MEDICAL PHYSICS / BIOMEDICAL ENGINEERING 574

Homework Set 1 - SOLUTIONS

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Due date: Friday, February 15, 2019
(Due by the end of the day)

1. (25 points) Vector Norms and Convexity

A. (5 points) Prove that the ℓ_1 norm is a valid vector norm.

SOLUTION: As covered in class, valid norms need to satisfy the following conditions:

1. A norm $\|\mathbf{x}\|$ is always ≥ 0, with $\|\mathbf{x}\| = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$
2. $\|c\mathbf{x}\| = |c|\|\mathbf{x}\|$ for any scalar c
3. $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)

So let us see if the ℓ_1 norm ($\|\mathbf{x}\|_1 = \sum_n |x_n|$) is a valid norm.

1. $\|\mathbf{x}\|_1 = \sum_n |x_n| \geq \sum_n 0 = 0$, with equality if and only if $x_1 = x_2 = \cdots = x_N = 0$, ie, if and only if $\mathbf{x} = \mathbf{0}$
2. $\|c\mathbf{x}\|_1 = \sum_n |cx_n| = |c| \sum_n |x_n| = |c|\|\mathbf{x}\|_1$
3. $\|\mathbf{x}+\mathbf{y}\|_1 = \sum_n |x_n + y_n| \leq \sum_n (|x_n| + |y_n|) = \sum_n |x_n| + \sum_n |y_n| = \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1$

B. (5 points) Prove that the ℓ_0 'metric' is not a valid vector norm.

Note: Remember that the ℓ_0 'metric' of a vector is defined as the number of non-zero entries in said vector.

SOLUTION:
In order to show this, we just need to demonstrate a counter-example to any of the three properties of norms. Let's focus on the second property: $\|c\mathbf{x}\|_0 = \|\mathbf{x}\|_0 \neq |c|\|c\mathbf{x}\|_0$, so this property does not hold.

C. (5 points) Prove that any valid vector norm $\|\cdot\|$ is a convex function

Note: In other words, prove that the function $f : \mathbb{R}^N \to \mathbb{R}$ defined as $f(\mathbf{x}) = \|\mathbf{x}\|$, where $\|\mathbf{x}\|$ calculates a given norm of $\mathbf{x}$ (for instance, the ℓ_2 aka Euclidean norm), is a convex function. This will be very important for optimization problems, as these problems (both the objective function as well as the inequality constraints) are often written as norms.

SOLUTION:
For an arbitrary choice of valid norm $\|\cdot\|$, two arbitrary vectors $\mathbf{x_1}$ and $\mathbf{x_2}$, and some $t \in [0,1]$, we can write:

$$\begin{aligned}
\|t\mathbf{x_1} + (1-t)\mathbf{x_2}\| &\leq \|t\mathbf{x_1}\| + \|(1-t)\mathbf{x_2}\| \text{ (triangle inequality property of norms)} \\
&= |t|\|\mathbf{x_1}\| + |(1-t)|\|\mathbf{x_2}\| \text{ (by the second property of norms)} \\
&= t\|\mathbf{x_1}\| + (1-t)\|\mathbf{x_2}\| \text{ (since } t \text{ is between 0 and 1)}
\end{aligned}$$

So norms are indeed convex functions.

D. (5 points) Prove that the set $\|\mathbf{x}\|_2 \leq b$, for some $b > 0$, is a convex set within $\mathbb{R}^N$.

SOLUTION:

We need to prove that, for any two points that are in this set, ie: $\|\mathbf{x_1}\|_2 \leq b$ and $\|\mathbf{x_2}\|_2 \leq b$, the entire segment that joins these two points is also entirely within the set (ie: all the points in the segment have ℓ_2 norm $\leq b$). For any two vectors with $\|\mathbf{x_1}\|_2 \leq b$ and $\|\mathbf{x_2}\|_2 \leq b$, and some $t \in [0,1]$, we can write:

$$
\begin{aligned}
\|t\mathbf{x_1} + (1-t)\mathbf{x_2}\|_2 &\leq \|t\mathbf{x_1}\| + \|(1-t)\mathbf{x_2}\|_2 \text{ (triangle inequality)} \\
&= |t|\|\mathbf{x_1}\|_2 + |(1-t)|\|\mathbf{x_2}\|_2 \text{ (by the second property of norms)} \\
&= t\|\mathbf{x_1}\|_2 + (1-t)\|\mathbf{x_2}\|_2 \text{ (since } t \text{ is between 0 and 1)} \\
&\leq tb + (1-t)b \\
&= b
\end{aligned}
$$

So this is indeed a convex set.

