Classical mechanics

1 | Motion in one dimension

Integration of Newton's 2nd law

Proposition 1.1 (Newton's 2nd law). Consider a particle with constant mass m that moves in one dimension. Then, it satisfies:

$$\ddot{x}(t) = \frac{1}{m}F(x(t), \dot{x}(t), t)$$

where we have supposed the force function F is known and $x(t)$ is the position of the particle as a function of time. We also suppose initial position and velocity, denoted by $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$ respectively, are known.

Proposition 1.2 (Integration of Newton's 2nd law). We consider a force that only depend on time, position and velocity.

- Time dependence:

$$\dot{x}(t) = \dot{x}(t) + \int_{t_0}^{t} \frac{F(t')}{m}\mathrm{d}t'$$
$$x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(t')\mathrm{d}t'$$

- Position dependence:

$$\dot{x}(x)^2 = \dot{x}(x_0)^2 + 2\int_{x_0}^{x} \frac{F(x')}{m}\mathrm{d}x'$$
$$x(t) = g^{-1}(t)$$

where $g(x) = \displaystyle\int_{x_0}^{x} \frac{1}{\dot{x}(x')}\mathrm{d}x' = t$.

- Velocity dependence:

$$\dot{x}(t) = h^{-1}(t)$$
$$x(t) = x(t_0) + \int_{t_0}^{t} h^{-1}(t')\mathrm{d}t'$$

where $h(\dot{x}) = \displaystyle\int_{\dot{x}_0}^{\dot{x}} \frac{m}{F(\dot{x}')}\mathrm{d}\dot{x}' = t$.

Variable mass

Proposition 1.3 (Mass accretion formula). Consider two objects of masses $m(t)$ and dm and velocities $\mathbf{v}(t)$ and $\mathbf{u}(t)$ respectively, which in an interval of time $\mathrm{d}t$ the second one collide with the first one and become a unique object. If $\mathbf{F}^{\text{ext}}$ is the external force acting to the system, we have:

$$\mathbf{F}^{\text{ext}} = m\dot{\mathbf{v}} + (\mathbf{v} - \mathbf{u})\dot{m} = \dot{\mathbf{p}} - \dot{m}\mathbf{u} \tag{1}$$

where $\dot{\mathbf{p}}$ is the momentum of the object that gains mass[1].

Rocket motion

Consider a rocket moving at a velocity $\mathbf{v}$ that expels gas at a velocity $\mathbf{c}$ with respect to the rocket to propel itself. Suppose the mass of the rocket is $m(t)$ and $m_0 := m(t_0)$. If $\mathbf{u} = \mathbf{v} + \mathbf{c}$ is the velocity of the gas with respect to an external frame of reference and $\mathbf{F}^{\text{ext}}$ is the net external force acting on the rocket, by equation (1) we have:

$$m\dot{\mathbf{v}} = \mathbf{F}^{\text{ext}} + \dot{m}\mathbf{c}.$$

Proposition 1.4 (Rocket without gravity). In this case we have $\mathbf{F}^{\text{ext}} = 0$ and if we suppose $\mathbf{v} = v\mathbf{j}$ and $\mathbf{c} = -c\mathbf{j}$, we have:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -c\frac{dm}{\mathrm{d}t} \implies v = c\log\frac{m_0}{m} \tag{2}$$

Consider now the discrete case, i.e. when the function $\dot{m}$ is not differentiable. For that we can consider instantaneous ejections of $\Delta m = (m_0 - m_f)/n$ amount of mass where m_f is the mass of the rocket after n ejections of mass. For this case, we have

$$v = c\sum_{k=1}^{n} \frac{(m_0 - m_f)/n}{m_f + k(m_0 - m_f)/n} = c\sum_{k=1}^{n} \frac{\Delta m}{m_f + k\Delta m}\text{[2]}$$

Proposition 1.5 (Rocket with gravity). In this case we have $\mathbf{F}^{\text{ext}} = -mg\mathbf{j}$. Suppose $\mathbf{v} = v\mathbf{j}$, $\mathbf{c} = -c\mathbf{j}$ and, for simplicity, consider only the case when $\dot{m} = -\beta$, $\beta > 0$. Therefore, we obtain:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg + c\beta \implies v = c\ln\frac{m_0}{m} - \frac{g}{\beta}(m_0 - m) \tag{3}$$

