Electricity and magnetism

1 | Vector calculus

	Formula (in Cartesian coordinates)
Gradient	$\boldsymbol{\nabla} f := \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y + \frac{\partial f}{\partial z}\mathbf{e}_z$
Divergence	$\mathbf{div}\,\mathbf{A} := \boldsymbol{\nabla}\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_z}{\partial z} + \frac{\partial A_z}{\partial z}$
Curl	$\mathbf{rot}\,\mathbf{A} := \boldsymbol{\nabla}\times\mathbf{A} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial z} - \frac{\partial A_y}{\partial y}\right)\mathbf{e}_x + \left(\frac{\partial A_x}{\partial x} - \frac{\partial A_z}{\partial z}\right)\mathbf{e}_y + \left(\frac{\partial A_y}{\partial y} - \frac{\partial A_x}{\partial x}\right)\mathbf{e}_z$
Laplacian	$\nabla^2 f := \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

2 | Electrostatics

Electric force

Proposition 1.1. The charge of any object is a multiple of the elementary charge e.

Law 1.2 (Charge conservation). The total electric charge in an isolated system never changes.

Law 1.3 (Coulomb's law). The force applied by a point charge q_1 over another point charge q_2 along a straight line is:

$$\mathbf{F}_1 = K\frac{q_1 q_2}{\|\mathbf{r}_{12}\|^2}\hat{\mathbf{r}}_{12}$$

where $\mathbf{r}_{12}$ is the vectorial distance between the charges, $\hat{\mathbf{r}}_{12} = \frac{\mathbf{r}_{12}}{\|\mathbf{r}_{12}\|}$ is the unit vector pointing from q_2 to q_1 and K is the Coulomb constant.

Principle 1.4 (Superposition principle). Consider a set of N point charges q_i which are at a distance $\mathbf{r}_i$ from another point charge Q. Then, the net force exerted by the N point charges to the charge Q is:

$$\mathbf{F}_Q = \sum_{i=1}^{N} K\frac{q_i Q}{\|\mathbf{r}_i\|^2}\hat{\mathbf{r}}_i$$

where $\hat{\mathbf{r}}_i$ is the unit vector pointing from q_i to Q.

Electric field

Definition 1.5. Given a point charge Q, the *electric field* created by this charge at a distance $\mathbf{r}$ from it is given by:

$$\mathbf{E} = K\frac{Q}{\|\mathbf{r}\|^2}\hat{\mathbf{r}}$$

Principle 1.6 (Superposition principle). Consider a set of N point charges q_i which are at a distance $\mathbf{r}_i$ from a point A. Then, the net electric field created by the N point charges at point A is:

$$\mathbf{E}_A = \sum_{i=1}^{N} K\frac{q_i}{\|\mathbf{r}_i\|^2}\hat{\mathbf{r}}_i$$

where $\hat{\mathbf{r}}_i$ is the unit vector pointing from q_i to A. In the case of a continuous distribution of charge we will have:

$$\mathbf{E} = \int \mathrm{d}\mathbf{E} = \int K\frac{\mathrm{d}q}{r^2}\hat{\mathbf{r}}$$

Note that $\mathrm{d}q = \lambda\mathrm{d}\ell$, $\mathrm{d}q = \sigma\mathrm{d}S$ or $\mathrm{d}q = \rho\mathrm{d}V$ depending on whether the distribution of charge is linear, superficial or volumetric. In each respective case, λ, σ and ρ represent the charge densities.

