MP574 Lecture 19: Degrees of Freedom and Strategies for Minimizing Registration Error

Learning Objectives

- Introduce Degrees of Freedom and Classes of Registration Complexity.

References:
Sotiras et al., IEEE Transactions on Medical Imaging, Vol. 32 (7): 1153-1190.

I. Introduction
   a. The degrees for freedom (DOF) are tailored to the complexity of the problem.
      i. Scale of deformation (rigid, affine, or deformable (i.e. elastic))?
         1. Small deformation problem => Rigid Fiducial (trusted reference points)
         2. Large deformation problem (e.g. Lungs) => Deformable
            a. Spline or Physical Model-Based methods
      ii. Mono or Multi-Modality?
         1. Feature-based => Mono-Modal anatomical markers
      iii. Automated or Supervised?
         1. Supervised => Control Point
         2. Automated => Normalized mutual information
         3. Off
   b. Example: Alignment of multiple moving frames within a dynamic stack of images is a registration problem.
      i. Registration of motion in a medical image setting is "small deformation, mono-modal, deformable registration" problem.
         1. Addressed by either affine or deformable depending on desired accuracy and organ system. Most often these methods are used in sequence, that is affine is used to converge to a reasonably close solution and a deformable method is used to refine the registration. This strategy typically works well since the main concern with deformable methods are: 1) Convergence to a local minimum, and 2) instability leading to different, non-unique, solutions.
         2. Another more subtle point has to do with the physical model of the organ system. This will impact the constraints that might be applied to the deformable registration algorithm. Is the organ system compressible (i.e. lungs), or mostly non-compressible (i.e. liver). In the former the Jacobian must be allowed to change for voxels between fixed and moving image, while for the latter the Jacobian can be fixed to always equal 1 so as to conserve volume in the transform.

II. Registration as a generalized optimization problem
   a. Optimization transformation

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$$\mathrm{M}_{i+1} = C(F, T \circ M_i) + R(T), \qquad [1]$$

Where:

- $T =$ Deformation or "Transform" operator
- $C =$ Objective cost function, or "Matching" function
- $R =$ Regularization

The transformation at every position, or control point, $\vec{x} \in \Omega$ is given as the addition of an identity transformation with the displacement field $\vec{u}$ such that $T(\vec{x}) = \vec{x} + \vec{u}(\vec{x})$.

Regularization and deformation models are closely related. In the case that the transformation is parameterized by a small number of variables and is inherently smooth, such as an interpolation basis with global support like B-spline, regularization serves to introduce prior knowledge regarding the solution (e.g. constrain the material to be incompressible, that is $\Delta J = 1$), or in the case of a non-parametric model in which every element can change, R may impose a smoothness constraint on the possible deformation. In general, R regularizes the transformation aiming to favor any specific properties in the solution that the user requires, and seeks to impose realistic physical constraints.

b. Main Classes of Geometric transforms
    i. Interpolation functions from approximation theory allow displacements that are considered known in a restricted set of locations in the image (i.e. at control points) and are interpolated for the rest of the image domain. In addition, the transformation smoothly approximates the known displacements at new coordinate positions.
        1. Radial basis functions
        2. Free form deformations
            a. B-splines
        3. Locally affine models
            a. Piecewise affine models
            b. Poly-affine models

In the interest of time, I will focus on B-splines as a widely used interpolation-based deformable registration approach used in medical image registration. Uses cubic B-splines as a basis and estimates the vector field across a grid of control points superimposed on the image (Figure). The deformation takes place locally under the influence of the control points in the grid mesh with the vector fields for the transform estimated by the summation of tensor products of univariate cubic-B splines (Eq.X). Recall that tensors takes in two (or three in 3D space) vectors, conceived of roughly as small arrows emanating from a specific point within a curved space, or manifold, and returns a local dot product of them relative to that particular point—an operation which encodes roughly the vectors' lengths as well as the angle between them. As the dot product is a scalar, the metric tensor provides a local deformation field between the fixed and moving images at each point of the manifold, and variation in the metric tensor thus encodes how that distance and angle change to map one into the other.

