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03/11/2019

MP574

# Homework 3

## Problem 1: Reconstruction from Partial Fourier Samples without Regularization

**Write a script that loads the data in `hw3_problem1.mat`, and reconstructs an image of size 320x320 pixels from the samples in array `b`**. The samples are DFT samples of the image, taken at the locations specified by the mask in array `mask`. In other words, the matrix $\mathbf{A}$ in our general formulation (equation 1 above) is

$$\mathbf{A} = \mathbf{MF}$$

where $\mathbf{F}$ performs a 2D DFT (including `fftshifts` and `ifftshifts`) and the matrix $\mathbf{M}$ is a sampling matrix that selects a subset of samples from the resulting 320x320 2D DFT. The selected subset of samples is given by the binary array `m`.

In this case, there is no regularization term (ie: $\lambda = 0$).

### Load Data

```
load('hw3_problem1.mat')

imageSize = [320 320]; % image dimensions
N = imageSize(1)*imageSize(2); % total number of pixels
dims = imageSize(1); % = 320
tol = 0.01; % tolerance (for approximate convergence)
iters = 100;

B(m) = b; % zero filled data
B = B(:); % vectorization
tru = reshape(B,imageSize); % original kSpace data
TRU = ifft2c(tru); % original undersampled image
```

### Steepest Descent

```
x = zeros(N,1); % set initial guess
f = zeros(1,iters); % initialize cost function values

lambda = 0; % no regularization
```

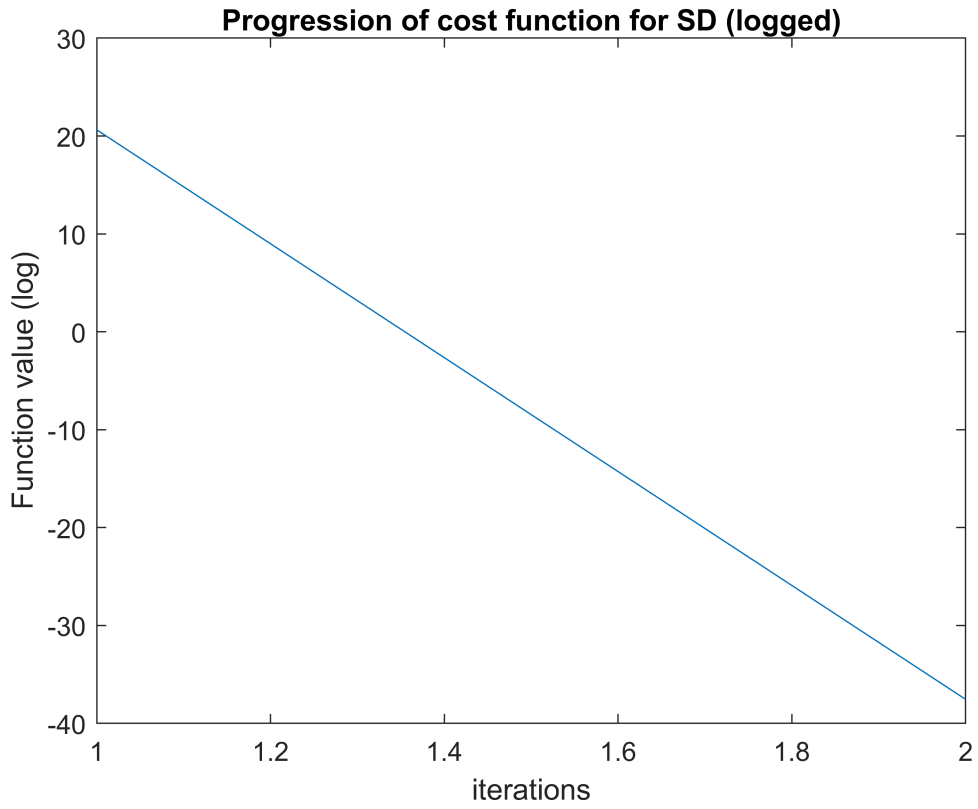
```
D = sparse(N,N); % dummy (all zeros, sparse) finite differences matrix
W = ones(dims,dims); % dummy (all ones) weighting matrix

tic
for ii=1:iters
    [f(ii),g] = evalGradients_L2( x,imageSize,b,m,lambda,D,W ); % get gradient and cost functio
    alpha = goldenSearch( x,g,imageSize,b,m,lambda,D,W ); % find alpha for this x
    x = x - alpha*g; % take a step (alpha) in grad direction, get new x
    if norm(g) < tol % if gradient is near 0
        break; % the solution converged
    end
end
toc
```

```
Elapsed time is 1.604990 seconds.
```

```
f = f(1:ii); % cutoff trailing zeros
solutionSD = x; % IMAGE SPACE SOLUTION
```

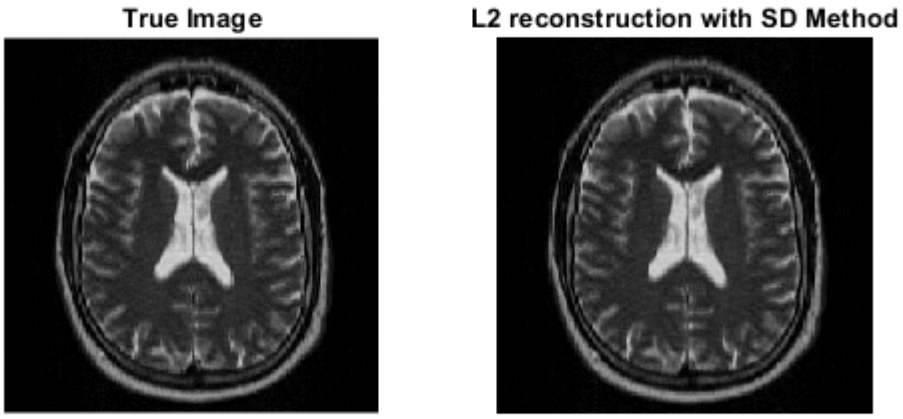
- **Plot the (logarithm of the) value of the cost function ($\|Ax - b\|_2^2 + \lambda \|Cx\|_2^2$) as a function of iterations (this should be decreasing for both algorithms, but different between algorithms). Since the value of the cost function will likely change a lot, make sure to plot its logarithm vs the iteration number.**

```
figure; plot(log(f));
title('Progression of cost function for SD (logged)'); xlabel('iterations'); ylabel('Function v
```




- **Plot the final image after the iterations are done for each of the algorithms (display it using a common colormap for both algorithms).**

```
% Show image solution
figure;
subplot(1,2,1); imshow(abs(TRU),[0 400]); title 'True Image';
subplot(1,2,2); imshow(reshape(abs(solutionSD),imageSize),[0 400]); title 'L2 reconstruction wi
```