Electric flux and Gauß' law

Definition 1.7. Let $\mathbf{A}$ be a vectorial field and $\mathrm{d}\mathbf{S}$ be a small surface area. The *flux* $\mathrm{d}\Phi$ of $\mathbf{A}$ through $\mathrm{d}\mathbf{S}$ is:

$$\mathrm{d}\Phi = \mathbf{A}\cdot\mathrm{d}\mathbf{S}$$

And the flux through a surface S will be:

$$\Phi = \iint_S \mathrm{d}\Phi = \iint_S \mathbf{A}\cdot\mathrm{d}\mathbf{S}$$

Corollary 1.8. The electric flux of a field $\mathbf{E}$ through a surface S is:

$$\Phi_E = \iint_S \mathbf{E}\cdot\mathrm{d}\mathbf{S}$$

Law 1.9 (Gauß' law). The net electric flux through a closed surface S is equal to $\frac{1}{\varepsilon_0}$ times the net electric charge Q_{int} within that closed surface.

$$\Phi_E = \oiint_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \frac{Q_{\text{int}}}{\varepsilon_0}$$

Electric potential

Proposition 1.10. The variation of the electrostatic potential energy that undergoes a point charge q when moving a distance $\mathrm{d}\boldsymbol{\ell}$ is:

$$\mathrm{d}U = -\mathbf{F}\cdot\mathrm{d}\boldsymbol{\ell} = -q\mathbf{E}\cdot\mathrm{d}\boldsymbol{\ell}$$

Therefore:

$$\Delta U = U(b) - U(a) = \int_a^b \mathrm{d}U = -\int_a^b q\mathbf{E}\cdot\mathrm{d}\boldsymbol{\ell}$$

Proposition 1.11. The work done by the electric field on a particle between two points a and b is $-\Delta U = U_a - U_b$, while the work done by the external forces on that particle in that interval is $\Delta U = U_b - U_a$.

Definition 1.12. The *potential difference* between two points a and b over a point charge q when an electric field $\mathbf{E}$ is applied to it is:

$$\mathrm{d}V := \frac{\mathrm{d}U}{q} = -\mathbf{E}\cdot\mathrm{d}\boldsymbol{\ell} \implies \Delta V = \frac{\Delta U}{q} = -\int_a^b \mathbf{E}\cdot\mathrm{d}\boldsymbol{\ell}$$

Definition 1.13. If we choose the infinite as an origin of potential (that is, $V = 0$ when $r = \infty$), we can define the *electric potential* at a distance r from a point charge q as:

$$V = K\frac{q}{r}$$

Principle 1.14 (Superposition principle). Consider a set of N point charges q_i which are at a distance $\mathbf{r}_i$ from a point A. Then, the total electric potential exerted by the N point charges on the point A is:

$$V_A = \sum_{i=1}^{N} K\frac{q_i}{\|\mathbf{r}_i\|}$$

In the case of a continuous distribution of charge we have:

$$\Delta V = V(b) - V(a) = -\int_a^b \mathbf{E}\cdot \mathrm{d}\ell$$

Electrostatic energy

Definition 1.15. The *electrostatic energy* between two charges q_1 and q_2 separated a distance r is:

$$U = K\frac{q_1 q_2}{r} = q_2 V_1 = q_1 V_2$$

where V_i is the electric potential created by the charge q_i at a distance r.

Proposition 1.16. Consider a set of N point charges q_i. Let r_{ij} be the distance between the charge q_i and q_j. Then, the total electrostatic energy of the set will be:

$$U = \sum_{i=1}^{N}\sum_{j=i+1}^{N} K\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N} K\frac{q_i q_j}{r_{ij}}$$

Conductors

Proposition 1.17. In a conductor, charges can move freely. In particular, if an external electric field is acting on a conductor, the charges move until they reach an electrostatic equilibrium.

Proposition 1.18. When a conductor is in electrostatic equilibrium:

- All the charges are in the surface and the total electric field inside the conductor is zero.
- The electric field just outside is perpendicular to the surface of the conductor and equal to σ/ε_0, where σ is the surface charge density.
- The volume enclosed in the conductor is an equipotential volume and its surface is an equipotential surface.

Capacitance and capacitors

Definition 1.19 (Capacitance). Consider a conductor with an electric charge Q. Then, if its potential is V, the *capacitance* of the conductor is defined as:

$$C := \frac{Q}{V}$$

The SI unit of the capacitance is the Farad ($1\ \mathrm{F} = \mathrm{C}\cdot\mathrm{V}^{-1}$).

