Constrained Reconstruction in MRI

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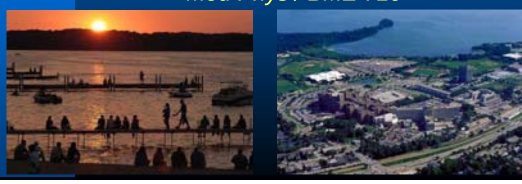
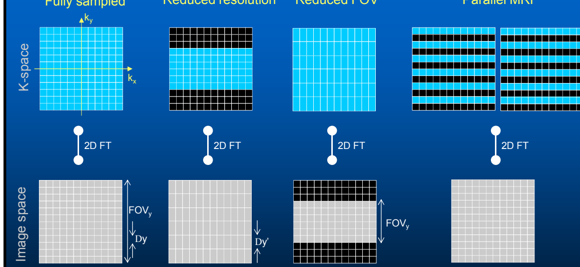
Med Phys / BME 710



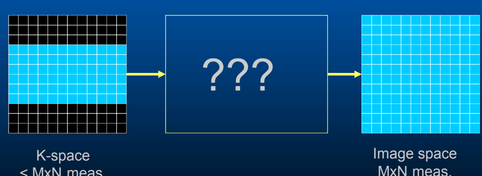
Reduced Sampling



M x N measurements in k-space AND image space

Constrained Reconstruction

How to obtain M x N images with < M x N measurements in k-space ??
-> Image reconstruction becomes ill-posed problem:
# of unknowns (pixels) > # of measurements
-> infinite number of possible solutions



Constrained Reconstruction

How to obtain M x N images with < M x N measurements in k-space ??
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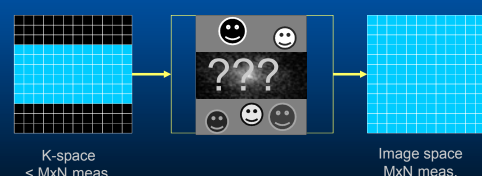
-> use constraints (a priori knowledge) to find a reasonable solution



Constrained Reconstruction

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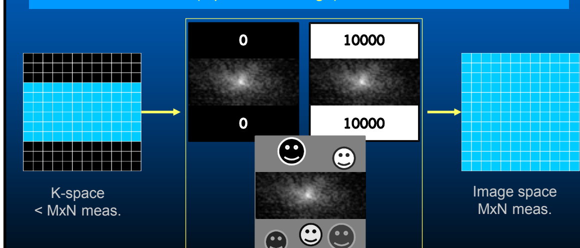

Constrained Reconstruction

How to obtain M x N images with < M x N measurements in k-space ??
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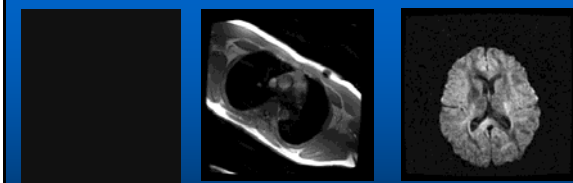
Single image – model based constraints
- trajectory design
  - radial, spiral, ... with undersampling
  - trade tolerable artefact / low SNR for speed
- phase (magnitude) constraints
  - partial Fourier imaging
- compressed sensing

Image series (e.g. dynamic series) – model based constraints
- sliding window, view sharing,
- keyhole,TRICKS, ...

Image series (e.g. dynamic series) – object based constraints
- k-t BLAST, k-t SENSE, k-t GRAPPA,
- RIGR, HYPR, ...

Correlations in Image Series



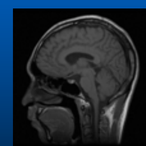
Dynamic Series
Time
Contrast-enhanced MRA
Perfusion Imaging
…

Periodic Series
cardiac cycle
Cardiac function
Phase contrast
…

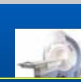
Parametric Series
Diffusion tensor
Diffusion weighting
Inversion recovery (TI)
Echo time (TE)
…

