CHAPTER-V

CONTINUITY AND DIFFERENTIABILITY

2 MARK QUESTIONS

1:Explain the continuity of the function f(x) = sin x . cos x

Solution:

We know that sin x and cos x are continuous functions. It is known that the product of two continuous functions is also a continuous function.

Hence, the function f(x) = sin x . cos x is a continuous function.

2:Determine the points of discontinuity of the composite function y = f[f(x)], given that, f(x) = 1/x-1.

Solution:

Given that, f(x) = 1/x-1

We know that the function f(x) = 1/x-1 is discontinuous at x = 1

Now, for x ≠1,

f[f(x)]= f(1/x-1)

= 1/[(1/x-1)-1]

= x-1/ 2-x, which is discontinuous at the point x = 2.

Therefore, the points of discontinuity are x = 1 and x=2.

3.Find the values of a and b such that the function defined as follows is continuous.

$$f(x) = \begin{cases} x+2, & x \le 2 \\ ax+b, & 2 < x < 5 \\ 3x-2, & x \ge 5 \end{cases}$$

Answer:
a = 3 and b = − 2

4 MARK QUESTIONS

1:If f (x) = |cos x|, find f'(3π/4)

Solution:

Given that, f(x) = |cos x|

When π/2 <x< π, cos x < 0,

Thus, |cos x| = -cos x

It means that, f(x) = -cos x

Hence, f'(x) = sin x

Therefore, f'(3π/4) = sin (3π/4) = 1/√2

f'(3π/4) = 1/√2

2.Determine the value of 'k' for which the following function is continuous at x = 3:

$$f(x)=\begin{cases}\dfrac{(x+3)^2-36}{x-3}, & x\neq 3\\ k, & x=3\end{cases}$$

Answer:

Given, $f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$

Let $f(x)$ is continuous at $x = 3$

Then, we have $\lim\limits_{x \to 3} f(x) = f(3)$

$\Rightarrow \lim\limits_{x \to 3} \dfrac{(x+3)^2 - 36}{x-3} = k$

$\Rightarrow \lim\limits_{x \to 3} \dfrac{(x+3)^2 - 6^2}{x-3} = k$

$\Rightarrow \lim\limits_{x \to 3} \dfrac{(x+3-6)(x+3+6)}{x-3} = k$

$[\because a^2 - b^2 = (a-b)(a+b)]$

$\Rightarrow \lim\limits_{x \to 3} \dfrac{(x-3)(x+9)}{(x-3)} = k$

$\Rightarrow \lim\limits_{x \to 3} (x+9) = k$

$\Rightarrow 3 + 9 = k \Rightarrow k = 12$

3.

Find the value of k, so that the following function is continuous at x = 2. (Delhi

$$f(x) = \begin{cases} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}, & x \neq 2 \\ k, & x = 2 \end{cases}$$

Answer:

Let $f(x) = \begin{cases} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}; & x \neq 2 \\ k; & x = 2 \end{cases}$

is continuous at $x = 2$.

Now, we have $f(2) = k$

and $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}$

$$= \lim_{x \to 2} \frac{(x-2)(x^2 + 3x - 10)}{(x-2)^2}$$

$$= \lim_{x \to 2} \frac{(x-2)(x+5)(x-2)}{(x-2)^2}$$

$$= \lim_{x \to 2} (x+5) = 2 + 5 = 7$$

Since, $f(x)$ is continuous at $x = 2$

$\therefore \lim\limits_{x \to 2} f(x) = f(2) \Rightarrow 7 = k \quad \Rightarrow k = 7$

4.

Find the value of k, so that the function f defined by

$$f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$$

is continuous at x = π2. (Delhi 2012C; Foregin 2011)

Answer:

Let $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$

is continuous at $x = \pi/2$

Then, at $x = \dfrac{\pi}{2}$, LHL = RHL = $f\left(\dfrac{\pi}{2}\right)$ ...(i)

Now, LHL = $\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \dfrac{k \cos x}{\pi - 2x}$

$$\Rightarrow \quad \text{LHL} = \lim_{h \to 0} \frac{k \cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)}.$$

$\left[\text{put } x = \dfrac{\pi}{2} - h; \text{ when } x \to \dfrac{\pi^-}{2}, \text{ then } h \to 0\right]$

$$= \lim_{h \to 0} \frac{k \sin h}{\pi - \pi + 2h} \quad \left[\because \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta\right]$$

$$= \lim_{h \to 0} \frac{k \sin h}{2h} = \frac{k}{2} \lim_{h \to 0} \frac{\sin h}{h}$$

$$\Rightarrow \text{LHL} = \frac{k}{2} \quad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

