Mechanics and special relativity

1 | Mechanics

Kinematics

Definition 1.1. The *equation of movement* of any particle is of the form:

$$\mathbf{r}(t)=x(t)\mathbf{e}_x+y(t)\mathbf{e}_y+z(t)\mathbf{e}_z$$

where $x(t)$, $y(t)$, $z(t)$ are the movements equations of the particle along x-, y- and z-axis, respectively.

Definition 1.2. Consider a particle with movement equation $\mathbf{r}(t)$. Then, the *average velocity over any time interval* $\Delta t=t_2-t_1$ is:

$$\mathbf{v}_{\text{avg}}=\frac{\Delta\mathbf{r}(t)}{\Delta t}=\frac{\mathbf{r}(t_2)-\mathbf{r}(t_1)}{t_2-t_1}$$

If we take the limit when $\Delta t\to 0$ (or $t_2\to t_1$), we get the *instantaneous velocity at time* t_1:

$$\mathbf{v}(t_1)=\lim_{\Delta t\to 0}\frac{\Delta\mathbf{r}(t)}{\Delta t}=\dot{\mathbf{r}}(t_1)=\dot{x}(t_1)\mathbf{e}_x+\dot{y}(t_1)\mathbf{e}_y+\dot{z}(t_1)\mathbf{e}_z$$

Definition 1.3. The *speed* of a particle moving at a velocity $\mathbf{v}(t)$ is:

$$v(t)=\|\mathbf{v}(t)\|$$

Definition 1.4. Consider a particle moving at a velocity $\mathbf{v}(t)$. Then the *average acceleration over any time interval* $\Delta t=t_2-t_1$ is:

$$\mathbf{a}_{\text{avg}}=\frac{\Delta\mathbf{v}(t)}{\Delta t}=\frac{\mathbf{v}(t_2)-\mathbf{v}(t_1)}{t_2-t_1}$$

If we take the limit when $\Delta t\to 0$ (or $t_2\to t_1$), we get the *instantaneous acceleration at time* t_1:

$$\mathbf{a}(t_1)=\lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}(t)}{\Delta t}=\ddot{\mathbf{r}}(t_1)=\ddot{x}(t_1)\mathbf{e}_x+\ddot{y}(t_1)\mathbf{e}_y+\ddot{z}(t_1)\mathbf{e}_z$$

Proposition 1.5 (Uniform linear motion). Consider a particle moving at a constant speed v along a straight line. If at time $t=0$ it is in the position x_0, then:

$$x(t)=x_0+vt$$

Proposition 1.6 (Accelerated linear motion). Consider a particle moving at a constant acceleration a along a straight line. If at time $t=0$ it is in the position x_0 with velocity v_0, then:

$$\dot{x}(t)=v_0+at\qquad x(t)=x_0+v_0t+\frac{1}{2}at^2$$

Definition 1.7. Suppose a particle is at Cartesian coordinates (x,y) and at polar coordinates (r,φ). Then, polar unit vectors are defined as:

$$\begin{aligned}\mathbf{e}_r&=\cos\varphi\mathbf{e}_x+\sin\varphi\mathbf{e}_y\\ \mathbf{e}_\varphi&=-\sin\varphi\mathbf{e}_x+\cos\varphi\mathbf{e}_y\end{aligned}$$

Definition 1.8. The equations of the circular movement are the following:

$$\mathbf{r}(t)=r\mathbf{e}_r\quad\dot{\mathbf{r}}(t)=r\dot{\varphi}(t)\mathbf{e}_\varphi\quad\ddot{\mathbf{r}}(t)=r\ddot{\varphi}(t)\mathbf{e}_\varphi-r\dot{\varphi}(t)^2\mathbf{e}_r$$

where we have supposed that r is constant. We define the *angular velocity* $\omega(t)$ as $\omega(t):=\dot{\varphi}(t)$ and the *angular acceleration* $\alpha(t)$ as $\alpha(t):=\ddot{\varphi}(t)$. The first term of $\ddot{\mathbf{r}}(t)$ is called *tangential acceleration* and its magnitude is $a_t:=r\alpha$. The second term is called *normal acceleration* and its magnitude is $a_n:=r\omega^2$.

