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IMAGE SEGMENTATION BY VARIATIONAL METHODS: MUMFORD AND SHAH FUNCTIONAL AND THE DISCRETE APPROXIMATIONS*

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Abstract. In this paper we discuss the links between Mumford and Shah's variational problem for (signal and) image segmentation, based on an energy functional of a continuous grey-level function, and the numerical algorithms proposed to solve it. These numerical approaches are based on a discrete functional. We recall that, in one dimension, this discrete functional is asymptotically equivalent to the continuous functional. This can be summarized in a Γ-convergence result. We show that the same result holds in dimension two, provided that the continuous energy is adapted to the anisotropy of the discrete approaches. We display a few experimental results in dimensions one and two.

Key words. theory and algorithms for image segmentation, variational problems, special bounded variation functions, Γ-convergence, Hausdorff measures

AMS subject classifications. 26A45, 49J45

1. Introduction. Variational methods in image segmentation have known a great success since Geman and Geman's paper [32] introducing (in a probabilistic setting) a segmentation energy. Many works in the past ten years, including Blake and Zisserman's book *Visual Reconstruction* [12], were devoted to this energy or equivalent formulations.

The discrete energy on which these algorithms are based has been transposed by Mumford and Shah [39], [40] in a continuous setting, raising a large number of very interesting—and difficult—mathematical problems. We quickly present those approaches:

Let $g = (g_{i,j})_{1\le i,j\le N}$ be an "image," i.e., a grey-level function defined on a (square) matrix of pixels. We'll always assume for simplicity's sake that $0 \le g_{i,j} \le 1$. Now suppose g represents the (noisy) corrupted version of a "clean" image $f_{i,j}$: the discrete problem, as presented by Blake and Zisserman, suggests looking for f as a minimizer of the "weak membrane" energy

$$E^d(f) = \sum_{i,j} W(|f_{i+1,j} - f_{i,j}|) + W(|f_{i,j+1} - f_{i,j}|) + |f_{i,j} - g_{i,j}|^2; \tag{1}$$

where the potential W, given by

$$W(x) = \min(\lambda^2 x^2, \alpha),$$

is a truncated quadratic potential. Here λ and α are parameters that must be adequately adjusted (see [12]) to fit the model to a particular case. Without loss of generality we'll usually set $\alpha = \lambda = 1$ in the mathematical study of the problem. In this energy the terms where function W appears are regularity terms: from their minimization it follows that the grey-level values at two adjacent pixels have to remain close, unless their difference is beyond a certain threshold, in which case the spring binding them is broken: This situation corresponds to the presence of a

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physical "edge" on the image between the two pixels, that should not be blurred. The other term, $\sum_{i,j} |f_{i,j} - g_{i,j}|^2$, ensures that f remains close to the original data g.

This problem is equivalent to minimizing Geman and Geman's energy:

$$
\begin{aligned}
E^{G-G}(f) = \sum_{i,j} (\lambda^2 |f_{i+1,j} - f_{i,j}|^2 (1 - v_{i,j}) \\
+ \lambda^2 |f_{i,j+1} - f_{i,j}|^2 (1 - h_{i,j}) \\
+ \alpha(h_{i,j} + v_{i,j}) + |f_{i,j} - g_{i,j}|^2).
\end{aligned} \tag{2}
$$

Here $h_{i,j} \in \{0,1\}$ and $v_{i,j} \in \{0,1\}$ are the "line process": $h_{i,j}$ represents a small piece of horizontal edge between pixel (i,j) and pixel $(i,j+1)$ and $v_{i,j}$ a small piece of vertical edge between pixel (i,j) and pixel $(i+1,j)$. The value of the line process is 1 when the piece of edge is "active," and 0 when there is no edge. Parameter α appears here as the cost for putting an edge between two adjacent pixels. This approach and several variants have been introduced in [4], [5], and [29]–[31]. (Many of the algorithms presented in these papers—for instance, Blake and Zisserman's "Graduate Non Convexity" and Geiger et al.'s "deterministic annealing"—are almost identical.)

Mumford and Shah noticed that the term $\alpha \sum_{i,j}(h_{i,j} + v_{i,j})$ could be a way of estimating the total length of edges found in the image, and that the remainder of the energy could then be written, in a continuous setting,

$$
\lambda^2 \int_{\Omega \setminus K} |\nabla f|^2 + \int_{\Omega} |f - g|^2,
$$

with $\Omega \subset \mathbf{R}^2$ as a rectangle open domain representing the screen; $f : \Omega \to [0,1]$ as a grey-level image; $g : \Omega \to [0,1]$ as the original datum (a Borel function); and $K \subset \Omega$ as a closed set of Hausdorff dimension one representing the edges. Thus they rewrote the problem as the following: minimize

$$
E(f,K) = \lambda^2 \int_{\Omega \setminus K} |\nabla f|^2 + \alpha \mathcal{H}^1(K) + \int_{\Omega} |f - g|^2, \tag{3}
$$

and stated the following conjecture.

CONJECTURE 1 (Mumford and Shah). *There exists at least one minimizer* (f,K) *of* (3), *where*

1. $f \in C^1(\Omega \setminus K)$.
2. *K is made up of a finite union of C^1-regular arcs (with special conditions when three of those arcs meet at the same point or one of them meets the boundary $\partial\Omega$. The intersection of four such arcs, or two with $\partial\Omega$, at the same point, is impossible).*

Until now it has been proved that

i. (3) has minimizers (f,K) with $f \in C^1(\Omega \setminus K)$ and K an $(\mathcal{H}^1, 1)$-rectifiable curve in the sense of Federer [26]; see [1], [2], [33], and [37].

ii. There exists a minimizing sequence (f_n, K_n) converging to a minimizer (f,K) ($K_n \to K$ in the Hausdorff sense) with K_n built of a finite number of Lipschitz-continuous[1] arcs [37].

[1] That is, the direct image of a Lipschitz-continuous function mapping $[0,1]$ to Ω.

iii. The minimizing set K is entirely included in one single regular (Lipschitz-continuous) curve (David and Semmes [18]—see also Dibos and Kœpfler [22], [23]).

However, no more is known about K's regularity.

The proof of the existence of a solution is done in two steps. A weak formulation is first stated, in the space of "special bounded variation" functions $\mathrm{SBV}(\Omega)$ (see Appendix A): we set for $u \in \mathrm{SBV}(\Omega)$,

$$E(u) = \int_\Omega |\nabla u|^2 + \mathcal{H}^1(S_u) + \int_\Omega |g-u|^2, \tag{4}$$

where ∇u is the approximate gradient of u (defined a.e.), and S_u the set of essential discontinuities of u ($\Omega \setminus S_u$ are the Lebesgue points of u). The existence of minimizers in $\mathrm{SBV}(\Omega)$ for this weak formulation of the problem is a straightforward consequence of the following compactness result, established by Ambrosio [1].

THEOREM 1.1 (Ambrosio). *If $(u_n)_{n\geq 0}$ is a sequence of $\mathrm{SBV}(\Omega)$ functions such that $E(u_n) \leq K < +\infty$ for all n, it is possible to find a subsequence u_{n_k} and $u \in \mathrm{SBV}(\Omega)$ with*

$$u_{n_k} \longrightarrow u \quad a.e.,$$

$$\nabla u_{n_k} \rightharpoonup \nabla u \quad \textit{weakly in } \mathrm{L}^1(\Omega);$$

and

$$\int_\Omega |\nabla u|^2 \leq \liminf_{k\to+\infty} \int_\Omega |\nabla u_{n_k}|^2,$$

$$\mathcal{H}^1(S_u) \leq \liminf_{k\to+\infty} \mathcal{H}^1(S_{u_{n_k}}).$$

It is then necessary to prove that (u, S_u) can be considered as a minimizer (u, K) of (3), for instance, by setting $K = \overline{S_u}$. This step is not easier than the first one. Checking that $u \in C^1(\Omega \setminus K)$ is simple; the main difficulty is the proof of the essential closedness of S_u (i.e., $\mathcal{H}^1(\overline{S_u} \setminus S_u) = 0$). It was given by De Giorgi, Carriero, and Leaci [33]. (See also Dal Maso, Morel, and Solimini [37].)

2. Questions and answers.

In this paper we address the following problem: In what sense is the discrete method (1) an approximation of the continuous problem (3)? Or, in other words, would the resolution of problem (1) give an approximate solution for (3)? This mathematical and numerical question is implicit in Mumford and Shah's papers, where both problems are considered as "equivalent."

Note that a different approach for solving (3) was proposed by Ambrosio and Tortorelli [3]. They considered a sequence of functionals (in any dimension)

$$E^n(f,v) = \int_\Omega (f-g)^2 + \int_\Omega (1-v^2)^n |\nabla f|^2 + \left(\int_\Omega (1-v^2)^n |\nabla v|^2 + \frac{n^2 v^2}{16} \right) \tag{5}$$

"Γ-converging" to E (see Appendix B), which means that as n goes to infinity a minimizer (f^n, v^n) of E^n will be close to a solution of (3) in the sense that $f^n(x)$ is close to $f(x)$ and $v^n(x)$ to $\chi_K(x)$.[2] This problem has been studied by Richardson [41], who developed an algorithm based on this result. Recently, Bellettini and Coscia [6] deduced from (5) a sequence of discrete energies which also Γ-converge to E, on a subset of $\mathrm{SBV}(\Omega)$ (made of functions u such that S_u is a piecewise C^2 manifold).

[2] $\chi_K(x)$ is the characteristic function of set K: it is equal to 1 if $x \in K$ and 0 otherwise.

2.1. A one-dimensional result. In dimension one, $\Omega \subset \mathbf{R}$ is an interval and we deal with signals. For $u \in \mathrm{SBV}(\Omega)$, let

$$E^0(u) = \lambda^2 \int_{\Omega \setminus S_u} |u'|^2 + \alpha \mathrm{card}(S_u) + \int_\Omega |u-g|^2 = F^0(u) + \int_\Omega |u-g|^2, \tag{6}$$

where $\mathrm{card}(S_u)$ denotes the cardinality of the discontinuity set S_u (i.e., its Hausdorff zero-dimensional measure); and, for $h > 0$ and $f^h = (f_k^h)_{kh \in \Omega}$ a given discrete signal at resolution h:

$$\begin{aligned} E^h(f^h) &= h \sum_{k\,:\,kh,(k+1)h \in \Omega} W_h\left(\frac{|f_{k+1}^h - f_k^h|}{h}\right) + h \sum_{k\,:\,kh\in\Omega} |f_k^h - g_k^h|^2 \\ &= F^h(f^h) + h \sum_{k\,:\,kh\in\Omega} |f_k^h - g_k^h|^2, \end{aligned} \tag{7}$$

where now

$$W_h(x) = \min(\lambda^2 x^2, \alpha/h), \tag{8}$$

and g^h is the approximation of g at resolution h (we set, for instance, for all $k \in \mathbf{Z}$,

$$g_k^h = \frac{1}{h}\int_{kh}^{(k+1)h} g(t)dt,$$

with $g = 0$ on $\mathbf{R} \setminus \Omega$). Note that in this case both studies of the continuous and the discrete problem are much simpler than in dimension two.

The figures show two examples of the segmentation of a signal $g(x)$ obtained by minimizing a functional such as (7). Figures 1 and 2 show, resp., a single step edge corrupted by some white noise, and the recovered signal. Figures 3 and 4 show a row of pixels extracted from a two-dimensional image, a smoothed version (corresponding to the minimization of (7) with a huge parameter α), and the minimum of (7) obtained

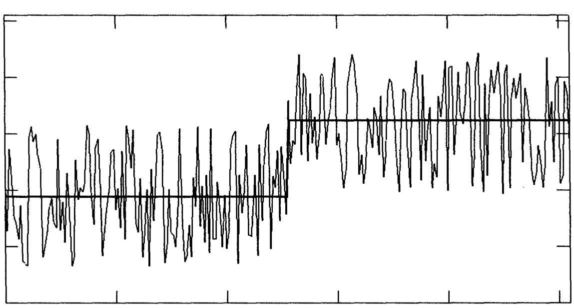

FIG. 1. *A step edge, and the corrupted version.*

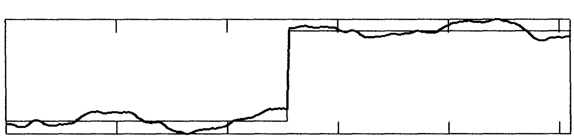

FIG. 2. *The recovered edge.*

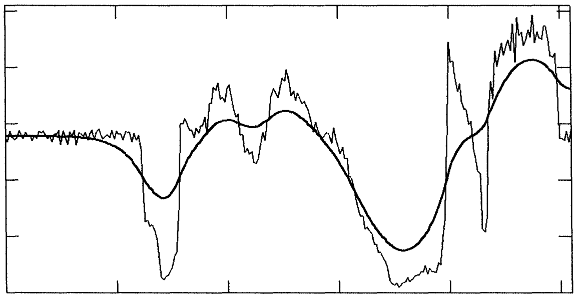

FIG. 3. *Line extracted from an image, and smoothed version.*

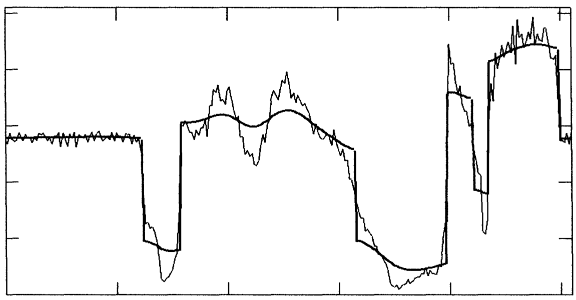

FIG. 4. *The minumum of* (7).

with a smaller α, where seven discontinuities appear. Note that in both cases the *true* minimum of (7) is reached through an algorithm proposed by Mumford and Shah [39] (see Appendix E; for more details see also [13] and [15]).

In the sequel, a discrete signal $f^h = (f_k^h)_{kh\in\Omega}$ at resolution h will also be considered as a function, more precisely a continuous piecewise affine function, linear on each interval $[kh,(k+1)h]\cap\Omega$ (for $k\in\mathbf{Z}$), and taking the value f_k^h at kh. It may simply be built as the function $\sum_k f_k^h\Delta(\frac{x}{h}-k)_{|\Omega}$, with $\Delta(x) = (1-|x|)\times\chi_{(-1,1)}(x)$, and letting f_k^h take any value (between 0 and 1), for instance 0, when $kh\notin\Omega$. We'll denote by V_h the space of such functions: this construction allows us to talk about the "limit" of a discrete signal f^h as h goes to 0.

THEOREM 2.1 (first version).