Definition 1.20 (Capacitor). A *capacitor* is a device that stores electric charge and electrical energy. It consists in two conductors close to each other and with equal and opposite charge.

Proposition 1.21. Consider a capacitor whose conductors are parallel plates of surface area S and are separated a distance d. If Q is the charge stored in one plate and the potential difference between the plates is ΔV, we have that the capacitance of the capacitor is:

$$C = \frac{Q}{\Delta V} = \varepsilon_0\frac{S}{d}$$

Definition 1.22. Consider two opposite point charges of charge q separated a distance $\mathbf{d}$ (electric dipole). We define the *electric dipole moment* as:

$$\mathbf{p} = q\mathbf{d}$$

Proposition 1.23. Consider an electric dipole of moment $\mathbf{p}$ that is immersed in an electric field $\mathbf{E}$. Then, the electric force creates a torque $\boldsymbol{\tau}$ on the dipole given by:

$$\boldsymbol{\tau} = \mathbf{p}\times\mathbf{E}$$

This torque tends to line up the dipole with the magnetic field $\mathbf{B}$, so that it takes its lowest energy configuration. The potential energy associated with the electric dipole moment is:

$$U = -\boldsymbol{\mu}\cdot\mathbf{E}$$

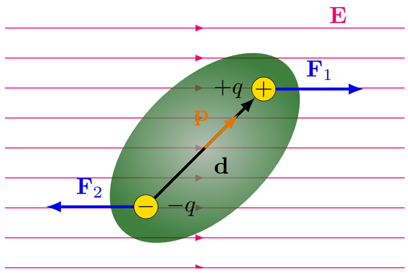

Figure 1: Electric dipole

Proposition 1.24. Consider a dielectric material with permittivity $\varepsilon = \kappa\varepsilon_0$ with $\kappa > 1$. Then, the capacitance of the capacitor with this material between their plates is:

$$C = \kappa C_0$$

where C_0 is the capacitance of the capacitor with no dielectric material (that is, in the vacuum).

Proposition 1.25. Consider a capacitor of capacitance C, charge Q and potential difference ΔV. Then, the potential energy stored in the capacitor is:

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

Proposition 1.26. Consider a capacitor with a dielectric material inside it of permittivity ε. If E is the magnitude of the electric field between the plates of the capacitor, the energy density η of the electric field will be:

$$\eta = \frac{1}{2}\varepsilon E^2$$

Proposition 1.27. Consider N capacitors of capacitance C_i. We can associate the capacitors in two ways:

- in series:
$$\frac{1}{C_{\text{total}}} = \sum_{i=1}^{N} \frac{1}{C_i}$$
- in parallel:
$$C_{\text{total}} = \sum_{i=1}^{N} C_i$$

Electric current

Definition 1.28. An *electric current* is a stream of charged particles moving through an electrical conductor or space. Mathematically, the electric current is:

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}$$

By agreement, the direction of the electric current is the one of the positive charges.

Definition 1.29. The current density $\mathbf{J}$ is the amount of charge per unit of time that flows through a unit area of a chosen cross section. Mathematically, we have the following relation:

$$I = \iint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$$

Proposition 1.30. Let n be the number of charge carriers per unit of volume (charge carrier density) of a conductor, q be the charge of these carriers, S be the section of the conductor and $\mathbf{v}_d$ be the drift velocity (average velocity attained by charged particles in a material due to an electric field). Then, we have:

$$I = \frac{\Delta Q}{\Delta t} = qn\|\mathbf{v}_d\|S$$

Moreover:

$$\mathbf{J} = qn\mathbf{v}_d$$

Law 1.31 (Microscopic Ohm's law). Let n be the charge carrier density of a conductor, τ be the average time between collisions of electrons and $\mathbf{E}$ be the electric field at which electrons are accelerated. Then:

