Fourier Transform & Some Applications

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- In SP, we try to express arbitrary signals as the sum of other basic signals.
- FS describes arbitrary signals as the sum of SBB.
- FS was named after Joseph Fourier (1768 - 1830).
- French mathematician & scientist.
- He studied heat flow problems.
- Key idea: He represented a signal $f(t)$, as a sum of harmonically related sinusoids.

Fourier Series (FS): Sinusoidal Building Blocks (SBB)

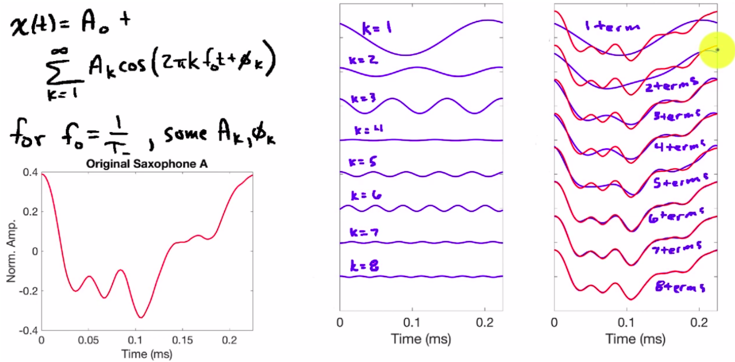


Figure: Representing the original signal with FS.

Fourier Series 4

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k)$$

use $A_k \cos(2\pi k f_0 t + \phi_k) = \frac{A_k}{2} e^{j\phi_k} e^{j2\pi k f_0 t} + \frac{A_k}{2} e^{-j\phi_k} e^{-j2\pi k f_0 t}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}; \quad a_0 = A_0, \quad a_\ell = \frac{A_\ell}{2} e^{j\phi_\ell} \quad \ell > 0$$

$$a_{-\ell} = a_\ell^*$$

exponential Fourier series

- $x(t)$ fund period $T_0 = \frac{1}{f_0}$
- or represent $x(t)$ on interval of length T_0

Comparison of FT and FS

Fourier Series: Used for periodic signals

Fourier Transform: Used for non-periodic signals (although we will see later that it can also be used for periodic signals)

	Synthesis	**Analysis**
Fourier Series	$x(t) = \sum_{n=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ **Fourier Series**	$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}\,dt$ **Fourier Coefficients**
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$ **Inverse Fourier Transform**	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$ **Fourier Transform**

FS coefficients c_k are a complex-valued function of integer k

FT $X(\omega)$ is a complex-valued function of the variable $\omega \in (-\infty, \infty)$

Applications of FT

- 1D FT is used in SP
- 2D FT is used in image processing
    1. image enhancement
    2. image restoration
    3. image encoding/decoding
    4. image description
- 3D FT is used in computer vision

One Dimensional Fourier Transform and its Inverse

- The Fourier transform F (u) of a single variable, continuous function f (x) is

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux}\, dx \qquad \text{where } j = \sqrt{-1}.$$

- Given F(u) we can obtain f (x) by means of the Inverse Fourier Transform

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux}\, du.$$

Discrete Fourier Transforms (DFT)

1-D DFT for M samples is given as

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} \text{ for } u = 0,1,2,\ldots\ldots, M-1$$

The inverse Fourier transform in 1-D is given as

$$f(x) = \sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M} \text{ for } x = 0,1,2,\ldots\ldots, M-1$$

Since $e^{j\theta} = \cos\theta + j\sin\theta$.

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)[\cos(2\pi ux/M) - j\sin(2\pi ux/M)].$$

$$F(u) = \underbrace{R(u)}_{\text{Real}} + \underbrace{jI(u)}_{\text{Imaginary}}$$

Two Dimensional Fourier Transform and its Inverse

- The Fourier transform F (u,v) of a two variable, continuous function f (x,y) is

$$F(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{-j2\pi(ux+vy)}\, dx\, dy$$

- Given F(u,v) we can obtain f (x,y) by means of the Inverse Fourier Transform

$$f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v)e^{j2\pi(ux+vy)}\, du\, dv$$

2-D DFT

2-D DFT for a total of MxN samples is given as

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

for $u = 0,1,2,\ldots\ldots, M-1$ and $v = 0,1,2,\ldots\ldots, N-1$

2-D inverse DFT is given as

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

for $x = 0,1,2,\ldots\ldots, M-1$ and $y = 0,1,2,\ldots\ldots, N-1$

FT used in Radio Astronomy

- Angular resolution refers to how well a telescope can distinguish between two objects in space which are separated by a small angular distance.
- $\theta_R \simeq \lambda/\text{dish diameter(D)}$.
- For a better resolution, we need to build a telescope with a very large diameter.

- On the other hand, radio interferometry uses a network of radio telescopes to observe the signal in order to obtain the resolution equivalent to a large single dish.
- $\theta_R \simeq \lambda/|\tilde{\mathrm{B}}|$.
- For a better resolution, we need to build a telescope with a very large diameter.

- Fourier Transformation (FT) has huge application in radio astronomy. Sky observed by radio telescope is recorded as the FT of true sky termed as visibility in radio astronomy language and this visibility goes through Inverse Fourier Transformation and deconvolution process to deduce the true sky image.

$$\Lambda_{\mathrm{ab}}^{vis}(u,v) = \iint A_{\mathrm{ab}}^{beam}(\gamma_1,\gamma_2) I_{\mathrm{ab}}^{sky}(\gamma_1,\gamma_2) \exp[-2\pi j(u\gamma_1+v\gamma_2)] d\gamma_1 d\gamma_2 \tag{1}$$

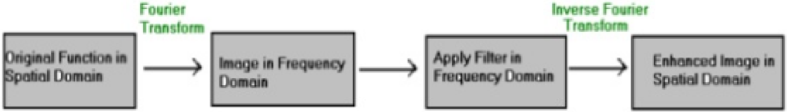
- In physical observation, the complex visibility $\Lambda_{\text{ab}}^{vis}(u, v)$ is not measured everywhere, but only finite in the uv domain.
- Let denote this sampling process as $S_{\text{ab}}(u, v)$ such that we record zeros if no data is captured.

$$I_{\text{ab}}^{dirty}(\gamma_1, \gamma_2) = A_{\text{ab}}^{beam}(\gamma_1, \gamma_2) I_{\text{ab}}^{sky}(\gamma_1, \gamma_2) \otimes B_{ab}(\gamma_1, \gamma_2) \quad (2)$$

- where

$$B_{ab}(\gamma_1, \gamma_2) = \iint S_{\text{ab}}(u, v) \exp[2\pi j(u\gamma_1 + v\gamma_2)] d\gamma_1 d\gamma_2 \quad (3)$$

Conclusion


Original Function in Spatial Domain
Fourier Transform
Image in Frequency Domain
Apply Filter in Frequency Domain
Inverse Fourier Transform
Enhanced Image in Spatial Domain