Part II

Image Reconstruction

Lecture 12

Basics of Image Reconstruction

12.1 Lecture Objectives

- Understand image reconstruction as a mathematical problem with important clinical and research implications
- Recognize the wide variety of available image reconstruction methods
- Understand the essential metrics we use to assess the quality of an image
- Place image quality metrics in the context of the big picture application

12.2 Image Reconstruction as an Inverse Problem

Inverse problems are mathematical problems where we seek to estimate an underlying object (eg: an image) from a set of measurements related to this object (eg: Fourier samples in the case of MRI, or projections in the case of CT). This is represented graphically in figure 12.1. When the object we are trying to estimate is an image (eg: a slice of a patient's brain), and the set of measurements arise from some imaging device (eg: MRI or CT), our inverse problem is the image reconstruction problem.

Note that we can represent this idea mathematically by expressing the measurements (written as a vector) $\mathbf{d}$ in terms of the underlying image (also written as a vector) $\mathbf{x}$:

$$\mathbf{d} = \mathcal{T}(\mathbf{x}) \tag{12.1}$$

where the operator $\mathcal{T}$ represents a general transformation (eg: a Fourier transform evaluated at various spatial frequencies, or a projection performed along various angles). Equation 12.1 above represents the forward problem in imaging (ie: predicting the measurements from the underlying image).

Consequently, the inverse problem can be viewed as applying the inverse of the operator $\mathcal{T}$, ie:

$$\hat{\mathbf{x}} = \mathcal{T}^{-1}(\mathbf{d}) \tag{12.2}$$

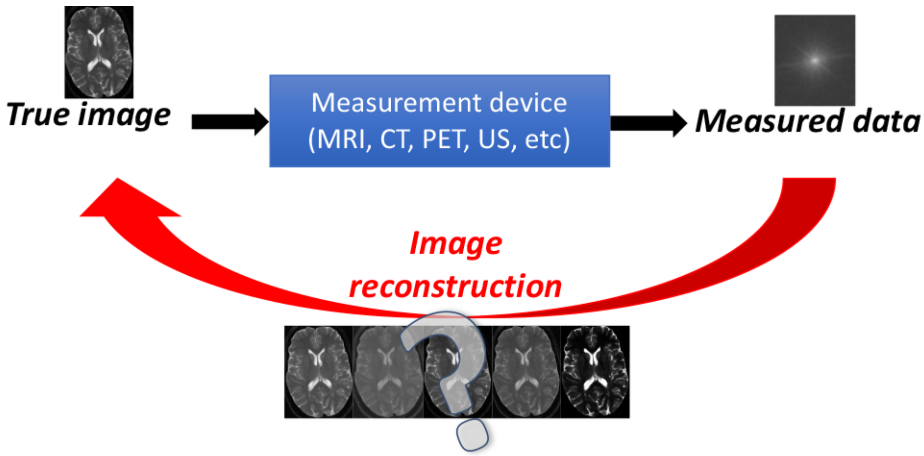

Figure 12.1: Image reconstruction as an inverse problem. For the purpose of this course, we will assume there is an underlying true image that we would like to obtain. However, we cannot access our true image directly. Instead, we can obtain measurements that are related to this true image using our imaging equipment (MRI, CT, etc). Further, we know from physics/statistics the relationship between our true image and our measured data (in other words, if we know the true image, we can predict the measured data). In other words, we have a good characterization of the forward problem. Then, the problem of image reconstruction is the corresponding inverse problem: knowing some measured data, can we estimate the underlying true image?

where we essentially try to find a solution $\hat{\mathbf{x}}$ that fits our data, ie: such that $\mathcal{T}(\hat{\mathbf{x}}) = \mathbf{d}$. However, as we will see in this course, one must approach this simple expression carefully. The reasons for this caution are that we may run into several pitfalls[1] when solving our image reconstruction inverse problem (as well as many other inverse problems):

i. A solution that matches our data exactly may not exist (particularly in the presence of noise, artifacts, or modeling mismatches).

ii. The solution, even if it exists, may not be unique (ie: multiple different candidate images may fit our data equally well).

iii. The solution may be 'unstable' in the sense that tiny changes in our data may result in huge changes in our reconstructed image. This is often undesirable too, since our data will always contain some level of noise and artifacts.

