Lie group notes

Giorgos

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Contents

Conventions

- By Lie group, we mean either real or complex.
- A closed Lie subgroup of a Lie group is a submanifold that is also a subgroup (see Theorem 3).
- A Lie subgroup is a subgroup that is also an immersed submanifold.

Chapter 1

Lie groups: basic definitions

We will have two notions of subgroups of Lie groups:

Definition 1.0.1. Let G be a complex or real Lie group and H be a subgroup of G.

(i) H is a Lie subgroup if it is an immersed submanifold of G with the multiplication and inverse maps being smooth (or analytic).

(ii) H is a closed Lie subgroup if it is an embedded submanifold of G.

Example 1.0.1. Any of the classical Lie groups are closed Lie subgroups of $\mathrm{GL}(n, \mathbb{K})$. For a Lie subgroup that is not closed, consider the irrational winding on the torus, that is $G = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and $H = f(\mathbb{R})$ with $f(t, s) = (t \mod \mathbb{Z}, \alpha s \mod \mathbb{Z})$, where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.

1.1 Lie subgroups and quotients

The following theorem allows us to reduce the study of Lie groups to the study of finite groups and connected Lie groups, since G_0 is a normal subgroup of G and G/G_0 is a discrete group. where G^0 is the identity component of G.

Theorem 1 (Theorem 2.6, [Kir08])**.** *Let G be a real or complex Lie group and G^0 its identity component. Then G^0 is a normal subgroup of G and a Lie group itself, while G/G^0 is a discrete group.*

In fact, we can reduce the case of connected Lie groups to simply connected Lie groups:

Theorem 2 (Theorem 2.7, [Kir08])**.** *Let G be a connected Lie group. Then its universal cover $\tilde{G}$ has a canonical structure of a Lie group such that the covering map $p : \tilde{G} \to G$ is a homomorphism of Lie groups whose kernel is isomorphic to the fundamental group of G. Moreover, in this case, $\ker p$ is a discrete central subgroup in $\tilde{G}$.*

Proof. The lifting property of the universal cover implies that $\tilde{G}$ is a Lie group. The kernel is discrete, being the fiber of a covering map. The fact that it is central follows from the more general fact that every discrete normal subgroup of a connected Lie group is central. To show the latter, one considers the map $G \to N, g \to gng^{-1}$ where N is the normal subgroup and $n \in N$ is some fixed element. Then the inverse image of any element $n' \in N$ is closed an open in G, so it is all of G in the case where $n' = n$ and empty otherwise. □

Example 1.1.1. In the case where $G = \mathbb{T}^2$ is the torus, the covering map $p : \mathbb{R}^2 \to \mathbb{T}^2$ is given by $p(t, s) = e^{2\pi it, 2\pi is}$, and $\ker p = \mathbb{N}^2 \simeq \pi_1(\mathbb{T}^2)$.

We have the following connection between subgroups and Lie subgroups (i.e. subgroups that are also submanifolds):

Theorem 3 (Theorem 2.8, [Kir08]). *(i) Any Lie subgroup of a Lie group is closed in the topology of the ambient group.*

(ii) Any closed subgroup of a Lie group is a real Lie subgroup.

Proof. For the first part of the theorem, we note that $\overline{H}$ is a subgroup of G as well. We claim that H (and thus Hx for every $x \in \overline{H}$) is open and dense in $\overline{H}$. To see this, note that $e \in \overline{H}$ implies that for every neighborhood $U \subseteq G$ of e in G, $U \cap H$ is nonempty. In particular there exists some open set U in G containing r such that $U \cap H \neq \emptyset$. Then $U \cap H$ will be a neighborhood of e in $\overline{H}$ and H is open because it can be written as the union of all subsets of the form $h(U \cap H)$ for $h \in H$.

To conclude, note that for $x, y \in \overline{H}$, $Hx \cap Hy$ is dense in $\overline{H}$, so it is nonempty. This implies that $Hx = Hy = H$ and H thus $H = \overline{H}$. □

1.2 Homogeneous spaces

We begin by describing coset spaces of Lie groups.

