

Lie group notes

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Contents

1	Lie groups: basic definitions	4
1.1	Lie subgroups and quotients	4
1.2	Homogeneous spaces	5
1.3	Classical Lie groups	6
2	Lie groups and Lie algebras	8
2.1	Exponential map	8
2.2	The commutator	9
2.3	Subalgebras, ideals and center	11
2.4	Lie algebra of vector fields	11
2.5	Stabilizers and the center	12
2.6	Campbell–Hausdorff formula	14
2.7	Fundamental theorems of Lie theory	14
2.8	Complex and real forms	16
3	Representations of Lie groups and Lie algebras	17
3.1	Basic definitions	17
3.2	Operations on representations	19
3.2.1	Subrepresentations and quotients	19
3.2.2	Direct sum and tensor product	19
3.2.3	Invariants	20
3.3	Irreducible representations	20
3.4	Intertwining operators and Schur’s lemma	21
3.5	Complete irreducibility of unitary representations, representations on finite groups	23
3.6	Haar measure on compact Lie groups	23
3.7	Orthogonality of characters and Peter–Weyl theorem	23
A	Covering theory reminder	27
A.1	Lifting properties	27
A.2	Universal covering	28
B	Exercises	30
B.1	Linear algebra exercises	31
C	Geometric realisation	33
D	$SO(d, d)$	34

E	Case studies	35
E.1	\mathbb{R}	35
E.2	\mathbb{S}^1	35
F	Fun facts	36

Conventions

- By Lie group, we mean either real or complex.
- A closed Lie subgroup of a Lie group is a submanifold that is also a subgroup (see Theorem 3).
- A Lie subgroup is a subgroup that is also an immersed submanifold.

Chapter 1

Lie groups: basic definitions

We will have two notions of subgroups of Lie groups:

Definition 1.0.1. Let G be a complex or real Lie group and H be a subgroup of G .

- (i) H is a Lie subgroup if it is an immersed submanifold of G with the multiplication and inverse maps being smooth (or analytic).
- (ii) H is a closed Lie subgroup if it is an embedded submanifold of G .

Example 1.0.1. Any of the classical Lie groups are closed Lie subgroups of $\mathrm{GL}(n, \mathbb{K})$. For a Lie subgroup that is not closed, consider the irrational winding on the torus, that is $G = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and $H = f(\mathbb{R})$ with $f(t, s) = (t \bmod \mathbb{Z}, \alpha s \bmod \mathbb{Z})$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

1.1 Lie subgroups and quotients

The following theorem allows us to reduce the study of Lie groups to the study of finite groups and connected Lie groups, since G_0 is a normal subgroup of G and G/G_0 is a discrete group, where G^0 is the identity component of G .

Theorem 1 (Theorem 2.6, [Kir08]). *Let G be a real or complex Lie group and G^0 its identity component. Then G^0 is a normal subgroup of G and a Lie group itself, while G/G^0 is a discrete group.*

In fact, we can reduce the case of connected Lie groups to simply connected Lie groups:

Theorem 2 (Theorem 2.7, [Kir08]). *Let G be a connected Lie group. Then its universal cover \tilde{G} has a canonical structure of a Lie group such that the covering map $p : \tilde{G} \rightarrow G$ is a homomorphism of Lie groups whose kernel is isomorphic to the fundamental group of G . Moreover, in this case, $\ker p$ is a discrete central subgroup in \tilde{G} .*

Proof. The lifting property of the universal cover implies that \tilde{G} is a Lie group. The kernel is discrete, being the fiber of a covering map. The fact that it is central follows from the more general fact that every discrete normal subgroup of a connected Lie group is central. To show the latter, one considers the map $G \rightarrow N, g \rightarrow gng^{-1}$ where N is the normal subgroup and $n \in N$ is some fixed element. Then the inverse image of any element $n' \in N$ is closed and open in G , so it is all of G in the case where $n' = n$ and empty otherwise. \square

Example 1.1.1. In the case where $G = \mathbb{T}^2$ is the torus, the covering map $p : \mathbb{R}^2 \rightarrow \mathbb{T}^2$ is given by $p(t, s) = e^{2\pi it}, e^{2\pi is}$, and $\ker p = \mathbb{N}^2 \simeq \pi_1(\mathbb{T}^2)$.

We have the following connection between subgroups and Lie subgroups (i.e. subgroups that are also submanifolds):

Theorem 3 (Theorem 2.8, [Kir08]). *(i) Any Lie subgroup of a Lie group is closed in the topology of the ambient group.*

(ii) Any closed subgroup of a Lie group is a real Lie subgroup.

Proof. For the first part of the theorem, we note that \overline{H} is a subgroup of G as well. We claim that H (and thus Hx for every $x \in \overline{H}$) is open and dense in \overline{H} . To see this, note that $e \in \overline{H}$ implies that for every neighborhood $U \subseteq G$ of e in G , $U \cap H$ is nonempty. In particular there exists some open set U in G containing e such that $U \cap H \neq \emptyset$. Then $U \cap H$ will be a neighborhood of e in \overline{H} and H is open because it can be written as the union of all subsets of the form $h(U \cap H)$ for $h \in H$.

To conclude, note that for $x, y \in \overline{H}$, $Hx \cap Hy$ is dense in \overline{H} , so it is nonempty. This implies that $Hx = Hy = H$ and H thus $H = \overline{H}$. \square

1.2 Homogeneous spaces

We begin by describing coset spaces of Lie groups.

Theorem 4 (Theorem 2.11, [Kir08]). *Let G be a Lie group of dimension n , $H \leq G$ a closed Lie subgroup of dimension k .*

(i) The coset space G/H has a natural structure of a manifold of dimension $n - k$ such that the canonical map $p : G \rightarrow G/H$ is a fiber bundle, with fiber diffeomorphic to H . The tangent space at the identity is isomorphic to the quotient space $T_H G/H \simeq T_e G/T_e H$.

(ii) If H is a normal closed Lie subgroup then G/H has a canonical Lie group structure.

The following is the analog of the homomorphism theorem for Lie groups:

Theorem 5 (Theorem 2.5, [Kir08]). *Let $f : G_1 \rightarrow G_2$ be a Lie group morphism.*

(i) $H = \ker f$ is a normal closed Lie subgroup of G_1 and f induces an injective homomorphism $G_1/H \rightarrow G_2$ that is an immersion

(ii) If moreover $\Im f$ is an embedded submanifold, then it is a closed Lie subgroup of G_2 and f induces an isomorphism $G_1/H \rightarrow \Im f$.

Theorem 6 (Theorem 2.20, [Kir08]). *Let G be a Lie group acting on a manifold M , and $m \in M$.*

(i) The stabilizer G_m is a closed Lie subgroup of G , with Lie algebra

$$\mathfrak{h} = \{x \in \mathfrak{g} \mid \rho_*(x)(m) = 0\}.$$

where $\rho : G \rightarrow \text{Diff}(M)$ is the action of G on M .

(ii) The orbit map $g \mapsto g \cdot m$ induces an injective immersion $G/G_m \hookrightarrow \mathcal{O}_m$ whose image coincides with \mathcal{O}_m .

(iii) The orbit \mathcal{O}_m is an immersed submanifold with tangent space $T_m\mathcal{O}_m = \mathfrak{g}/\mathfrak{h}$.

(iv) If the orbit is a submanifold, then the orbit map is a diffeomorphism.

Proof. See [Kir08, Theorem 2.20, Theorem 3.29]. □

The case of one orbit gives rise to G -homogeneous spaces:

Theorem 7 (Theorem 2.22 [Kir08]). *Let M be a G -homogeneous space and $m \in M$. Then the orbit map $G \rightarrow M$ is a fiber bundle over M with fiber G_m .*

The following proposition gives us a way to check whether a Lie group morphism is a covering map:

Proposition 1.2.1 (Lie morphism covering map criterion). *Let $f : G_1 \rightarrow G_2$ be a Lie group morphism such that $f_* : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is an isomorphism. Then f is a covering map and $\ker f$ is a discrete central group.*

Proof. The inverse mapping theorem tells us that there exist neighborhoods U_1, U_2 of the identities of G_1 and G_2 respectively such that $f : U_1 \rightarrow U_2$ is a diffeomorphism. In particular, f is surjective since its image contains U_2 which generates G_2 , being a neighborhood of the identity. Moreover, for every $g \in G_1$ and $z_1, z_2 \in \ker f$:

$$gU_1z_1 \cap gU_2z_2 = \{e_1\} \text{ unless } z_1 = z_2.$$

Indeed, if $guz_1 = gvz_2$ for some $u, v \in U$, then $u = vz_2z_1^{-1}$, so $f(u) = f(v)$. But f being injective on U_1 , this implies $u = v$ and $z_1 = z_2$.

Thus $f(g)U_1$ is an evenly covered neighborhood for every $f(g) \in G_2$.

$$f^{-1}(f(g)U_2) = gU_1 \ker f = \bigsqcup_{z \in \ker f} gU_1z.$$

To see that $\ker f$ is discrete, note that the injectivity of f implies $U_1 \cap \ker f = \{e_1\}$. Being a discrete normal subgroup, it must also be central. □

1.3 Classical Lie groups

Definition 1.3.1. We define

$$Sp(n, \mathbb{K}) = \{A \in GL(2n, \mathbb{K}) \mid \omega(Ax, Ay) = \omega(x, y)\}.$$

where ω is the standard symplectic form on \mathbb{K}^{2n} , which is given by $\omega(x, y) = x^* J y = \sum_{i=1}^n x_i y_{i+n} - x_{i+n} y_i$, and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

We also define the group of unitary quaternionic transformations by

$$Sp(n) = Sp(n, \mathbb{C}) \cap SU(2n).$$

The following theorem tells us that the logarithmic map behaves well when restricted to a neighborhood of the identity in each classical group.

G	$O(n, \mathbb{R})$	$SO(n, \mathbb{R})$	$U(n)$	$SU(n)$	$Sp(n)$
\mathfrak{g}	$x^t = -x$	$x^t = -x$	$x^* = -x$	$x^* = -x, \text{tr } x = 0$	$J^{-1}x^*J = -x, x^* = -x$
$\dim G$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	n^2	$n^2 - 1$	$n(2n + 1)$
$\pi_0(G)$	\mathbb{Z}_2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}_2 \ (n \geq 3)$	$\mathbb{Z}_2 \ (n \geq 3)$	\mathbb{Z}	$\{1\}$	$\{1\}$
$\mathfrak{g}_{\mathbb{C}}$			$\mathfrak{gl}(n, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C})$	

Table 1.1: Compact classical groups

G	$GL(n, \mathbb{R})$	$SL(n, \mathbb{R})$	$Sp(n, \mathbb{R})$
\mathfrak{g}	$\mathfrak{gl}(n, \mathbb{R})$	$\text{tr } x = 0$	$x + J^{-1}x'J = 0$
$\dim G$	n^2	$n^2 - 1$	$n(2n + 1)$
$\pi_0(G)$	\mathbb{Z}_2	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}_2 \ (n \geq 3)$	$\mathbb{Z}_2 \ (n \geq 3)$	\mathbb{Z}
$\mathfrak{g}_{\mathbb{C}}$		$\mathfrak{sl}(n, \mathbb{C})$	

Table 1.2: Noncompact real classical groups.

