

Limit sets of Anosov representations

Giorgos

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Chapter 1

Introduction

Definition 1.0.1. Let Γ be a discrete group of isometries of a metric space (X, d) . We define the critical exponent of Γ to be the asymptotic exponential growth of its orbits, i.e. the following limit:

$$\delta_\Gamma = \limsup_{n \rightarrow \infty} \frac{\log \#\{\gamma \in \Gamma : d(x, \gamma x) \leq n\}}{n}$$

for some fixed $x \in X$.

Remark 1.0.1. It is not hard to show that the critical exponent is independent of the choice of x .

1.1 Lie group preliminaries

We fix the Cartan subalgebra \mathfrak{a} of $\mathrm{SL}(d, \mathbb{R})$:

$$\mathfrak{a} = \{\mathrm{diag}(\alpha_1, \dots, \alpha_d) : \alpha_1 + \dots + \alpha_d = 0\}$$

and the Weyl chamber \mathfrak{a}^+ of $\mathrm{SL}(d, \mathbb{R})$

$$\mathfrak{a}^+ = \{\mathrm{diag}(\alpha_1, \dots, \alpha_d) : \alpha_1 \geq \dots \geq \alpha_d\}.$$

Denoting with $K = \mathrm{SO}(d, \mathbb{R})$, $A^+ = e^{\mathfrak{a}^+}$, we have the Cartan decomposition:

$$\begin{aligned} \mathfrak{sl}(d, \mathbb{R}) &\rightarrow K \times A^+ \times K \\ g &\mapsto (k_g, a_g, l_g) \end{aligned}$$

such that $g = k_g a_g l_g$. In particular $a_g = \mathrm{diag}(\sigma_1(g), \dots, \sigma_d(g))$ with $\sigma_1 \geq \dots \geq \sigma_d(g)$, where $\sigma_i(g)$ is the i -th singular value of g , i.e. eigenvalue of $g^t \cdot g$.

We will use the spaces

$$U_p(g) = \mathbb{R}u_1(g) \oplus \dots \oplus \mathbb{R}u_p(g)$$

where $u_i(g) = k_g \cdot e_i$. One can easily show that the decomposition

$$g^{-1} \cdot U_p(g) \oplus U_{d-p}(g^{-1})$$

is orthogonal with respect to the standard inner product and that $u_p(g^{-1}) = l_g^{-1} e_{d-p+1}(g)$.

1.2 Limit set preliminaries

Definition 1.2.1. For $p \in \{2, \dots, d\}$, $s \in \mathbb{R}$ and $g \in SL(d, \mathbb{R})$ we denote with $\tilde{\Psi}_s^p(g), \Psi_s^p(g) : SL(d, \mathbb{R}) \rightarrow \mathbb{R}$ the functional:

$$\begin{aligned}\Psi_s^p(g) &= \alpha_{12}(a(g)) + \dots + \alpha_{1(p-1)}(a(g)) + (s - (p-2))\alpha_{1p}(a(g)) \\ \tilde{\Psi}_s^p(g) &= \left(\frac{\sigma_2}{\sigma_1} \dots \frac{\sigma_{p-1}}{\sigma_1}(g) \right) \left(\frac{\sigma_{p-1}}{\sigma_1}(g) \right)^{s-(p-2)}\end{aligned}$$

Remark 1.2.1. We have $\alpha_{ij}(a) = a_i - a_j$, $a_i(g) = \log(\sigma_i(g))$ and

$$\Psi_s^p(g) = \log \tilde{\Psi}_s^p(g)$$

and that

$$\min_{p \in \llbracket 2, d \rrbracket} \left\{ \sum_{|\gamma|=T} \frac{\sigma_2}{\sigma_1} \dots \frac{\sigma_{p-1}}{\sigma_1}(g) \left(\frac{\sigma_{p-1}}{\sigma_1}(g) \right)^{s-(p-2)} \right\} = \sum_{|\gamma|=T} e^{-\max_{p \in \llbracket 2, d \rrbracket} \Psi_s^p(g)}$$

Remark 1.2.2. For any $g \in SL(d, \mathbb{R})$ we have that:

$$\max_{p \in \llbracket 2, d \rrbracket} \Psi_s^p(g) = \Psi_s^{p_0}(g) \text{ for } s \in [p_0 - 2, p_0 - 1].$$

Indeed, a quick calculation shows that for $s \geq 0$ and $p \in \llbracket 2, d \rrbracket$:

$$\Psi_s^p(g) \leq \Psi_s^p(g) \text{ if and only if } s \geq p - 1.$$

and that equality holds in the case $s = p - 1$. Thus for $s \in [p - 2, p - 1]$ we have that

$$\begin{aligned}s \geq p - 2, \dots, 1 \text{ implies that } \Psi_s^p(g) &\geq \dots \geq \Psi_s^2(g) \\ s \leq p, \dots, d - 1 \text{ implies that } \Psi_s^p(g) &\leq \dots \leq \Psi_s^d(g)\end{aligned}$$

Another way to see this (refer to Figure 1.1) is to note that $\Psi_s^2(g), \dots, \Psi_s^d(g)$ is a sequence of functions that are affine in s , with slopes $\alpha_{12}(g) \leq \dots \leq \alpha_{1d}(g)$ and that they satisfy $\Psi_1^2(g) = \Psi_2^2(g), \Psi_2^3(g) = \Psi_3^4(g) \dots, \Psi_{d-2}^{d-1}(g) = \Psi_{d-2}^d(g)$.