MR Data Acquisition

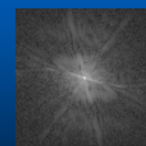
Ef = b



f = image



Encoding



b = k-space data

Forward problem is linear

MR Reconstruction

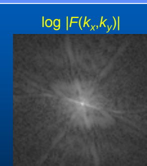
Ef = b

Invertible E ⇨ unique f — e.g. Fourier transform

2D Fourier Transform

$$F(k_x,k_y) = \frac{1}{M\cdot N}\sum_{0}^{M-1}\sum_{0}^{N-1} f(x,y)e^{-j(2\pi/M)k_x x}e^{-j(2\pi/N)k_y y}$$

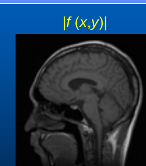
log $|F(k_x,k_y)|$



M x N data
M: # of readout points
N: # of phase encodes
Scan time: N x TR

2D FT

$|f(x,y)|$



M x N data

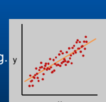
$$\Delta x = \frac{1}{2k_{max,x}} \qquad FOV_x = \frac{1}{\Delta k_x}$$

MR Reconstruction

Ef = b

Invertible E ⇨ unique f — e.g. Fourier transform

Overdetermined E ⇨ unique or no f — e.g.



Underdetermined E ⇨ Infinite number of f

Constrained Reconstruction to narrow down solution

quick aside . . .

- There is no convention for how linear equations are written in MRI literature
- Encoding matrix: A, E, F, $\mathcal{F}_U$
- Image vector: x, m, f
- Data vector: b, y, s

$$Ax = b$$
$$Ex = y$$

Linear Systems

$$Ax = b$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Fourier Transform (for traditional MRI)

Unknown (image, N voxels)

Known (acquired data, M points)

Reconstruction From Undersampled Data

Simplest way to obtain the solution is least squares (aka zero-filling) – undersampling artifacts, low SNR

Estimated Image

$$\min_f \| Ef - b \|_2^2$$

Encoding matrix *data*

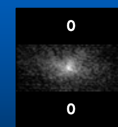


Image quality can be improved by regularization

$$\min_f \left(\| Ef - b \|_2^2 + \lambda \| f \|_X \right)$$

Norm X corresponds to theoretical assumptions.

$$X = \ell_p, \; \| f \|_p = \left(\sum | f_i |^p \right)^{1/p}$$

Choice of the norm

l_2 -norm (Tikhonov regularization) minimizes noise

$$\min_f \left(\| Ef - b \|_2^2 + \lambda \| f \|_2^2 \right)$$

Unregularized Regularized

F.-H. Lin et al, "Parallel Imaging Reconstruction Using Automatic Regularization", MRM, Vol. 51 (3), pp. 559-567

l_1 -norm promotes sparsity

What is a measure of sparsity?

Count the number of non-zero pixels

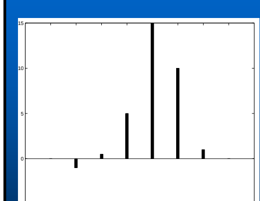
l_0 - ~~norm~~

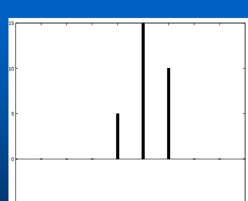
- Doesn't satisfy convexity axiom
- Non-convex problems are hard to deal with computationally

l_p - norm is a norm for p >=1

$$\| f \|_1 = \sum_{m,n} | f_{m,n} | \qquad \| f \|_2 = \left(\sum_{m,n} | f_{m,n} |^2 \right)^{1/2}$$

l_0, l_1, l_2, …

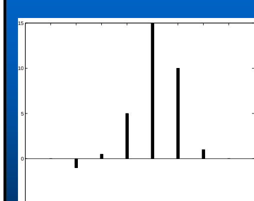


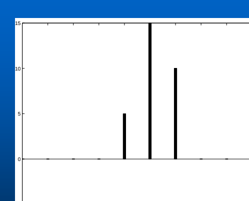


$\| f \|_0 =$ [redacted] $\| f \|_0 =$ [redacted]

