CHAPTER-IX

DIFFERENTIAL EQUATIONS

2 MARK QUESTIONS

1: Find the differential equation of the family of lines through the origin.

Solution:

Let $y = mx$ be the family of lines through the origin.

Therefore, $dy/dx = m$

Eliminating m, (substituting $m = y/x$)

$y = (dy/dx) . x$

or

$x. dy/dx - y = 0$

2.Find the differential equation of the family of circles in the first quadrant which touch the coordinate axes.

Answer:

Equation of family of circles in the first quadrant which touch the coordinate axes is $(x - a)^2 + (y - a)^2 = a^2$
$(x - y)^2 [(y')^2 + 1] = (x + yy')^2$

3.Solve the differential equation
cos(dydx) = a,($a \in R$)

Answer:

Given equation is cos(dydx) = a
which can be rewritten as dydx = $\cos^{-1}a$
⇒ dy = $\cos^{-1}a$
⇒ $\int dy = \int \cos^{-1}a \, dx$
⇒ $y = \cos^{-1}a.\ x + C$
which is the required solution.

4 MARK QUESTIONS

1: Determine order and degree (if defined) of differential equation $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$

Solution:

Given differential equation is $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$

The highest order derivative present in the differential equation is y'''.

Therefore, its order is 3.

The given differential equation is a polynomial equation in y''', y'', and y'.

The highest power raised to y''' is 2.

Hence, its degree is 2.

2: Verify that the function $y = a \cos x + b \sin x$, where, $a, b \in R$ is a solution of the differential equation $d^2y/dx^2 + y=0$.

Solution:

The given function is $y = a \cos x + b \sin x$ ... (1)

Differentiating both sides of equation (1) with respect to x,

$dy/dx = - a \sin x + b \cos x$

$d^2y/dx^2 = - a \cos x - b \sin x$

$LHS = d^2y/dx^2 + y$

$= - a \cos x - b \sin x + a \cos x + b \sin x$

$= 0$

$= RHS$

Hence, the given function is a solution to the given differential equation.

3: The number of arbitrary constants in the general solution of a differential equation of fourth order is:

(A) 0 (B) 2 (C) 3 (D) 4

Solution:

We know that the number of constants in the general solution of a differential equation of order n is equal to its order.

Therefore, the number of constants in the general equation of the fourth-order differential equation is four.

Hence, the correct answer is D.

Note: The number of constants in the general solution of a differential equation of order n is equal to zero.

4: Form the differential equation representing the family of curves y = a sin (x + b), where a, b are arbitrary constants.

Solution:

Given,

$y = a \sin (x + b)$ … (1)

Differentiating both sides of equation (1) with respect to x,

$dy/dx = a \cos (x + b)$ … (2)

Differentiating again on both sides with respect to x,

$d^2y/dx^2 = -a \sin (x + b)$ … (3)

Eliminating a and b from equations (1), (2) and (3),

$d^2y/dx^2 + y = 0$ ... (4)

The above equation is free from the arbitrary constants a and b.

This the required differential equation.

5: Form the differential equation of the family of circles having a centre on y-axis and radius 3 units.

Solution:

The general equation of the family of circles having a centre on the y-axis is $x^2 + (y - b)^2 = r^2$

Given the radius of the circle is 3 units.

The differential; equation of the family of circles having a centre on the y-axis and radius 3 units is as below:

$x^2 + (y - b)^2 = 3^2$

$x^2 + (y - b)^2 = 9$ ......(i)

Differentiating (i) with respect to x,

$2x + 2(y - b).y' = 0$

$\Rightarrow (y - b).\ y' = -x$

$\Rightarrow (y - b) = -x/y'$ ....(ii)

Substituting (ii) in (i),

$x^2 + (-x/y')^2 = 9$

$\Rightarrow x^2[1 + 1/(y')^2] = 9$

$\Rightarrow x^2\ [(y')^2 + 1) = 9\ (y')^2$

$\Rightarrow (x^2 - 9)\ (y')^2 + x^2 = 0$

Hence, this is the required differential equation.

