Mathematical analysis

1 | Numeric series

Series convergence

Definition 1.1. Let (a_n) be a sequence of real numbers. A *numeric series* is an expression of the form

$$\sum_{n=1}^{\infty} a_n.$$

We call a_n *general term of the series* and $S_N = \sum_{n=1}^{N} a_n$, for all $N \in \mathbb{N}$, *N-th partial sum of the series*[1].

Definition 1.2. We say the series $\sum a_n$ is *convergent* if the sequence of partial sums is convergent, that is, if $S = \lim_{N\to\infty} S_N$ exist and it's finite. In that case, S is called the *sum of the series*. If the previous limit doesn't exists or it is infinite we say the series is *divergent*[2].

Proposition 1.3. Let (a_n) be a sequence such that $\sum a_n < \infty$. Then $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$ such that

$$\left| \sum_{n=1}^{N} a_n - \sum_{n=1}^{\infty} a_n \right| < \varepsilon$$

if $N \geq n_0$.

Theorem 1.4 (Cauchy's test). Let (a_n) be a sequence. $\sum a_n < \infty$ if and only if $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$ such that

$$\left| \sum_{n=N}^{M} a_n \right| < \varepsilon$$

if $M \geq N \geq n_0$.

Corollary 1.5. Changing a finite number of terms in a series has no effect on the convergence or divergence of the series.

Corollary 1.6. If $\sum a_n < \infty$, then $\lim_{n\to\infty} a_n = 0$.

Theorem 1.7 (Linearity). Let $\sum a_n, \sum b_n$ be two convergent series with sums A and B respectively and let λ be a real number. The series

$$\sum_{n=1}^{\infty} (a_n + \lambda b_n)$$

is convergent and has sum $A + \lambda B$.

Theorem 1.8 (Associative property). Let $\sum a_n$ be a convergent series with sum A. Suppose (n_k) is a strictly increasing sequence of natural numbers. The series $\sum b_n$, with $b_k = a_{n_{k-1}+1} + \cdots + a_{n_k}$ for all $i \in \mathbb{N}$, is convergent and its sum is A.

Non-negative terms series

Theorem 1.9. Let $\sum a_n$ be a series of non-negative terms $a_n \geq 0$[3]. The series converges if and only if the sequence (S_N) of partial sums is bounded.

Theorem 1.10 (Comparison test). Let $(a_n), (b_n) \geq 0$ be two sequences of real numbers. Suppose that exists a constant $C > 0$ and a number $n_0 \in \mathbb{N}$ such that $a_n \leq C b_n$ for all $n \geq n_0$.

1. If $\sum b_n < \infty \implies \sum a_n < \infty$.
2. If $\sum a_n = +\infty \implies \sum b_n = +\infty$.

Theorem 1.11 (Limit comparison test). Let (a_n), $(b_n) \geq 0$ be two sequences of real numbers. Suppose that the limit $\ell = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists.

1. If $0 < \ell < \infty \implies \sum a_n < \infty \iff \sum b_n < \infty$.
2. If $\ell = 0$ and $\sum b_n < \infty \implies \sum a_n < \infty$.
3. If $\ell = \infty$ and $\sum a_n < \infty \implies \sum b_n < \infty$.

Theorem 1.12 (Root test). Let $(a_n) \geq 0$. Suppose that the limit $\ell = \lim_{n\to\infty} \sqrt[n]{a_n}$ exists.

1. If $\ell < 1 \implies \sum a_n < \infty$.
2. If $\ell > 1 \implies \sum a_n = +\infty$.

Theorem 1.13 (Ratio test). Let $(a_n) \geq 0$. Suppose that the limit $\ell = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists.

1. If $\ell < 1 \implies \sum a_n < \infty$.
2. If $\ell > 1 \implies \sum a_n = +\infty$.

Theorem 1.14 (Raabe's test). Let $(a_n) \geq 0$. Suppose that the limit $\ell = \lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right)$ exists.

1. If $\ell > 1 \implies \sum a_n < \infty$.
2. If $\ell < 1 \implies \sum a_n = +\infty$.

Theorem 1.15 (Condensation test). Let $(a_n) \geq 0$ be a decreasing sequence. Then:

$$\sum a_n < \infty \iff \sum 2^n a_{2^n} < \infty.$$

Theorem 1.16 (Logarithmic test). Let $(a_n) \geq 0$. Suppose that the limit $\ell = \lim_{n\to\infty} \frac{\log \frac{1}{a_n}}{\log n}$ exists.

1. If $\ell > 1 \implies \sum a_n < \infty$.
2. If $\ell < 1 \implies \sum a_n = +\infty$.

[1] From now on we will write $\sum a_n$ to refer $\sum_{n=1}^{\infty} a_n$.

[2] We will use the notation $\sum a_n < \infty$ or $\sum a_n = +\infty$ to express that the series converges or diverges, respectively.

[3] Obviously the following results are also valid if the series is of non-positive terms or has a finite number of negative or positive terms.

Theorem 1.17 (Integral test). Let $f : [1, \infty) \to (0, \infty)$ be a decreasing function. Then:

$$\sum f(n) < \infty \iff \iff \exists C > 0 \text{ such that } \int_1^n f(x)\mathrm{d}x \leq C \ \forall n.$$

Alternating series

Definition 1.18. An *alternating series* is a series of the form $\sum(-1)^n a_n$, with $(a_n) \geq 0$.

Theorem 1.19 (Leibnitz's test). Let $(a_n) \geq 0$ be a decreasing sequence such that $\lim_{n\to\infty} a_n = 0$. Then, $\sum(-1)^n a_n$ is convergent.

