Lie group notes

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Contents

Conventions

- By Lie group, we mean either real or complex.
- A closed Lie subgroup of a Lie group is a submanifold that is also a subgroup (see Theorem 3).
- A Lie subgroup is a subgroup that is also an immersed submanifold.

Chapter 1

Lie groups: basic definitions

The following theorem allows us to reduce the study of Lie groups to the study of finite groups and connected Lie groups, since G_0 is a normal subgroup of G and G/G_0 is a discrete group. where G^0 is the identity component of G.

Theorem 1 (Theorem 2.6, [Kir08])**.** *Let G be a real or complex Lie group and G^0 its identity component. Then G^0 is a normal subgroup of G and a Lie group itself, while G/G^0 is a discrete group.*

In fact, we can reduce the case of connected Lie groups to simply connected Lie groups:

Theorem 2 (Theorem 2.7, [Kir08])**.** *Let G be a connected Lie group. Then its universal cover $\tilde{G}$ has a canonical structure of a Lie group such that the covering map $p : \tilde{G} \to G$ is a homomorphism of Lie groups whose kernel is isomorphic to the fundamental group of G. Moreover, in this case, $\ker p$ is a discrete central subgroup in $\tilde{G}$.*

We have the following connection between subgroups and Lie subgroups (i.e. subgroups that are also submanifolds):

Theorem 3 (Theorem 2.8, [Kir08])**.**

(i) Any Lie subgroup of a Lie group is closed in the topology of the ambient group.

(ii) Any closed subgroup of a Lie group is a real Lie subgroup.

1.0.1 Homogeneous spaces

We begin by describing coset spaces of Lie groups.

Theorem 4 (Theorem 2.11, [Kir08])**.** *Let G be a Lie group of dimension n, $H \leq G$ a closed Lie subgroup of dimension k.*

(i) The coset space G/H has a natural structure of a manifold of dimension $n - k$ such that the canonical map $p : G \to G/H$ is a fiber bundle, with fiber diffeomorphic to H. The tangent space at the identity is isomorphic to the quotient space $T_H G/H \simeq T_e G/T_e H$.

(ii) If H is a normal closed Lie subgroup then G/H has a canonical Lie group structure.

The following is the analog of the homomorphism theorem for Lie groups:

Theorem 5 (Theorem 2.5, [Kir08])**.** *Let $f : G_1 \to G_2$ be a Lie group morphism.*

(i) $H = \ker f$ *is a normal closed Lie subgroup of* G_1 *and* f *induces an injective homomorphism* $G_1/H \to G_2$ *that is an immersion*

(ii) *If moreover* $\Im f$ *is an embedded submanifold, then it is a closed Lie subgroup of* G_2 *and* f *induces an isomorphism* $G_1/H \to \Im f$.

Theorem 6 (Theorem 2.20, [Kir08])**.** *Let* G *be a Lie group acting on a manifold* M, *and* $m \in M$.

(i) *The stabilizer is a closed Lie subgroup of* G, *and the orbit map* $g \mapsto g \cdot m$ *induces an injective immersion* $G/G_m \hookrightarrow \mathcal{O}_m$ *whose image coincides with* $\mathcal{O}_m$.

(ii) *The orbit* $\mathcal{O}_m$ *is an immersed submanifold with tangent space* $T_m\mathcal{O}_m = T_1G/T_1G_m$.

(iii) *If the orbit is a submanifold, then the orbit map is a diffeomorphism.*

The case of one orbit gives rise to G-homogeneous spaces:

Theorem 7 (Theorem 2.22 [Kir08])**.** *Let* M *be a* G*-homogeneous space and* $m \in M$. *Then the orbit map* $G \to M$ *is a fiber bundle over* M *with fiber* G_m.