[redacted]

l_0, l_1, l_2, …

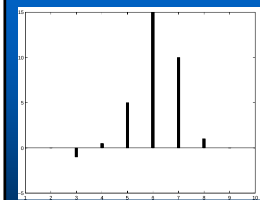


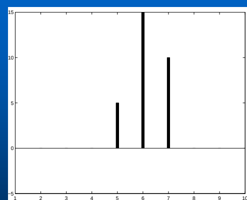


$\| f \|_0 = 6$ $\| f \|_0 = 3$ 50%

$\| f \|_1 =$ [redacted] $\| f \|_1 =$ [redacted]

l_0, l_1, l_2, …





$\|f\|_0 = 6$ | $\|f\|_0 = 3$ | 50%
$\|f\|_1 = 30.4$ | $\|f\|_1 = 30$ | 1.33%
$\|f\|_2 = 18.7084$ | $\|f\|_2 = 18.7083$ | 0.0005%

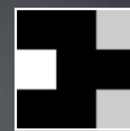
---

x1 =

$$\begin{matrix} 1.6991 & 1.5472 & 1.2575 \\ 1.8909 & 1.1386 & 1.8407 \\ 1.9593 & 1.1493 & 1.2543 \end{matrix}$$

x2 =

$$\begin{matrix} 0 & 0 & 3 \\ 4 & 0 & 0 \\ 0 & 0 & 3 \end{matrix}$$



$\|x\|_2 = 4.68$ | $\|x\|_2 = 5.83$
$\|x\|_1 = 13.74$ | $\|x\|_1 = 10$
$\|x\|_0 = 9$ | $\|x\|_0 = 3$

---

x1 =

$$\begin{matrix} 1.6991 & 1.5472 & 1.2575 \\ 1.8909 & 1.1386 & 1.8407 \\ 1.9593 & 1.1493 & 1.2543 \end{matrix}$$

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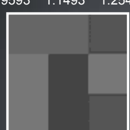
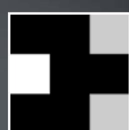




$\boxed{\|x\|_2 = 4.68}$ min | $\|x\|_2 = 5.83$
$\|x\|_1 = 13.74$ | $\boxed{\|x\|_1 = 10}$ min
$\|x\|_0 = 9$ | $\boxed{\|x\|_0 = 3}$ min

Minimizing l1 and l0 promotes sparsity

Compressed Sensing Theory

aka Compressive Sampling

CS theory states that if E satisfies certain conditions (Restricted Isometry Principle), then solutions of l_1 and l_0 problems are the same

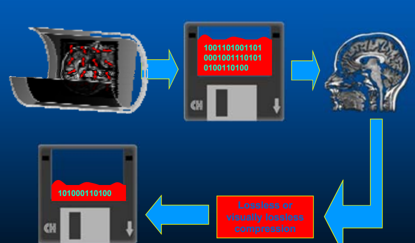
CS theory gives number of necessary samples: if an *NxN* image has S non-zero pixels, then it can be reconstructed from *O(S* log *N)* measurements

The measurements should be incoherent and provide unstructured artifacts

Possible trajectories: random Cartesian, radial

Compression

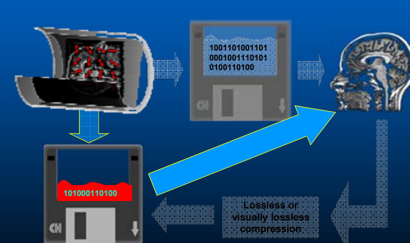
Standard approach:
First collect, then compress

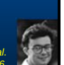


Courtesy of Miki Lustig, Stanford University

Compressed Sensing

Instead: Compressed Sensing (CS)
First Compress, then reconstruct.



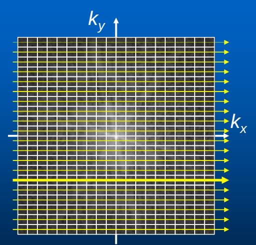


Candes et al.
IEEE TIF '06

Donoho
IEEE TIF '06

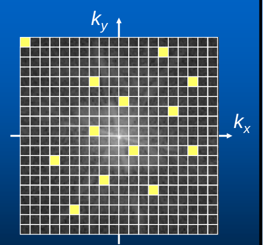
Courtesy of Miki Lustig, Stanford University