G	$GL(n, \mathbb{C})$	$SL(n, \mathbb{C})$	$O(n, \mathbb{C})$	$SO(n, \mathbb{C})$
$\pi_0(G)$	$\{1\}$	$\{1\}$	\mathbb{Z}_2	$\{1\}$
$\pi_1(G)$	\mathbb{Z}	$\{1\}$	\mathbb{Z}_2	\mathbb{Z}_2

Table 1.3: Complex classical groups.

Chapter 2

Lie groups and Lie algebras

2.1 Exponential map

Definition 2.1.1 (Proposition 3.1, [Kir08]). Let G be a Lie group and $x \in \mathfrak{g}$. The one-parameter subgroup $\gamma_x : \mathbb{K} \rightarrow G$ is the unique Lie group morphism such that $\gamma'_x(0) = x$. We define the exponential map of G as

$$\exp(x) = \gamma_x(1)$$

Remark 2.1.1. By looking at the proof of the statements in the above definition, one can see that for $x \in \mathfrak{g}$, the curve

$$\exp(tx) = \gamma_x(t) = \gamma_{tx}(1).$$

is an integral curve of the left-invariant vector field $X \in \mathcal{X}(G)$ that satisfies

$$X_e = x.$$

The following are some properties of the exponential map:

Theorem 8 (Theorems 3.7 and 3.36, [Kir08]). *Let G be a Lie group.*

- (i) $d_e \exp = \text{id}_{\mathfrak{g}}$
- (ii) *The exponential map is a local diffeomorphism at 0.*
- (iii) *For any Lie group morphism $\phi : G_1 \rightarrow G_2$, we have $d_e \phi(\exp(x)) = \exp d_e(\phi(x))$ for all $x \in \mathfrak{g}$.*
- (iv) *For any $g \in G, x \in \mathfrak{g}$*
$$g \exp(x) g^{-1} = \exp(\text{Ad}_g x).$$
- (v) *For $x, y \in \mathfrak{g}$, we have:*

$$\text{If } [x, y] = 0 \text{ then } e^x e^y = e^y e^x = e^{x+y}.$$

Remark 2.1.2. The exponential map is not surjective in general. It is however for compact Lie groups.

Lemma 2.1.1. *The exponential map of a connected Lie group G sends generators of \mathfrak{g} to generators of G . That is, if $\{x_1, \dots, x_n\}$ is a basis of \mathfrak{g} , then $\{\exp(tx_i) : t \in \mathbb{R}, i \in \llbracket 1, n \rrbracket\}$ is a basis of G .*

Proof. Consider the map $f : \mathbb{R}^n \rightarrow G$ given by

$$f(t_1, \dots, t_n) = \exp(t_1 x_1) \cdots \exp(t_n x_n).$$

Then $d_0 f = I_n$ in the basis with respect to the standard basis for \mathbb{R}^n and $\{x_1, \dots, x_n\}$ for \mathfrak{g} . In particular it is surjective and by the constant rank theorem, $f(\mathbb{R}^n)$ contains a neighborhood of the identity of G . \square

Example 2.1.1. Let $G = SO(3, \mathbb{R})$. Then $\mathfrak{so}(3, \mathbb{R})$ consists of skew-symmetric matrices, with basis:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The exponential matrix is given by

$$e^{tJ_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix},$$

i.e. rotation around x-axis by angle t ; similarly, J_y, J_z generate rotations around y, z axes. Elements of the form $\exp(tJ_x), \exp(tJ_y), \exp(tJ_z)$ generate a neighborhood of identity in $SO(3, \mathbb{R})$. Since $SO(3, \mathbb{R})$ is connected, these elements generate the whole group. For this reason, it is common to refer to J_x, J_y, J_z as “infinitesimal generators” of $SO(3, \mathbb{R})$. Thus, in a certain sense $SO(3, \mathbb{R})$ is generated by three elements.

Remark 2.1.3. To motivate the term “infinitesimal generators”, one can think of them as directions in which one can move from the identity (using the exponential map) in order to generate the whole group.

2.2 The commutator

In literature, one has at least three (and four in the case of vector fields) ways to define the commutator of some Lie group, Namely

- (i) Using left-invariant vector fields (see [Lee18]).
- (ii) As the lowest order term in the logarithm of the multiplication of exponentials (see Lemma 2.2.1).
- (iii) As the differential of the adjoint representation.
- (iv) In the case of vector fields, as the differentiation of the second vector field along the first.

For completeness, we recall that in [Lee18], the commutator of a Lie group is defined as

$$[x, y] = [X, Y]_e \text{ for } x, y \in \mathfrak{g}$$

where X, Y are the left-invariant vector fields for which $X_e = x, Y_e = y$, and the bracket of two vector fields is defined as $[X, Y] = XY - YX$.

In [Kir08], the commutator is defined as the lowest order term in the logarithm of the multiplication of exponentials (see Lemma 2.2.1). We say that a map $Q : \mathfrak{g} \rightarrow \mathfrak{g}$ is of order k if $Q(tx) = t^k Q(X)$ for $t \in \mathbb{R}, x \in \mathfrak{g}$.

Lemma 2.2.1 (Lemma 3.11, [Kir08]). *Let G be a Lie group. Then there exists a neighborhood of $0 \in \mathfrak{g}$ and smooth (or analytic in the complex case) functions $[\cdot, \cdot], \mu : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ such that in that neighborhood:*

$$e^x e^y = e^{\mu(x, y)}$$

$$\mu(x, y) = x + y + \frac{1}{2}[x, y] + \cdots$$

with $[\cdot, \cdot]$ being a bilinear skew-symmetric form and the dots denoting terms of order higher than 3.

The commutator has the following properties:

Proposition 2.2.1 (Proposition 3.12, [Kir08]). *Let G, H be Lie groups and $\phi : G \rightarrow H$ a Lie group morphism. Then for $x, y \in \mathfrak{g}$ we have*

- (i) $\phi_*[x, y] = [\phi_*x, \phi_*y]$.
- (ii) $\text{Ad}_g[x, y] = [\text{Ad}_g x, \text{Ad}_g y]$.
- (iii) $e^x e^y e^{-x} e^{-y} = e^{[x, y] + \cdots}$, where dots stand for degrees higher than 2.
- (iv) $\text{ad}_x y = [x, y]$, where $\text{ad} = d_e \text{Ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$.
- (v) $\text{Ad}_{e^x} = e^{\text{ad}_x} \in \mathfrak{gl}(\mathfrak{g})$

For instance when G is commutative, the commutator is zero (and the exponential map is a homomorphism).

Example 2.2.1. (i) For $\mathfrak{g} = \mathfrak{gl}(d, \mathbb{K})$, the commutator is given by

$$[x, y] = xy - yx.$$

- (ii) For a general associative algebra A , the commutator over \mathcal{K} , is given by the same formula.
- (iii) Any vector space can be made into a commutative Lie algebra by defining the commutator to be zero.

Theorem 9 (Jacobi identity). *Let G be a real or complex Lie group and $x, y, z \in \mathfrak{g}$. Then*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Proof. See [Kir08, Theorem 3.16]. □

Proposition 2.2.2. *Differentiation at the identity induces a map*

$$\text{hom}(G_1, G_2) \rightarrow \text{hom}(\mathfrak{g}_1, \mathfrak{g}_2)$$

which is injective when G_1 is connected.

2.3 Subalgebras, ideals and center

Definition 2.3.1. A subalgebra of a Lie algebra \mathfrak{g} is a subspace $\mathfrak{h} \subseteq \mathfrak{g}$ that is closed under the Lie bracket: $[x, y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$. An ideal is a subalgebra \mathfrak{h} such that $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{h}, y \in \mathfrak{g}$.

It is easy to see that one can take quotients of Lie algebras by ideals and thus obtain Lie algebras.

The next theorem tells us that Lie subgroups correspond to subalgebras, and normal closed Lie subgroups correspond to ideals. In the case of the latter, a converse statement gives us a way to check if a subgroup is normal by checking whether the corresponding subalgebra is an ideal. In this way we obtain a link between an algebraic condition (being a normal subgroup) and a linear one (being an ideal).

Theorem 10 (Theorem 3.22, [Kir08]). *Let G be a real or complex Lie group with Lie algebra \mathfrak{g} . Then*

- (i) *If H is a Lie subgroup (not necessarily closed) of G , then \mathfrak{h} is a subalgebra of \mathfrak{g} .*
- (ii) *If H is a normal closed Lie subgroup of G , then \mathfrak{h} is an ideal of \mathfrak{g} .*
- (iii) *If H is a closed Lie subgroup of G , both G and H are connected, then \mathfrak{h} is an ideal of \mathfrak{g} if and only if H is a normal subgroup of G .*

2.4 Lie algebra of vector fields

For a manifold M , the space $\text{Vect}(M)$ can be made into a Lie algebra by directly defining the bracket of two vector fields and showing that it is skew-symmetric and bilinear. However, one can think of $\text{Diff}(M)$ as similar to a Lie group (but not quite, since it is infinite-dimensional), whose Lie algebra is $\text{Vect}(M)$.

Definition 2.4.1. Let M be a smooth manifold and G be a Lie group.

- (i) A smooth map $\rho : G \rightarrow \text{Diff}(M)$ is a map that arises from a smooth action of G on M .
- (ii) The Lie algebra of $\text{Diff}(M)$ is the space of vector fields $\text{Vect}(M)$.
- (iii) The exponential map of $\text{Diff}(M)$ is the flow (whenever well-defined) of a vector field: $(e^X)_m = \Phi_X^1(m)$.
- (iv) The differential of $\rho : G \rightarrow \text{Diff}(M)$ is the map $\rho_* : \mathfrak{g} \rightarrow \text{Vect}(M)$ given by

$$(\rho_*(x))_m = \left. \frac{d}{dt} \right|_{t=0} \rho(e^{tx})(m).$$

- (v) The commutator $[\xi, \eta]$ of two vector fields $\xi, \eta \in \text{Vect}(M)$ is the unique vector field that satisfies

$$\Phi_\xi^t \Phi_\eta^s \Phi_\xi^{-t} \Phi_\eta^{-s} = \Phi_{[\xi, \eta]}^{ts} + \dots$$

where the dots stand for terms of order 3 and higher in s, t .

Remark 2.4.1. To motivate why $\text{Vect}(M) = \mathfrak{diff}(M)$, we note that $\mathfrak{diff}(M)$ should be given by the derivatives of the one-parameter subgroups of $\text{Diff}(M)$. If $\phi^t \in \text{Diff}(M)$ is a one-parameter subgroup of M , then its derivative defines a vector field $\partial_{t=0}\phi^t \in \text{Vect}(M)$:

$$\left. \frac{d}{dt} \right|_{t=0} \phi^t(m) \in T_m(M), \text{ for } m \in M.$$

Similarly, to motivate the differential of $\rho : G \rightarrow \text{Diff}(M)$, we note that ρ maps one-parameter subgroups to one-parameter subgroups.