$$\mathbf{J} = \frac{ne^2\tau}{m_e}\mathbf{E} =: \sigma\mathbf{E}$$

Here, σ is called *conductivity.*

Law 1.32 (Macroscopic Ohm's law). Suppose a conductor has a resistance R and carries an electric current I. If the conductor is subjected to a potential difference ΔV, then:

$$I = \frac{\Delta V}{R}$$

Definition 1.33 (Resistivity). Consider a conductor with conductivity σ that has length ℓ, section S and electric resistance R. Then, the *resistivity* of the conductor is:

$$\rho = R\frac{S}{\ell} = \frac{1}{\sigma}$$

Moreover, this resistivity varies with the temperature in the following way:

$$\rho(T) = \rho_0 \left[1 + \alpha(T - T_0)\right]$$

where ρ_0 is the resistivity of the material at temperature T_0 and α is the *temperature coefficient of resistivity.*

Proposition 1.34 (Joule effect). Suppose that a conductor of resistance R carries an electric current I. If it is subjected to a potential difference ΔV, then the power dissipated by heat is:

$$P = IV = RI^2 = \frac{V^2}{R}$$

Proposition 1.35. Consider N resistors of resistance R_i. We can associate the resistors in two ways:

- in series:
$$R_{\text{total}} = \sum_{i=1}^{N} R_i$$
- in parallel:
$$\frac{1}{R_{\text{total}}} = \sum_{i=1}^{N} \frac{1}{R_i}$$

Kirchhoff's laws and RC circuits

Definition 1.36. A battery is a device that maintains a constant potential difference while charges move along the circuit. The *electromotive force (emf)* ξ of a battery describes the work done per unit of charge. Generally, batteries have an internal resistance r and therefore the potential difference between their terminals is:

$$\Delta V = \xi - Ir$$

where I is the electric current passing through it. Finally, the total energy stored in the battery is:

$$W = Q\xi$$

where Q is the charge of the battery.

Law 1.37 (Kirchhoff's laws).

1. Kirchhoff's junction rule: In a node (junction), the sum of currents flowing into that node is equal to the sum of currents flowing out of that node.
2. Kirchhoff's loop rule: The directed sum of the potential differences around any closed loop is zero.

[1]Sometimes RC is denoted by τ and it is called the *RC time constant.*

Proposition 1.38 (Capacitor discharging). Suppose we have a circuit consisting of a resistor of resistance R and a charged capacitor of capacitance C and charge Q. Then, the charge of the capacitor as a function of time will be:

$$q(t) = Q\mathrm{e}^{-\frac{t}{RC}}$$

And, therefore, the electric current will be:

$$i(t) = I_0\mathrm{e}^{-\frac{t}{RC}}$$

where I_0 is the electric current at $t = 0$[1].

Proposition 1.39 (Capacitor charging). Suppose we have a circuit consisting of a battery of emf ξ, a resistor of resistance R and a discharged capacitor of capacitance C. Then, the charge of the capacitor as a function of time will be:

$$q(t) = Q_f(1 - \mathrm{e}^{-\frac{t}{RC}})$$

where Q_f is the final charge of the capacitor. Therefore the electric current will be:

$$i(t) = \frac{\xi}{R}\mathrm{e}^{-\frac{t}{RC}}$$

3 | Magnetostatics

Magnetic force

Proposition 1.40. Consider a point charge q moving at a velocity $\mathbf{v}$. If we apply a magnetic field $\mathbf{B}$ to it, a magnetic force acting on the particle is created:

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

The SI unit of the magnetic field is the Tesla ($1\ \mathrm{T} = 1\ \mathrm{N} \cdot \mathrm{A}^{-1} \cdot \mathrm{m}^{-1}$).