‘Random’ Sampling - Cartesian



2D : $k_x - k_y$ space

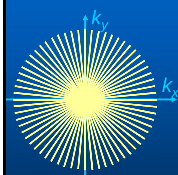
Random sampling impractical



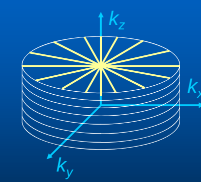
3D : $k_y - k_z$ space

Random sampling possible

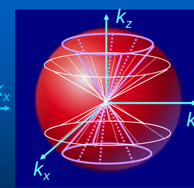
Radial Sampling



2D



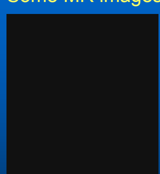
3D hybrid – Stack of stars



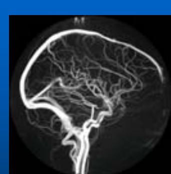
3D radial - VIPR

CS – Sparse Image

Some MR images are sparse



CE MRA



PC MR Angiogram

$$\min_f \left(\|Ef - b\|_2^2 + \lambda \|f\|_1 \right)$$

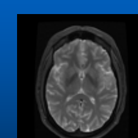
J. Velikina, MR Angio Club 2004
M. Lustig, MRM, 2007
K. T. Block, MRM 2007

CS - Sparsifying Imaging Transforms

Many MR images are not sparse!

We can include a sparsifying transform into compressed sensing framework to promote sparsity

$$\min_f \left(\|Ef - b\|_2^2 + \lambda \|\Phi f\|_1 \right)$$

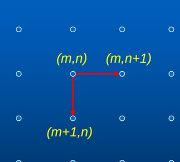


Popular choices of sparsifying transforms

- discrete gradient
- wavelet transform

Total Variation

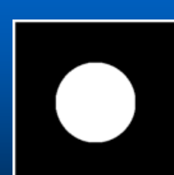
Most popular sparsifying transform is discrete gradient, whose l_1 norm is called Total Variation



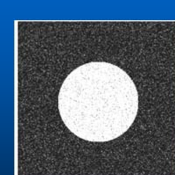
$$\nabla f_{m,n} := \sqrt{\left|f_{m,n+1} - f_{m,n}\right|^2 + \left|f_{m+1,n} - f_{m,n}\right|^2}$$