Theorem 1.20 (Abel's summation formula). Let $(a_n), (b_n)$ be two sequences of real numbers. Then,

$$\sum_{n=N}^{M} a_n(b_{n+1} - b_n) = a_{M+1}b_{M+1} - a_N b_N - \sum_{n=N}^{M} b_{n+1}(a_{n+1} - a_n).$$

Theorem 1.21 (Dirichlet's test). Let $(a_n), (b_n)$ be two sequences of real numbers such that:

1. $\exists C > 0$ such that $\left|\sum_{n=1}^{N} a_n\right| \leq C$ for all $N \in \mathbb{N}$.
2. (b_n) is monotone and $\lim_{n\to\infty} b_n = 0$.

Then, $\sum a_n b_n$ is convergent.

Theorem 1.22 (Abel's test). Let $(a_n), (b_n)$ be two sequences of real numbers such that:

1. The series $\sum a_n$ is convergent.
2. (b_n) is monotone and bounded.

Then, $\sum a_n b_n$ is convergent.

Absolute convergence and rearrangement of series

Definition 1.23. We say a series $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ is convergent.

Theorem 1.24. If a series converges absolutely, it converges.

Definition 1.25. We say a sequence (b_n) is a *rearrangement of the sequence* (a_n) if exists a bijective map $\sigma : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{\sigma(n)}$. A *rearrangement of the series* $\sum a_n$ is the series $\sum a_{\sigma(n)}$ for some bijection $\sigma : \mathbb{N} \to \mathbb{N}$.

Definition 1.26. Let $x \in \mathbb{R}$. We define the *positive part of* x as

$$x^+ = \begin{cases} x & \text{si } x \geq 0 \\ 0 & \text{si } x < 0 \end{cases}$$

Analogously, we define the *negative part of* x as

$$x^- = \begin{cases} 0 & \text{si } x \geq 0 \\ -x & \text{si } x < 0 \end{cases}$$

Note that we can write $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

Theorem 1.27. A series $\sum a_n$ is absolutely convergent if and only if positive and negative terms series, $\sum a_n^+$ and $\sum a_n^-$, converge. In this case,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-.$$

Theorem 1.28. Let $\sum a_n$ be an absolutely convergent series. Then, for all bijection $\sigma : \mathbb{N} \to \mathbb{N}$, the rearranged series $\sum a_{\sigma(n)}$ is absolutely convergent and $\sum a_n = \sum a_{\sigma(n)}$.

Theorem 1.29 (Riemann's theorem). Let $\sum a_n$ be a convergent series but not absolutely convergent. Then, $\forall \alpha \in \mathbb{R} \cup \{\infty\}$, there exists a bijective map $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum a_{\sigma(n)}$ converges and $\sum a_{\sigma(n)} = \alpha$.

Theorem 1.30. A series $\sum a_n$ is absolutely convergent if and only if any rearranged series converges to the same value of $\sum a_n$.

2 | Sequences and series of functions

Sequences of functions

Definition 1.31. Let $D \subseteq \mathbb{R}$. A set

$$(f_n(x)) = \{f_1(x), f_2(x), \ldots, f_n(x), \ldots\}$$

is a *sequence of real functions* if $f_i : D \to \mathbb{R}$ is a real-valued function. In this case we say the sequence $(f_n(x))$, or simply (f_n), is well-defined on D.

Definition 1.32. Let (f_n) be a sequence of functions defined on $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. We say (f_n) *converges pointwise to* f *on* D if $\forall x \in D$, $\lim_{n\to\infty} f_n(x) = f(x)$

Definition 1.33. Let (f_n) be a sequence of functions defined on $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. We say (f_n) *converges uniformly to* f *on* D if $\forall \varepsilon > 0$, $\exists n_0 : |f_n(x) - f(x)| < \varepsilon$ $\forall n \geq n_0$ and $\forall x \in D$.

Lemma 1.34. Let (f_n) be an uniform convergent sequence of functions defined on $D \subseteq \mathbb{R}$ and let f be a function such that (f_n) converges pointwise to f. Then, (f_n) converges uniformly f on D.

Lemma 1.35. Let (f_n) be a sequence of functions defined on $D \subseteq \mathbb{R}$. (f_n) converges uniformly a f en D if and only if $\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in D\} = 0$.

Corollary 1.36. A sequence of functions (f_n) converges uniformly to f on $D \subseteq \mathbb{R}$ if and only if there is a sequence (a_n), with $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, and a number $n_0 \in \mathbb{N}$ such that $\sup\{|f_n(x) - f(x)| : x \in D\} \leq a_n$, $\forall n \geq n_0$.

Theorem 1.37 (Cauchy's test). A sequence of functions (f_n) converges uniformly to f on $D \subseteq \mathbb{R}$ if and only if $\forall \varepsilon > 0$ $\exists n_0 : \sup\{|f_n(x) - f_m(x)| : x \in D\} < \varepsilon$ if $n, m \geq n_0$.

Theorem 1.38. Let (f_n) be a sequence of continuous functions defined on $D \subseteq \mathbb{R}$. If (f_n) converges uniformly to f on D, then f is continuous on D, that is, for any $x_0 \in D$, it satisfies:

$$\lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right) = \lim_{x\to x_0} f(x).$$

Theorem 1.39. Let (f_n) be a sequence of functions defined on $I = [a, b] \subseteq \mathbb{R}$. If (f_n) are Riemann-integrable on I and (f_n) converges uniformly to f on I, then f is Riemann-integrable on I and

$$\int_a^b \lim_{n\to\infty} f_n(x)\mathrm{d}x = \lim_{n\to\infty}\int_a^b f_n(x)\mathrm{d}x.$$

Theorem 1.40. Let (f_n) be a sequence of functions defined on $I = (a, b) \subset \mathbb{R}$. If (f_n) are derivable on I, $(f'_n(x))$ converges uniformly on I and $\exists x_0 \in I : \lim_{n\to\infty} f_n(x_0) \in \mathbb{R}$, then there is a function f such that (f_n) converges uniformly to f on I, f is derivable on I and $(f'_n(x))$ converges uniformly to f' on I.

Series of functions

Definition 1.41. Let (f_n) be a sequence of functions defined on $D \subseteq \mathbb{R}$. The expression

$$\sum_{n=1}^{\infty} f_n(x)$$

is the *series of functions associated with* (f_n).