Definition 1.9. Consider a particle moving along the trajectory $\mathbf{r}(t)$. We define Frenet vectors as:

1. First Frenet vector: $\mathbf{e}_1(t)=\dfrac{\dot{\mathbf{r}}(t)}{\|\dot{\mathbf{r}}(t)\|}$
2. Second Frenet vector: $\mathbf{e}_2(t)=\dfrac{\dot{\mathbf{e}}_1(t)}{\|\dot{\mathbf{e}}_1(t)\|}$

Note that the first vector is tangent to the trajectory at each point and the second one is normal to the trajectory at each point.
From this definition we have:

$$\dot{\mathbf{r}}(t)=v(t)\mathbf{e}_1\qquad\ddot{\mathbf{r}}(t)=a_t(t)\mathbf{e}_1+a_n(t)\mathbf{e}_2(t)$$

We also define the *curvature* $\kappa(t)$ and *radius of curvature* $R(t)$ as:

$$\frac{1}{\kappa(t)}=R(t):=\frac{\|\dot{\mathbf{r}}(t)\|}{\|\dot{\mathbf{e}}_1(t)\|}$$

Finally, the normal acceleration is:

$$a_n(t)=\frac{v(t)^2}{R(t)}$$

Proposition 1.10 (Curvature). Consider a particle moving along a two-dimensional trajectory and let $\Delta\varphi$ be the angle the trajectory has curved when traveling a distance Δs. Then the *average curvature along* Δs is:

$$\kappa_{\text{avg}}=\frac{\Delta\varphi}{\Delta s}$$

If we take the limit when $\Delta s\to 0$ we have:

$$\kappa=\lim_{\Delta s\to 0}\frac{\Delta\varphi}{\Delta s}=\frac{\mathrm{d}\varphi}{\mathrm{d}s}$$

Proposition 1.11 (Arc length). The total distance traveled by a particle moving along a curve $\mathbf{r}(t)$ between the instants t_1 and t_2 is:

$$\int_{t_1}^{t_2}\|\dot{\mathbf{r}}(t)\|\mathrm{d}t$$

Proposition 1.12 (Projectile motion). The equations of a projectile motion like the one in figure 1 are:

$$\begin{aligned}x(t)&=x_0+v_0\cos\theta t\qquad & y(t)&=y_0+v_0\sin\theta t-\frac{1}{2}gt^2\\ v_x(t)&=v_0\cos\theta & v_y(t)&=v_0\sin\theta-gt\end{aligned}$$

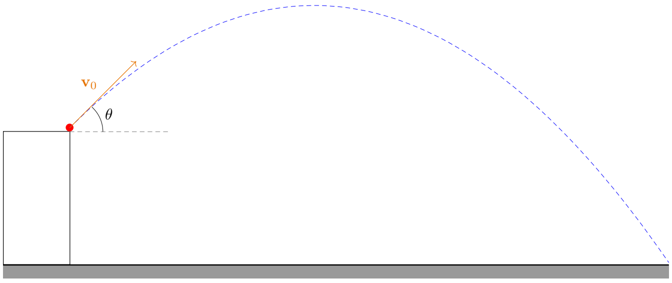

Figure 1

Dynamics

Law 1.13 (Newton's laws).

1. An object at rest will stay at rest and an object in motion will stay in motion unless acted on by a net external force. That is:

$$\sum \mathbf{F} = 0 \iff \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 0$$

2. The rate of change of momentum of a body over time is directly proportional to the force applied, and occurs in the same direction as the applied force. That is:

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

3. If one object A exerts a force $\mathbf{F}_A$ on a second object B, then B simultaneously exerts a force $\mathbf{F}_B$ on A and the two forces are equal in magnitude and opposite in direction:

$$\mathbf{F}_A = -\mathbf{F}_B$$

Proposition 1.14 (Gravity force). Any two object with mass m_1 and m_2 exerts an attracting force called *gravity*:

$$\mathbf{F}_{21} = -G\frac{m_1 m_2}{|\mathbf{r}_{12}|^3}\mathbf{r}_{12}$$

where $\mathbf{F}_{21}$ is the force applied on object 2 exerted by object 1, $\mathbf{r}_{12}$ is the vector distance from object 1 to object 2.

Proposition 1.15 (Elastic force). Consider an object attached to a string of natural length x_0 as shown in the figure 2.

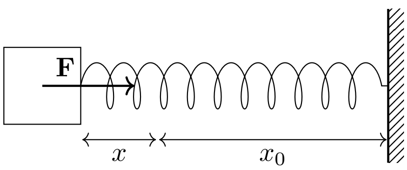

Figure 2

If we displace the object a distance of x from its equilibrium position, the resulting elastic force is:

$$\mathbf{F} = -k\mathbf{x}$$

where k is the spring constant. Moreover, ignoring the friction, the mass starts to oscillate and this oscillation have the following equations:

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi) \\ \dot{x}(t) &= -\omega A \sin(\omega t + \phi) \\ \ddot{x}(t) &= -\omega^2 A\cos(\omega t + \phi) = -\omega^2 x(t) \end{aligned}$$

$$\omega = \sqrt{\frac{k}{m}} \qquad T = \frac{2\pi}{\omega} \qquad \nu = \frac{1}{T}$$

where A is the amplitude, ϕ is the initial phase, ω is the angular frequency, T is the period and ν is the frequency.