Theorem 4 (Theorem 2.11, [Kir08]). *Let G be a Lie group of dimension n, $H \leq G$ a closed Lie subgroup of dimension k.*

(i) The coset space G/H has a natural structure of a manifold of dimension $n - k$ such that the canonical map $p : G \to G/H$ is a fiber bundle, with fiber diffeomorphic to H. The tangent space at the identity is isomorphic to the quotient space $T_H G/H \simeq T_e G/T_e H$.

(ii) If H is a normal closed Lie subgroup then G/H has a canonical Lie group structure.

The following is the analog of the homomorphism theorem for Lie groups:

Theorem 5 (Theorem 2.5, [Kir08]). *Let $f : G_1 \to G_2$ be a Lie group morphism.*

(i) $H = \ker f$ is a normal closed Lie subgroup of G_1 and f induces an injective homomorphism $G_1/H \to G_2$ that is an immersion

(ii) If moreover $\Im f$ is an embedded submanifold, then it is a closed Lie subgroup of G_2 and f induces an isomorphism $G_1/H \to \Im f$.

Theorem 6 (Theorem 2.20, [Kir08]). *Let G be a Lie group acting on a manifold M, and $m \in M$.*

(i) The stabilizer G_m is a closed Lie subgroup of G, with Lie algebra

$$\mathfrak{h} = \{x \in \mathfrak{g} \mid \rho_*(x)(m) = 0\}.$$

where $\rho : G \to \mathrm{Diff}(M)$ is the action of G on M.

(ii) The orbit map $g \mapsto g \cdot m$ induces an injective immersion $G/G_m \hookrightarrow \mathcal{O}_m$ whose image coincides with $\mathcal{O}_m$.

(iii) The orbit $\mathcal{O}_m$ is an immersed submanifold with tangent space $T_m\mathcal{O}_m = \mathfrak{g}/\mathfrak{h}$.

(iv) If the orbit is a submanifold, then the orbit map is a diffeomorphism.

Proof. See [Kir08, Theorem 2.20, Theorem 3.29]. □

The case of one orbit gives rise to G-homogeneous spaces:

Theorem 7 (Theorem 2.22 [Kir08])**.** *Let M be a G-homogeneous space and $m \in M$. Then the orbit map $G \to M$ is a fiber bundle over M with fiber G_m.*

1.3 Classical Lie groups

Definition 1.3.1. We define

$$Sp(n, \mathbb{K}) = \{A \in GL(2n, \mathbb{K}) \mid \omega(Ax, Ay) = \omega(x, y)\}.$$

where ω is the standard symplectic form on $\mathbb{K}^{2n}$, which is given by $\omega(x, y) = x^* Jy = \sum_{i=1}^n x_i y_{i+n} - x_{i+n} y_i$, and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

We also define the gorup of unitary quaternionic transformations by

$$\mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{SU}(2n).$$

The following theorem tells us that the logarithmic map behaves well when restricted to a neighborhood of the identity in each classical group.

G	$O(n,\mathbb{R})$	$SO(n,\mathbb{R})$	$U(n)$	$SU(n)$	$Sp(n)$
$\mathfrak{g}$	$x^t = -x$	$x^t = -x$	$x^* = -x$	$x^* = -x,\ \mathrm{tr}, x = 0$	$J^{-1}x^*J = -x, x^* = -x$
$\dim G$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	n^2	$n^2 - 1$	$n(2n+1)$
$\pi_0(G)$	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}_2\ (n \geq 3)$	$\mathbb{Z}_2\ (n \geq 3)$	$\mathbb{Z}$	$\{1\}$	$\{1\}$

Table 1.1: Compact classical groups

G	$GL(n,\mathbb{R})$	$SL(n,\mathbb{R})$	$Sp(n,\mathbb{R})$
$\mathfrak{g}$	$\mathfrak{gl}(n,\mathbb{R})$	$\mathrm{tr}\, x = 0$	$x + J^{-1}x'J = 0$
$\dim G$	n^2	$n^2 - 1$	$n(2n+1)$
$\pi_0(G)$	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}_2\ (n \geq 3)$	$\mathbb{Z}_2\ (n \geq 3)$	$\mathbb{Z}$

Table 1.2: Noncompact real classical groups.