- *For all u in* $\mathrm{SBV}(\Omega)$,
  i. *If $f^h\in V_h$ [or a sequence $f^{h_n}\in V_{h_n}$, $n\in\mathbf{N}$ (for $h_n\searrow 0$ as n goes to infinity)] is such that*

$$f^h \overset{h\searrow 0}{\longrightarrow} u \quad a.e.\ in\ \Omega \quad [or\ f^{h_n} \overset{n\to+\infty}{\longrightarrow} u],$$

  then

$$\liminf_{h\searrow 0} F^h(f^h) \geq F^0(u) \quad [\liminf_{n\to+\infty} F^{h_n}(f^{h_n}) \geq F^0(u)].$$

  ii. *For every (small) $h>0$, there exists $f^h\in V_h$ such that*

$$f^h \longrightarrow u \quad a.e.\ in\ \Omega\ as\ h\searrow 0,$$

$$and \quad \limsup_{h\searrow 0} F^h(f^h) \leq F^0(u).$$

- *Moreover,*
  iii. *For all families* $(f^h)_{h>0}$, $(f^h \in V_h)$ [*or, again, as in* i, *for any sequence* ...], *if*

$$F^h(f^h) \le K < +\infty,$$

there exists a subsequence f^{h_n} *and* $u \in \mathrm{SBV}(\Omega)$ *with*

$$f^{h_n} \overset{n\to+\infty}{\longrightarrow} u \quad a.e.\ in\ \Omega.$$

[i. *then implies* $\liminf_{n\to+\infty} F^{h_n}(f^{h_n}) \ge F^0(u)$.]

(This theorem still holds with E^h and E^0 instead of F^h and F^0.)

As Ω and all functions are bounded, the above limits could also be replaced by $\mathrm{L}^p(\Omega)$-limits ($p < \infty$), or by limits in the sense of the distance

$$d_\Omega(f,g) = \int_\Omega \frac{|f-g|}{1+|f-g|} \tag{9}$$

that makes the space $\mathcal{B}(\Omega)$ of Borel functions [or rather classes of a.e. equal Borel functions] over Ω a metric space (see [1] and [3]).

Then, if we consider F^h (E^h) and F^0 (E^0) as functionals over $\mathcal{B}(\Omega)$ by letting $F^h(f) = E^h(f) = +\infty$, and $F^0(u) = E^0(u) = +\infty$, whenever $f \notin V_h$ and $u \notin \mathrm{SBV}(\Omega)$, we can summarize the previous theorem in a Γ-convergence result (see Definition 2).

THEOREM 2.1.bis. *As h goes to* 0,

$$F^h \ (resp.,\ E^h)\ \Gamma(\mathcal{B}(\Omega))\text{-converges to } F^0 \ (resp.,\ E^0).$$

The (extensive) proof of this theorem is given in [14] and [15]. It can also be adapted from the proof of the result in dimension two, given in the sequel.

2.2. The result in dimension two. Let Ω be an open rectangular domain in $\mathbf{R}^2$. Without loss of generality we'll assume $\Omega = (0,1) \times (0,1)$.

A good reason for which we cannot expect to get a result as simple as Theorem 2.1 bis for the two-dimensional Mumford–Shah functional is the fact that the discrete energy (1) is highly rotationally noninvariant, as it measures the total length of edges made of small horizontal and vertical pieces; while energy (3) is rotationally invariant. Thus, if one image g and the corresponding segmentation (u, K) are rotated by an angle of 45 degrees their continuous energy will remain unchanged, unlike the discrete energy. Therefore we have to introduce a nonisotropic Hausdorff one-dimensional measure, denoted by Λ, that behaves closer to the way of measuring the length of the discrete functional. (Basically, Λ measures the length of a curve only through its projections along the horizontal and vertical axes, and for a regular C^1 curve $C = \gamma([0,1])$, with $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \in \Omega$, we have

$$\Lambda(C) = \int_0^1 |\gamma_1'(t)| + |\gamma_2'(t)| dt;$$

see (17) for the general definition.) Let us replace in (3) $\mathcal{H}^1$ by this new measure:

$$E(f,K) = \lambda^2 \int_{\Omega\setminus K} |\nabla f|^2 + \alpha\Lambda(K) + \int_\Omega |f-g|^2 = F(f,K) + \int_\Omega |f-g|^2. \tag{10}$$

For $u \in \mathrm{SBV}(\Omega)$, we'll also denote by E (and F) the "weak energy"

$$E(u) = \lambda^2 \int_\Omega |\nabla u|^2 + \alpha \Lambda(S_u) + \int_\Omega |u-g|^2 = F(u) + \int_\Omega |u-g|^2. \tag{11}$$

These two energies are in some sense equivalent, just like (3) is equivalent to (4). As in dimension one, a discrete energy $E^h(f^h)$ depending on a scale step h has to be introduced. We will restrict ourselves to $h = \frac{1}{N}$, $N \in \mathbf{N} \setminus \{0\}$ (remember $\Omega = (0,1) \times (0,1)$). An "image" at resolution h will thus be a square matrix $f^h = (f^h_{i,j})_{0 \le i,j < N}$. We'll also define a *pixel:* $P^h_{i,j} = [ih, (i+1)h] \times [jh, (j+1)h]$; and $x^h_{i,j}$ (or $x_{i,j}$) will denote the center $((i+\frac{1}{2})h, (j+\frac{1}{2})h)$ of pixel $P^h_{i,j}$. As in dimension one we'll still consider f^h as a function over Ω. The construction of this function is explained in § 4.

Let $E^h(f^h)$ be the discrete two-dimensional "weak membrane" energy at resolution h—the same energy function as (1) or (2), with just different parameters:

$$E^h(f^h) = F^h(f^h) + h^2 \sum_{k,l} |f^h_{k,l} - g^h_{k,l}|^2, \tag{12}$$

with

$$\begin{aligned} F^h(f^h) &= h^2 \sum_k \sum_l W_h\left(\frac{f^h_{k,l} - f^h_{k-1,l}}{h}\right) + W_h\left(\frac{f^h_{k,l} - f^h_{k,l-1}}{h}\right) \\ &= \left\{ \sum_{0<k<N} \sum_{0 \le l < N} \min\{\lambda^2 |f^h_{k,l} - f^h_{k-1,l}|^2, \alpha h\} \right. \\ &\quad \left. + \sum_{0<l<N} \sum_{0 \le k < N} \min\{\lambda^2 |f^h_{k,l} - f^h_{k,l-1}|^2, \alpha h\} \right\} \end{aligned} \tag{13}$$

and

$$g^h_{k,l} = \frac{1}{h^2} \int_{P^h_{k,l}} g(x)dx \in [0,1] \tag{14}$$

(W_h is given by (8)). The figure (Fig. 5) shows an example of a segmentation in dimension two obtained by minimizing (12) (see Appendix E.2 for details). On the left we have shown the original 256×256 pixels image, and on the right the piecewise regular result, with the discontinuities appearing in white. The result presented here is not a real minimizer of the energy, as no algorithm has been found until now to really reach the minimum of (12). However, it corresponds to a low energy image.

THEOREM 2.2.

- *For any bounded u in* $\mathrm{SBV}(\Omega)$,

  i. *For all $(f^h)_{h^{-1}=N\ge 1}$ with*

$$\lim_{h \searrow 0} f^h = u \quad a.e. \ in\ \Omega,$$

[*or for a sequence f^{h_n}, $h_n^{-1} \in \mathbf{N}$, $h_n \searrow 0 \ldots$*]

$$\liminf_{h \searrow 0} E^h(f^h) \ge E(u).$$

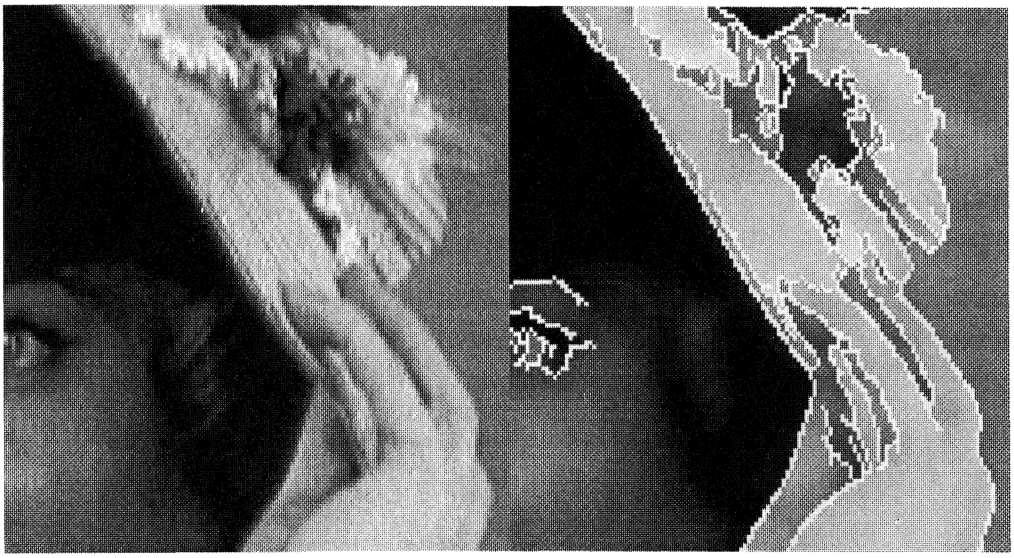

FIG. 5. *Example of a two-dimensional segmentation.*

ii. a. *If* $K = \overline{S_u}$ *is regular (here "regular" is intended as needed in Proposition* 5.3, § 5.3*), there exists a sequence* $(f^h)_{h^{-1}=N\geq 1}$ *such that*

$$\lim_{h\searrow 0} f^h = u \quad a.e.$$

$$and \quad \limsup_{h\searrow 0} E^h(f^h) \leq E(u,K)$$

and when $\mathcal{H}^1(K\setminus S_u) = 0$*, this simplifies as* $\limsup_{h\searrow 0} E^h(f^h) \leq E(u)$.[3]

b. *If* u *can be approached by functions* $u_n \in H^1(\Omega \setminus K_n)$ *with* K_n *a regular one-dimensional closed set such that* $u_n \to u$ *a.e. and* $E(u_n, K_n) \to E(u)$*, then there exists a sequence* $(f^h)_{h^{-1}=N\geq 1}$ *such that*

$$\lim_{h\searrow 0} f^h = u \quad a.e.$$

$$and \quad \limsup_{h\searrow 0} E^h(f^h) \leq E(u).$$

- *Moreover,*

iii. *If a sequence* $(f^h)_{h^{-1}=N\geq 1}$*, [or, again, a sequence* $(f^{h_n})\ldots$*], is such that for all* h

$$E^h(f^h) \leq K < +\infty,$$

there exists a subsequence f^{h_n} *and* $u \in \mathrm{SBV}(\Omega)$ *with*

$$f^{h_n} \overset{n\to+\infty}{\longrightarrow} u \quad a.e.\ in\ \Omega$$

[and i *then implies* $\liminf_{n\to+\infty} E^{h_n}(f^{h_n}) \geq E(u)$*].*

[3] Remember the difference between $E(u)$ and $E(u,K)$: The first one is the weak formulation of the latter, and their values coincide only when $\Lambda(K) = \Lambda(S_u)$.

Dibos and Séré [24] have just proved that any minimizer of the Mumford and Shah functional, or any $u \in \mathrm{SBV}(\Omega)$ with $\mathcal{H}^1(\overline{S_u} \setminus S_u) = 0$, can be approached by functions u^ε with S_{u^ε} made of a finite number of *segments* and such that the limit as ε goes to zero of the Mumford and Shah energies of u^ε is less than the energy of u. (In fact this also holds with our energy (11).) Their proof involves the fact that the set S_u is essentially compact, but Dal Maso extended it to any SBV function. His demonstration is reproduced in Appendix G at the end of this paper. This means that *any* $u \in \mathrm{SBV}(\Omega)$ satisfies the assumptions of point (b) of Proposition 5.3 (§ 5.3) and thus also Theorem 2.2ii.a, and the following fundamental result follows (extending again, as in one dimension, functionals E^h and E—the weak functional defined on $\mathrm{SBV}(\Omega)$—to $\mathcal{B}(\Omega)$, and using the appropriate distance (9)).

THEOREM 2.2.bis. *As h goes to* 0,

$$F^h \text{ (resp., } E^h) \ \Gamma(\mathcal{B}(\Omega))\text{-converges to } F \text{ (resp., } E).$$

In particular, if for all $h = 1/N$, (f^h) minimizes the "weak membrane" discrete energy (12), then (f^h) has subsequences that converge almost everywhere in Ω to functions $u \in \mathrm{SBV}(\Omega)$ minimizing the continuous energy (11) (and—see Appendix C—such that $(u, \overline{S_u})$ minimizes (10)).

Remark. If (11) has a unique minimizer u, then (f^h) converges to u.

This theorem therefore is the main result of this paper: it establishes that the discrete approaches to Mumford and Shah's continuous problem should lead to a correct solution—up to the substitution of $\mathcal{H}^1$ with Λ in the continuous energy.

Note. In points i and iii of Theorem 2.2 it is important to notice that the following result doesn't hold: we don't necessarily get

$$\liminf_{h \searrow 0} E^h(f^h) \geq E(u, K)$$

with $K = \overline{S_u}$. As a matter of fact, it is possible to build a sequence of "images" (f^{h_n}) satisfying the assumptions of point iii. and converging to a function $u \in \mathrm{SBV}(\Omega)$ such that $\Lambda(S_u)$ (and $E(u)$) is small but $\overline{S_u} = \Omega$ and thus $\Lambda(\overline{S_u}) = +\infty$. See Appendix F.

3. The "cab driver" length Λ.

3.1. Definitions. In this section $|X|$ denotes the diameter of a set $X \subset \mathbf{R}^2$, i.e., $|X| = \sup\{d(x,y);\ x,y \in X\} = \sup\{|x-y|;\ x,y \in X\}$. For $\rho > 0$, a *ρ-covering* of X is a countable or finite collection of sets $(X_i)_{i \in I}$ with

- $X \subset \bigcup_{i \in I} X_i$,
- $\forall i \in I, |X_i| \leq \rho$.

The Hausdorff one-dimensional measure of a set $E \subset \mathbf{R}^2$ is

$$\mathcal{H}^1(E) = \lim_{\rho \searrow 0^+} \mathcal{H}^1_\rho(E) \text{ with}$$

$$\mathcal{H}^1_\rho(E) = \inf\left\{\sum_{i \in I} |C_i|;\ (C_i)_{i \in I}\ \rho\text{-covering of } E \text{ by convex open sets}^4\right\}. \tag{15}$$

Let $\|\cdot\|_1$ be the following norm over $\mathbf{R}^2$:

$$\forall x = (x_1, x_2) \in \mathbf{R}^2,\ \|x\|_1 = |x_1| + |x_2|. \tag{16}$$

[4] The C_i should theoretically be sets of any kind, but it is easy to check that the value of $\mathcal{H}^1$ doesn't change if we restrict ourselves to convex open sets.

The measure Λ is the "spherical" Hausdorff-type measure defined by

$$\Lambda(E) = \lim_{\rho \searrow 0^+} \Lambda_\rho(E) \text{ with}$$

$$\Lambda_\rho(E) = \inf\left\{\sum_{i\in I} |B_i|;\ (B_i)_{i\in I}\ \rho\text{-covering of } E \text{ by open } \|\cdot\|_1\text{–balls}\right\}. \tag{17}$$

Remark. By introducing $|X|_1 = \sup\{\|x-y\|_1;\ x,y \in X\}$, we could also consider

$$\tilde\Lambda(E) = \lim_{\rho \searrow 0^+} \tilde\Lambda_\rho(E) \text{ with}$$

$$\tilde\Lambda_\rho(E) = \inf\left\{\sum_{i\in I} |C_i|_1;\ (C_i)_{i\in I}\ \rho\text{-covering of } E \text{ by open convex sets}\right\}.$$

The results would be similar; for this particular norm $\|\cdot\|_1$ we have $\tilde\Lambda = \Lambda$.

