Furthermore, if the receiver coil has a homogeneous reception field over the region of interest, as is often assumed, the signal expression in Eq. (3.149) can be further simplified to

$$S(t) = \int_{\text{object}} M_{xy}(\boldsymbol{r}, 0)e^{-i\Delta\omega(\boldsymbol{r})t}d\boldsymbol{r} \tag{3.150}$$

Note that in the preceding derivation, it is implicitly assumed that the object sees a static inhomogeneous magnetic field during the free precession period. Expressing the field distribution as

$$B(\boldsymbol{r}) = B_0 + \Delta B(\boldsymbol{r}) \tag{3.151}$$

we have

$$\Delta\omega(\boldsymbol{r}) = \gamma\Delta B(\boldsymbol{r}) \tag{3.152}$$

and Eq. (3.150) becomes

$$S(t) = \int_{\text{object}} M_{xy}(\boldsymbol{r}, 0)e^{-i\gamma\Delta B(\boldsymbol{r})t}d\boldsymbol{r} \tag{3.153}$$

If the inhomogeneous field is time-varying, that is, ΔB is a function of both space and time, then all the foregoing signal expressions need to be modified accordingly. Specifically, denoting the inhomogeneous field component as $\Delta B(\boldsymbol{r}, t)$, $\Delta\omega(\boldsymbol{r})t$ should be replaced by $\gamma\int_0^t \Delta B(\boldsymbol{r}, \tau)d\tau$. For example, Eq. (3.148) should be rewritten as

$$S(t) = \omega_0 e^{i\pi/2} \int_{\text{object}} B^*_{r,xy}(\boldsymbol{r}) M_{xy}(\boldsymbol{r}, 0)e^{-i\gamma\int_0^t \Delta B(\boldsymbol{r},\tau)d\tau}d\boldsymbol{r} \tag{3.154}$$

■ **Example 3.9**

We calculate the signal generated by an α pulse in this example.

Assume that the object has a thermal equilibrium magnetization $M_z^0(\boldsymbol{r})$. The transverse magnetization generated by the pulse is

$$M_{xy}(\boldsymbol{r}, t = 0_+) = M_z^0(\boldsymbol{r}) \sin\alpha e^{i\phi_e(\boldsymbol{r})}$$

Substituting the result into Eq. (3.153) yields

$$S(t) = \sin\alpha \int_{\text{object}} M_z^0(\boldsymbol{r})e^{i\phi_e(\boldsymbol{r})}e^{-i\gamma\Delta B(\boldsymbol{r})t}d\boldsymbol{r}$$

which is the desired expression for the signal generated by an arbitrary α pulse in the presence of an inhomogeneous static field.

Exercises

3.1 For the following nuclei, does their spin quantum number take an integral, half-integral, or zero value? For each case, discuss whether the nucleus is NMR-active.

(a) ^{1}H, ^{2}H

(b) ^{16}O, ^{17}O

(c) ^{12}C, ^{13}C

(d) ^{31}P, ^{23}Na

3.2 In the absence of an external magnetic field, a bulk object exhibits no net nuclear magnetism because:

(a) Nuclear magnetic moments for all nuclei are zero.

(b) Nuclear magnetic moments cancel out each other.

(c) The bulk magnetization vector is too small to be detected.

(d) All of the above.

3.3 What are the primary functions of the static magnetic field $\vec{B}_0$ in MR imaging?

3.4 What is the Zeeman splitting phenomenon?

3.5 Why is MRI known as a low-sensitivity imaging technique?

3.6 What is the primary function of the oscillating $\vec{B}_1(t)$ field?

3.7 What is the resonance condition?

3.8 Why does a spin system often have more than one resonance frequency? If you place a cup of water in a perfectly homogeneous magnetic field, do you expect to detect more than one resonance frequency from the protons? Why?

3.9 What is an isochromat?

3.10 Justify the last two equations in Eq. (3.57).

3.11 Given a fixed flip angle, the larger the $\vec{M}$ the stronger the $\vec{B}_1$ needed because a stronger force is required to flip a larger $\vec{M}$. True or false?

3.12 Briefly discuss how one can selectively elicit the NMR phenomenon from one spin system of a biological sample (such as protons) without affecting the others (such as ^{31}P)?