Definition 1.42. A series of functions $\sum f_n(x)$ defined on $D \subseteq \mathbb{R}$ *converges pointwise on* D if the sequence of partials sums

$$F_N(x) = \sum_{n=1}^{N} f_n(x)$$

converges pointwise. If the pointwise limit of (F_N) is $F(x)$, we say F is the *sum of the series in a pointwise sense.*

Definition 1.43. A series of functions $\sum f_n(x)$ defined on $D \subseteq \mathbb{R}$ *converges uniformly on* D if the sequence of partials sums

$$F_N(x) = \sum_{n=1}^{N} f_n(x)$$

converges uniformly. If the uniform limit of (F_N) is $F(x)$, we say F is the *sum of the series in an uniform sense.*

Theorem 1.44 (Cauchy's test). A series of functions $\sum f_n(x)$ defined on $D \subseteq \mathbb{R}$ converges uniformly if and only if $\forall \varepsilon > 0\ \exists n_0$ such that

$$\sup\left\{\left|\sum_{n=N}^{M} f_n(x)\right| : x \in D\right\} < \varepsilon$$

if $M \geq N \geq n_0$.

Corollary 1.45. If $\sum f_n(x)$ is a series of continuous functions on $D \subseteq \mathbb{R}$, then (f_n) converges uniformly to zero on D.

Theorem 1.46. If $\sum f_n(x)$ is uniformly convergent series of functions on $D \subseteq \mathbb{R}$, then its sum function is also continuous on D.

Theorem 1.47. Let (f_n) be a sequence of functions defined on $I = [a, b] \subseteq \mathbb{R}$. If (f_n) are Riemann-integrable on I and $\sum f_n(x)$ converges uniformly on I, then $\sum f_n(x)$ is Riemann-integrable on I and

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\mathrm{d}x = \sum_{n=1}^{\infty}\int_a^b f_n(x)\mathrm{d}x.$$

Theorem 1.48. Let (f_n) be a sequence of functions defined on $I = (a, b) \subset \mathbb{R}$. If (f_n) are derivable on I, $\sum f'_n(x)$ converges uniformly on I and $\exists c \in I : \sum f_n(c) < \infty$, then $\sum f_n(x)$ converges uniformly on I, $\sum f_n(x)$ is derivable on I and

$$\left(\sum_{n=1}^{\infty} f_n(x)\right)' = \sum_{n=1}^{\infty} f'_n(x).$$

Theorem 1.49 (Weierstraß M-test). Let (f_n) be a sequence of functions defined on $D \subseteq \mathbb{R}$ such that $|f_n(x)| \leq M_n\ \forall x \in D$ and suppose that $\sum M_n$ is a convergent numeric series. Then, $\sum f_n(x)$ is converges uniformly on D.

Theorem 1.50 (Dirichlet's test). Let $(f_n), (g_n)$ be two sequences of functions defined on $D \subseteq \mathbb{R}$. Suppose:

1. $\exists C > 0 : \sup\left\{\left|\sum_{n=1}^{N} f_n(x)\right| : x \in D\right\} \leq C,\ \forall N.$
2. $(g_n(x))$ is a monotone sequence for all $x \in D$ and $\lim_{n\to\infty} \sup\{|g_n(x)| : x \in D\} = 0$.

Then, $\sum f_n(x)g_n(x)$ converges uniformly on D.

Theorem 1.51 (Abel's test). Let $(f_n), (g_n)$ be two sequences of functions defined on $D \subseteq \mathbb{R}$. Suppose:

1. The series $\sum f_n(x)$ converges uniformly on D.
2. $(g_n(x))$ is a monotone and bounded sequence for all $x \in D$.

Then, $\sum f_n(x)g_n(x)$ converges uniformly on D.

Power series

Definition 1.52. Let (a_n) be a sequence of real numbers and $x_0 \in \mathbb{R}$. A *power series centred on* x_0 is a series of functions of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

Proposition 1.53. Let $\sum a_n(x - x_0)^n$ be a power series. Suppose there exists an $x_1 \in \mathbb{R}$ such that $\sum a_n(x_1 - x_0)^n < \infty$. Then, $\sum a_n(x - x_0)^n$ converges uniformly on any closed interval $I \subset A = \{x \in \mathbb{R} : |x - x_0| < |x_1 - x_0|\}$.

Theorem 1.54. Let $\sum a_n(x - x_0)^n$ be a power series and consider

$$R = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1} \in [0, \infty].$$

Then:

1. If $|x-x_0|<R \implies \sum a_n(x-x_0)^n$ converges absolutely.

2. If $0 \leq r < R \implies \sum a_n(x-x_0)^n$ converges uniformly on $[x_0-r, x_0+r]$.

3. If $|x-x_0|>R \implies \sum a_n(x-x_0)^n$ diverges.

The number R is called *radius of convergence of the power series*.

Theorem 1.55 (Abel's theorem). Let $\sum a_n x^n$ be a power series[4] with radius of convergence R satisfying $\sum a_n R^n < \infty$. Then the series $\sum a_n x^n$ converges uniformly on $[0,R]$. In particular, if $f(x)=\sum a_n x^n$,

$$\lim_{x\to R^-} f(x) = \sum_{n=0}^{\infty} a_n R^n.$$

Corollary 1.56. Let f be the sum function of a power series $\sum a_n x^n$. Then f is continuous on the domain of convergence of the series.

Corollary 1.57. If the series $\sum a_n x^n$ has radius of convergence $R \neq 0$ and f is its sum function, then f is Riemann-integrable on any closed subinterval on the domain of convergence of the series. In particular, for $|x|<R$,

$$\int_0^x f(t)\mathrm{d}t = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}\text{[5]}.$$

Corollary 1.58. Let f be the sum function of the power series $\sum a_n x^n$. Then f is derivable within the domain of convergence of the series and

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

Corollary 1.59. Any function f defined as a sum of a power series $\sum a_n x^n$ is indefinitely derivable within the domain of convergence of the series and

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n x^{n-k},$$

for all $k \in \mathbb{N}\cup\{0\}$. In particular $f^{(k)}(0)=k!a_k$.