Also, from the given function, we get

$$f\left(\frac{\pi}{2}\right) = 3$$

Now, from Eq. (i), we have

$$\text{LHL} = f\left(\frac{\pi}{2}\right) \Rightarrow \frac{k}{2} = 3$$

$$\therefore \quad k = 6$$

7 MARK QUESTIONS

1:Verify the mean value theorem for the following function f (x) = (x – 3) (x – 6) (x – 9) in [3, 5]

Solution:

$f(x)=(x-3)(x-6)(x-9)$

$=(x-3)(x^2-15x+54)$

$=x^3-18x^2+99x-162$

$fc\in(3,5)$

$f'(c)=f(5)-f(3)/5-3$

$f(5)=(5-3)(5-6)(5-9)$

$=2(-1)(-4)=-8$

$f(3)=(3-3)(3-6)(3-9)=0$

$f'(c)=8-0/2=4$

$\therefore f'(c)=3c^2-36c+99$

$3c^2-36c+99=4$

$3c^2-36c+95=0$

$ax^2+bx+c=0$

$a=3$

$b=-36$

$c=95$

$c=36\pm\sqrt{(36)^2-4(3)(95)}/2(3)$

$=36\pm\sqrt{1296-1140}/6$

$=36\pm12.496$

$c=8.8$ & $c=4.8$

$c\in(3,5)$

$f(x)=(x-3)(x-6)(x-9)$ on $[3,5]$

2.Explain the continuity of the function $f = |x|$ at $x = 0$.

Solution:

From the given function, we define that,

$f(x) = \{-x$, if $x<0$ and x, if $x\geq0$

It is clearly mentioned that the function is defined at 0 and $f(0) = 0$. Then the left-hand limit of f at 0 is

$\lim_{x\to0^-} f(x)= \lim_{x\to0^-} (-x) = 0$

Similarly for the right hand side,

$\lim_{x\to0^+} f(x)= \lim_{x\to0^+} (x) = 0$

Therefore, for the both left hand and the right hand limit, the value of the function coincide at the point $x = 0$.

Therefore, the function f is continuous at the point $x =0$.

3:If $y= \tan x + \sec x$, then show that $d^2y / dx^2 = \cos x / (1-\sin x)^2$

Solution:

Given that, $y= \tan x + \sec x$

Now, the differentiate wih respect to x, we get

$dy/dx = \sec^2 x + \sec x \tan x$

$= (1/ \cos^2 x) + (\sin x/ \cos^2 x)$

$= (1+\sin x)/ (1+\sin x)(1-\sin x)$

Thus, we get.

$dy/dx = 1/(1-\sin x)$

Now, again differentiate with respect to x, we will get

$d^2y / dx^2 = -(-\cos x)/(1- \sin x)^2$

$d^2y / dx^2 = \cos x / (1-\sin x)^2$.

4.Find the values of p and q for which

$$f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x<\dfrac{\pi}{2}\\ p, & \text{if } x=\dfrac{\pi}{2}\\ \dfrac{q\,(1-\sin x)}{(\pi-2x)^2}, & \text{if } x>\dfrac{\pi}{2}\end{cases}$$

is continuous at $x = \pi 2$. (Delhi 2016)
Answer:

Let $f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x<\dfrac{\pi}{2}\\ p, & \text{if } x=\dfrac{\pi}{2}\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & \text{if } x>\dfrac{\pi}{2}\end{cases}$

is continuous at $x=\dfrac{\pi}{2}$.

Then, $(\text{LHL})_{x=\frac{\pi}{2}}=(\text{RHL})_{x=\frac{\pi}{2}}=f\left(\dfrac{\pi}{2}\right)$ ...(i)

Now, $\text{LHL}=\lim\limits_{x\to\frac{\pi}{2}^-} f(x)=\lim\limits_{h\to 0} f\left(\dfrac{\pi}{2}-h\right)$

$\left[\text{put } x=\dfrac{\pi}{2}-h;\text{ when } x\to\dfrac{\pi^-}{2},\text{ then } h\to 0\right]$

$$=\lim_{h\to 0}\frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)}=\lim_{h\to 0}\frac{1-\cos^3 h}{3\sin^2 h}$$

$\left[\because \cos\left(\dfrac{\pi}{2}-\theta\right)=\sin\theta,\ \sin\left(\dfrac{\pi}{2}-\theta\right)=\cos\theta\right]$

$$=\lim_{h\to 0}\frac{(1-\cos h)(1^2+\cos^2 h+1\times\cos h)}{3(1-\cos^2 h)}$$

$$=\lim_{h\to 0}\frac{(1-\cos h)(1+\cos^2 h+\cos h)}{3(1-\cos h)(1+\cos h)}$$

$$=\lim_{h\to 0}\frac{(1+\cos^2 h+\cos h)}{3(1+\cos h)}$$

$$=\frac{1+\cos^2 0+\cos 0}{3(1+\cos 0)}=\frac{1+1+1}{3(1+1)}=\frac{3}{3\times 2}=\frac{1}{2}$$

