Limit sets of Anosov representations

Giorgos

June 23, 2024

Contents

Chapter 1

Introduction

1.1 Lie group preliminaries

We fix the Cartan subalgebra $\mathfrak{a}$ of $\mathrm{SL}(d,\mathbb{R})$:

$$\mathfrak{a} = \{\mathrm{diag}(\alpha_1,\dots,\alpha_d) : \alpha_1 + \cdots \alpha_d = 0\}$$

and the Weyl chamber $\mathfrak{a}^+$ of $\mathrm{SL}(d,\mathbb{R})$

$$\mathfrak{a}^+ = \{\mathrm{diag}(\alpha_1,\dots,\alpha_d) : \alpha_1 \geq \cdots \geq \alpha_d\}.$$

Denoting with $K = \mathrm{SO}(d,\mathbb{R})$, $A^+ = e^{\mathfrak{a}^+}$, we have the Cartan decomposition:

$$\begin{aligned} \mathfrak{sl}(d,\mathbb{R}) &\to K \times A^+ \times K \\ g &\mapsto (k_g, a_g, l_g) \end{aligned}$$

such that $g = k_g a_g l_g$. In particular $a_g = \mathrm{diag}(\sigma_1(g),\dots,\sigma_d(g))$ with $\sigma_1 \geq \cdots \geq \sigma_d(g)$, where $\sigma_i(g)$ is the i-th singular value of g, i.e. eigenvalue of $g^t \cdot g$.

We will use the spaces

$$U_p(g) = \mathbb{R}u_1(g) \oplus \cdots \oplus \mathbb{R}u_p(g)$$

where $u_i(g) = k_g \cdot e_i$. One can easily show that the decomposition

$$g^{-1} \cdot U_p(g) \oplus U_{d-p}(g^{-1})$$

is orthogonal with respect to the standard inner product and that $u_p(g^{-1}) = l_g^{-1} e_{d-p+1}(g)$.

1.2 Limit set preliminaries

Definition 1.2.1. For $p \in \{2,\dots,d\}$, $s \in \mathbb{R}$ and $g \in SL(d,\mathbb{R})$ we denote with $\tilde{\Psi}_s^p(g), \Psi_s^p(g)$: $\mathrm{SL}(d,\mathbb{R}) \to \mathbb{R}$ the functional:

$$\Psi_s^p(g) = \alpha_{12}(a(g)) + \cdots + \alpha_{1(p-1)}(a(g)) + (s-(p-2))\alpha_{1p}(a(g))$$
$$\tilde{\Psi}_s^p(g) = \left(\frac{\sigma_2}{\sigma_1} \cdots \frac{\sigma_{p-1}}{\sigma_1}(g)\right)\left(\frac{\sigma_{p-1}}{\sigma_1}(g)\right)^{s-(p-2)}$$

Remark 1.2.1. We have $\alpha_{ij}(a) = a_i - a_j, a_i(g) = \log(\sigma_i(g))$ and

$$\Psi_s^p(g) = \log \tilde{\Psi}_s^p(g)$$

and that

$$\min_{p \in [\![2,d]\!]} \left\{ \sum_{|\gamma|=T} \frac{\sigma_2}{\sigma_1} \cdots \frac{\sigma_{p-1}}{\sigma_1}(g) \left(\frac{\sigma_{p-1}}{\sigma_1}(g) \right)^{s-(p-2)} \right\} = \sum_{|\gamma|=T} e^{-\max_{p \in [\![2,d]\!]} \Psi_s^p(g)}$$

Remark 1.2.2. For any $g \in \mathrm{SL}(d, \mathbb{R})$ we have that:

$$\max_{p \in [\![2,d]\!]} \Psi_s^p(g) = \Psi_s^{p_0}(g) \text{ for } s \in [p_0 - 2, p_0 - 1].$$

Indeed, a quick calculation shows that for $s \geq 0$ and $p \in [\![2,d]\!]$:

$$\Psi_s^p(g) \leq \Psi_s^p(g) \text{ if and only if } s \geq p - 1.$$

and that equality holds in the case $s = p - 1$. Thus for $s \in [p-2, p-1]$ we have that

$$s \geq p - 2, \ldots, 1 \text{ implies that } \Psi_s^p(g) \geq \ldots \geq \Psi_s^2(g)$$
$$s \leq p, \ldots, d - 1 \text{ implies that } \Psi_s^p(g) \leq \ldots \leq \Psi_s^d(g)$$

Another way to see this (refer to Figure 1.1) is to note that $\Psi_s^2(g), \cdots, \Psi_s^d(g)$ is a sequence of functions that are affine in s, with slopes $\alpha_{12}(g) \leq \cdots \leq \alpha_{1d}(g)$ and that they satisfy $\Psi_1^2(g) = \Psi_2^2(g), \Psi_2^3(g) = \Psi_3^4(g) \ldots, \Psi_{d-2}^{d-1}(g) = \Psi_{d-2}^d(g)$.

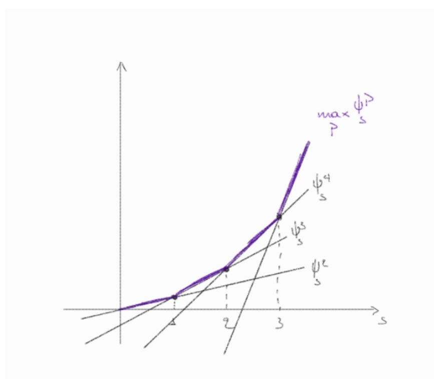


Figure 1.1: Visual illustration that $\max_p \Psi_s^p(g) = \Psi_s^{p_0}(g)$ for $s \in [p_0 - 2, p_0 - 1]$.

The following definition comes from [LL23], in the special case of projective Anosov representations ($P = 1$):

Definition 1.2.2. For $s \geq 0$ we consider the Falconer functional $F_s : \mathrm{SL}(d, \mathbb{R}) \to \mathbb{R}$ by:

$$F_s(g) = \min \left\{ \sum_{j=2}^{d} s_j \alpha_{1j}(a(g)) : s_j \in (0, 1], \sum_{j=2}^{d} s_j = s \right\},$$

and define the Falconer dimension $\dim_F(\rho)$ of ρ to be its critical exponent:

$$\dim_F(\rho) = \inf\left\{ s > 0 : \sum_{\gamma\in\Gamma} e^{-F_s(\rho(\gamma))} < \infty \right\}.$$

Remark 1.2.3. Using elementary computations one may prove that for all $s \geq 0$:

$$F_s(g) = \max_{p\in[\![2,d]\!]} \Psi_s^p(g)$$

Definition 1.2.3. Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a linear representation and $p \in [\![1, d-1]\!]$. We say that ρ is p-Anosov if there exist constants $\mu, C > 0$ such that for all $\gamma \in \Gamma$:

$$\frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma)) \leq Ce^{-\mu|\gamma|}.$$

One can show that in that case there exist equivariant continuous maps $\xi^p : \hat{\Gamma} \to \mathcal{G}_p(\mathbb{R}^d), \xi^{d-p} : \hat{\Gamma} \to \mathcal{G}_{d-p}(\mathbb{R}^d)$ that are transverse and restrict to

$$\xi^p(\gamma) = U_p(\rho(\gamma)), \xi^{d-p}(\gamma) = U_{d-p}(\rho(\gamma))$$

for $\gamma \in \Gamma$, where $U_p(\gamma), U_{d-p}(\gamma)$ denote the flags corresponding to $\rho(\gamma)$.

Figure out what this exactly means

Chapter 2

Upper bound

2.1 Proof of bound

Lemma 2.1.1 (Upper bound for dimension)**.** *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. Then:*

$$\dim_H(\xi^1(\partial\Gamma)) \leq \dim_F(\rho).$$

Remark 2.1.1. The idea of the proof of Lemma 2.1.1 is to find a covering whose Hausdorff content is dominated by the Dirichlet series of some functional Ψ_s^p, which will in turn imply that $\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(\Psi^p)$. Choosing then the most "effective" cover (i.e. the one which yields the smallest Hausdorff content up to a constant) we obtain that

$$\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(\max_p \Psi^p)$$

To obtain this we first cover $\xi^1(\partial\Gamma)$ by the bassins of attraction $\rho(\gamma) \cdot B_{\alpha_1,\alpha}(\rho(\gamma))$ for $\gamma \in \Gamma$ satisfying $|\gamma| = T$. Then we cover each bassin by an ellipsoid of axes lengths

$$\frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \ldots, \frac{1}{\sin(\alpha)} \frac{\sigma_d}{\sigma_1}(\rho(\gamma)).$$

Finally we cover each ellipsoid by balls of some fixed radius $r > 0$. It can be shown by comparing the series appearing in the Hausdorff content of each resulting cover that the most "effective" choice of r depends only on the Hausdorff exponent $s > 0$ and in any case will be to have r equal (up to a constant) to the the length of an axis of the ellipsoid, i.e.

$$r \in \left\{ \frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \ldots, \frac{1}{\sin(\alpha)} \frac{\sigma_d}{\sigma_1}(\rho(\gamma)). \right\}$$

In particular, when $s \in [p-2, p-1]$, the most effective choice is $r = \sigma_p(\rho(\gamma))/\sigma_1(\rho(\gamma))$, whose Hausdorff content is dominated by the Dirichlet series of Ψ_s^p.