Proposition 1.16. Consider an object on a surface that undergo a normal force $\mathbf{F}_N$ and it is pulled by a net force of magnitude F. Then the magnitude of the frictional force is:

$$F_f = \begin{cases} F & \text{if} \quad F \leq \mu_s F_N \\ \mu_k F_N & \text{if} \quad F > \mu_s F_N \end{cases}$$

where μ_s is the *static coefficient of friction* and μ_k is the *kinetic coefficient of friction.*

Proposition 1.17 (Inertial forces). Consider two general reference frames $\mathcal{R}$ and $\mathcal{R}'$ (separated by $\mathbf{R}(t)$) and suppose that we observe a particle of mass m at position $\mathbf{r}(t)$ from $\mathcal{R}$ and at position $\mathbf{r}'(t)$ from $\mathcal{R}'$, as shown in the figure:

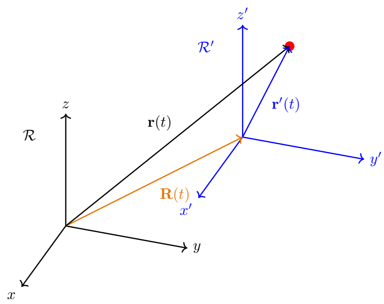

Figure 3

Then for a general $\mathbf{R}(t)$ we have $\mathbf{r}'(t) = \mathbf{r}(t) - \mathbf{R}(t)$ and therefore $\ddot{\mathbf{r}}'(t) = \ddot{\mathbf{r}}(t) - \ddot{\mathbf{R}}(t)$. If we assume that $\mathcal{R}$ is inertial, then

$$\mathbf{F}(t) - m\ddot{\mathbf{R}}(t) = m\ddot{\mathbf{r}}'(t)$$

If $\mathcal{R}'$ is not inertial, Newton's second law is not satisfied. In this case, we denote the term $-m\ddot{\mathbf{R}}(t)$ as an *inertial force* or *fictitious force*: $\mathbf{F}_{\text{iner}}(t) := -m\ddot{\mathbf{R}}(t)$[1].

Proposition 1.18 (Galilean transformation). Consider two reference frames $\mathcal{R}$ and $\mathcal{R}'$. Using the previous notation, suppose $\mathbf{R}(t) = Vt\mathbf{e}_x$. Then:

$$\begin{aligned} x' &= x - Vt & \qquad v'_x &= v_x - V \\ y' &= y & v'_y &= v_y \\ z' &= z & v'_z &= v_z \end{aligned}$$

$$t' = t$$

[1]Note that if $\mathcal{R}'$ is inertial, Newton's second law is still satisfied because $\mathbf{R}(t) = \mathbf{V}t$ and therefore $-m\ddot{\mathbf{R}}(t) = 0$.

Statics

Definition 1.19 (Linear momentum of a particle). Consider a particle of mass m moving at a velocity of $\mathbf{v}$. We define its *linear momentum* as:

$$\mathbf{p} = m\mathbf{v}$$

Proposition 1.20 (Linear momentum of a system of particles). Consider a system of N particles which interact with themselves (internal forces) and also with external forces. The *linear momentum of the system* is:

$$\mathbf{P} = \sum_{a=1}^{N} \mathbf{p}_a$$

Moreover if the net external force is $\mathbf{F}_{\text{ext}}$ we have:

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \mathbf{F}_{\text{ext}}$$

Proposition 1.21 (Center of masses). The *center of masses* (CM) of a system of N particles is:

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{r}_i$$

where $M = \displaystyle\sum_{i=1}^{N} m_i$. Differentiating the last equality we get

$$M\dot{\mathbf{R}} = \mathbf{P} \qquad M\ddot{\mathbf{R}} = \dot{\mathbf{P}} = \mathbf{F}_{\text{ext}}$$

If the mass distribution is continuous with the density $\rho(\mathbf{r})$ within a solid Ω, the center of mass is:

$$\mathbf{R} = \frac{1}{M}\iiint_{\Omega} \rho(\mathbf{r})\mathbf{r}\,\mathrm{d}V$$

where $M = \displaystyle\iiint_{\Omega} \rho(\mathbf{r})\mathrm{d}V$.