G	$GL(n,\mathbb{C})$	$SL(n,\mathbb{C})$	$O(n,\mathbb{C})$	$SO(n,\mathbb{C})$
$\pi_0(G)$	$\{1\}$	$\{1\}$	$\mathbb{Z}_2$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}$	$\{1\}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Table 1.3: Complex classical groups.

Chapter 2

Lie groups and Lie algebras

2.1 Exponential map

Definition 2.1.1 (Proposition 3.1, [Kir08]). Let G be a Lie group and $x \in \mathfrak{g}$. The one-parameter subgroup $\gamma_x : \mathbb{K} \to G$ is the unique Lie group morphism such that $\gamma_x'(0) = x$. We define the exponential map of G as

$$\exp(x) = \gamma_x(1)$$

Remark 2.1.1. By looking at the proof of the statements in the above definition, one can see that for $x \in \mathfrak{g}$, the curve

$$\exp(tx) = \gamma_x(t) = \gamma_{tx}(1).$$

integral curve of the left-invariant vector field $X \in \mathcal{X}(G)$ that satisfies

$$X_e = x.$$

The following are some properties of the exponential map:

Theorem 8 (Theorems 3.7 and 3.36, [Kir08]). *Let G be a Lie group.*

(i) $d_e \exp = \mathrm{id}_{\mathfrak{g}}$

(ii) The exponential map is a local diffeomorphism at 0.

(iii) For any Lie group morphism $\phi : G_1 \to G_2$, we have $d_e\phi(exp(x)) = \exp d_e(\phi(x))$ for all $x \in \mathfrak{g}$.

(iv) For any $g \in G, x \in \mathfrak{g}$

$$g \exp(x) g^{-1} = \exp(\mathrm{Ad}_g x).$$

(v) For $x, y \in \mathfrak{g}$, we have:

$$\text{If } [x, y] = 0 \text{ then } e^x e^y = e^y e^x = e^{x+y}.$$

Remark 2.1.2. The exponential map is not surjective in general. It is however for compact Lie groups.

Lemma 2.1.1. *The exponential map of a connected Lie group G sends generators of $\}$ to generators of G. That is, if $\{x_1, \ldots, x_n\}$ is a basis of $\mathfrak{g}$, then $\{\exp(tx_i) : t \in \mathbb{R}, i \in [\![1, n]\!]\}$ is a basis of G.*

Proof. Consider the map $f : \mathbb{R}^n \to \mathbb{G}$ given by

$$f(t_1, \ldots, t_n) = \exp(t_1 x_1) \cdots \exp(t_n x_n).$$

Then $d_0 f = I_n$ in the basis with respect to the standard basis for $\mathbb{R}^n$ and $\{x_1, \ldots, x_n\}$ for $\mathfrak{g}$. In particular it is surjective and by the constant rank theorem, $f(\mathbb{R}^n)$ contatins a neighborhood of the identity of G. $\square$

Example 2.1.1. Let $G = SO(3, \mathbb{R})$. Then $\mathfrak{so}(3, \mathbb{R})$ consists of skew-symmetric mattrices, with basis:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The exponential matrix is given by

$$e^{tJ_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix},$$

i.e. rotation around x-axis by angle t; similarly, J_y, J_z generate rotations around y, z axes. Elements of the form $exp(tJx), exp(tJ_y), exp(tJ_z)$ generate a neighborhood of identity in SO(3, $\mathbb{R}$). Since $SO(3, \mathbb{R})$ is connected, these elements generate the whole group. For this reason, it is common to refer to J_x, J_y, J_z as "infinitesimal generators" of $SO(3, \mathbb{R})$. Thus, in a certain sense $SO(3, \mathbb{R})$ is generated by three elements.

Remark 2.1.3. To motivate the term "infinitesimal generators", one can think of them as directions in which one can move from the identity (using the exponential map) in order to generate the whole group.