Proposition 2.4.1 (Proposition 3.23, [Kir08]). *Let M be a smooth manifold and G be a Lie group acting smoothly on M .*

- (i) *The commutator of vector fields makes $\text{Vect}(M)$ into a Lie algebra.*
- (ii) *The commutator can be also defined by the following formulas:*

$$\begin{aligned} [\xi, \eta] &= \frac{d}{dt} (\Phi_\xi^t)_* \eta \\ \partial_{[\xi, \eta]} f &= \partial_\eta \partial_\xi f - \partial_\xi \partial_\eta f \\ [f^i \partial_i, g^j \partial_j] &= (g^i \partial_i f^j - f^i \partial_i g^j) \partial_j \end{aligned}$$

Note that the first expression from the second point of the above theorem should be read as

$$[\xi, \eta]_p = \left. \frac{d}{dt} \right|_{t=0} d_{\Phi_\xi^{-t}(p)} \Phi_\xi^t(\eta_{\Phi_\xi^{-t}(p)})$$

which allows us to interpret the Lie bracket of two vector fields as differentiation of the one along (the integral curves of) the other.

Theorem 11 (Theorem 3.25, [Kir08]). *The transformation $\rho_* : \mathfrak{g} \rightarrow \text{Vect}(M)$ is a Lie algebra morphism.*

Example 2.4.1. (i) Considering the action of G on itself by left-multiplication $L : G \rightarrow \text{Diff}(G)$, the Lie algebra isomorphism

$$L_* : \mathfrak{g} \leftrightarrow \{\text{right invariant vector fields on } G\}$$

associates to $x \in \mathfrak{g}$ the right-invariant vector field X such that $X_e = x$.

2.5 Stabilizers and the center

Corollary 2.5.1. *Let $f : G_1 \rightarrow G_2$ be a morphism of real or complex Lie groups. Then*

- (i) *$\ker f$ is a closed Lie subgroup with Lie algebra $\ker f_*$.*
- (ii) *The map $G_1 / \ker f \rightarrow \text{Im} f$ is an immersion.*
- (iii) *When $\text{Im} f$ is a submanifold, the map is a $G_1 / \ker f \rightarrow \text{Im} f$ diffeomorphism.*

Example 2.5.1. (i) Let V be a vector space over \mathbb{K} with a bilinear form B . Then

$$\text{O}(V, B) = \{g \in \text{GL}(V) : B(gu, gv) = B(u, v) \text{ for all } u, v \in V\}$$

is a closed Lie group with Lie algebra:

$$\mathfrak{o}(V, B) = \{g \in \mathfrak{gl}(V) : B(xu, xv) = B(u, v) \text{ for all } u, v \in V\}.$$

Indeed, define an action of $\mathrm{GL}(V)$ on the space \mathcal{F} of bilinear forms on V by $gF(u, v) = F(g^{-1}u, g^{-1}v)$. This defines a representation $\rho : \mathrm{GL}(V) \rightarrow \mathrm{GL}(\mathcal{F})$ and $\mathrm{O}(V, B)$ is the stabilizer of B under this action. The space \mathcal{F} is a vector space, so $\mathrm{Vect}(\mathcal{F})$ can be identified with the space smooth maps from \mathcal{F} to itself. Then for $x \in \mathfrak{gl}(V)$

$$\begin{aligned} (\rho_*(x))_B(u, v) &= \left(\frac{d}{dt} \Big|_{t=0} \rho(e^{tx})B \right)(u, v) = \left(\frac{d}{dt} \Big|_{t=0} B(e^{-tx}u, e^{-tx}v) \right) \\ &= -B(u, xv) - B(xu, v). \end{aligned}$$

- (ii) Let A be a finite-dimensional associative algebra over \mathbb{K} . Then the group of all automorphisms of A

$$\mathrm{Aut}(A) = \{g \in \mathrm{GL}(A) \mid (ga) \cdot (gb) = g(a \cdot b) \text{ for all } a, b \in A\}$$

is a Lie group with Lie algebra

$$\mathrm{Der}(A) = \{x \in \mathfrak{gl}(A) \mid (xa)b + a(xb) = x(ab) \text{ for all } a, b \in A\}$$

(this Lie algebra is called the *algebra of derivations of A*).

Indeed, if we consider the space W of all linear maps $A \otimes A \rightarrow A$ and define the action of G by $(gf)(a \otimes b) = gf(g^{-1}a \otimes g^{-1}b)$ then $\mathrm{Aut} A = G_\mu$, where $\mu : A \otimes A \rightarrow A$ is the multiplication. So $\mathrm{Aut}(A)$ is a Lie group with Lie algebra $\mathrm{Der}(A)$.

- (iii) The same argument also shows that for a finite-dimensional Lie algebra \mathfrak{g} , the group

$$\mathrm{Aut}(\mathfrak{g}) = \{g \in \mathrm{GL}(\mathfrak{g}) \mid [ga, gb] = g[a, b] \text{ for all } a, b \in \mathfrak{g}\}$$

is a Lie group with Lie algebra

$$\mathrm{Der}(\mathfrak{g}) = \{x \in \mathfrak{gl}(\mathfrak{g}) \mid [x.a, b] + [a, x.b] = x.[a, b] \text{ for all } a, b \in \mathfrak{g}\}$$

called the Lie algebra of derivations of \mathfrak{g} .

Finally, we can show that the center of G is a closed Lie subgroup.

Theorem 12. *Let G be a connected Lie group. Then its center $Z(G)$ is a closed Lie subgroup with Lie algebra the ideal*

$$\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \ \forall y \in \mathfrak{g}\}.$$

The quotient group $G/Z(G)$ is usually called the *adjoint group* associated with G and denoted $\mathrm{Ad} G$:

$$\mathrm{Ad} G = G/Z(G) = \mathrm{Im}(\mathrm{Ad} : G \rightarrow \mathrm{GL}(\mathfrak{g})) \quad (\text{for connected } G).$$

The corresponding Lie algebra is

$$\mathrm{ad} \mathfrak{g} = \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) = \mathrm{Im}(\mathrm{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})).$$

2.6 Campbell–Hausdorff formula

So far, we have shown that the multiplication in a Lie group G defines the commutator in $\mathfrak{g} = T_1G$. However, the definition of commutator only used the lowest non-trivial term of the group law in logarithmic coordinates. Thus, it might be expected that higher terms give more operations on \mathfrak{g} . However, it turns out that it is not so: the whole group law is completely determined by the lowest term, i.e. by the commutator, as follows from the Campbell–Hausdorff formula:

Theorem 13 (Campbell–Hausdorff Formula). *For small enough $x, y \in \mathfrak{g}$ one has*

$$\exp(x)\exp(y) = \exp(\mu(x, y))$$

for some \mathfrak{g} -valued function $\mu(x, y)$ which is given by the following series convergent in some neighborhood of $(0, 0)$:

$$\mu(x, y) = x + y + \sum_{n \geq 2} \mu_n(x, y),$$

where $\mu_n(x, y)$ is a Lie polynomial in x, y of degree n , i.e. an expression consisting of commutators of x, y , their commutators, etc., of total degree n in x, y . This expression is universal: it does not depend on the Lie algebra \mathfrak{g} or on the choice of x, y . In particular, the group operation in a connected Lie group G can be recovered from the commutator in \mathfrak{g} .

It is possible to write the expression for μ explicitly. However, this is rarely useful, so we will only write the first few terms:

$$\mu(x, y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + \cdots$$

Although it follows from the Campbell–Hausdorff formula, the commutation criterion of the exponential can be also proven directly:

Theorem 14 (Theorem 3.36, [Kir08]). *Let $x, y \in \mathfrak{g}$. Then $e^x e^y = e^y e^x$ if and only if $[x, y] = 0$. In that case $e^x e^y = e^y e^x = e^{x+y}$.*

Proof. The if direction follows from the definition of the commutator via Taylor series. For the other direction we follow the method suggested in Exercise 3.11 of [Kir08]:

$$\text{Ad}(e^x)y = e^{\text{ad } x}y = \sum_{n \geq 0} \frac{(\text{ad } x)^n}{n!}y = y$$

But $C_g e^z = e^{\text{Ad}_g z}$, so the above implies that $e^x e^y e^{-x} = C_{e^x} e^y = e^{\text{Ad}(e^x)y} = e^y$. \square

2.7 Fundamental theorems of Lie theory

Let us summarize the results we have so far about the relation between Lie groups and Lie algebras.

1. Every real or complex Lie group G defines a Lie algebra $\mathfrak{g} = T_1G$ (respectively, real or complex), with commutator defined by (3.2); we will write $\mathfrak{g} = \text{Lie}(G)$. Every morphism of Lie groups $\varphi : G_1 \rightarrow G_2$ defines a morphism of Lie algebras $\varphi_* : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$. For connected G_1 , the map

$$\text{Hom}(G_1, G_2) \rightarrow \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$$

$$\varphi \mapsto \varphi_*$$

is injective. (Here $\text{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$ is the set of Lie algebra morphisms.)

2. As a special case of the previous, every Lie subgroup $H \subset G$ defines a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$.
3. The group law in a connected Lie group G can be recovered from the commutator in \mathfrak{g} ; however, we do not yet know whether we can also recover the topology of G from \mathfrak{g} .

However, this still leaves a number of questions.

Question 1. Given a morphism of Lie algebras $\mathfrak{g}_1 \rightarrow \mathfrak{g}_2$, where $\mathfrak{g}_1 = \text{Lie}(G_1)$, $\mathfrak{g}_2 = \text{Lie}(G_2)$, can this morphism always be lifted to a morphism of the Lie groups?

Question 2. Given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g} = \text{Lie}(G)$, does there always exist a corresponding Lie subgroup $H \subset G$?

Question 3. Can every Lie algebra be obtained as a Lie algebra of a Lie group?

As the following example shows, in this form the answer to question 1 is negative.

Example 2.7.1. Let $G_1 = S^1 = \mathbb{R}/\mathbb{Z}$, $G_2 = \mathbb{R}$. Then the Lie algebras are $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathbb{R}$ with zero commutator. Consider the identity map $\mathfrak{g}_1 \rightarrow \mathfrak{g}_2 : a \mapsto a$. Then the lift $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ of the corresponding morphism to the universal covering group (provided that the underlying morphism $f : S^1 \rightarrow \mathbb{R}$ exists), should be given by $\tilde{f}(\theta) = \theta + c$; on the other hand, it must also satisfy $\tilde{f}(\mathbb{Z}) = \{0\}$. Thus, this morphism of Lie algebras can not be lifted to a morphism of Lie groups.

In this example the difficulty arose because G_1 was not simply-connected. It turns out that this is the only difficulty: after taking care of this, the answers to all the questions posed above are positive. The following theorems give precise statements.

