

Lie group notes

Giorgos

June 17, 2024

Contents

1	Lie groups: basic definitions	3
1.0.1	Homogeneous spaces	3
1.0.2	Classical Lie groups	4
2	Lie groups and Lie algebras	6
2.1	Exponential map	6

Conventions

- By Lie group, we mean either real or complex.
- A closed Lie subgroup of a Lie group is a submanifold that is also a subgroup (see Theorem 3).
- A Lie subgroup is a subgroup that is also an immersed submanifold.

Chapter 1

Lie groups: basic definitions

The following theorem allows us to reduce the study of Lie groups to the study of finite groups and connected Lie groups, since for a general Lie group G we have

$$G = G^0 \times G/G^0,$$

where G^0 is the identity component of G .

Theorem 1 (Theorem 2.6, [Kir08]). *Let G be a real or complex Lie group and G^0 its identity component. Then G^0 is a normal subgroup of G and G/G^0 is a discrete group.*

In fact, we can reduce the case of connected Lie groups to simply connected Lie groups:

Theorem 2 (Theorem 2.7, [Kir08]). *Let G be a connected Lie group. Then its universal cover \tilde{G} has a canonical structure of a Lie group such that the covering map $p : \tilde{G} \rightarrow G$ is a homomorphism of Lie groups whose kernel is isomorphic to the fundamental group of G . Moreover, in this case, $\ker p$ is a discrete central subgroup in \tilde{G} .*

We have the following connection between subgroups and Lie subgroups (i.e. subgroups that are also submanifolds):

Theorem 3 (Theorem 2.8, [Kir08]). *(i) Any Lie subgroup of a Lie group is closed in the topology of the ambient group.*

(ii) Any closed subgroup of a Lie group is a real Lie subgroup.

1.0.1 Homogeneous spaces

We begin by describing coset spaces of Lie groups.

Theorem 4 (Theorem 2.11, [Kir08]). *Let G be a Lie group of dimension n , $H \leq G$ a closed Lie subgroup of dimension k .*

(i) The coset space G/H has a natural structure of a manifold of dimension $n - k$ such that the canonical map $p : G \rightarrow G/H$ is a fiber bundle, with fiber diffeomorphic to H . The tangent space at the identity is isomorphic to the quotient space $T_H G/H \simeq T_e G/T_e H$.

(ii) If H is a normal closed Lie subgroup then G/H has a canonical Lie group structure.

The following is the analog of the homomorphism theorem for Lie groups:

Theorem 5 (Theorem 2.5, [Kir08]). *Let $f : G_1 \rightarrow G_2$ be a Lie group morphism.*

- (i) *$H = \ker f$ is a normal closed Lie subgroup of G_1 and f induces an injective homomorphism $G_1/H \rightarrow G_2$ that is an immersion*
- (ii) *If moreover $\Im f$ is an embedded submanifold, then it is a closed Lie subgroup of G_2 and f induces an isomorphism $G_1/H \rightarrow \Im f$.*

Theorem 6 (Theorem 2.20, [Kir08]). *Let G be a Lie group acting on a manifold M , and $m \in M$.*

- (i) *The stabilizer is a closed Lie subgroup of G , and the orbit map $g \mapsto g \cdot m$ induces an injective immersion $G/G_m \hookrightarrow \mathcal{O}_m$ whose image coincides with \mathcal{O}_m .*
- (ii) *The orbit \mathcal{O}_m is an immersed submanifold with tangent space $T_m \mathcal{O}_m = T_1 G / T_1 G_m$.*
- (iii) *If the orbit is a submanifold, then the orbit map is a diffeomorphism.*

The case of one orbit gives rise to G -homogeneous spaces:

Theorem 7 (Theorem 2.22 [Kir08]). *Let M be a G -homogeneous space and $m \in M$. Then the orbit map $G \rightarrow M$ is a fiber bundle over M with fiber G_m .*

1.0.2 Classical Lie groups

Definition 1.0.1. We define

$$Sp(n, \mathbb{K}) = \{A \in GL(2n, \mathbb{K}) \mid \omega(Ax, Ay) = \omega(x, y)\}.$$

where ω is the standard symplectic form on \mathbb{K}^{2n} , which is given by $\omega(x, y) = x^* J y = \sum_{i=1}^n x_i y_{i+n} - x_{i+n} y_i$, and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

We also define the group of unitary quaternionic transformations by

$$Sp(n) = Sp(n, \mathbb{C}) \cap SU(2n).$$

The following theorem tells us that the logarithmic map behaves well when restricted to a neighborhood of the identity in each classical group.