Definition 1.60. A function is *analytic* if it can be expressed locally as a power series.

Stone-Weierstraß approximation theorem

Definition 1.61. Let f be a real-valued function. We say f has *compact support*[6] if exists an $M>0$ such that $f(x)=0$ for all $x\in\mathbb{R}\setminus[-M,M]$.

Definition 1.62. Let f, g be real-valued functions with compact support. We define the convolution of f with g as

$$(f*g)(x) = \int_{\mathbb{R}} f(t)g(x-t)\mathrm{d}t\text{[7]}.$$

Definition 1.63. We sap a sequence of functions (ϕ_ε) with compact support is an *approximation of unity* if

1. $\phi_\varepsilon \geq 0$.

2. $\displaystyle\int_{\mathbb{R}} \phi_\varepsilon = 1$.

3. For all $\delta>0$, $\phi_\varepsilon(t)$ converges uniformly to zero when $\varepsilon\to 0$ if $|t|>\delta$.

Lemma 1.64. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function with compact support. Let (ϕ_ε) be an approximation of unity. Then $(f*\phi_\varepsilon)$ converges uniformly to f on $\mathbb{R}$ when $\varepsilon\to 0$.

Theorem 1.65 (Stone-Weierstraß theorem). Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then, there exists polynomials $p_n\in\mathbb{R}[x]$ such that the sequence (p_n) converge uniformly to f on $[a,b]$.

3 | Improper integrals

Locally integrable functions

Definition 1.66. Let $f:[a,b)\to\mathbb{R}$, with $b\in\mathbb{R}\cup\{\infty\}$. We say f is *locally integrable on* $[a,b)$ if f is Riemann-integrable on $[a,x]$ for all $a\leq x<b$.

Definition 1.67. Let $f:[a,b)\to\mathbb{R}$ be a locally integrable function. If there exists

$$\lim_{x\to b^-}\int_a^x f$$

and it's finite, we say that the *improper integral of f on* $[a,b)$, $\displaystyle\int_a^b f$, is *convergent*.

Theorem 1.68 (Cauchy's test). Let $f:[a,b)\to\mathbb{R}$ be a locally integrable function. The improper integral $\displaystyle\int_a^b f$ is convergent if and only if $\forall\varepsilon>0\ \exists b_0,\ a<b_0<b$, such that

$$\left|\int_{x_1}^{x_2} f\right| < \varepsilon$$

if $b_0<x_1<x_2<b$.

[4] From now on we will suppose, for simplicity, $x_0=0$.
[5] The formula is also valid for $|x|=R$ if the series $\sum a_n R^n$ (or $\sum a_n(-R)^n$) is convergent.
[6] In general, the support of a function is the adherence of the set of points which are not mapped to zero.
[7] Alternatively if f,g are Riemann-integrable functions on $[a,b]$ we can define the convolution of f and g as

$$(f*g)(x) = \int_a^b f(t)g(x-t)\mathrm{d}t.$$

Improper integrals of non-negative functions

Theorem 1.69. Let $f : [a, b) \to \mathbb{R}$ be a locally integrable non-negative function. A necessary and sufficient condition for $\int_a^b f$ to be convergent is that the function

$$F(x) = \int_a^x f(t)\mathrm{d}t$$

must be bounded for all $x < b$.

Theorem 1.70 (Comparison test). Let $f, g : [a, b) \to [0, +\infty)$ be two locally integrable non-negative functions. Then:

1. If $\exists C > 0$ such that $f(x) \leq Cg(x)$ $\forall x$ on a neighborhood of b and $\int_a^b g < \infty \implies \int_a^b f < \infty$.
2. Suppose the limit $\ell = \lim_{x\to b} \frac{f(x)}{g(x)}$ exists. Then,
   i) If $\ell \in (0, \infty) \implies \int_a^b f < \infty \iff \int_a^b g < \infty$.
   ii) If $\ell = 0$ and $\int_a^b g < \infty \implies \int_a^b f < \infty$.
   iii) If $\ell = \infty$ and $\int_a^b f < \infty \implies \int_a^b g < \infty$.

Theorem 1.71 (Integral test). Let $f : [1, \infty) \to (0, \infty)$ be a locally integrable decreasing function. Then:

$$\sum f(n) < \infty \iff \int_1^\infty f(x)\mathrm{d}x < \infty^8.$$

Absolute convergence of improper integrals

Definition 1.72. Let $f : [a, b) \to (0, \infty)$ be a locally integrable function. We say $\int_a^b f$ *converges absolutely* if $\int_a^b |f|$ is convergent.

Theorem 1.73 (Dirichlet's test). Let $f, g : [a, b) \to \mathbb{R}$ be two locally integrable functions Suppose:

1. $\exists C > 0$ such that $\left|\int_a^x f(t)\mathrm{d}t\right| \leq C$ for all $x \in [a, b)$.
2. g is monotone and $\lim_{x\to b} g(x) = 0$.

Then, $\int_a^b fg$ is convergent.

Theorem 1.74 (Abel's test). Let $f, g : [a, b) \to \mathbb{R}$ be two locally integrable functions. Suppose:

1. $\int_a^b f$ is convergent.
2. g is monotone and bounded.

Then, $\int_a^b fg$ is convergent.

[8] This is another way of formulating theorem 1.17.