2.2 The commutator

In literature, one has at least three ways to define the commutator of some Lie group, Namely

(i) Using left-invariant vector fields (see [Lee18]).

(ii) As the lowest order term in the logarithm of the multiplication of exponentials (see Lemma 2.2.1).

(iii) As the differential of the adjoint representation.

For completeness, we recall that in [Lee18], the commutator of a Lie group is defined as

$$[x, y] = [X, Y]_e \text{ for } x, y \in \mathfrak{g}$$

where X, Y are the left-invariant vector fields for which $X_e = x, Y_e = y$, and the bracket of two vector fields is defined as $[X, Y] = XY - YX$.

In [Kir08], the commutator is defined as the lowest order term in the logarithm of the multiplication of exponentials (see Lemma 2.2.1). We say that a map $Q : \mathfrak{g} \to \mathfrak{g}$ is of order k if $Q(tx) = t^k Q(X)$ for $t \in \mathbb{R}, x \in \mathfrak{g}$.

Lemma 2.2.1 (Lemma 3.11, [Kir08])**.** *Let G be a Lie group. Then there exists a neighborhood of $0 \in \mathfrak{g}$ and smooth (or analytic in the complex case) functions $[\cdot,\cdot], \mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that in that neighborhood:*

$$e^x e^y = e^{\mu(x,y)}$$
$$\mu(x,y) = x + y + \frac{1}{2}[x.y] + \cdots$$

with $[\cdot,\cdot]$ being a bilinear skew-symmetric form and the dots denoting terms of order higher than 3.

The commutator has the following properties:

Proposition 2.2.1 (Proposition 3.12, [Kir08])**.** *Let G, H be Lie groups and $\phi : G \to H$ a Lie group morphism. Then for $x, y \in \mathfrak{g}$ we have*

(i) $\phi_*[x,y] = [\phi_* x, \phi_* y]$.

(ii) $\mathrm{Ad}_g[x,y] = [\mathrm{Ad}_g x, \mathrm{Ad}_g y]$.

(iii) $e^x e^y e^{-x} e^{-y} = e^{[x,y]+\cdots}$*, where dots stand for degrees higher than* 2.

(iv) $\mathrm{ad}_x y = [x,y]$*, where* $\mathrm{ad} = d_e \mathrm{Ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

(v) $Ad_{e^x} = e^{\mathrm{ad}_x} \in \mathfrak{gl}(\mathfrak{g})$

For instance when G is commutative, the commutator is zero (and the exponential map is a homomorphism).

Example 2.2.1. (i) For $\mathfrak{g} = \mathfrak{gl}(d, \mathbb{K})$, the commutator is given by

$$[x,y] = xy - yx.$$

(ii) For a general associative algebra A, the commutator over $\mathcal{K}$, is given by the same formula.

(iii) Any vector space can be made into a commutative Lie algebra by defining the commutator to be zero.

Theorem 9 (Jacobi identity)**.** *Let G be a real or complex Lie group and $x, y, z \in \mathfrak{g}$. Then*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

Proof. See [Kir08, Theorem 3.16]. □

Proposition 2.2.2. *Differentiation at the identity induces a map*

$$\hom(G_1, G_2) \to \hom(\mathfrak{g}_1, \mathfrak{g}_2)$$

which is injective when G_1 is connected.

2.3 Subalgebras, ideals and center

Definition 2.3.1. A subalgebra of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subseteq \mathfrak{g}$ that is closed under the Lie bracket: $[x, y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$. An ideal is a subalgebra $\mathfrak{h}$ such that $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{h}, y \in \mathfrak{g}$.

It is easy to see that one can take quotients of Lie algebras by ideals and thus obtain Lie algebras.

The next theorem tells us that Lie subgroups correspond to subalgebras, and normal closed Lie subgroups correspond to ideals. In the case of the latter, a converse statement gives us a way to check if a subgroup is normal by checking whether the corresponding subalgebra is an ideal. In this way we obtain a link between an algebraic condition (being a normal subgroup) and a linear one (being an ideal).