Proposition 1.41. Consider a wire of length $\boldsymbol{\ell}$ transporting an electric current I. If we apply a magnetic field $\mathbf{B}$ to the wire and $\boldsymbol{\ell}$ is the vector pointing at the direction of the current and whose magnitude is ℓ, then the magnetic force created by the wire is:

$$\mathbf{F} = I(\boldsymbol{\ell} \times \mathbf{B})$$

If the we take a differential element of length $\mathrm{d}\boldsymbol{\ell}$, then:

$$\mathrm{d}\mathbf{F} = I(\mathrm{d}\boldsymbol{\ell} \times \mathbf{B})$$

Lemma 1.42. The work done by the magnetic field on a particle is zero.

Proposition 1.43. Consider a particle of mass m, charge q and velocity $\mathbf{v}$. If there is a magnetic field $\mathbf{B}$ applied to it, we have two possibilities for its trajectory:

- If $\mathbf{v} \perp \mathbf{B}$, the trajectory will be circular with radius:

$$r = \frac{mv}{qB}$$

- If $\mathbf{v} \not\perp \mathbf{B}$, then $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel$ (where $\mathbf{v}_\perp \perp \mathbf{B}$ and $\mathbf{v}_\parallel \parallel \mathbf{B}$) and the trajectory will be an helicoidal with radius:

$$r = \frac{mv_\perp}{qB}$$

Proposition 1.44. If there is a charge particle q moving at a velocity $\mathbf{v}$ in a region where there is an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$, the particle experiences a force called *Lorentz force*:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Magnetic moment

Definition 1.45. We define the *magnetic moment* of a coil as:

$$\boldsymbol{\mu} = I\mathbf{S}$$

where I is the electric current passing through it and $\mathbf{S}$ is the surface vector. The magnetic moment of a solenoid of N turns (each of are S) is:

$$\boldsymbol{\mu} = NI\mathbf{S}$$

Proposition 1.46. The torque done when a magnetic field $\mathbf{B}$ is applied to an object of magnetic moment $\boldsymbol{\mu}$ is:

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$$

This torque tends to line up the magnetic moment with the magnetic field $\mathbf{B}$, so that it takes its lowest energy configuration. The potential energy associated with the magnetic moment is:

$$U = -\boldsymbol{\mu} \cdot \mathbf{B}$$

Proposition 1.47. Consider a magnetic dipole of magnetic moment $\boldsymbol{\mu}$ that cannot rotate over itself within a magnetic field $\mathbf{B}$. The external force necessary to move the dipole a distance $\mathrm{d}y$ is:

$$F_{\text{ext}} = \frac{\mathrm{d}(\boldsymbol{\mu} \cdot \mathbf{B})}{\mathrm{d}y}$$

Proposition 1.48 (Hall effect). The *Hall effect* is the production of a voltage difference V_{H} across an electrical conductor of width d that is transverse to an electric current I in the conductor and to an applied magnetic field B perpendicular to the current. It is used for:

- determine the density n of charge carriers:

$$n = \frac{IB}{qdV_{\mathrm{H}}}$$

  where q is the charge of the charge carriers.

- measure the magnitude of the magnetic field:

$$B = \frac{nqd}{I}V_{\mathrm{H}}$$

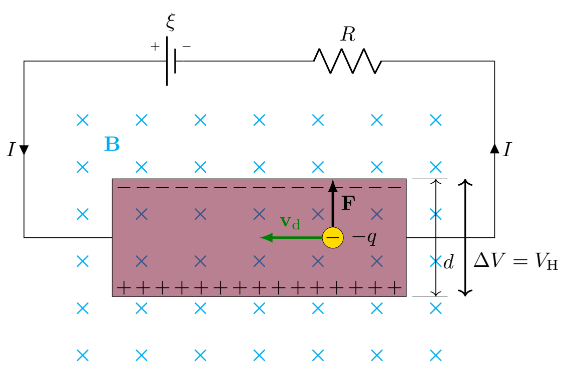

Figure 2: Hall effect when negative charge carriers are flowing through the circuit

Biot-Savart law

Proposition 1.49. The magnetic field created by a point charge q moving at velocity $\mathbf{v}$ at a distance $\mathbf{r}$ from it is:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q\mathbf{v}\times\hat{\mathbf{r}}}{\|\mathbf{r}\|^2}$$

where μ_0 is the *vacuum permeability* and $\hat{\mathbf{r}}$ is the unit vector pointing from the charge to the point where we calculate the magnetic field.

