Sept 15: Hamiltonian Cycles

Last lecture, we studied walks that use no edge more than once (trails) and walks that use each edge exactly once (eulerian trails). Today, we study trails that use no vertex more than once (paths) and trails that use each vertex exactly once (hamiltonian paths).

Definition: A **path** is a trail that uses each vertex at most once, (except possibly the first and last vertex). A **cycle** is a closed trail. A path or cycle is **Hamiltonian** if it uses every vertex.

Last lecture, we learned that a train enthusiast might plan a vacation by finding an Eulerian circuit in the graph of train routes so that she could ride on all the train tracks and end where she started. Her friend that loves train *stations* would want to find Hamiltonian cycle. If you represent your social friend network as a graph (where vertices represent people and edges represent friendship), then a Hamiltonian cycle in this graph would represent a way to seat your friends around a circular dinner table so that everyone is friends with the two people sitting next to them.

In general, Hamiltonian paths and cycles are *much* harder to find than Eulerian trails and circuits. We will see one kind of graph (complete graphs) where it is always possible to find Hamiltonian cycles, then prove two results about Hamiltonian cycles.

Definition: The **complete graph on n vertices**, written K_n, is the graph that has n vertices and each vertex is connected to every other vertex by an edge.

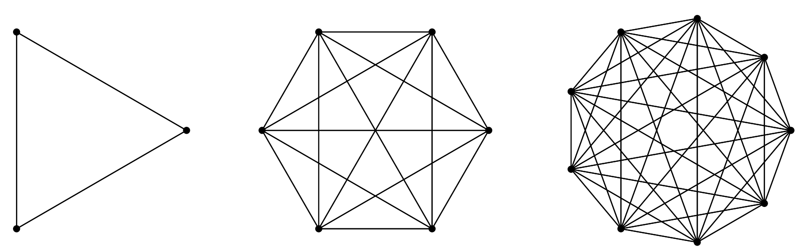

Figure 1: $K_3, K_6,$ and K_9

Easy Remark: For every $n \geq 3$, the graph K_n has $n!$ Hamiltonian cycles: there are n choices for where to begin, then $(n-1)$ choices for which vertex to visit next, then $(n-2)$ choices for which vertex to visit after that, and so on. Because the graph is complete, there is always guaranteed to be an edge that will take you to whichever vertex you want to go to next. Once you've visited the final vertex, take the edge that returns to your starting vertex is guaranteed to be unused[1]

In general, having more edges in a graph makes it more likely that there's a Hamiltonian cycle. The next theorem says that if all vertices in a graph are connected to *at least half* of the other vertices, there is guaranteed to be a Hamiltonian cycle.

[1] Ask yourself: where did we use the condition that $n \geq 3$? Try and see what happens if $n = 2$.

Theorem: Let G be a simple graph with at least 3 vertices. If every vertex of G has degree $\geq |V(G)|/2$, then G has a Hamiltonian cycle.

Proof: Assume that G satisisfies the condition, but does *not* have a Hamiltonian cycle. If it is possible to add edges to G so that the result still a simple graph with no Hamiltonian cycle, do so. Continue adding edges until it becomes impossible to add edges without creating a cycle. Call this new graph G'.

Because G' has no Hamiltonian cycle and has ≥ 3 vertices, it cannot be the case every vertex of G' is connected by an edge to every other vertex – we showed above that the complete graph has a Hamiltonian cycle. This means that there are vertices $v, w \in V(G')$ that are *not* connected by an edge. Adding the edge vw to G' will result in a graph having a Hamiltonian cycle; deleting the edge vw from this cycle produces a Hamiltonian path in G' from v to w. Let $(v, v_2, v_3, \ldots, v_{n-1}, w)$ be the vertices in this path in order (so $|V(G')| = n$).

Define two subsets of the set $\{2, 3, \ldots, n-2\}$ as follows

$$\begin{aligned} A &= \text{all numbers } i \in \{2,3,\ldots,n-2\} \text{ such that } vv_{i+1} \text{ is an edge of } G' \\ B &= \text{all numbers } i \in \{2,3,\ldots,n-2\} \text{ such that } v_i w \text{ is an edge of } G' \end{aligned}$$

Notice that every edge with endpoint v is accounted for in the set A except for the edge vv_2. Because $d(v) \geq n/2$, this means $|A| \geq n/2 - 1$. Similarly, $|B| \geq n/2 - 1$. Because the set $\{2, 3, \ldots, n-2\}$ has $n-3$ elements in it, and$|A| + |B| \geq n-2$, it must be the case that at least one element if of $\{2, 3, \ldots, n-2\}$ is in $A \cap B$. Then there is a Hamiltonian cycle

$$(v, v_2, v_3, \ldots, v_i, w, v_{n-1}, v_{n-2}, \ldots, v_{i+1}, v)$$

which gives the desired contradiction.

There is one last definition that you'll need for your homework:

Definition: The definitions of a **walk, trail, Eulerian circuit, path, cycle, Hamiltonian cycle** generalize to the context of *directed graphs* by insisting that edges may only be traversed in the direction in which they are directed.

You can now solve all problems on homework 1