Proof of Lemma 2.1.1. Let $p \in [\![2, d]\!]$. Then using Proposition 2.2.1, Lemma 2.2.2, and Lemma 2.2.3 we have that for $T > 0$ large enough, $\xi^1(\partial\Gamma)$ is covered by the family

$$\mathcal{U}_T = \{\rho(\gamma) B_{\alpha_1,\alpha}(\rho(\gamma)) : |\gamma| = T\},$$

and that each basin $\rho(\gamma) B_{\alpha_1,\alpha}(\rho(\gamma))$ is in turn covered by

$$2^{p-2} \cdot \frac{\sigma_p(g)^{p-2}}{\sigma_2(g) \cdots \sigma_{p-1}(g)}$$

many balls of radius

$$\sqrt{d-1}\frac{1}{\sin\alpha}\frac{\sigma_p(g)}{\sigma_1(g)}.$$

By the definition of the Hausdorff measure, for $s \geq 0$:

$$\begin{aligned}
\mathcal{H}^s(\xi^1(\partial\Gamma)) &\leq \sum_{|\gamma|=T} 2^{2p+1}\cdot\frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\cdots\frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\left(\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\right)^{-(p-2)}\left(\sqrt{d-1}\frac{1}{\sin\alpha}\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\right)^s = \\
&= 2^{2p+1}\cdot\left(\frac{\sqrt{d-1}}{\sin\alpha}\right)^s\sum_{|\gamma|=T}\frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\cdots\frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\left(\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\right)^{s-(p-2)} = \\
&= 2^{2p+1}\cdot\left(\frac{\sqrt{d-1}}{\sin\alpha}\right)^s\sum_{|\gamma|=T} e^{-\left(\alpha_{12}+\ldots+\alpha_{1(p-1)}+(s-(p-2))\alpha_{1p}\right)\rho(\gamma)} \\
&= 2^{2p+1}\cdot\left(\frac{\sqrt{d-1}}{\sin\alpha}\right)^s\sum_{|\gamma|=T} e^{-\Psi_s^p(\rho(\gamma))}
\end{aligned}$$

and thus

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq 2^{2p+1}\cdot\left(\frac{\sqrt{d-1}}{\sin\alpha}\right)^s\sum_{|\gamma|=T} e^{-\max_p\Psi_s^p(\rho(\gamma))} \lesssim \sum_{|\gamma|=T} e^{-F_s(\rho(\gamma))}.$$

To see that the above implies the upper bound, consider some $s > \dim_F(\rho)$. By the definition of the Falconer dimension, this implies that the Dirichlet series corresponding to the Falconer functional converges:

$$\sum_{\gamma\in\Gamma} e^{-F_s(\rho(\gamma))} < \infty$$

and in particular

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq \lim_{T\to\infty} e^{-F_s(\rho(\gamma))} = 0.$$

□

2.2 Lemmata

Definition 2.2.1. Let V be a finite-dimensional $\mathbb{R}$-vector space. We consider a decomposition

$$V = \mathbb{R}u_1 \bigoplus\cdots\bigoplus \mathbb{R}u_d$$

be a direct decomposition that is orthogonal with respect to a fixed inner-product over V. Given $\beta_2 \geq \ldots\beta_d > 0$, we define an ellipsoid with axes $u_1 \oplus u_p(g)$ and lengths β_p to be the image of

$$\left\{v = \sum_1^d v_i u_i \in V : \sum_2^d\left(\frac{v_j}{\beta_j}\right)^2 \leq 1\right\}$$

through the projection $V \to \mathbb{P}(V)$.

The following aims to be something along the lines of [PSW23, Lemma 2.4]:

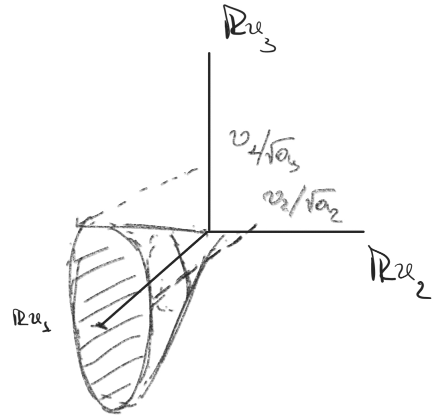


Figure 2.1: Depiction in $\mathbb{R}^3$ of an ellipsoid of $\mathbb{P}(\mathbb{R}^2)$

Lemma 2.2.1. *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. For $\alpha > 0$ small enough, there exists $L > 0$ such that for any geodesic ray $(a_j)_j$ through e we have:*

$$\angle(U_1(\rho(a_i)), U_{d-1}(\rho(a_0))) > \alpha$$

when $|a_i|, |a_0| > T$.

Proof. Assume the contrary for the shake of contradiction. Then (see Figure 2.2) for each $n > 0$ there exists a geodesic ray a^n through e such that

$$|a_n^n|, |a_0^n| > n \text{ and } \angle(U_1(\rho(a_n^n)), U_1(\rho(a_0^n))) < \frac{1}{n}.$$

Due to compactness of $\Gamma \cup \partial\Gamma$ we can find some subsequence k_n and $x, y \in \partial\Gamma$ such that $a_{k_n}^{k_n} \to x$, $a_0^{-k_n} \to y$ and $x \neq y$. Since the limit map is dynamics preserving, we have that

$$\angle(\xi^1(x), \xi^{d-1}(y)) = 0,$$

which contradicts its transversality property. □

The following is [PSW23, Proposition 3.5].

Lemma 2.2.2. *Let $\rho : \Gamma \to SL(d, \mathbb{R})$ be projective Anosov. Then for $\alpha > 0$ small enough, there exists some $T_0 > 0$ such that for all $T \geq T_0$ the family*

$$\mathcal{U}_T = \{\rho(\gamma) B_{\alpha_1, \alpha}(\rho(\gamma)) : |\gamma| = T\}$$

is an open covering of $\xi^1(\partial\Gamma)$.

Proof. Let $\alpha, T > 0$ be as in the statement of Lemma 2.2.1 and $x \in \partial\Gamma$ be represented by a geodesic ray $(\gamma_j)_{j \geq 0}$ starting from e. Then $(\gamma_T^{-1}\gamma_j)_j$ is a geodesic ray starting from $(\gamma_T)^{-1}$ that passes through e, so

$$\angle(U_1(\rho(\gamma_T^{-1}\gamma_j)), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

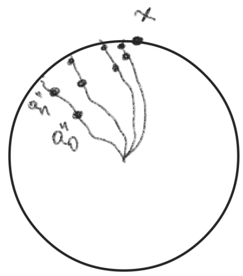


Figure 2.2: Situation in Lemma 2.2.1

as implied by Lemma 2.2.1. Taking the limit $j \to \infty$ and using the equivariance of the limit map, we obtain

$$\angle(\rho(\gamma_T^{-1})\xi^1(x), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

and thus $\xi^1(x) \in \rho(\gamma_T) \cdot B_{\alpha_1,\alpha}(\rho(\gamma_T))$. □

The following is [PSW23, Proposition 3.8].

Proposition 2.2.1. *For each $g \in \mathrm{SL}(d, \mathbb{R}), \alpha > 0$, the basin of attraction $g \cdot B_{\alpha_1,\alpha}(g)$ lies in the ellipsoid with axes $u_1(g) \oplus u_p(g)$ with lengths*

$$\frac{1}{\sin \alpha} \cdot \frac{\sigma_p(g)}{\sigma_1(g)}$$

Proof. Using the definition of the basin of attraction (see Figure 2.3), we have that $w = w_1 u_1(g^{-1}) + \cdots + w_d u_d(g^{-1}) \in B_{\alpha_1,\alpha}(g)$ if and only if

$$w_d^2 \geq (\sin \alpha)^2 \sum_1^d w_i^2.$$

Considering now some $v = v_1 u_1(g) + \cdots + v_d u_d(g) \in g \cdot B_{\alpha_1,\alpha}(g)$ we have that

$$\begin{aligned} w = g^{-1} v &= v_1 \sigma_1(g)^{-1} l_g^{-1} e_1(g) + \cdots v_d \sigma_d(g)^{-1} l_g^{-1} e_d(g) \\ &= v_1 \sigma_1(g)^{-1} u_d(g^{-1}) + \cdots v_d \sigma_d(g)^{-1} u_1(g^{-1}) \end{aligned}$$

where we used that $u_p(g^{-1}) = l_g^{-1} e_{d+1-p}$. Hence

$$\sigma_1(g)^{-2} \cdot v_1^2 \geq (\sin a)^2 \sum_1^d \sigma_i(g)^{-2} v_i^2.$$

□

The following is [PSW23, Lemma 3.7]:

Lemma 2.2.3. *For any $p \in [\![2, d]\!]$, an ellipsoid in $\mathbb{P}(\mathbb{R}^d)$ of axes lengths $\beta_2, \cdots, \beta_d$ is covered by*

$$2^{p-2} \frac{\beta_2 \cdots \beta_{p-1}}{\beta_p^{p-2}}$$

many (projected) balls of radius $\sqrt{d-1}\beta_p$.