6: Find the general solution of the differential equation $dy/dx = 1+y^2/1+x^2$.

Solution:

Given differential equation is $dy/dx = 1+y^2/1+x^2$

Since $1 + y^2 \neq 0$, therefore by separating the variables, the given differential equation can be written as:

$dy/1+y^2 = dx/1+x^2$ ....(i)

Integrating equation (i) on both sides,

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$\tan^{-1}y = \tan^{-1}x + C$

This is the general solution of the given differential equation.

7 MARK QUESTIONS

1: For each of the given differential equation, find a particular solution satisfying the given condition:

dy/dx = y tan x ; y = 1 when x = 0

Solution:

dy/dx = y tan x

dy/y = tan x dx

Integrating on both sides,

$$\int \frac{dy}{y} = \int tan\, x\, dx$$

log y = log (sec x) +C

log y = log (C sec x)

⇒ y = C sec x ……..(i)

Now consider y = 1 when x= 0.

1 = C sec 0

1 = C (1)

C = 1

Substituting C = 1 in (i)

y = sec x

Hence, this is the required particular solution of the given differential equation.

2: Find the equation of a curve passing through $(1, \pi/4)$ if the slope of the tangent to the curve at any point P (x, y) is $y/x - \cos^2(y/x)$.

Solution:

According to the given condition,

$dy/dx = y/x - \cos^2(y/x)$ ………….(i)

This is a homogeneous differential equation.

Substituting $y = vx$ in (i),

$v + (x)\, dv/dx = v - \cos^2 v$

$\Rightarrow (x) dv/dx = -\cos^2 v$

$\Rightarrow \sec^2 v\, dv = -\, dx/x$

By integrating on both the sides,

$\Rightarrow \int \sec^2 v\, dv = -\int dx/x$

$\Rightarrow \tan v = -\log x + c$

$\Rightarrow \tan (y/x) + \log x = c$ ……….(ii)

Substituting $x = 1$ and $y = \pi/4$,

$\Rightarrow \tan (\pi/4) + \log 1 = c$

$\Rightarrow 1 + 0 = c$

$\Rightarrow c = 1$

Substituting $c = 1$ in (ii),

$\tan (y/x) + \log x = 1$

3: Integrating factor of the differential equation $(1 - x^2)dy/dx - xy = 1$ is

(A) -x

(B) $x/(1 + x^2)$

(C) $\sqrt{(1 - x^2)}$

(D) ½ $\log(1 - x^2)$

Solution:

Given differential equation is $(1 - x^2)dy/dx - xy = 1$

$(1 - x^2)dy/dx = 1 + xy$

$dy/dx = (1/1 - x^2) + (x/1 - x^2)y$

$dy/dx - (x/1 - x^2)y = 1/1\text{-}x^2$

This is of the form $dy/dx + Py = Q$

We can get the integrating factor as below:

$$I.F = e^{\int P\,dx}$$

$$= e^{\int \frac{-x}{1-x^2} dx}$$

Let $1 - x^2 = t$

Differentiating with respect to x

$-2x\,dx = dt$

$-x\,dx = dt/2$

Now,

$$I.F = e^{\int \frac{dt}{2t}}$$

$$= e^{\frac{1}{2}\int \frac{dt}{t}}$$

$$= e^{\frac{1}{2}\log t}$$

$$= e^{\log \sqrt{t}}$$

I.F = $\sqrt{t} = \sqrt{(1- x^2)}$

Hence, option C is the correct answer.

4.Solve the differential equation
$(1 + x)^2 + 2xy - 4x^2 = 0$, subject to the initial condition $y(0) = 0$.