Law 1.50 (Biot-Savart law). The magnetic field created by a wire of length $\mathrm{d}\boldsymbol{\ell}$ carrying an electric current I at a distance $\mathbf{r}$ from the wire is:

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}\boldsymbol{\ell}\times\hat{\mathbf{r}}}{\|\mathbf{r}\|^2}$$

Proposition 1.51. Magnetic field created by:

- a coil of radius R when it carries a current I:
  - on its center:
  $$\mathbf{B} = \frac{\mu_0 I}{2R}\mathbf{e}_x$$
  - at a distance x from its center in the same axis:
  $$\mathbf{B} = \frac{\mu_0}{2}\frac{R^2 I}{(x^2+R^2)^{3/2}}\mathbf{e}_x$$
- a solenoid of N turns, length ℓ and radius R when it carries a current I:
  - at a distance x from its center and over its axis:
  $$\mathbf{B} = \frac{\mu_0}{2}nI\left(\frac{x-a}{\sqrt{(x-a)^2+R^2}}-\right.$$
  $$\left.-\frac{x-b}{\sqrt{(x-b)^2+R^2}}\right)\mathbf{e}_x$$
  where $n=\frac{N}{\ell}$.

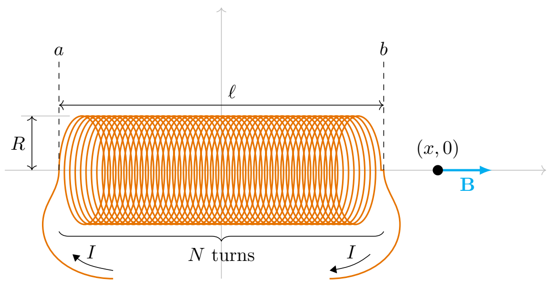

Figure 3

  - inside the solenoid ($|a|,|b|\gg R$) and far from its ends:
  $$\mathbf{B} = \mu_0 nI\mathbf{e}_x$$
- a finite wire at a point P situated at distance R from the axis of the wire and angles θ_1 and θ_2 from the point to the ends of the wire:
$$B = \frac{\mu_0 I}{4\pi R}(\sin\theta_1+\sin\theta_2)$$

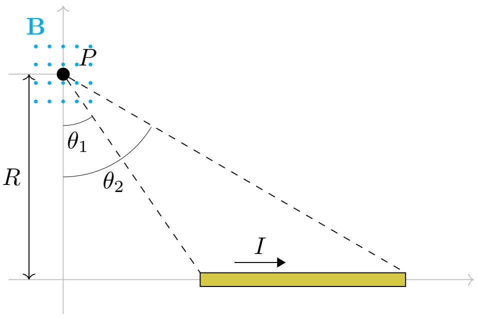

Figure 4

- an infinite wire at a distance R from it:
$$B = \frac{\mu_0 I}{2\pi R}$$

Gauß' law and Ampère's law

Proposition 1.52. The magnetic force per unit of length ℓ between two straight parallel conductors carrying electric currents I_1 and I_2 and separated a distance r from each other is:

$$\frac{F}{\ell} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{r}$$

Law 1.53 (Gauß' law for magnetism). The magnetic flux through any closed surface S is zero.

$$\oiint_S \mathbf{B}\cdot\mathrm{d}\mathbf{S} = 0$$

Law 1.54 (Ampère's law). The line integral of a magnetic field $\mathbf{B}$ around a closed curve C is proportional to the total current I_{enc} passing through a surface S enclosed by C.