### Conjugate Gradients

```
x = zeros(N,1); % set initial guess
f = zeros(1,iters); % initialize cost function values

tic

[f(1),g] = evalGradients_L2( x,imageSize,b,m,lambda,D,W ); % calculate gradient and cost at ini
d = -g;

solution_exists = 0; % set to false
jj = 0;
while(jj < iters && solution_exists == 0)
    jj = jj+1;
        % Create Qd vector
        dK = fft2c(reshape(d,imageSize)); % kSpace d vector (Fd)
        dKm = dK(:).*m; % masked kSpace d vector (M'MFd)
        Qd = ifft2c(reshape(dKm,imageSize)); % back to image space (F'M'MFd)
    alpha = -(g'*d)/(d'*Qd(:)); % find next stepsize
    x = x + alpha*d; % move that stepsize in direction of gradient (d).
    [f(jj+1),g] = evalGradients_L2( x,imageSize,b,m,lambda,D,W ); % redefine gradient
    if norm(g) < tol % if gradient is near 0
        solution_exists = 1; % BREAK
    else
        beta = (g'*Qd(:))/(d'*Qd(:)); % find next-next step
        d = -g + beta*d; % second gradient move
```

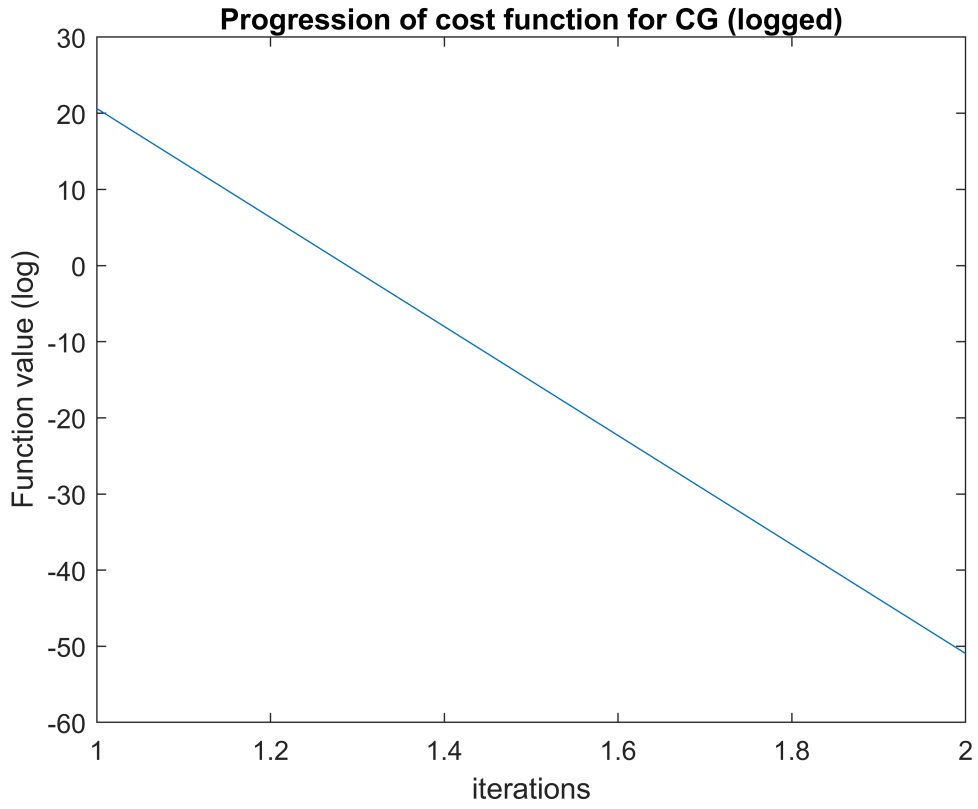
```
    end
end
toc
```

```
Elapsed time is 0.070816 seconds.
```

```
f = f(1:jj+1); % cutoff trailing zeros
solutionCG = x; % IMAGE SPACE SOLUTION
```

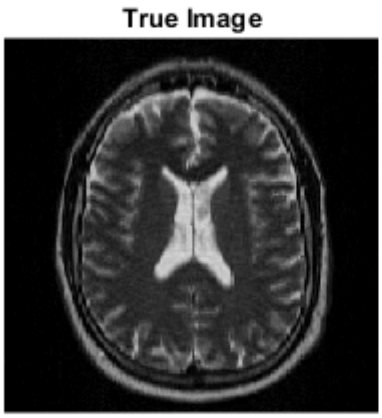
- **Plot the (logarithm of the) value of the cost function ($\|Ax - b\|_2^2 + \lambda \|Cx\|_2^2$) as a function of iterations (this should be decreasing for both algorithms, but different between algorithms). Since the value of the cost function will likely change a lot, make sure to plot its logarithm vs the iteration number.**

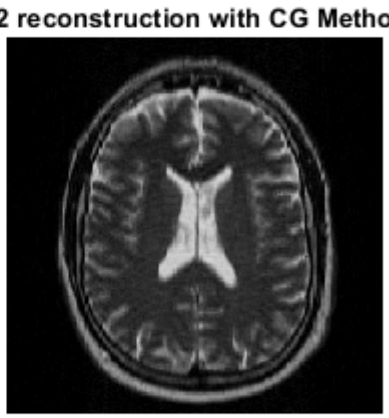
```
figure; plot(log(f));
title('Progression of cost function for CG (logged)'); xlabel('iterations'); ylabel('Function v
```




- **Plot the final image after the iterations are done for each of the algorithms (display it using a common colormap for both algorithms).**

```
figure;
subplot(1,2,1); imshow(abs(TRU),[0 400]); title 'True Image';
subplot(1,2,2); imshow(reshape(abs(solutionCG),imageSize),[0 400]); title 'L2 reconstruction wi
```




- **Report the relative performance of SD and CG (ie: how rapidly do they converge).**

```
disp(['Steepest Descent method converged in ' num2str(ii) ' iterations']);
```

```
Steepest Descent method converged in 2 iterations
```

```
disp(['Conjugate Gradient method converged in ' num2str(jj+1) ' iterations']);
```

```
Conjugate Gradient method converged in 2 iterations
```

Both the CG and SD method converged quickly in 2 iterations. The SD method took longer (1.7 s) to converge, while the CG method took approximately 83 ms to converge.

## Problem 2: Reconstruction from Partial Fourier Samples with Smoothness Regularization

**Write a script that loads the data in `hw3_problem2.mat`, and reconstructs an image (320x320 pixels) from the samples in array `b`**. Next we describe the data and smoothness terms:

• Data term: Similar to the previous case, the samples in `b` are DFT samples of the image, taken at the locations specified by the mask in array `mask`. In other words, the matrix $\mathbf{A}$ in our general formulation (equation 1 above) is

$$\mathbf{A} = \mathbf{MF}$$

where $\mathbf{F}$ performs a DFT (including `fftshifts`) and the matrix $\mathbf{M}$ is a sampling matrix that selects a subset of samples from the resulting 320x320 DFT. The selected subset of samples is given by the binary array `m`.

• Smoothness term: The smoothness term is given by $\lambda \|\mathbf{Cx}\|_2^2$, where $\mathbf{C} = \mathbf{D}$ calculates finite differences in 2D, as described in lecture 14 of this course (where we denoted it as `D2` in the Matlab code to indicate its 2D nature).