Answer:

Given differential equation is
$(1 + x)^2 + 2xy - 4x^2 = 0$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{4x^2}{1+x^2}$$

which is the equation of the form

$$\frac{dy}{dx} + Py = Q,$$

where $P = \frac{2x}{1+x^2}$ and $Q = \frac{4x^2}{1+x^2}$

Now, $\text{IF} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = 1 + x^2$

The general solution is

$$y \cdot (1 + x^2) = \int (1 + x^2) \frac{4x^2}{(1 + x^2)} dx + C$$

$$\Rightarrow \quad (1 + x^2)\, y = \int 4x^2 dx + C$$

$$\Rightarrow \quad (1 + x^2)\, y = \frac{4x^3}{3} + C$$

$$\Rightarrow \quad y = \frac{4x^3}{3(1 + x^2)} + C(1 + x^2)^{-1} \quad \text{...(i)}$$

Now, $y(0) = 0$

$$\Rightarrow \quad 0 = \frac{4 \cdot 0^3}{3(1 + 0^2)} + C(1 + 0^2)^{-1} \Rightarrow C = 0$$

Put the value of C in Eq. (i), we get

$$y = \frac{4x^3}{3(1 + x^2)},$$

which is the required solution.

5.

Solve the following differential equation.

x dydx = y – x tan(yx). (All India 2019)

Answer:

Given differential equation is

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right) \Rightarrow \frac{dy}{dx} = \frac{y - x\tan\left(\frac{y}{x}\right)}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right) \quad \ldots\text{(i)}$$

which is a homogeneous differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i),

we get

$$v + x\frac{dv}{dx} = v - \tan v \Rightarrow x\frac{dv}{dx} = -\tan v$$

$$\Rightarrow \quad \frac{dv}{\tan v} = -\frac{dx}{x}$$

$$\Rightarrow \quad \cot v\, dv = -\frac{dx}{x} \quad \left[\because \frac{1}{\tan v} = \cot v\right]$$

On integrating both sides, we get

$$\int \cot v\, dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \log|\sin v| = -\log|x| + C$$

$$[\because \int \cot v\, dv = \log|\sin v|]$$

$$\Rightarrow \quad \log|\sin v| + \log|x| = C$$

$$\Rightarrow \quad \log|x\sin v| = C$$

$$[\because \log m + \log n = \log mn]$$

$$\therefore \quad \log\left|x\sin\frac{y}{x}\right| = C \quad \left[\text{put } v = \frac{y}{x}\right]$$

$$\Rightarrow x\sin\frac{y}{x} = e^C$$

$$\Rightarrow x\sin\frac{y}{x} = A \quad [\because e^C = A]$$

$$\Rightarrow \sin\frac{y}{x} = \frac{A}{x} \Rightarrow y = x\sin^{-1}\left(\frac{A}{x}\right).$$

which is the required solution.

6.Solve the differential equation. (All India 2019)
dydx=−[x+ycosx1+sinx]

Answer:

Given, $\frac{dy}{dx} = -\frac{x}{1+\sin x} - \frac{y\cos x}{1+\sin x}$

or $\frac{dy}{dx} + \frac{y\cos x}{1+\sin x} = -\frac{x}{1+\sin x}$ ...(i)

which is in the linear form, $\frac{dy}{dx} + Py = Q$, where

$$P = \frac{\cos x}{1+\sin x}, Q = -\frac{x}{1+\sin x}$$

Now, $\text{IF} = e^{\int \frac{\cos x}{1+\sin x} dx} = e^{\log(1+\sin x)} = 1 + \sin x$

and the general solution is

$$y(1+\sin x) = \int -x\, dx + C$$

$$[\because y\cdot(\text{IF}) = \int Q\cdot(\text{IF})\, dx + C]$$

$\Rightarrow y(1+\sin x) = -\frac{x^2}{2} + C$

7.Solve the following differential equation
cosec x log |y| dydx + x^2y^2 = 0.