Differentiation under integral sign

Theorem 1.75. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function on $[a, b] \times [c, d]$. Consider the function $F(y) = \int_a^b f(x, y)\mathrm{d}x$ defined on $[c, d]$. Then, F is continuous, that is, if $c < y_0 < d$,

$$\lim_{y\to y_0} F(y) = \lim_{y\to y_0} \int_a^b f(x, y)\mathrm{d}x = \int_a^b \lim_{y\to y_0} f(x, y)\mathrm{d}x = \\ = \int_a^b f(x, y_0)\mathrm{d}x = F(y_0).$$

Theorem 1.76. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a Riemann-integrable function and let $F(y) = \int_a^b f(x, y)\mathrm{d}x$. If f is differentiable with respect to y and $\partial f/\partial y$ is continuous on $[a, b] \times [c, d]$, then $F(y)$ is derivable on (c, d) and its derivative is

$$F'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y)\mathrm{d}x,$$

for all $y \in (c, d)$.

Theorem 1.77. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function on $[a, b] \times [c, d]$. Let $a, b : [c, d] \to \mathbb{R}$ be to differentiable functions satisfying $a \leq a(y) \leq b(y) \leq b$ for every $y \in [c, d]$. Suppose that $\partial f/\partial y$ is continuous on $\{(x, y) \in \mathbb{R}^2 : a(y) \leq x \leq b(y),\ c \leq y \leq d\}$. Then $F(y) = \int_{a(y)}^{b(y)} f(x, y)\mathrm{d}x$ is derivable on (c, d) and its derivative is

$$F'(y) = b'(y)f(b(y), y) - a'(y)f(a(y), y) + \\ + \int_{a(y)}^{b(y)} \frac{\partial f}{\partial y}(x, y)\mathrm{d}x,$$

for all $y \in (c, d)$.

Theorem 1.78. Let $f : [a, b) \times [c, d] \to \mathbb{R}$ be a continuous function on $[a, b) \times [c, d]$. We consider $F(y) = \int_a^b f(x, y)\mathrm{d}x$. Suppose that:

1. $\frac{\partial f}{\partial y}$ is continuous on $[a, b) \times [c, d]$.
2. Given $y_0 \in [c, d]$, $\exists \delta > 0$ such that the integral
   $$\int_a^b \sup\left\{\left|\frac{\partial f}{\partial y}(x, y)\right| : y \in (y_0 - \delta, y_0 + \delta)\right\}\mathrm{d}x$$
   exists and it's finite on $[a, b)$.

Then, $F(y)$ is derivable at y_0 and

$$F'(y_0) = \int_a^b \frac{\partial f}{\partial y}(x, y_0)\mathrm{d}x.$$

Theorem 1.79. Let $f : [a,b) \times [c,d] \to \mathbb{R}$ be a continuous function on $[a,b) \times [c,d]$. Let $a, b : [c,d] \to \mathbb{R}$ be two differentiable functions satisfying $a \le a(y) \le b(y) \le b$ for every $y \in [c,d]$. We consider $F(y) = \displaystyle\int_{a(y)}^{b(y)} f(x,y)\mathrm{d}x$. Suppose that:

1. $\dfrac{\partial f}{\partial y}$ is continuous on $\{(x,y) \in \mathbb{R}^2 : a(y) \le x \le b(y),\ c \le y \le d\}$.
2. Given $y_0 \in [c,d]$, $\exists \delta > 0$ such that the integral
$$\int_{a(y)}^{b(y)} \sup\left\{\left|\frac{\partial f}{\partial y}(x,y)\right| : y \in (y_0-\delta, y_0+\delta)\right\}\mathrm{d}x$$
exists and it's finite on $[a,b)$.

The, $F(y)$ is derivable at y_0 and

$$F'(y_0) = b'(y_0)f(b(y_0),y_0) - a'(y_0)f(a(y_0),y_0) + \\ + \int_{a(y_0)}^{b(y_0)} \frac{\partial f}{\partial y}(x,y_0)\mathrm{d}x.$$

Gamma function

Definition 1.80. For $x > 0$, *Gamma function* is defined as
$$\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t.$$

Theorem 1.81. Gamma function is a generalization of the factorial. In fact, for $x > 0$ we have
$$\Gamma(x+1) = x\Gamma(x).$$
In particular, $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

Theorem 1.82. Gamma function satisfies:
$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{(x/e)^x\sqrt{2\pi x}} = 1.$$

Corollary 1.83 (Stirling's formula).
$$\lim_{n\to\infty} \frac{n!}{n^n\mathrm{e}^{-n}\sqrt{2\pi n}} = 1.$$

4 | Fourier series

Periodic functions

Definition 1.84. Let $f : \mathbb{R} \to \mathbb{C}$ be a function. We say that f is *T-periodic*, or is *periodic with period T*, being $T > 0$, if $f(x+T) = f(x)$ for all $x \in \mathbb{R}$.

Lemma 1.85. Let $f : \mathbb{R} \to \mathbb{C}$ be a T-periodic function. Then $f(x+T') = f(x)$ for all $x \in \mathbb{R}$ if and only if $T' = kT$ for some $k \in \mathbb{Z}$.

Proposition 1.86. Let $f : \mathbb{R} \to \mathbb{C}$ be a T-periodic function. Then
$$\int_a^{a+T} f(x)\mathrm{d}x = \int_0^T f(x)\mathrm{d}x,$$
where $a \in \mathbb{R}$. In particular,
$$\int_a^{a+kT} f(x)\mathrm{d}x = k\int_0^T f(x)\mathrm{d}x.$$

Lemma 1.87. Let $f : \mathbb{R} \to \mathbb{C}$ be a T-periodic continuous function. Then, $|f|$ is bounded.

Proposition 1.88. Given a T-periodic function f, there is no power series uniformly convergent to f on $\mathbb{R}$.