E. (5 points) Plot or draw the following sets for a two-dimensional vector $\mathbf{x} \in \mathbb{R}^2$

i. $\|\mathbf{x}\|_\infty < 1$ (where $\|\mathbf{x}\|_\infty = \max_n |x_n|$

ii. $\|\mathbf{x}\|_2 < 1$

iii. $\|\mathbf{x}\|_1 < 1$

iv. $\|\mathbf{x}\|_{\frac{1}{2}} < 1$ (note that this one is not a norm, but we can define and plot the set just as well using the expression $\|\mathbf{x}\|_{\frac{1}{2}} = \left(\sum_n |x_n|^{1/2}\right)^2$))

Feel free to do this analytically (eg: drawing by hand for easy shapes), or computationally. For each of these plots, graphically indicate whether the corresponding set is convex or not (ie: if the set is convex, the line segment that joins any two points within the set will be completely inside the set, whereas if the set is non-convex you will be able to find two points within the set such that the segment that joins them is not completely within the set).

SOLUTION:

There are several ways to draw these, either analytically or computationally. In my case, I drew the first three analytically, and the fourth one using Matlab as follows:

```
x = linspace(-1.6,1.6,501);
[X,Y] = meshgrid(x,x);
f = (sqrt(abs(X)) + sqrt(abs(Y))).^2;
im(:,:,1) = 255*(f>1);
im(:,:,2) = 255*(f>1);
```

```
im(:,:,3) = 255;
imagesc(im);axis equal tight off
```

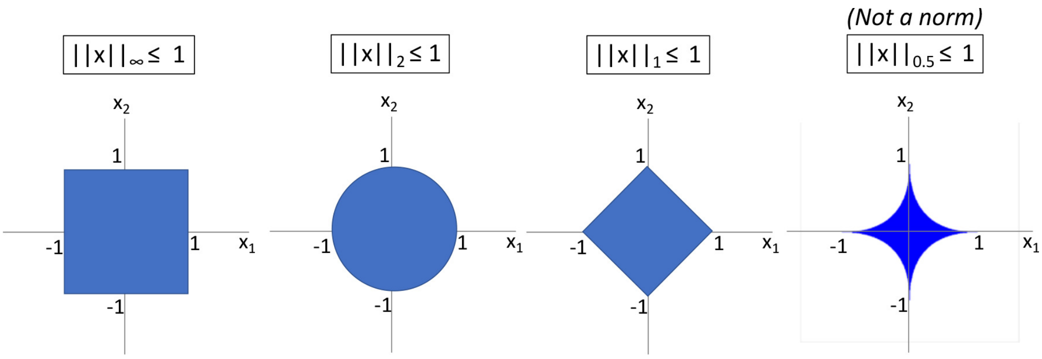

Figure 1.1: Regions of $\mathbf{x}$ with $\|\mathbf{x}\|_p \leq 1$ for various choices of p.

2. Solving Optimization Problems by Plotting and Staring (15 points)

A. (7.5 points) Minimum-norm solutions to a linear system of equations

Consider the following constrained optimization problem:

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_p, \text{ subject to } \mathbf{A}\mathbf{x} = \mathbf{b} \tag{2.1}$$

where $\|\cdot\|_p$ represents the ℓ_p norm (if $p \geq 1$) or metric (if $p < 1$), and

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \end{bmatrix} \tag{2.2}$$

Solve this problem graphically (ie: find the optimum $\hat{\mathbf{x}}$ by plotting, on the space (x_1, x_2), the constraint region $\mathbf{A}\mathbf{x} = \mathbf{b}$ allowed by our equality constraints and finding the points with minimum ℓ_p norm for various p) for the cases where:

i. $p = 0$

ii. $p = 1$

iii. $p = 2$

iv. $p = \infty$

Note that, for some values of p, there may be several solutions - make sure to indicate all of them on the plot. You do not need to specify all the numerical values of the solutions (although this is OK too, in addition to the graphical display), but should display graphically the location of each solution $\hat{\mathbf{x}}$, labeling them with the corresponding value of p. *Hint: Feel free to use the shape of the sets* $\|\mathbf{x}\|_p \leq 1$ *derived in the previous problem as a guide to solving this problem.*

Additional question: As we will see later in the course, the ℓ_0 metric is the ideal metric for minimization if we seek sparse solutions to linear systems (which is often the case in image reconstruction and processing). Remember that a sparse solution is one where many (or most) of the entries in the solution $\hat{\mathbf{x}}$ are zero. In our two-dimensional case, we only have two entries so the sparsest solutions may have one non-zero entry, and one zero entry. However, minimizing the ℓ_0 metric subject to some constraints is computationally very challenging (as we will also see later in the course). Based on your experience with this problem, if you had to pick an actual norm to minimize and try to mimic the behavior of ℓ_0 minimization, would you pick the ℓ_1, ℓ_2, or ℓ_∞ norm?