Theorem 10 (Theorem 3.22, [Kir08]). *Let G be a real or complex Lie group with Lie algebra $\mathfrak{g}$. Then*

(i) If H is a Lie subgroup (not necessarily closed) of G, then $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$.

(ii) If H is a normal closed Lie subgroup of G, then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

(iii) If H is a closed Lie subgroup of G, both G and H are connected, then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ if and only if H is a normal subgroup of G.

2.4 Lie algebra of vector fields

For a manifold M, the space $\mathrm{Vect}(M)$ can be made into a Lie algebra by directly defining the bracket of two vector fields and showing that it is skew-symmetric and bilinear. However, one can think of $\mathrm{Diff}(M)$ as similar to a Lie group (but not quite, since it is infinite-dimensional), whose Lie algebra is $\mathrm{Vect}(M)$.

Definition 2.4.1. Let M be a smooth manifold and G be a Lie group.

(i) A smooth map $\rho : G \to \mathrm{Diff}(M)$ is a map that arises from a smooth action of G on M.

(ii) The Lie algebra of $\mathrm{Diff}(M)$ is the space of vector fields $\mathrm{Vect}(M)$.

(iii) The exponential map of $\mathrm{Diff}(M)$ is the flow (whenever well-defined) of a vector field: $\left(e^X\right)_m = \Phi_X^1(m)$.

(iv) The differential of $\rho : G \to \mathrm{Diff}(M)$ is the map $\rho_* : \mathfrak{g} \to \mathrm{Vect}(M)$ given by

$$(\rho_*(x))_m = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho\left(e^{tx}\right)(m).$$

(v) The commutator $[\xi, \eta]$ of two vector fields $\xi, \eta \in \mathrm{Vect}(M)$ is the unique vector field that satisfies

$$\Phi_\xi^t \Phi_\eta^s \Phi_\xi^{-t} \Phi_\eta^{-s} = \Phi_{[\xi,\eta]}^{ts} + \cdots$$

where the dots stand for terms of order 3 and higher in s, t.

Remark 2.4.1. To motivate why $\mathrm{Vect}(M) = \mathfrak{diff}(M)$, we note that $\mathfrak{diff}(M)$ should be given by the derivatives of the one-parameter subgroups of $\mathrm{Diff}(M)$. If $\phi^t \in \mathrm{Diff}(M)$ is a one-parameter subgroup of M, then its derivative defines a vector field $\partial_{t=0}\phi^t \in \mathrm{Vect}(M)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \phi^t(m) \in T_m(M), \text{ for } m \in M.$$

Similarly, to motivate the differential of $\rho : G \to \mathrm{Diff}(M)$, we note that ρ maps one-parameter subgroups to one-parameter subgroups.

Proposition 2.4.1 (Proposition 3.23, [Kir08])**.** *Let M be a smooth manifold and G be a Lie group acting smoothly on M.*

(i) The commutator of vector fields makes $\mathrm{Vect}(M)$ *into a Lie algebra.*

(ii) The commutator can be also defined by the following formulas:

$$\begin{aligned}[\xi, \eta] &= \frac{\mathrm{d}}{\mathrm{d}t}(\Phi_\xi^t)_*\eta \\ \partial_{[\xi,\eta]} f &= \partial_\eta \partial_\xi f - \partial_\xi \partial_\eta f \\ \left[f^i \partial_i, g^j \partial_j\right] &= \left(g^i \partial_i f^j - f^i \partial_i g^j\right) \partial_j\end{aligned}$$

Theorem 11 (Theorem 3.25, [Kir08])**.** *The transformation $\rho_* : \mathfrak{g} \to \mathrm{Vect}(M)$ is a Lie algebra morphism.*

Example 2.4.1. (i) Considering the action of G on itself by left-multiplication $L : G \to \mathrm{Diff}(G)$, the Lie algebra isomorphism

$$L_* : \mathfrak{g} \leftrightarrow \{\text{right invariant vector fields on }\}G$$

associates to $x \in \mathfrak{g}$ the right-invariant vector field X such that $X_e = x$.