1.0.2 Classical Lie groups

Definition 1.0.1. We define

$$Sp(n, \mathbb{K}) = \{A \in GL(2n, \mathbb{K}) \mid \omega(Ax, Ay) = \omega(x, y)\}.$$

where ω is the standard symplectic form on $\mathbb{K}^{2n}$, which is given by $\omega(x, y) = x^* Jy = \sum_{i=1}^n x_i y_{i+n} - x_{i+n} y_i$, and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

We also define the gorup of unitary quaternionic transformations by

$$\mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{SU}(2n).$$

The following theorem tells us that the logarithmic map behaves well when restricted to a neighborhood of the identity in each classical group.

G	$O(n,\mathbb{R})$	$SO(n,\mathbb{R})$	$U(n)$	$SU(n)$	$Sp(n)$
$\mathfrak{g}$	$x + x' = 0$	$x + x' = 0$	$x + x^* = 0$	$x + x^* = 0,\ \operatorname{tr} x = 0$	$x + J^{-1}x'J = 0\ x + x^* = 0$
$\dim G$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	n^2	$n^2 - 1$	$n(2n+1)$
$\pi_0(G)$	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}_2\ (n \geq 3)$	$\mathbb{Z}_2\ (n \geq 3)$	$\mathbb{Z}$	$\{1\}$	$\{1\}$

Table 1.1: Compact classical groups

G	$GL(n,\mathbb{R})$	$SL(n,\mathbb{R})$	$Sp(n,\mathbb{R})$
$\mathfrak{g}$	$\mathfrak{gl}(n,\mathbb{R})$	$\operatorname{tr} x = 0$	$x + J^{-1}x'J = 0$
$\dim G$	n^2	$n^2 - 1$	$n(2n+1)$
$\pi_0(G)$	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}_2$ $(n \geq 3)$	$\mathbb{Z}_2$ $(n \geq 3)$	$\mathbb{Z}$

Table 1.2: Noncompact real classical groups.

G	$GL(n,\mathbb{C})$	$SL(n,\mathbb{C})$	$O(n,\mathbb{C})$	$SO(n,\mathbb{C})$
$\pi_0(G)$	$\{1\}$	$\{1\}$	$\mathbb{Z}_2$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}$	$\{1\}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Table 1.3: Complex classical groups.

Chapter 2

Lie groups and Lie algebras

2.1 Exponential map

Definition 2.1.1 (Proposition 3.1, [Kir08]). Let G be a Lie group and $x \in \mathfrak{g}$. The one-parameter subgroup $\gamma_x : \mathbb{K} \to G$ is the unique Lie group morphism such that $\gamma_x'(0) = x$. We define the exponential map of G as

$$\exp(x) = \gamma_x(1)$$

Remark 2.1.1. By looking at the proof of the statements in the above definition, one can see that for $x \in \mathfrak{g}$, the curve

$$\exp(tx) = \gamma_x(t) = \gamma_{tx}(1).$$

integral curve of the left-invariant vector field $X \in \mathcal{X}(G)$ that satisfies

$$X_e = x.$$

The following are some properties of the exponential map:

Theorem 8 (Theorems 3.7 and 3.36, [Kir08]). *Let G be a Lie group.*

(i) $d_e \exp = \mathrm{id}_{\mathfrak{g}}$

(ii) The exponential map is a local diffeomorphism at 0.

(iii) For any Lie group morphism $\phi : G_1 \to G_2$, we have $d_e\phi(exp(x)) = \exp d_e(\phi(x))$ for all $x \in \mathfrak{g}$.

(iv) For any $g \in G, x \in \mathfrak{g}$

$$g \exp(x) g^{-1} = \exp(\mathrm{Ad}_g\, x).$$

(v) For $x, y \in \mathfrak{g}$, we have:

$$\textit{If } [x, y] = 0 \textit{ then } e^x e^y = e^y e^x = e^{x+y}.$$

Remark 2.1.2. The exponential map is not surjective in general. It is however for compact Lie groups.