Theorem 15 (First fundamental theorem of Lie theory). *For any real or complex Lie group G , there is a bijection between connected Lie subgroups $H \subset G$ and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$, given by $H \mapsto \mathfrak{h} = \text{Lie}(H) = T_1 H$.*

Theorem 16 (Second fundamental theorem of Lie theory). *If G_1, G_2 are Lie groups (real or complex) and G_1 is connected and simply connected, then $\text{Hom}(G_1, G_2) = \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$, where $\mathfrak{g}_1, \mathfrak{g}_2$ are Lie algebras of G_1, G_2 respectively.*

Theorem 17 (Lie's third theorem). *Any finite-dimensional real or complex Lie algebra is isomorphic to a Lie algebra of a Lie group (respectively, real or complex).*

These are the fundamental theorems of Lie theory and combining these with the previous results, we get the following important corollary.

Corollary 2.7.1 (Lie group and algebra equivalence). *The categories of finite-dimensional Lie algebras and connected, simply-connected Lie groups are equivalent. In other words, for any real or complex finite-dimensional Lie algebra \mathfrak{g} , there is a unique (up to isomorphism) connected simply-connected Lie group G (respectively, real or complex) with $\text{Lie}(G) = \mathfrak{g}$. Any other connected Lie group G' with Lie algebra \mathfrak{g} must be of the form G/Z for some discrete central subgroup $Z \subset G$.*

2.8 Complex and real forms

The operation of complexification, which is trivial at the level of Lie algebras, is highly non-trivial at the level of Lie groups. Real forms and their complexifications can be topologically quite different: for example, $SU(n)$ is compact while $SL(n, \mathbb{C})$ is not. On the other hand, it is natural to expect that their algebras share many algebraic properties, such as semisimplicity. Thus we may use the properties of the complex Lie algebra to study the real Lie algebra, and vice versa, or use the properties of one real form to study another real form. Thus, we may use, for example, use the compact group $SU(n)$ to prove some results about the non-compact group $SL(n, \mathbb{C})$. Moreover, since $\mathfrak{sl}(n, \mathbb{R})_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$, this will also give us results about the non-compact real group $SL(n, \mathbb{R})$.

Definition 2.8.1. (i) The complexification $\mathfrak{g}_{\mathbb{C}}$ of a real Lie algebra \mathfrak{g} is the complex Lie algebra

$$\mathfrak{g}_{\mathbb{C}} \stackrel{\text{def}}{=} \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}.$$

We say that \mathfrak{g} is a real form of $\mathfrak{g}_{\mathbb{C}}$.

(ii) Let G be a connected complex Lie group with Lie algebra \mathfrak{g} . A closed real Lie subgroup $K \leq G$ is a real form of G if its Lie algebra \mathfrak{k} is a real form of \mathfrak{g} .

In some cases, complexification is obvious, as in $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. Others however, are less obvious, as $\mathfrak{g} = \mathfrak{u}(n)$. Finding a real form for a complex group is can be often direct:

Proposition 2.8.1. *Let G be a connected simply-connected complex Lie group with Lie algebra \mathfrak{g} . Then every real form \mathfrak{k} of \mathfrak{g} is realised as the Lie algebra of a real form K of G .*

Proof. Considering $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$, we define the real Lie algebra automorphism

$$\theta : \mathfrak{g} \rightarrow \mathfrak{g}, \quad x + iy \mapsto x - iy.$$

Since G is connected and simply connected, the second fundamental theorem of Lie groups tells us that θ can be lifted to a Lie group automorphism $\Theta : G \rightarrow G$. Then $K = G^{\Theta}$ is a real form of G with Lie algebra $\ker \theta = \mathfrak{k}$. \square

Going in the opposite direction, from a real Lie group to a complex one, is more subtle: there are real Lie groups that can not be obtained as real forms of a complex Lie group (for example, it is known that the universal cover of $SL(2, \mathbb{R})$ is not a real form of any complex Lie group). It is still possible to define a complexification for any real Lie group G ; however, in general the former does not admit the latter as a subgroup.

Chapter 3

Representations of Lie groups and Lie algebras

In this chapter, we will discuss the representation theory of Lie groups and Lie algebras. Unless specified otherwise, all Lie groups, algebras, and representations are finite-dimensional, and all representations are complex. Lie groups and Lie algebras can be either real or complex; unless specified otherwise, all results are valid both for the real and complex case.

3.1 Basic definitions

Definition 3.1.1. A representation of a Lie group G is a vector space V together with a morphism $\rho : G \rightarrow \mathrm{GL}(V)$.

A representation of a Lie algebra \mathfrak{g} is a vector space V together with a morphism $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$.

A morphism between two representations V, W of the same group G is a linear map $f : V \rightarrow W$ which commutes with the action of G : $f\rho(g) = \rho(g)f$. In a similar way, one defines a morphism of representations of a Lie algebra. The space of all G -morphisms (respectively, \mathfrak{g} -morphisms) between V and W will be denoted by $\mathrm{Hom}_G(V, W)$ (respectively, $\mathrm{Hom}_{\mathfrak{g}}(V, W)$).

Remark 3.1.1. Morphisms between representations are also frequently called *intertwining operators* because they “intertwine” action of G in V and W .

The notion of a representation is completely parallel to the notion of module over an associative ring or algebra; the difference of terminology is due to historical reasons. In fact, it is also usual to use the word “module” rather than “representation” for Lie algebras: a module over Lie algebra \mathfrak{g} is the same as a representation of \mathfrak{g} . We will use both terms interchangeably.

Note that in this definition we did not specify whether V and G are real or complex. Usually if G (respectively, \mathfrak{g}) is complex, then V should also be taken a complex vector space. However, it also makes sense to take complex V even if G is real: in this case we require that the morphism $G \rightarrow \mathrm{GL}(V)$ be smooth, considering $\mathrm{GL}(V)$ as $2n^2$ -dimensional real manifold. Similarly, for real Lie algebras we can consider complex representations requiring that $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be \mathbb{R} -linear.

Of course, we could also restrict ourselves to consideration of real representations of real groups. However, it turns out that the introduction of complex representations significantly simplifies the theory even for real groups and algebras. Thus, from now on, all representations will be complex unless specified otherwise.

Example 3.1.1. For a Lie group G or a Lie algebra \mathfrak{g} we always have the following two representations:

- (i) Trivial representation: $V = \mathbb{C}, \rho(g) = \text{id}$ for all $g \in G$ (respectively $\rho(x) = 0$ for all $x \in \mathfrak{g}$).
- (ii) Adjoint representation: $V = \mathfrak{g}, \rho(g) = \text{Ad}_g$ for all $g \in G$ (respectively $\rho(x) = \text{ad}_x$ for all $x \in \mathfrak{g}$).
- (iii) Coadjoint representation: $V = \mathfrak{g}^*, \rho(x) = \text{ad}_x^*$ for all $x \in \mathfrak{g}$.

The first important result about representations of Lie groups and Lie algebras is the following theorem. It tells us that from a representation (or a morphism of representations) of a Lie group we can obtain a representation (or a morphism of representations) of its Lie algebra, and that when G is connected, simply connected, then representing G and representing its Lie algebra is equivalent.

Theorem 18. *Let G be a Lie group (real or complex) with Lie algebra \mathfrak{g} .*

1. *Every representation $\rho : G \rightarrow \text{GL}(V)$ defines a representation $\rho_* : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, and every morphism of representations of G is automatically a morphism of representations of \mathfrak{g} .*
2. *If G is connected, simply-connected, then $\rho \mapsto \rho_*$ gives an equivalence of categories of representations of G and representations of \mathfrak{g} . In particular, every representation of \mathfrak{g} can be uniquely lifted to a representation of G , and $\text{Hom}_G(V, W) = \text{Hom}_{\mathfrak{g}}(V, W)$.*

This is an important result, as Lie algebras are, after all, finite dimensional vector spaces, so they are easier to deal with.

Example 3.1.2. A representation of $\text{SU}(2)$ is the same as a representation of $\mathfrak{su}(2)$, i.e. a vector space with three endomorphisms X, Y, Z , satisfying commutation relations $XY - YX = Z, YZ - ZY = X, ZX - XZ = Y$.

Remark 3.1.2. This theorem can also be used to describe representations of a group which is connected but not simply-connected, in the sense that representations of G are the same as representations of its universal cover that are trivial over the fundamental group of G . More concretely, any such group G can be written as $G = \tilde{G}/Z$ for some simply-connected group \tilde{G} and a discrete central subgroup $Z \subset G$. Thus, representations of G are the same as representations of \tilde{G} satisfying $\rho(Z) = \text{id}$. An important example of this is when $G = \text{SO}(3, \mathbb{R})$, $\tilde{G} = \text{SU}(2)$.

Lemma 3.1.1. *Let \mathfrak{g} be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then the categories of complex representations of \mathfrak{g} and $\mathfrak{g}_{\mathbb{C}}$ are equivalent.*

The next example shows that we can reduce the problem of study of representations of a non-compact Lie group $\text{SL}(2, \mathbb{C})$ to

Example 3.1.3. The categories of finite-dimensional representations of $\text{SL}(2, \mathbb{C})$, $\text{SU}(2)$, $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{su}(2)$ are all equivalent. Indeed, $\mathfrak{sl}(2, \mathbb{C}) = (\mathfrak{su}(2))_{\mathbb{C}}$, so categories of their finite-dimensional representations are equivalent; since the groups $\text{SU}(2), \text{SL}(2, \mathbb{C})$ are simply-connected, they have the same representations as the corresponding Lie algebras.

3.2 Operations on representations

3.2.1 Subrepresentations and quotients

Definition 3.2.1. Let V be a representation of G (respectively \mathfrak{g}). A subrepresentation is a vector subspace $W \subseteq V$ stable under the action: $\rho(g)W \subseteq W$ for all $g \in G$ (respectively, $\rho(x)W \subseteq W$ for all $x \in \mathfrak{g}$).

When $W \subseteq V$ is a subrepresentation, then one can obtain the factor representation or quotient representation V/W given by

$$\rho_{V/W}(g) = \rho_V(g) + W.$$

3.2.2 Direct sum and tensor product

Lemma 3.2.1. Let V, W be representations of G (respectively, \mathfrak{g}). Then there is a canonical structure of a representation on $V^*, V \oplus W, V \otimes W$.

Proof. It is given by

$$\begin{aligned} \rho(g)(v + w) &= \rho(g)(v) + \rho(g)(w) \text{ on } V \oplus W \text{ for } G, \\ \rho(x)(v + w) &= \rho(x)(v) + \rho(x)(w) \text{ on } V \oplus W \text{ for } \mathfrak{g}, \\ \rho(g)(v \otimes w) &= \rho(g)(v) \otimes \rho(g)(w) \text{ on } V \otimes W \text{ for } G, \\ \rho(x)(v \otimes w) &= \rho(x)(v) \otimes w + v \otimes \rho(x)(w) \text{ on } V \otimes W \text{ for } \mathfrak{g}, \\ \rho_{V^*}(g) &= \rho(g^{-1})^t \text{ on } V^* \text{ for } G, \\ \rho_{V^*}(x) &= -\rho(x)^t \text{ on } V^* \text{ for } \mathfrak{g}, \end{aligned}$$

where we denote with A^t the adjoint operator $A^t : V^* \rightarrow V^*$ for $A : V \rightarrow V$. \square

For example, the coadjoint representation is obtained by dualizing the adjoint representation of \mathfrak{g} .