You should **repeat this optimization for several different values of** $\lambda$:

• $\lambda = 10^0$

• $\lambda = 10^2$

• $\lambda = 10^4$

• $\lambda = 10^6$

• $\lambda = 10^8$

### Load data

```
lambda = [10^0 10^2 10^4 10^6 10^8]; % array of each lambda

% Create D (finite differences matrix)
D = 2*eye(dims) - circshift(eye(dims),[0, -1]) - circshift(eye(dims),[0, 1]);
D = sparse(D);
I = speye(dims);
D2 = kron(I,D) + kron(D,I); % 2D finite differences matrix
```

### Steepest Descent

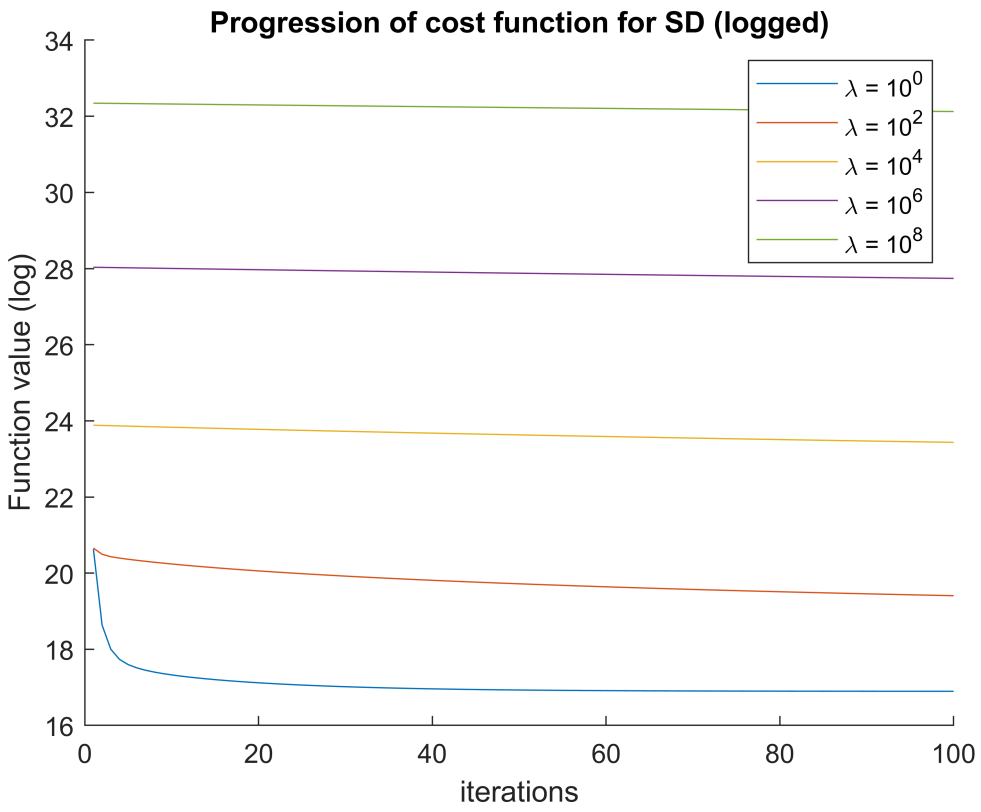
```
x = zeros(N,1); % set initial guess
f = zeros(1,iters); % initialize cost function values

tic
for ll=1:length(lambda)
    for ii=1:iters
        [f(ii),g] = evalGradients_L2( x,imageSize,b,m,lambda(ll),D2,W ); % calculate gradient a
        alpha = goldenSearch( x,g,imageSize,b,m,lambda(ll),D2,W ); % find alpha for this x
        x = x - alpha*g; % find new x
        if norm(g) < tol % if gradient is near 0
            break; % the solution converged
        end
    end
```

```
    fCell{ll} = f(1:ii); % cutoff trailing zeros and store in cell
    solutionCell{ll} = x; % grab solution and store in cell
    itsSD(ll) = ii; % get iteration count for each lambda
    toc
end
```

```
Elapsed time is 90.899970 seconds.
Elapsed time is 181.127889 seconds.
Elapsed time is 271.908272 seconds.
Elapsed time is 362.998973 seconds.
Elapsed time is 454.030823 seconds.
```

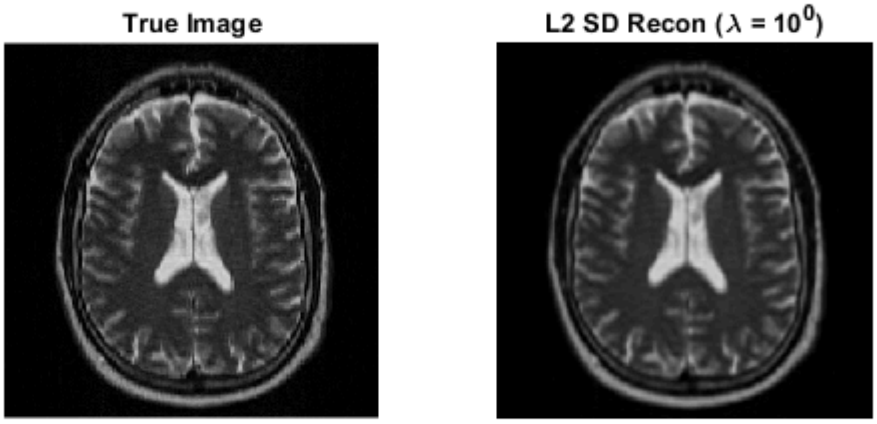
- **Plot the (logarithm of the) value of the cost function ($\|Ax - b\|_2^2 + \lambda \|Cx\|_2^2$) as a function of iterations (this should be decreasing for both algorithms, but different between algorithms). Since the value of the cost function will likely change a lot, make sure to plot its logarithm vs the iteration number.**

```
figure; hold on;
for lf=1:length(lambda)
    plot(log(fCell{lf}));
end
title('Progression of cost function for SD (logged)'); xlabel('iterations'); ylabel('Function v
legend({'\lambda = 10^0','\lambda = 10^2','\lambda = 10^4','\lambda = 10^6','\lambda = 10^8'})
```

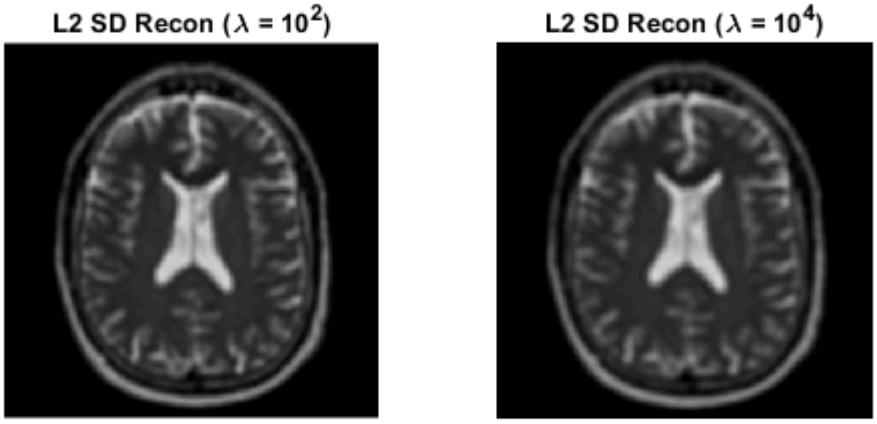


- **Plot the final image after the iterations are done for each of the algorithms (display it using a common colormap for both algorithms).**