SOLUTION:

See Figure 2.1 for a graphical solution.

Note that ℓ_0 minimization promotes sparsity (by definition), and ℓ_1 minimization appears to be a better approximation to ℓ_0 minimization compared to ℓ_p with $p > 1$. There is

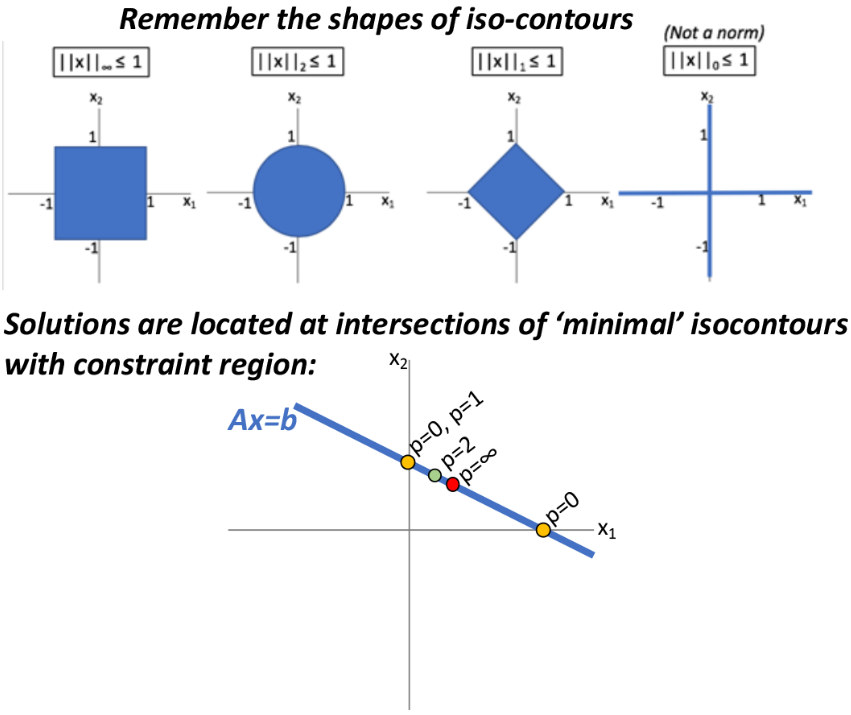

Figure 2.1: Minimum norm solutions, ie: $\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_p$, subject to $\mathbf{Ax} = \mathbf{b}$, for various choices of p.

theory (compressed sensing), as well as substantial empirical evidence that backs up this observation. Geometric intuition for this observation can be derived from the 'pointy' shape of the iso-contours of the ℓ_1 norm, with corners pointing along the axes, which make it likely that the minimum ℓ_1 solution to a linear system of equations will be found along the axis.

B. (7.5 points) Approximate solutions based on ℓ_p norms with additional constraints

Consider the following constrained optimization problem:

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|_p, \text{ subject to } \mathbf{Cx} = \mathbf{d} \tag{2.3}$$

where $\|\cdot\|_p$ represents the ℓ_p norm (if $p \geq 1$) or metric (if $p < 1$), and

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 3 & -1 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} 0 \end{bmatrix} \tag{2.4}$$

Solve this problem graphically (ie: find the optimum $\hat{\mathbf{x}}$ by plotting, on the space (b_1, b_2), the 'data' point $\mathbf{b}$ in 2D, as well as the space of possible vectors $\mathbf{Ax}$ allowed by our equality constraints $\mathbf{Cx} = \mathbf{d}$) for the cases where:

i. $p = 0$

ii. $p = 1$

iii. $p = 2$

iv. $p = \infty$

Note that, for some values of p, there may be several solutions - make sure to indicate all of them on the plot. You do not need to specify all the numerical values of the solutions (although this is OK too, in addition to the graphical display), but should display graphically the location of each solution $\mathbf{A}\hat{\mathbf{x}}$, labeling them with the corresponding value of p.

