Lecture 25: Segmentation: Level Sets and Graph Cut

Learning Objectives:

- Introduce new topological methods
    - Level sets

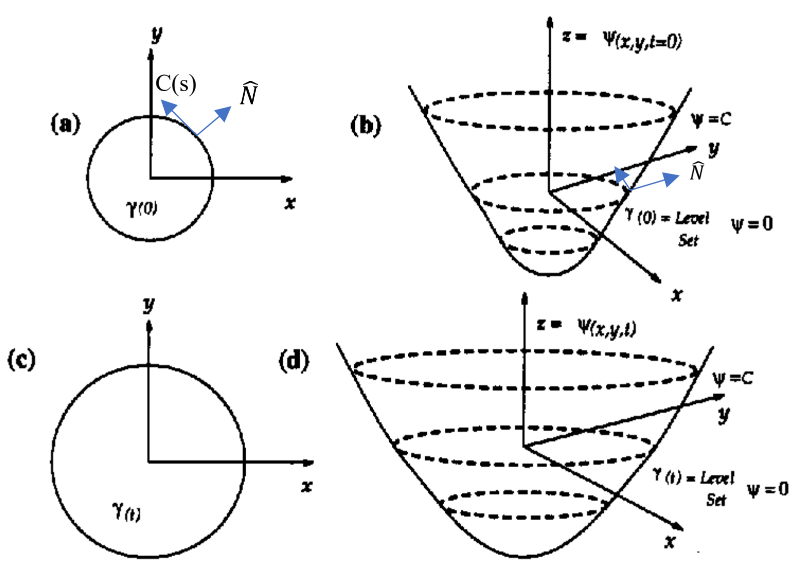
References:

1. Osher S, Paragios N. Geometric Level Set Methods in Imaging, Vision, and Graphics, Springer 2003 (https://www.springer.com/us/book/9780387954882).
2. Professor Herve Lombaert, Canada Research Chair, Montreal ETS, Research Site: https://profs.etsmtl.ca/hlombaert/levelset/

$$\varphi(C(x,y;t),t)$$

$$\varphi_s = \frac{\partial \varphi}{\partial s} = 0 = \frac{\partial \varphi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \varphi}{\partial y}\frac{\partial y}{\partial s} = \nabla\varphi \cdot C(s)$$

$$\hat{N} = \frac{\nabla\varphi}{|\nabla\varphi|}$$



$$\varphi(C(x,y;t),t) = 0 \text{ at the level set.}$$

We start with the time derivative to estimate the function $\varphi(C(x,y;t),t)$.

Start with the time derivative:

$$\frac{\partial\varphi(C(x,y;t),t)}{\partial t} = 0$$

Chain rule:

$$\frac{\partial\varphi(C(t))}{\partial t} = \frac{\partial\varphi}{\partial C}\frac{\partial C}{\partial t} + \frac{\partial\varphi}{\partial t}\frac{\partial t}{\partial t} = 0$$

Defining terms and simplifying:

$$\underbrace{\frac{\partial\varphi}{\partial C(x,y;t)}}_{\nabla\varphi} + \underbrace{\frac{\partial C(x,y;t)}{\partial t}}_{\text{Speed}} + \underbrace{\frac{\partial\varphi}{\partial t}}_{\text{Time Derivative}} = 0$$

We can model the speed as force, $F \cdot \widehat{N} = F \cdot \frac{\nabla\varphi}{|\nabla\varphi|}$, that is the normal to the surface $\varphi(C(x,y;t))$

$\frac{\partial\varphi}{\partial t} = -\nabla\varphi \cdot F \cdot \widehat{N} = -\nabla\varphi \cdot \frac{\nabla\varphi}{|\nabla\varphi|}F = -|\nabla\varphi|F$

Therefore, the level set equation is stated succinctly as:

$\frac{\partial\varphi}{\partial t} = -|\nabla\varphi|F,$

Where F is a force normal to the surface $\varphi(C(x,y;t))$ that acts to deform the function in time.

The degree of change can be modified based on the curvature, which is expressed as:

$\kappa = \nabla\frac{\nabla\varphi}{|\nabla\varphi|} = \frac{\varphi_{xx}{\varphi_y}^2 - 2\varphi_{xy}\varphi_x\varphi_y + \varphi_{yy}{\varphi_x}^2}{({\varphi_x}^2+{\varphi_y}^2)^{3/2}},$

which can be used as a constraint on the implementation of the function.

Implementation as a finite difference equation:

$$\frac{\varphi(i,j,t+\Delta t) - \varphi(i,j,t)}{\Delta t} + \max[F,0]\,\nabla^{+C}(i,j) + \min[F,0]\,\nabla^{-C}(i,j) = 0$$

Updating the surface:

$$\varphi(i,j,t+\Delta t) = \varphi(i,j,t) - \Delta t[\max[F,0]\,\nabla^{+C}(i,j) + \min[F,0]\,\nabla^{-C}(i,j)]$$

Updating with curvature term included:

$$\varphi(i,j,t+\Delta t) = \varphi(i,j,t) - \Delta t[\max[F,0]\,\nabla^{+C}(i,j) + \min[F,0]\,\nabla^{-C}(i,j)] + \Delta t[\kappa(i,j)].$$

As can be seen with careful study, the curve is updated based on the strength of the gradient weighted by the force acting either inward or outward on the surface. The curvature term can be used to penalize solutions with high curvature.

The choice of the force model is important; most common choices weight the force based on the strength of the gradient.

$$F(i,j) = \frac{1}{1+\lambda|\nabla I(i,j)|}$$

$$F(i,j) = e^{-\frac{|\nabla I(i,j)|^2}{2\sigma^2}}$$

These functions are large when far from a large gradient and small when near a large gradient. Here λ can act as a weighting to determine rate of convergence.

Summary:

- Level sets have the advantage of being adaptive to multiple objects based on physical model and optimization with respect to an application specific cost function.
    - Particularly strong for automated segmentation of time-dependent functions.