Orthogonal systems

Definition 1.89. Let $f : \mathbb{R} \to \mathbb{C}$ be a function. Then $f \in L^p(I)$, $p \ge 1$, if
$$\|f\|_p := \left(\int_I |f(t)|^p\mathrm{d}t\right)^{1/p} < \infty.$$

Definition 1.90. Let $f, g : [a,b] \to \mathbb{C}$ be Riemann-integrable functions. We define the *inner product of f and g* as
$$\langle f, g\rangle := \int_a^b f(x)\overline{g(x)}\mathrm{d}x,$$
where $\overline{g}$ is the complex conjugate of g. Now, it's natural to define the *norm of f* as
$$\|f\| := \langle f, f\rangle^{1/2} = \left(\int_a^b |f(x)|^2\mathrm{d}x\right)^{1/2} = \|f\|_2.$$
And the *distance between f and g* as
$$d(f,g) := \|f-g\|.$$

Proposition 1.91. Let $f, g : [a,b] \to \mathbb{C}$ be Riemann-integrable functions and let $\alpha \in \mathbb{C}$. Then we have:

1. $\langle f, f\rangle \ge 0$.
2. $\langle f+h, g\rangle = \langle f, g\rangle + \langle h, g\rangle$ and $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$.
3. $\langle f, g\rangle = \overline{\langle g, f\rangle}$.
4. $\langle \alpha f, g\rangle = \alpha\langle f, g\rangle$ and $\langle f, \alpha g\rangle = \overline{\alpha}\langle f, g\rangle$.

Theorem 1.92 (Cauchy–Schwarz inequality). Let $f, g : [a,b] \to \mathbb{C}$ be Riemann-integrable functions. Then,
$$|\langle f, g\rangle| \le \|f\| \cdot \|g\|,$$
which can be written as
$$\int_a^b f\overline{g} \le \left(\int_a^b |f|^2\right)^{1/2}\left(\int_a^b |g|^2\right)^{1/2}.$$

Theorem 1.93 (Minkowski inequality). Let $f, g : [a,b] \to \mathbb{C}$ be Riemann-integrable functions. Then,
$$\|f+g\| \le \|f\| + \|g\|.$$

Definition 1.94. Let $f, g : [a,b] \to \mathbb{C}$ be Riemann-integrable functions with $f \ne g$. We say f and g are *orthogonal* if $\langle f, g\rangle = 0$. We say f and g are *orthonormal* if they are orthogonal and $\|f\| = \|g\| = 1$.

Definition 1.95. Let $S=\{\phi_0,\phi_1,\dots\}$ be a collection of Riemann-integrable functions on $[a,b]$. We say S is an *orthonormal system* if $\|\phi_n\|=1$ $\forall n$ and $\langle\phi_n,\phi_m\rangle=0$ $\forall n\neq m$.

Proposition 1.96. Let

$$S_1=\left\{\frac{1}{T}\mathrm{e}^{\frac{2\pi i n x}{T}}, n\in\mathbb{Z}\right\},$$

$$S_2=\left\{\frac{1}{\sqrt{T}},\frac{\cos\left(\frac{2\pi n x}{T}\right)}{\sqrt{T/2}},\frac{\sin\left(\frac{2\pi m x}{T}\right)}{\sqrt{T/2}}, n,m\in\mathbb{N}\right\}.$$

Then S_1 and S_2 orthonormal systems on $[-T/2,T/2]$.

Definition 1.97. A collection of functions $S=\{\phi_0,\phi_1,\dots,\phi_n\}$ is *linearly dependent* on $[a,b]$ if there exist $c_0,c_1,\dots,c_n\in\mathbb{R}$ not all zero, such that

$$c_0\phi_0+c_1\phi_1+\cdots+c_n\phi_n=0,\quad \forall x\in[a,b].$$

Otherwise we say S is *linearly independent*. If the collection S has an infinity number of functions, we say S is linearly independent on $[a,b]$ if any finite subset of S is linearly independent on $[a,b]$.

Theorem 1.98. Let $S=\{\phi_0,\phi_1,\dots\}$ be an orthonormal system on $[a,b]$. Suppose that $\sum c_n\phi_n(x)$ converges uniformly to a function f on $[a,b]$. Then, f is Riemann-integrable on $[a,b]$ and, moreover,

$$c_n=\langle f,\phi_n\rangle=\int_a^b f(x)\overline{\phi_n(x)}\mathrm{d}x,\quad \forall n\geq 0.$$

Fourier coefficients and Fourier series

Definition 1.99. Let $S=\left\{\frac{1}{T}\mathrm{e}^{\frac{2\pi i n x}{T}}, n\in\mathbb{Z}\right\}$ be an orthonormal system on $[-T/2,T/2]$ and let $f\in L^1([-T/2,T/2])$[9] be a T-periodic function[10]. We define the *n-th Fourier coefficient of f* as

$$\widehat{f}(n)=\left\langle f,\frac{1}{T}\mathrm{e}^{\frac{2\pi i n x}{T}}\right\rangle=\frac{1}{T}\int_{-T/2}^{T/2}f(x)\mathrm{e}^{-\frac{2\pi i n x}{T}}\mathrm{d}x,$$

for all $n\in\mathbb{Z}$.

Proposition 1.100. Let $f,g\in L^1([-T/2,T/2])$. The following properties are satisfied:

1. For all $\lambda,\mu\in\mathbb{C}$,
$$\widehat{\lambda f+\mu g}(n)=\lambda\widehat{f}(n)+\mu\widehat{g}(n).$$
2. Let $\tau\in\mathbb{R}$. We define $f_\tau(x)=f(x-\tau)$. Then,
$$\widehat{f_\tau}(n)=\mathrm{e}^{-\frac{2\pi i n\tau}{T}}\widehat{f}(n).$$
3. If f is even, then $\widehat{f}(n)=\widehat{f}(-n)$, $\forall n\in\mathbb{Z}$.
If f is odd, then $\widehat{f}(n)=-\widehat{f}(-n)$, $\forall n\in\mathbb{Z}$.
4. If $f\in\mathcal{C}^k$, then
$$\widehat{f^{(k)}}(n)=\left(\frac{2\pi \mathrm{i}n}{T}\right)^k\widehat{f}(n).$$
5. $\widehat{(f*g)}(n)=\widehat{f}(n)\widehat{g}(n)$.