Observe that if $m_0 g > \beta c$ then $\mathrm{d}v/\mathrm{d}t$ will be negative, which is not possible. Therefore in this case the formula is not correct if we are considering the rocket launch. In this case the formula becomes

$$v = c\ln\frac{\beta c}{mg} - \frac{g}{\beta}\left(\frac{\beta c}{g} - m\right) \tag{4}$$

Because of $\dot{m} = -\beta \implies m(t) = m_0 - \beta t$, we can express formulas (3), (4) respectively as:

$$v(t) = c\ln\frac{m_0}{m_0 - \beta t} - gt$$
$$v(t) = c\ln\frac{\beta c}{m_0 g - g\beta t} - gt - \frac{g}{\beta}\left(\frac{\beta c}{g} - m_0\right)$$

2 | Oscillations

Simple harmonic oscillator

Proposition 1.6. Consider the differential equation

$$\ddot{x} + \omega_0^2 x = 0$$

with initial values $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$. The general solution is:

$$x(t) = x_0\cos\omega_0 t + \frac{\dot{x}_0}{\omega_0}\sin\omega_0 t = A\cos(\omega_0 t + \phi) \tag{5}$$

[1] The formula is also valid for the case when the object is losing mass, i.e. $\dot{m} < 0$.
[2] Obviously if we tend n to infinity we get the equation (2).

where $A = \sqrt{{x_0}^2 + \left(\frac{\dot{x}_0}{\omega_0}\right)^2}$ and $\phi = -\arctan\frac{\dot{x}_0}{\omega_0 x_0}$. Such constants ω_0 ($[\omega_0] = \mathrm{rad}\cdot\mathrm{s}^{-1}$), A ($[A] = \mathrm{m}$) and ϕ ($[\phi] = \mathrm{rad}$) are called *angular frequency*, *amplitude* and *initial phase*, respectively. Observe that the function in equation (5) is periodic with period $T = \frac{2\pi}{\omega_0}$ and frequency $\nu = T^{-1} = \frac{\omega_0}{2\pi}$[3].

Definition 1.7. Let $U(x)$ be a potential function of class $\mathcal{C}^2(\mathbb{R})$. We say x_0 is a *point of stable equilibrium* if U attains a minimum in x_0. Analogously, we say x_0 is a *point of unstable equilibrium* if U attains a maximum in x_0.

Proposition 1.8 (Behaviour near a minimum). Suppose x_0 is a point of stable equilibrium an let $U(x)$ be the potential function associated with a particle of mass m. Then if we disturb slightly the particle, it will start to oscillate at a frequency

$$\omega_0 = \sqrt{\frac{U''(x_0)}{m}}.$$

Proposition 1.9 (Examples).

- Mass hanging from a spring: Let $y(t)$ be the position of the mass measured from initial string's length (without the mass) to the position of the mass at time t. If we disturb the system with an external force so that the mass starts to oscillate, we have

$$y(t) = \frac{mg}{k} + A\cos(\omega_0 t + \phi), \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

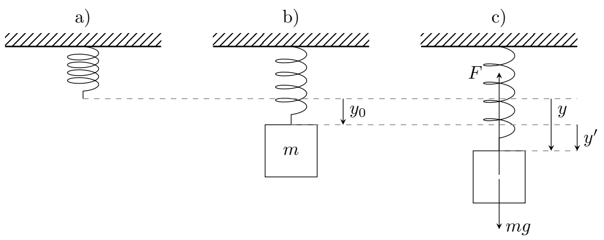

Figure 1: Mass hanging from a spring.

- Simple pendulum:

$$\theta(t) = A\cos(\omega_0 t + \phi), \quad \omega_0 = \sqrt{\frac{g}{l}}.$$

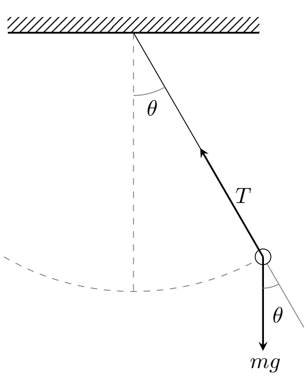

Figure 2: Simple pendulum.

- Physical pendulum:

$$\theta(t) = A\cos(\omega_0 t + \phi), \quad \omega_0 = \sqrt{\frac{mgD}{I_e}}.$$

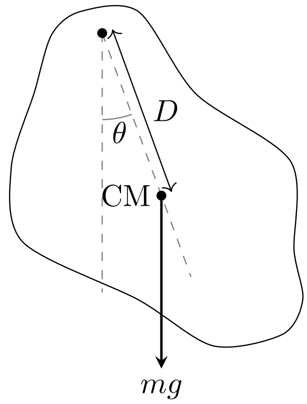

Figure 3: Physical pendulum.