SOLUTION:

See Figure 2.2 for a graphical solution.

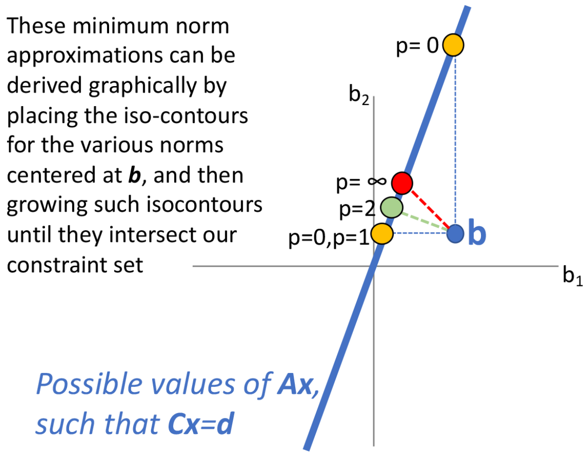

Figure 2.2: Minimum norm approximations, ie: $\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|_p$, subject to $\mathbf{Cx} = \mathbf{d}$, for various choices of p.

3. Reformulate LS Problems (15 points)

A. (7.5 points) Simplifying a two-component objective function

Reformulate the following formulation with two components

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \left\{ \|\mathbf{Ax} - \mathbf{b}\|_2^2 + \lambda \|\mathbf{Cx}\|_2^2 \right\} \tag{3.1}$$

for some $\lambda > 0$, as a single matrix, single vector LS formulation.
In other words, rewrite the expression in Equation 3.1 above as the equivalent formulation $\min \|\mathbf{Dx} - \mathbf{e}\|_2^2$, and determine the matrix $\mathbf{D}$ and the vector $\mathbf{e}$ in terms of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{C}$, and λ in order for the equivalence to hold.

SOLUTION:

There are several ways of approaching this problem. I personally think the 'visual' approach of formulating a longer vector that is built from both of our components is a rather intuitive solution. This is the approach I present below.

Our two-term expression can be equivalently written as the squared ℓ_2 norm of a 'longer' vector formed by concatenating $\mathbf{Ax} - \mathbf{b}$ and $\lambda\mathbf{Cx}$ (the latter of which can be rewritten as $\lambda\mathbf{Cx} - \mathbf{0}$):

$$\begin{aligned}
\|\mathbf{Ax} - \mathbf{b}\|_2^2 + \lambda \|\mathbf{Cx}\|_2^2 &= \left\| \begin{bmatrix} \mathbf{A} \\ \lambda\mathbf{C} \end{bmatrix} \mathbf{x} - \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} \right\|_2^2 \\
&= \|\mathbf{Dx} - \mathbf{e}\|_2^2
\end{aligned}$$

where

$$\mathbf{D} = \begin{bmatrix} \mathbf{A} \\ \lambda\mathbf{C} \end{bmatrix}, \text{ and } \mathbf{e} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$$

And so our minimization problem can be rewritten as $\min \|\mathbf{Dx} - \mathbf{e}\|_2^2$

B. (7.5 points) Simplifying a constrained optimization problem

Consider the following constrained optimization problem

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|_2^2 \tag{3.2}$$

$$\text{where } \mathbf{x} = \mathbf{x_0} + \mathbf{Cy} \tag{3.3}$$

for some vector $\mathbf{y}$, where $\mathbf{x} \in \mathbb{R}^{20}$ and $\mathbf{y} \in \mathbb{R}^{10}$, and the remaining matrices and vectors have suitable dimensions. Note that we do not know $\mathbf{y}$ - all we know is that there needs to exist a $\mathbf{y}$ such that $\mathbf{x}$ satisfies the condition above.

Reformulate the optimization problem above as an unconstrained optimization problem in terms of $\mathbf{y}$ instead of $\mathbf{x}$, written as $\min \|\mathbf{Dy} - \mathbf{e}\|_2^2$ for some matrix $\mathbf{D}$ and vector $\mathbf{e}$, such that $\hat{\mathbf{x}}$ can be obtained readily from the optimum $\hat{\mathbf{y}}$. Express $\mathbf{D}$ and $\mathbf{e}$ in terms of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{x_0}$, and $\mathbf{C}$.