...(ii)

and $\text{RHL}=\lim\limits_{x\to\frac{\pi}{2}^+} f(x)=\lim\limits_{h\to 0} f\left(\dfrac{\pi}{2}+h\right)$

$\left[\text{put } x=\dfrac{\pi}{2}+h;\text{ when } x\to\dfrac{\pi^+}{2},\text{ then } h\to 0\right]$

$$=\lim_{h\to 0}\frac{q\left[1-\sin\left(\frac{\pi}{2}+h\right)\right]}{\left[\pi-2\left(\frac{\pi}{2}+h\right)\right]^2}$$

$$=\lim_{h\to 0}\frac{q(1-\cos h)}{(\pi-\pi-2h)^2}=\lim_{h\to 0}\frac{q(1-\cos h)}{4h^2}$$

$$= \frac{q}{8} \lim_{h \to 0} \left[\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right]^2$$

$$= \frac{q}{8} \times 1 = \frac{q}{8} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right] \ldots\text{(iii)}$$

On substituting the values from Eqs. (ii) and (iii) to Eq. (i), we get

$$\frac{1}{2} = \frac{q}{8} = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \qquad \frac{1}{2} = \frac{q}{8} = p \qquad \left[\because f\left(\frac{\pi}{2}\right) = p \text{ (given)}\right]$$

$$\Rightarrow \qquad \frac{1}{2} = \frac{q}{8} \text{ and } \frac{1}{2} = p$$

$$\therefore \qquad q = 4 \text{ and } p = \frac{1}{2}$$

5.Find the value of k, so that the function

$$f(x) = \begin{cases} \left(\dfrac{1 - \cos 4x}{8x^2}\right), & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

is continuous at x = 0. (All India 2014C).

Let $f(x) = \begin{cases} \left(\dfrac{1-\cos 4x}{8x^2}\right), & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$

is continuous at $x = 0$.

Then, $(\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0)$ ...(i)

Now, $\text{LHL} = \lim_{x \to 0^-} f(x)$

$$= \lim_{x \to 0^-} \frac{1-\cos 4x}{8x^2}$$

$$= \lim_{h \to 0} \frac{1-\cos 4(0-h)}{8(0-h)^2}$$

[put $x = 0-h$; when $x \to 0^-$, then $h \to 0$]

$$= \lim_{h \to 0} \frac{1-\cos 4h}{8h^2} \quad [\because \cos(-\theta) = \cos\theta]$$

$$= \lim_{h \to 0} \frac{2\sin^2 2h}{8h^2} \quad [\because 1-\cos 2\theta = 2\sin^2\theta]$$

$$= \lim_{h \to 0} \frac{\sin^2 2h}{4h^2} = \lim_{h \to 0} \left(\frac{\sin 2h}{2h}\right)^2 = 1$$

$$\left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

On substituting this value in Eq. (i), we get

$$1 = f(0) \Rightarrow 1 = k \quad [\because f(0) = k, \text{(given)}]$$

Hence, for $k = 1$, the given function $f(x)$ is continuous at $x = 0$.

Answer:

Let $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0\\ \dfrac{2x+1}{x-1}, & \text{if } 0\le x<1\end{cases}$

is continuous at $x = 0$.

Now, $f(0)=\dfrac{2\cdot 0+1}{0-1}=\dfrac{1}{-1}=-1$

and $\text{LHL}=\lim_{h\to 0} f(0-h)$

$$=\lim_{h\to 0}\frac{\sqrt{1-kh}-\sqrt{1+kh}}{-h}$$

$$=\lim_{h\to 0}\frac{\sqrt{1-kh}-\sqrt{1+kh}}{-h}\times\frac{(\sqrt{1-kh}+\sqrt{1+kh})}{(\sqrt{1-kh}+\sqrt{1+kh})}$$

$$=\lim_{h\to 0}\frac{(1-kh)-(1+kh)}{-h(\sqrt{1-kh}+\sqrt{1+kh})}$$

$[\because (a+b)(a-b)=a^2-b^2]$

$$=\lim_{h\to 0}\frac{-2kh}{-h(\sqrt{1-kh}+\sqrt{1+kh})}$$

$$=\lim_{h\to 0}\frac{2k}{\sqrt{1-kh}+\sqrt{1+kh}}=\frac{2k}{1+1}=\frac{2k}{2}=k$$

Since, $f(x)$ is continuous at $x = 0$.

$\therefore \quad f(0)=\text{LHL} \Rightarrow -1=k$

$\Rightarrow \quad k=-1$