Λ is like $\mathcal{H}^1$ a metric outer measure (or Caratheodory measure). In particular, this implies that it defines a regular measure on the σ-field of its "measurable sets," which contains the Borel σ-field. It is easy to check that for any $E \subset \mathbf{R}^2$,

$$\mathcal{H}^1(E) \le \Lambda(E) \le \sqrt{2}\mathcal{H}^1(E). \tag{18}$$

3.2. Derivative of the measure Λ. More generally, we have the following theorem, that still holds for any other norm $\|\cdot\|_1$ in $\mathbf{R}^2$ (i.e., not necessarily defined by (16)).

THEOREM 3.1. *If Λ is the measure defined by formula* (17)*, then we have the following.*

i. *There exists $c > 0$ such that for all $E \subset \mathbf{R}^2$,*

$$\mathcal{H}^1(E) \le \frac{1}{\kappa}\Lambda(E) \le c\mathcal{H}^1(E), \tag{19}$$

where $\kappa = \frac{1}{2}|\{x\ :\ \|x\|_1 < 1\}|$ is the radius of the $\|\cdot\|_1$-unit ball.[5]

ii. *The Λ-measurable sets of finite Λ measure are the* 1*-sets, i.e., the $\mathcal{H}^1$-measurable sets of finite $\mathcal{H}^1$ measure.*

iii. *If E is a regular* 1*-set,*

$$d\Lambda_{|_E}(x) = \|\tau_E(x)\|_1 d\mathcal{H}^1_{|_E}(x), \tag{20}$$

$\tau_E(x)$ being the approximate tangent vector to E at x (or Besicovitch tangent, see the papers by Besicovitch or [25]*), defined $\mathcal{H}^1$-almost everywhere on E.*

Using definition (16), we may rewrite (20) for all regular sets E:

$$\Lambda(E) = \int_E |\langle \tau_E(x), e_1\rangle| + |\langle \tau_E(x), e_2\rangle| d\mathcal{H}^1 = \int_E |\langle \nu_E, e_1\rangle| + |\langle \nu_E, e_2\rangle| d\mathcal{H}^1 \tag{21}$$

with $e_1 = (1,0)$, $e_2 = (0,1)$, and $\|\nu_E(x)\| = 1$, $\nu_E(x) \perp \tau_E(x)$ for all x.

Remember that the upper and lower densities of a 1-set E at x are, resp.,

$$\overline{D}^1(E,x) = \limsup_{r\to 0^+} \frac{\mathcal{H}^1(E\cap B_r(x))}{2r} \text{ and } \underline{D}^1(E,x) = \liminf_{r\to 0^+} \frac{\mathcal{H}^1(E\cap B_r(x))}{2r},$$

[5] With norm (16) $\kappa = 1$.

and that a *regular* 1-set is a 1-set where almost all $\mathcal{H}^1$ points have density 1, i.e., for $\mathcal{H}^1$-*almost all* $x \in E$

$$\overline{D}^1(E,x) = \underline{D}^1(E,x) = 1 = \lim_{r\to 0^+} \frac{\mathcal{H}^1(E\cap B_r(x))}{2r}.$$

In this case, E is $(\mathcal{H}^1, 1)$-rectifiable in the sense of Federer, i.e., included, up to an $\mathcal{H}^1$-negligible set, in an countable union of rectifiable curves. This holds, for instance, for the jump set S_u of an SBV(Ω) function u. For the general theory, see [7], [8], [25], and [26]. For other properties or discussions about more general Caratheodory measures, see also [9]–[11], [19], [20], [28], [38], and [27].

Points i and ii are quite easy to prove; ii is generalized in [38]. Point iii is more difficult; most of the demonstration can be found in [10] and [11], or in [26, Def. 2.8.16, Thms. 2.9.2, 2.9.6, 2.9.7]. For the complete demonstration—including the value of the Radon–Nikodým derivative $\|\tau_E(x)\|_1$—see [15].

Now that the measure Λ is defined, the meaning of energies (10) and (11) is clear. We discuss, in Appendix C, the problem of the existence of a minimizer for these energies.

4. The proof of points i and iii in Theorem 2.2. In this section we're going to explain how we extend the discrete "images" f^h into functions in order to show points i and iii in Theorem 2.2.

For a given scale step $h = 1/N$, let $\mathcal{Z}_h$ be the set of closed subsets S of $\overline{\Omega} \cap (h\mathbf{Z}\times\mathbf{R}\cup\mathbf{R}\times h\mathbf{Z})$ only made of segments $[kh,(k+1)h]\times\{lh\}$ with $0\le k<N$ and $1\le l<N$ or $\{kh\}\times[lh,(l+1)h]$ with $1\le k<N$ and $0\le l<N$. Obviously for all $S\in\mathcal{Z}_h$, $\mathcal{H}^1(S\cap\Omega) = \Lambda(S\cap\Omega)$ (from (21)).

When $S\in\mathcal{Z}_h$ and $f = (f^h_{k,l})_{0\le k,l<N}$ is a discrete "image" at resolution h, we define a function $\tilde f = (f,S)$ in SBV(Ω), continuous over $\Omega\setminus S$, in the way described as follows:

If the segment $[x_{k,l}, x_{k+1,l}]$ (resp., $[x_{k,l}, x_{k,l+1}]$)[6] doesn't meet $S\cup\partial\Omega$, the extension $\tilde f$ is linear on this segment, taking the values $a = f_{k,l}$ at $x_{k,l}$ and $b = f_{k+1,l}$ at $x_{k+1,l}$ (resp., $f_{k,l}$ at $x_{k,l}$ and $d = f_{k,l+1}$ at $x_{k,l+1}$).

On the other hand, if this segment meets $S\cup\partial\Omega$, $\tilde f$ is piecewise constant on the segment, with one discontinuity at the point where it meets S. (If it just meets $\partial\Omega$, only half of it is inside the domain Ω and we just have to define $\tilde f$ on this half: It is chosen to be constant, taking the value of f at the end of the segment lying inside Ω.)

Inside a square $C_{k,l} = (x_{k,l}\; x_{k+1,l}\; x_{k+1,l+1}\; x_{k,l+1})$ (those squares are different from the above defined "pixels"!), or rather in $C_{k,l}\cap\Omega$, $\tilde f$ is the continuous (even C^∞) function on $C_{k,l}\cap\Omega\setminus S$ minimizing

$$\int_{C_{k,l}\cap\Omega\setminus S} |\nabla \tilde f|^2$$

with the previously prescribed boundary conditions (nothing is prescribed on $S\cup\partial\Omega$).

[6] We still denote by $x_{k,l}$ a point $\left(\left(k+\frac12\right)h, \left(l+\frac12\right)h\right)$ that is not necessarily inside Ω (with $k,l\in\mathbf{Z}$).

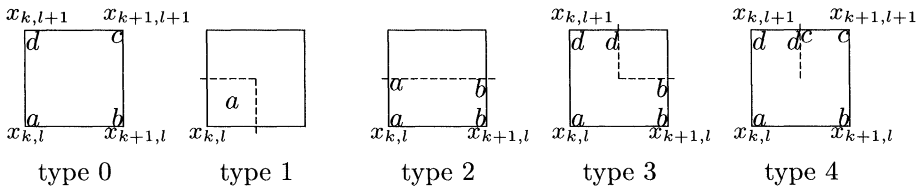

FIG. 6. *Construction of* $\tilde{f}$.

Figure 6 shows the five different situations that may occur. The dotted line represents $S \cup \partial\Omega$, and we let $c = f_{k+1,l+1}$. At each corner of the domain we wrote the value of $\tilde{f}$; however in the "type 1" case $\tilde{f}$ has the same constant value a over the whole small lower left square delimited by the dotted line.

Such a function is indeed in SBV(Ω); moreover, we obviously have $S_{\tilde{f}} \subseteq S$. In what follows we'll mainly consider two extensions of a discrete "image" $f^h = (f^h_{k,l})_{0 \le k,l < N}$:

- The continuous extension corresponding to an empty set S: $\overline{f}^h = (f, \emptyset)$.
- The extension $\tilde{f} = \tilde{f}^h = (f, S_0)$ where $S_0 \in \mathcal{Z}_h$ is made of the segments $[kh,(k+1)h] \times \{lh\}$ which separate two points $x_{k,l-1}$ and $x_{k,l}$ such that $|f^h_{k,l-1} - f^h_{k,l}| > \sqrt{h}$ and the segments $\{kh\} \times [lh,(l+1)h]$ which separate two points $x_{k-1,l}$ and $x_{k,l}$ such that $|f^h_{k-1,l} - f^h_{k,l}| > \sqrt{h}$: in this case it is easy to check that $S_{\tilde{f}} = S_0$ (or rather $\overline{S_{\tilde{f}}} = S_0$), all points of S_0 (except the possible ends) being indeed discontinuity points of $\tilde{f}$.

  We have in this case

$$\Lambda(S_{\tilde{f}}) = h\big(\mathrm{card}\{(k,l) \ : \ |f^h_{k,l} - f^h_{k+1,l}| > \sqrt{h}\} \\ + \mathrm{card}\{(k,l) \ : \ |f^h_{k,l} - f^h_{k,l+1}| > \sqrt{h}\}\big) \tag{22}$$

(where card(X) is the cardinality of a set X).

Remark that for a given f^h and $S \in \mathcal{Z}_h$, if we let[7]

$$\begin{aligned} E'^h(f^h,S) = &\textstyle\sum_{[x_{k,l},x_{k+1,l}]\cap S=\emptyset} |f^h_{k,l} - f^h_{k+1,l}|^2 \\ &+ \textstyle\sum_{[x_{k,l},x_{k,l+1}]\cap S=\emptyset} |f^h_{k,l} - f^h_{k,l+1}|^2 \\ &+ h\mathrm{card}\{(k,l) \ : \ [x_{k,l},x_{k+1,l}] \cap S \neq \emptyset\} \\ &+ h\mathrm{card}\{(k,l) \ : \ [x_{k,l},x_{k,l+1}] \cap S \neq \emptyset\} \\ &+ h^2 \textstyle\sum_{k,l} |f^h_{k,l} - g^h_{k,l}|^2 \end{aligned}$$

(this corresponds to Geman and Geman's point of view, the set S being a representation of the "line process" $h_{i,j}$ and $v_{i,j}$ of energy (2)), we get

$$\min_{S\in\mathcal{Z}_h} E'^h(f^h,S) = E'^h(f^h,S_0) = E^h(f^h)$$

[7] In this section and the following, $\alpha = \lambda = 1$.

and therefore

$$\min_{f^h,S} E'^h(f^h,S) = \min_{f^h} E^h(f^h);$$

this obviously makes the extension $\tilde{f}^h = (f^h, S_0)$ of f^h the "right" extension to consider.

PROPOSITION 4.1. *If $f^h = (f^h_{k,l})_{0\le k,l<N}$ is a sequence of "images" (for $h = 1/N$, $N \geq 1$) and if there exists $C \geq 0$ such that*

$$\forall h = \frac{1}{N}, \quad F^h(f^h) \leq C < +\infty \tag{23}$$

then the sequence $(\tilde{f}^h)$ of the extensions of f^h (or any subsequence of this sequence) is relatively compact, in the sense that there exists a subsequence $(\tilde{f}^{h_p})_{p\geq 1}$ and $u \in$ SBV(Ω) such that

$$\tilde{f}^{h_p} \longrightarrow u \quad \textit{a.e. in } \Omega;$$

$$\nabla \tilde{f}^{h_p} \rightharpoonup \nabla u \textit{ weakly in } \mathrm{L}^1 \textit{ and } \mathrm{L}^2(\Omega)$$

as $p \to +\infty$. Moreover,

$$F(u) \leq \liminf_{p\to+\infty} F^{h_p}(f^{h_p}),$$

(where $F(u) = \int_\Omega |\nabla u|^2 + \Lambda(S_u)$).

Remark. Considering an appropriate subsequence of (f^h), we can replace the last formula with

$$F(u) \leq \liminf_{h^{-1}=N\to+\infty} F^h(f^h). \tag{24}$$

PROPOSITION 4.2. *Under the same assumptions, it is also true that*

$$\sum_{k,l} f^{h_p}_{k,l} \cdot \chi_{P^{h_p}_{k,l}} \overset{\mathrm{L}^2(\Omega)}{\longrightarrow} u, \tag{25}$$

thus

$$\lim_{p\to\infty} h_p^2 \sum_{k,l} |f^{h_p}_{k,l} - g^{h_p}_{k,l}|^2 = \int_\Omega |u-g|^2; \tag{26}$$

and therefore, we also have

$$E(u) \leq \liminf_{h^{-1}=N\to+\infty} E^h(f^h). \tag{27}$$

4.1. Proof of Proposition 4.1.

Proof. In order to prove that proposition, we first must estimate for all h the integral $\int_\Omega |\nabla \tilde{f}^h|^2$. Here $\nabla \tilde{f}^h$ is the (a.e. defined) gradient of $\tilde{f}^h$, and $\nabla \tilde{f}^h(x)dx$ is the absolutely continuous (w.r.t. the Lebesgue measure) part of the distributional derivative of $\tilde{f}^h \in$ SBV(Ω).

In the sequel we'll often write $\tilde{f}$ instead of $\tilde{f}^h$ and $S = \overline{S_{\tilde{f}^h}} = S_0$ when there is no ambiguity.

Ω is made of pieces of the five types of Fig. 6; we estimate $\int |\nabla \tilde{f}^h|^2$ for each case.

- Type 0: In this situation we have an explicit formulation for $\tilde{f}$ inside the square $C_{k,l}$. For all $\xi, \eta \in [0,1]$,

$$\tilde{f}(x_{k,l} + (\xi,\eta)\cdot h) = a + \xi(b-a) + \eta(d-a) + \xi\eta(c-b-d+a). \tag{28}$$

Therefore we have

$$\int_{C_{k,l}} \left|\frac{\partial \tilde{f}}{\partial x}\right|^2 = \frac{1}{3}((b-a)^2 + (c-d)^2 + (b-a)(c-d)), \text{ and thus}$$

$$\frac{1}{6}((b-a)^2 + (c-d)^2) \le \int_{C_{k,l}} \left|\frac{\partial \tilde{f}}{\partial x}\right|^2 \le \frac{1}{2}((b-a)^2 + (c-d)^2). \tag{29}$$

We get similar estimates for the value of $\int_{C_{k,l}} |\frac{\partial \tilde{f}}{\partial y}|^2$.