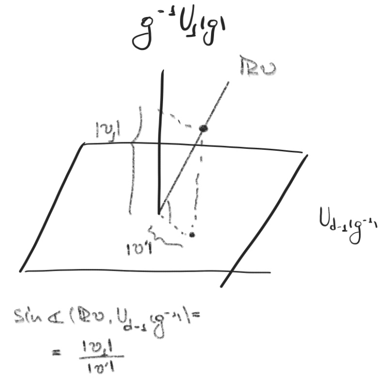


Figure 2.3: Aid for Proposition 2.2.1

Proof. We assume that E is an ellipsoid about $\mathbb{R}e_1$, so it suffice to cover its intersection $E_1 = E \cap U_1 \subseteq \mathbb{R}^{d-1}$ with the affine chart $U_1 = \{[x_1 : \ldots, x_d] \in \mathbb{P}(\mathbb{R}^d) : x_1 \neq 0\}$. Clearly $E_1 \subseteq [-\beta_2, \beta_2] \times \ldots \times [-\beta_d, \beta_d]$, so we proceed by covering the rectangle with side-lengths $2\beta_2, \ldots, 2\beta_d$. Clearly each interval $(-\beta_j, \beta_j)$ is contained in the union of $\lceil \beta_j / \beta_p \rceil$ intervals of length $2\beta_p$, thus E_1 is contained in the union of

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil \cdots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil = \left\lceil \frac{\beta_2}{\beta_p} \right\rceil \cdots \left\lceil \frac{\beta_d}{\beta_p} \right\rceil$$

many squares of side-length $2\beta_p$. Since each such product is contained in a $(d-1)$-ball of radius $\sqrt{d-1}\beta_p$ we may use at most

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil \cdots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil \leq \sum_{i \in \{0,1\}^{p-2}} \prod_{j=2}^{p-1} \left(\frac{\beta_j}{\beta_p} \right)^{i_j} \leq 2^{p-2} \frac{\beta_2}{\beta_p} \cdots \frac{\beta_{p-1}}{\beta_p}$$

many $(d-1)$-balls of radius $\sqrt{d-1}\beta_p$ to cover E_1. □

The following can be found in [PSW23, Proposition 3.3]:

Proposition 2.2.2. *Let $\rho : \Gamma \to SL(d, \mathbb{R})$ be projective Anosov and $\alpha > 0$ Then there exist $c_0, c_1 > 0$ that depends only on α and ρ such that for all $\gamma \in \Gamma$:*

$$(\xi^1)^{-1}(B_{\alpha_1,\alpha}(\rho(\gamma))) \subseteq C^{\infty}_{c_0,c_1}(\gamma)$$

Proof. We begin by noting that it suffices to show this for all but finitely many $\gamma \in \Gamma$, since then we may alter the constants to satisfy the wanted inclusion for the finitely many remaining $\gamma \in \Gamma$ as well. Hence, we may assume that $|\gamma| \geq l_0$ where $l_0 > 0$ is such that

$$Ce^{-\mu l_0} < 1 \text{ and } \mathsf{a}_1(\gamma) \geq C|\gamma| - L.$$

Suppose $x \in \partial\Gamma$ such that $\xi^1(x) \in B_{\alpha_1,\alpha}(\rho(\gamma))$, and consider a geodesic ray $a_j \to x$ starting from $a_0 = e$. To prove the result, it suffices to find constants c_0, c_1 independent of γ and for which there exists a (c_0, c_1)-quasi-geodesic from γ^{-1} to x that passes through e and stays at a bounded distance from $(a_j)_{j=0}^{\infty}$.

Using the exponential convergence rate of $\xi^1(a_j) \to \xi^1(x)$ and the definition of $B_{\alpha_1,\alpha}(\rho(\gamma))$ we have that:

$$d(\xi^1(a_j), \xi^1(\gamma)) \geq d(\xi^1(x), U_1(\rho(\gamma^{-1})) - d(\xi^1(a_j), \xi^1(x))) \geq \\ \geq d(\xi^1(x), U_{d-1}(\rho(\gamma^{-1})) - d(\xi^1(a_j), \xi^1(x))) \geq \sin\alpha - Ce^{-\mu j}$$

which along with the uniform continuity of $\xi^1 : \Gamma \cup \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ this implies there exists some $\alpha' > 0$ and $L > 0$ such that for all $j \geq L$:

$$d(a_j, \gamma^{-1}) \geq \alpha'.$$

Upon considering a large L, we may also assume that $|a_L| = L > l_0$. Note that both α' and L do not depend on each γ but only on ρ and α.

Using a coarse geometric arguement, we can show that for all $j \geq L$

$$d(\gamma^{-1}, a_j) > \alpha' \Rightarrow d([\gamma^{-1}, a_j], e) < \alpha''$$

for some α'' that depends only on Γ and α', where $[a_j, \gamma^{-1}]$ denotes any geodesic segment connecting γ^{-1} and a_j. Indeed, [GH13, Lemme 2.17] states that $d([\gamma^{-1}, a_j]) \leq (\gamma_j^{-1}, a_j)_e + \delta$ where δ is the hyperbolicity constant of Γ. Thus

$$d([\gamma^{-1}, a_j]) \leq \delta + \frac{d(a_j, e) + d(\gamma^{-1}, e) + d(a_j, \gamma^{-1})}{2} \leq \delta + \frac{L + d(\gamma^{-1}, e) + \alpha'}{2}.$$

Consider the concatenation $(a'_j)_{j=L-K}^{\infty}$ of $[\gamma^{-1}, a_L]$ and $[a_L, x]$. To find quasi-geodesic-constants that are uniform in γ, we note that for any $c_0 \geq 1, c_1 \geq 0$:

$$c_0^{-1}|i-j| - c_1 \leq d(a'_i, a'_j) = d(a_i, a_j) \leq c_0|i-j| + c_1 \text{ when } i, j \geq L \text{ or } i, j \leq L$$

and that the upper bound follows trivially by the triangle inequality.

For the lower bound we proceed in two steps. First we bound the distance of $\gamma^{-1} = a'_{L-K}$ to a_{L+j} for $j \geq 0$:

$$d(a'_{L-K}, a'_{L+j}) \geq \nu(|a_{L+j}| - |\gamma^{-1}|) - c'_0 - c'_1|\log(d(U_1(\rho(a_{L+j})), U_1(\rho(\gamma^{-1}))))| \geq \\ \geq \nu((L+j) + (K-L)) - c'_0 - c'_1|\log(\sin a)| \geq \\ = c_0^{-1}(j+K) - c_1$$

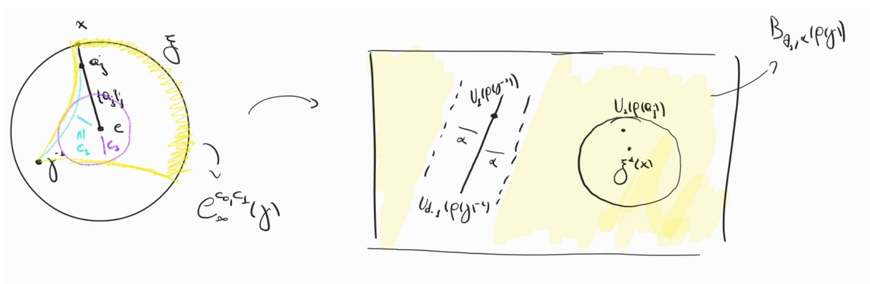
for $c_0 = \nu^{-1}, c_1 = c'_0 + c'_1|\log(\sin\alpha)|$. The first inequality comes from [PSW23, Lemma 3.9]. For the second inequality we estimate $|\gamma^{-1}|$ from below using the triangle inequality. We are now ready to show that the concatenation $(a'_j)_j$ is indeed a $(c_0,\, c_1)$-geodesic:

$$d(a_{L+j}, a_{L-i}) \geq d(a_{L+j}, a_{L_K}) - d(a_{L_K}, a_{L-i}) \geq c_0^{-1}(j+K) - c_1 - (K-i) \geq \\ \geq c_0^{-1}(j+i) - c_1.$$

Note however that (a'_j) does not necessarily lie in $C_\infty^{c_0,c_1}$ since it may not pass through e. For this reason we some $L - K \leq i_0 \leq L$ such that $|a_{i_0}| < \alpha''$, the existence of which is guaranteed by the fact that $d([\gamma^{-1}, a_L], \epsilon) < \alpha''$. We then consider alter (a'_j) at i_0 so that it passes through e to obtain

$$a''_j = \begin{cases} a_j & \text{for } j \neq i_0 \\ e & \text{for } j = i_0 \end{cases}$$

which is a $(c_0, c_1 + \alpha'')$-quasigeodesic passing from e and converging to x. $\square$