SOLUTION:

Since we can express $\mathbf{x}$ in terms of $\mathbf{y}$, as $\mathbf{x} = \mathbf{x_0} + \mathbf{Cy}$, we can rewrite our optimization problem as the following unconstrained optimization problem in terms of $\mathbf{y}$:

$$\begin{aligned}
\hat{\mathbf{y}} &= \arg\min_{\mathbf{y}} \|\mathbf{A}(\mathbf{x_0} + \mathbf{Cy}) - \mathbf{b}\|_2^2 \\
&= \arg\min_{\mathbf{y}} \|\mathbf{Ax_0} + \mathbf{ACy} - \mathbf{b}\|_2^2 \\
&= \arg\min_{\mathbf{y}} \|\mathbf{ACy} - (\mathbf{b} - \mathbf{Ax_0})\|_2^2 \\
&= \arg\min_{\mathbf{y}} \|\mathbf{Dy} - \mathbf{e}\|_2^2
\end{aligned}$$

where $\mathbf{D} = \mathbf{AC}$ and $\mathbf{e} = \mathbf{b} - \mathbf{Ax_0}$. And so our minimization problem can be rewritten in terms of $\mathbf{y}$, and once we solve for $\hat{\mathbf{y}}$ we can obtain the desired solution as $\hat{\mathbf{x}} = \mathbf{x_0} + \mathbf{C}\hat{\mathbf{y}}$.

4. Convexity (20 points)

Which of the following functions are convex? Prove it or disprove it for each function, based on the definition of convexity and/or its properties (eg: operations that preserve convexity):

i. $f(x) = 0$

ii. $f(x) = \log x$ (for $x > 0$)

iii. $f(x) = |x|^3$

iv. $f(x) = f_1(x) + f_2(x)$, where both f_1 and f_2 are convex.

v. $f(x) = ag(x) + b$, where g is convex, and a and b are scalars (which can be positive or negative).

vi. $f(x) = g(ax + b)$, where g is convex, and a and b are scalars (which can be positive or negative).

vii. $f(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$

viii. $f(x) = |e^{ix} - b|^2$ (for $x \in \mathbb{R}$, and where $i = \sqrt{-1}$)

ix. $f(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \|\mathbf{F}\mathbf{x}\|_0$

Note: In the cases that include unspecified matrices and vectors, you can assume that the matrices are identity matrices and the vectors are all ones.

SOLUTION:

For arbitrary x_1 and x_2, and some $t \in [0, 1]$:

i. $f(x) = 0$: $f(tx_1 + (1 - t)x_2) = f(x_1) = f(x_2) = 0$ (**CONVEX, although not strictly convex**)

ii. $f(x) = \log x$ (for $x > 0$): Consider $x_1 = 1$, $x_2 = 3$, $t = 0.5$, then $f(2) = 1.61 > 0.5f(1) + 0.5f(3) = 1.15$ (**NON-CONVEX**)

iii. $f(x) = x^3$: Consider $x_1 = 0$, $x_2 = -2$, $t = 0.5$, then $f(-1) = -1 > 0.5f(0) + 0.5f(-2) = -4$ (**NON-CONVEX**)

iv. $f(x) = f_1(x) + f_2(x)$, where both f_1 and f_2 are convex. (**CONVEX**, since weighted sum of convex functions with non-negative weights remains convex, and the weights here are both 1).

v. $f(x) = ag(x) + b$, where g is convex, and a and b are scalars (which can be positive or negative). (generally **NON-CONVEX**, since a negative a would turn a convex function into a concave function. Only exception is if $g(x)$ is a straight line, ie: both convex and concave, and then $f(x)$ will also be both convex and concave)

vi. $f(x) = g(ax+b)$, where g is convex, and a and b are scalars (which can be positive or negative). (**CONVEX**, since composition of a convex function with an affine function (such as $ax+b$ for a and b constants).

vii. $f(\mathbf{x}) = \|\mathbf{Ax} - \mathbf{b}\|_2^2$ (**CONVEX**, since this is a composition of an ℓ_2 norm squared, which is convex, with an affine function).

viii. $f(x) = |e^{ix} - b|^2$ (for $x \in \mathbb{R}$): Consider $b = 1$, $x_1 = 0$, $x_2 = 2\pi$, $t = 0.5$, then $f(\pi) = 4 > 0.5f(0) + 0.5f(2\pi) = 0$. (**NON-CONVEX**, since this is an oscillatory function which can be clearly seen to have multiple maximizers and minimizers).

ix. $f(\mathbf{x}) = \|\mathbf{Ax} - \mathbf{b}\|_2^2 + \|\mathbf{Fx}\|_0$: Consider the 1D case, ie: $\mathbf{A} = 1$, $\mathbf{x}$ a scalar, $\mathbf{b} = 1$, $\mathbf{F} = 1$, ie: $f(x) = |x-1|^2 + \|x\|_0$, and consider $x_1 = 0$, $x_2 = 0.1$, $t = 0.5$. Then, $f(0.05) = 0.95^2 + 1 = 1.9 > 0.5f(0) + 0.5f(0.1) = 1.41$. (**NON-CONVEX**, given the non-convex nature of the ℓ_0 metric).