$$TV(f) := \sum_{m,n} \nabla f_{m,n}$$

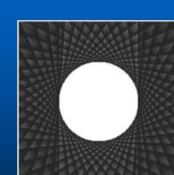
Examples



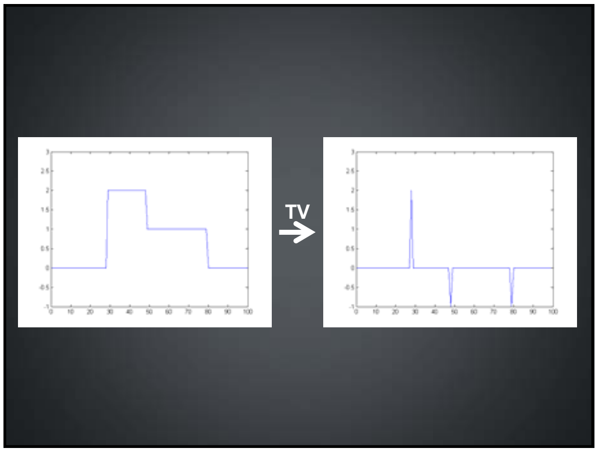
TV = 473.23

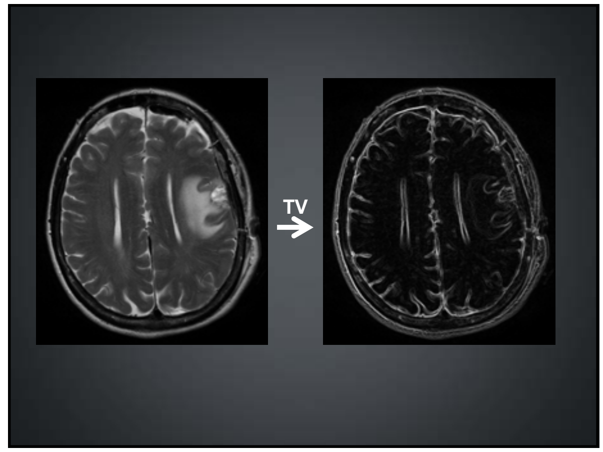


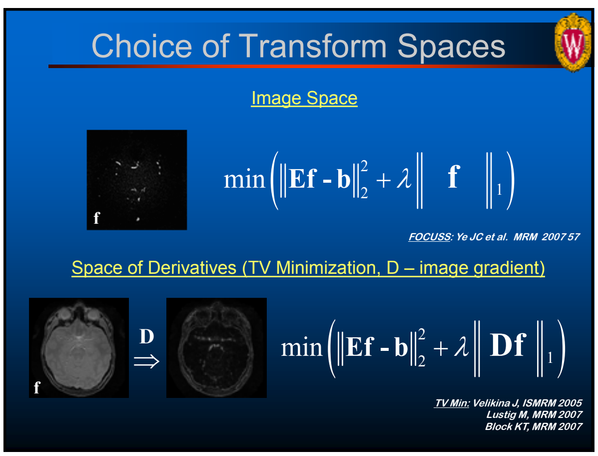
TV = 28719

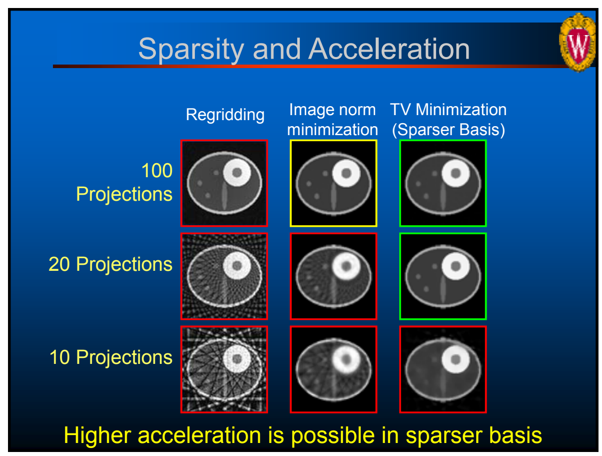


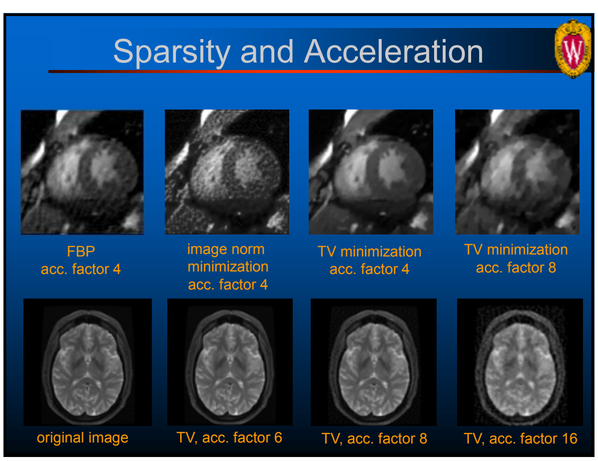
TV = 6266.8


TV

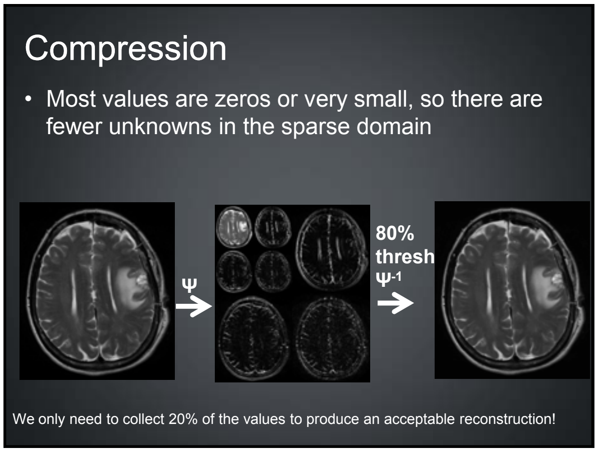

TV

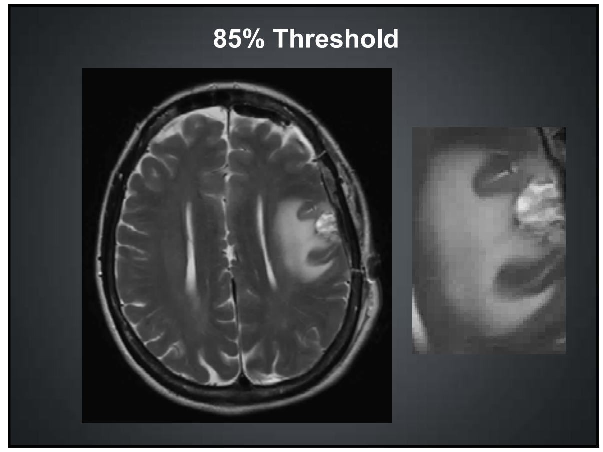

Choice of Transform Spaces
Image Space
f
$\min\left(\|\mathbf{Ef}-\mathbf{b}\|_2^2+\lambda\|\ \mathbf{f}\ \|_1\right)$
FOCUSS: Ye JC et al. MRM 2007 57
Space of Derivatives (TV Minimization, D – image gradient)
f