- Type 1: Let C' be the small square (in the lower left corner on our figure). On C' we have $\nabla \tilde{f} = 0$ and

$$\int_{C'} |\nabla \tilde{f}|^2 = 0. \tag{30}$$

- Type 2: Let R be the rectangle delimited by the borders of the square $C_{k,l}$ and the dotted line. $\tilde{f}$ is linear on R and we have

$$\int_R \left|\frac{\partial \tilde{f}}{\partial x}\right|^2 = \frac{h^2}{2}\cdot\frac{(b-a)^2}{h^2} = \frac{(b-a)^2}{2}, \quad \int_R \left|\frac{\partial \tilde{f}}{\partial y}\right|^2 = 0. \tag{31}$$

Types 3 and 4 are less easy to deal with, as the domain on which $\tilde{f}$ must be estimated is more complex (Figs. 7 and 8).

- Type 3: On L (as defined on Fig. 7), $\tilde{f} \in C^1(L)$ is the solution of the following problem:

$$\begin{cases} \Delta \tilde{f} = 0 \text{ inside } L \text{ and} \\ \tilde{f}(x_{k,l} + (\xi,\eta)\cdot h) = \begin{cases} a + \xi(b-a) & \text{if } 0 \le \xi \le 1, \eta = 0, \\ a + \eta(d-a) & \text{if } \xi = 0, 0 \le \eta \le 1, \\ b & \text{if } \xi = 1, 0 \le \eta \le 1/2, \\ d & \text{if } 0 \le \xi \le 1/2, \eta = 1, \end{cases} \\ \frac{\partial \tilde{f}}{\partial n} = 0 \text{ on } S \cap \partial L. \end{cases} \tag{32}$$

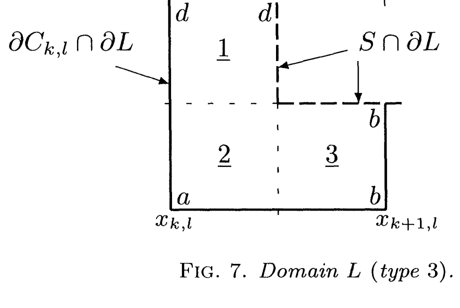

FIG. 7. *Domain L (type 3).*

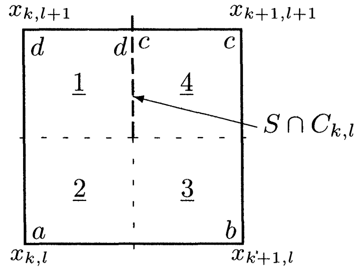

FIG. 8. *Domain* $C_{k,l}$, *type* 4.

This is, of course, equivalent to saying that $\tilde{f}$ minimizes $\int_\Omega |\nabla \tilde{f}|^2$ among the functions taking the same values on $\partial L \setminus S$. Using this fact and estimates such as (29), we can show that

$$\int_L |\nabla \tilde{f}|^2 \le \frac{11}{12}((b-a)^2 + (d-a)^2), \tag{33}$$

but it is not true, in general, that $\int_L |\nabla \tilde{f}|^2 \le \frac{1}{2}((b-a)^2 + (d-a)^2)$.

- Type 4: This time $\tilde{f} \in C^1(C_{k,l} \setminus S)$ is the solution, discontinuous at S, of

$$\begin{cases} \Delta \tilde{f} = 0 \text{ inside } C_{k,l} \setminus S \text{ and} \\ \tilde{f}(x_{k,l} + (\xi,\eta)\cdot h) = \begin{cases} a + \xi(b-a) & \text{if } 0 \le \xi \le 1, \eta = 0, \\ a + \eta(d-a) & \text{if } \xi = 0, 0 \le \eta \le 1, \\ b + \eta(c-a) & \text{if } \xi = 1, 0 \le \eta \le 1, \\ d & \text{if } 0 \le \xi < 1/2, \eta = 1, \\ b & \text{if } 1/2 < \xi \le 1, \eta = 1, \end{cases} \\ \frac{\partial \tilde{f}}{\partial n} = 0 \text{ on both sides of } S \cap C_{k,l}. \end{cases} \tag{34}$$

In this case we can check that

$$\int_{C_{k,l}} |\nabla \tilde{f}|^2 \le \frac{5}{3}(b-a)^2 + \frac{5}{4}((c-b)^2 + (d-a)^2), \tag{35}$$

but again, we don't have $\int_{C_{k,l}} |\nabla \tilde{f}|^2 \le \frac{1}{2}((b-a)^2 + (c-b)^2 + (d-a)^2)$.

Now, let

$$A_h = \bigcup_{C_{k,l} \cap S = \emptyset} C_{k,l} \cap \Omega \quad \text{and} \quad B_h = \Omega \setminus A_h.$$

A_h is (at a given scale h), the reunion of all squares $C_{k,l}$ (whose vertices are the points $x_{k,l}$) that don't meet the discontinuity set S, and B_h the reunion of the other squares. In particular, A_h is only made up of pieces of types 0, 1 or 2: equations (29), (30), and (31) (and the definition of $S = S_0$) therefore allow us to write

$$\begin{aligned} \int_{A_h} |\nabla \tilde{f}^h|^2 &\le \sum_{k,l\,:\,|f^h_{k,l} - f^h_{k+1,l}| \le \sqrt{h}} |f^h_{k,l} - f^h_{k+1,l}|^2 + \sum_{k,l\,:\,|f^h_{k,l} - f^h_{k,l+1}| \le \sqrt{h}} |f^h_{k,l} - f^h_{k,l+1}|^2 \\ &= F^h(f^h) - \Lambda(S_{\tilde{f}^h}), \end{aligned} \tag{36}$$

with the last equality a consequence of (22).

Now, if we also consider inequalities (33) and (35), we get

$$\int_\Omega |\nabla \tilde{f}^h|^2 \leq c(F^h(f^h) - \Lambda(S_{\tilde{f}^h})), \tag{37}$$

where constant c is something like twice the largest constant appearing in (33) and (35) (i.e., 10/3).

From this, we deduce that for all $h = 1/N$,

$$\int_\Omega |\nabla \tilde{f}^h|^2 + c\mathcal{H}^1(S_{\tilde{f}^h}) = \int_\Omega |\nabla \tilde{f}^h|^2 + c\Lambda(S_{\tilde{f}^h}) \leq cC < +\infty,$$

and we now are in a position to apply the fundamental compactness result of Ambrosio [1] (Theorem 1.1) to the uniformly bounded functions $\tilde{f}^h$. It shows that there exist a subsequence $(\tilde{f}^{h_p})$ and a function $u \in \mathrm{SBV}(\Omega)$ such that, as $p \to \infty$,

$$\tilde{f}^{h_p} \to u \text{ a.e. in } \Omega \text{ and}$$

$$\nabla \tilde{f}^{h_p} \rightharpoonup \nabla u \text{ weakly in } \mathrm{L}^1(\Omega).$$

The first convergence also holds in L^q for $q < \infty$ as for all h, $0 \leq \tilde{f}^h \leq 1$ — and these inequalities are also satisfied almost everywhere by u.

Moreover, we get

$$\Lambda(S_u) \leq \liminf_{p\to\infty} \Lambda(S_{\tilde{f}^{h_p}}).$$

This last inequality is not a straightforward consequence of Ambrosio's theorem: we don't want to explain in detail here how it is established. This is done in Appendix C, where the same property is used.

We still have to show that

$$\int_\Omega |\nabla u|^2 + \Lambda(S_u) \leq \liminf_{p\to\infty} F^{h_p}(f^{h_p}),$$

knowing that for all h (36),

$$\int_{A_h} |\nabla \tilde{f}^h|^2 + \Lambda(S_{\tilde{f}^h}) \leq F^h(f^h).$$

Therefore we must prove, for instance, that

$$\int_\Omega |\nabla u|^2 \leq \liminf_{p\to\infty} \int_{A_{h_p}} |\nabla \tilde{f}^{h_p}|^2.$$

This is true if we show that the subsequence $(\tilde{f}^{h_p})$ may be chosen such that

$$\chi_{A_{h_p}} \cdot \nabla \tilde{f}^{h_p} \rightharpoonup \nabla u \text{ weakly in } \mathrm{L}^2(\Omega),$$

which comes from the following two remarks.

- By definition,

$$\begin{aligned} \text{meas}(B_h) &\leq h^2 \text{card}\{C_{k,l} \ : \ C_{k,l} \cap S_{\tilde{f}^h} \neq \emptyset\} \\ &\leq 2h^2 \big(\text{card}\{(k,l) \ : \ |f^h_{k,l} - f^h_{k+1,l}| > \sqrt{h}\,\} \\ &\qquad + \ \text{card}\{(k,l) \ : \ |f^h_{k,l} - f^h_{k,l+1}| > \sqrt{h}\,\}\big) \\ &= 2h\Lambda(S_{\tilde{f}^h}) \\ &\leq 2hF^h(f^h) \leq 2hC, \end{aligned} \tag{38}$$

  and therefore $\lim_{h \to 0} \text{meas}(B_h) = 0$.
- As $\int_\Omega |\nabla \tilde{f}^h|^2 \leq cC < +\infty$, the sequence $(\nabla \tilde{f}^h)$ is relatively weakly compact in $L^2(\Omega, \mathbf{R}^2)$ and we thus may assume the existence of two functions $v_1, v_2 \in L^2(\Omega)$, that are the weak limits in L^2 of $\partial \tilde{f}^{h_p}/\partial x$ and $\partial \tilde{f}^{h_p}/\partial y$.

  As ∇u is the L^1-weak limit of $\nabla \tilde{f}^{h_p}$, we necessarily also have $(v_1, v_2) = \nabla u$.

These two points ensure that for all $\phi \in \mathrm{L}^2(\Omega)$, $\chi_{B_h} \cdot \phi$ strongly converges to zero in L^2 and

$$\int_\Omega \chi_{B_{h_p}} \cdot \phi \cdot \nabla \tilde{f}^{h_p} \to 0$$

as p goes to infinity. In other words, $\chi_{B_{h_p}} \cdot \nabla \tilde{f}^{h_p}$ weakly converges to zero and

$$\chi_{A_{h_p}} \cdot \nabla \tilde{f}^{h_p} = \nabla \tilde{f}^{h_p} - \chi_{B_{h_p}} \cdot \nabla \tilde{f}^{h_p} \rightharpoonup \nabla u$$

weakly in L^2. This achieves the proof of Proposition 4.1. □

Remarks.

1. If $\overline{f}^h$ is the continuous extension of (f^h), $\overline{f}^h$ and $\tilde{f}^h$ take the same values on A_h and we therefore have $\|\overline{f}^h - \tilde{f}^h\|_{\mathrm{L}^q} \leq 2\text{meas}(B_h)^{1/q}$. This implies that $\overline{f}^{h_p}$ converges to u in L^q for $q < \infty$ (and also, up to a subsequence, almost everywhere). As $|\nabla \overline{f}^h| \leq c/h$ (with $c = \sqrt{2}$), we also have

$$\begin{aligned} \int_\Omega |\nabla \overline{f}^h| &= \int_{A_h} |\nabla \overline{f}^h| + \int_{B_h} |\nabla \overline{f}^h| \\ &\leq \text{meas}(A_h)^{\frac{1}{2}} \left(\int_{A_h} |\nabla \overline{f}^h|^2 \right)^{\frac{1}{2}} + \text{meas}(B_h) \cdot \frac{c}{h} \\ &\leq \sqrt{C} + 2hC \cdot \frac{c}{h} \ (\text{cf. } (38)) = \text{ a constant}, \end{aligned}$$

   which means that we can also choose (h_p) such that

$$\nabla \overline{f}^{h_p} \cdot dx \rightharpoonup Du \ \text{ weakly in } \mathrm{M}^1(\Omega)$$

   where Du is the distributional derivative of u, and $\mathrm{M}^1(\Omega)$ the space of Radon measures μ over Ω with $|\mu|(\Omega) < +\infty$ (we have $Du = \nabla u \cdot dx + (u^+ - u^-)\nu_u d\mathcal{H}^1_{|S_u} \in \mathrm{M}^1(\Omega)$).
2. Here we proved the existence of f^{h_p} and u such that f^{h_p} converges to u and

$$F(u) \leq \liminf_{p \to \infty} F^{h_p}(f^{h_p}).$$

In order to find a limit function u such that (24) holds, i.e.,

$$F(u) \le \liminf_{h^{-1}=N\to\infty} F^h(f^h),$$

we must apply this result to a subsequence f^{h_k} of f^h that satisfies

$$\lim_{k\to\infty} F^{h_k}(f^{h_k}) = \liminf_{h^{-1}=N\to\infty} F^h(f^h).$$

4.2. Proof of Proposition 4.2.

Notation. Let us first extend $(f^h_{k,l})_{0\le k,l<N}$ to $\mathbf{Z}\times\mathbf{Z}$, by simply letting $f^h_{k,l}=0$ when $x_{k,l}\notin\Omega$. The functions $\tilde{f}^h$, $\overline{f}^h$, u can be extended the same way outside Ω by the value 0. Now we define a function $\hat{f}^h$ by

$$\hat{f}^h(x) = \sum_{k,l\in\mathbf{Z}} f^h_{k,l}\cdot\chi_{P^h_{k,l}}(x)$$

for all $x\in\mathbf{R}^2$. For all $k,l\in\mathbf{Z}$ we recall that $P^h_{k,l} = (kh,lh)+hP^1$ with $P^1 = [0,1]\times[0,1]$. Proposition 4.2 holds if we establish that $\hat{f}^{h_p}$ tends to u in $\mathrm{L}^2(\mathbf{R}^2)$.

Define

$$w(x) = w(x_1,x_2) = \Delta\left(x_1-\frac{1}{2}\right)\Delta\left(x_2-\frac{1}{2}\right) \tag{39}$$

with $\Delta(x)=1-|x|$ in $[-1,1]$ and 0 outside. If we set

$$\check{f}^h(x) = \sum_{k,l\in\mathbf{Z}} f^h_{k,l}\cdot w\Big(\frac{x}{h}-(k,l)\Big),\ x\in\mathbf{R}^2,$$

then, by construction, $\check{f}^h$ and $\overline{f}^h$ take the same values everywhere but on the set $\{x\ :\ d(x,\partial\Omega)<h/2\}$, therefore $\|\check{f}^h-\overline{f}^h\|_{\mathrm{L}^2}\le c\sqrt{h}$ and $\check{f}^{h_p}$ converges to u in $\mathrm{L}^2(\mathbf{R}^2)$ as p goes to infinity. Thus we have to check that $\|\check{f}^{h_p}-\hat{f}^{h_p}\|_{\mathrm{L}^2}$ converges to zero.