2.5 Stabilizers and the center

Corollary 2.5.1. *Let $f : G_1 \to G_2$ be a morphism of real or complex Lie groups. Then*

(i) $\ker f$ *is a closed Lie subgroup with Lie algebra* $\ker f_*$.

(ii) The map $G_1/\ker f \to \mathrm{Im} f$ is an immersion.

(iii) When $\mathrm{Im} f$ is a submanifold, the map is a $G_1/\ker f \to \mathrm{Im} f$ diffeomorphism.

Example 2.5.1. (i) Let V be a vector space over $\mathbb{K}$ with a bilinear form B. Then

$$\mathrm{O}(V, B) = \{g \in \mathrm{GL}(V) : B(gu, gv) = B(u, v) \text{ for all } u, v \in V\}$$

is a closed Lie group with Lie algebra:

$$\mathfrak{o}(V, B) = \{g \in \mathfrak{gl}(V) : B(xu, xv) = B(u, v) \text{ for all } u, v \in V\}.$$

Indeed, define an action of $\mathrm{GL}(V)$ on the space $\mathcal{F}$ of bilinear forms on V by $gF(u, v) = F(g^{-1}u, g^{-1}v)$. This defines a representation $\rho : \mathrm{GL}(V) \to \mathrm{GL}(\mathcal{F})$ and $\mathrm{O}(V, B)$ is the stabilizer of B under this action. The space $\mathcal{F}$ is a vector space, so $\mathrm{Vect}(\mathcal{F})$ can be identifies with the space smooth maps from $\mathcal{F}$ to itself. Then for $x \in \mathfrak{gl}(V)$

$$\begin{aligned}(\rho_*(x))_B\,(u,v) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\rho(e^{tx})B\right)(u,v) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}B(e^{-tx}u, e^{-tx}v)\right)\\ &= -B(u,xv) - B(xu,v).\end{aligned}$$

(ii) Let A be a finite-dimensional associative algebra over $\mathbb{K}$. Then the group of all automorphisms of A

$$\mathrm{Aut}(A) = \{g \in \mathrm{GL}(A) \mid (ga)\cdot(gb) = g(a\cdot b) \text{ for all } a,b \in A\}$$

is a Lie group with Lie algebra

$$\mathrm{Der}(A) = \{x \in \mathfrak{gl}(A) \mid (x.a)b + a(x.b) = x(ab) \text{ for all } a,b \in A\}$$

(this Lie algebra is called the *algebra of derivations of* A).

Indeed, if we consider the space W of all linear maps $A \otimes A \to A$ and define the action of G by $(gf)(a\otimes b) = gf(g^{-1}a \otimes g^{-1}b)$ then $\mathrm{Aut}A = G_\mu$, where $\mu : A \otimes A \to A$ is the multiplication. So $\mathrm{Aut}(A)$ is a Lie group with Lie algebra $\mathrm{Der}(A)$.

(iii) The same argument also shows that for a finite-dimensional Lie algebra $\mathfrak{g}$, the group

$$\mathrm{Aut}(\mathfrak{g}) = \{g \in \mathrm{GL}(\mathfrak{g}) \mid [ga, gb] = g[a,b] \text{ for all } a,b \in \mathfrak{g}\}$$

is a Lie group with Lie algebra

$$\mathrm{Der}(\mathfrak{g}) = \{x \in \mathfrak{gl}(\mathfrak{g}) \mid [x.a, b] + [a, x.b] = x.[a,b] \text{ for all } a,b \in \mathfrak{g}\}$$

called the Lie algebra of derivations of $\mathfrak{g}$.

Finally, we can show that the center of G is a closed Lie subgroup.