Answer:

First, separate the variables, then integrate by using integration by parts.
Given differential equation is
cosec x log |y| dydx + x^2y^2 = 0

$$\operatorname{cosec} x \log|y| \frac{dy}{dx} = -x^2y^2$$

On separating the variables, we get

$$\frac{\log|y|}{y^2}\,dy = \frac{-x^2}{\operatorname{cosec} x}\,dx$$

On integrating both sides, we get

$$\int \frac{\log|y|}{y^2}\,dy = -\int \frac{x^2}{\operatorname{cosec} x}\,dx$$

$\Rightarrow$ $\qquad I_1 = -I_2 \qquad$ ...(ii)

where, $I_1 = \int \frac{\log|y|}{y^2}\,dy$

and $\quad I_2 = \int \frac{x^2}{\operatorname{cosec} x}\,dx = \int x^2 \sin x\,dx$

Consider, $I_1 = \int \frac{\log|y|}{y^2}\,dy$

Put $\quad \log y = t \Rightarrow y = e^t$, then $\frac{dy}{y} = dt$

$\therefore\ I_1 = \int \underset{\text{I}}{t}\,\underset{\text{II}}{e^{-t}}\,dt = t\int e^{-t}\,dt - \int\left[\frac{d}{dt}(t)\int e^{-t}\,dt\right]dt$

[using integration by parts]

$$= -t\,e^{-t} - \int(-e^{-t})\,dt$$

$$= -t\,e^{-t} + \int e^{-t}dt = -t\,e^{-t} - e^{-t} + C_1$$

$$= -\frac{\log|y|}{y} - \frac{1}{y} + C_1 \qquad \text{...(iii)}$$

$\left[\because t = \log|y| \text{ and } e^{-t} = \frac{1}{y}\right]$

and $I_2 = \int \underset{\text{I}}{x^2}\,\underset{\text{II}}{\sin x}\,dx$

$$= x^2\int \sin x\,dx - \int\left[\frac{d}{dx}(x^2)\int \sin x\,dx\right]dx$$

[using integration by parts]

$$= x^2(-\cos x) - \int[2x(-\cos x)]\,dx$$

$$= -x^2\cos x + 2\int \underset{\text{I}}{x}\,\underset{\text{II}}{\cos x}\,dx$$

$$= -x^2\cos x + 2\left[x\int \cos x\,dx\right.$$

$$\left.-\int\left\{\frac{d}{dx}(x)\int \cos x\,dx\right\}dx\right.$$

$= - x^2 \cos x + 2 [x \sin x - \int \sin x \, dx]$
$= - x^2 \cos x + 2x \sin x + 2 \cos x + C_2$ ...(iv)

On putting the values of I_1 and I_2 from Eqs.(iii) and (iv) in Eq. (ii), we get

$$-\frac{\log|y|}{y} - \frac{1}{y} + C_1 = x^2 \cos x - 2x \sin x - 2\cos x - C_2$$

$$\Rightarrow \quad -\frac{(1 + \log|y|)}{y} = x^2 \cos x - 2x \sin x - 2\cos x - C_2 - C_1$$

$$\Rightarrow \quad -\frac{(1 + \log|y|)}{y} = x^2 \cos x - 2x \sin x - 2\cos x + C$$

where, $C = - C_2 - C_1$
which is the required solution of given differential equation.

8.Solve the following differential equation
$2x^2$ dydx – 2xy + y^2 = 0.

Answer:
Given differential equation is

$$2x^2\frac{dy}{dx} - 2xy + y^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{2x^2} \quad \ldots(i)$$

which is a homogeneous differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i),

we get

$$\Rightarrow v + x\frac{dv}{dx} = v - \frac{v^2}{2} \Rightarrow x\frac{dv}{dx} = \frac{-v^2}{2} \Rightarrow \frac{2dv}{v^2} = -\frac{1}{x}dx \quad \textbf{(1)}$$

On integrating both sides, we get

$$2\int v^{-2}dv = -\log|x| + C$$

$$\Rightarrow \quad \frac{2v^{-1}}{-1} = -\log|x| + C$$

$$\Rightarrow \quad \frac{-2}{v} = -\log|x| + C$$

$$\Rightarrow \quad \frac{-2x}{y} = -\log|x| + C \qquad \left[\text{put } v = \frac{y}{x}\right]$$

$$\Rightarrow \quad -2x = y(-\log|x| + C)$$

$$\therefore \quad y = \frac{-2x}{-\log|x| + C}$$