Proof. Simple algebra leads to the relation

$$w\Big(\frac{x}{h}-(k,l)\Big) = h^{-2}\chi_{[-\frac{h}{2},\frac{h}{2}]\times[-\frac{h}{2},\frac{h}{2}]} * \chi_{P^h_{k,l}}(x), \tag{40}$$

which implies that the two functions $\check{f}^h$ and $\hat{f}^h$ satisfy

$$\check{f}^h(x) = h^{-2}\chi_{[-h/2,h/2]^2} * \hat{f}^h(x) = h^{-2}\int_{[-h/2,h/2]^2}\hat{f}^h(x-\tau)d\tau.$$

Using the Cauchy–Schwarz inequality and Fubini's theorem, we can write

$$\begin{aligned}
\|\hat{f}^h-\check{f}^h\|^2_{\mathrm{L}^2} &= \int\left|\hat{f}^h(x)-h^{-2}\int_{[-h/2,h/2]^2}\hat{f}^h(x-\tau)d\tau\right|^2 dx \\
&= h^{-4}\int\left|\int_{[-h/2,h/2]^2}\hat{f}^h(x)-\hat{f}^h(x-\tau)d\tau\right|^2 dx \\
&\le h^{-4}\int h^2\cdot\int_{[-h/2,h/2]^2}|\hat{f}^h(x)-\hat{f}^h(x-\tau)|^2 d\tau dx \\
&= h^{-2}\int_{[-h/2,h/2]^2} d\tau\|\hat{f}^h - {}^{t_\tau}\hat{f}^h\|^2_{\mathrm{L}^2},
\end{aligned} \tag{41}$$

where ${}^{t_\tau}\hat{f}^h$ denotes the translation of the function $\hat{f}^h$ by a vector $\tau\in\mathbf{R}^2$.

Let's now choose $\varepsilon > 0$. As $\hat{f}^{h_p}$ converges to u, and as the mapping from $\mathbf{R}^2$ into $\mathrm{L}^2(\mathbf{R}^2)$, $\tau \mapsto {}^{t_\tau}u$ is (uniformly) continuous, there exists $\alpha > 0$ and $N \in \mathbf{N}$ such that

(a) $|\tau|_\infty \leq \alpha \Rightarrow \|u - {}^{t_\tau} u\|_{\mathrm{L}^2} \leq \varepsilon$

(b) $p \geq N \Rightarrow \|\hat{f}^{h_p} - u\|_{\mathrm{L}^2} \leq \varepsilon$

(and therefore $\|{}^{t_\tau}\hat{f}^{h_p} - {}^{t_\tau} u\|_{\mathrm{L}^2} = \|\hat{f}^{h_p} - u\|_{\mathrm{L}^2} \leq \varepsilon$).

If p is large enough, such that we get simultaneously $p \geq N$ and $h_p \leq \alpha$, then for all τ with $|\tau|_\infty \leq h_p/2$,

$$\|\hat{f}^{h_p} - {}^{t_\tau} \hat{f}^{h_p}\|_{\mathrm{L}^2} \leq \|\hat{f}^{h_p} - u\|_{\mathrm{L}^2} + \|u - {}^{t_\tau} u\|_{\mathrm{L}^2} + \|{}^{t_\tau}u - {}^{t_\tau} \hat{f}^{h_p}\|_{\mathrm{L}^2} \leq 3\varepsilon,$$

and

$$\|\hat{f}^{h_p} - \check{f}^{h_p}\|^2_{\mathrm{L}^2} \leq h_p^{-2} \int_{[-h_p/2, h_p/2]^2} d\tau \|\hat{f}^{h_p} - {}^{t_\tau} \hat{f}^{h_p}\|^2_{\mathrm{L}^2} \leq 9\varepsilon^2.$$

This proves that $\|\hat{f}^{h_p} - \check{f}^{h_p}\|_{\mathrm{L}^2}$ converges to 0 and establishes (25) of Proposition 4.2. The other two relations are a straightforward consequence of (25) and of the fact that

$$\sum_{k,l} g^h_{k,l} \cdot \chi_{P^h_{k,l}} \xrightarrow{h \to 0} g$$

in $\mathrm{L}^2(\Omega)$ ($g_{k,l}$ is defined by (14)), which is a classical result. □

4.3. Conclusion. Point iii of Theorem 2.2 is a straightforward consequence of the previously shown propositions. Consider now f^h and u with the assumptions of point i of the theorem. Suppose we extract a subsequence f^{h_k} with

$$E = \lim_{k \to \infty} E^{h_k}(f^{h_k}) = \liminf_{h^{-1} = N \to \infty} E^h(f^h).$$

If $E = +\infty$, then point i is obviously true. Otherwise, there exists a constant $C < \infty$ such that $F^{h_k}(f^{h_k}) \leq E^{h_k}(f^{h_k}) \leq C$ and we conclude again with the propositions.

5. The proof of Theorem 2.2ii.

We would like to prove that for any (bounded) $u \in \mathrm{SBV}(\Omega)$, or at least for all u that satisfy some properties of the minimizers of (11), it is possible to build a sequence (f^h) of "images" at resolution h, with $h^{-1} = N \in \mathbf{N} \setminus \{0\}$, such that f^h converges to u, meaning for instance[8] that

$$\sum_{0 \leq k,l < N} f^h_{k,l} \chi_{P^h_{k,l}}(x) \to u(x) \text{ almost everywhere in } \Omega, \tag{42}$$

and with the property

$$\limsup_{h^{-1} = N \to +\infty} E^h(f^h) \leq E(u). \tag{43}$$

(We could also write the same equations with F^h and F instead of E^h and E.)

Such a result has been first proved only up to some assumptions on the regularity of $K = \overline{S_u}$, the closure of the jump set of u. However, we could state an approximation lemma that implies that these regularity assumptions need not be satisfied by the function u itself: The result still holds if we can merely find a sequence of functions converging to u and satisfying the requested properties. Dibos and Séré proved then that this is possible for any bounded $u \in \mathrm{SBV}(\Omega)$ with $\mathcal{H}^1(\overline{S_u} \backslash S_u) = 0$, and Dal Maso extended their result to any $u \in \mathrm{SBV}(\Omega)$ (see Appendix G).

[8] It is here equivalent to consider more elaborate extensions of f^h into functions such as the ones proposed in § 4.

5.1. The approximation lemma. In this section we just state the following simple result:

LEMMA 5.1. *Given* $u \in \mathrm{SBV}(\Omega)$, *suppose*

$$E(u) = \int_\Omega |\nabla u|^2 + \Lambda(S_u) + \int_\Omega |u-g|^2 < +\infty.$$

Also assume that for all (small) $\varepsilon > 0$, *there exists* $(u^\varepsilon, K^\varepsilon)$ *such that*

- K^ε *is closed and* $u^\varepsilon \in C^1(\Omega \setminus K^\varepsilon) \cap \mathrm{SBV}(\Omega)$,[9]
- $E(u^\varepsilon, K^\varepsilon) \le E(u) + \varepsilon$, *and*
- $u^\varepsilon(x) \xrightarrow{\varepsilon \searrow 0} u(x)$ *almost everywhere in* Ω.

Assume at last that for all ε *and for all* $h = 1/N$, $N \ge 1$, *there exists* $f^{h,\varepsilon} = (f^{h,\varepsilon}_{k,l})_{0\le k,l<N}$ *that satisfy*

$$\sum_{0\le k,l<N} f^{h,\varepsilon}_{k,l}\chi_{P^h_{k,l}}(x) \to u^\varepsilon(x) \ \text{a.e. in } \Omega,$$

$$\text{and} \quad \limsup_{h^{-1}=N\to+\infty} E^h(f^{h,\varepsilon}) \le E(u^\varepsilon, K^\varepsilon).$$

Then there exists (f^h) *that satisfies* (42) *and* (43)

$$\sum_{0\le k,l<N} f^h_{k,l}\chi_{P^h_{k,l}}(x) \to u(x) \ \text{a.e. in } \Omega;$$

$$\limsup_{h^{-1}=N\to+\infty} E^h(f^h) \le E(u).$$

(*We could in the whole lemma replace* E^h *and* E *with* F^h *and* F.)

The proof of this lemma is quite technical. It is given in Appendix D.

5.2. "Monotonous" curves in the two-dimensional plane. We define in this section what will be called in the sequel "monotonous" curves with respect to a basis (e_1, e_2) of $\mathbf{R}^2$.

DEFINITION 1. *Let* (e_1, e_2) *be an orthonormal basis in* $\mathbf{R}^2$. *We will call a "nondecreasing" curve a continuum* C (*i.e., a connected compact set*) *such that*

$$\forall x, y \in C, \quad \langle \vec{xy}, e_1\rangle \cdot \langle \vec{xy}, e_2\rangle \ge 0;$$

which means that for all $x \in C$ *the curve* C *entirely lies in what could be called the "northeast" and the "southwest" of* x. *We define the same way a "nonincreasing" curve, with the reverse inequality. A curve is "monotonous" if it is either nondecreasing or nonincreasing.*

We can easily check that such a curve can be described by a mapping $\xi_2 = \phi(\xi_1)$, (ξ_1, ξ_2) being the coordinates of a point of the plane in the basis $(\frac{1}{\sqrt{2}}(e_1+e_2), \frac{1}{\sqrt{2}}(e_2-e_1))$ [thus we could also write $-x_1 + x_2 = \sqrt{2}\phi(\frac{x_1+x_2}{\sqrt{2}})$]. Here ϕ is a Lipschitz-continuous function with $\|\phi'\|_\infty \le 1$. The Hausdorff length along the curve satisfies the following equality:

$$d\mathcal{H}^1(\xi, \phi(\xi)) = \sqrt{1+\phi'(\xi)^2}\,d\xi, \tag{44}$$

[9] In fact, $u^\varepsilon \in \mathrm{W}^{1,1}(\Omega \setminus K^\varepsilon) \cap \mathrm{SBV}(\Omega)$—or $S_{u^\varepsilon} \subset K^\varepsilon$—is enough.

and similarly, we have

$$d\Lambda(\xi, \phi(\xi)) = \sqrt{2}d\xi. \tag{45}$$

As a consequence of (45), we have the following proposition.

PROPOSITION 5.2. *The length $\Lambda(C)$ of a monotonous curve C only depends on its endpoints A and B. If the coordinates of A and B are, resp., (a_1, a_2) and (b_1, b_2) in the basis (e_1, e_2),*

$$\Lambda(C) = |a_1 - b_1| + |a_2 - b_2|.$$

5.3. The result. We are now in order to state and prove the main result of § 5.

PROPOSITION 5.3. *Let $u \in \mathrm{SBV}(\Omega)$ be bounded and let $K \supseteq \overline{S_u}$ be a closed set. We have $u \in W^{1,1}(\Omega \setminus K)$, and if $E(u, K) < +\infty$, then*

(a) *if K is made up of a finite union of monotonous curves which don't intersect each other but at their endpoints, there exists a sequence $(f^h)_{h=N^{-1}>0}$ with* (42) *and*

$$\limsup_{h^{-1}=N\to+\infty} E^h(f^h) \le E(u, K); \tag{46}$$

of course, if $\mathcal{H}^1(K \setminus S_u) = 0$, (f^h) also satisfies (43).

Using Lemma 5.1 *leads to the following result: let $u \in \mathrm{SBV}(\Omega)$*

(b) *if u is the limit (in the sense of the lemma) of bounded functions $(u^\varepsilon, K^\varepsilon)$ such that K^ε has the properties of K in point* (a), *then there exists a sequence $(f^h)_{h=N^{-1}>0}$ with* (42) *and* (43).

Remark. As proved by Dibos, Séré, and Dal Maso, any $u \in \mathrm{SBV}(\Omega)$ satisfies (b).

We just need to prove point (a) of Proposition 5.3.

Proof. Suppose u and K satisfy the assumptions of point (a). We have that $u \in W^{1,1}(\Omega \setminus K)$ and $K \supseteq \overline{S_u}$ is a finite union of monotonous curves, say $L_1, \ldots, L_n$. Notice that it is enough to build a sequence $(f^h)_{h=N^{-1}>0}$ with (42) and

$$\limsup_{h^{-1}=N\to+\infty} F^h(f^h) \le F(u, K) = \int_\Omega |\nabla u|^2 + \Lambda(K) < +\infty. \tag{47}$$

Condition (42) indeed implies that

$$\lim_{h^{-1}=N\to+\infty} h^2 \sum_{0\le k,l<N} |f^h_{k,l} - g^h_{k,l}|^2 = \int_\Omega |u - g|^2.$$

Let $\xi_1, \ldots, \xi_p$ be the endpoints of curves L_j (remember $L_j \cap L_{j'}$ contains nothing but 0, 1, or 2 points ξ_i), and choose $\varepsilon > 0$. Each ξ_i is the center of a square Γ_i of perimeter ε/p and whose sides are parallel to the coordinate axes. The total length $\Lambda(\partial(\cup_{i=1}^p \Gamma_i)) = \mathcal{H}^1(\partial(\cup_{i=1}^p \Gamma_i))$ doesn't exceed ε.

As the n curves[10] $L_j^\varepsilon = L_j \backslash (\cup_{i=1}^p \mathrm{Int}(\Gamma_i))$ are disjoint compact sets, we may choose h small enough to ensure that if $i \neq j$,

$$\{x \ : \ d(x, L_i^\varepsilon) < 3h\} \cap \{x \ : \ d(x, L_j^\varepsilon) < 3h\} = \emptyset. \tag{48}$$

We now decompose domain Ω in three subsets, say, $\Omega = A_h \cup B_h \cup C_h$, where

- A_h is the union of pixels $P_{k,l}^h$ that meet the interior of a square Γ_i,
- B_h are the pixels that meet a curve L_j but no square Γ_i,
- C_h are the other pixels of Ω.

We then set

- $f_{k,l}^h = 0$ when $x_{k,l} \in A_h$ ($\Leftrightarrow P_{k,l}^h \subset A_h$),
- $f_{k,l}^h = h^{-2} \int_{P_{k,l}^h} u(x,y)dxdy$ when $x_{k,l} \in C_h$,
- and for $x_{k,l} \in B_h$ we'll explain the construction of $f_{k,l}^h$ later.

The energy $F^h(f^h)$ is to be split into three parts. It is the sum, for all pairs of adjacent pixels of $\overline{\Omega}$, for instance $P_{k,l}^h$ and $P_{k+1,l}^h$, of

$$\min\{|f_{k,l}^h - f_{k+1,l}^h|^2, h\},$$

which is, by definition, lower than $h = \Lambda(P_{k,l}^h \cap P_{k+1,l}^h)$ and $|f_{k,l}^h - f_{k+1,l}^h|^2$. We will estimate independently the contribution to the energy of

(1) adjacent pixels with one of those (or both) included in A_h,
(2) adjacent pixels both included in C_h,
(3) adjacent pixels with one included in B_h and the other in $B_h \cup C_h = \overline{\Omega} \backslash \mathrm{Int}(A_h)$.

Following this classification we write the energy $F^h(f^h) = F_1 + F_2 + F_3$.

An estimate of F_1. We have $F_1 \leq \varepsilon + 8ph$, as A_h is made up of p rectangles whose sides are parallel to the coordinate axes and with length at most $\varepsilon/4p + 2h$, and as $f_{k,l}^h = 0$ inside A_h.