Proposition 1.22 (Angular momentum). Consider a particle with linear momentum $\mathbf{p}$ situated at position $\mathbf{r}$ with respect to the origin O. We define its *angular momentum* as:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

The angular momentum of a system of N particles is:

$$\mathbf{L}_{\text{sys}} = \sum_{i=1}^{N} \mathbf{L}_i$$

Proposition 1.23 (Torque). Consider a particle at position $\mathbf{r}$ with respect to the origin O and let $\mathbf{F}$ be a force acting on the particle. We define the *torque* as:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

The torque of a system of N particles is:

$$\boldsymbol{\tau}_{\text{ext}} = \sum_{i=1}^{N} \boldsymbol{\tau}_i$$

Proposition 1.24. Relating the torque and angular momentum of a particle and a system of particles we have:

$$\dot{\mathbf{L}} = \boldsymbol{\tau} \qquad \dot{\mathbf{L}}_{\text{sys}} = \boldsymbol{\tau}_{\text{ext}}$$

Therefore, if $\boldsymbol{\tau}_{\text{ext}} = 0$, then $\mathbf{L}_{\text{sys}} = \text{const.}$

Definition 1.25 (Mechanical equilibrium). The conditions of mechanical equilibrium are:

$$\mathbf{F}_{\text{ext}} = 0 \quad \text{and} \quad \boldsymbol{\tau}_{\text{ext}} = 0$$

Work and energy

Definition 1.26 (Work). The *work* of a constant force $\mathbf{F}$ acting on a particle that moves throughout a straight distance $\Delta\mathbf{r}$ is:

$$W = \mathbf{F} \cdot \Delta\mathbf{r}$$

If the force is not necessary constant and the particle moves along a curve c, we have:

$$W = \int_c \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Definition 1.27 (Power). The *power* is defined as

$$P = \frac{\mathrm{d}W}{\mathrm{d}t}$$

If ΔW is the amount of work performed during a period of time of duration Δt, the *average power* is:

$$P = \frac{\Delta W}{\Delta t}$$

From the first definition we can deduce the following general formula:

$$P = \mathbf{F} \cdot \mathbf{v}$$

Definition 1.28 (Kinetic energy). The *kinetic energy* of a particle of mass m moving at a speed v is:

$$K = \frac{1}{2}mv^2$$

Theorem 1.29. The total work done on a particle is:

$$W = \Delta K$$

Definition 1.30 (Conservative forces). A force is *conservative* if for any path c connecting points A and B, the work necessary to move a particle from A to B does not depend on c.

Proposition 1.31. The work done by a conservative force can be expressed as a variation of a function called *potential energy.*

Proposition 1.32 (Potential energy). If a force $\mathbf{F}$ is conservative, we define the potential energy as:

$$U(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{r}_0$ is a reference point and can be chosen arbitrarily. It can be easily seen that:

$$W = -\Delta U$$

Proposition 1.33 (Mechanical energy). The mechanical energy of a particle (with kinetic energy K) subjected to a conservative force of potential energy U is:

$$E = K + U$$

Theorem 1.34 (Conservation of mechanical energy). For a particle subjected to a conservative force we have:

$$\Delta E = 0$$

That is, E is constant. If there are non-conservative forces acting on the particle we have:

$$\Delta E = W_{\text{nc}}$$

where W_{nc} is the work done by non-conservative forces.

Proposition 1.35 (Examples of potential energies).

1. Elastic potential energy of a spring:

   $$U = \frac{1}{2}kx^2$$

   where x is the distance the spring has been stretched.

2. Gravitational potential energy of a solid of mass m:

   $$U = -\frac{GM_T m}{r}$$

   where M_T is the Earth mass, r is the distance form the center of the earth to the position of the solid and G is the gravitational constant. Note that if $r = R_T + h$, $h > 0$ and $\frac{r}{R_T} = 1 + \frac{h}{R_T} \approx 1$, then:

   $$U = mgh$$

   where R_T is the radius of earth and g is the surface gravity.

Rotation

Definition 1.36. Consider a system of N particles that spin around a reference axis at an angular velocity $\boldsymbol{\omega}$. The *moment of inertia* I with respect to the axis is:

$$I = \sum_{i=1}^{N} m_i {r_i}^2$$

where m_i is the mass of the i-th particle and r_i is the distance between that particle and the axis. Moreover we have:

$$\mathbf{L}_{\text{sys}} = I\boldsymbol{\omega}$$

Proposition 1.37. For a rigid body of moment of inertia I that spins around a reference axis at an angular velocity $\boldsymbol{\omega}$ we have:

$$\boldsymbol{\tau}_{\text{ext}} = I\dot{\boldsymbol{\omega}} = I\boldsymbol{\alpha}$$