3.13 Justify that the two representations of nuclear precession in Eq. (3.25) and Eq. (3.16) are equivalent.

3.14 Prove the following relationships for the rotation matrices $\mathbf{R}_{x'}(\alpha)$, $\mathbf{R}_{y'}(\alpha)$, and $\mathbf{R}_{z'}(\alpha)$:

(a) $\mathbf{R}_{x'}^{-1}(\alpha) = \mathbf{R}_{x'}(-\alpha) = \mathbf{R}_{-x'}(\alpha)$

(b) $\mathbf{R}_{y'}^{-1}(\alpha) = \mathbf{R}_{y'}(-\alpha) = \mathbf{R}_{-y'}(\alpha)$

(c) $\mathbf{R}_{z'}^{-1}(\alpha) = \mathbf{R}_{z'}(-\alpha) = \mathbf{R}_{-z'}(\alpha)$

3.15 In which plane does the receiver coil pick up the activated MR signal? Is the received signal dependent on the time evolution of the longitudinal component after an RF pulse? Why?

3.16 Calculate and sketch $\vec{B}_{1,\mathrm{rot}}(t)$ assuming that

$$\vec{B}_1(t) = B_1\cos(\omega_{\mathrm{rf}}t + \pi/4)\vec{i} - B_1\sin(\omega_{\mathrm{rf}}t + \pi/4)\vec{j}$$

3.17 The bulk magnetization of a proton spin system is flipped 90° by a rectangular RF pulse of width 1.0 ms.

(a) What is the magnitude of the B_1 field required?

(b) How many precession cycles take place in the laboratory frame during the pulse, assuming B_0= 0.5, 1.0, and 1.5 T, respectively.

3.18 Assume that $\vec{B}_1(t) = B_1\cos\omega_{\mathrm{rf}}t\vec{i} - B_1\sin\omega_{\mathrm{rf}}t\vec{j}$ is a stationary vector in the ω_{rf}-rotating frame; namely, $\vec{B}_{1,\mathrm{rot}}(t) = B_1\vec{i}'$. Derive an expression for $\vec{B}_1(t)$ such that

(a) $\vec{B}_{1,\mathrm{rot}}(t) = B_1\vec{j}'$

(b) $\vec{B}_{1,\mathrm{rot}}(t) = B_1\vec{i}' + B_1\vec{j}'$

3.19 Derive the closed-form solution given in Eq. (3.82) for the Bloch equation for on-resonance excitation with an arbitrary pulse.

3.20 Calculate and depict the bulk magnetization vector of a spin system relative to the prepulse reference frame after a $90^\circ_{x'}$ pulse. Assume that the

Larmor frequency of the spin system is 10 MHz, the pulse lasts 1.0 ms, and the prepulse condition is $M_{x'}(0_-) = M_{y'}(0_-) = 0$ and $M_{z'}(0_-) = M_z^0$.

3.21 Calculate the resulting magnetization in the laboratory frame immediately after a $90^\circ_{x'}$ pulse with duration of τ and 2τ, respectively.

3.22 Assume that a spin system with a single resonance component was at thermal equilibrium. Calculate the transverse magnetization resulting from the following excitation sequences:

(a) $90^\circ_{x'}90^\circ_{y'}$

(b) $90^\circ_{x'} - \tau - 90^\circ_{y'}$

(c) $45^\circ_{x'}90^\circ_{y'}$

(d) $30^\circ_{x'}(-15^\circ_{z'})80^\circ_{x'}15^\circ_{z'}$

3.23 Calculate the effects of the following excitation sequences on a spin system with two isochromats at resonance frequencies ω_0 and $\omega_0 - \delta\omega_0$. It is assumed that the spin system is at thermal equilibrium and $\tau = \frac{\pi}{3}$.

(a) $90^\circ_{x'} - \tau - 180^\circ_{y'}$

(b) $45^\circ_{x'} - \tau/2 - 90^\circ_{y'}$

3.24 Derive the closed-form solution given in Eq. (3.107) for the Bloch equation for off-resonance excitation with a rectangular pulse.

3.25 Prove the relationships given in Eq. (3.57).

3.26 Prove the result in Eq. (3.60).

3.27 Assume that a known RF pulse $\vec{B}_1(t) = B_1\cos(\omega_0 t)\vec{i} - B_1\sin(\omega_0 t)\vec{j}$ flips the bulk magnetization vector onto the y'-axis (of the rotating frame) immediately after the pulse. Modify this $\vec{B}_1$ field such that the bulk magnetization vector ends up in the following positions immediately after the pulse:

(a) Lying along the $-y'$-axis

(b) Lying along the x'-axis

(c) Lying along the $-x'$-axis

(d) Lying along a vector 45° away from the y'-axis toward the x' in the transverse plane

3.28 Specify two pulses that will convert $M_{x'y'}$ to $M^*_{x'y'}$ and $-M^*_{x'y'}$, respectively.

3.29 Use a vector model to schematically show the effects of a $90^\circ_{x'}$, $90^\circ_{-x'}$, $90^\circ_{y'}$, $90^\circ_{-y'}$, $180^\circ_{x'}$, $180^\circ_{y'}$ pulse on the bulk magnetization vector originally pointing along the z'-axis.