An estimate of F_2. Here more computation is needed. We have

$$F_2 \leq \sum_{x_{k,l}, x_{k+1,l} \in C_h} |f_{k,l}^h - f_{k+1,l}^h|^2 + \sum_{x_{k,l}, x_{k,l+1} \in C_h} |f_{k,l}^h - f_{k,l+1}^h|^2,$$

and when $x_{k,l} \in C_h$ and $x_{k+1,l} \in C_h$,

$$\begin{aligned} |f_{k,l}^h - f_{k+1,l}^h|^2 &= h^{-4} \left| \int_{P_{k,l}^h} u(x+h,y) - u(x,y) dxdy \right|^2 \\ &= h^{-4} \left| \int_{P_{k,l}^h} \left(\int_{t=0}^h \frac{\partial u}{\partial x}(x+t,y)dt \right) dxdy \right|^2 \\ &\leq h^{-4} \cdot h^3 \int_{P_{k,l}^h} \int_{t=0}^h \left| \frac{\partial u}{\partial x}(x+t,y) \right|^2 dtdxdy \\ &= h^{-1} \int_{t=0}^h \left(\int_{P_{k,l}^h} \left| \frac{\partial u}{\partial x}(x+t,y) \right|^2 dxdy \right) dt. \end{aligned}$$

[10] It may happen that one of those curves is split in two or more pieces by one or several squares Γ_i, in which case we have to consider more than just n curves. Taking into account this possibility only makes the demonstration heavier.

Consequently,

$$\begin{aligned}\sum_{x_{k,l},x_{k+1,l}\in C_h} &|f^h_{k,l}-f^h_{k+1,l}|^2\\ &\le h^{-1}\int_{t=0}^{h}\Bigg(\sum_{x_{k,l},x_{k+1,l}\in C_h}\int_{P^h_{k,l}}\left|\frac{\partial u}{\partial x}(x+t,y)\right|^2 dxdy\Bigg)dt\\ &\le h^{-1}\int_{t=0}^{h}\left(\int_\Omega\left|\frac{\partial u}{\partial x}\right|^2\right)dt\\ &=\int_\Omega\left|\frac{\partial u}{\partial x}\right|^2.\end{aligned}$$

This proves that

$$F_2 \le \int_\Omega |\nabla u|^2. \tag{49}$$

An estimate of F_3. Condition (48) ensures that domain B_h is made up of n disjoint pieces B^i_h, $i=1,\dots,n$, each piece B^i_h being a connected union of pixels meeting the monotonous curve L_i and no square Γ_j, $j=1,\dots,p$ (see Fig. 9).

These pixels can be split into two groups separated by a monotonous curve $L^h_i \in \mathcal{Z}_h$ included in B^i_h that runs from one "end"[11] to the other of the set B^i_h—see Fig. 10. Moreover, Proposition 5.2 implies that

$$\Lambda(L^h_i) \le \Lambda(L_i).$$

Remark. L^h_i could basically be any monotonous curve in $\mathcal{Z}_h \cap B^i_h$. For some inappropriate choice we might have $\Lambda(L^h_i) \le \Lambda(L_i) + ch$, with c a small constant instead

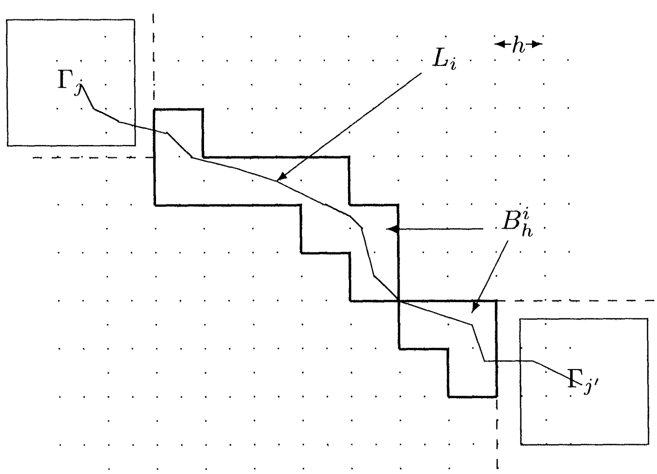

FIG. 9. *Domain B^i_h between two squares Γ_j and $\Gamma_{j'}$.*

[11] We mean that the curve L^h_i ends on the squares Γ_j—or rather the set A_h—that terminate B^i_h on both sides.

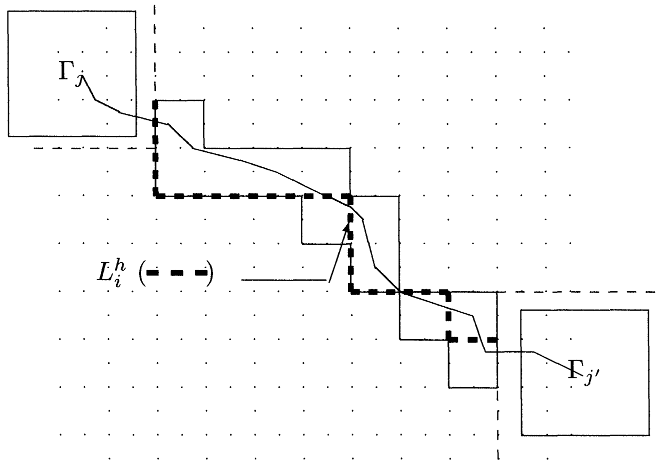

FIG. 10. *A possible example of curve L_i^h associated with B_h^i.*

of the previous formula, and there should be a "best choice" that minimizes constant c in (50), but this discussion is irrelevant for the present demonstration.

The total contribution to F_3 of the pairs of adjacent pixels separated by the curve L_i^h (one being thus in B_h^i, and the other in B_h^i or C_h, the pixels of A_h being already considered in F_1, and the other domains $B_h^{i'}$ being "too far" because of condition (48)) is less than $\Lambda(L_i^h)$ and therefore less than $\Lambda(L_i)$.

We just need to show that it is possible to choose the values of $f_{k,l}^h$ inside B_h^i in such a way that the contribution to F_3 of adjacent pixels, which are not separated by the curve L_i^h, is small. A simple but rather technical study would establish that we can choose for $f_{k,l}^h$ (i.e., at pixel $P_{k,l}^h \subset B_h^i$) the mean of the values of the adjacent pixels not separated from $P_{k,l}^h$ by L_i^h and that are in C_h, or, in the case where no such pixel exists, the mean of the ones that are in B_h. With these values, the total contribution to F_3 of B_h^i is less than

$$\Lambda(L_i) + c\sum |f_{k,l}^h - f_{k',l'}^h|^2$$

where the sum is taken over the pairs $(P_{k,l}^h, P_{k',l'}^h)$ of adjacent pixels that are both included in C_h and in the set $\{x : d(x, L_i^\varepsilon) < 3h\}$, and c is a universal constant. As seen in the previous paragraph, we may show that this sum is not greater than $\int_{\{x\ :\ d(x,L_i^\varepsilon)<3h\}} |\nabla u|^2$, and consequently

$$F_3 \le \Lambda(K) + c\int_{\{x\ :\ d(x,K)<3h\}} |\nabla u|^2. \tag{50}$$

Conclusion. All this shows that it is possible to build a sequence (f^h) which converges almost everywhere to u on $\Omega \setminus (\cup_{i=1}^p \Gamma_i)$ as h goes to zero, and such that

$$F^h(f^h) \le F(u,K) + \varepsilon + 8ph + c\int_{\{x\ :\ d(x,K)<3h\}} |\nabla u|^2,$$

and therefore

$$\limsup_{h^{-1}=N\to+\infty} F^h(f^h) \le F(u,K) + \varepsilon.$$

It is now easy to conclude (by building a diagonal sequence or applying directly Lemma 5.1) and achieve the proof of Proposition 5.3. ☐

Appendix A. The "special bounded variation" functions. Here is a short introduction to SBV functions. See also, for instance, [26], [35], and [41].

Recall that if $I = (a, b) \subset \mathbf{R}$ is a bounded interval, a bounded variation function $u : I \to \mathbf{R}$ is a function which satisfies

$$V_I(u) = \sup\left\{\sum_{i=1}^{k-1} |u(t_{i+1}) - u(t_i)| \ : \ a < t_1 < \cdots < t_k < b\right\} < +\infty.$$

$V_I(u)$ is the *total variation* of u on I, and $\mathrm{BV}(I)$ is the space of Borel functions $u : I \to \mathbf{R}$ such that the *essential total variation* of u is bounded:

$$\text{ess--}V_I(u) = \inf\{V_I(v) \ : \ v = u \text{ almost everywhere } \} < +\infty.$$

A fundamental property of $\mathrm{BV}(I)$ is that $u \in \mathrm{BV}(I)$ if and only if $u \in \mathrm{L}^1(I)$ and the distributional derivative $u' = Du$ of u defines a bounded Radon measure over I, $Du \in \mathrm{M}^1(I)$. Moreover, ess–$V_I(u) = \int_a^b |Du|$.

This allows one to generalize this definition to higher dimensions. For a Borel function u defined over a n-dimensional open domain $\Omega \subset \mathbf{R}^n$, we define

$$u \in \mathrm{BV}(\Omega) \Leftrightarrow u \in \mathrm{L}^1(\Omega) \text{ and } Du \in \mathrm{M}^1(\Omega).$$

For all x in Ω, we define the *approximate upper limit* of u at x as the greater lower bound of the set of real numbers $t \in [-\infty, +\infty]$ such that $\{x \in \Omega \ : \ u(x) > t\}$ has 0 density at x, i.e.,

$$u^+(x) = \inf\left\{t \in [-\infty, +\infty] \ : \ \lim_{r \searrow 0^+} \frac{\operatorname{meas}(\{u > t\} \cap B_r(x))}{r^n} = 0\right\},$$

and we define in the same way the *approximate lower limit* $u^-(x)$. We set

$$S_u = \{x \in \Omega, u^-(x) < u^+(x)\}.$$

This is the set of essential discontinuities of u, and when $x \notin S_u$ we say that u is *approximately continuous* at x. In this case we write $\tilde{u}(x) = u^-(x) = u^+(x)$, and as $\mathrm{BV}(\Omega) \subset \mathrm{L}^\infty(\Omega)$ it may be shown that x is a Lebesgue point of u, i.e., such that

$$\exists z \ [= \tilde{u}(x)], \ \lim_{r \searrow 0^+} \frac{1}{r^n} \int_{B_r(x)} |u(y) - z| dy = 0.$$

As almost all points are Lebesgue points, $\operatorname{meas}(S_u) = 0$.

But there is more: S_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable in the sense of Federer [26] and therefore has an *approximate tangent hyperplane* (*or an approximate normal vector*) in $\mathcal{H}^{n-1}$-almost all point. The normal vector is the unique $\nu \in \mathbf{S}^{n-1}$ such that for all $\epsilon > 0$,

$$\lim_{r \searrow 0^+} \frac{\operatorname{meas}\{y \in B_r(x) \ : \ \langle y - x, \nu\rangle > 0, |u(y) - u^+(x)| > \epsilon\}}{r^n} = 0$$

(or equivalently,

$$\lim_{r \searrow 0^+} \frac{\operatorname{meas}\{y \in B_r(x) \ : \ \langle y - x, \nu\rangle < 0, |u(y) - u^-(x)| > \epsilon\}}{r^n} = 0 \ ).$$

In dimension two, this definition coincides almost everywhere with the definition of the tangent vector to a "regular 1-set," as given by Besicovitch, already mentioned in § 3.2.

When u is approximately continuous at x, we say it is *approximately differentiable* if there exists a vector $\nabla u(x)$ such that

$$\{y \in \Omega \ : \ |u(y) - \tilde{u}(x) - \langle \nabla u(x), y - x \rangle| > \epsilon |y - x|\}$$

has zero density at x for all $\epsilon > 0$; or equivalently, if there exists a function g on an open domain $U \ni x$ that is differentiable at x and such that $\{g \neq u\}$ has zero density at x. If $u \in \mathrm{BV}(\Omega)$, then ∇u exists almost everywhere.

For all $u \in \mathrm{BV}(\Omega)$, the distributional derivative Du may be decomposed in three parts [1],

$$Du = \nabla u dx + Ju + Cu,$$

where $\nabla u dx$ is the absolutely continuous part with respect to Lebesgue measure dx (and ∇u therefore appears as the Radon–Nikodým derivative of Du w.r.t. dx),

$$Ju(B) = \int_{B \cap S_u} (u^+ - u^-) \nu_u d\mathcal{H}^{n-1}$$

for all Borel subset $B \subset \Omega$, and Cu is a bounded Radon measure over Ω such that $\mathcal{H}^{n-1}(B) < +\infty \Rightarrow Cu(B) = 0$ for all Borel subset B.

$\mathrm{SBV}(\Omega)$ is then defined by

$$\mathrm{SBV}(\Omega) = \{u \in \mathrm{BV}(\Omega) \ : \ Cu = 0\}.$$

We thus have $u \in \mathrm{SBV}(\Omega)$ if and only if $u \in \mathrm{BV}(\Omega)$ and

$$Du \prec dx + \mathcal{H}^{n-1}|_{S_u}.$$

Ambrosio [1] established his Theorem 1.1 of compactness and lower semicontinuity in $\mathrm{SBV}(\Omega)$, which appears to be the right space for dealing with functional (3)–(4).

Appendix B. The Γ-convergence. Γ-convergence for variational problems was invented by De Giorgi [34] to deal with sequences and limits of variational problems (see [3], [36], and [41]).

DEFINITION 2. *Let (S, d) be a separable metric space and $F_n : S \to [0, +\infty]$ a sequence of functions. (F_n) $\Gamma(S)$ converges to $F : S \to [0, +\infty]$ if for all $x \in S$,*

a. $\forall (x_n), \lim_{n\to\infty} x_n = x$, *we have* $\liminf_{n\to\infty} F_n(x_n) \geq F(x)$,

b. $\exists (x_n)$ *such that* $\lim_{n\to\infty} x_n = x$ *and* $\limsup_{n\to\infty} F_n(x_n) \leq F(x)$.

When the limit F exists, it is unique and lower semicontinuous. It may be useful to join this definition to the following compactness property—which is not part of the definition of Γ-convergence itself, but whose utility is made clear in what follows:

c. *If the sequence (x_n) is such that for all n, $F_n(x_n) \leq K < +\infty$, it is relatively compact in S.*

We have the next proposition (see, for instance, [3]).

PROPOSITION B.1. *If F_n $\Gamma(S)$ converges to F, then*

- *$F_n + G$ $\Gamma(S)$ converges to $F + G$ for all continuous $G : S \to \mathbf{R}$.*
- *If $\varepsilon_n \searrow 0$, the cluster points of the sets*

$$\{x \in S \ : \ F_n(x) \leq \inf_S F_n + \varepsilon_n\}$$

are minimizers of F.

- *If the functions F_n are lower semicontinuous and if for all $t \geq 0$, there is a compact subset K_t of S such that for all n,*

$$\{x \in S \ : \ F_n(x) \leq t\} \subset K_t$$

(for instance if c is true), then the functions F_n have minimizers in S and any sequence (x_n) of minimizers of F_n admits subsequences that converge to a minimizer of F.

