Lecture 4

Constrained vs Unconstrained Formulations

4.1 Lecture Objectives

- Understand the basic types of optimization problems we face, in terms of the presence of absence of constraints.
- Understand how constraints arise naturally in many optimization applications
- Understand the process by which hard constraints can be approximately included as additional terms in the objective function

4.2 Unconstrained Optimization

An unconstrained optimization problem can be written as follows

$$\text{minimize } f(\mathbf{x}) \tag{4.1}$$

where $\mathbf{x} = [x_1, \cdots, x_N]^T \in \mathbb{R}^N$ is the vector we are trying to optimize.

4.3 Constrained Optimization

4.3.1 Hard Constraints

A constrained optimization problem can be written as follows

$$\begin{aligned} &\text{minimize } f(\mathbf{x}) \\ &\text{such that } g_k(\mathbf{x}) \leq b_k, \text{ for } k = 1, \ldots, K \\ &\text{such that } h_l(\mathbf{x}) = c_l, \text{ for } l = 1, \ldots, L \end{aligned} \tag{4.2}$$

where $\mathbf{x} = [x_1, \cdots, x_N]^T \in \mathbb{R}^N$ is the vector we are trying to optimize. However, even though in principle $\mathbf{x}$ can take values all over $\mathbb{R}^N$, in reality it is constrained to the intersection of two regions: the region defined by our inequality constraints ($g_k(\mathbf{x}) \leq b_k,$ for $k = 1, \ldots, K$), and the region defined by our equality constraints ($h_l(\mathbf{x}) = c_l,$ for $l = 1, \ldots, L$).

A few comments:

- Equality and inequality constraints arise naturally in a variety of applications. Can you think of a few of each kind?
- Constrained optimization problems are generally a lot harder to solve than unconstrained problems
- General constrained optimization problems can be written in the standard form shown in Equation 4.2 above.

4.3.2 Soft Constraints

A constrained optimization problem (as described above) can be modified to become an unconstrained optimization problem, as follows:

$$\text{minimize } f(\mathbf{x}) + \phi_k(g_k(\mathbf{x}) - b_k) + \psi_l(h_l(\mathbf{x}) - c_l) \tag{4.3}$$

where ϕ_k is a function that penalizes positive values of its argument, eg: $\phi_k(x) = \lambda_k \max(x, 0)^2$, and ψ_l is a function that penalizes nonzero values of its argument, eg: $\psi_l(x) = \mu_l x^2$. Note that this unconstrained formulation will promote a solution similar to that corresponding to the hard-constraint formulation described above. However, the solutions to Equations 4.2 and 4.3 are generally not the same.

A few comments:

- How can we pick the functions ϕ_k and ψ_l beyond the (somewhat arbitrary) examples shown above? Note that if we want results very similar to the constrained formulation we are approximating, we may want to pick functions that are zero in the allowed area, and quickly increase towards infinity otherwise. However, this type of very sharp function may not lend itself to easy algorithmic handling (eg: it may not be differentiable everywhere, which can complicate the algorithm).
- The parameters λ_k and μ_k are known as regularization parameters in many applications. These parameters often control the tradeoff between data fidelity and agreement with prior knowledge.