Definition 1.101. Let $f\in L^1([-T/2,T/2])$. We define the *Fourier series of f* as

$$Sf(x)=\sum_{n\in\mathbb{Z}}\widehat{f}(n)\mathrm{e}^{\frac{2\pi i n x}{T}}.$$

Definition 1.102. Let $f\in L^1([-T/2,T/2])$ and Sf be the Fourier series of f. The *N-th partial sum of Sf* is

$$S_Nf(x)=\sum_{n=-N}^{N}\widehat{f}(n)\mathrm{e}^{\frac{2\pi i n x}{T}}.$$

Proposition 1.103. Let $f\in L^1([-T/2,T/2])$. Then

$$Sf(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{2\pi n x}{T}\right)+b_n\sin\left(\frac{2\pi n x}{T}\right),$$

where

$$a_n=\frac{2}{T}\int_{-T/2}^{T/2}f(x)\cos\left(\frac{2\pi n x}{T}\right)\mathrm{d}x,$$
$$b_n=\frac{2}{T}\int_{-T/2}^{T/2}f(x)\sin\left(\frac{2\pi n x}{T}\right)\mathrm{d}x,$$

for $n\geq 0$[11]. In particular, if f is even we have

$$Sf(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{2\pi n x}{T}\right),$$

and if f is odd we have

$$Sf(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{2\pi n x}{T}\right).$$

Definition 1.104. Let $f:(0,L)\to\mathbb{C}$ be a function. We define the *even extension of f* as

$$\tilde{f}(x)=\begin{cases} f(x) & \text{si} \quad x\in(0,L)\\ f(-x) & \text{si} \quad x\in(-L,0)\end{cases}$$

Analogously, we define the *odd extension of f* as

$$\hat{f}(x)=\begin{cases} f(x) & \text{si} \quad x\in(0,L)\\ -f(-x) & \text{si} \quad x\in(-L,0)\end{cases}$$

[9] Saying that $f\in L^1([-T/2,T/2])$ is equivalent to say that f is integrable on $[-T/2,T/2]$.
[10] From now on, we will work only with functions defined on $[-T/2,T/2]$ and extended periodically on $\mathbb{R}$.
[11] The relation between a_n,b_n and $\widehat{f}(n)$ is given by:

$$a_n=\widehat{f}(n)+\widehat{f}(-n)\quad\text{and}\quad b_n=\mathrm{i}\left[\widehat{f}(n)-\widehat{f}(-n)\right],\quad \forall n\in\mathbb{N}\cup\{0\}.$$

Proposition 1.105. Let $f \in L^1([0,T/2])$. If we make the even extension of f[12], then

$$Sf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right),$$

where $a_n = \dfrac{4}{T}\displaystyle\int_0^{T/2} f(x)\cos\left(\frac{2\pi n x}{T}\right)\mathrm{d}x$ for $n \geq 0$. If we make the odd extension of f, then

$$Sf(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{T}\right),$$

where $b_n = \dfrac{4}{T}\displaystyle\int_0^{T/2} f(x)\sin\left(\frac{2\pi n x}{T}\right)\mathrm{d}x$ for $n \geq 1$.

Pointwise convergence

Definition 1.106 (Dirichlet kernel). We define the *Dirichlet kernel of order N* as

$$D_N(t) = \frac{1}{T}\sum_{n=-N}^{N} \mathrm{e}^{\frac{2\pi i n t}{T}} = \frac{1}{T}\frac{\sin\left(\frac{(2N+1)\pi t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}.$$

Proposition 1.107. The Dirichlet kernel has the following properties:

1. D_N is a T-periodic and even function.
2. $\displaystyle\int_0^T D_N(t)\mathrm{d}t = 1,\ \forall N.$

Proposition 1.108. Let $f \in L^1([-T/2,T/2])$. Then

$$S_N f(x) = (f * D_N)(x) = \int_{-T/2}^{T/2} f(x-t)D_N(t)\mathrm{d}t = \\ = \int_0^{T/2} [f(x+t)+f(x-t)]D_N(t)\mathrm{d}t.$$

Lemma 1.109 (Riemann-Lebesgue lemma). Let $f \in L^1([-T/2,T/2])$ and $\lambda \in \mathbb{R}$. Then:

$$\lim_{\lambda\to\infty}\int_{-T/2}^{T/2} f(t)\sin(\lambda t)\mathrm{d}t = \lim_{\lambda\to\infty}\int_{-T/2}^{T/2} f(t)\cos(\lambda t)\mathrm{d}t = 0.$$

In particular, $\displaystyle\lim_{|n|\to\infty}\widehat{f}(n) = 0$.

Theorem 1.110. Let $f \in L^1([-T/2,T/2])$ be a function left and right differentiable at x_0, that is, there exists the following limits

$$f'(x_0^+) = \lim_{t\to 0^+}\frac{f(x_0+t)-f(x_0^+)}{t},$$
$$f'(x_0^-) = \lim_{t\to 0^-}\frac{f(x_0+t)-f(x_0^-)}{t},$$

(supposing the existence of left- and right-sided limits). Then,

$$\lim_{N\to\infty} S_N f(x_0) = \frac{f(x_0^+)+f(x_0^-)}{2}.$$

Theorem 1.111 (Dini's theorem). Let $f \in L^1([-T/2,T/2])$, $x_0 \in (-T/2,T/2)$ and $\ell \in \mathbb{R}$ such that

$$\int_0^{\delta}\frac{|f(x_0+t)+f(x_0-t)-2\ell|}{t}\mathrm{d}t < \infty$$

for some $\delta > 0$. Then $\displaystyle\lim_{N\to\infty} S_N f(x_0) = \ell$.

Theorem 1.112 (Lipschitz's theorem). Let $f \in L^1([-T/2,T/2])$ such that at a point $x_0 \in (-T/2,T/2)$ it satisfies

$$|f(x_0+t)-f(x_0)| \leq k|t|$$

for some constant $k \in \mathbb{R}$ and for $|t| < \delta$. Then $\displaystyle\lim_{N\to\infty} S_N f(x_0) = f(x_0)$.