- LC circuit:

$$q(t) = A\cos(\omega_0 t + \phi), \quad \omega_0 = \frac{1}{\sqrt{LC}}.$$

Damped harmonic oscillator

Proposition 1.10 (Movement equation). Consider the following differential equation:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

with initials values of $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$. Then we have three cases for the general solution:

- If $\beta < \omega_0$,
$$x(t) = e^{-\beta t}\left(c_1\cos\tilde{\omega}t + c_2\sin\tilde{\omega}t\right). \tag{6}$$
- If $\beta = \omega_0$,
$$x(t) = e^{-\beta t}\left(c_1 + c_2 t\right). \tag{7}$$
- If $\beta > \omega_0$,
$$x(t) = c_1 e^{-(\beta+\tilde{\omega})t} + c_2 e^{-(\beta-\tilde{\omega})t}. \tag{8}$$

Here c_1, c_2 are constants depending on the initial values and we have defined $\tilde{\omega} = \sqrt{|\omega_0^2 - \beta^2|}$.

Proposition 1.11 (Energy of damped harmonic oscillator).

$$E = \frac{\mu}{2}\left(\dot{x}^2 + \omega_0^2 x^2\right),$$

where μ is a constant.

Proposition 1.12 (Underdamped harmonic oscillator: $\beta < \omega_0$). Coefficients c_1, c_2 of the general solution (6) are:

$$c_1 = x_0, \quad c_2 = \frac{\dot{x}_0 + \beta x_0}{\tilde{\omega}}.$$

The equation, can be simplified to

$$x(t) = Ae^{-\beta t}\cos(\tilde{\omega}t + \phi),$$

where $A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0 + \beta x_0}{\tilde{\omega}}\right)^2}$ and $\phi = -\arctan\frac{\dot{x}_0 + \beta x_0}{\tilde{\omega}x_0}$.

[3]Note that $[T] = \mathrm{s}$ and $[\nu] = \mathrm{s}^{-1} = \mathrm{Hz}$.

Definition 1.13 (Quality factor). The *quality factor* is defined as follows:
$$Q := \frac{\omega_0}{2\beta}.$$
From that, we can rewrite the expression of $\tilde{\omega}$ to get:
$$\tilde{\omega} = \omega_0\sqrt{1 - \frac{1}{4Q^2}}.$$

Proposition 1.14 (Energy of underdamped harmonic oscillator). For
$$E(t) = \frac{\mu\omega_0^2 A^2}{2}e^{-2\beta t} = E_0 e^{-2\beta t}.$$
The rate at which the energy is dissipated is
$$\left|\frac{dE}{\mathrm{d}t}(t)\right| = 2\beta E(t) \implies \frac{E}{|dE/\mathrm{d}t|} = \frac{1}{2\beta}.$$
If $\beta \ll \omega_0$, then
$$Q = 2\pi\frac{E}{\Delta E},$$
where ΔE is the energy dissipated in a pseudo-period $\tilde{T} = 2\pi/\tilde{\omega} \approx 2\pi/\omega_0$.

Proposition 1.15 (Critically damped harmonic oscillator: $\beta = \omega_0$). Coefficients c_1, c_2 of the general solution (7) are:
$$c_1 = x_0, \quad c_2 = x_0\omega_0 + \dot{x}_0.$$
This harmonic oscillator is the one that returns to balance more quickly.

Proposition 1.16 (Overdamped harmonic oscillator: $\beta < \omega_0$). Coefficients c_1, c_2 of the general solution (8) are:
$$c_1 = \frac{x_0(\tilde{\omega} - \beta) - \dot{x}_0}{2\tilde{\omega}}, \quad c_2 = \frac{x_0(\tilde{\omega} + \beta) + \dot{x}_0}{2\tilde{\omega}}.$$

Driven harmonic oscillators

Proposition 1.17 (Movement equation). Consider the following differential equation:
$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) = f_0\cos(\omega t + \psi),$$
with initials values of $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$. Then the particular solution is:
$$x_p(t) = A\cos(\omega t + \psi - \phi),$$
where $A = \dfrac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}}$ and $\phi = \arctan\dfrac{2\beta\omega}{\omega_0^2 - \omega^2}$. Therefore for the general solution we have three cases to consider:

- If $\beta < \omega_0$,
$$x(t) = e^{-\beta t}(c_1\cos\tilde{\omega}t + c_2\sin\tilde{\omega}t) + A\cos(\omega t + \psi - \phi). \quad (9)$$
- If $\beta = \omega_0$,
$$x(t) = e^{-\beta t}(c_1 + c_2 t) + A\cos(\omega t + \psi - \phi). \quad (10)$$
- If $\beta > \omega_0$,
$$x(t) = c_1 e^{-(\beta+\tilde{\omega})t} + c_2 e^{-(\beta-\tilde{\omega})t} + A\cos(\omega t + \psi - \phi). \quad (11)$$

Here c_1, c_2 are constants depending on the initial values.