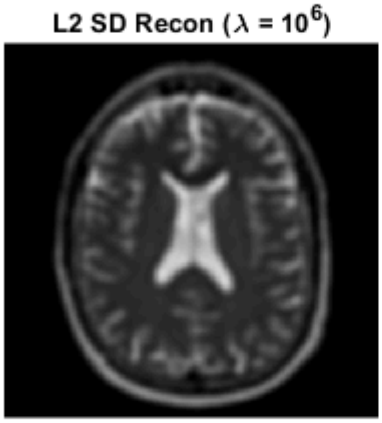
```
% Show image solution
figure;
subplot(1,2,1); imshow(abs(TRU),[0 400]); title 'True Image';
subplot(1,2,2); imshow(reshape(abs(solutionCell{1}),imageSize),[0 400]); title 'L2 SD Recon (\
```

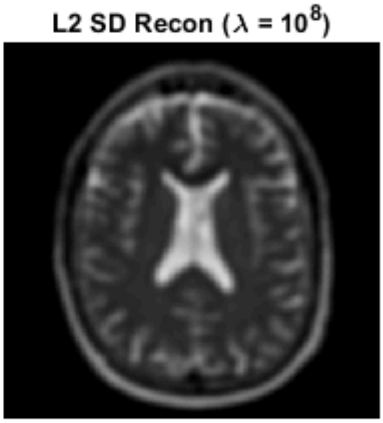



```
figure;
subplot(1,2,1); imshow(reshape(abs(solutionCell{2}),imageSize),[0 400]); title 'L2 SD Recon (\
subplot(1,2,2); imshow(reshape(abs(solutionCell{3}),imageSize),[0 400]); title 'L2 SD Recon (\
```




```
figure;
subplot(1,2,1); imshow(reshape(abs(solutionCell{4}),imageSize),[0 400]); title 'L2 SD Recon (\
subplot(1,2,2); imshow(reshape(abs(solutionCell{5}),imageSize),[0 400]); title 'L2 SD Recon (\
```




### Conjugate Gradients

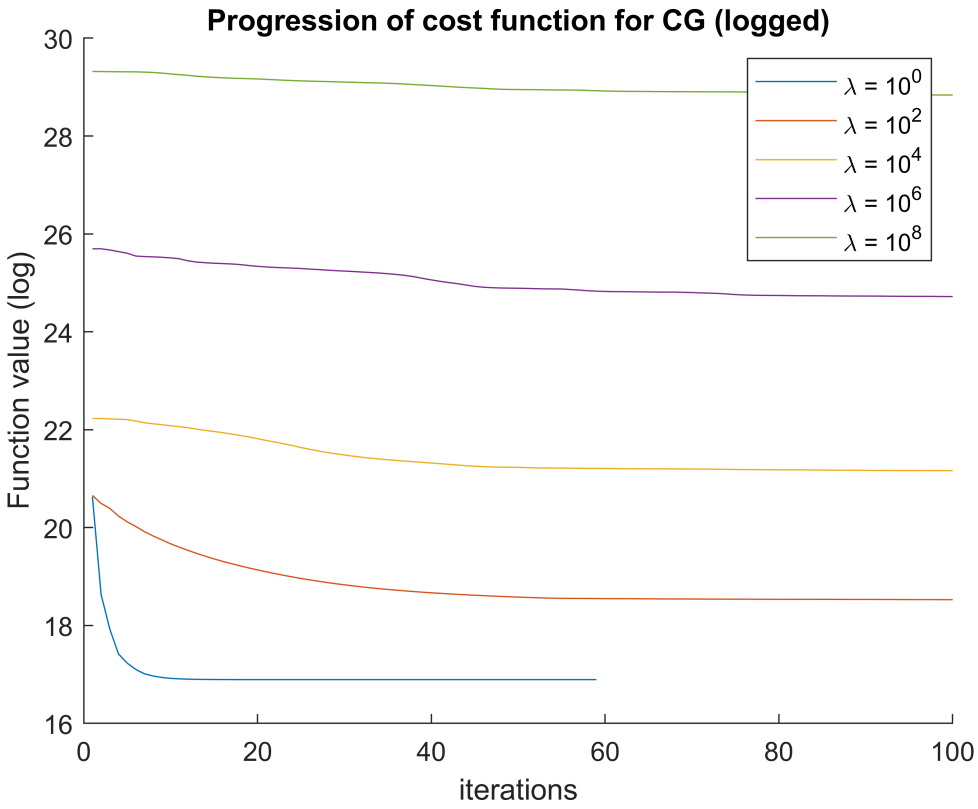
```
x = zeros(N,1); % set initial guess
f = zeros(1,iters); % initialize cost function values

tic
for ll=1:length(lambda)
    [f(1),g] = evalGradients_L2( x,imageSize,b,m,lambda(ll),D2,W ); % calculate gradient and co
    d = -g;

    solution_exists = 0; % set to false
    jj = 0;
    while(jj < iters && solution_exists == 0)
        jj = jj+1;
            % Get Q matrix
            dK = fft2c(reshape(d,imageSize)); % kSpace d vector (Fd)
            dK = dK(:).*m; % masked kSpace d vector (M'MFd)
            fxfd = ifft2c(reshape(dK,imageSize));
            Qd = fxfd(:) + lambda(ll)*D2'*(conj(W(:)).*W(:).*(D2*d)); % back to image space (F
        alpha = -(g'*d)/(d'*Qd(:)); % find next stepsize
        x = x + alpha*d; % move that stepsize in direction of gradient (d).
        [f(jj+1),g] = evalGradients_L2( x,imageSize,b,m,lambda(ll),D2,W ); % redefine gradient
        if norm(g) < tol % if gradient is near 0
            solution_exists = 1; % BREAK
        else
            beta = (g'*Qd(:))/(d'*Qd(:)); % find next-next step
```

```
            d = -g + beta*d; % second gradient move
        end
    end
    fCell{ll} = f(1:jj); % cutoff trailing zeros
    solutionCell{ll} = x; % grab solution
    itsCG(ll) = jj; % get iteration count for each lambda
    toc

end
```

```
Elapsed time is 3.740817 seconds.
Elapsed time is 10.005595 seconds.
Elapsed time is 16.316510 seconds.
Elapsed time is 22.534831 seconds.
Elapsed time is 28.757092 seconds.
```