Chapter 3

Lower bound

We denote with Π the set of simple positive roots, and for $\Theta \subseteq \Pi$ we consider the Levi-Anosov subspace of $\mathfrak{a}$

$$\mathfrak{a}_\Theta = \bigcap_{\alpha \notin \Theta} \ker \alpha,$$

which in particular admits $\{\omega_{\alpha_i} : \alpha_i \in \Theta\}$ as a basis. Finally, we shall consider the Busemann cocycle

$$b_\Theta : \mathrm{PSL}(d, \mathbb{R}) \times \mathcal{F}_\Theta \to \mathfrak{a}_\Theta$$

which might as well be defined as

$$\omega_{\alpha_i}(b_\Theta(g, x)) = \log \frac{\|gv_1 \wedge \cdots gv_i\|}{\|v_1 \wedge \cdots v_i\|} \text{ for all } \alpha_i \in \Theta$$

for any basis $v_1, \ldots, v_i$ of $x^i \in \mathcal{G}_i(\mathbb{R}^d)$, where $\| \cdot \|$ denotes the norm on $\bigwedge^i \mathbb{R}^d$ induced by the euclidean inner product on $\mathbb{R}^d$, i.e. $\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle := \det(\langle v_i, w_j \rangle)$.

Definition 3.0.1. For a discrete subgroup $\Gamma < \mathrm{PSL}(d, \mathbb{R}), \phi \in (\alpha_\Theta)^*$, a (Γ, ϕ)-Patterson Sullivan measure on $\mathcal{F}_\Theta$ is a finite Radon measure μ such that for every $\gamma \in \Gamma$

$$\frac{\mathrm{d}\gamma_* \mu}{\mathrm{d}\mu}(x) = e^{-\phi(b_\Theta(g^{-1}, x))}, \text{ for all } x \in \mathcal{F}_\Theta(\mathbb{R}^d).$$

Lemma 3.0.1. *Let $\alpha > 0, \Theta \subseteq \Pi$. There exists $K = K(\alpha) > 0$ such that for each $g \in \mathrm{SL}(d, \mathbb{R}), \mathsf{a_i} \in \Theta, \mathsf{y} \in \mathsf{B}_{\Theta,\alpha}(\mathsf{g})$*

$$|\omega_i(a(g) - b(g, y))| \leq K.$$

How to prove this?

Recalling that $\{\omega_i\}_{\mathsf{a_i} \in \Theta}$ is a basis for $\mathfrak{a}_\Theta$, the above implies in particular that for each $\phi \in \mathfrak{a}_\Theta^*$ there exists $K = K(\alpha, \phi) > 0$ such that for all $g \in \mathrm{SL}(d, \mathbb{R}), y \in B_{\Theta,\alpha}(g)$

$$|\phi(a(g) - b(g, y))| \leq K.$$

3.1 Proof strategy

Denoting with $d_\Gamma = \dim_H \xi^1_\rho(\partial\Gamma)$ the Hausdorff dimension of the limit set, we shall outline how to obtain its lower bound by the Falconer dimension

$$d_\Gamma \geq h_\rho(F).$$

First we recall that $F_s(a) = \max\{\Psi_s^p(a) : p \in [\![2, d]\!]\}$ and in particular $h_\rho(F) \leq h_\rho(\Psi^{d_\Gamma+1})$. Thus the lower bound will follow once we have shown that

$$d_\Gamma \geq h_\rho(\Psi^{d_\Gamma+1}).$$

Noting that $(s+1)J_{d_\Gamma^u} \leq \Psi_{s+d_\Gamma}^{d_\Gamma+1}$, the above bound will follow as soon as we have shown that

$$h_\rho(J_{d_\Gamma}) \leq 1. \tag{LB}$$

To obtain inequality (LB), one uses the method of Patterson-Sullivan-Quint, which may be summed up in

$$\text{There exists a } (\phi, \rho)\text{-Patterson-Sullivan measure on } \mathcal{F}_\Theta(\mathbb{R}^d) \Rightarrow h_\rho(\phi) \leq 1,$$

where $\phi \in \mathfrak{a}_\Theta$ and $\Theta \subseteq \Pi$. The property that we will need of our measure is that there exists a collection of open sets $U_{\gamma_\gamma} \in \Gamma$ such that

$$\mu(U_\gamma) \sim e^{-J_{d_\Gamma}^u(a(\rho(\gamma)))} \text{ and } M := \sup_{n\in\mathbb{N}} \max\left\{\sharp A : A \subseteq \Gamma_n, \bigcap_{\gamma\in A} U_\gamma \neq \emptyset\right\} < \infty \tag{MP}$$

where $\Gamma_n = \{\gamma \in \Gamma : |\gamma| = n\}$. For the proof of the existence of a $(J_{d_\Gamma}^u, \rho)$-Patterson-Sullivan measure that satisfies the property (MP) we refer to Section 3.2, noting that the Zariski-density assumption is necessary only for the equivalence appearing on the left hand side of Equation (MP). Assuming that for the time being, below we outline the Patterson-Sullivan-Quint method of obtaining Equation (LB).

Indeed, we first obtain the uniform in n bound:

$$\sum_{\gamma\in\Gamma_n} e^{-J_{d_\Gamma}^u(a(\rho(\gamma)))} \lesssim \sum_{\gamma\in\Gamma_n} \mu(U_\gamma) \leq \frac{1}{M}\mu(\mathcal{F}_\Theta(\mathbb{R}^d)) < \infty$$

along with the bound implied by the Anosov property of ρ:

$$J_{d_\Gamma}(a(\rho(\gamma))) \geq \mathsf{a}_{12}(a(\rho(\gamma))) \geq C|\gamma| - b$$

to conclude that

$$\sum_{\gamma\in\Gamma} e^{-(s+1)J_{d_\Gamma}^u(a(\rho(\gamma)))} = \sum_{n\geq 0}\sum_{\gamma\in\Gamma_n} e^{-sJ_{d_\Gamma}^u(a(\rho(\gamma)))} e^{J_{d_\Gamma}^u(a(\rho(\gamma)))} \lesssim \sum_{n\geq 0} e^{-s(Cn-b)} < \infty$$

which holds for any $s > 0$, and thus Equation (LB) holds.

3.2 Existence of Patterson-Sullivan measure

Definition 3.2.1. For $p \in [\![2, d]\!]$, we denote the p-th unstable Jacobian $J_p^u \in \mathfrak{a}^*$ by

$$J_p^u = (p+1)\omega_{\mathsf{a}_1} - \omega_{\mathsf{a}_{p+1}} = \mathsf{a}_{12} + \cdots + \mathsf{a}_{1(p+1)}.$$

Definition 3.2.2. Let $V \in \mathcal{G}_{p+1}\mathbb{R}^d$ and $l \in \mathbb{P}(V)$. Using the canonical identification $T_l\mathbb{P}(V) \simeq \hom(l, V/l)$, we define the density $|\Omega_{l,V}|$ on $\bigwedge^p T_l\mathbb{P}(V)$ by

$$|\Omega_{l,V}|(\phi_1, \ldots, \phi_p) = \frac{\|v \wedge \tilde{\phi}_1(v) \wedge \cdots \wedge \tilde{\phi}_p(v)\|}{\|v\|^{p+1}}$$

for any $v \in l - \{0\}$, where $\tilde{\phi}_1, \ldots \tilde{\phi}_p \in \hom(l, V)$ are such that $\phi_i = \tilde{\phi}_i + \hom(l, l)$ and $\|\cdot\|$ denotes the norm on $\bigwedge^{p+1}\mathbb{R}^d$ induced by the euclidean inner product.