Theorem 12. *Let G be a connected Lie group. Then its center $Z(G)$ is a closed Lie subgroup with Lie algebra the ideal*

$$\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x,y] = 0\ \forall y \in \mathfrak{g}\}.$$

The quotient group $G/Z(G)$ is usually called the *adjoint group* associated with G and denoted $\mathrm{Ad}\,G$:

$$\mathrm{Ad}\,G = G/Z(G) = \mathrm{Im}(\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})) \quad \text{(for connected } G\text{)}.$$

The corresponding Lie algebra is

$$\mathrm{ad}\,\mathfrak{g} = \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) = \mathrm{Im}(\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})).$$

Appendix A

Covering theory reminder

In this chapter we recall certain facts and definitions from basic covering theory. A nice reference for these is the Chapter 2 from [Hat02]. We begin by defining covering spaces.

Definition A.0.1. A covering map is a continuous surjective map $p : \tilde{X} \to X$ such that for every $x \in X$ there exists an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U by p.

The prototypical example of a covering map is $p : \mathbb{S}^1 \to \mathbb{S}^1, p(z) = z^n$.

A.1 Lifting properties

One of the particular characteristics of covering spaces are their lifting properties, that we will recall below.

Proposition A.1.1 (Homotopy lifting property)**.** *Let $p : \tilde{X} \to X$ be a covering space and a homotopy $f_t : Y \to X$. Then every lift $\tilde{f}_0 : Y \to \tilde{X}$ of f_0 extends to a unique homotopy $\tilde{f}_t$ lifting f_t.*

Proof. See [Hat02, Proposition 1.30]. □

This in particular implies the path lifting property of covering spaces:

Corollary A.1.1. *Let $p : \tilde{X} \to X$ be a covering space. Then for every path $\gamma : I \to X$ and every lift $\tilde{x}_0$ of some point $x_0 \in X$ admits a unique lift $\tilde{\gamma} : I \to \tilde{X}$ of γ starting at $\tilde{x}_0$.*

Both of the results above imply that path-homotopies lift to pathhomotopies, where we require for a path homotopy to keep the endpoints of paths fixed. We also have the following corollary that is useful in proving the lifting criterion below.

Corollary A.1.2. *The image subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are loops.*

Proof. See [Hat02, Corollary 1.31]. □

If we care about lifting maps and not homotopies, we have the following criterion that tells us when a lift exists. Namely when f sends loops to loops that lift to loops.

Proposition A.1.2 (Lifting criterion)**.** *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space and $f : (Y, y_0) \to (X, x_0)$ with Y being path-connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ exists if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, x_0))$.*

Proof. See [Hat02, Proposition 1.33]. □

And regarding uniqueness of lifts:

Proposition A.1.3. *Let $p : \tilde{X} \to X$ be a covering space and $f : Y \to X$ be a map. If Y is connected, then any two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ of f that coincide at one point will coincide everywhere on Y.*

Proof. See [Hat02, Proposition 1.34]. □

A.2 Universal covering

In this section we will be concerned with proving that under mild conditions, a space has a universal covering, i.e. a simply connected covering space.

Assumption 1. We consider a topological space X that will be path-connected, locally connected and semi-locally simply connected.

While the first two assummptions may seem rather natural, we will now explain the third one.

Definition A.2.1. A space X is semi-locally simply connected if for every $x \in X$ there exists an open neighborhood U of x such that every loop in U based at x is homotopic in X to a constant loop in U. In other words, the homomorphism

$$\pi_1(U, x) \to \pi_1(X, x)$$

induced by the inclusion is trivial.

To motivate this assumption, we note that it is necessary for the existence of a universal covering. Indeed, if $p : \tilde{X} \to X$ is a universal covering, $x_0 \in X$ and U an evenly covered neighborhood of x_0, then any loop in U based at x lifts to a loop in $\tilde{X}$ based at some $\tilde{x}_0$. Since $\tilde{X}$ is simply connected, this loop is homotopic to a constant loop in $\tilde{X}$ and the homotopy projects down to a homotopy in X.

Before constructing the universal cover, we remark that every universal cover $\tilde{X}$ can be thought of as homotopy classes of paths in X starting at some fixed point x_0.