5. ML and MAP Estimation (25 points)

A. (8 points) Formulate a ML estimation problem in the presence of Gaussian noise as a LS optimization problem

Suppose we have a measuring device (eg: an imaging system) that takes measurements from an unknown object determined by a vector $\mathbf{x}$, as follows:

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \eta \tag{5.1}$$

where $\mathbf{b} \in \mathbb{R}^M$ is our measurement vector, $\mathbf{A} \in \mathbb{R}^{M\times N}$ is the system matrix, $\mathbf{x} \in \mathbb{R}^N$ is the unknown vector that we would like to estimate, and $\eta \in \mathbb{R}^M$ is a noise vector containing independent, identically distributed (iid) noise in each of its entries. Further, assume that the noise distribution is Gaussian with mean zero and standard deviation σ_1.
Derive the maximum-likelihood (ML) estimate for $\mathbf{x}$ *as a least-squares (LS) optimization problem, and describe whether this optimization problem depends on* σ *(ie: will the ML solution* $\hat{\mathbf{x}}_{ML}$ *change based upon the noise level?).*
Hint: Note that the ML estimate $\hat{\mathbf{x}}_{ML}$ corresponds to maximizing the probability of observing $\mathbf{b}$ given $\mathbf{x}$, ie:

$$\hat{\mathbf{x}}_{ML} = \arg\max_{\mathbf{x}} P(\mathbf{b}|\mathbf{x}) \tag{5.2}$$

where, in the presence of Gaussian noise, $P(\mathbf{b}|\mathbf{x})$ is given by:

$$P(\mathbf{b}|\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^M \sigma_1^{2M}}} e^{-\frac{\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2}{2\sigma_1^2}} \tag{5.3}$$

SOLUTION:

$$\begin{aligned}
\hat{\mathbf{x}}_{ML} &= \arg\max_{\mathbf{x}} P(\mathbf{b}|\mathbf{x}) \\
&= \arg\max_{\mathbf{x}} \frac{1}{\sqrt{(2\pi)^M \sigma_1^{2M}}} e^{-\frac{\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2}{2\sigma_1^2}} \\
&= \arg\max_{\mathbf{x}} \ \log\left[\frac{1}{\sqrt{(2\pi)^M \sigma_1^{2M}}} e^{-\frac{\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2}{2\sigma_1^2}}\right] \\
&= \arg\max_{\mathbf{x}} \left[\log\left(\frac{1}{\sqrt{(2\pi)^M \sigma_1^{2M}}}\right) - \frac{\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2}{2\sigma_1^2}\right] \\
&= \arg\max_{\mathbf{x}} \left[-\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2\right] \\
&= \arg\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2
\end{aligned}$$

where we can apply a log transformation to our objective function, since the log function is monotonically increasing for arguments greater than zero, so it will preserve the location of the minimizer. We can also remove constant offset terms ($\log\left(\frac{1}{\sqrt{(2\pi)^M\sigma_1^{2M}}}\right)$) from the maximization since they do not depend on $\mathbf{x}$. Subsequently, we can remove constant multiplicative terms ($\frac{1}{2\sigma_1^2}$) since they do not depend on $\mathbf{x}$ either. Finally, maximizing a function $-f(x)$ is equivalent to minimizing $f(x)$, as applied in the last step.

Importantly, the approach to obtain an ML solution to this estimation problem is independent of the noise level. In the presence of iid Gaussian noise, ML is equivalent to LS fitting, ie: minimizing the ℓ_2 norm (or ℓ_2 norm squared) of the data fitting residual $\mathbf{Ax}-\mathbf{b}$.