Proposition 1.38. Consider a system of particles whose CM is at a distance $\mathbf{R}(t)$ from a fixed point O. If $\mathbf{P}$ is the linear momentum of the CM, we have:

$$\mathbf{L}_O = \mathbf{L}_{\text{CM}} + \mathbf{R} \times \mathbf{P}$$

where $\mathbf{L}_O$ is the angular momentum of the system with respect to the point O and $\mathbf{L}_{\text{CM}}$ is the angular momentum of the system with respect to the CM. Moreover if $\mathbf{F}_{\text{ext}}$ is the total external force applied onto the system, $\boldsymbol{\tau}_{\text{O,ext}}$ is the torque done by the forces with respect to the point O and $\boldsymbol{\tau}_{\text{CM,ext}}$ is the torque done by the forces with respect to the CM, we have:

$$\boldsymbol{\tau}_{\text{O,ext}} = \boldsymbol{\tau}_{\text{CM,ext}} + \mathbf{R} \times \mathbf{F}_{\text{ext}}$$

Finally, we deduce:

$$\dot{\mathbf{L}}_{CM} = \boldsymbol{\tau}_{\text{CM,ext}}$$

Proposition 1.39. Consider a system of particles with total mass M. Suppose the moment of inertia of the system with respect to the CM is I_{CM} and that the speed of the CM is V. If the angular velocity of the system around the CM is ω, the kinetic energy of rotation will be:

$$K = \frac{1}{2}MV^2 + \frac{1}{2}I_{\text{CM}}\omega^2$$

Theorem 1.40 (Parallel axis theorem). Consider a body of mass m that is rotating around an axis that passes through the body's center of mass. Let I_{CM} be the moment of inertia with respect of that axis. Suppose there is another axis parallel to the previous one and separated each other a distance of d. Then, the moment of inertia of the body with respect to this latter axis I will be:

$$I = I_{\text{CM}} + md^2$$

2 | Special relativity

Definition 1.41. A *inertial frame of reference* is a frame of reference in which a particle remains at rest or in uniform linear motion.

Principle 1.42 (First postulate). The laws of physics take the same form in all inertial frames of reference.

Principle 1.43 (Second postulate). The speed of light, c, is a constant, independent of the relative motion of the source.

Definition 1.44 (Lorentz factor). For an object moving at speed v, *Lorentz factor* is defined as:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

where $\beta = v/c$.

Proposition 1.45 (Time dilation). Consider two frames of reference in uniform relative motion with velocity v such that one of them has a clock. If Δt_0 is the time interval between two events made in the same location and measured in the frame in which the clock is at rest (*proper time*), then the time measured by the other frame is:

$$\Delta t = \gamma \Delta t_0$$

Proposition 1.46 (Length contraction). Consider two frames of reference in uniform relative motion with velocity v such that one of them has an object. If L_0 is length of the object measured instantaneously in the frame in which the object is at rest (*proper length*), then the length measured by the other frame is:

$$L = \frac{L_0}{\gamma}$$

Proposition 1.47 (Lorentz transformations). Consider coordinates (x, y, z, t) and (x', y', z', t') of a single arbitrary event measured in two coordinate systems S and S', in uniform relative motion (S' is moving at velocity $\mathbf{v} = (v, 0, 0)$ with respect to S) in their common x and x' directions and with their spatial origins coinciding at time $t = t' = 0$. Then:

$$\begin{aligned} x' &= \gamma(x - \beta ct) & x &= \gamma(x' + \beta ct') \\ y' &= y & y &= y' \\ z' &= z & z &= z' \\ ct' &= \gamma(ct - \beta x) & ct &= \gamma(ct' + \beta x') \end{aligned}$$

Proposition 1.48 (Lorentz transformations of velocities). In a situation similar to the previous one, if an object is moving at a velocity $\mathbf{u} = (u_x, u_y, u_z)$ in S and $\mathbf{u}' = (u'_x, u'_y, u'_z)$ in S', we have:

$$\begin{aligned} u'_x &= \frac{u_x - v}{1 - u_x v/c^2} & u_x &= \frac{u'_x + v}{1 + u'_x v/c^2} \\ u'_y &= \frac{u_y}{\gamma\,(1 - u_x v/c^2)} & u_y &= \frac{u'_y}{\gamma\,(1 + u_x v/c^2)} \\ u'_z &= \frac{u_z}{\gamma\,(1 - u_x v/c^2)} & u_z &= \frac{u'_z}{\gamma\,(1 + u_x v/c^2)} \end{aligned}$$

Proposition 1.49 (Matrix form of Lorentz transformations). We can write the Lorentz transformations as:

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

If

$$\Lambda := \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}, \text{ then } \Lambda^{-1} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}$$

and we obtain the inverse transformations.

Proposition 1.50 (Lorentz invariant). The factor s^2, defined as follows, is invariant in any inertial frame of reference.

$$s^2 = c^2t^2 - x^2 - y^2 - z^2$$

Proposition 1.51 (Types of events). There are three types of events: *timelike*, *lightlike* and *spacelike*.