Uniform convergence

Definition 1.113. Let $\sum a_n$ be a series with partial sums S_k. The series $\sum a_n$ is called *Cesàro summable* with sum S if

$$\lim_{N\to\infty}\frac{S_1+\cdots+S_N}{N} = S.$$

Definition 1.114 (Fejer kernel). We define the *Fejer kernel of order N* as

$$K_N(t) = \frac{1}{N+1}\sum_{k=0}^{N} D_k(t) = \frac{1}{T(N+1)}\frac{\sin^2\left(\frac{(N+1)\pi t}{T}\right)}{\sin^2\left(\frac{\pi t}{T}\right)},$$

being $D_k(t)$ the Dirichlet kernel of order k, $0 \leq k \leq N$.

Proposition 1.115. The Fejer kernel has the following properties:

1. K_N is a T-periodic, even and non-negative function.
2. $\displaystyle\int_{-T/2}^{T/2} K_N(t)\mathrm{d}t = 1,\ \forall N.$
3. $\forall\delta > 0,\ \displaystyle\lim_{N\to\infty}\sup\{|K_N(t)| : \delta \leq |t| \leq T/2\} = 0.$

Definition 1.116. Let $f \in L^1([-T/2,T/2])$. We define the *Fejér means $\sigma_N f$*, for all $N \in \mathbb{N}$, as

$$\sigma_N f(x) = \frac{S_0 f(x)+\cdots+S_N f(x)}{N+1}.$$

Proposition 1.117. Let $f \in L^1([-T/2,T/2])$. Then

$$\sigma_N f(x) = (f * K_N)(x) = \int_{-T/2}^{T/2} f(x-t)K_N(t)\mathrm{d}t = \\ = \int_0^{T/2} [f(x+t)+f(x-t)]K_N(t)\mathrm{d}t.$$

Theorem 1.118 (Fejér's theorem). Let $f \in L^1([-T/2,T/2])$ be a function having left- and right-sided limits at point x_0. Then,

$$\lim_{N\to\infty}\sigma_N f(x_0) = \frac{f(x_0^+)+f(x_0^-)}{2}.$$

In particular, if f is continuous at x_0, $\displaystyle\lim_{N\to\infty}\sigma_N f(x_0) = f(x_0)$.

[12] For simplicity, when we have a function f and make its even or odd extension, we will still call its even or odd extension f instead of $\tilde{f}$ or $\hat{f}$.

Theorem 1.119 (Fejér's theorem). Let f be a continuous function on $[-T/2, T/2]$. Then $\sigma_N f$ converges uniformly to f on $[-T/2, T/2]$.

Corollary 1.120. Let f be a continuous function on $[-T/2, T/2]$. Then there exists a sequence of trigonometric polynomials that converge uniformly to f on $[-T/2, T/2]$. In fact,

$$\sigma_N f(x) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) \widehat{f}(k) \mathrm{e}^{2\pi \mathrm{i} kx}.$$

Corollary 1.121. Let f and g be continuous functions on $[-T/2, T/2]$ such that $Sf(x) = Sg(x)$. Then $f = g$.

Convergence in norm

Definition 1.122. We say a sequence (f_N) *converge to f in norm L^p* if $\lim_{N\to\infty} \|f_N - f\|_p = 0$.

Theorem 1.123. Let $f \in L^2([-T/2, T/2])$. Then, $\lim_{N\to\infty} \|\sigma_N f - f\| = 0$.

Corollary 1.124. Let $f \in L^1([-T/2, T/2])$. Then $\lim_{N\to\infty} \|\sigma_N f - f\|_1 = 0$.

Corollary 1.125. Let $f, g \in L^1([-T/2, T/2])$ be functions such that $Sf(x) = Sg(x)$. Then $\lim_{N\to\infty} \|g - f\|_1 = 0$.

Theorem 1.126 (Bessel's inequality). Let $f \in L^2(I)$, where I is any interval on the real line. Then:

$$T \sum_{n=-N}^{N} |\widehat{f}(n)|^2 \leq \|f\|^2,$$

$$\frac{T}{2} \left(\frac{|a_0|^2}{2} + \sum_{n=1}^{N} |a_n|^2 + |b_n|^2 \right) \leq \|f\|^2,$$

for all $N \in \mathbb{N}$.

Theorem 1.127. $S_N f$ is the trigonometric polynomial of degree N that best approximates f in norm L^2.

Corollary 1.128. Let $f \in L^2([-T/2, T/2])$. Then, $\lim_{N\to\infty} \|S_N f - f\| = 0$.

Theorem 1.129 (Parseval's identity). Let $f, g \in L^2([-T/2, T/2])$ be bounded functions. Then

$$\langle f, g \rangle = T \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(n)}.$$

In particular, if $f = g$:

$$\|f\|^2 = T \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2,$$

$$\|f\|^2 = \frac{T}{2} \left(\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2 \right).$$

Applications of Fourier series

Theorem 1.130 (Wirtinger's inequality). Let f be a function such that $f(0) = f(T)$, $f' \in L^2([0, T])$ and $\int_a^b f(t)\mathrm{d}t = 0$. Then,

$$\int_0^T |f(x)|^2 \mathrm{d}x \leq \frac{T^2}{4\pi^2} \int_0^T |f'(x)|^2 \mathrm{d}x,$$

with equality if and only if

$$f(x) = A\cos\left(\frac{2\pi x}{T}\right) + B\sin\left(\frac{2\pi x}{T}\right).$$

Theorem 1.131 (Wirtinger's inequality). Let $f \in \mathcal{C}^1([a, b])$ with $f(a) = f(b) = 0$. Then,

$$\int_a^b |f(x)|^2 \mathrm{d}x \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 \mathrm{d}x.$$

Theorem 1.132 (Isoperimetric inequality). Let c be a simple and closed curve of class $\mathcal{C}^1$ whose length is L. If A_c is the area enclosed by c, then

$$A_c \leq \frac{L^2}{4\pi},$$

with equality if and only if c is a circle.