Before giving the next series of examples, recall that for any finite dimensional vector spaces V, W , we have the following canonical isomorphism:

$$\begin{aligned} V^* \otimes W &\simeq \text{hom}(V, W) \\ f \otimes w &\mapsto (u \mapsto f(u)w) \\ e^i \otimes (Ae_i) &\leftrightarrow A \end{aligned}$$

where e_1, \dots, e_n are any basis of V and e_1, \dots, e^n are its dual basis for V^* .

Example 3.2.1. (i) Coadjoint representation on \mathfrak{g}^* : Dualizing the adjoint representation of \mathfrak{g} we obtain the coadjoint representation given by $\rho(x) = \text{ad}_x^*$

(ii) $\text{hom}(V, W)$: Given two representations V, W of G , the space of linear maps $\text{hom}(V, W)$ is a representation by:

$$\begin{aligned} \rho(g)_{\text{hom}(V, W)}(A) &= \rho_V(g)A\rho_W(g^{-1}) \text{ for } g \in G, \\ \rho(x)_{\text{hom}(V, W)}(A) &= \rho_V(x)A - A\rho_W(x) \text{ for } x \in \mathfrak{g}. \end{aligned}$$

This is obtained by dualizing and tensoring as in the lemma above, i.e. the diagram below is commutative

$$\begin{array}{ccc}
V \otimes W^* & \xrightarrow{\simeq} & \text{hom}(V, W) \\
\downarrow \rho(g) & & \downarrow \rho(g) \\
V \otimes W^* & \xrightarrow{\simeq} & \text{hom}(V, W)
\end{array}$$

(iii) Bilinear forms: Given a representation V , the space of bilinear forms on V becomes a representation when we consider it as a subspace of $V^* \otimes V^*$. It is given by:

$$\begin{aligned}
gB(u, v) &= B(g^{-1}u, g^{-1}v), \text{ for } g \in G \\
xB(u, v) &= -B(xu, v) - B(u, xv), \text{ for } x \in \mathfrak{g}.
\end{aligned}$$

3.2.3 Invariants

Definition 3.2.2. Let V be a representation of a Lie group G . A vector $v \in V$ is called invariant if $\rho(g)v = v$ for all $g \in G$. The subspace of invariant vectors in V is denoted by V^G . Similarly, let V be a representation of a Lie algebra \mathfrak{g} . A vector $v \in V$ is called invariant if $\rho(x)v = 0$ for all $x \in \mathfrak{g}$. The subspace of invariant vectors in V is denoted by $V^{\mathfrak{g}}$.

Remark 3.2.1. When G is connected, and V is a representation of G , then

$$V^G = V^{\mathfrak{g}}$$

Example 3.2.2. (i) $\text{hom}(V, W)$: Considering $\text{hom}(V, W)$ as the representation arising from two representations V, W , we have $\text{hom}(V, W)^G = \text{hom}_G(V, W)$.

(ii) Bilinear forms: We have $B \in V^G$ if and only if the linear map $V \rightarrow V^*$, given by $v \rightarrow B(v, -)$, is a representations morphism.

3.3 Irreducible representations

One of the main problems of the representation theory is the problem of classification of all representations of a Lie group or a Lie algebra. In this generality, it is an extremely difficult problem and for a general Lie group, no satisfactory answer is known. We will later show that for some special classes of Lie groups (namely compact Lie groups and semisimple Lie groups, to be defined later) this problem does have a good answer. The most natural approach to this problem is to start by studying the simplest possible representations, which would serve as building blocks for more complicated representations.

Definition 3.3.1. (i) A non-zero representation V of G or \mathfrak{g} is called irreducible or simple if it has no subrepresentations other than 0 and V . Otherwise V is called reducible.

(ii) V is a completely reducible or semisimple representation if it is isomorphic to a direct sum of irreducible representations: $V = \oplus_i V_i$, with V_i irreducible.

(iii) If V is completely reducible, one usually groups together isomorphic summands writing $V = \oplus_i n_i V_i$, where V_i are pair-wise non-isomorphic irreducible representations and the numbers n_i are called multiplicities.

Remark 3.3.1. Sometimes, it will be useful to consider the isomorphic expressions:

$$nV \simeq \oplus_1^n V \simeq \mathbb{C}^n \otimes V$$

where the latter isomorphism is given by the map $(v_1, \dots, v_n) \mapsto e_1 \otimes v_1 + \dots e_n \otimes v_n$ and in the other direction by the map $z \otimes v \mapsto (z_1 v, \dots, z_n v)$.

Thus, more modest goals of the representation theory would be answering the following questions:

- (i) For a given Lie group G , classify all irreducible representations of G .
- (ii) For a given representation V of a Lie group G , given that it is completely reducible, find explicitly the decomposition of V into direct sum of irreducibles.
- (iii) For which Lie groups G all representations are completely reducible?

Example 3.3.1 (Irreducible representations). (i) \mathbb{C}^n : It is irreducible as a representation of $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{GL}(n, \mathbb{C})$.

Example 3.3.2 (Reducible representations). (i) \mathbb{R} : It is reducible but not completely reducible as a representation of \mathbb{R} (either seen as a group, or a Lie algebra).

(ii) $\mathbb{C}^n \otimes \mathbb{C}^n$: It is completely reducible as a representation of $\mathrm{GL}(n, \mathbb{C})$, since $\mathbb{C}^n \otimes \mathbb{C}^n = S^2\mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n$, where $S^2\mathbb{C}^n$ is the symmetric tensor product and $\Lambda^2\mathbb{C}^n$ is the antisymmetric tensor product.

(iii) Unitary representations are completely irreducible (see Theorem 19). This includes in particular representations of compact groups (see Theorem 22).

Proof. (i) Looking at the Lie algebra $\mathfrak{g} = \mathbb{R}$, representations on V are linear maps $\rho_* : \mathbb{R} \rightarrow \mathfrak{gl}(V)$ who will have the form $t \mapsto tA$, where $A = \rho_*(1)$. Hence representations of $G = \mathbb{R}$ are the Lie group morphisms $\mathbb{R} \rightarrow \mathrm{GL}(d, \mathbb{C})$ that have the form $t \mapsto e^{tA}$, where $A = \rho_*(1) \in \mathfrak{gl}(\mathbb{C}, d)$. If $v \in V$ is an eigenvector of A , then $\mathbb{C}v$ is a subrepresentation in V , so the only irreducible representations of \mathbb{R} are one-dimensional. In fact, $t \mapsto e^{tA}$ is completely irreducible if and only if A is diagonalizable. Hence, the irreducible ones are given as the linear maps $\rho_* : \mathbb{R} \rightarrow \mathfrak{gl}(V) \simeq \mathbb{C}, t \mapsto t\lambda$, for some $\lambda = \lambda_{\rho_*} \in \mathbb{C}$, and the corresponding representation of G is given by $\rho : \mathbb{R} \rightarrow \mathrm{GL}(V) \simeq \mathbb{C}^\times, t \mapsto e^{t\lambda}$. □

One tool which can be used in decomposing representations into direct sum is the use of central elements.

Lemma 3.3.1. *Let V be a representation of a Lie group G (or a Lie algebra \mathfrak{g} respectively).*

- (i) *Let $A \in \mathrm{hom}_G(V, V)$ be a diagonalizable interwining operator. Then each eigenspace V_λ of A is an invariant subspace and we have the decomposition $V = \oplus V_\lambda$ into subrepresentations.*
- (ii) *If $Z \in Z(G)$ is a central element such that $\rho(Z)$ is diagonalizable, then as a representation of G , V decomposes into a direct sum of eigenspaces of $\rho(Z)$: $V = \oplus V_\lambda$.*

Of course, there is no guarantee that V_λ will be an irreducible representation; moreover, in many cases the Lie groups we consider have no central elements at all.

3.4 Intertwining operators and Schur's lemma

In this section we consider the following questions:

Question 4. How can G -invariance of an operator $A \in \mathfrak{gl}(V)$ helps computing eigenvalues and eigenspaces?

Question 5. What is an easy way to compute the centre of classical Lie groups?

The following Lemma can be seen as a way to answer these questions:

Lemma 3.4.1 (Schur's lemma). *(i) Let V be an irreducible representation of a Lie group G (similar result holds for representations of a Lie algebra \mathfrak{g}). Then the space of intertwining operators is one-dimensional:*

$$\text{hom}_G(V, V) = \mathbb{C} \text{id}.$$

(ii) Let V, W be irreducible representations of a Lie group G which are not isomorphic. Then $\text{hom}_G(V, W) = 0$.

Example 3.4.1 (Answer to Question 5). Consider the group $\text{GL}(n, \mathbb{C})$. Since \mathbb{C}^n is irreducible as a representation of $\text{GL}(n, \mathbb{C})$, every operator commuting with $\text{GL}(n, \mathbb{C})$ must be scalar. Thus, the center $Z(\text{GL}(n, \mathbb{C})) = \{\lambda \cdot \text{id}, \lambda \in \mathbb{C}^\times\}$; similarly, the center of the Lie algebra is $Z(\mathfrak{gl}(n, \mathbb{C})) = \{\lambda \cdot \text{id}, \lambda \in \mathbb{C}\}$.

Since \mathbb{C}^n is also irreducible as a representation of $\text{SL}(n, \mathbb{C})$, $\text{U}(n)$, $\text{SU}(n)$, $\text{SO}(n, \mathbb{C})$, a similar argument can be used to compute the center of each of these groups. The answer is

$$\begin{aligned} Z(\text{SL}(n, \mathbb{C})) &= Z(\text{SU}(n)) = \{\lambda \cdot \text{id}, \lambda^n = 1\} & \mathfrak{z}(\text{U}(n)) &= \{\lambda \cdot \text{id}, |\lambda| = 1\} \\ Z(\mathfrak{sl}(n, \mathbb{C})) &= Z(\mathfrak{su}(n)) = 0 & \mathfrak{z}(\mathfrak{u}(n)) &= \{\lambda \cdot \text{id}, \lambda \in i\mathbb{R}\} \\ Z(\text{SO}(n, \mathbb{C})) &= Z(\text{SO}(n, \mathbb{R})) = \{\pm 1\} & \mathfrak{z}(\mathfrak{so}(n, \mathbb{C})) &= \mathfrak{z}(\mathfrak{so}(n, \mathbb{R})) = 0. \end{aligned}$$

To answer Question 4, we can use the following corollary of Schur's lemma:

Corollary 3.4.1. *Let V be a completely reducible representation of Lie group G (respectively, Lie algebra \mathfrak{g}). Then*

1. *If $V = \bigoplus V_i$, V_i irreducible, pairwise non-isomorphic, then any intertwining operator $\Phi : V \rightarrow V$ is of the form $\Phi = \bigoplus \lambda_i \text{id}_{V_i}$.*
2. *If $V = \bigoplus n_i V_i = \bigoplus n_i \mathbb{C}^{n_i} \otimes V_i$, V_i irreducible, pairwise non-isomorphic, then any intertwining operator $\Phi : V \rightarrow V$ is of the form $\Phi = \bigoplus (A_i \otimes \text{id}_{V_i})$, where $A_i \in \text{End}(\mathbb{C}^{n_i})$.*

Proof. For part (1), notice that any operator $V \rightarrow V$ can be written in a block form: $\Phi = \bigoplus \Phi_{ij}$, $\Phi_{ij} : V_i \rightarrow V_j$. By Schur's lemma, $\Phi_{ij} = 0$ for $i \neq j$ and $\Phi_{ii} = \lambda_i \text{id}_{V_i}$. Part (2) is proved similarly. \square

This result shows that indeed, if we can decompose a representation V into irreducible ones, this will give us a very effective tool for analyzing intertwining operators. For example, if $V = \bigoplus V_i$, $V_i \neq V_j$, then $\Phi = \bigoplus \lambda_i \text{id}_{V_i}$, so one can find λ_i by computing $\Phi(v)$ for just one vector in V_i . It also shows that each eigenvalue λ_i will appear with multiplicity equal to $\dim V_i$. This is exactly what we did in the baby example above, only now V_i was one-dimensional, where we had $G = \mathbb{Z}_n$.