The following is [PSW23, Proposition 6.4]:

Proposition 3.2.1. *Assume that $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold of dimension d_Γ. Then there exists a $(\rho(\Gamma), J^u_{d_\Gamma})$-Patterson-Sullivan measure on $\mathcal{F}_{1,d_\Gamma+1}$.*

Proof. By Rademacher's theorem, $\xi^1_\rho(\partial\Gamma)$ has a well-defined Lebesgue measure class, and Lebesgue-almost every $\xi^1_\rho(x) \in \xi^1_\rho(\partial\Gamma)$ admits a well-defined tangent space $T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma)$. Considering such a $\xi^1_\rho(x)$ we let

$$\pi : \hom(\xi^1_\rho(x), \mathbb{R}^d) \to \hom(\xi^1_\rho(x), \mathbb{R}^d/\xi^1_\rho(x)) \simeq T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma),$$

and

$$x^{d_\Gamma+1} = \pi^{-1}(T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma))\xi^1_\rho(x) \in \mathcal{G}_{d_\Gamma+1}(\mathbb{R}^d),$$

for which one can show that

$$T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma) \simeq \hom(\xi^1_\rho(x), \mathbb{R}^d/\xi^1_\rho(x)) \simeq \hom(\xi^1_\rho(x), x^{d_\Gamma+1}/\xi^1_\rho(x)).$$

In this notation, we shall define (Lebesgue-almost eeverywhere) the map

$$\zeta_\rho : \xi^1_\rho(\partial\Gamma) \to \mathcal{F}_{1,d_\Gamma+1}(R^d), \quad \zeta_\rho(\xi^1_\rho(x)) = (\xi^1_\rho(x), x^{d_\Gamma+1}).$$

We now define the non-negative density on $\xi^1_\rho(\partial\Gamma)$

$$\mu_{\xi^1_\rho(x)} = |\Omega_{\zeta_\rho(\xi^1_\rho(x))}|$$

which satisfies

$$\frac{\mathrm{d}\gamma_*\mu}{\mathrm{d}\mu}(\xi) = \frac{\mathrm{d}(\rho(\gamma)^{-1})^*\mu}{\mathrm{d}\mu}(\xi) = e^{-J^u_{d_\Gamma+1}(b_\Theta(\rho(\gamma)^{-1},\zeta(x))))},$$

where the term on the left-hand side involves measures, the term between the two equalities involves densities, and $\Theta = \{1, d_\Gamma+1\}$. Indeed, for $\phi_1, \ldots, \phi_{d_\Gamma} \in T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma)$:

$$\begin{aligned}
&(\rho(\gamma)^*\mu)_{\xi^1_\rho(x)}(\phi_1, \ldots, \phi_{d_\Gamma}) \\
&= \mu_{\rho(\gamma)\xi^1_\rho(x)}(\rho(\gamma)\phi_1\rho(\gamma)^{-1}, \ldots, \rho(\gamma)\phi_{d_\Gamma}\rho(\gamma)^{-1}) = \\
&= \frac{\|\rho(\gamma)\xi^1_\rho(x) \wedge \rho(\gamma)\phi_1(\xi^1_\rho(x)) \wedge \cdots \wedge \rho(\gamma)\phi_{d_\Gamma}(\xi^1_\rho(x))\|}{\|\rho(\gamma)\xi^1_\rho(x)\|^{d_\Gamma+1}} = \\
&= \frac{\|\rho(\gamma)\xi^1_\rho(x) \wedge \rho(\gamma)\phi_1(\xi^1_\rho(x)) \wedge \cdots \wedge \rho(\gamma)\phi_{d_\Gamma}(\xi^1_\rho(x))\|}{\|\xi^1_\rho(x) \wedge \phi_1(\xi^1_\rho(x)) \wedge \cdots \wedge \phi_{d_\Gamma}(\xi^1_\rho(x))\|} \cdot \frac{\|\xi^1_\rho(x) \wedge \phi_1(\xi^1_\rho(x)) \wedge \cdots \wedge \phi_{d_\Gamma}(\xi^1_\rho(x))\|}{\|\rho(\gamma)\xi^1_\rho(x)\|^{d_\Gamma+1}} \cdot \frac{\|\rho(\gamma)\xi^1_\rho(x)\|^{d_\Gamma+1}}{\|\xi^1_\rho(x)\|^{d_\Gamma+1}} = \\
&= e^{\omega_{d_\Gamma}(b_\Theta(\rho(\gamma),\zeta_\rho(\xi^1_\rho(x))))} \cdot \mu_{\xi^1_\rho(x)}(\phi_1, \ldots, \phi_{d_\Gamma}) \cdot e^{-(p+1)\omega_1(b_\Theta(\rho(\gamma),\zeta_\rho(\xi^1_\rho(x))))} = \\
&= e^{-J^u_{d_\Gamma}(b_\Theta(\rho(\gamma)^{-1},\zeta(\xi^1_\rho(x)))}\mu_{\xi^1_\rho(x)}(\phi_1, \ldots, \phi_{d_\Gamma}).
\end{aligned}$$

Finally, we let $\nu = \zeta_{\rho*}\mu$, which is the wanted Patterson-Sullivan measure on $\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d)$, since for $f \in C_c(\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d))$:

$$\begin{aligned}
\int_{\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d)} f\,\mathrm{d}(\gamma_*\zeta_{\rho*}\mu) &= \int_{\xi^1_\rho(\partial\Gamma)} f \circ \gamma \circ \zeta_\rho\,\mathrm{d}\mu = \int_{\xi^1_\rho(\partial\Gamma)} f \circ \zeta_\rho \circ \gamma\,\mathrm{d}\mu = \\
&= \int_{\xi^1_\rho(\partial\Gamma)} f \circ \zeta_\rho(\xi) e^{-J^u_{d_\Gamma}(b_\Theta(\rho(\gamma)^{-1},\zeta(\xi^1_\rho(x)))}\,\mathrm{d}\mu(\xi^1_\rho(x)) = \\
&= \int_{\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d)} f(y) e^{-J^u_{d_\Gamma}(b_\Theta(\rho(\gamma)^{-1},y)}\,\mathrm{d}(\zeta_{\rho*}\mu)(y)
\end{aligned}$$

∎

Before giving the next definition, we recall that the annihilator annihilator of an element $y \in \mathcal{F}_F i\Theta(\mathbb{R}^d)$ is the set of partial flags that are not transverse to y, that is:

$$\operatorname{Ann}(y) = \left\{x \in \mathcal{F}_\Theta(\mathbb{R}^d) : x^\theta \cap y^{d-\theta} \neq 0 \text{ for some } \theta \in \Theta\right\}.$$

Definition 3.2.3. Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a linear representation, $\Theta \subseteq \Pi$ and μ a measure over $\mathcal{F}_\Theta(\mathbb{R}^d)$. We say that ρ is μ-irreducible there is no element in $\mathcal{F}_{i\Theta}(\mathbb{R}^d)$, whose annihilator is of full measure, i.e. for all $y \in \mathcal{F}_{i\Theta}(\mathbb{R}^d)$:

$$\mu(\operatorname{Ann}(y)) < \mu(\mathcal{F}_\Theta(\mathbb{R}^d)).$$

Example 3.2.1. If $\rho(\Gamma)$ is Zariski-dense in $\mathrm{SL}(d, \mathbb{R})$, then ρ is μ-irreducible for any ρ-quasi-equivariant measure μ, and in particular for any $(\rho(\Gamma), \phi)$-Patterson-Sullivan measure.

Remark 3.2.1. The reason that we introduce the concept of μ-irreducibility is that for any μ-irreducible representation $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$, there exist $\alpha, \kappa > 0$ such that $\mu(B_{\Theta,\alpha}(\rho(\gamma))) \geq k$ for all $\gamma \in \Gamma$.

Indeed, if this were not the case, then there would exists a sequence $\alpha_n \searrow 0$ and $\gamma_n \in \Gamma$ such that

$$\mu(B_{\theta,\alpha}(\rho(\gamma))) \leq \frac{1}{n}.$$

Due to the compactness of $\mathcal{F}_\Theta(\mathbb{R}^d)$, up to considering a subsequence, we may assume that the reppeling flags or $\rho(\gamma_n)$ converge to some $\xi \in \mathcal{F}_\Theta(\mathbb{R}^d)$:

$$(U_{d-i}(\rho(\gamma_n)^{-1}))_{\mathsf{a}_i \in \Theta} \to \xi$$

In that case, the complements $B^c_{\Theta,\alpha_n}(\rho(\gamma_n))$ will converge to the annihilator of ξ, in the sense:

$$\limsup_n B^c_{\Theta,\alpha_n}(\rho(\gamma_n)) \subseteq \operatorname{Ann}(\xi).$$

Indeed, let $y \in \limsup_n B^c_{\Theta,\alpha_n}(\rho(\gamma_n))$ and consider a subsequence k_n such that $y \in B^c_{\Theta,\alpha_n}(\rho(\gamma_{k_n}))$. By the very definition of $B_{\Theta,\alpha_n}(\rho(\gamma_n))$, there exists some p such that up to considering a subsequence of k_n,

$$\angle(y^p, U_{d-p}(\rho(\gamma_n)^{-1})) \leq \alpha_n$$

holds. Taking the limit as $n \to \infty$, we have that $y^p \cap \xi^{d-p} \neq 0$ and hence $y \in \operatorname{Ann}(\xi)$.

Using a measure-theoretic arguement we conclude that $\operatorname{Ann}(\xi)$ is of full measure, which contradicts the μ-irreducibility of ρ:

$$\mu(\operatorname{Ann}(\xi)) \geq \mu(\limsup_n B^c_{\Theta,\alpha_n}(\rho(\gamma_{k_n}))) \geq \limsup_n \mu(B^c_{\Theta,\alpha_n}(\rho(\gamma_{k_n}))) = \mu(\mathcal{F}_\Theta(\mathbb{R}^d)).$$

Lemma 3.2.1. *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a representation and μ^ϕ be a $(\rho(\Gamma), \phi)$-Patterson-Sullivan measure. If $\rho(\Gamma)$ is μ-irreducible, then there exists some $\alpha_0 > 0$, such that for any $\alpha \in (0, \alpha_0)$, there's some $k = k(\alpha) > 0$ for which*

$$\frac{1}{k} e^{-\phi(a(\rho(\gamma)))} \leq \mu^\phi(\rho(\gamma) B_{\Theta,\alpha}(\rho(\gamma))) \leq k e^{-\phi(a(\rho(\gamma)))}$$

for all $\gamma \in \Gamma$.