B. (8 points) Let's turn our ML estimation into a MAP estimation problem

Now let us assume that we know the a priori statistical distribution of $\mathbf{x}$, which happens to be a Gaussian iid distribution with standard deviation σ_2. In other words, in the absence of any data, the probability density function of $\mathbf{x}$ is given by:

$$P(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^N\sigma_2^{2N}}}e^{-\frac{\|\mathbf{x}\|_2^2}{2\sigma_2^2}} \tag{5.4}$$

Now that we have both an a priori distribution as well as some noisy data, let's formulate the maximum-a-posteriori (MAP) estimation of $\mathbf{x}$ (which combines both sources of information), as follows:

$$\begin{aligned}\hat{\mathbf{x}}_{MAP} &= \arg\max_{\mathbf{x}} P(\mathbf{x}|\mathbf{b}) && (5.5)\\ &= \arg\max_{\mathbf{x}} \frac{P(\mathbf{b}|\mathbf{x})P(\mathbf{x})}{P(\mathbf{b})} && (5.6)\\ &= \arg\max_{\mathbf{x}} P(\mathbf{b}|\mathbf{x})P(\mathbf{x}) && (5.7)\end{aligned}$$

where the last step is warranted since $P(\mathbf{b})$ does not depend on $\mathbf{x}$.
Formulate the MAP estimation above as a LS optimization problem of the form:

$$\hat{\mathbf{x}}_{MAP} = \arg\min_{\mathbf{x}} \|\mathbf{Ex}-\mathbf{f}\|_2^2 + \lambda\|\mathbf{Gx}\|_2^2 \tag{5.8}$$

and determine the matrices, vectors, and scalar in this LS optimization ($\mathbf{E}$, $\mathbf{f}$, λ, $\mathbf{G}$), in terms of the parameters of our problem ($\mathbf{A}$, $\mathbf{b}$, σ_1, σ_2, etc).

SOLUTION:

$$
\begin{aligned}
\hat{\mathbf{x}}_{MAP} &= \arg\max_{\mathbf{x}} P(\mathbf{b}|\mathbf{x})P(\mathbf{x}) \\
&= \arg\max_{\mathbf{x}} \frac{1}{\sqrt{(2\pi)^M \sigma_1^{2M}}} e^{-\frac{\|\mathbf{Ax}-\mathbf{b}\|_2^2}{2\sigma_1^2}} \frac{1}{\sqrt{(2\pi)^N \sigma_2^{2N}}} e^{-\frac{\|\mathbf{x}\|_2^2}{2\sigma_2^2}} \\
&= \arg\max_{\mathbf{x}} \log\left[\frac{1}{\sqrt{(2\pi)^M \sigma_1^{2M}}} e^{-\frac{\|\mathbf{Ax}-\mathbf{b}\|_2^2}{2\sigma_1^2}} \frac{1}{\sqrt{(2\pi)^N \sigma_2^{2N}}} e^{-\frac{\|\mathbf{x}\|_2^2}{2\sigma_2^2}}\right] \\
&= \arg\max_{\mathbf{x}} \left[\log\left(\frac{1}{\sqrt{(2\pi)^M \sigma_1^{2M}}}\right) - \frac{\|\mathbf{Ax}-\mathbf{b}\|_2^2}{2\sigma_1^2} + \log\left(\frac{1}{(\sqrt{2\pi)^N \sigma_2^{2N}}}\right) - \frac{\|\mathbf{x}\|_2^2}{2\sigma_2^2}\right] \\
&= \arg\max_{\mathbf{x}} \left[-\frac{\|\mathbf{Ax}-\mathbf{b}\|_2^2}{2\sigma_1^2} - \frac{\|\mathbf{x}\|_2^2}{2\sigma_2^2}\right] \\
&= \arg\max_{\mathbf{x}} \left[-\|\mathbf{Ax}-\mathbf{b}\|_2^2 - \frac{2\sigma_1^2}{2\sigma_2^2}\|\mathbf{x}\|_2^2\right] \\
&= \arg\max_{\mathbf{x}} \left[-\|\mathbf{Ax}-\mathbf{b}\|_2^2 - \frac{\sigma_1^2}{\sigma_2^2}\|\mathbf{x}\|_2^2\right] \\
&= \arg\min_{\mathbf{x}} \left[\|\mathbf{Ax}-\mathbf{b}\|_2^2 + \frac{\sigma_1^2}{\sigma_2^2}\|\mathbf{x}\|_2^2\right] \\
&= \arg\min_{\mathbf{x}} \left[\|\mathbf{Ax}-\mathbf{b}\|_2^2 + \lambda\|\mathbf{x}\|_2^2\right]
\end{aligned}
$$

which is a minimization problem with the desired form, with $\mathbf{E} = \mathbf{A}$, $\mathbf{f} = \mathbf{b}$, $\mathbf{G} = \mathbf{I}$ (the identity matrix), and $\lambda = \frac{\sigma_1^2}{\sigma_2^2}$. Importantly, the parameter λ is the ratio of the data noise variance to the variance of the prior distribution of $\mathbf{x}$. This parameter controls the tradeoff between how much we trust our data (if σ_1^2/σ_2^2 is small, then the optimization problem is dominated by the data term $\|\mathbf{Ax}-\mathbf{b}\|_2^2$) and how much we trust our prior distribution (if σ_1^2/σ_2^2 is large, then the optimization problem is dominated by the 'prior' (or 'regularization') term $\|\mathbf{x}\|_2^2$).