We also have the following corollary on commutative groups and algebras.

Proposition 3.4.1. *If G is a commutative group, then any irreducible complex representation of G is one-dimensional. Similarly, if \mathfrak{g} is a commutative Lie algebra, then any irreducible complex representation of \mathfrak{g} is one-dimensional.*

Now we are ready to give a classification of irreducible representations of \mathbb{R} and \mathbb{S}^1

Example 3.4.2. Let $G = \mathbb{R}$. Then the only irreducible representations of G are one-dimensional (which follows either from the proposition above, or from Example 3.4.2), and are given by $t \mapsto e^{\lambda t}$ for some $\lambda \in \mathbb{C}$. Similarly, the only irreducible representations of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ are one-dimensional complex spaces V_k for $k \in \mathbb{Z}$, and are given by $t \mapsto e^{2\pi i k t}$.

3.5 Complete irreducibility of unitary representations, representations on finite groups

Definition 3.5.1. A complex representation V of a real group G is unitary if there exists a G -invariant inner-product (i.e. positive Hermitian form) on V : $\rho(G) \subseteq U(V)$. Similarly, a representation V of a Lie algebra \mathfrak{g} is unitary if there exists a \mathfrak{g} -invariant inner-product on V : $\rho(\mathfrak{g}) \subseteq \mathfrak{u}(V)$, or equivalently $(\rho(x)v, w) + (v, \rho(x)w) = 0$ for all $x \in \mathfrak{g}, v, w \in V$.

Theorem 19. *Unitary representations are completely reducible.*

Example 3.5.1 (Unitary representations). (i) If G acts by permutations on a finite set S , then the representation $\rho(g)f \stackrel{\text{def}}{=} f \circ g^{-1}$ on the space $F(S)$ of complex functions over S is unitary.

(ii) Representations of finite groups.

(iii) Representations of compact groups (see Theorem 22).

3.6 Haar measure on compact Lie groups

Since the proof that finite group representations are unitary is based on the existence of a right-invariant integral on the group, it is natural to ask whether a similar result holds for a broader class of Lie groups.

Definition 3.6.1. A right Haar measure on a Lie group G is a measure μ on G that is invariant under the right action of G on itself: $R_g^* \mu = \mu$. Equivalently $\int_G f(g) d\mu(g) = \int_G f(gh) d\mu(g)$ for all $h \in G$ and integrable functions f on G .

Theorem 20. *Let G be a real Lie group.*

1. *Then G is orientable and admits an orientation that is invariant under the right action of G on itself.*
2. *If G is compact, then for a fixed choice of right-invariant orientation on G , there exists a unique right-invariant top degree non-vanishing differential form ω such that $\int_G \omega = 1$. This form is left-invariant and left-invariant and invariant up to a sign under $i : g \mapsto g^{-1}, i^* \omega = (-1)^{\dim G} \omega$.*

Theorem 21. *Let G be a compact Lie group. Then G admits a right Haar measure, which is a Borel measure, and invariant under the right and left actions of G on itself, and under inversion $i : g \mapsto g^{-1}$.*

Example 3.6.1. The Haar measure for $G = S^1 = \mathbb{R}/\mathbb{Z}$ is the Lebesgue measure dx on $[0, 1]$.

Theorem 22. *Any finite-dimensional representation of a compact Lie group is unitary and thus completely reducible.*

3.7 Orthogonality of characters and Peter–Weyl theorem

Throughout this section, we will assume that G is a compact real Lie group with Haar measure dg . In the previous section, we saw that it is completely reducible, but we have no way of

knowing what the factors of G are, or what the multiplicities are. This is what we will try to discuss in this section.

When V is a representation of G and v_i a basis for V , we can consider the matrix corresponding to $\rho(g) \in \text{GL}(V)$ in this basis, and denote with ρ_{ij} its coefficients. In this way we obtain scalar-valued functions $\rho_{ij}(g)$ on G .

We will make use of the following inner product on functions over G :

$$(f, g) = \int_G f(g) \overline{g(g)} dg. \quad (3.1)$$

The next theorem tells us that irreducible representations of a compact Lie group G can give us an orthogonal set of functions on G .

Theorem 23. *Let G be a compact real Lie group with Haar measure dg .*

- (i) *Let V, W be non-isomorphic irreducible representations of G with basis v_i, w_a . Then the corresponding matrix coefficients are orthogonal:*

$$(\rho_{ij}^V, \rho_{ab}^W) = 0$$

for all i, j, a, b , where the inner-product is defined as in Equation (3.1).

- (ii) *Let V be an irreducible representation of G , and v_i be an orthonormal basis with respect to a G -invariant inner product. Then the matrix coefficients $\rho_{ij}(g)$ are orthogonal:*

$$(\rho_{ij}, \rho_{kl}) = \frac{\delta_{ik} \delta_{jl}}{\dim V}.$$

It is evident though that the set of matrix coefficients does depend on the choice of basis for the vector spaces. For this reason, one considers a linear combination of coefficients that is basis-independent.

Definition 3.7.1. A character of a representation V of a real Lie group G is the scalar function on the group defined by

$$\chi_V(g) = \text{tr}_V \rho(g) = \sum_i \rho_{ii}(g),$$

where $\rho_{ij}(g)$ are the matrix coefficients of $\rho(g)$ in some basis of V .

We have the following properties:

Proposition 3.7.1. *Let V be a representation of a real Lie group G .*

- (i) *If $V = \mathbb{C}$ is the trivial representation, then $\chi_V(g) = 1$.*
- (ii) *$\chi_{V \oplus W} = \chi_V + \chi_W$.*
- (iii) *$\chi_{V \otimes W} = \chi_V \chi_W$.*
- (iv) *Characters are conjugation invariant: $\chi_V(hgh^{-1}) = \chi_V(g)$.*
- (v) *If G is compact, then $\chi_{V^*} = \overline{\chi_V}$.*

The following theorem which follows from the orthogonality of matrix coefficients, tells us that characters of irreducible representations of a compact Lie group form an orthogonal family of functions on G .

Theorem 24. *Let G be a compact real Lie group with Haar measure dg .*

(i) *Characters of non-isomorphic irreducible representations of G are orthogonal:*

$$(\chi_V, \chi_W) = 0$$

for all non-isomorphic irreducible representations V, W .

(ii) *For any representation V of G we have*

$$(\chi_V, \chi_V) = 1 \text{ if and only if } V \text{ is irreducible.}$$

(iii) *The set $\{\chi_V : V \in \hat{G}\}$ is an orthonormal family of functions of G , where \hat{G} is the set of isomorphism classes of irreducible representations of G .*

(iv) *For any representation V of G we can compute its multiplicities via:*

$$n_i = (\chi_V, \chi_{V_i}),$$

where $V = \oplus_i n_i V_i$, and V_i are pair-wise non-isomorphic irreducible representations.

Note that the last part of the theorem implies in particular that the factors and the multiplicities of a completely reducible representation of a compact Lie group can be computed from its character and are unique.

Returning to the matrix coefficients of representations, one might ask whether it is possible to give a formulation of their orthogonality in a way that does not require a choice of basis. The answer is “yes”. Indeed, let $v \in V$, $v^* \in V^*$. Then we can define a function on the group $\rho_{v^*,v}(g)$ by

$$\rho_{v^*,v}(g) = \langle v^*, \rho(g)v \rangle,$$

and this correspondence extends to a mapping $V^* \otimes V \rightarrow C^\infty(G, \mathbb{C})$. This is a generalization of a matrix coefficient: if $v = v_j$, $v^* = v_j^*$, we recover matrix coefficient $\rho_{ij}(g)$.

The space $V^* \otimes V$ has additional structure. First, we have two commuting actions of G on it, given by action on the first factor and on the second one; in other words, $V^* \otimes V$ is a G -bimodule. In addition, if V is unitary, then the inner product defines an inner product on V^* (the simplest way to define it is to say that if v_i is an orthonormal basis in V , then the dual basis v_i^* is an orthonormal basis in V^*). Define an inner product on $V^* \otimes V$ by

$$\langle v_1^* \otimes w_1, v_2^* \otimes w_2 \rangle = \frac{1}{\dim V} \langle v_1^*, v_2^* \rangle \langle w_1, w_2 \rangle. \quad (3.2)$$

Theorem 25 (Peter-Weyl theorem). *Let \hat{G} be the set of isomorphism classes of irreducible representations of G . Define the map*

$$m : \bigoplus_{V_i \in \hat{G}} V_i^* \otimes V_i \rightarrow C^\infty(G, \mathbb{C}), \text{ by}$$

$$v^* \otimes v \mapsto \langle v^*, \rho(\cdot)v \rangle.$$

Then

(i) m is an isomorphism of G -bimodules, i.e. it is a linear isomorphism that commutes with the actions of G on the domain and the codomain:

$$\begin{aligned} m((gv^*) \otimes v) &= L_g(m(v^* \otimes v)), \\ m(v^* \otimes (gv)) &= R_g(m(v^* \otimes v)), \end{aligned}$$

where R_g, L_g are the right and left actions of G on $C^\infty(G, \mathbb{C})$ given by $(R_g f)(x) = f(xg)$ and $(L_g f)(x) = f(g^{-1}x)$.

(ii) The map m extends to a Hilbert spaces isomorphism:

$$\hat{\bigoplus}_{V_i \in \hat{G}} V_i^* \otimes V_i \xrightarrow{\sim} L^2(G, dg),$$

where $\hat{\bigoplus}$ denotes the Hilbert space direct sum i.e. the completion of the algebraic direct sum with respect to the metric given by inner product defined by Equation (3.2), and $L^2(G, dg)$ is the Hilbert space of complex-valued square-integrable functions on G with respect to the Haar measure, with the inner product defined by Equation (3.1).

(iii) The set of characters

$$\{\chi_V : V \in \hat{G}\}$$

is an orthonormal basis of the space $(L^2(G, dg))^G$ of conjugation invariant functions on G , where \hat{G} denotes the isomorphism classes of irreducible representations of G .