- $s^2 > 0 \implies$ *timelike*
- $s^2 = 0 \implies$ *lightlike*
- $s^2 < 0 \implies$ *spacelike*

Timelike and lightlike events are in causal relation with the origin (that is, it is possible to send a light signal from the origin to the point or vice versa), while *spacelike* events are not.

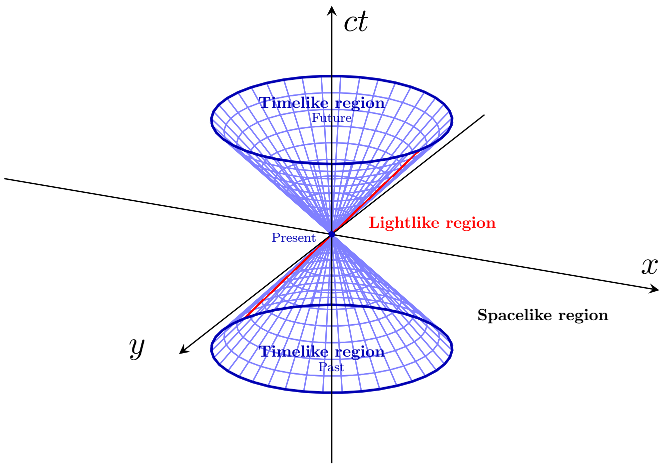

Figure 4: Minkowski diagram

Proposition 1.52 (Relativistic Doppler effect). Suppose a frame of reference where the receiver is at rest and the source is moving at speed β forming an angle ϕ with the light direction (measured in receiver frame). Then:

$$\nu_R = \frac{\nu_S}{\gamma(1 - \beta\cos\phi)} \tag{1}$$

$$\lambda_R = \gamma\lambda_S(1 - \beta\cos\phi) \tag{2}$$

where ν_S is the frequency measured by the source and ν_R is the frequency measured by the receiver, and analogously with wavelengths λ_S and λ_R.
Relation between the angles ϕ and ϕ', where ϕ' is the angle between the velocity and the light direction measured in source frame:

$$\tan\frac{\phi'}{2} = \sqrt{\frac{1+\beta}{1-\beta}}\tan\frac{\phi}{2}$$

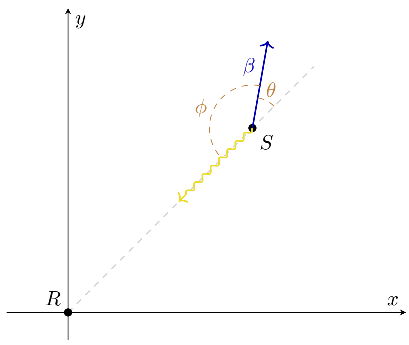

Figure 5: General case of Doppler effect

Corollary 1.53. There are three important cases to consider:

- The source moves away, that is making $\phi = \pi$ in equation (1) (*Redshift*):
$$\nu_R = \nu_S\sqrt{\frac{1-\beta}{1+\beta}}$$
- The source gets close, that is making $\phi = 0$ in equation (1) (*Blueshift*):
$$\nu_R = \nu_S\sqrt{\frac{1+\beta}{1-\beta}}$$

- The source moves transversely, that is making $\phi = \pi/2$ in equation (1):

$$\nu_R = \nu_S/\gamma$$

Proposition 1.54 (Relativistic mass). If m_0 is the mass of an object at rest, then the mass of an object at a velocity β is:

$$m = \gamma m_0$$

The mass m_0 is invariant.

Proposition 1.55 (Relativistic momentum). The relativistic momentum for a particle with mass at rest m_0 and moving at a velocity of $\mathbf{v}$ is given by:

$$\mathbf{p} = \gamma m_0 \mathbf{v}$$

Proposition 1.56 (Relativistic energy). The relativistic energy of a particle is:

$$E = mc^2 = \gamma m_0 c^2$$

On the other hand, $E = K + m_0c^2$, where K is the kinetic energy of a particle and m_0c^2 its rest energy. Moreover we can express the energy of a particle in terms of its momentum:

$$E = mc^2 = \sqrt{p^2c^2 + m_0^2c^4}$$

Proposition 1.57 (Photon energy and momentum). For a photon of frequency ν, energy E and linear momentum p, we have:

$$E = h\nu \qquad p = \frac{h\nu}{c}$$

Proposition 1.58 (Lorentz transformations of energy and momentum). Consider a particle that have energy E and momentum $\mathbf{p} = (p_x, p_y, p_z)$ in a frame of reference S and have energy E' and momentum $\mathbf{p}' = (p'_x, p'_y, p'_z)$ in frame of reference S'. These frames are in uniform relative motion (S' is moving at velocity $\mathbf{v} = (v, 0, 0)$ with respect to S) and their spatial origins coincide at time $t = t' = 0$. Then:

$$\begin{aligned} E' &= \gamma(E - \beta c p_x) & E &= \gamma(E' + \beta c p'_x) \\ cp'_x &= \gamma(cp_x - \beta E) & cp_x &= \gamma(cp'_x + \beta E') \\ p'_y &= p_y & p_y &= p'_y \\ p'_z &= p_z & p_z &= p'_z \end{aligned}$$

