CHAPTER-VI

APPLICATIONS AND DERIVATIVES

2 MARK QUESTIONS

1:For the given curve: $y = 5x - 2x^3$, when x increases at the rate of 2 units/sec, then how fast is the slope of curve changes when $x = 3$?

Solution:

Given that, $y = 5x - 2x^3$

Then, the slope of the curve, $dy/dx = 5-6x^2$

$\Rightarrow d/dt\ [dy/dx] = -12x.\ dx/dt$

$= -12(3)(2)$

$= -72$ units per second

Hence, the slope of the curve is decreasing at the rate of 72 units per second when x is increasing at the rate of 2 units per second.

2.The total revenue received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$ in rupees. Find the marginal revenue when $x = 5$, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.

Answer:

Marginal Rcvcnuc (MR) = dRdx=ddx$(3x^2 + 36x + 5)$
$= 6x + 36$
$\therefore$ When $x = 5$
Marginai Revenue (MR) $= 6 \times 5 + 36 = 66$

3.The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of its edge is 12 cm?

Answer:

Let x be the length of an edge of the cube, V be the volume and S be the surface area at any time t.

Then, $V = x^3$ and $S = 6x^2$.

It is given that,

$$\frac{dV}{dt} = 8 \text{ cm}^3/\text{sec} \Rightarrow \frac{d}{dt}(x^3) = 8$$

$$\Rightarrow \quad 3x^2 \frac{dx}{dt} = 8 \Rightarrow \frac{dx}{dt} = \frac{8}{3x^2}$$

Now, $S = 6x^2$

$$\Rightarrow \quad \frac{dS}{dt} = 12x \frac{dx}{dt} \Rightarrow \frac{dS}{dt} = 12x \times \frac{8}{3x^2}$$

$$\Rightarrow \quad \frac{dS}{dt} = \frac{32}{x}$$

$$\Rightarrow \left(\frac{dS}{dt}\right)_{x=12} = \frac{32}{12} \text{ cm}^2/\text{sec} = \frac{8}{3} \text{ cm}^2/\text{sec}$$

4.Find the intervals in which the function given by ;
$f(x) = 310\, x^4 - 45x^3 - 3x^2 + 36x5 + 11$ is
(i) strictly increasing.
(ii) strictly decreasing.

Answer:

(i) Strictly increasing in (-2, 1) and (3, ∞).
(ii) Strictly decreasing in (-∞,- 2) and (1, 3).

5.The sides of an equilateral triangle are increasing at the rate of 2 cm/s. Find the rate at which the area increases, when the side is 10 cm? (All India 2014C)

Answer:

$10\sqrt{3}\ cm^2/s$

4 MARK QUESTIONS

1:Show that the function f(x) = tan x – 4x is strictly decreasing on [-π/3, π/3]

Solution:

Given that, $f(x) = \tan x - 4x$

Then, the differentiation of the function is given by:

$f'(x)= \sec^2 x - 4$

When $-\pi/3 < x\ \pi/3$, $1<\sec x <2$

Then, $1<\sec^2 x <4$

Hence, it becomes $-3 < (\sec^2 x-4)<0$

Hence, for $-\pi/3 < x\ \pi/3$, $f'(x)<0$

Therefore, the function "f" is strictly decreasing on [-π/3, π/3]

2:A stone is dropped into a quiet lake and waves move in the form of circles at a speed of 4 cm/sec. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

Solution:

We know that the area of a circle with radius "r" is given by $A = \pi r^2$.

Hence, the rate of change of area "A' with respect to the time "t" is given by:

$dA/dt = (d/dt)\ \pi r^2$

By using the chain rule, we get:

$(d/dr)(\pi r^2).\ (dr/dt) = 2\pi r.(dr/dt)$

It is given that, $dr/dt = 4$ cm/sec

Therefore, when r = 10 cm,

dA/dt = 2π. (10). (4)

dA.dt = 80 π

Hence, when r = 10 cm, the enclosing area is increasing at a rate of 80π cm^2/sec.