The following example suggests that the study of $L^2(G)$ can be considered as a generalisation of harmonic analysis:

Example 3.7.1. Let $G = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. As we have already discussed, the Haar measure on G is given by dx and the irreducible representations are parametrized by \mathbb{Z} : for any $k \in \mathbb{Z}$, we have one-dimensional representation V_k with the action of \mathbb{S}^1 given by $\rho(a) = e^{2\pi i k a}$. The corresponding matrix coefficient is the same as character and is given by $\chi_k(a) = e^{2\pi i k a}$.

Then the orthogonality relation of matrix coefficients gives

$$\int_0^1 e^{2\pi i k x} e^{2\pi i l x} dx = \delta_{kl},$$

which is the usual orthogonality relation for exponents. The Peter–Weyl theorem in this case just says that the exponents $e^{2\pi i k x}$, $k \in \mathbb{Z}$, form an orthonormal basis of $L^2(\mathbb{S}^1, dx)$ which is one of the main statements of the theory of Fourier series: every L^2 function on \mathbb{S}^1 can be written as a series $f(x) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k x}$ which converges in L^2 metric. For this reason, the study of the structure of $L^2(G)$ can be considered as a generalization of harmonic analysis.

Appendix A

Covering theory reminder

In this chapter we recall certain facts and definitions from basic covering theory. A nice reference for these, is the Chapter 2 from [Hat02]. We begin by defining covering spaces.

Definition A.0.1. A covering map is a continuous surjective map $p : \tilde{X} \rightarrow X$ such that for every $x \in X$ there exists an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets in \tilde{X} , each of which is mapped homeomorphically onto U by p .

The prototypical example of a covering map is $p : \mathbb{S}^1 \rightarrow \mathbb{S}^1, p(z) = z^n$.

A.1 Lifting properties

One of the particular characteristics of covering spaces are their lifting properties, that we will recall below.

Proposition A.1.1 (Homotopy lifting property). *Let $p : \tilde{X} \rightarrow X$ be a covering space and a homotopy $f_t : Y \rightarrow X$. Then every lift $\tilde{f}_0 : Y \rightarrow \tilde{X}$ of f_0 extends to a unique homotopy \tilde{f}_t lifting f_t .*

Proof. See [Hat02, Proposition 1.30]. □

This in particular implies the path lifting property of covering spaces:

Corollary A.1.1. *Let $p : \tilde{X} \rightarrow X$ be a covering space. Then for every path $\gamma : I \rightarrow X$ and every lift \tilde{x}_0 of some point $x_0 \in X$ admits a unique lift $\tilde{\gamma} : I \rightarrow \tilde{X}$ of γ starting at \tilde{x}_0 .*

Both of the results above imply that path-homotopies lift to pathhomotopies, where we require for a path homotopy to keep the endpoints of paths fixed. We also have the following corollary that is useful in proving the lifting criterion below.

Corollary A.1.2. *The image subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ consists of the homotopy classes of loops in X based at x_0 whose lifts to \tilde{X} starting at \tilde{x}_0 are loops.*

Proof. See [Hat02, Corollary 1.31]. □

If we care about lifting maps and not homotopies, we have the following criterion that tells us when a lift exists. Namely when f sends loops to loops that lift to loops.

Proposition A.1.2 (Lifting criterion). *Let $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ be a covering space and $f : (Y, y_0) \rightarrow (X, x_0)$ with Y being path-connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ exists if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.*

Proof. See [Hat02, Proposition 1.33]. □

And regarding uniqueness of lifts:

Proposition A.1.3. *Let $p : \tilde{X} \rightarrow X$ be a covering space and $f : Y \rightarrow X$ be a map. If Y is connected, then any two lifts $\tilde{f}_1, \tilde{f}_2 : Y \rightarrow \tilde{X}$ of f that coincide at one point will coincide everywhere on Y .*

Proof. See [Hat02, Proposition 1.34]. □

A.2 Universal covering

In this section we will be concerned with proving that under mild conditions, a space has a universal covering, i.e. a simply connected covering space.

Assumption 1. We consider a topological space X that will be path-connected, locally connected and semi-locally simply connected.

While the first two assumptions may seem rather natural, we will now explain the third one.

Definition A.2.1. A space X is semi-locally simply connected if for every $x \in X$ there exists an open neighborhood U of x such that every loop in U based at x is homotopic in X to a constant loop in U . In other words, the homomorphism

$$\pi_1(U, x) \rightarrow \pi_1(X, x)$$

induced by the inclusion is trivial.

To motivate this assumption, we note that it is necessary for the existence of a universal covering. Indeed, if $p : \tilde{X} \rightarrow X$ is a universal covering, $x_0 \in X$ and U an evenly covered neighborhood of x_0 , then any loop in U based at x lifts to a loop in \tilde{X} based at some \tilde{x}_0 . Since \tilde{X} is simply connected, this loop is homotopic to a constant loop in \tilde{X} and the homotopy projects down to a homotopy in X .

Before constructing the universal cover, we remark that every universal cover \tilde{X} can be thought of as homotopy classes of paths in X starting at some fixed point x_0 .

Remark A.2.1. Let $p : \tilde{X} \rightarrow X$ be a universal covering and $x_0 \in X$. Then the set of homotopy classes of paths in X starting at x_0 is in bijection with \tilde{X} . Indeed, given a path $\gamma : I \rightarrow X$ starting at x_0 , we associate it to its endpoint, which defines a map $\tilde{\gamma} : I \rightarrow \tilde{X}$. This map is well-defined since homotopic paths have the same endpoint. It is surjective, because X is path-connected, while injectivity follows from the fact that \tilde{X} is simply connected.

We now proceed with the construction of the universal cover by fixing some $x_0 \in X$ and letting

$$\tilde{X} \stackrel{\text{def}}{=} \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\},$$

and defining the projection map $p : \tilde{X} \rightarrow X$ by $p([\gamma]) = \gamma(1)$.

To define a topology on \tilde{X} , we first consider a convenient basis of open sets for X :

$$\mathcal{U} \stackrel{\text{def}}{=} \{U \subseteq X \text{ open and path-connected} \mid \pi_1(U, x) \rightarrow \pi_1(X, x) \text{ is trivial for some } x \in U\}.$$

Note that the above set is well-defined since if there exists some $x \in U$ such that $\pi_1(U, x) \rightarrow \pi_1(X, x)$ is trivial, then it is trivial for all $x' \in U$ because U is path-connected. To see that it is a basis for X , we note that $V \in \mathcal{U}$ for every a path-connected open subset $V \subseteq U$.

To each homotopy class $[\gamma] \in \tilde{X}$ and $U \in \mathcal{U}$ we associate the set

$$U_{[\gamma]} \stackrel{\text{def}}{=} \left\{ [\gamma \cdot \eta] \in \tilde{X} \mid \eta \text{ is a path in } U \text{ such that } \eta(0) = \gamma(1) \right\}.$$

Then, given $x \in X$ and a neighborhood U of x , the collection

$$U_{[\gamma]} \text{ for paths } \gamma \text{ going from } x_0 \text{ to } x$$

will be the sheets of U . The following property is crucial into showing that the topology defined by the above basis is well-defined:

$$U_{[\gamma]} = U_{[\gamma']} \text{ for all } [\gamma'] \in U_{[\gamma]}.$$

The proofs that $p : U_{[\gamma]} \rightarrow U$ is a homeomorphism for all $U \in \mathcal{U}$ and $[\gamma] \in \tilde{X}$ and that \tilde{X} is simply connected can be found in [Hat02].

An elementary example is the torus:

Example A.2.1. For $X = \mathbb{T}^2$ being the torus, the universal covering $p : \mathbb{R}^2 \rightarrow \mathbb{T}^2$ and given by $p(t, s) = e^{2\pi i t}, e^{2\pi i s}$. Under the identification $\pi_1(\mathbb{T}^2) \simeq \mathbb{N}^2$, which acts on \mathbb{R}^2 by translations, we have $\mathbb{T}^2 \simeq \mathbb{R}^2 / \mathbb{N}^2$.

Having now constructed the universal cover, we now move on to discussing deck transformations.

Definition A.2.2. A deck transformation of a covering space $p : \tilde{X} \rightarrow X$ is a homeomorphism $\phi : \tilde{X} \rightarrow \tilde{X}$ such that $p \circ \phi = p$. In other words, the following triangular diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\phi} & \tilde{X} \\ & \searrow p & \swarrow p \\ & X & \end{array}$$

For instance, the fundamental group $\pi_1(X, x_0)$ of X acts by deck transformations on the universal cover \tilde{X} . To each $[\alpha] \in \pi_1(X, x_0)$ we associate the deck transformation $\phi_{[\alpha]}$ defined by

$$\phi_{[\alpha]}([\eta]) = [\alpha \cdot \eta].$$

Considering the quotient space $\tilde{X} / \pi_1(X, x_0)$, we have the following result:

Theorem 26. *The quotient space $\tilde{X} / \pi_1(X, x_0)$ is homeomorphic to X where the homeomorphism is induced by the projection map $p : \tilde{X} \rightarrow X$ as in the following diagram*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{p} & X \\ \downarrow & \nearrow \simeq & \\ \tilde{X} / \pi_1(X, x_0) & & \end{array}$$

Appendix B

Exercises

In this chapter, we have solutions for certain exercises of [Kir08].

Exercise 1 (Exercise 2.7). Define a bilinear form on $\mathfrak{su}(2)$ by $(a, b) = \text{tr}(ab^*)/2$. Show that this form is symmetric, positive definite, and invariant under the adjoint action of $\text{SU}(2)$.

Solution. It is symmetric since in $\mathfrak{su}(2)$ we have $b^* = -b$, so $(a, b) = \text{tr}(ab^*)/2 = -\text{tr}(ab)/2 = -\text{tr}(ba)/2 = (b, a)$. It is positive definite since $(a, a) = \frac{1}{2} \sum_{i,j} |a_{ij}|^2$. It is also invariant under the adjoint action of $\text{SU}(2)$ since the trace is invariant under conjugation, and the adjoint representation is given by conjugation. \square

Exercise 2 (Exercise 2.8). Consider the usual basis for $\mathfrak{su}(2)$:

$$i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Show that the adjoint representation

$$\begin{aligned} \text{Ad} : \text{SU}(2) &\rightarrow \text{GL}(3, \mathbb{R}) \\ g &\mapsto \text{matrix of } \text{Ad}_g \text{ in the basis } i\sigma_1, i\sigma_2, i\sigma_3 \end{aligned}$$

gives a morphism of Lie groups $\text{SU}(2) \rightarrow \text{SO}(3)$.

Solution. Note that the above exercise gave us an isometry between $\mathfrak{su}(2)$ and \mathbb{R}^3 , where the first is equipped with the inner product $(a, b) = \text{tr}(ab^*)/2$ and the second with the standard euclidean inner product. Denoting with $[\text{Ad}_g]$ the matrix of Ad_g in the basis $i\sigma_1, i\sigma_2, i\sigma_3$, the above isometry tells us Ad_g preserves the given inner product if and only if the matrix $[\text{Ad}_g]$ preserves the standard euclidean inner product, i.e. $[\text{Ad}_g] \in \text{SO}(3, \mathbb{R})$. Indeed, denoting the isometry with $\Phi : \text{SU}(2) \rightarrow \mathbb{R}^3$, it gives us an isomorphism

$$\begin{aligned} \text{GL}(\Phi) : \text{GL}(\text{SU}(2)) &\xrightarrow{\sim} \text{GL}(3, \mathbb{R}) \\ T &\mapsto \Phi T \Phi^{-1} \end{aligned}$$

and $T \in \text{U}(2)$ if and only if $\text{GL}(\Phi) \in \text{O}(3)$. But we saw in the above exercise that the inner product is preserved, so the adjoint representation lies in $\text{O}(3)$.

