event of getting 2 heads.

n (A) = 3

∴P(B) = n(B)/n(S)

= 3/8

(iii) at least 2 heads

Let us assume ‘C’ be the event of getting at least 2 heads.

n(C) = 4

∴P(C) = n(C)/n(S)

= 4/8

= ½

(iv) at most 2 heads

Let us assume ‘D’ be the event of getting at most 2 heads.

n(D) = 7

∴P(D) = n(D)/n(S)

= 7/8

(v) no head

Let us assume ‘E’ be the event of getting no heads.

n(E) = 1

∴P(E) = n(E)/n(S)

= 1/8

(vi) 3 tails

Let us assume ‘F’ be the event of getting 3 tails.

n(F) = 1

∴P(F) = n(F)/n(S)

= 1/8

(vii) Exactly two tails

Let us assume ‘G’ be the event of getting exactly 2 tails.

n(G) = 3

∴P(G) = n(G)/n(S)

= 3/8

(viii) no tail

Let us assume ‘H’ be the event of getting no tails.

n(H) = 1

∴P(H) = n(H)/n(S)

= 1/8

(ix) at most two tails

Let us assume ‘I’ be the event of getting at most 2 tails.

n(I) = 7

$\therefore P(I) = n(I)/n(S)$

$= 7/8$

SUMMARY

Probability can be considered as a numerical measure of the likelihood that an event occurs relative to a set of alternative events that do not occur. The set of all possible events must be known.