Example 2.1.1. Let $G = SO(3,\mathbb{R})$. Then $\mathfrak{so}(3,\mathbb{R})$ consists of skew-symmetric mattrices, with basis:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The exponential matrix is given by

$$e^{tJ_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix},$$

i.e. rotation around x-axis by angle t; similarly, J_y, J_z generate rotations around y, z axes. Elements of the form $exp(tJx), exp(tJ_y), exp(tJ_z)$ generate a neighborhood of identity in SO$(3,\mathbb{R})$. Since $SO(3,\mathbb{R})$ is connected, these elements generate the whole group. For this reason, it is common to refer to J_x, J_y, J_z as "infinitesimal generators" of $SO(3,\mathbb{R})$. Thus, in a certain sense $SO(3,\mathbb{R})$ is generated by three elements.

Appendix A

Covering theory reminder

In this chapter we recall certain facts and definitions from basic covering theory. A nice reference for these is the Chapter 2 from [Hat02]. We begin by defining covering spaces.

Definition A.0.1. A covering map is a continuous surjective map $p : \tilde{X} \to X$ such that for every $x \in X$ there exists an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U by p.

The prototypical example of a covering map is $p : \mathbb{S}^1 \to \mathbb{S}^1, p(z) = z^n$.

A.1 Lifting properties

One of the particular characteristics of covering spaces are their lifting properties, that we will recall below.

Proposition A.1.1 (Homotopy lifting property)**.** *Let $p : \tilde{X} \to X$ be a covering space and a homotopy $f_t : Y \to X$. Then every lift $\tilde{f}_0 : Y \to \tilde{X}$ of f_0 extends to a unique homotopy $\tilde{f}_t$ lifting f_t.*

Proof. See [Hat02, Proposition 1.30]. □

This in particular implies the path lifting property of covering spaces:

Corollary A.1.1. *Let $p : \tilde{X} \to X$ be a covering space. Then for every path $\gamma : I \to X$ and every lift $\tilde{x}_0$ of some point $x_0 \in X$ admits a unique lift $\tilde{\gamma} : I \to \tilde{X}$ of γ starting at $\tilde{x}_0$.*

Both of the results above imply that path-homotopies lift to pathhomotopies, where we require for a path homotopy to keep the endpoints of paths fixed. We also have the following corollary that is useful in proving the lifting criterion below.

Corollary A.1.2. *The image subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are loops.*

Proof. See [Hat02, Corollary 1.31]. □

If we care about lifting maps and not homotopies, we have the following criterion that tells us when a lift exists. Namely when f sends loops to loops that lift to loops.

Proposition A.1.2 (Lifting criterion)**.** *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space and $f : (Y, y_0) \to (X, x_0)$ with Y being path-connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ exists if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, x_0))$.*

Proof. See [Hat02, Proposition 1.33]. □

And regarding uniqueness of lifts:

Proposition A.1.3. *Let $p : \tilde{X} \to X$ be a covering space and $f : Y \to X$ be a map. If Y is connected, then any two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ of f that coincide at one point will coincide everywhere on Y.*

Proof. See [Hat02, Proposition 1.34]. □

A.2 Universal covering

In this section we will be concerned with proving that under mild conditions, a space has a universal covering, i.e. a simply connected covering space.

Assumption 1. We consider a topological space X that will be path-connected, locally connected and semi-locally simply connected.

While the first two assummptions may seem rather natural, we will now explain the third one.

Definition A.2.1. A space X is semi-locally simply connected if for every $x \in X$ there exists an open neighborhood U of x such that every loop in U based at x is homotopic in X to a constant loop in U. In other words, the homomorphism

$$\pi_1(U, x) \to \pi_1(X, x)$$

induced by the inclusion is trivial.

To motivate this assumption, we note that it is necessary for the existence of a universal covering. Indeed, if $p : \tilde{X} \to X$ is a universal covering, $x_0 \in X$ and U an evenly covered neighborhood of x_0, then any loop in U based at x lifts to a loop in $\tilde{X}$ based at some $\tilde{x}_0$. Since $\tilde{X}$ is simply connected, this loop is homotopic to a constant loop in $\tilde{X}$ and the homotopy projects down to a homotopy in X.

Before constructing the universal cover, we remark that every universal cover $\tilde{X}$ can be thought of as homotopy classes of paths in X starting at some fixed point x_0.