Proof. Let $\alpha_0, k > 0$ be as in the remark preceeding the statement of the lemma. As noted in Lemma 3.0.1, there exists some $K = K(\alpha_0, \phi) > 0$ such that for any $\alpha \in (0, \alpha_0)$ and $y \in B_{\Theta,\alpha}(\rho(\gamma))$:

$$|\phi(a(\rho(\gamma)) - b(\rho(\gamma), y))| \leq K,$$

from which we obtain the upper bound

$$\mu^\phi(\rho(\gamma)B_{\Theta,\alpha}(\rho(\gamma))) = (\rho(\gamma^{-1})_*\mu^\phi)(B_{\Theta,\alpha}(\rho(\gamma))) = \int_{\mathcal{F}_\Theta(\mathbb{R}^d)} e^{-\phi(b(\rho(\gamma),y))} d\mu^\phi(y) \leq$$
$$\leq e^{-K}\mu^\phi(\mathcal{F}_\Theta(\mathbb{R}^d))e^{-\phi(a(\rho(\gamma)))}.$$

Similarly we obtain the lower bound □

Appendix A

Tangent space to the Grassmanian

Let V be a d-dimensional real vector space. We denote with $\mathcal{G}_k(V)$ the Grassmanian of k-dimensional subspaces of V. Our first objective is to find a convenient way to express its tangent space.

Proposition A.0.1. *We have the following canonical identification:*

$$\begin{aligned} \hom(W, V/W) &\simeq T_W \mathcal{G}_k(V) \\ \phi &\mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Gamma(t\phi) \end{aligned}$$

where $\Gamma(\phi) = (Id + \phi)(W)$ *is the graph of* ϕ.

Proof. We will consider the map

$$F : \mathrm{Injhom}(W, V) \to \mathcal{G}_k(V), \quad \phi \mapsto \phi(W).$$

whose derivative is given by:

$$d_I F(\phi) = F\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (I + t\phi)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (I + t\phi(W)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Gamma(t\phi).$$

The result will follow as soon as we have shown that $d_I F$ is surjective and that $\ker d_I F = \hom(W, W)$.

To show that it is surjective, we consider a $(d-k)$-dimensional subspace $W' \in \mathcal{G}_{d-k}(V)$ that is complementary to W, i.e. $V = W \oplus W'$. Denoting with $U_{W'} = \{Z \in \mathcal{G}_k(V) : Z \cap W' = 0\}$, we recall the corresponding chart:

$$\begin{aligned} \Psi : \hom(W, W') &\to U_{W'} \\ \phi &\mapsto \Gamma(\phi). \end{aligned}$$

Surjectivity of $d_I F$ now follows by the fact that

$$d_I F(\phi) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Gamma(t\phi) = d_0 \Psi(\phi).$$

To show that $\ker d_I F = \hom(W, W)$, we first note that clearly $\ker \mathrm{d}_I F \supseteq \hom(W, W)$. Equality then follows by the fact that $\dim \hom(W, W) = \dim \ker \mathrm{d}_I F$, which is a direct consequence of the surjectivity. $\square$

Note that another way to prove the above identification throught the fact that the Grassmanian is a homogeneous space of $\mathrm{GL}(d,\mathbb{R})$, giving us the diffeomorphism

$$\begin{aligned} \mathrm{GL}(V)/\mathrm{St}_{GL(V)}W &\to \mathcal{G}_k(V) \\ [g] &\mapsto gW, \end{aligned}$$

where $\mathrm{St}_{GL(V)}W = \{g \in \mathrm{GL}(V) : gW = W\}$ is the stabilizer of W. Thus an expression for the tangent space at W may be obtained by differentiating the map above at the identity coset:

$$\hom(W, V/W) \simeq \hom(V,V)/\hom(W,W) \simeq T_W\mathcal{G}_k(V).$$

In particular the above gives us a direct proof that the kernel of the map constructed in the previous proof is indeed $\hom(W,W)$.

Our second objective is to identify subspaces of $T_l\mathbb{P}(V)$ with subspaces of V, by considering the first as projectivisation of the second. More concretely, we shall consider the space

$$\mathcal{P} = \{(l,P) : l \in \mathbb{P}(V), P \in \mathcal{G}_k(T_l\mathbb{P}(V))\}$$

as a homogenous space of $\mathrm{SL}(V)$, where the action is given by

$$g \cdot (l,P) = (gl,\, \mathrm{d}_l g(P) = g\pi^{-1}(P)g^{-1} + \hom(gl,gl)).$$

where we use the identification of $T_l\mathbb{P}(V)$ with $\hom(l,V/l)$ as above and denote with $\pi : \hom(l,V) \to \hom(l,V/l)$ the canonical projection. For the sake of completeness, we outline the calculation of the differential:

$$\begin{aligned} \hom(l,V/l) \to T_l\mathbb{P}(V) \qquad &\to T_{gl}\mathbb{P}(V) &&\to \hom(gl,V/gl) \\ \phi \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g(I+t\tilde{\phi})(l) \quad &\mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(I+tg\tilde{\phi}g^{-1})(gl) &&\mapsto g\tilde{\phi}g^{-1} + \hom(gl,gl) \end{aligned}$$

where $\phi \in \hom(l,V/l), \tilde{\phi} \in \hom(l,V)$ such that $\tilde{\phi} + \hom(l,l) = \phi$.

We are now ready to express the needed identification:

Proposition A.0.2. *We have the following* $\mathrm{SL}(V)$ *equivariant identification:*

$$\begin{aligned} \mathcal{P} &\to \mathcal{F}_{1,k+1}(V) \\ (l,P) &\mapsto (l, \pi^{-1}(P)l) \\ (l, \hom(l,Q/l)) &\leftarrow\!\shortmid (l,Q). \end{aligned}$$

where $\pi : \hom(l,V) \to \hom(l,V/l)$ *is the canonical projection.*

Proof. We begin by showing that the left-to-right direction of the map is well-defined. For this, we first need to check that for $(l,P) \in \mathcal{P}$, we have that $\dim \pi^{-1}(P)l = k+1$. Indeed, we have that $\dim \pi^{-1}(P) = k+1$ as implied by the rank-nullity theorem for $\pi : \pi^{-1}(P) \to P$. The result then follows by the fact that $\pi^{-1}(P)l = T_1(l)\oplus\cdots\oplus T_{k+1}(l)$ for any base $T_1,\ldots,T_{k+1}$ of $\pi^{-1}(P)$. The second thing to check is that $l \le \pi^{-1}(P)l$, which holds since $\ker\pi = \hom(l,l) \le \pi^{-1}(P)$.

To see that the two directions above are inverse to each other, we begin by examining the right-to-left-to-right composition:

$$(l,Q) \mapsto (l, \pi(\hom(l,Q))) \mapsto (l, \pi^{-1}\pi(\hom(l,Q))) = (l, \hom(l,Q)l) = (l,Q).$$

and for the left-to-right-to-left composition

$$(l,P) \mapsto (l,\pi^{-1}(P)l) \mapsto (l, \hom(l, \pi^{-1}(P)l)),$$

so it suffices to show that $\hom(l, \pi^{-1}(P)l/l) = P$. Indeed, for $\pi^{-1}(P) = \mathbb{R}T_1 \oplus \cdots \oplus \mathbb{R}T_k$, we have that

$$\begin{aligned}\hom(l, \pi^{-1}(P)l/l) &= \hom(l, \pi^{-1}(P)l)/\hom(l,l) = (\oplus_i \hom(l, T_i(l))/\hom(l,l)) = \\ &= (\oplus_i \mathbb{R}T_i)/\hom(l,l) = \pi^{-1}(P)/\hom(l,l) = P.\end{aligned}$$

For the equivariance, the calculations has as follows:

$$\begin{tikzcd}
(l, P) \arrow[r, mapsto] \arrow[d, mapsto, "g"] & (l, \pi^{-1}(P)l) \arrow[d, mapsto, "g"] \\
(gl, g\pi^{-1}(P)g^{-1} + \hom(gl, gl)) \arrow[r, mapsto] & (gl, (g\pi^{-1}(P)g^{-1})(gl)) = (gl, g\pi^{-1}(P)l)
\end{tikzcd}$$

□

Appendix B

Irreducible actions problem

The matter of this chapter has to do with an obstruction, found in the proof of this lemma:

Lemma B.0.1 (Lemma 6.8 in [PSW23])**.** *Let* Γ *be a hyperbolic group and* $\eta : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ *be a strongly irreducible projective Anosov representation such that* $\xi_\eta(\partial\Gamma)$ *is homeomorphic to* S^{d_Γ}*, and which admits a measurable* η*-equivariant section* $\zeta : \partial\Gamma \to \mathcal{F}_{\{\mathsf{a}_1, \mathsf{a}_{d_\Gamma+1}\}}(\mathbb{R}^d)$*. Then* η *is* μ*-irreducible for any* $(\eta(\Gamma), \phi)$*-Patterson-Sullivan measure* μ *on* $\mathcal{F}_{\{\mathsf{a}_1, \mathsf{a}_{d_\Gamma+1}\}}(\mathbb{R}^d)$*.*

For convenience, we recall that a linear representation $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is strongly irreducible if there is no proper $\rho(\Gamma)$-invariant subspace of $\mathbb{R}^d$, and it is μ-irreducible if there is no element in $\mathcal{F}_{i\Theta}(\mathbb{R}^d)$, whose annihilator is of full measure.