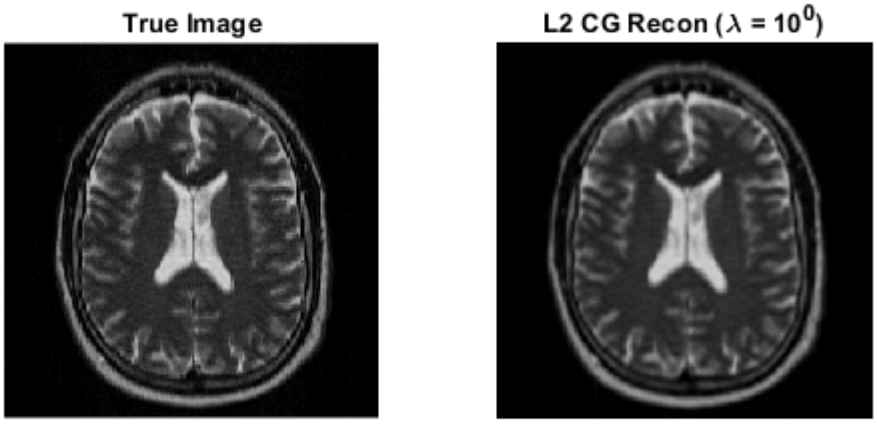
- **Plot the (logarithm of the) value of the cost function ($\|Ax - b\|_2^2 + \lambda \|Cx\|_2^2$) as a function of iterations (this should be decreasing for both algorithms, but different between algorithms). Since the value of the cost function will likely change a lot, make sure to plot its logarithm vs the iteration number.**

```
figure; hold on;
for lf=1:length(lambda)
    plot(log(fCell{lf}));
end
title('Progression of cost function for CG (logged)'); xlabel('iterations'); ylabel('Function v
legend({'\lambda = 10^0','\lambda = 10^2','\lambda = 10^4','\lambda = 10^6','\lambda = 10^8'})
```

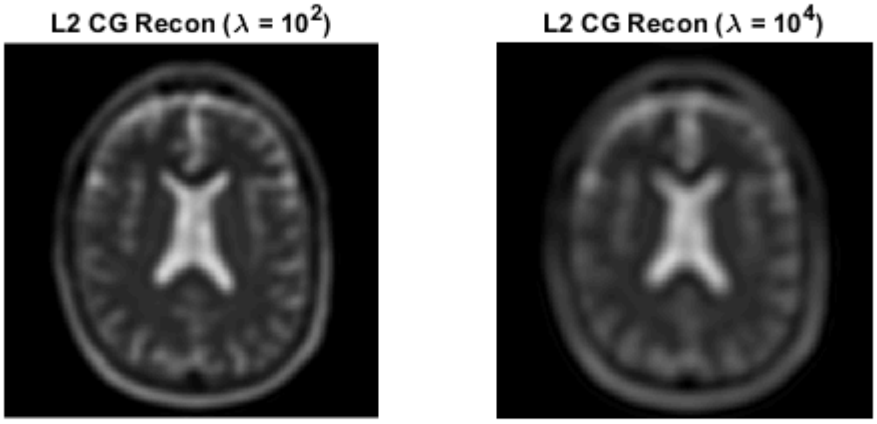



- **Plot the final image after the iterations are done for each of the algorithms (display it using a common colormap for both algorithms).**

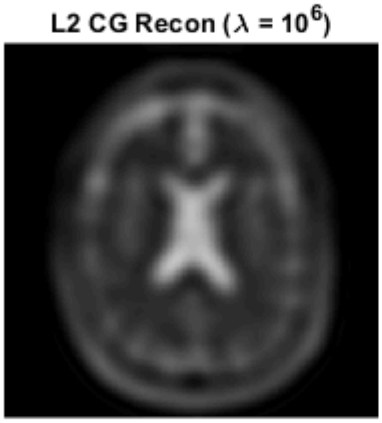
```
% Show image solution
figure;
subplot(1,2,1); imshow(abs(TRU),[0 400]); title 'True Image';
subplot(1,2,2); imshow(reshape(abs(solutionCell{1}),imageSize),[0 400]); title 'L2 CG Recon (\
```

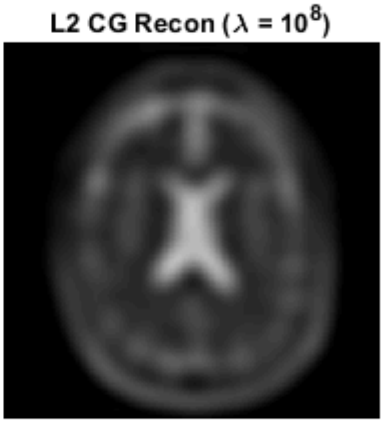



```
figure;
subplot(1,2,1); imshow(reshape(abs(solutionCell{2}),imageSize),[0 400]); title 'L2 CG Recon (\
subplot(1,2,2); imshow(reshape(abs(solutionCell{3}),imageSize),[0 400]); title 'L2 CG Recon (\
```




```
figure;
subplot(1,2,1); imshow(reshape(abs(solutionCell{4}),imageSize),[0 400]); title 'L2 CG Recon (\
subplot(1,2,2); imshow(reshape(abs(solutionCell{5}),imageSize),[0 400]); title 'L2 CG Recon (\
```




- **Report the relative performance of SD and CG (ie: how rapidly do they converge).**

```
disp(['For lambda = [10^0 10^2 10^4 10^6 10^8], Steepest Descent Method converged in [' num2str
```

```
For lambda = [10^0 10^2 10^4 10^6 10^8], Steepest Descent Method converged in 100  100  100  100  100 iterations
```

```
disp(['For lambda = [10^0 10^2 10^4 10^6 10^8], Conjugate Gradient Method converged in [' num2s
```

```
For lambda = [10^0 10^2 10^4 10^6 10^8], Conjugate Gradient Method converged in 59  100  100  100  100 iterations
```

Both methods did not converge before 100 iterations (except for lambda=1 in CG which took 59 iterations). The CG method was much quicker in converging, taking around 2 s for each lambda. The SD method took, on average, around 30 s to converge for each lambda.

- **How does the final value of the cost function change as $\lambda$ increases? How does the visual appearance of the image change as $\lambda$ increases?**

The cost function increases as lambda increases. This is because the finite difference regularization term is being multiplied by increasingly higher lambda values. It is quite apparent that the images become blurrier as lambda increases.

## Problem 3: Reconstruction from Partial Fourier Samples with Anatomically-Informed Smoothness Regularization

**Write a script that loads the data in `hw3_problem3.mat`, and reconstructs an image (320x320 pixels) from the samples in array `b`**. Next we describe the data and smoothness terms:

• Data term: Similar to the previous case, the samples in `b` are DFT samples of the image, taken at the locations specified by the mask in array `mask`. In other words, the matrix $\mathbf{A}$ in our general formulation (equation 1 above) is

$$\mathbf{A} = \mathbf{MF}$$

where $\mathbf{F}$ performs a DFT (including `fftshifts`) and the matrix $\mathbf{M}$ is a sampling matrix that selects a subset of samples from the resulting 320x320 DFT. The selected subset of samples is given by the binary array `m`.

• Smoothness term: The smoothness term is given by $\lambda \|\mathbf{Cx}\|_2^2$, where $\mathbf{C} = \mathbf{WD}$ calculates weighted finite differences in 2D. The matrix $\mathbf{D}$, similarly to the previous problem, is as described in lecture 14 of this course (where we denoted it as `D2` in the Matlab code to indicate its 2D nature). The weighting matrix $\mathbf{W}$ is a diagonal matrix with weights derived from an image of the same anatomy as our desired image: it specifies in which locations we will penalize roughness, and in which locations (those with expected edges) we will not penalize roughness. The diagonal elements of this matrix can be found in array `w`.