[1] These pitfalls reflect the conditions for an inverse problem to be well-posed in the sense of Hadamard, as described in these URLs: `https://math.la.asu.edu/~platte/apm598/intro_inv_problems.pdf`, and `https://en.wikipedia.org/wiki/Well-posed_problem`

12.3 Types of Measurements

Medical imaging measurements can arise in many forms. For the sake of illustration, here we will focus on two types of measurements, Fourier and projections, which are representative of various imaging modalities including MRI and CT.

12.3.1 Fourier measurements

In several imaging modalities (most notably MRI), there is a Fourier relationship between measurements and image, eg in the 2D case:

$$d(k_1, k_2) = \int \int x(r_1, r_2) e^{-i2\pi(k_1 r_1 + k_2 r_2)} dr_1 \, dr_2 \tag{12.3}$$

where r_1 and r_2 represent the spatial coordinates in image space, $x(r_1, r_2)$ is the corresponding image, k_1 and k_2 represent the Fourier-space coordinates (termed k-space in MRI lingo), and $d(k_1, k_2)$ represents the k-space measurements as a 2D array.

Note that, upon discretization of our image and Fourier spaces, we can represent this relationship using matrix-vector notation, as:

$$\mathbf{d} = \mathbf{F}\mathbf{x} \tag{12.4}$$

where $\mathbf{F}$ is a matrix that performs a 2D DFT (or 3D DFT if performing 3D imaging) on the image that is represented (in vector form) by vector $\mathbf{x}$.

Also, note that we can use our matrix-vector representation above to express the case where we do not obtain all the samples in the 2D DFT but only a subset of these samples, or the case where we obtain samples that are not located on a Cartesian grid (eg: if we are using radial or other non-Cartesian k-space trajectories in our MRI pulse sequence). In some of these cases, we may not be able to directly use FFTs to obtain fast computational implementations of our transform (there are tricks that enable us to still use FFTs, but we will not cover these in depth in this class), however, we should still be able to express it as a linear matrix-vector relationship.

12.3.2 Projections

For tomographic imaging modalities (eg: CT), we can model our data as projections along a certain angle θ. Specifically, for parallel beam CT, we can model our data (after appropriate transformation) as follows:

$$d(s, \theta) = \int \int x(r_1, r_2) \delta(r_1 \cos\theta + r_2 \sin\theta - s) dr_1 \, dr_2 \tag{12.5}$$

as depicted graphically in Figure 12.2.

Similarly to the Fourier imaging example above, upon discretization we can represent this projection relationship using matrix-vector notation, as:

$$\mathbf{d} = \mathbf{P}\mathbf{x} \tag{12.6}$$

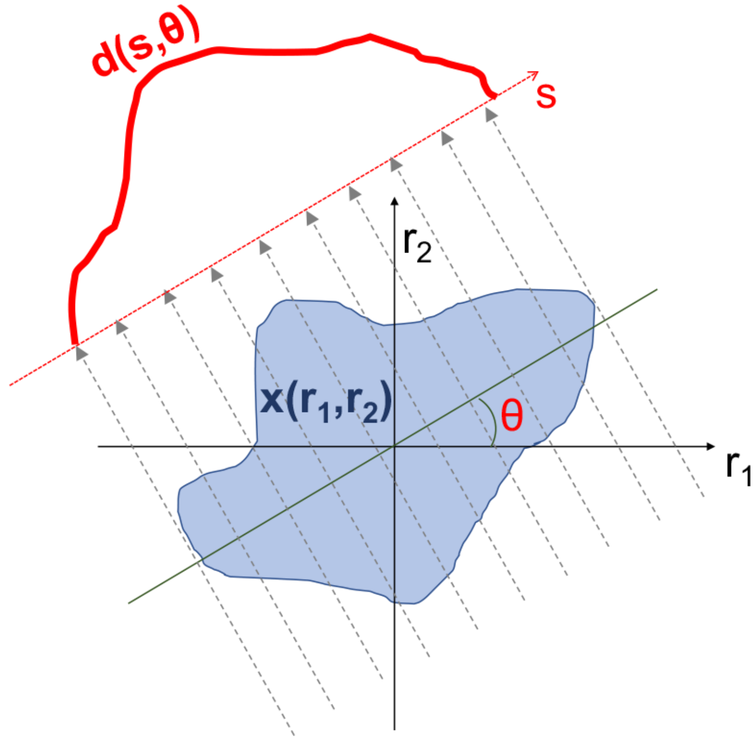

Figure 12.2: Graphical representation of a projection as used in tomographic imaging techniques (eg: CT). If we obtain sufficient projections of our object (image), we are able to reconstruct an accurate depiction of the object.

where $\mathbf{P}$ is a matrix that performs a set of projections on our discretized image, along a set of different angles.

12.4 Types of Image Reconstruction Methods

12.4.1 Direct Methods

Sometimes we can directly express a solution to the image reconstruction problem using a closed-form operation on the acquired data.

- *Fourier measurements:* For instance, for Fourier imaging, if we acquire samples on a Cartesian grid with sufficiently dense sampling, we can simply perform image reconstruction by applying an inverse DFT (in 2D or 3D as appropriate) to the acquired data:

$$\hat{\mathbf{x}} = \mathbf{F}^{-1}\mathbf{d} \tag{12.7}$$

where $\mathbf{F}^{-1}$ represents an inverse DFT matrix (in 2D or 3D, as appropriate).

- *Projection measurements:* If we acquire sufficient projections, we can also recover our image using so-called filtered backprojection (FBP). FBP can be understood based on the central slice theorem. In FBP, each projection is first filtered using a ramp filter

$$p_R(s, \theta) = p(s, \theta) * R(s) \tag{12.8}$$

where $R(s)$ is a 'high-pass' filter such that its Fourier transform is a ramp function, and then a backprojection operation is applied:

$$\hat{x}(r_1, r_2) = \int_0^{\pi} p_R(s, \theta)|_{s=r_1 \cos\theta + r_2 \sin\theta} \, d\theta \tag{12.9}$$

leading to the reconstructed image $\hat{x}(r_1, r_2)$. Upon discretization of these two operations (filtering and backprojection), a direct reconstruction method for projection data can be obtained.

12.4.2 Optimization-Based Methods

Oftentimes direct algorithms are not appropriate for medical image reconstruction. For instance, in the presence of high noise, substantial artifacts, or missing data, direct reconstructions may be noisy, artifact-prone, or infeasible. In these cases, optimization-based methods have the potential to enable improved image reconstruction. Over the next few lectures, we will describe optimization-based image reconstruction in more detail. In general, these methods seek a reconstructed image that is optimal in the sense of combining data fidelity and agreement with certain prior knowledge. A typical formulation is as follows

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} f(\mathbf{x}; \mathbf{d}) + \lambda R(\mathbf{x}) \tag{12.10}$$

where the first term in the cost function $f(\mathbf{x}; \mathbf{d})$ is the data fidelity term, and penalizes mismatch between our reconstructed image and the measured data $\mathbf{d}$. The second term in the cost function, $R(\mathbf{x})$, promotes certain properties (eg: smoothness, sparsity, compressibility) of the image that are known a priori. Finally, the "regularization" parameter λ balances the importance of the data fidelity term and the prior term. Note that, as we covered in homework 1, sometimes this type of formulation can be interpreted as a Maximum a Posteriori (MAP) estimation including a log-likelihood term ($f(\mathbf{x}; \mathbf{d})$), and a prior term ($\lambda R(\mathbf{x})$).