Remark A.2.1. Let $p : \tilde{X} \to X$ be a universal covering and $x_0 \in X$. Then the set of homotopy classes of paths in X starting at x_0 is in bijection with $\tilde{X}$. Indeed, given a path $\gamma : I \to X$ starting at x_0, we associate it to its endpoint, which defines a map $\tilde{\gamma} : I \to \tilde{X}$. This map is well-defined since homotopic paths have the same endpoint. It is surjective, beccause X is path-connected, while injectivity follows from the fact that $\tilde{X}$ is simply connected.

We now proceed with the construction of the universal cover by fixing some $x_0 \in X$ and letting

$$\tilde{X} \overset{\text{def}}{=} \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\},$$

and defining the projection map $p : \tilde{X} \to X$ by $p([\gamma]) = \gamma(1)$.

To define a topology on $\tilde{X}$, we first consider a convenient basis of open sets for X:

$$\mathcal{U} \overset{\text{def}}{=} \{U \subseteq X \text{ open and path-connected} \mid \pi_1(U, x) \to \pi_1(X, x) \text{ is trivial for some } x \in U\}.$$

Note that the above set is well-defined since if there exists some $x \in U$ such that $\pi_1(U, x) \to \pi_1(X, x)$ is trivial, then it is trivial for all $x' \in U$ because U is path-connected. To see that it is a basis for X, we note that $V \in \mathcal{U}$ for every a path-connected open subset $V \subseteq U$.

To each homotopy class $[\gamma] \in \tilde{X}$ and $U \in \mathcal{U}$ we associate the set

$$U_{[\gamma]} \overset{\text{def}}{=} \left\{ [\gamma \cdot \eta] \in \tilde{X} \mid \eta \text{ is a path in } U \text{ such that } \eta(0) = \gamma(1) \right\}.$$

Then, given $x \in X$ and a neighborhood U of x, the collection

$$U_{[\gamma]} \text{ for paths } \gamma \text{ going from } x_0 \text{ to } x$$

will be the sheets of U. The following property is crucial into showing that the topology defined by the above basis is well-defined:

$$U[\gamma] = U_[\gamma'] \text{ for all } [\gamma'] \in U_{[\gamma]}.$$

The proofs that $p : U_[\gamma] \to U$ is a homeomorphism for all $U \in \mathcal{U}$ and $[\gamma] \in \tilde{X}$ and that $\tilde{X}$ is simply connected can be found in [Hat02].

An elementary example is the torus:

Example A.2.1. For $X = \mathbb{T}^2$ being the torus, the universal covering $p : \mathbb{R}^2 \to \mathbb{T}^2$ and given by $p(t, s) = e^{2\pi it, 2\pi is}$. Under the identification $\pi_1(\mathbb{T}^2) \simeq \mathbb{N}^2$, which acts on $\mathbb{R}^2$ by translations, we have $\mathbb{T}^2 \simeq \mathbb{R}^2/\mathbb{N}^2$.

Having now constructed the universal cover, we now move on to discussing deck transformations.

Definition A.2.2. A deck transformation of a covering space $p : \tilde{X} \to X$ is a homeomorphism $\phi : \tilde{X} \to \tilde{X}$ such that $p \circ \phi = p$. In other words, the following triangular diagram commutes:

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\phi"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

For instance, the fundamental group $\pi_1(X, x_0)$ of X acts by deck transformations on the universal cover $\tilde{X}$. To each $[\alpha] \in \pi_1(X, x_0)$ we associate the deck transformation $\phi_{[\alpha]}$ defined by

$$\phi_{[\alpha]}([\eta]) = [\alpha \cdot \eta].$$

Considering the quotient space $\tilde{X}/\pi_1(X, x_0)$, we have the following result:

Theorem 13. *The quotient space $\tilde{X}/\pi_1(X, x_0)$ is homeomorphic to X where the homeomorphism is induced by the projection map $p : \tilde{X} \to X$ as in the following diagram*

$$\begin{tikzcd} \tilde{X} \arrow[r, "p"] \arrow[d] & X \\ \tilde{X}/\pi_1(X, x_0) \arrow[ur, dashed, "\simeq"'] & \end{tikzcd}$$

Bibliography

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