Proposition 1.18 (Underdamped driven oscillator). Coefficients c_1, c_2 of the general solution (9) are:
$$c_1 = x_0 - A\cos(\psi - \phi),$$
$$c_2 = \frac{\dot{x}_0 - \omega A\sin(\psi - \phi) + \beta\left[x_0 - A\cos(\psi - \phi)\right]}{\tilde{\omega}}.$$

Proposition 1.19 (Critically damped driven oscillator). Coefficients c_1, c_2 of the general solution (10) are:
$$c_1 = x_0 - A\cos(\psi - \phi),$$
$$c_2 = \dot{x}_0 A + \omega_0 x_0 + A\left(\omega\sin(\phi - \psi) - \omega_0\cos(\psi - \phi)\right).$$

Proposition 1.20 (Overdamped driven oscillator). Coefficients c_1, c_2 of the general solution (11) are:
$$c_1 = A\frac{(\beta - \tilde{\omega})\cos(\psi - \phi) - \omega\sin(\psi - \phi)}{2\tilde{\omega}} + \frac{-(\beta - \tilde{\omega})x_0 - \dot{x}_0}{2\tilde{\omega}},$$
$$c_2 = A\frac{-(\beta + \tilde{\omega})\cos(\psi - \phi) + \omega\sin(\psi - \phi)}{2\tilde{\omega}} + \frac{(\beta + \tilde{\omega})x_0 + \dot{x}_0}{2\tilde{\omega}}.$$

Definition 1.21. Given a driven oscillator, we say it is in the *steady-state part* if $t \gg 1/\beta$. In that case $x(t)$ become:
$$x(t) = A\cos(\omega t + \psi - \phi).$$
While the dependency on c_1, c_2 is non-negligible, we say the driven oscillator is in the *transient part.*

Proposition 1.22 (Resonance in amplitude). If $\omega = \omega_r := \sqrt{\omega_0^2 - 2\beta^2}$ we say the oscillator is in *resonance in amplitude.* For $\omega = \omega_r$ we have
$$A_r = \frac{f_0}{2\beta\sqrt{\omega_0^2 - \beta^2}}.$$

Proposition 1.23 (Energy in steady-state part).
$$E = \frac{\mu A^2}{2}\left[\omega^2\sin^2(\omega t + \psi - \phi) + \omega_0^2\cos^2(\omega t + \psi - \phi)\right].$$
If $\omega \approx \omega_0$ and $\beta \ll \omega_0$ then
$$E = \frac{\mu f_0^2}{8}\frac{1}{(\omega - \omega_0)^2 + \beta^2}.$$
Observe E has a maximum at $\omega = \omega_0$ with the value of $E^{\max} = \frac{\mu f_0^2}{8\beta^2}$.

Definition 1.24. We define the *cutoff frequencies* as this two frequencies:

$$\omega_1 = \omega_0 - \beta, \quad \omega_2 = \omega_0 + \beta.$$

The value $\Delta\omega = \omega_2 - \omega_1 = 2\beta$ is called the *bandwidth*. Therefore, we can redefined the quality factor as:

$$Q = \frac{\omega_0}{2\beta} = \frac{\omega_0}{\Delta\omega} = \frac{\nu_0}{\Delta\nu}$$

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Proposition 1.25 (Impulsive forces). Consider a driven oscillator of equation $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f$, where

$$f(t) = \begin{cases} 0 & \text{if } t < t' \\ f_0 & \text{if } t' \leq t \leq t' + \Delta t \\ 0 & \text{if } t > t' + \Delta t \end{cases}$$

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3 | Central forces

Definition and properties

Definition 1.26 (Central force). A *central force* is a force of the form

$$\mathbf{F}(\boldsymbol{r}) = f(r)\boldsymbol{e}_r,$$

where $r = \|\boldsymbol{r}\|$ and $\boldsymbol{e}_r = \boldsymbol{r}/r$ is the unit radial vector.