G	$O(n, \mathbb{R})$	$SO(n, \mathbb{R})$	$U(n)$	$SU(n)$	$Sp(n)$
\mathfrak{g}	$x + x' = 0$	$x + x' = 0$	$x + x^* = 0$	$x + x^* = 0, \operatorname{tr} x = 0$	$x + J^{-1}x'J = 0, x + x^* = 0$
$\dim G$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	n^2	$n^2 - 1$	$n(2n + 1)$
$\pi_0(G)$	\mathbb{Z}_2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}_2 \ (n \geq 3)$	$\mathbb{Z}_2 \ (n \geq 3)$	\mathbb{Z}	$\{1\}$	$\{1\}$

Table 1.1: Compact classical groups

G	$GL(n, \mathbb{R})$	$SL(n, \mathbb{R})$	$Sp(n, \mathbb{R})$
\mathfrak{g}	$\mathfrak{gl}(n, \mathbb{R})$	$\text{tr } x = 0$	$x + J^{-1}x'J = 0$
$\dim G$	n^2	$n^2 - 1$	$n(2n + 1)$
$\pi_0(G)$	\mathbb{Z}_2	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}_2 \ (n \geq 3)$	$\mathbb{Z}_2 \ (n \geq 3)$	\mathbb{Z}

Table 1.2: Noncompact real classical groups.

G	$GL(n, \mathbb{C})$	$SL(n, \mathbb{C})$	$O(n, \mathbb{C})$	$SO(n, \mathbb{C})$
$\pi_0(G)$	$\{1\}$	$\{1\}$	\mathbb{Z}_2	$\{1\}$
$\pi_1(G)$	\mathbb{Z}	$\{1\}$	\mathbb{Z}_2	\mathbb{Z}_2

Table 1.3: Complex classical groups.

Chapter 2

Lie groups and Lie algebras

2.1 Exponential map

Definition 2.1.1 (Proposition 3.1, [Kir08]). Let G be a Lie group and $x \in \mathfrak{g}$. The one-parameter subgroup $\gamma_x : \mathbb{K} \rightarrow G$ is the unique Lie group morphism such that $\gamma'_x(0) = x$. We define the exponential map of G as

$$\exp(x) = \gamma_x(1)$$

Remark 2.1.1. By looking at the proof of the statements in the above definition, one can see that for $x \in \mathfrak{g}$, the curve

$$\exp(tx) = \gamma_x(t) = \gamma_{tx}(1).$$

integral curve of the left-invariant vector field $X \in \mathcal{X}(G)$ that satisfies

$$X_e = x.$$

The following are some properties of the exponential map:

Theorem 8 (Theorems 3.7 and 3.36, [Kir08]). *Let G be a Lie group.*

- (i) $d_e \exp = \text{id}_{\mathfrak{g}}$
- (ii) *The exponential map is a local diffeomorphism at 0.*
- (iii) *For any Lie group morphism $\phi : G_1 \rightarrow G_2$, we have $d_e \phi(\exp(x)) = \exp d_e(\phi(x))$ for all $x \in \mathfrak{g}$.*
- (iv) *For any $g \in G, x \in \mathfrak{g}$*
$$g \exp(x) g^{-1} = \exp(\text{Ad}_g x).$$
- (v) *For $x, y \in \mathfrak{g}$, we have:*

$$\text{If } [x, y] = 0 \text{ then } e^x e^y = e^y e^x = e^{x+y}.$$

Remark 2.1.2. The exponential map is not surjective in general. It is however for compact Lie groups.

Example 2.1.1. Let $G = SO(3, \mathbb{R})$. Then $\mathfrak{so}(3, \mathbb{R})$ consists of skew-symmetric matrices, with basis:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The exponential matrix is given by

$$e^{tJ_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix},$$

i.e. rotation around x-axis by angle t ; similarly, J_y, J_z generate rotations around y, z axes. Elements of the form $\exp(tJ_x), \exp(tJ_y), \exp(tJ_z)$ generate a neighborhood of identity in $SO(3, \mathbb{R})$. Since $SO(3, \mathbb{R})$ is connected, these elements generate the whole group. For this reason, it is common to refer to J_x, J_y, J_z as “infinitesimal generators” of $SO(3, \mathbb{R})$. Thus, in a certain sense $SO(3, \mathbb{R})$ is generated by three elements.

Bibliography

- [Kir08] Alexander A Kirillov. *An introduction to Lie groups and Lie algebras*. Vol. 113. Cambridge University Press, 2008 (cit. on pp. 3, 4, 6).