Remark A.2.1. Let $p : \tilde{X} \to X$ be a universal covering and $x_0 \in X$. Then the set of homotopy classes of paths in X starting at x_0 is in bijection with $\tilde{X}$. Indeed, given a path $\gamma : I \to X$ starting at x_0, we associate it to its endpoint, which defines a map $\tilde{\gamma} : I \to \tilde{X}$. This map is well-defined since homotopic paths have the same endpoint. It is surjective, beccause X is path-connected, while injectivity follows from the fact that $\tilde{X}$ is simply connected.

We now proceed with the construction of the universal cover by fixing some $x_0 \in X$ and letting

$$\tilde{X} \overset{\text{def}}{=} \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\},$$

and defining the projection map $p : \tilde{X} \to X$ by $p([\gamma]) = \gamma(1)$.

To define a topology on $\tilde{X}$, we first consider a convenient basis of open sets for X:

$$\mathcal{U} \overset{\text{def}}{=} \{U \subseteq X \text{ open and path-connected} \mid \pi_1(U, x) \to \pi_1(X, x) \text{ is trivial for some } x \in U\}.$$

Note that the above set is well-defined since if there exists some $x \in U$ such that $\pi_1(U, x) \to \pi_1(X, x)$ is trivial, then it is trivial for all $x' \in U$ because U is path-connected. To see that it is a basis for X, we note that $V \in \mathcal{U}$ for every a path-connected open subset $V \subseteq U$.

To each homotopy class $[\gamma] \in \tilde{X}$ and $U \in \mathcal{U}$ we associate the set

$$U_{[\gamma]} \stackrel{\text{def}}{=} \left\{ [\gamma \cdot \eta] \in \tilde{X} \mid \eta \text{ is a path in } U \text{ such that } \eta(0) = \gamma(1) \right\}.$$

Then, given $x \in X$ and a neighborhood U of x, the collection

$$U_{[\gamma]} \text{ for paths } \gamma \text{ going from } x_0 \text{ to } x$$

will be the sheets of U. The following property is crucial into showing that the topology defined by the above basis is well-defined:

$$U[\gamma] = U_[\gamma'] \text{ for all } [\gamma'] \in U_{[\gamma]}.$$

The proofs that $p : U_[\gamma] \to U$ is a homeomorphism for all $U \in \mathcal{U}$ and $[\gamma] \in \tilde{X}$ and that $\tilde{X}$ is simply connected can be found in [Hat02].

Having now constructed the universal cover, we now move on to discussing deck transformations.

Definition A.2.2. A deck transformation of a covering space $p : \tilde{X} \to X$ is a homeomorphism $\phi : \tilde{X} \to \tilde{X}$ such that $p \circ \phi = p$. In other words, the following triangular diagram commutes:

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\phi"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

For instance, the fundamental group $\pi_1(X, x_0)$ of X acts by deck transformations on the universal cover $\tilde{X}$. To each $[\alpha] \in \pi_1(X, x_0)$ we associate the deck transformation $\phi_{[\alpha]}$ defined by

$$\phi_{[\alpha]}([\eta]) = [\alpha \cdot \eta].$$

Considering the quotient space $\tilde{X}/\pi_1(X, x_0)$, we have the following result:

Theorem 9. *The quotient space $\tilde{X}/\pi_1(X, x_0)$ is homeomorphic to X where the homeomorphism is induced by the projection map $p : \tilde{X} \to X$ as in the following diagram*

$$\begin{tikzcd} \tilde{X} \arrow[r, "p"] \arrow[d] & X \\ \tilde{X}/\pi_1(X, x_0) \arrow[ur, dashed, "\simeq"'] & \end{tikzcd}$$

Bibliography

[Kir08] Alexander A Kirillov. *An introduction to Lie groups and Lie algebras*. Vol. 113. Cambridge University Press, 2008 (cit. on pp. 3, 4, 6).

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002, pp. xii+544. ISBN: 0-521-79160-X; 0-521-79540-0 (cit. on pp. 8–10).