3.30 The excitation property of an RF pulse is derived from the *inverse* Fourier transform of its envelope function. How is it related to the *forward* transform?

3.31 An RF pulse applied along the x'-axis for 100 μs flips an "on-resonance" magnetization by 90° onto the y'-axis. How much magnetization is tipped onto the (x', y')-plane if the excitation is "off-resonance" by 10 kHz?

3.32 Describe what is meant by "hard" and "soft" pulses.

3.33 The frequency distribution of an RF pulse can presumably be calculated from its Fourier transform. Compare the situation pertaining to Problem 3.31 with the result you expect from the Fourier transform.

3.34 A spin system has three isochromats with resonance frequencies at ω_0, $\omega_0 + \Delta$, and $\omega_0 - \Delta$, where $\omega_0 = 42$ MHz and $\Delta = 0.25$ kHz. We next assume that an RF pulse defined by $B_1(t) = B_1^e(t)e^{-i\omega_0 t}$, where the Fourier transform of $B_1^e(t)$ is given in the following figure, will flip the isochromats by 90°, 67.5°, and 67.5°, respectively. Calculate the flip angles of all the isochromats for the following pulses based on the Fourier theory.

(a) $B_1^e(2t)e^{-i\omega_0 t}$

(b) $2B_1^e(2t)e^{-i\omega_0 t}$

(c) $B_1^e(t/2)e^{-i\omega_o t}$

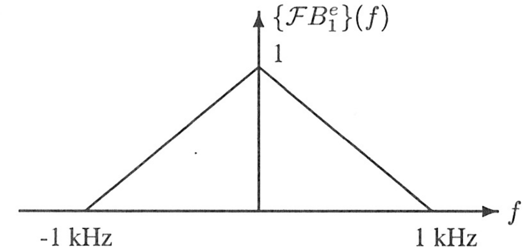
(d) $\frac{1}{2}B_1^e(t/2)e^{-i\omega_0 t}$

(e) $2B_1^e(2t)e^{-i(\omega_0+\Delta)t}$

(f) $2B_1^e(2t)e^{-i(\omega_0-\Delta)t}$

(g) $\frac{1}{2}B_1^e(t/2)e^{-i(\omega_0+\Delta)t}$

(h) $\frac{1}{2}B_1^e(t/2)e^{-i(\omega_0-\Delta)t}$



3.35 Design an RF pulse that will selectively excite a 10 kHz wide frequency band centered at 42 MHz with 45° flip angle for a spin system of protons.

3.36 After an RF pulse, M_{xy} decays to zero and M_z returns to M_z^0. During this relaxation process, the amount of M_{xy} lost is equal to the amount of M_z that is gained. True or false?

3.37 During the *excitation* period, the magnitude of $\vec{M}(t)$ stays constant while $\vec{M}(t)$ spirals down. Give an example to demonstrate that this statement is not true during the *relaxation* period when $\vec{M}(t)$ spirals up.

3.38 How long does it take for the longitudinal magnetization M_z of a spin system with longitudinal relaxation time constant T_1 to recover 63% of its thermal equilibrium value after (a) a 90° pulse and (b) a 75° pulse?

3.39 A spin system is excited by a $180^\circ_{x'} - \tau - 90^\circ_{x'}$ sequence with $\tau \sim 2T_1$.

(a) Plot the time evolution of the $M_{z'}$ component in the τ time interval.

(b) Calculate the magnitude of $M_{x'y'}$ immediately after the $90^\circ_{x'}$ pulse and plot its time evolution after this pulse.

3.40 An imaging sequence often involves a series of excitation pulses to generate signals to cover k-space. Since a 90° pulse completely rotates any available M_z component onto the transverse plane, magnetization along the z-axis is always zero immediately after a 90° pulse in any imaging sequence with 90° excitation pulses. True or false?

3.41 Why is forced precession much slower than free precession?