Appendix C. Existence for the anisotropic problem. In this section we deal with the anisotropic version of Mumford and Shah's variational problem. That is to say, problem (3), with Λ instead of $\mathcal{H}^1$: for a given Borel function $g : \Omega \to [0,1]$ (the input image), find a pair (u, K) with

- $K \subset \overline{\Omega}$ a closed one-dimensional set (or a closed 1-set),
- $u \in C^1(\Omega \setminus K)$,

minimizing

$$E(u,K) = \lambda^2 \int_{\Omega\setminus K} |\nabla u|^2 + \alpha\Lambda(K) + \int_\Omega |u-g|^2 = F(u,K) + \int_\Omega |u-g|^2. \tag{51}$$

For the original problem, we know that there is a solution (u, K); see De Giorgi, Carriero, and Leaci [33] (in any dimension), and also Dal Maso, Morel, and Solimini [37]. (These authors also proved that there exists a minimizing sequence (u_n, K_n), where K_n is a finite union of Lipschitz-regular curves.[12])

Most of these proofs are based on the compactness result of Ambrosio [1] (quoted in Theorem 1.1), and can be adapted for problem (51).[13] Let's first consider the weak formulation of (51), defined for any $u \in \mathrm{SBV}(\Omega)$ (we set $\lambda = \alpha = 1$):

$$E(u) = \int_\Omega |\nabla u|^2 + \Lambda(S_u) + \int_\Omega |u-g|^2 = F(u) + \int_\Omega |u-g|^2. \tag{52}$$

We must check that [37, Thm. 0.4] still holds for (51) and (52).

THEOREM C.1 (Existence theorem). *The functionals* (51) *and* (52) *reach their minimum. Moreover, their minimum values are the same, and the corresponding functions satisfy the following items:*

(a) *if $u \in \mathrm{SBV}(\Omega)$ minimizes* (52)*, then $(u, \overline{S}_u)$ minimizes* (51) *and $\mathcal{H}^1(\overline{S}_u \setminus S_u) = 0$;*

(b) *if (u, K) minimizes* (51) *then $u \in \mathrm{SBV}(\Omega)$ and u minimizes* (52)*, moreover $\overline{S}_u \subseteq K$ and $\mathcal{H}^1(K \setminus S_u) = 0$.*

Proof. First we need to show that (52) has a minimizer in $\mathrm{SBV}(\Omega)$. It is a direct consequence of Ambrosio's compactness result. It shows that if (u_n) are $\mathrm{SBV}(\Omega)$ functions such that

$$\int_\Omega |\nabla u_n|^2 + \mathcal{H}^1(S_{u_n}) + \int_\Omega |g-u_n|^2 \leq K < +\infty,$$

then there exists a subsequence (u_{n_k}) of (u_n) and $u \in \mathrm{SBV}(\Omega)$ with $u_{n_k} \to u$ a.e. in Ω, moreover the sequence of the approximate gradients (∇u_{n_k}) weakly converges to

[12] See footnote 1.

[13] For instance all the results of Chapters 1, 2, and 3 of [37] also hold for this problem, just because Λ satisfies (18).

∇u in $\mathrm{L}^1(\Omega, \mathbf{R}^2)$. It is not difficult to see that $\nabla u_{n_k} \rightharpoonup \nabla u$ also weakly in $\mathrm{L}^2(\Omega)$, and if this result is applied to a minimizing sequence of (52) it follows that

$$\int_\Omega |\nabla u|^2 \leq \liminf_{k\to+\infty} \int_\Omega |\nabla u_{n_k}|^2$$

and by Fatou's lemma

$$\int_\Omega |g-u|^2 \leq \liminf_{k\to+\infty} \int_\Omega |g-u_{n_k}|^2.$$

Now we must check that

$$\Lambda(S_u) \leq \liminf_{k\to+\infty} \Lambda(S_{u_{n_k}}).$$

It is a consequence of Ambrosio's intermediate result ((4.4) in [1, p. 875])

$$\int_{S_u} |\langle \nu_u, \nu\rangle| \Theta(|u^+ - u^-|) d\mathcal{H}^1 \leq \liminf_{k\to+\infty} \int_{S_{u_{n_k}}} |\langle \nu_{u_{n_k}}, \nu\rangle| \Theta(|u^+_{n_k} - u^-_{n_k}|) d\mathcal{H}^1,$$

where ν_u is the approximate normal vector to S_u, ν is any vector with $\|\nu\| = 1$, and Θ is any concave nonincreasing function defined on $]0, +\infty]$ with

$$\Theta(+\infty) = \lim_{+\infty} \Theta, \ \lim_{t\to 0^+} \frac{\Theta(t)}{t} = +\infty, \ \lim_{0^+} \Theta = 0, \quad \text{and } \forall t < +\infty,\ \Theta(t) < +\infty.$$

If we choose for Θ the sequence $\theta_h(t) = \min\{1, h\sqrt{t}\}$, we get, applying the result to the vectors $\nu = (1,0)$ and $(0,1)$

$$\begin{aligned}\int_{S_u} \|\nu_u\|_1 \theta_h(|u^+ - u^-|) d\mathcal{H}^1 &\leq \liminf_{k\to+\infty} \int_{S_{u_{n_k}}} \|\nu_{u_{n_k}}\|_1 \theta_h(|u^+ - u^-|) d\mathcal{H}^1 \\ &\leq \liminf_{k\to+\infty} \int_{S_{u_{n_k}}} \|\nu_{u_{n_k}}\|_1 d\mathcal{H}^1 \\ &= \liminf_{k\to+\infty} \Lambda(S_{u_{n_k}}).\end{aligned}$$

Since by definition, $u^+ - u^- > 0$ on S_u, $\theta_h(|u^+ - u^-|) \nearrow 1$ as $h \to +\infty$ in any point of S_u, and we eventually get

$$\int_\Omega |\nabla u|^2 + \Lambda(S_u) + \int_\Omega |g-u|^2 \leq \liminf_{k\to+\infty} \int_\Omega |\nabla u_{n_k}|^2 + \Lambda(S_{u_{n_k}}) + \int_\Omega |g - u_{n_k}|^2.$$

This is enough to show that (52) has a minimizer in $\mathrm{SBV}(\Omega)$.

We now must check that if u minimizes (52), then $\mathcal{H}^1(\overline{S}_u \setminus S_u) = 0$. This follows from the fact that when $\mathcal{H}^1(E) < +\infty$,

$$\lim_{\rho \searrow 0^+} \frac{\mathcal{H}^1(E \cap B_\rho(x))}{2\rho} = 0,$$

for $\mathcal{H}^1$-almost all $x \in \mathbf{R}^2 \setminus E$ (see [26, 2.10.19(4)]), and from the following *elimination lemma* established by Dal Maso, Morel, and Solimini ([37, Lem. 0.7]).

LEMMA C.2. *There exists an absolute constant $\beta > 0$ (not depending on g and Ω) such that if u is a minimizer of* (52) *in* $\mathrm{SBV}(\Omega)$, *for all discs $D_R = B_R(x_0)$ with $x_0 \in \overline{\Omega}$, $0 < R < \min\{1, \sigma\}$, and*

$$\mathcal{H}^1(S_u \cap D_R) < \beta R,$$

we have $S_u \cap B_{R/2}(x_0) = \emptyset$.

Here σ is the length of the shortest side of the rectangle Ω. This lemma is shown for the original Mumford and Shah functional, but the proof still holds when $\mathcal{H}^1$ is replaced with Λ or any Hausdorff-type length, satisfying inequalities such as (18).

The remainder of the proof is the same as in [37]. We need in particular to notice that if (u, K) minimizes (51),

$$\int_\Omega |\nabla u| \leq \sqrt{\mathrm{meas}(\Omega)} \left(\int_\Omega |\nabla u|^2 \right)^{\frac{1}{2}} < +\infty,$$

thus $u \in \mathrm{C}^1(\Omega \setminus K) \cap \mathrm{W}^{1,1}(\Omega \setminus K) \cap \mathrm{L}^\infty(\Omega)$ and (after [33, Lem. 2.3]), $u \in \mathrm{SBV}(\Omega)$ and $\overline{S_u} \subset K$. □

Appendix D. Proof of Lemma 5.1.

Proof. Let ε_n be the sequence 2^{-n}, or any convergent and nonincreasing series. Let also $N_0 = 0$. By induction we can build an increasing sequence of integers N_n such that

$$N = h^{-1} \geq N_n \Rightarrow E^h(f^{h,\varepsilon_n}) \leq E(u^{\varepsilon_n}, K^{\varepsilon_n}) + \varepsilon_n \leq E(u) + 2\varepsilon_n = E(u) + 2^{-n+1}.$$

If for all $N = h^{-1}$, we let $f^h = f^{h,\varepsilon_n}$ when $N_n \leq N < N_{n+1}$, then the sequence (f^h) satisfies (43).

This is not enough to ensure that (f^h) converges to u.[14] For all $\varepsilon > 0$ and all h, let

$$U_h^\varepsilon = \left\{ x \in \Omega \ : \ \sup_{h' \leq h} |f^{h',\varepsilon}(x) - u^\varepsilon(x)| \geq \varepsilon \right\}.$$

This is a decreasing sequence and if we let $U^\varepsilon = \cap_h U_h^\varepsilon$, we have $\lim_{h \searrow 0} \mathrm{meas}(U_h^\varepsilon) = \mathrm{meas}(U^\varepsilon)$ (as $\mathrm{meas}(\Omega) < +\infty$). But if $x \in U^\varepsilon$,

$$\limsup_{h \searrow 0} |f^{h,\varepsilon}(x) - u^\varepsilon(x)| \geq \varepsilon,$$

therefore as $f^{h,\varepsilon}$ converges to u^ε, $\mathrm{meas}(U^\varepsilon) = 0$, i.e., $\lim_{h \searrow 0} \mathrm{meas}(U_h^\varepsilon) = 0$.

If we also ask that N_n satisfy

$$N = h^{-1} \geq N_n \Rightarrow \mathrm{meas}(U_h^{\varepsilon_n}) \leq \varepsilon_n,$$

then we will show that $f^h(x)$ converges almost everywhere in Ω to $u(x)$. For every integer $k \geq 1$ let

$$\omega_k = \left\{ x \in \Omega \ : \ \limsup_{h \searrow 0} |f^h(x) - u(x)| \geq k^{-1} \right\}$$

[14] For simplicity's sake f^h will also denote here the function $\sum_{0 \leq k,l < N} f_{k,l}^h \chi_{P_{k,l}^h}(x)$.

and

$$\omega = \bigcup_{k\geq 1} \omega_k = \Big\{ x \in \Omega \ : \ \limsup_{h\searrow 0} |f^h(x) - u(x)| > 0 \Big\}.$$

We need to show that $\mathrm{meas}(\omega) = 0$, or that for any $k \geq 1$ $\mathrm{meas}(\omega_k) = 0$. Let's choose an integer k, $\delta > 0$ and n_0 an integer such that $\varepsilon_{n_0} < k^{-1}/2$ and $\sum_{n\geq n_0} \varepsilon_n < \delta$. We can check that

$$\begin{aligned}
&\{x \in \Omega \ : \sup_{n\geq n_0, N_n \leq h^{-1} < N_{n+1}} |f^{h,\varepsilon_n}(x) - u^{\varepsilon_n}(x)| \geq k^{-1}/2\} \\
\subset &\bigcup_{n\geq n_0} \{x \in \Omega \ : \max_{N_n \leq h^{-1} < N_{n+1}} |f^{h,\varepsilon_n}(x) - u^{\varepsilon_n}(x)| \geq \varepsilon_{n_0}\} \\
\subset &\bigcup_{n\geq n_0} \{x \in \Omega \ : \max_{N_n \leq h^{-1} < N_{n+1}} |f^{h,\varepsilon_n}(x) - u^{\varepsilon_n}(x)| \geq \varepsilon_{n}\} \\
\subset &\bigcup_{n\geq n_0} U^{\varepsilon_n}_{N_n^{-1}},
\end{aligned}$$

and therefore

$$\mathrm{meas}\Big(\Big\{x \in \Omega \ : \sup_{n\geq n_0, N_n \leq h^{-1} < N_{n+1}} |f^{h,\varepsilon_n}(x) - u^{\varepsilon_n}(x)| \geq k^{-1}/2\Big\}\Big) \leq \sum_{n\geq n_0} \varepsilon_n < \delta. \tag{53}$$

Moreover, as $\lim_{n\to\infty} u^{\varepsilon_n}(x) = u(x)$ almost everywhere in Ω, n_0 may be chosen large enough so that

$$\mathrm{meas}\Big(\Big\{x \in \Omega \ : \ \sup_{n\geq n_0} |u^{\varepsilon_n}(x) - u(x)| \geq k^{-1}/2\Big\}\Big) < \delta. \tag{54}$$

Then for any h' with $h'^{-1} \geq N_{n_0}$, if $n' \geq n_0$ is such that $N_{n'} \leq h'^{-1} < N_{n'+1}$,

$$\begin{aligned}
|f^{h'}(x) - u(x)| &\leq |f^{h',\varepsilon_{n'}}(x) - u^{\varepsilon_{n'}}(x)| + |u^{\varepsilon_{n'}}(x) - u(x)| \\
&\leq \Big(\max_{N_{n'} \leq h^{-1} < N_{n'+1}} |f^{h,\varepsilon_{n'}}(x) - u^{\varepsilon_{n'}}(x)|\Big) + |u^{\varepsilon_{n'}}(x) - u(x)| \\
&\leq \sup_{n\geq n_0, N_n \leq h^{-1} < N_{n+1}} |f^{h,\varepsilon_n}(x) - u^{\varepsilon_n}(x)| + \sup_{n\geq n_0} |u^{\varepsilon_n}(x) - u(x)|.
\end{aligned}$$

Thus,

$$\sup_{h^{-1}\geq N_{n_0}} |f^h(x) - u(x)| \leq \sup_{n\geq n_0, N_n \leq h^{-1} < N_{n+1}} |f^{h,\varepsilon_n}(x) - u^{\varepsilon_n}(x)| + \sup_{n\geq n_0} |u^{\varepsilon_n}(x) - u(x)|$$

and if $x \in \omega_k$, we get $\sup_{h^{-1}\geq N_{n_0}} |f^h(x) - u(x)| \geq k^{-1}$, and therefore either

$$\sup_{n\geq n_0, N_n \leq h^{-1} < N_{n+1}} |f^{h,\varepsilon_n}(x) - u^{\varepsilon_n}(x)| \geq k^{-1}/2,$$

or

$$\sup_{n\geq n_0} |u^{\varepsilon_n}(x) - u(x)| \geq k^{-1}/2.$$

It follows that

$$\omega_k \subset \left\{x \in \Omega \ : \sup_{n \geq n_0, N_n \leq h^{-1} < N_{n+1}} |f^{h,\varepsilon_n}(x) - u^{\varepsilon_n}(x)| \geq k^{-1}/2\right\}$$
$$\bigcup \left\{x \in \Omega \ : \ \sup_{n \geq n_0} |u^{\varepsilon_n}(x) - u(x)| \geq k^{-1}/2\right\}$$

and from (53) and (54) we get $\mathrm{meas}(\omega_k) < 2\delta$. This proves the lemma, since δ may be taken as small as we want. □

Appendix E. Algorithms for segmentation. In this section, we explain the algorithms we used to obtain the results shown in Figs. 2, 4, and 5.

Appendix E.1. Segmentation in one dimension. Figs. 2 and 4 (§ 2.1) show the segmentations of two signals. They are true minimizers of Mumford and Shah's one-dimensional functional, computed for some values of the parameters α and λ. Of course, the signals presented in these figures are discrete signals and the energy which was minimized is a discrete functional such as (7). We used a dynamic programming method proposed by Mumford and Shah [39], and explained in the sequel.