In what follows, we will show that the above lemma is false, by providing a counterexample. Let Γ be a uniform lattice of $\mathrm{SU}(2, 1)$ (i.e. acts convex cocompactly), and $\eta : \Gamma \to \mathrm{SL}(\mathfrak{su}(2, 1))$ be the restriction of the adjoint representation, i.e. $\eta(\gamma) = \mathrm{Ad}_\gamma$ for all $\gamma \in \Gamma$.

For convenience, we recall the definition of a uniform lattice:

Definition B.0.1. Let G be a locally compact group. A uniform lattice is a discrete subgroup $\Gamma \leq G$ that is co-compact, i.e. G/Γ is compact.

Remark B.0.1. When $G = \mathrm{Isom}(X)$ is the isometry group of a complete Riemannian manifold X, and Γ is a uniform lattice of G,then it acts properly discontinuously and cocompactly on X.

We begin by showing proving the Anosov property of η.

Proposition B.0.1. *η is projective Anosov.*

Proof. Let $\gamma \in \Gamma$. Since $\gamma \in \mathrm{SU}(2, 1)$, we have that

$$\gamma = k_1 \exp\left(r(\gamma)x_0\right) k_2$$

for x_0 a fixed non-zero in the Weyl-chamber $\mathbb{R}x_0$ of $\mathfrak{su}(2, 1), r(\gamma) \in \mathbb{R}$ and $k_1, k_2 \in \mathrm{U}(2)$. Then by the definition of a uniform lattice, we have that Γ acts properly discontinuously and cocompactly, which means that the inclusion $\Gamma \hookrightarrow \mathrm{SU}(2, 1)$ is projective Anosov (since $\mathrm{SU}(2, 1)$ is of rank 1). Thus there exist constants $L \geq 1, b \geq 0$ such that for all $\gamma \in \Gamma$:

$$r(\gamma) \geq \mathsf{a}_1(x_0)^{-1}(L|\gamma| - b) = L'|\gamma| - b'.$$

Note that $\mathsf{a}_1(x_0) > 0$ since x_0 is in the interior of the Weyl-chamber $\mathbb{R}x_0$.

Letting $k_1' = Ad_{k_1}, k_2' = Ad_{k_2}$ and $K' \leq \mathrm{SL}(\mathfrak{su}(2,1))$ be a maximal comapct subgroup containing them, we have that:

$$\eta(\gamma) = \mathrm{Ad}_\gamma = k_1' \,\mathrm{Ad}_{\exp(r(\gamma)x_0)}\, k_2' = k_1' \exp(r(\gamma)\,\mathrm{ad}_{x_0})k_2'.$$

Thus

$$\mathsf{a}_1(\mu(\eta(\gamma))) = r(\gamma)\mathsf{a}_1(\mathrm{ad}_{x_0}) \geq (L'|\gamma| - b')\mathsf{a}_1(x_0)$$

which is Anosov because $\mathsf{a}_1(\mathrm{ad}_{x_0}) > 0$, as can be seen by concrete calculations. □

Before giving an expression for the projective part of the limit map of η, we make a few observations regarding Gromov boundary of Γ. In particular, we claim that since Γ is a uniform lattice os $\mathrm{SU}(2,1)$, we have that $\partial\Gamma$ is homeomorphic to $\mathrm{SU}(2,1)/P$, where P is a parabolic subgroup of $\mathrm{SU}(2,1)$, and it coincides with the stabilizer of some isotropic line $l \in \partial_\infty \mathbb{H}^2_{\mathbb{C}}$.

Indeed, for a uniform lattice Γ of the isometry group G of a homogenous G-space X, the Milnor-Švarc lemma implies that for any $x_0 \in X$, the map $\Gamma \to X, \gamma \mapsto \gamma x_0$ is a quasi-isometry. In our case $G = \mathrm{SU}(2,1)$ and $X = \mathbb{H}^2_{\mathbb{C}}$ is a hyperbolic metric space, so it the quasi-isometry extends to a homeomorphism $\partial\Gamma \to \partial H^2_{\mathbb{C}}$ of the Gromov-boundaries. On the other hand, the action of $\mathrm{SU}(2,1)$ on $\partial_\infty \mathbb{H}^2_{\mathbb{C}}$ is transitive, so we have that $\partial\mathbb{H}^2_{\mathbb{C}} \simeq \mathrm{SU}(2,1)/P$ where P is the stabilizer of a point in $\partial\mathbb{H}^2_{\mathbb{C}}$. In fact, we have that P is a parabolic subgroup of $SU(2,1)$. The combination of the above, along with the fact that the geometric and the Gromov boundaries agree in the case of $\mathbb{H}^2_{\mathbb{C}}$, we deduce that $\partial\Gamma \simeq \mathrm{SU}(2,1)/P$.

Add proofs of these.

To calculate the projective part of the limit map, we shall show that there exists a unique $\mathrm{SU}(2,1)$-equivariant map from $\partial\Gamma$ to $\mathbb{P}(\mathbb{R}^d)$. The uniqueness follows from the following characterisation of limit maps:

Lemma B.0.2. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a strongly irreducible projective Anosov representation, and denote with $\xi_\rho : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ its limit map. Then ξ^1_ρ is the unique continuous, $\rho(\Gamma)$-equivariant map from $\partial\Gamma$ to $\mathbb{P}(\mathbb{R}^d)$.*

Proof. Let $\eta^1 : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ be a continuous, $\rho(\Gamma)$-equivariant map. Since the action of Γ on its boundary $\partial\Gamma$ has dense orbits, it suffices to show that it agrees with ξ^1_ρ on at least one boundary point.

Suppose for the shake of contradiction that η^1 does not coincide with ξ^1 and let $z \in \partial\Gamma, y \in \partial\Gamma\backslash\{z\}$. Then for any $x \in \partial\Gamma\backslash\{y\}$ we may find some quasi-geodesic $\{\gamma_n\}_n$ such that $\gamma_n \to x, \gamma_{-n} \to y$ as $n \to \infty$. Then since $z \neq y$ we know that $\gamma_n z \to z$ as $n \to \infty$ and continuity of η^1 implies that $\eta^1(\gamma_n z) \to \eta^1(z)$. But equivariance of η^1 and the fact that ξ^1 is dynamics-preserving implies that $\eta(\gamma_n z) = \rho(\gamma_n)\eta(z) \to \xi^1(x)$, unless $\eta^1(z) \in \xi^{d-1}(y)$. But if in fact $\eta^1(z) \notin \xi^{d-1}(y)$, then the limits would agree, i.e. $\eta^1(x) = \xi^1(x)$ which is a contradiction. Thus we have that $\eta^1(z) \in \xi^{d-1}(y)$ and since y was an arbitrary points of $\partial\Gamma\backslash\{y\}$, we have that

$$\eta^1(z) \in \bigcap_{y \in \partial\Gamma\backslash\{z\}} \xi^{d-1}(y) \subseteq \bigcap_{y \in \partial\Gamma\backslash\Gamma\cdot z} \xi^{d-1}(y).$$

In particular, the set appearing on the right hand side is ρ-equivariant, non-empty proper subset of $\mathbb{R}^d$, which contradicts the strong irreducibility assumption of ρ. □

Given the lemma above, it suffices to find an $\mathrm{SU}(2,1)$-equivariant map from $\partial\Gamma$ to $\mathbb{P}(\mathbb{R}^d)$.

Proposition B.0.2. *The projective part of the limit map of η is given by*

$$\xi_\eta : \partial\Gamma = \mathrm{SU}(2,1)/P_0 \to \mathbb{P}(\mathfrak{su}(2,1)), \quad \xi_\eta(gP_0) = \mathbb{R}\,\mathrm{Ad}_\gamma\, x_0.$$

where $P_0 = \mathrm{St}_{\mathrm{SU}(2,1)}[1:0:0]$ *and*

$$x_0 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{su}(2,1).$$

Its derivative satisfies:

$$d_x\xi(T_x \,\mathrm{SU}(2,1)/P_0) = \pi(\mathrm{ad}_{\xi^1(x)} \,\mathfrak{su}(2,1))$$

where $\pi : \hom(\xi^1(x), \mathfrak{su}(2,1)) \to \hom(\xi^1(x), \mathfrak{su}(2,1)/\xi^1(x))$ *is the canonical projection.*

Proof. Since by Lemma B.0.2 the limit map is the unique continuous ρ-equivariant from the boundary of Γ to the projective space, it suffices to show that there exists an η-equivariant map $\xi^1 : \mathrm{SU}(2,1)/P_0 \to \mathbb{P}(\mathfrak{su}(2,1))$, since it will then restrict to the limit map on $\partial\Gamma$.