You should **repeat this optimization for several different values of $\lambda$**:

• $\lambda = 10^0$

• $\lambda = 10^2$

• $\lambda = 10^4$

• $\lambda = 10^6$

• $\lambda = 10^8$

and, in each case, **report the cost function as a function of iterations (plot), as well as the final estimated image (display)**. This should be done **for both the SD and the CG algorithms**.

### Load data

```
load('hw3_problem3.mat')
```

### Steepest Descent

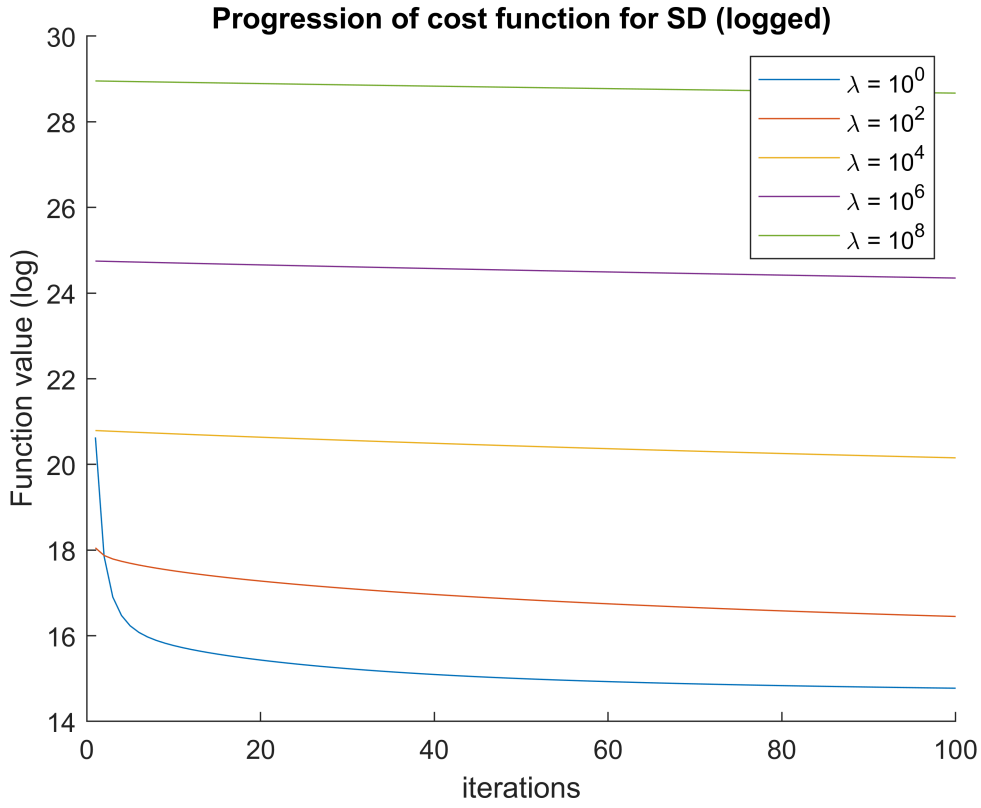
```
x = zeros(N,1); % set initial guess
f = zeros(1,iters); % initialize cost function values

tic
for ll=1:length(lambda)
    for ii=1:iters
        [f(ii),g] = evalGradients_L2( x,imageSize,b,m,lambda(ll),D2,w ); % calculate gradient a
        alpha = goldenSearch( x,g,imageSize,b,m,lambda(ll),D2,w ); % find alpha for this x
        x = x - alpha*g; % find new x
        if norm(g) < tol % if gradient is near 0
```

```
            break; % the solution converged
        end
    end
    fCell{ll} = f(1:ii); % cutoff trailing zeros and store in cell
    solutionCell{ll} = x; % grab solution and store in cell
    itsSD(ll) = ii; % get iteration count for each lambda
    toc
end
```

```
Elapsed time is 92.434604 seconds.
Elapsed time is 183.205780 seconds.
Elapsed time is 273.273670 seconds.
Elapsed time is 364.489620 seconds.
Elapsed time is 455.080258 seconds.
```

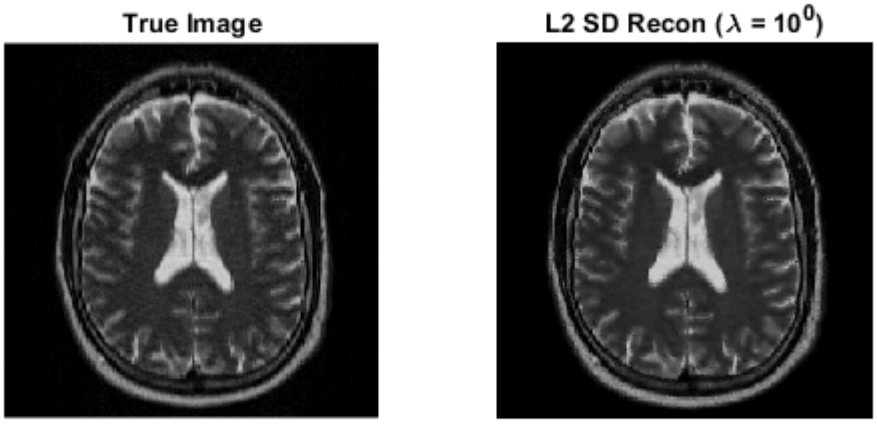
- **Plot the (logarithm of the) value of the cost function ($\|Ax - b\|_2^2 + \lambda \|Cx\|_2^2$) as a function of iterations (this should be decreasing for both algorithms, but different between algorithms). Since the value of the cost function will likely change a lot, make sure to plot its logarithm vs the iteration number.**

```
figure; hold on;
for lf=1:length(lambda)
    plot(log(fCell{lf}));
end
title('Progression of cost function for SD (logged)'); xlabel('iterations'); ylabel('Function v
legend({'\lambda = 10^0','\lambda = 10^2','\lambda = 10^4','\lambda = 10^6','\lambda = 10^8'})
```

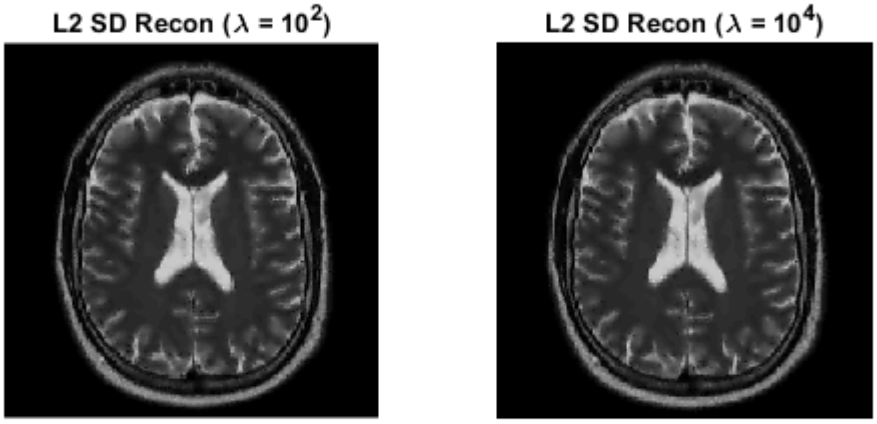


- **Plot the final image after the iterations are done for each of the algorithms (display it using a common colormap for both algorithms).**