Proposition 1.59 (Compton scattering). Consider a photon with wavelength λ colliding with a particle at rest of mass m_0 (usually an electron). As a result of the collision, the photon energy decrease and therefore its wavelength increase (let's say the scattered photon has wavelength λ'). If the scattered photon is moving at an angle θ with respect to initial direction, we have:

$$\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\theta)$$

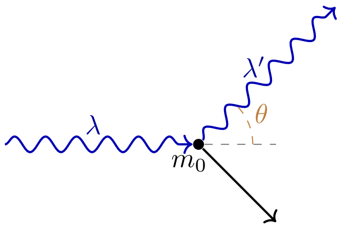

Figure 6: Compton scattering

3 | Fluids

Definition 1.60. A *fluid* is a substance that continually flows under an applied external force.

Definition 1.61. The *viscosity* of a fluid is a measure of its resistance to deformation at a given rate. We say a fluid is *ideal* if we don't consider viscosity.

Proposition 1.62 (Density). The density of a fluid of mass m that occupies a volume V is:

$$\rho = \frac{m}{V}$$

The density depends on temperature and pressure[2].

Definition 1.63. A fluid is said to be *incompressible* if its density doesn't varies with the pressure.

Proposition 1.64 (Pressure). Consider a point x and a small sphere centered at x. Then, the pressure $p(x)$ at point x is:

$$p(x) = \frac{\sum F_N}{S}$$

where $\sum F_N$ is the sum of normal forces and S is the surface which the forces are applied to. The SI unit of pressure is the Pascal: $1\ \text{Pa} = 1\ \text{N/m}^2$.

Proposition 1.65 (Hydrostatic pressure). Consider a static fluid with constant density ρ and let p_0 be the pressure on its surface. Then, the pressure p on a depth h is

$$p = p_0 + \rho g h$$

Proposition 1.66 (Pascal's principle). Any pressure applied to the surface of a fluid is transmitted uniformly throughout the fluid in all directions, in such a way that initial variations in pressure are not changed.

$$p_1 = \frac{F_1}{S_1} = \frac{F_2}{S_2} = p_2$$

Proposition 1.67 (Archimedes' principle). Any object (of mass m), totally or partially immersed in a fluid of density ρ, is buoyed up by a force equal to the weight of the fluid displaced by the object, that is:

$$F_B := \rho g V_{\text{dis}}$$

where F_E is called the *buoyancy* and V_{dis} is the volume of the liquid displaced[3].

[2] This variation is typically small for solids and liquids but much greater for gases.

[3] Note that if $F_B - mg > 0$, the object rises to the surface of the liquid; if $F_B - mg < 0$, the object sinks, and if $F_B - mg = 0$, the object is neutrally buoyant, that is, it remains in place without either rising or sinking.

Definition 1.68. We define the *discharge of a fluid* as:

$$Q = Sv$$

where S is the cross-sectional area of the portion of the channel occupied by the flow and v is the average flow velocity. If the velocity is not constant, then:

$$Q = \iint_S \mathbf{v} \cdot d\mathbf{S}$$

Proposition 1.69 (Continuity equation). Consider an incompressible fluid moving throughout a channel. Then, the volume per unit of time is conserved, that is, the discharge is conserved. Mathematically:

$$Q_1 = S_1 v_1 = S_2 v_2 = Q_2$$

Definition 1.70. *Laminar flow* is a fluid motion that occurs when a fluid flows in parallel layers, with no disruption between those layers. *Turbulent flow* is a fluid motion characterized by chaotic changes in pressure and flow velocity.

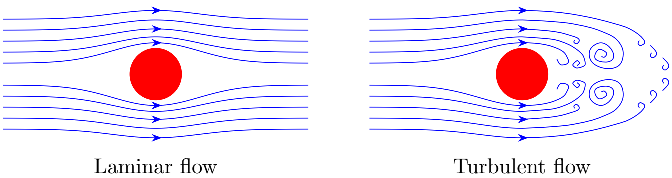

Figure 7

Proposition 1.71 (Bernolli's principle). Consider an incompressible and ideal fluid of density ρ with steady laminar flow. Then:

$$p + \rho g h + \frac{1}{2}\rho v^2 = \text{const.}$$

where p is the pressure at a point on a streamline; h, the elevation of the point from a reference frame, and v, the fluid flow speed at the chosen point.