Suppose we consider energy (7) with, for simplification, $h = 1$ and the index k running from 0 to $N-1$—signal g_k having N samples. Our problem is to minimize

$$E(f) = \sum_{k=1}^{N-1} W(|f_k - f_{k-1}|) + \sum_{k=0}^{N-1} |f_k - g_k|^2 \tag{55}$$

over $f = (f_0, \ldots, f_{N-1})$. Remember that for $h = 1$, $W(x) = \min\{\lambda^2 x^2, \alpha\}$.

Notice that once we find the number and positions of the discontinuities of the solution (i.e., the indices k such that $|f_k - f_{k-1}| > \sqrt{\alpha}/\lambda$ and therefore $W(|f_k - f_{k-1}|) = \alpha < \lambda^2 |f_k - f_{k-1}|^2$) for some given values of λ and α, it is very easy to compute the solution itself. Actually, between two discontinuities k_l and k_{l+1}—i.e., two jumps of f between $k_l - 1$ and k_l and between $k_{l+1} - 1$ and k_{l+1}—the vector $(f_k)_{k_l \leq k < k_{l+1}}$ is given by

$$\text{minimize} \ \left\{\lambda^2 \sum_{k_l < k < k_{l+1}} |f_k - f_{k-1}|^2 + \sum_{k_l \leq k < k_{l+1}} |g_k - f_k|^2\right\} \tag{56}$$

over $(f_k)_{k_l \leq k < k_{l+1}} \in \mathbf{R}^m$, where $m = k_{l+1} - k_l$. This is a simple convex problem, which may be written

$$(I_m + \lambda^2 A_m) F_{k_l}^{k_{l+1}-1} = G_{k_l}^{k_{l+1}-1}, \tag{57}$$

where I_m, $A_m \in \mathcal{M}_m(\mathbf{R})$ are the following mth-order matrices

$$I_m = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}, A_m = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix},$$

and $X_{k_l}^{k_{l+1}-1}$ is the vector $(x_{k_l}, \cdots, x_{k_{l+1}-1})^{\mathrm{T}}$.

Mumford and Shah's method for finding the positions of the breaks is the following: first define a "partial" Mumford and Shah energy, depending on a position k_0 and a number of breaks n

$$E^n_{k_0}((f_k)_{k\ge k_0};k_1;\cdots;k_n) = \lambda^2\sum_{l=0}^{n}\sum_{k_l<k<k_{l+1}}|f_k-f_{k-1}|^2+\sum_{k=k_0}^{N-1}|f_k-g_k|^2, \tag{58}$$

where the n breaks are located just before k_1, k_2, $\ldots$, k_n and we set $k_{n+1}=N$ (we obviously assume $k_0<k_1<\cdots<k_{n+1}$).

Notice that for a given f with n breaks, i.e., n positions $0<k_1<\cdots<k_n<N$ where $|f_{k_l}-f_{k_l-1}|>\sqrt{\alpha}/\lambda$, we have

$$E(f)=E^n_0((f_k)_{k\ge 0};k_1;\cdots;k_n)+n\alpha. \tag{59}$$

Consequently, once the solution of the problem of minimizing E^n_0 is found for every n, then the minimization of E is performed. Notice also that it is easy to find an upper bound of the maximum number of breaks n to explore, for instance $n\le N-1$ or $n\le \|g\|_{L^2}/\alpha$. In practice, the minimum of (59) for a fixed n roughly decreases (as n increases) until the right number of breaks n is reached, and increases after that.

Mumford and Shah propose to minimize $E^n_{k_0}$ for every k_0 and n. This is done easily by induction. First compute this minimum for $n=0$ (in this case the problem is convex) and every $k_0=0,\cdots,N-1$ and set

$$U^0_{k_0}=\min_{f_k,k\ge k_0}E^0_{k_0}((f_k)_{k\ge k_0}).$$

Then, to compute $E^n_{k_0}$ for $n=1$ and every k_0, we just notice that

$$\begin{aligned} &E^n_{k_0}((f_k)_{k\ge k_0};k_1;\cdots;k_n)\\ &\quad=\lambda^2\sum_{k_0<k<k_1}|f_k-f_{k-1}|^2+\sum_{k_0\le k<k_1}|g_k-f_k|^2+E^{n-1}_{k_1}((f_k)_{k\ge k_1};k_2;\cdots;k_n)\end{aligned} \tag{60}$$

and therefore

$$\begin{aligned} U^n_{k_0}&=\min_{(f_k)_{k\ge k_0};k_1;\cdots;k_n}E^n_{k_0}\\ &=\min_{k_1}\left\{\min_{f_k,k_0\le k<k_1}\left(\lambda^2\sum_{k_0<k<k_1}|f_k-f_{k-1}|^2+\sum_{k_0\le k<k_1}|g_k-f_k|^2\right)+U^{n-1}_{k_1}\right\}.\end{aligned} \tag{61}$$

This method is quite efficient, provided the value of

$$\min_{f_k,k'\le k<k''}\left(\lambda^2\sum_{k'<k<k''}|f_k-f_{k-1}|^2+\sum_{k'\le k<k''}|g_k-f_k|^2\right)$$

has been computed initially and stored in an array for every pair (k',k''), $k'<k''$.

Appendix E.2. Segmentation in dimension two. In dimension two, many algorithms exist to minimize the discrete energy (1) (see [4], [5], and [29]–[32]). However, none of these algorithms converges in any situation to the right solution. The dynamic programming method that works in dimension one—and is just a nice way of scanning the whole space over which we need to minimize the energy—has no equivalent in two dimensions and we don't know any efficient algorithm that finds the absolute minimum of (1). To obtain Fig. 5 we experimented with a different method, based on the following fact.

Consider the convex part of Mumford and Shah's functional

$$E^c(f) = \int_\Omega \lambda^2 |\nabla f(x)|^2 + |f(x) - g(x)|^2 dx. \tag{62}$$

When $\Omega = \mathbf{R}^2$, or if the functions f and g are properly extended outside Ω, then the function f minimizing E^c is given by

$$f = g * \psi \quad \text{with} \quad \widehat{\psi}(k_x, k_y) = \frac{1}{1 + \lambda^2 |\mathbf{k}|^2} \tag{63}$$

($\widehat{\psi}$ denotes the Fourier transform of ψ and (k_x, k_y) is the vector in the Fourier domain, with $|\mathbf{k}|^2 = k_x^2 + k_y^2$). The filter ψ is a good edge detector. It may also be shown (see [16] and [17]) that the filter $\psi' = \psi * \psi$, that corresponds to the minimization of

$$E^{c\prime}(f) = \int_\Omega \lambda^4 |\Delta f(x)|^2 + 2\lambda^2 |\nabla f(x)|^2 + |f(x) - g(x)|^2 dx,$$

is the best two-dimensional extension of Deriche's optimal filter for edge detection [21].

The main interest of these filters is that the variational formulation makes them easy to implement on any kind of domain, for instance it is simple to replace Ω in (62) with $\Omega \setminus K$ where K is a set of already computed edges. A possible algorithm of segmentation is thus the following:

- Set $K_0 = \emptyset$, and $n = 0$.
- Minimize $\int_{\Omega \setminus K_n} \lambda^2 |\nabla f|^2 + |f - g|^2$ and get a function f_n.
- Then, detect edges on f_n, for instance by tracking the "best significant" zero-crossings of the operator $d^2 f_n(\nabla f_n, \nabla f_n)$, that detects the extrema of $|\nabla f_n|$ along the direction of the gradient.
- Set $K_{n+1} = K_n \bigcup \{\text{the detected edges}\}$ and increment n by one.
- Start the minimization again until, for instance, the total energy of Mumford and Shah stops decreasing.

Figure 5 was computed by an algorithm of this kind; for more details see [16].

Appendix F. A strange function. Here we build a counterexample that shows that in point i of Theorem 2.2, it is not possible to replace the weak energy of u, $E(u)$, by the energy $E(u, K)$, for any closed set $K \supseteq S_u$.

Proof. Consider $\Omega = (0,1) \times (0,1)$ and let $(r_n)_{n \in \mathbf{N}}$ be the sequence of points of Ω with rational coordinates. Now let $\varepsilon > 0$ and for all integer $n > 1$ let ϕ_n be the function

$$\frac{1}{2^{n+1}} \chi_{B_{\frac{\varepsilon}{2^{n+1}}}(r_n)}(x),$$

which takes the value $1/2^{n+1}$ inside the disc of center r_n and radius $\varepsilon/2^{n+1}$ and 0 outside. Let $B_n = B_{\frac{\varepsilon}{2^{n+1}}}(r_n)$.

We define a sequence of $\mathrm{SBV}(\Omega)$ functions

$$u_n(x) = \sum_{k=1}^{n} \phi_k(x) \leq 1$$

for all $x \in \Omega$. It is clear that $S_{u_n} = \Omega \cap \bigcup_{k=1}^{n} \partial B_k$.

Call u the limit of these functions as n goes to infinity. This is an SBV function and we have

$$0 \le u < 1,$$

$$\nabla u = 0,$$

$$\mathcal{H}^1(S_u) \le \liminf_{n\to+\infty} \mathcal{H}^1\left(\bigcup_{k=1}^{n} \partial B_k\right) \le 2\pi\varepsilon.$$

(This follows from Ambrosio's Theorem 1.1.) It is quite easy to prove that $S_u \supseteq \Omega \cap \bigcup_{n=1}^{\infty} \partial B_n$ and that, if $x \notin \partial B_n$ for all n, x is a Lebesgue point of u and therefore $x \notin S_u$. This means that

$$S_u = \Omega \cap \bigcup_{n=1}^{\infty} \partial B_n,$$

and makes obvious the fact that $\mathcal{H}^1(S_u) \le 2\pi\varepsilon$ and $\Lambda(S_u) \le 8\varepsilon$, but also implies that $K = \overline{S_u} = \overline{\Omega}$ and therefore $\mathcal{H}^1(K) = \Lambda(K) = +\infty$!

It is then possible to build a sequence of "images" (f^h) with $F^h(f^h) \le C < +\infty$, but such that f^h converges to u and therefore

$$F(u) \le \liminf_{h\searrow 0} F^h(f^h) < F(u,K) = +\infty.$$

(Here we recall that $F(u) = \int_\Omega |\nabla u|^2 + \Lambda(S_u)$; $F(u,K) = \int_\Omega |\nabla u|^2 + \Lambda(K)$ and $F^h(f^h)$ is given by formula (12).) This is done in the following way.

For all $h = 1/N$, $N \ge 1$, simply let $f_{k,l}^{h,n} = u_n((k+\frac{1}{2})h, (l+\frac{1}{2})h)$ (with $0 \le k,l < N$). It may be shown that any extension of $(f^{h,n})$ (as defined in § 4) converges to u_n a.e. in Ω, and that

$$\lim_{h\searrow 0} F^h(f^{h,n}) = \Lambda\left(\Omega \cap \bigcup_{i=1}^{n} \partial B_i\right) = F(u_n).$$

As we also have

$$\lim_{n\to+\infty} F(u_n) = F(u),$$

one then just has to build from the $(f^{h,n})$ a diagonal sequence f^h that converges to u and such that

$$\lim_{h\searrow 0} F^h(f^h) = F(u)$$

(which also implies that $F^h(f^h) \le C < +\infty$). □

Appendix G. Dal Maso's generalization of Dibos and Séré's approximation result to any SBV function. Françoise Dibos and Eric Séré proved the following result.

LEMMA G.1 (Dibos–Séré [24]). *Let u be a minimizer of the functional*

$$\int_\Omega |\nabla u|^2 + \Lambda(S_u) + \int_\Omega |u-g|^2,$$

with $g \in \mathrm{L}^\infty(\Omega)$. Then for every $\varepsilon > 0$ there exists $u^\varepsilon \in \mathrm{SBV}(\Omega)$ such that

$$\|u^\varepsilon - u\|_{\mathrm{L}^1(\Omega)} \le \varepsilon, \tag{64}$$

$$S_{u^\varepsilon} \text{ is made of a finite number of segments,} \tag{65}$$

$$F(u^\varepsilon) \le F(u) + \varepsilon, \tag{66}$$

where $F(u) = \int_\Omega |\nabla u| + \Lambda(S_u)$.

Dal Maso could then deduce the following lemma, which is more general.

LEMMA G.2 (Dal Maso). *Let u be a function in* SBV(Ω). *Then the conclusions of Lemma* G.1 *hold for u.*

Proof. If $E(u) = +\infty$ then the assertion is trivial (take any continuous function satisfying (64)). Therefore we assume $E(u) < +\infty$. Assume also that $u \in \mathrm{L}^\infty$. For every $\beta > 0$ let v_β be the minimum point in SBV(Ω) of

$$\int_\Omega |\nabla v|^2 + \Lambda(S_v) + \beta^2 \int_\Omega |v-u|^2. \tag{67}$$

We have $F(v_\beta) + \beta^2 \int_\Omega |v_\beta - u|^2 \le F(u)$, thus

$$F(v_\beta) \le F(u) \tag{68}$$

and

$$\|v_\beta - u\|_{\mathrm{L}^2} \le \sqrt{E(u)}/\beta, \tag{69}$$

hence

$$\|v_\beta - u\|_{\mathrm{L}^1} \le c/\beta, \tag{70}$$

with $c = \sqrt{\mathrm{meas}(\Omega) E(u)}$. Since v_β is a minimum point of (67), with $u \in \mathrm{L}^\infty(\Omega)$ playing the role of g in Lemma G.1, we can find for $\varepsilon > 0$ a function $v_\beta^\varepsilon \in \mathrm{SBV}(\Omega)$ with

$$\|v_\beta^\varepsilon - v_\beta\|_{\mathrm{L}^1(\Omega)} \le \varepsilon/2, \tag{71}$$

$$S_{v_\beta^\varepsilon} \text{ is made of a finite number of segments,} \tag{72}$$

$$F(v_\beta^\varepsilon) \le F(v_\beta) + \varepsilon. \tag{73}$$

Taking $\beta = 2c/\varepsilon$ from (70) and (71), (68) and (73), we get

$$\|v_\beta^\varepsilon - u\|_{\mathrm{L}^1(\Omega)} \le \varepsilon$$

and

$$F(v_\beta^\varepsilon) \le F(u) + \varepsilon,$$

therefore $u^\varepsilon = v_\beta^\varepsilon$ satisfies (64), (65), (66). (This proof is based on a penalization method described in [36].)

Now, if $u \in \mathrm{SBV}(\Omega)$ is not bounded, we consider the truncated function u_t for every $t > 0$

$$u_t(x) = \begin{cases} -t, & \text{if } u(x) < -t, \\ u(x), & \text{if } -t \leq u(x) \leq t, \\ t, & \text{if } u(x) > t. \end{cases}$$

$u_t \in \mathrm{SBV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and $E(u_t) \leq E(u)$; therefore to conclude we just use the fact that

$$\lim_{t \to +\infty} u_t = u \text{ in } \mathrm{L}^1(\Omega). \qquad \square$$

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