12.5 What Makes a Good Image?

12.5.1 Spatial Resolution

Spatial resolution refers to the ability to distinguish small objects in our image. For many imaging modalities, we can characterize the reconstructed image as a convolution of the true underlying image with a specific spatial kernel:

$$\hat{x}(r_1, r_2) = [x * h](r_1, r_2) \tag{12.11}$$

where the kernel $h(r_1, r_2)$ depends on the imaging modality and the choice of acquisition parameters, as well as the reconstruction approach (eg: the choice of regularization $\lambda R(\mathbf{x})$ in optimization-based reconstruction.

Spatial resolution is usually given as a single number (eg: 1mm) or as a number along each of the relevant spatial dimensions (eg: $1x1\text{mm}^2$ in 2D imaging). Although spatial resolution cannot capture every aspect of the kernel h above, it is generally viewed as a measure of the width of the main lobe of h. In practice, our measure of spatial resolution refers to our ability to distinguish two small objects that are separated by a certain distance.

For many imaging modalities, there is a permanent quest for higher spatial resolution. However, achieving higher spatial resolution typically means requiring more expensive equipment, or requiring longer acquisition times, or giving up some signal-to-noise ratio (SNR), or a combination of these.

12.5.2 SNR

Signal-to-noise ratio (SNR) is an important measure of image quality, typically quantified as:

$$\text{SNR} = \frac{\mu}{\sigma} \tag{12.12}$$

where μ is the mean image intensity measured at some location, and σ is the standard deviation of the image intensity. Note that measuring SNR for in vivo imaging applications is often difficult since it generally requires the acquisition of multiple replicates of the image, in order to extract voxel-wise statistics. However, there are approximated methods to calculate SNR in many cases. Further, relative SNR can often be calculated analytically when comparing different acquisition protocols or reconstruction methods.

12.5.3 Contrast

Ultimately, contrast is the key feature of many clinical imaging techniques. Contrast relates to the ability to distinguish different types of tissues, and is often measured in terms of the contrast to noise ratio between tissues 1 and 2:

$$\text{CNR} = \frac{|\mu_1 - \mu_2|}{\sigma} \tag{12.13}$$

where μ_1 and μ_2 are the mean signal intensities between two different (but typically nearby) tissues, and σ is the noise standard deviation.

12.5.4 Artifacts

A variety of artifacts appear in medical images. A few common sources of artifacts include:

- Motion
- Lack of sufficient data acquired
- System imperfections or inaccurate forward models

Over the course of the next few lectures, we will review examples of typical artifacts observed in various imaging modalities.

12.5.5 Placing image quality in the context of the application

Why do we often measure spatial resolution, SNR, or CNR when characterizing the quality of our medical images? These are important, helpful metrics. However, ultimately the true measure of image quality is the image's ability to help solve the corresponding clinical or research question.

This is often a source of confusion for physicists and engineers when beginning to work with clinicians and radiologists. There are many instances when a radiologist will prefer a noisy image with substantial artifacts to a less noisy-looking image or an image with less obvious artifacts. The reason for this, generally speaking, is that the noisy, artifactual image, may still clearly depict the pathology in question - and the radiologist is able to "read through" (ie: ignore) many common artifacts, even if they are large or have high intensity.[2]

For these reasons, when evaluating the performance of novel imaging techniques, it is helpful to combine objective measures of image quality (resolution, SNR, CNR), with subjective measures of image quality (to what degree does it help answer the clinical question?).

[2] An event where this discrepancy became obvious to the instructor was the 2010 Meeting of the International Society for Magnetic Resonance in Medicine. This conference included an MR image reconstruction challenge (https://zenodo.org/record/47801.XGuGps9Kib8), which led to fascinating discussion between physicists/engineers and radiologists about the meaning of image quality.