Definition 1.27. The origin $\boldsymbol{r} = 0$ is called *center of forces*.

Proposition 1.28. All central forces are conservative and

$$f(r) = -U'(r),$$

where $U(r)$ is the potential energy of the central force.

Conservation of angular momentum and areal velocity

Proposition 1.29. The angular momentum with respect to the center of forces is conserved, that is, $\dot{\boldsymbol{L}} = 0$.

Proposition 1.30 (Kepler's 2nd law). The areal velocity $dA/\mathrm{d}t$ is constant. In fact,

$$\frac{dA}{\mathrm{d}t} = \frac{L}{2m}.$$

Proposition 1.31 (Unit vectors). Remember we have

$$\boldsymbol{e}_r = \boldsymbol{i}\cos\theta + \mathbf{j}\sin\theta, \quad \boldsymbol{e}_\theta = -\boldsymbol{i}\sin\theta + \mathbf{j}\cos\theta.$$

Therefore we obtain,

$$\boldsymbol{r} = r\boldsymbol{e}_r, \quad \dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta, \quad \ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{e}_\theta. \tag{12}$$

Proposition 1.32 (Trajectory equation). From (12), Newton's second law can be written as:

$$\ddot{r} - r\dot{\theta}^2 = \frac{f(r)}{m}, \quad 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0.$$

And we can obtain the following differential equations:

$$\dot{\theta} = \frac{L}{mr^2} := \frac{l}{r^2}, \quad \ddot{r} - \frac{l^2}{r^3} = \frac{f(r)}{m},$$

where we have defined the magnitude $l := L/m$. Finally, we get the *trajectory equation*:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{1}{ml^2}r^2 f(r).$$

Conservation of energy and orbits

Proposition 1.33 (Kinetic energy).

$$K = \frac{1}{2}m|\dot{\boldsymbol{r}}^2 = \frac{1}{2}m\dot{r}^2 + \frac{ml^2}{2r^2}.$$

Definition 1.34. We define the effective potential as

$$U_{\text{eff}} = U(r) + \frac{ml^2}{2r^2}.$$

The term $ml^2/(2r^2)$ gives the centripetal force:

$$f_{\text{centr}} = -\frac{d}{dr}\left(\frac{ml^2}{2r^2}\right) \implies \boldsymbol{f}_{\text{centr}} = mr\dot{\theta}^2\boldsymbol{e}_r$$

Proposition 1.35 (Energy).

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}} = \frac{1}{2}m\dot{r}^2 + \frac{ml^2}{2r^2} + U(r). \tag{13}$$

Proposition 1.36 (Orbits). The minimum of $U_{\text{eff}}(r)$ determines the radius of the *stable circular orbits*. If $E < 0$, the orbits are in a range of radius $r_{\min}$ and $r_{\max}$, and so are *bounded orbits*. If $E > 0$, the orbits are called *unbounded orbits*. Equating equation (13) to, we can obtain the angle $\Delta\theta$ in a *radial oscillation*:[4]

$$\Delta\theta = 2\int_{r_{\min}}^{r_{\max}} \frac{l/r^2}{\sqrt{(2/m)(E - U_{\text{eff}}}} dr.$$[5]

Note, that the orbits are *closed orbits* if

$$\Delta\theta = 2\pi\frac{p}{q}, \quad p, q \in \mathbb{N}.$$

Theorem 1.37 (Bertrand's theorem). The two unique potentials for which every bounded orbit is closed are:

$$U(r) = -\frac{k}{r}, \quad U(r) = \frac{k}{2}r^2, \quad k > 0.$$

Potential $-k/r$

Proposition 1.38 (Movement equation).

$$r(\theta) = \frac{\alpha}{\varepsilon\cos\theta + \operatorname{sgn} k},$$

where $\alpha = \dfrac{L^2}{m|k|}$ and $\varepsilon = \sqrt{1 + \dfrac{2EL^2}{mk^2}}$. This is the equation of a conic.

Definition 1.39. We define the *pericenter of an orbit* as the minimum value of $r(\theta)$, that is, $r = r_{\min}$. Analogously, we define the *apocenter of an orbit* as the maximum value of $r(\theta)$, that is, $r = r_{\max}$.

Conics

[4] A radial oscillation is the trajectory when r moves from $r_{\min}$ to $r_{\max}$ and the comes back to $r_{\min}$.

[5] Here we have taken the positive orientation, that is, $\boldsymbol{L}$ pointing to the positive z-axis.