D
$\Rightarrow$
$\min\left(\|\mathbf{Ef}-\mathbf{b}\|_2^2+\lambda\|\mathbf{Df}\|_1\right)$
TV Min: Velikina J, ISMRM 2005
Lustig M, MRM 2007
Block KT, MRM 2007

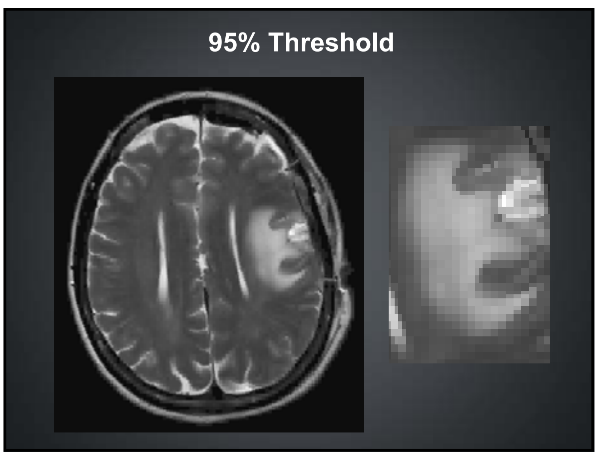

Sparsity and Acceleration
Regridding
Image norm minimization
TV Minimization (Sparser Basis)
100 Projections
20 Projections
10 Projections
Higher acceleration is possible in sparser basis

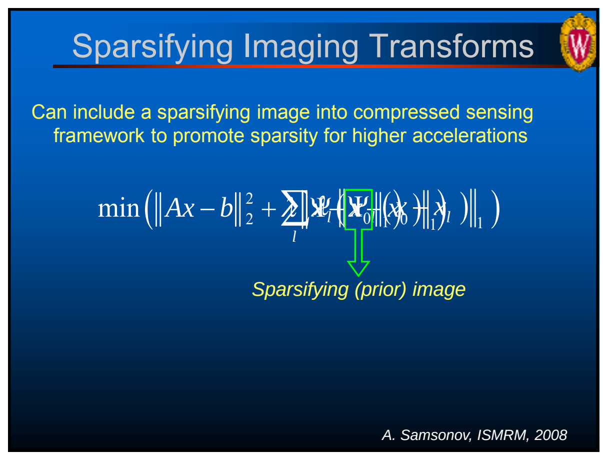

Sparsity and Acceleration
FBP acc. factor 4
image norm minimization acc. factor 4
TV minimization acc. factor 4
TV minimization acc. factor 8
original image
TV, acc. factor 6
TV, acc. factor 8
TV, acc. factor 16