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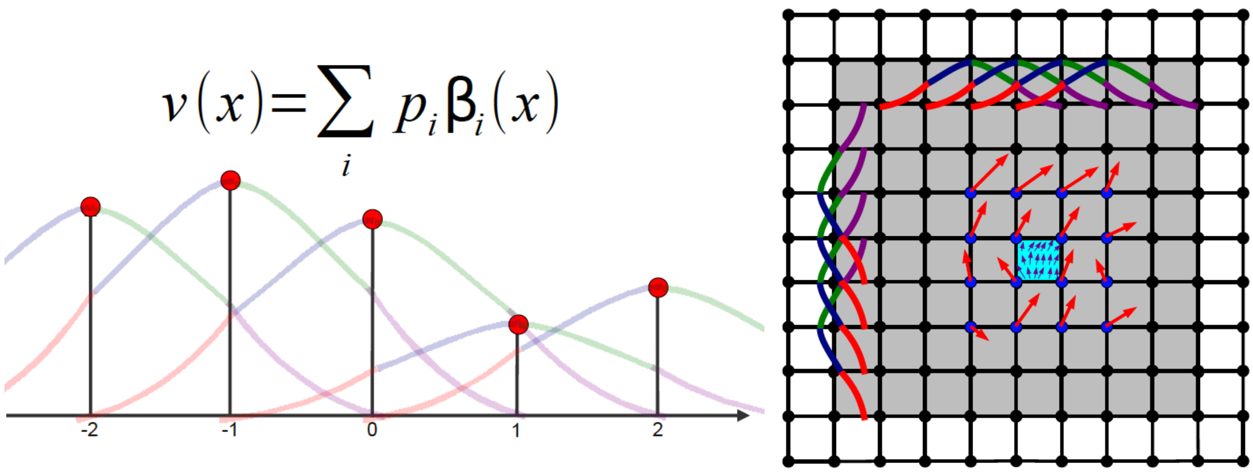

Figure courtesy Greg Sharp, MGH.

$$u(\vec{x}) = \sum_{l=0}^{3}\sum_{m=0}^{3}\sum_{n=0}^{3} B_l(x')B_m(y')B_n(z')\, d_{i+l,j+m,k+n}$$

i,j,k are normalized indices over the grid Nx X Ny X Nz.

$$B_0(u) = (1-u)^3/6 \quad , \quad B_1(u) = (3u^3 - 6u^2 + 4)/6$$
$$B_2(u) = (-3u^3 + 3u^2 + 3u + 1)/6 \, , B_3(u) = u^3/6 \; .$$

ii. Physical models generally allow deformations that mimic processes that achieve a steady-state balance of forces or flux, F, to constrain allowable conformations. These include:
   4. Elastic body models
   5. Fluid flow models
   6. Diffusion models.

In the interest of time, I will focus on diffusion models, a subset of which, called the “Demons” algorithm, and its variants are widely used in medical image registration.

$$\Delta\vec{u} + F = 0.$$

Where $\Delta\vec{u}$ takes on the analogy of the physical concentration gradient and F is in this model not a force, but a flux determining the “diffusion” from one coordinate to another.

c. Important tradeoffs between physical and interpolation-based models
   i. Interpolation-based models are generally rich enough to describe the transformation while having low DOF and thus simplifying the computational complexity and constraining the allowable deformation
   ii. Interpolation bases generally have the property of global support such that interpolation of motion fields in sparsely populated areas is feasible.
   iii. The main tradeoff with respect to the physical model-based approaches is the inability of interpolation bases to accommodate local transformations in highly deformable regions.
      1. Hybrid approaches have use modified interpolation models with control points that present non-uniformly over the image or have spatially limited influence.

III. Strategies for Minimizing Error
   a. General topological constraints that contributed to the “break-through” in deformable registration that began around 2000.
      i. Partly driven the robustness of entropy-based cost functions (next lecture)
      ii. But also by the implementation of specific geometric constraints that were largely made possible by rapidly increasing computational power.
         1. Symmetry – Most registration algorithms are asymmetric such that when interchanging the order of the input images the registration does not estimate the inverse transformation.
            a. Inverse consistency and symmetry are closely related geometric constraints that are generally imposed by implementing the transform small steps and cross-checking at each step to enforce that the forward and inverse transforms are converge at each step. By daisy-chaining in this way the total transform can be made to be inverse consistent. Obviously, this increases computational workload and reduced efficiency and is enabled by parallel-process/threaded architectures.
         2. Topological constraints – These constraints are very important because they impose limits to deformation that conform to real-world physical systems. These include: 1) the transform must be locally one-to-one, which is analogous to enforcing physical consistency, that is the object cannot fold or wrap into itself; and 2) the transformation must be differentiable, which insures continuity across control points, such that 3) the Jacobian, J, exists. Note that 1 and 2 also imply that J>0, which is a shorthand for stating that your transform is topology preserving.
         3. Diffeomorphism – diffeomorphic transformations also embody a kind of “topological shorthand.” If a transform is diffeomorphic, then it preserves topology (J>0), is invertible, and differentiable. The term refers to the mapping of a differentiable manifold into

another. A manifold is a mathematical construct to deal with complex shapes that are constrained such that they locally resemble Euclidean space near each control point.

IV. Summary
  a. The degrees for freedom (DOF) are tailored to the complexity of the registration problem.
      1. Affine is used to converge to a reasonably close solution and a deformable method is used to refine the registration.
      2. This strategy typically works well since the main concern with deformable methods are: 1) Convergence to a local minimum, and 2) instability leading to different, non-unique, solutions.
  b. Registration is a general optimization problem
    i. Regularization is critical to constrain the deformation to conform to realistic physiology
    ii. Main classes of deformable registration are encompassed by physical model-based and interpolation-based
      1. B-spline is a commonly used interpolation-based approach with favorable properties
    iii. Geometric constraints of deformation transforms are critical
      1. Symmetry
      2. Topology
      3. Diffeomorphism