We consider the parabolic subgroup $P_0 = \mathrm{St}_{\mathrm{SU}(2,1)}[1:0:0]$ of $\mathrm{SU}(2,1)$. Then its Lie algebra is given by:

$$\mathfrak{p}_0 = \mathrm{St}_{\mathfrak{su}(2,1)}[1:0:0] = \left\{ \begin{pmatrix} u-is & a & it \\ 0 & 2is & -\bar{a} \\ 0 & 0 & -u-is \end{pmatrix} : a \in \mathbb{C}, u,s,t \in \mathbb{R} \right\}.$$

Since for $\mathbb{R}x \in \mathbb{P}(\mathfrak{su}(2,1))$ we have that P_0 fixes $\mathbb{R}x$ if and only if $\mathfrak{p}_0$ fixes $\mathbb{R}x$. But a quick calculation shows that the only element of $\mathfrak{su}(2,1)$ fixed by $\mathfrak{p}_0$ is x_0.

For the calculation of the image of the differential at the identity coset P, we differentiate the commutative diagram:

$$\begin{array}{ccc} \mathrm{SU}(2,1) & \xrightarrow{\mathrm{Ad}.\, x_0} & \mathfrak{su}(2,1) \\ \downarrow & & \downarrow \\ \mathrm{SU}(2,1)/P_0 & \xrightarrow{\xi^1} & \mathbb{P}(\mathfrak{su}(2,1)) \end{array} \quad \text{to get} \quad \begin{array}{ccc} \mathfrak{su}(2,1) & \xrightarrow{\mathrm{ad}.\, x_0} & \mathfrak{su}(2,1) \\ \downarrow & & \downarrow \pi \\ \mathfrak{su}(2,1)/\mathfrak{p}_0 & \xrightarrow{\mathrm{d}_P\xi^1} & T_{\xi^1(P)}\mathbb{P}(\mathfrak{su}(2,1)) \end{array}$$

In the general case we use the equivariance of the limit map

$$\begin{aligned} \mathrm{d}_{gP}\xi^1(T_{gP}\,\mathrm{SU}(2,1)/P_0) &= d_{gP}\xi^1 d_P g(T_P\,\mathrm{SU}(2,1)/P_0) = d_{\xi^1(P)} g d_P \xi^1(T_P\,\mathrm{SU}(2,1)/P_0) = \\ &= d_{\xi^1(P)} g \pi(\mathrm{ad}_{\xi^1(P)}\,\mathfrak{su}(2,1)) = \\ &= \pi(Ad_g(\mathrm{ad}_{\xi^1(P)}\,\mathfrak{su}(2,1))) = \pi(\mathrm{ad}_{Ad_g\xi^1(P)}\,\mathfrak{su}(2,1)) = \\ &= \pi(\mathrm{ad}_{\xi^1(gP)}\,\mathfrak{su}(2,1)). \end{aligned}$$

□

Recall that all parabolic subgroups of $\mathrm{SU}(2,1)$ are conjugate to each other, so we have the following identification:

$$\begin{array}{lll} \mathrm{SU}(2,1)/P_0 \leftrightarrow \{ \text{ Parabolic subgroups of } \mathrm{SU}(2,1)\} & \leftrightarrow \{ \text{ Parabolic subalgebras of } \mathfrak{su}(2,1)\} \\ gP_0 \leftrightarrow gP_0g^{-1} & \leftrightarrow \mathrm{Ad}_g(\mathfrak{p}_0) \end{array}$$

Lemma B.0.3. *Let* $\mathfrak{p}, \mathfrak{p}' \leq \mathfrak{su}(2,1)$ *be two distinct parabolic subalgebras. Then there exists some* $g \in \mathrm{SU}(2,1)$ *such that* $\mathrm{Ad}_g(\mathfrak{p}) = \mathfrak{p}_0$ *and* $\mathrm{Ad}_g\,\mathfrak{p}' = \mathfrak{p}_0^t$.

The following proposition implies that the falsehood of the lemma in the beginning of this chapter.

Proposition B.0.3. *Let $\Gamma \leq \mathrm{SU}(2,1)$ be a uniform lattice and $\eta : \Gamma \to \mathrm{SL}(\mathfrak{su}(2,1))$ be the restriction of the adjoint representation. Then*

(i) η is strongly irreducible,

(ii) η is projective Anosov

(iii) η admits a measurable η-equivariant section:

$$\begin{aligned}\zeta : \partial\Gamma &\to \mathcal{F}_{\{1,4\}}(\mathfrak{su}(2,1)) \simeq \mathcal{P}\\ x &\mapsto (\xi^1(x), T_{\xi^1(x)}\xi^1(\partial\Gamma)) \simeq (\xi^1(x), (\,\mathrm{d}_{\xi^1(x)}p)^{-1}(T_{\xi^1(x)}\xi^1(\partial\Gamma))\xi^1(x)).\end{aligned}$$

where $d_{\xi^1(x)}p : \hom(\xi^1(x), \mathfrak{su}(2,1)) \to \hom(\xi^1(x), \mathfrak{su}(2,1)/\xi^1(x))$ is the canonical projection.

(iv) For all $x, y \in \partial\Gamma : \zeta(x)^4 \cap \zeta(y)^4 \neq 0$.

(v) For any $y_0 \in \mathrm{SU}(2,1)/P_0$ and $W_0 \in \mathcal{G}_7(\mathbb{R}^4)$ that contains $\zeta(y_0)^4$, we have that $\mathrm{Ann}(\zeta(y_0)^4, W_0) \supseteq \zeta(\mathrm{SU}(2,1)/P_0)$ and is in particular of full μ-measure, for any $(\eta(\Gamma), \phi)$-Patterson-Sullivan measure μ supported over $\zeta(\partial\Gamma)$.

Proof. (i) Follows from the fact that $\mathrm{SU}(2,1)$ is a simple Lie group.

(ii) Shown in Proposition B.0.1.

(iii) Follows from the fact that ξ^1 is $\mathrm{SU}(2,1)$-equivariant and the equivariant identification of $\mathcal{F}_{\{1,4\}}(\mathfrak{su}(2,1)) \simeq \mathcal{P}$.

(iv) Letting $g \in \mathrm{SU}(2,1)$ be as in Lemma B.0.3, we have that $\mathrm{Ad}_g(\mathfrak{p}_0) = \mathfrak{p}$ and $\mathrm{Ad}_g(\mathfrak{p}_0^t) = \mathfrak{p}'$. Thus $\zeta(x)^4 \cap \zeta(y)^4 \neq \emptyset$ if and only if

$$\emptyset \neq Ad_g(\zeta(x)^4 \cap \zeta(y)^4) = \mathrm{Ad}_g\,\zeta(x)^4 \cap \mathrm{Ad}_g\,\zeta(y)^4 = \zeta(gx)^4 \cap \zeta(gy)^4 = \zeta(\mathfrak{p}_0)^4 \cap \zeta(\mathfrak{p}_0^t)^4 =$$
$$= \pi\left(\mathbb{R}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\right).$$

For the last equality, we use Proposition B.0.2 and the fact that $\mathfrak{p}_0^t = \mathrm{Ad}_g\,\mathfrak{p}_0$ for

$$g = \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix},$$

to conclude that

$$\zeta(\mathfrak{p}_0) = \zeta(P_0) = \pi\left(\left\{\begin{pmatrix}u & a & it\\ 0 & 0 & -\bar{a}\\ 0 & 0 & -u\end{pmatrix} : a \in \mathbb{C}, u, t \in \mathbb{R}\right\}\right),$$

and

$$\zeta(\mathfrak{p}_0^t) = \zeta(gP_0) = \mathrm{Ad}_g\,\zeta(P_0) = \pi\left(\left\{\begin{pmatrix}u & 0 & 0\\ a & 0 & 0\\ it & -\bar{a} & 0\end{pmatrix} : a \in \mathbb{C}, u, t \in \mathbb{R}\right\}\right).$$

□

Bibliography

[LL23] François Ledrappier and Pablo Lessa. "Dimension gap and variational principle for Anosov representations". In: (Dec. 2023). arXiv:2310.13465 [math] version: 3 (cit. on p. 3).

[PSW23] Beatrice Pozzetti, Andrés Sambarino, and Anna Wienhard. "Anosov representations with Lipschitz limit set". In: *Geometry & Topology* 27.8 (Nov. 2023). arXiv:1910.06627 [math], pp. 3303–3360. ISSN: 1364-0380, 1465-3060 (cit. on pp. 6–10, 14, 20).

[GH13] Etienne Ghys and Pierre da la Harpe. *Sur les groupes hyperboliques d'apres Mikhael Gromov*. Vol. 83. Springer Science & Business Media, 2013 (cit. on p. 10).