3:What is the equation of the normal to the curve y = sin x at (0, 0)?

(a)x =0 (b) y=0 (c)x+y =0 (d)x-y=0

Solution:

A correct answer is an option (c)

Explanation:

Given that, y = sin x

Hence, dy/dx = cos x

Thus, the slope of the normal = $(-1/\cos x)_{x=0}$ = -1

Therefore, the equation of the normal is y-0 = -1(x-0) or x+y=0

Hence, the correct solution is option c.

4.Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on R.

Answer:

We have, $f(x) = 4x^3 - 18x^2 + 27x - 7$
On differentiating both sides w.r.t. x, we get
$f(x) = 12x^2 - 36x + 27$
$\Rightarrow f'(x) = 3(4x^2 - 12x + 9)$
$\Rightarrow f'(x) = 3(2x - 3)^2$

$\Rightarrow f(x) > 0$
$\Rightarrow$ For any $x \in R$, $(2x - 3)^2 > 0$
Since, a perfect square number cannot be negative.
$\therefore$ Given function f(x) is an increasing function on R.

5.Using differentials, find the approximate value of $(3.968)^{3/2}$. (Delhi 2014C)

Answer:

Let $y = f(x) = (x)^{3/2}$
On differentiating both sides w.r.t. x, we get
Let $x = 4$ and $x + \Delta x = 3.968$
Then, $\Delta x = -0.032$
Now, $f(x + \Delta x)^{3/2} \approx f(x) + f'(x)\Delta x$
$(x + \Delta x)^{3/2} \approx (x)^{3/2} + 32.(x)^{1/2}.(-0.032)$
$\Rightarrow (4 - 0.032)^{3/2} \approx (4)^{3/2} + 32(4)^{1/2}(-0.032)$ [put x = 4]
$\Rightarrow (3.968)^{3/2} \approx 8 + 32\ .2.(-0.032)$
$\Rightarrow (39368)^{3/2} \approx 8 - 0.096$
$\Rightarrow (3.968)^{3/2} \approx 7.904$

6.Find the approximate value of f(3.02), upto 2 places of decimal, where $f(x) = 3x^2 + 15x + 3$. (Foreign 2014)

Answer:

First, split 3.02 into two parts x and Ax, so that $x + \Delta x = 3.02$ and $f(x + \Delta x) = f(3.02)$
Now, write $f(x + \Delta x) = f(x) + \Delta x\ .\ f'(x)$ and use this result to find the required value.
Given function is $f(x) = 3x^2 + 15x + 3$
On differentiating both sides w.r.t. x, we get $f'(x) = 6x + 15$
Let $x = 3$ and $\Delta x = 0.02$
So that $f(x + \Delta x) - f(302)$
By using $f(x + \Delta x) \sim f(x) + \Delta x\ f'(x)$, we get
$f(x + \Delta x) = 3x^2 + 15x + 3 + (6x + 15)\ \Delta x$

$f(3 + 0.02) = 3(3)^2 + 15(3) + 3+ [6(3) + 15] (0.02)$
$= 27 + 45 + 3 + 33(0.02)$
$= 75 + 0.66$
$= 75.66$
Hence, $f(3.02) \approx 75.66$

7.If the radius of sphere is measured as t 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area.

Answer:

Let S be the surface area, r be the radius of the sphere.
Given, $r = 9$ cm
Then, dr = Approximate error in radius $r = 0.03$ cm
and dS = Approximate error in surface area
Now, we know that surface area of sphere is given by
$S = 4\pi r^2$
On differentiating both sides w.r.t. r, we get
$dSdr = 4\pi \times 2r = 8\pi r$
$dS = 8\pi r \times dr$
$\Rightarrow dS = 8\pi \times 9 \times 0.03$ [$\because r = 9$ cm and $dr = 0.03$ cm]
$\Rightarrow dS = 72 \times 0.03\pi$
$\therefore dS = 2.16\pi$ cm²/cm
Hence, approximate error in surface area is 2.16π cm²/cm.