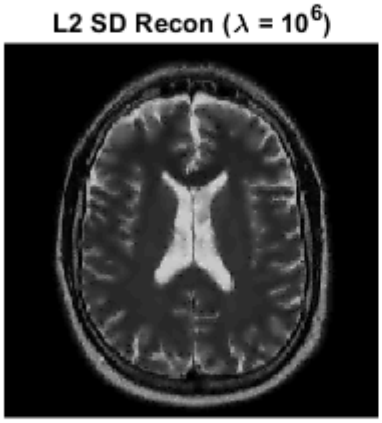
```
% Show image solution
figure;
subplot(1,2,1); imshow(abs(TRU),[0 400]); title 'True Image';
subplot(1,2,2); imshow(reshape(abs(solutionCell{1}),imageSize),[0 400]); title 'L2 SD Recon (\
```

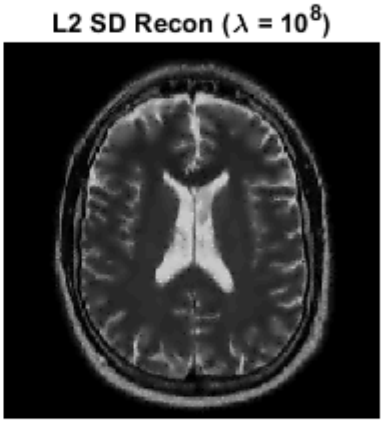



```
figure;
subplot(1,2,1); imshow(reshape(abs(solutionCell{2}),imageSize),[0 400]); title 'L2 SD Recon (\
subplot(1,2,2); imshow(reshape(abs(solutionCell{3}),imageSize),[0 400]); title 'L2 SD Recon (\
```




```
figure;
subplot(1,2,1); imshow(reshape(abs(solutionCell{4}),imageSize),[0 400]); title 'L2 SD Recon (\
subplot(1,2,2); imshow(reshape(abs(solutionCell{5}),imageSize),[0 400]); title 'L2 SD Recon (\
```




### Conjugate Gradients

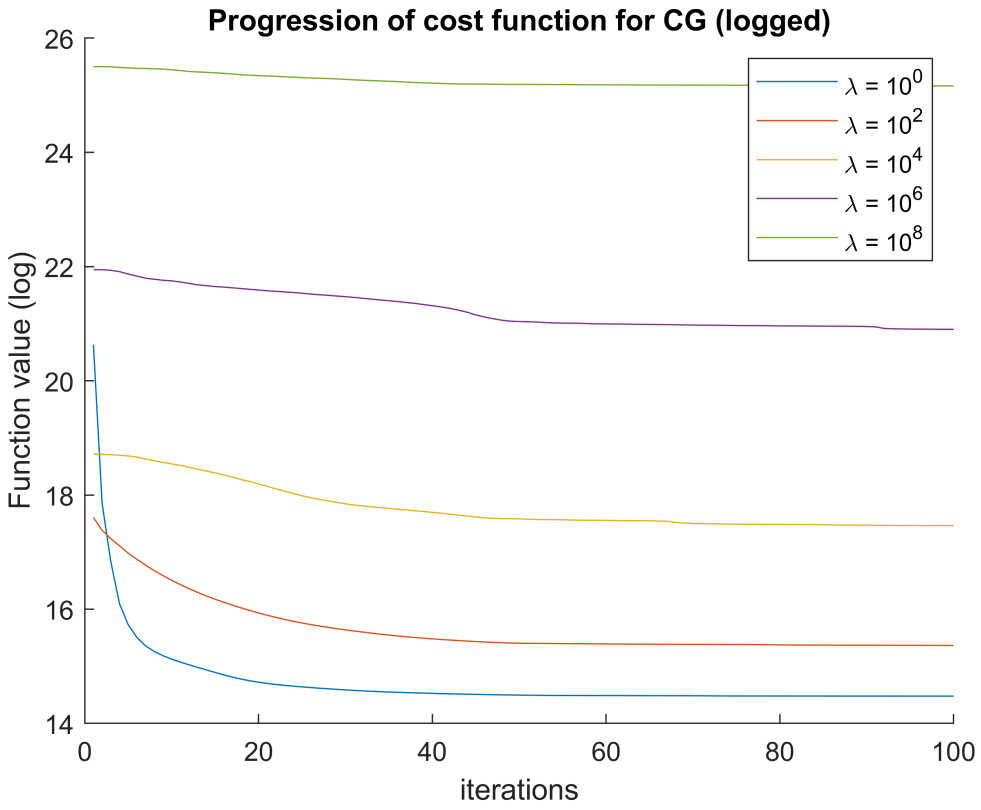
```
x = zeros(N,1); % set initial guess
f = zeros(1,iters); % initialize cost function values

tic
for ll=1:length(lambda)
    [f(1),g] = evalGradients_L2( x,imageSize,b,m,lambda(ll),D2,w ); % calculate gradient and co
    d = -g; % create new d variable

    solution_exists = 0; % set to false
    jj = 0;
    while(jj < iters && solution_exists == 0)
        jj = jj+1;
            % Get Q matrix
            dK = fft2c(reshape(d,imageSize)); % kSpace d vector (Fd)
            dK = dK(:).*m; % masked kSpace d vector (M'MFd)
            fxfd = ifft2c(reshape(dK,imageSize));
            Qd = fxfd(:) + lambda(ll)*D2'*(conj(w(:)).*w(:).*(D2*d)); % back to image space (F
        alpha = -(g'*d)/(d'*Qd(:)); % find next stepsize
        x = x + alpha*d; % move that stepsize in direction of gradient (d).
        [f(jj+1),g] = evalGradients_L2( x,imageSize,b,m,lambda(ll),D2,w ); % redefine gradient
        if norm(g) < tol % if gradient is near 0
            solution_exists = 1; % BREAK
        else
            beta = (g'*Qd(:))/(d'*Qd(:)); % find next-next step
```

```
            d = -g + beta*d; % second gradient move
        end
    end
    fCell{ll} = f(1:jj); % cutoff trailing zeros
    solutionCell{ll} = x; % grab solution
    itsCG(ll) = jj; % get iteration count for each lambda
    toc
end
```

```
Elapsed time is 6.250245 seconds.
Elapsed time is 12.535248 seconds.
Elapsed time is 18.888125 seconds.
Elapsed time is 25.111843 seconds.
Elapsed time is 31.380187 seconds.
```