$$\oint_C \mathbf{B}\cdot\mathrm{d}\boldsymbol{\ell} = \mu_0 I_{\mathrm{enc}}$$

Magnetism of the matter

Proposition 1.55. Consider a particle of mass m, charge q, angular momentum $\mathbf{L}$ and magnetic moment $\boldsymbol{\mu}$. The relation between $\mathbf{L}$ and $\boldsymbol{\mu}$ is:

$$\boldsymbol{\mu} = \frac{q}{2m}\mathbf{L}$$

Proposition 1.56. The angular momentum is quantized. For an electron the quantum unit of the magnetic moment is called *Bohr magneton* and has a value of:

$$\mu_{\mathrm{B}} = \frac{e\hbar}{2m_e}$$

Therefore:

$$\boldsymbol{\mu}_L = -\mu_{\mathrm{B}}\frac{\mathbf{L}}{\hbar}\quad\text{and}\quad\boldsymbol{\mu}_S = -2\mu_{\mathrm{B}}\frac{\mathbf{S}}{\hbar}$$

where $\boldsymbol{\mu}_L$ is the magnetic moment due to the orbital angular momentum and $\boldsymbol{\mu}_S$ is the magnetic moment due to the spin. The total angular momentum is: $\mathbf{j}=\mathbf{L}+\mathbf{S}$

Definition 1.57. The *magnetization* $\mathbf{M}$ is defined as:

$$\mathbf{M} = \frac{\mathrm{d}\boldsymbol{\mu}}{\mathrm{d}V}$$

where $\mathrm{d}V$ is the volume element. Moreover if a section of a cylinder of length $\mathrm{d}\ell$ carries a current $\mathrm{d}i$, then:

$$M = \frac{\mathrm{d}i}{\mathrm{d}\ell}$$

Proposition 1.58. Suppose we place a cylinder of magnetic material inside a long solenoid that has n turns per unit of length and carries a current I. Then, the applied field of the solenoid $\mathbf{B}_{\text{app}}$ ($B_{\text{app}} = \mu_0 n I$) magnetizes the material so that it acquires a magnetization $\mathbf{M}$. The resultant magnetic field at a point inside the solenoid is:

$$\mathbf{B} = \mathbf{B}_{\text{app}} + \mu_0 \mathbf{M}$$

Proposition 1.59. The magnetization $\mathbf{M}$ of a material is found to be proportional to the applied magnetic field that produces the alignment of the magnetic dipoles in the material. So, using the previous notation, we can write:

$$\mathbf{M} = \chi_{\text{m}} \frac{\mathbf{B}_{\text{app}}}{\mu_0}$$

where the constant χ_{m} is called *magnetic susceptibility*. Based on the value of χ_{m}, materials can be classified in three groups: *ferromagnetic*, *paramagnetic* and *diamagnetic*.

Material	χ_{m}	Attraction
Ferromagnetic	$(10^2, 10^5)$	Strong attraction
Paramagnetic	$(10^{-5}, 10^{-2})$	Weak attraction
Diamagnetic	$(-10^{-6}, -10^{-4})$	Weak repulsion

Definition 1.60. The permeability μ of a material is defined as:

$$\mu = (1 + \chi_{\text{m}})\mu_0$$

Electromagnetic induction

Definition 1.61. We define the *magnetic flux* as:

$$\Phi_B = \iint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$$

Law 1.62 (Faraday's law). The emf ξ induced on a circuit is equal to the time rate of change of the magnetic flux Φ_B through the circuit.

$$\xi = \oint \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

Law 1.63 (Lenz's law). The emf and induced electric current tend to oppose the change in flux and to exert a mechanical force which opposes the motion.

Proposition 1.64. Consider a coil of radius r and a magnetic field B applied to it. Then, this induces an electric field of magnitude:

$$E = -\frac{r}{2}\frac{\mathrm{d}B}{\mathrm{d}t}$$

Proposition 1.65. The emf induced on a circuit by the relative motion between a magnetic field B and a segment of length ℓ of electric current is:

$$\xi = -B\ell v$$

where v is the velocity of the segment relative to the magnetic field.