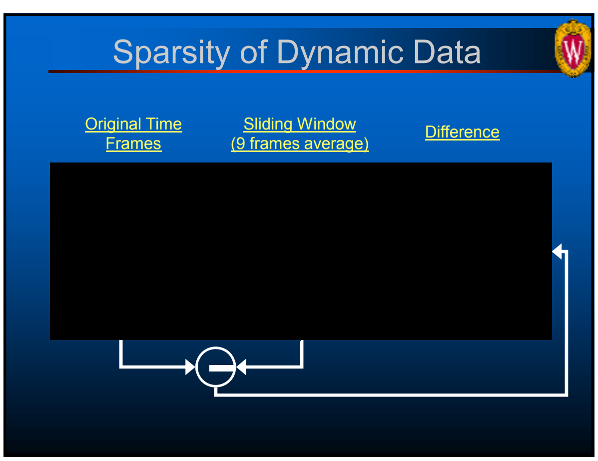

Wavelet Transform

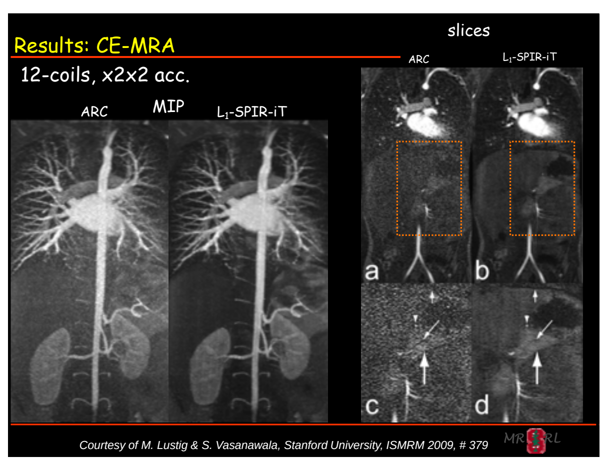

Compression
Most values are zeros or very small, so there are fewer unknowns in the sparse domain
Ψ
80% thresh
Ψ^{-1}
We only need to collect 20% of the values to produce an acceptable reconstruction!


85% Threshold


95% Threshold


Sparsifying Imaging Transforms
Can include a sparsifying image into compressed sensing framework to promote sparsity for higher accelerations
$\min\left(\|Ax-b\|_2^2+\sum_i \lambda_i \|\Psi_i(x - x_0)\|_1\right)$
Sparsifying (prior) image
A. Samsonov, ISMRM, 2008


Sparsity of Dynamic Data
Original Time Frames
Sliding Window (9 frames average)
Difference


slices
Results: CE-MRA
12-coils, x2x2 acc.
ARC
L_1-SPIR-iT
ARC
MIP
L_1-SPIR-iT
a
b
c
d
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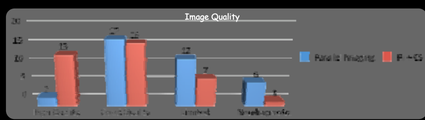
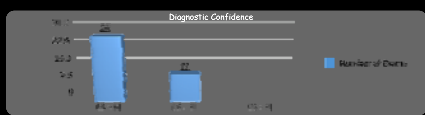
Clinical Study

- 3D contrast enhanced, T1-weighted SPGR
- 37 pediatric patients





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Regularization Parameter

Crucial to performance of CS regularization is the choice of regularization parameter

$$J(\mathbf{f}) = \|\mathbf{E}\mathbf{f} - \tilde{\mathbf{s}}\|_2^2 + \boxed{\lambda}\|\Psi\mathbf{f}\|_1$$

Regularization parameter (Balances two sources of information about **f**)

Reference image approach
- Acquire reference image
- Find λ giving the best fit of reconstructed image to the reference
- Fast processing (around 4 sec for 2D problem)

Ying L, et al. MRM, 2008, 60:414–421

CS – Practical Considerations

Choose Sampling Pattern
- 3D: 'Randomized' Cartesian
- 2D/3D: Radial

Choose sparsity transform
- May / may not be needed
- May / may not be supported by image subtraction
- Many choices, most common
  - wavelet
  - discrete gradient

Choose λ
- Reference image
- Discrepancy principle – residual is at noise level
- In practice: empirically

When to stop iterations
- Usually follow differences between successive iterations

Compressed Sensing - Summary

Compressed Sensing in MRI
- New Sampling Theorem
  - # of required samples depends on sparsity of image
- MR Image can be
  - sparse or
  - 'sparsified'
    - Transform
    - (Prior) Image subtraction
- Enabling approach for accelerated imaging
  - Can be combined with other approaches, e.g. parallel imaging
- Too much regularization degrades image quality
- Many applications are being explored