7 MARK QUESTIONS

1:Determine all the points of local maxima and local minima of the following function: $f(x) = (-\frac{3}{4})x^4 - 8x^3 - (45/2)x^2 + 105$

Solution:

Given function: $f(x) = (-\frac{3}{4})x^4 - 8x^3 - (45/2)x^2 + 105$

Thus, differentiate the function with respect to x, we get

$f'(x) = -3x^3 - 24x^2 - 45x$

Now take, -3x as common:

$= -3x(x^2 + 8x + 15)$

Factorise the expression inside the bracket, then we have:

$= -3x(x+5)(x+3)$

$f'(x) = 0$

$\Rightarrow x = -5, x = -3, x = 0$

Now, again differentiate the function:

$f''(x) = -9x^2 - 48x - 45$

Take -3 outside,

$= -3(3x^2 + 16x + 15)$

Now, substitue the value of x in the second derivative function.

$f''(0) = -45 < 0$. Hence, $x = 0$ is point of local maxima

$f''(-3) = 18 > 0$. Hence, $x = -3$ is point of local minima

$f''(-5) = -30 < 0$. Hence, $x = -5$ is point of local maxima.

2:A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at a rate of 0.05 cm per second. Find the rate at which its area is increasing if the radius is 3.2 cm.

Solution:

Let us assume that "r" be the radius of the given disc and "A" be the area, then the area is given as:

$A = \pi r^2$

By using the chain rule,

Then $dA/dt = 2\pi r(dr/dt)$

Thus, the approximate rate of increase of radius = $dr = (dr/dt)\, \Delta t = 0.05$ cm per second

Hence, the approximate rate of increase in area is:

$dA = (dA/dt)(\Delta t) = 2\pi r[(dr/dt)\, \Delta t\,]$

$= 2\pi\ (3.2)\ (0.05)$

$= 0.320\pi$ cm^2 per second.

Therefore, when $r= 3.2$ cm, then the area is increasing at a rate of 0.320π cm^2/second.

3.Find the points on the curve
$y = [x(x-2)]^2$, where the tangent is parallel to X-axis.

Answer:

We have to find the points on the given curve where the tangent is parallel to X-axis. We know that, when a tangent is parallel to X-axis, then

dydx = 0

$\Rightarrow$ ddx$(x^2 - 2x)^2 = 0$

$\Rightarrow 2(x^2 - 2x)(2x - 2) = 0$

$\Rightarrow x = 0, 1, 2$

When x = 0, then $y = [0(-2)]^2 = 0$

When x = 1, then $y = [1 - 2(1)]^2 = 1$

When x = 2, then $y = [2^2 - 2 \times 2]^2 = 0$

Hence, the tangent is parallel to X-axis at the points (0, 0), (1, 1) and (2, 0).

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x^2 - 5x + 6)\cdot 1 - (x-7)(2x-5)}{(x^2 - 5x + 6)^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left[(x^2 - 5x + 6) - y\,(x^2 - 5x + 6)(2x-5)\right]}{(x^2 - 5x + 6)^2}$$

$$\left[\begin{array}{l}\because \quad y = \dfrac{x-7}{x^2 - 5x + 6} \\ \therefore (x-7) = y\,(x^2 - 5x + 6)\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 - (2x-5)\,y}{x^2 - 5x + 6} \qquad \text{...(ii)}$$

[dividing numerator and denominator by $x^2 - 5x + 6$]

Also, given that curve cuts X-axis, so its y-coordinate is zero.
Put $y = 0$ in Eq. (i), we get
$x-7x2-5x+6 = 0$
$\Rightarrow x = 7$
So, curve passes through the point (7, 0).
Now, slope of tangent at (7, 0) is
$m = (dydx)(70)=1-049-35+6=120$
Hence, the required equation of tangent passing through the point (7, 0) having slope 1/20 is
$y - o = 120(x - 7)$
$\Rightarrow 20y = x - 7$
$\therefore x - 20y = 7$