Proposition 1.66. Due to the rotation at angular velocity ω of a solenoid of N turns and section S in a magnetic field B, the potential difference induced between the ends of the solenoid is:

$$V = NBS\omega \sin(\omega t) =: V_0 \sin(\omega t)$$ [2]

Moreover if we connect the solenoid to a circuit of resistance R, we will produce an intensity I given by:

$$I = \frac{V_0}{R}\sin(\omega t)$$

Definition 1.67 (Eddy current). *Eddy currents* are loops of electrical current induced within conductors by a changing magnetic field. These currents induce a magnetic force that opposes the movement.

Inductance

Definition 1.68. Consider a solenoid of N turns, length ℓ and section S carrying an electric current I. Then, the magnetic flux Φ_B that passes through it is

$$\Phi_B = LI$$

where $L = \mu_0 n^2 S\ell$ and $n = \frac{N}{\ell}$. The coefficient L is called inductance. The SI unit of the inductance is the Henry ($1\text{ H} = \text{Wb} \cdot \text{A}^{-1}$).

Definition 1.69. An *inductor* is a solenoid with many turns.

Proposition 1.70. Consider a solenoid of inductance L and internal resistance r carrying an electric current I. Then, Faraday-Lenz law can be written as:

$$\xi = -L\frac{\mathrm{d}I}{\mathrm{d}t}$$

Therefore, the potential difference between the two ends of the solenoid is:

$$\Delta V = -L\frac{\mathrm{d}I}{\mathrm{d}t} - Ir$$

Definition 1.71. Consider two circuits close to each other so that the magnetic flux across a circuit depends also on the electric current that carries the other circuit. This dependance is given by:

$$\Phi_{B,1} = L_1 I_1 + M_{12} I_2 \qquad \Phi_{B,2} = L_2 I_2 + M_{21} I_1$$

where $\Phi_{B,i}$ is the flux that passes across the circuit i, I_i is the electric current flowing in the circuit i, L_i is the inductance coefficient of the circuit i and M_{ij} is the *mutual inductance* between the circuit i and j. Relating to the latter point, in general we have $M_{12} = M_{21}$.

Proposition 1.72. Consider an inductor of inductance L carrying an electric current I. Then, the potential energy stored in the inductor is:

$$U = \frac{1}{2}LI^2$$

Proposition 1.73. Consider an inductor that produces a magnetic field B inside it. Then, the energy density η of the magnetic field will be:

$$\eta = \frac{1}{2}\frac{B^2}{\mu_0}$$

[2] With this method, the energy isn't used at all. To solve this, three-phase electric power are used instead. This method consist in three coils separated by 120° between them.

Generalized Ampère's law

Definition 1.74. The *displacement current* is defined as:

$$I_{\mathrm{d}} = \varepsilon_0 \frac{\mathrm{d}\Phi_E}{\mathrm{d}t}$$

where Φ_E is the flux of the electric field through the surface where the current is flowing.

Law 1.75. The generalized Ampère's law (Ampère-Maxwell law) which takes into account displacement currents is:

$$\oint_C \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}$$

Definition 1.76. The speed of the electromagnetic waves in the vacuum is:

$$v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} =: c$$

Law	**Differential form**	**Integral form**
Gauß' law	$\boldsymbol{\nabla} \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$	$\oiint_S \mathbf{E} \cdot d\mathbf{S} = \dfrac{Q_{\text{int}}}{\varepsilon_0}$
Gauß' law for magnetism	$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$	$\oiint_S \mathbf{B} \cdot d\mathbf{S} = 0$
Faraday-Lenz law	$\boldsymbol{\nabla} \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\dfrac{\mathrm{d}}{\mathrm{d}t} \iint_S \mathbf{B} \cdot d\mathbf{S}$
Ampère-Maxwell law	$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$	$\oint_C \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 I + \mu_0 \varepsilon_0 \dfrac{\mathrm{d}}{\mathrm{d}t} \iint_S \mathbf{E} \cdot d\mathbf{S}$

Figure 5: Maxwell equations