C. (9 points) A specific example of ML and MAP estimation.

Assuming that $\sigma_1 = 1$, $\sigma_2 = 1$, and:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{5.9}$$

$$\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \tag{5.10}$$

find the ML and MAP estimates $\hat{\mathbf{x}}_{ML}$ and $\hat{\mathbf{x}}_{MAP}$.

Next, assume that the a priori distribution of $\mathbf{x}$ is 'tighter', eg: $\sigma_2 = 0.5$. What is the new value of $\hat{\mathbf{x}}_{MAP}$? How about if $\sigma_2 = 0.2$? Can you guess what is the limit of $\hat{\mathbf{x}}_{MAP}$ as $\sigma_2 \to 0$?

Next, assume that the a priori distribution of $\mathbf{x}$ is'broader', eg: $\sigma_2 = 2$. What is the new value of $\hat{\mathbf{x}}_{MAP}$? How about if $\sigma_2 = 5$? Can you guess what is the limit of $\hat{\mathbf{x}}_{MAP}$ as $\sigma_2 \to \infty$?
MatLab note: For the LS problems considered in this exercise, if we want to solve

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 \tag{5.11}$$

we can simply run "`xhat = A\b`" on MatLab. Note that for the problems that have several terms (eg: the MAP estimation in Equation 5.8), we can rewrite them as a single-term LS problem as shown in Question 3 in this homework set.

SOLUTION:

This is an example MatLab script that solves this exercise:

```
A = [ 1 1 0; 1 0 1; 0 1 1];
b = [3 4 5]';
G = eye(3);

%ML solution
x_ML = A\textbackslash b;

s1 = 1;
s2 = 1;

A2 = [A;s1^2/s2^2*G];
b2 = [b;zeros(3,1)];

k=0; % Let's keep track of how many MAP solutions we generate
k=k+1;
% MAP solution
x_MAP(:,k) = A2\textbackslash b2;

% Tighter prior
s2 = 0.5;
A2 = [A;s1^2/s2^2*G];
b2 = [b;zeros(3,1)];
k=k+1;
x_MAP(:,k) = A2\textbackslash b2;

s2 = 0.2;
A2 = [A;s1^2/s2^2*G];
b2 = [b;zeros(3,1)];
k=k+1;
```

```
x_MAP(:,k) = A2\textbackslash b2;

s2 = 0.01;
A2 = [A;s1^2/s2^2*G];
b2 = [b;zeros(3,1)];
k=k+1;
x_MAP(:,k) = A2\textbackslash b2;

% Broader prior
s2 = 2;
A2 = [A;s1^2/s2^2*G];
b2 = [b;zeros(3,1)];
k=k+1;
x_MAP(:,k) = A2\textbackslash b2;

s2 = 5;
A2 = [A;s1^2/s2^2*G];
b2 = [b;zeros(3,1)];
k=k+1;
x_MAP(:,k) = A2\textbackslash b2;

s2 = 100;
A2 = [A;s1^2/s2^2*G];
b2 = [b;zeros(3,1)];
k=k+1;
x_MAP(:,k) = A2\textbackslash b2;
```

and these are corresponding solutions (where columns in the $\mathbf{x}_{MAP}$ array indicate the seven different values of σ_2 I tried: $1, 0.5, 0.2, 0.01, 2, 5, 100$).

```
x_ML =
     1
     2
     3

x_MAP =
    1.1000    0.3412    0.0111    0.0000    1.0281    1.0008    1.0000
    1.6000    0.4000    0.0127    0.0000    1.9692    1.9992    2.0000
    2.1000    0.4588    0.0143    0.0000    2.9104    2.9976    3.0000
```

Note that, as $\sigma_2 \to 0$ the MAP solution approaches an all-zero vector (since we are basically enforcing a very firm prior knowledge that our solution should be zero). In contrast, as $\sigma_2 \to \infty$ the MAP solution approaches the ML solution (since we are basically not enforcing any prior knowledge on our solution).