To show that it lies actually in $\text{SO}(3)$, note that the determinant of every matrix in $\text{O}(3)$ is either 1 or -1 . However, $\text{SU}(2)$ is connected, so the same is true for $\phi(\text{SU}(2))$. Since it is connected and contains I_3 , so the determinant of the adjoint representation must be 1. \square

Exercise 3 (Exercise 2.9). Let $\phi : \mathrm{SU}(2) \rightarrow \mathrm{SO}(3, \mathbb{R})$ be the morphism of Lie groups defined in the previous exercise. Compute explicitly the map of the tangent spaces $\phi_* : \mathfrak{su}(2) \rightarrow \mathfrak{so}(3, \mathbb{R})$ and show that it is an isomorphism. Deduce from this that $\ker \phi$ is a discrete normal subgroup in $\mathrm{SU}(2)$ and that $\Im \phi$ is an open subgroup in $\mathrm{SO}(3, \mathbb{R})$.

Solution. Let $\Phi : \mathrm{SU}(2) \rightarrow \mathbb{R}^3$ be the isometry that maps $i\sigma_1, i\sigma_2, i\sigma_3$ to the standard basis of \mathbb{R}^3 and the induced isomorphism $\mathrm{GL}(\Phi) : \mathrm{GL}(\mathrm{SU}(2)) \rightarrow \mathrm{GL}(3, \mathbb{R})$, $\mathrm{GL}(\Phi)(T) = \Phi T \Phi^{-1}$. Then Φ is given as the composition:

$$\begin{array}{ccccc} \mathrm{SU}(2) & \rightarrow & \mathrm{GL}(\mathrm{SU}(2)) & \xrightarrow{\mathrm{GL}(\Phi)} & \mathrm{GL}(3, \mathbb{R}) \\ g & \mapsto & \mathrm{Ad}_g & \mapsto & \Phi \mathrm{Ad}_g \Phi^{-1} \end{array}$$

Differentiating the above, we see that $\phi_* = \mathrm{gl}(\Phi) \circ \mathrm{ad}$ which is simply the matrix of ad in the base $i\sigma_1, i\sigma_2, i\sigma_3$. Hence we calculate

$$\mathrm{ad}_{i\sigma_1} = -2J_x, \quad \mathrm{ad}_{i\sigma_2} = -2J_y, \quad \mathrm{ad}_{i\sigma_3} = -2J_z.$$

and thus the matrix of $\phi_* = -2 \mathrm{diag}(1, 1, 1)$ which is clearly an isomorphism. Hence ϕ is a covering map and its kernel is a discrete central normal subgroup of $\mathrm{SU}(2)$, and its image is an open subgroup of $\mathrm{SO}(3, \mathbb{R})$. \square

Exercise 4. Prove that the map ϕ used in the two previous exercises establishes an isomorphism $\mathrm{SU}(2)/\mathbb{Z}^2 \rightarrow \mathrm{SO}(3, \mathbb{R})$, and thus, since $\mathrm{SU}(2) \simeq \mathbb{S}^3$, we have $\mathbb{RP}^3 \simeq \mathrm{SO}(3, \mathbb{R})$.

Exercise 5 (Exercise 3.12). Let $x, y \in \mathfrak{g}$. Then $e^x e^y = e^y e^x$ if and only if $[x, y] = 0$. In that case $e^x e^y = e^y e^x = e^{x+y}$.

Solution. The if direction follows from the definition of the commutator via Taylor series. For the other direction we follow the method suggested in Exercise 3.11 of [Kir08]:

$$\mathrm{Ad}(e^x)y = e^{\mathrm{ad}_x}y = \sum_{n \geq 0} \frac{(\mathrm{ad}_x)^n}{n!}y = y$$

But $C_g e^z = e^{\mathrm{Ad}_g z}$, so the above implies that $e^x e^y e^{-x} = C_{e^x} e^y = e^{\mathrm{Ad}(e^x)y} = e^y$. \square

Exercise 6 (Exercise 3.15). Let G be a connected simply-connected complex Lie group with Lie algebra \mathfrak{g} . Then every real form \mathfrak{k} of \mathfrak{g} is realised as the Lie algebra of a real form K of G .

Solution. Considering $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$, we define the real Lie algebra automorphism

$$\theta : \mathfrak{g} \rightarrow \mathfrak{g}, \quad x + iy \mapsto x - iy.$$

Since G is connected and simply connected, the second fundamental theorem of Lie groups tells us that θ can be lifted to a Lie group automorphism $\Theta : G \rightarrow G$. Then $K = G^\Theta$ is a real form of G with Lie algebra $\ker \theta = \mathfrak{k}$. \square

B.1 Linear algebra exercises

Exercise 7. Let A be a diagonalizable operator with distinct eigenvalues. Then the invariant subspaces of A are exactly the ones that are spanned by some family of its eigenvectors.

First Solution. Let n be the ambient dimension and W be an invariant subspace of A and consider some complement W^\perp of W . Considering some basis e_1, \dots, e_k of W and e_{k+1}, \dots, e_n of W^\perp , we have that in this basis A is block-diagonal:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where A_1 is the restriction of A to W and A_2 is the restriction to W^\perp . In particular, the characteristic polynomial of A is the product of the characteristic polynomials of A_1 and A_2 . In particular, A_1 and A_2 will be diagonalisable with distinct eigenvalues (that are also eigenvalues of A). Letting w_1, \dots, w_k be a basis of eigenvectors of A_1 , we see that these must be eigenvectors of A as well, so W is indeed generated by eigenvectors of A . \square

Second Solution. Let e_1, \dots, e_d be a basis of eigenvectors of A . If the result were not true, then there would exist an invariant subspace W , some element $w = a_1 e_{i_1} + \dots + a_k e_{i_k} \in W$ with $a_1, \dots, a_k \neq 0$ and some $j_0 \in \llbracket 1, k \rrbracket$ such that $e_{i_{j_0}}$ is contained in W . Without loss of generality assume that $j_0 = k$. Then

$$(A - \lambda_1 I)w = a_2(\lambda_1 - \lambda_2)e_{i_2} + \dots + a_k(\lambda_1 - \lambda_k)e_{i_k} = a'_2 e_{i_2} + \dots + a'_{k-1} e_{i_{k-1}}$$

which is contained in W and $a'_2, \dots, a'_{k-1} \neq 0$. Keeping on like that, we arrive at $e_k \in W$, which is a contradiction. \square

Appendix C

Geometric realisation

Proposition C.0.1. *$SU(2)$ is homeomorphic to S^3 . In particular, it is simply connected.*

Appendix D

$SO(d, d)$

The goal of this chapter is to show that $SO(d, d)$ has 2 connected components. First we show that a Lie group G has the same number of components as its maximal compact subgroup.

Proposition D.0.1. *Let G be a Lie group and K its maximal compact subgroup. Then G has the same number of connected components as K .*

Proof. The Cartan decomposition of G tells us that

$$\begin{aligned} K \times \mathfrak{p} &\xrightarrow{\cong} G \\ (k, X) &\mapsto ke^X \end{aligned}$$

is a diffeomorphism. Since $K \times \mathfrak{p}$ is contractible to K , they have the same number of components. \square

Given this, we just need to calculate a maximal compact subgroup of $SO(d, d)$.

Proposition D.0.2. *A maximal compact subgroup of $SO(d, d)$ is $S(O(d) \times O(d))$.*

Proof. Clearly it suffices to show that each $g \in SO(d, d)$ has the form

$$g = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

for certain square matrices A, D , since then they will both necessarily lie in $O(d)$. Since the maximal compact subgroup of an algebraic subgroup G of $GL(d, \mathbb{C})$ is given by the set of fixed points of the involution $A \mapsto (A^*)^{-1}$, we see that its Lie algebra \mathfrak{k} is the space of antisymmetric matrices that commute with $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. These are then matrices of the form

$$x = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

with A, D square and antisymmetric. Thus $e^{\mathfrak{k}}$ has matrices of the desired form and is a generating set for K . \square

Appendix E

Case studies

E.1 \mathbb{R}

The representations of $\mathfrak{g} = \mathbb{R}$ are given as $\rho_* : \mathbb{R} \rightarrow \mathfrak{gl}(V)$, $t \mapsto tA$, where $A = \rho_*(1)$, and the representations of $G = \mathbb{R}$ are given by $\rho : \mathbb{R} \rightarrow \mathrm{GL}(V)$, $t \mapsto e^{tA}$ for $A = \rho_*(1)$. The irreducible representations of \mathbb{R} are one-dimensional and are given by $\rho_* : \mathbb{R} \rightarrow \mathfrak{gl}(V) \simeq \mathbb{C}$, $t \mapsto t\lambda$ for some $\lambda \in \mathbb{C}$, and the corresponding representation of G is given by $\rho : \mathbb{R} \rightarrow \mathrm{GL}(V) \simeq \mathbb{C}^\times$, $t \mapsto e^{t\lambda}$ (see Example 3.4.2).

E.2 \mathbb{S}^1

The representations $\rho : \mathbb{S}^1 \rightarrow \mathrm{GL}(V)$ are in bijective correspondence with the ones $\tilde{\rho} : \mathbb{R} \rightarrow \mathrm{GL}(V)$ such that $\rho(\mathbb{Z}) = \mathrm{id}$, since $p : \mathbb{R} \rightarrow \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is a covering group.

$$\begin{array}{ccc} \mathbb{R} & & \\ p \downarrow & \searrow \tilde{\rho} & \\ \mathbb{S}^1 & \xrightarrow{\rho} & \mathrm{GL}(V) \end{array}$$

The irreducible ones of \mathbb{S}^1 are in (the same) bijective correspondence with the irreducible ones of \mathbb{R} such that $\tilde{\rho}(1) = 1$, where we recall that the irreducible representations of \mathbb{R} are exactly the one-dimensional ones and they have the form $\rho(t) = e^{t\lambda} \in \mathrm{GL}(V) \simeq \mathbb{C}^\times$. Hence the irreducible representations of \mathbb{S}^1 are exactly one-dimensional complex vector spaces V_k and are given by $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z} \rightarrow \mathrm{GL}(V_k) \simeq \mathbb{C}^\times$, $t + \mathbb{Z} \mapsto e^{2\pi i k t}$.

Appendix F

Fun facts

Question 6. What do Lie groups have to do with harmonic analysis?

Answer. The study of the space $L^2(G, dg)$ where G is a compact real Lie group and dg denotes its Haar measure, can be considered as a generalisation of the classical Fourier analysis on the circle \mathbb{S}^1 , as can be seen by setting $G = \mathbb{S}^1$ (see Example 3.7.1). \square

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