Homework 1

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Problem 1

A. Prove that the ℓ_1 norm is a valid vector norm.

B. Prove that the ℓ_0 'metric' is not a valid vector norm.

Note: Remember that the ℓ_0 'metric' of a vector is defined as the number of non-zero entries in said vector.

C. Prove that any valid vector norm $\|\cdot\|$ is a convex function.

Note: In other words, prove that the function $f : \mathbb{R}^N \rightarrow R$ defined as $f(\mathbf{x}) \in \|\mathbf{x}\|$, where $\|\mathbf{x}\|$ calculates a given norm of $\mathbf{x}$ (for instance, the ℓ_2 aka Euclidean norm), is a convex function. This will be very important for optimization problems, as these problems (both the objective function as well as the inequality constraints) are often written as norms.

D. Prove that the set $\|\mathbf{x}\|_2 \leq b$, for some $b > 0$, is a convex set within $\mathbb{R}^N$.

E. Plot or draw the following sets for a two-dimensional vector $\mathbf{x} \in \mathbb{R}^2$

i. $\|\mathbf{x}\|_\infty < 1$ (where $\|\mathbf{x}\|_\infty = \max_n |x_n|$
ii. $\|\mathbf{x}\|_2 < 1$
iii. $\|\mathbf{x}\|_1 < 1$
iv. $\|\mathbf{x}\|_{\frac{1}{2}} < 1$ (note that this one is not a norm, but we can define and plot the set just as well using the expression $\|\mathbf{x}\|_{\frac{1}{2}} = (\sum_n |x_n|^{\frac{1}{2}})^2$).

Feel free to do this analytically (eg: drawing by hand for easy shapes), or computationally. For each of these plots, graphically indicate whether the corresponding set is convex or not (ie: if the set is convex, the line segment that joins any two points within the set will be completely inside the set, whereas if the set is non-convex you will be able to find two points within the set such that the segment that joins them is not completely within the set).