Proposition 1.72 (Lift force). If the air has density ρ and an object of cross-sectional area S is moving at a velocity of v relative to the air, then the lift force is:

$$F_L = \frac{1}{2} C_L \rho S v^2$$

where C_L is the *lift coefficient*. From that we deduce that the minimum velocity for lifting is:

$$F_L = mg \implies v_{\min} = \sqrt{\frac{2mg}{C_L \rho S}}$$

Proposition 1.73 (Viscosity). Consider a fluid trapped between two plates of area S, one fixed and the other one in parallel motion at constant speed v. If we suppose a laminar flow, each layer of fluid moves faster than the one just below it and so this creates a friction force resisting their relative motion. An external force F is therefore required in order to keep the top plate moving at constant speed. This force is given by:

$$F = \eta \frac{vS}{z}$$

where z is the separation between the plates and η is the viscosity of the fluid ($[\eta] = \text{Pa} \cdot \text{s}$).

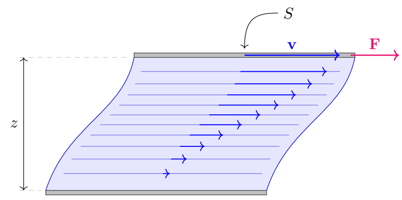

Figure 8

Proposition 1.74 (Velocity of a fluid in a channel). Consider a fluid with viscosity η in laminar flow so that the layer in contact with the wall of the channel (of radius r) is at rest. Let p_1 be the pressure at one point of the channel and p_2 be the pressure at another point separated a distance L along the x-axis from the previous point. Then, the speed of each layer of fluid at a distance x from the center of the channel is:

$$v(y) = \frac{p_1 - p_2}{4\eta L}(r^2 - y^2)$$

The average speed and maximal speed of the fluid are:

$$v_{\text{avg}} = \frac{p_1 - p_2}{8\eta L} r^2 \qquad v_{\max} = \frac{p_1 - p_2}{4\eta L} r^2 \tag{3}$$

Proposition 1.75 (Poiseuille's law). In conditions of the equation (3), we have:

$$Q = S v_{\text{avg}} = \frac{\pi}{8\eta} \frac{p_1 - p_2}{L} r^4 \implies \Delta p = \frac{8\eta}{\pi} \frac{L}{r^4} Q$$

If we denote $R_f := \frac{8\eta}{\pi} \frac{L}{r^4}$ the hydrodynamic resistance, we can write Poiseuille's law as follows:

$$\Delta p = R_f Q,$$

which is an analogy of Ohm's law[4].

Proposition 1.76 (Resistance in fluids). Consider n channels each of resistance R_i. The total resistance will be:

- Connected in series:

$$R_T = \sum_{i=1}^{n} R_i$$

- Connected in parallel:

$$\frac{1}{R_T} = \sum_{i=1}^{n} \frac{1}{R_i}$$

Proposition 1.77 (Dissipated power). Consider a fluid that passes throughout a channel of resistance R_f. If the discharge of the fluid is Q in a section where the pressure difference is Δp, the *dissipated power* will be:

$$P = \Delta p Q = R_f Q^2$$

[4] In that case, R_f would play the role of electric resistance; Q, the role of intensity of the current, and Δp, the role of electric potential difference.

Proposition 1.78 (Drag forces). An object moving at a velocity v in a fluid of density ρ and viscosity η creates drag forces:

- For low speeds and high viscosity, viscous forces predominate:

$$F = k\eta vr$$

  where $k = 6\pi$ if the object is an sphere and r is its radius.

- For high speeds and low viscosity, inertial forces predominate:

$$F = \frac{1}{2}C_a\rho Sv^2$$

  where C_a is the aerodynamic coefficient and S the cross-sectional area.

Proposition 1.79 (Terminal velocity). An object falling (by gravity) inside a fluid attains a maximum velocity (terminal velocity) when its weight equals the drag force. We have two cases to consider:

- For viscous forces:

$$v_t = \frac{mg}{k\eta r}$$

- For inertial forces:

$$v_t = \sqrt{\frac{2mg}{C_a\rho S}}$$

Proposition 1.80 (Reynolds number). The Reynolds number helps to predict flow patterns in different fluid flow situations.

$$\mathrm{Re} = \frac{\rho vD}{\eta} \approx \frac{F_{\text{inertial}}}{F_{\text{viscous}}}$$

where v is the flow speed and D is the diameter of the object.

$$\mathrm{Re} < 2000 \implies \text{laminar flow}$$
$$\mathrm{Re} > 3000 \implies \text{turbulent flow}$$