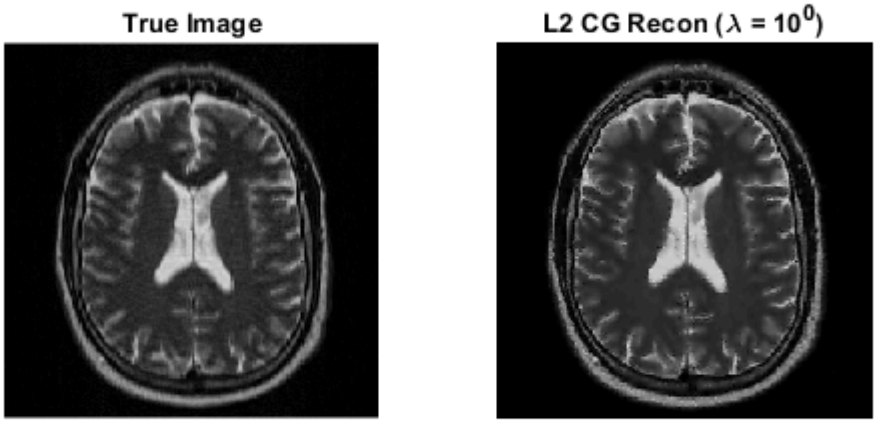
- **Plot the (logarithm of the) value of the cost function ($\|Ax - b\|_2^2 + \lambda \|Cx\|_2^2$) as a function of iterations (this should be decreasing for both algorithms, but different between algorithms). Since the value of the cost function will likely change a lot, make sure to plot its logarithm vs the iteration number.**

```
figure; hold on;
for lf=1:length(lambda)
    plot(log(fCell{lf}));
end
title('Progression of cost function for CG (logged)'); xlabel('iterations'); ylabel('Function v
legend({'\lambda = 10^0','\lambda = 10^2','\lambda = 10^4','\lambda = 10^6','\lambda = 10^8'})
```

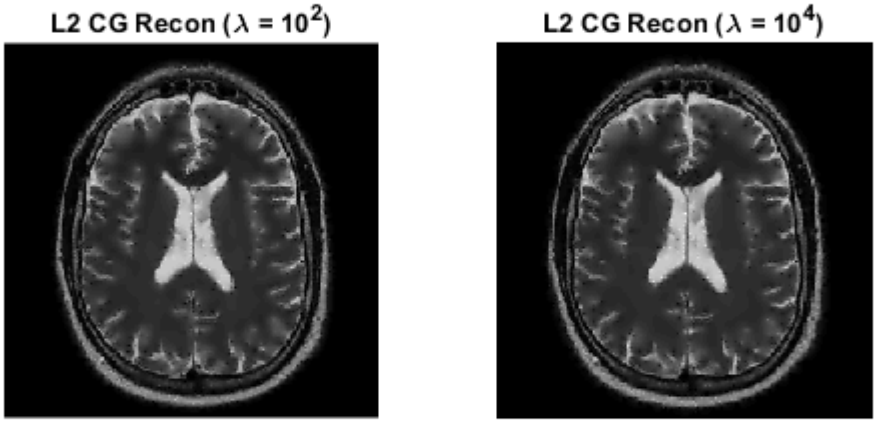


- **Plot the final image after the iterations are done for each of the algorithms (display it using a common colormap for both algorithms).**

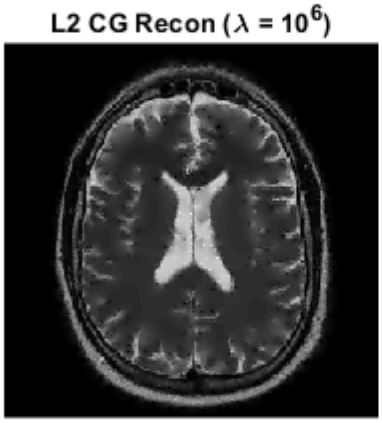
```
% Show image solution
figure;
subplot(1,2,1); imshow(abs(TRU),[0 400]); title 'True Image';
subplot(1,2,2); imshow(reshape(abs(solutionCell{1}),imageSize),[0 400]); title 'L2 CG Recon (\
```

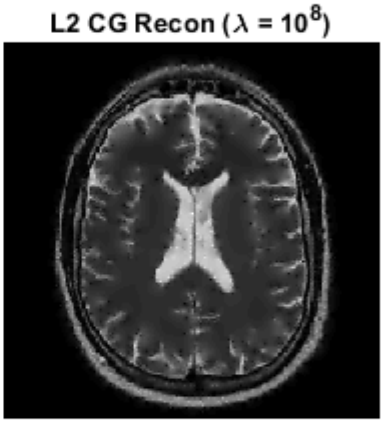



```
figure;
subplot(1,2,1); imshow(reshape(abs(solutionCell{2}),imageSize),[0 400]); title 'L2 CG Recon (\
subplot(1,2,2); imshow(reshape(abs(solutionCell{3}),imageSize),[0 400]); title 'L2 CG Recon (\
```




```
figure;
subplot(1,2,1); imshow(reshape(abs(solutionCell{4}),imageSize),[0 400]); title 'L2 CG Recon (\
subplot(1,2,2); imshow(reshape(abs(solutionCell{5}),imageSize),[0 400]); title 'L2 CG Recon (\
```




- **Report the relative performance of SD and CG (ie: how rapidly do they converge).**

```
disp(['For lambda = [10^0 10^2 10^4 10^6 10^8], Steepest Descent Method converged in [' num2str
```

```
For lambda = [10^0 10^2 10^4 10^6 10^8], Steepest Descent Method converged in 100  100  100  100  100 iterations
```

```
disp(['For lambda = [10^0 10^2 10^4 10^6 10^8], Conjugate Gradient Method converged in [' num2s
```

```
For lambda = [10^0 10^2 10^4 10^6 10^8], Conjugate Gradient Method converged in 100  100  100  100  100 iterations
```

Both methods did not converge before 100 iterations. The CG method was much quicker in converging, taking around 6 s for each lambda. The SD method took, on average, around 90 s to converge for each lambda.

- **How does the final value of the cost function change as $\lambda$ increases? How does the visual appearance of the image change as $\lambda$ increases? How do the cost function and visual appearance of the image in this problem compare to those in the previous problem?**

The cost function increases as lambda increases. This is because the regularization term is still heavily dependent on the finite differences, despite the inclusion of the weighting matrix (w) which avoids penalizing edge structures. As lambda increases, the non-edge regions are smoothed while the edges are preserved. This acts to reduce (smooth) the artifacts that are seen in the white matter regions of the brain, making the images look much better. However, as lambda increases to higher values (10^6 and 10^8), we see that the non-edge regions are over-smoothed.

The cost function is less than that of problem 2, likely because there is less penalty incurred from the removal of the edges (very high finite differences values) in the regularization term. The images appear much more like a fully-sampled image. In problem 2, there was a global blurring, whereas in this problem, the blurring is only localized to areas in which the artifact is visually present.

## ANCILLARY FUNCTIONS

Inverse Fourier transform with shifts

```
% function x = ifft2c(k)
%     x = ifftshift(ifft2(fftshift(k)));
% end
```

Forward Fourier transform with shifts

```
% function k = fft2c(x)
%     k = fftshift(fft2(ifftshift(x)));
% end
```