The following definition comes from [LL23], in the special case of projective Anosov representations ($P = 1$):

Definition 1.2.2. For $s \geq 0$ we consider the Falconer functional $F_s : SL(d, \mathbb{R}) \rightarrow \mathbb{R}$ by:

$$F_s(g) = \min \left\{ \sum_{j=2}^d s_j \alpha_{1j}(a(g)) : s_j \in (0, 1], \sum_{j=2}^d s_j = s \right\},$$

and define the Falconer dimension $\dim_F(\rho)$ of ρ to be its critical exponent:

$$\dim_F(\rho) = \inf \left\{ s > 0 : \sum_{\gamma \in \Gamma} e^{-F_s(\rho(\gamma))} < \infty \right\}.$$

Remark 1.2.3. Using elementary computations one may prove that for all $s \geq 0$:

$$F_s(g) = \max_{p \in \llbracket 2, d \rrbracket} \Psi_s^p(g)$$

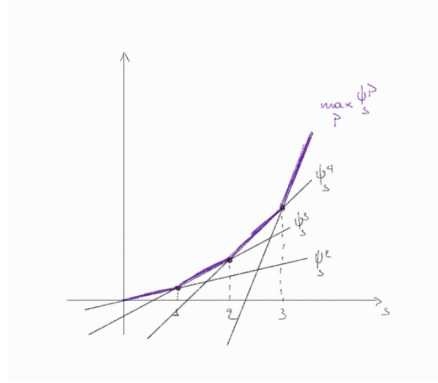


Figure 1.1: Visual illustration that $\max_p \Psi_s^p(g) = \Psi_s^{p_0}(g)$ for $s \in [p_0 - 2, p_0 - 1]$.

Definition 1.2.3. Let $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$ be a linear representation and $p \in \llbracket 1, d - 1 \rrbracket$. We say that ρ is p -Anosov if there exist constants $\mu, C > 0$ such that for all $\gamma \in \Gamma$:

$$\frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma)) \leq C e^{-\mu|\gamma|}.$$

One can show that in that case there exist equivariant continuous maps $\xi^p : \hat{\Gamma} \rightarrow \mathcal{G}_p(\mathbb{R}^d), \xi^{d-p} : \hat{\Gamma} \rightarrow \mathcal{G}_{d-p}(\mathbb{R}^d)$ that are transverse and restrict to

$$\xi^p(\gamma) = U_p(\rho(\gamma)), \xi^{d-p}(\gamma) = U_{d-p}(\rho(\gamma))$$

for $\gamma \in \Gamma$, where $U_p(\gamma), U_{d-p}(\gamma)$ denote the flags corresponding to $\rho(\gamma)$.

Figure out what this exactly means

Chapter 2

Upper bound

2.1 Proof of bound

Lemma 2.1.1 (Upper bound for dimension). *Let $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. Then:*

$$\dim_H(\xi^1(\partial\Gamma)) \leq \dim_F(\rho).$$

Remark 2.1.1. The idea of the proof of Lemma 2.1.1 is to find a covering whose Hausdorff content is dominated by the Dirichlet series of some functional Ψ_s^p , which will in turn imply that $\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(\Psi^p)$. Choosing then the most "effective" cover (i.e. the one which yields the smallest Hausdorff content up to a constant) we obtain that

$$\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(\max_p \Psi^p)$$

To obtain this we first cover $\xi^1(\partial\Gamma)$ by the bassins of attraction $\rho(\gamma) \cdot B_{\alpha_1, \alpha}(\rho(\gamma))$ for $\gamma \in \Gamma$ satisfying $|\gamma| = T$. Then we cover each bassin by an ellipsoid of axes lengths

$$\frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \dots, \frac{1}{\sin(\alpha)} \frac{\sigma_d}{\sigma_1}(\rho(\gamma)).$$

Finally we cover each ellipsoid by balls of some fixed radius $r > 0$. It can be shown by comparing the series appearing in the Hausdorff content of each resulting cover that the most "effective" choice of r depends only on the Hausdorff exponent $s > 0$ and in any case will be to have r equal (up to a constant) to the the length of an axis of the ellipsoid, i.e.

$$r \in \left\{ \frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \dots, \frac{1}{\sin(\alpha)} \frac{\sigma_d}{\sigma_1}(\rho(\gamma)) \right\}$$

In particular, when $s \in [p-2, p-1]$, the most effective choice is $r = \sigma_p(\rho(\gamma))/\sigma_1(\rho(\gamma))$, whose Hausdorff content is dominated by the Dirichlet series of Ψ_s^p .

Proof of Lemma 2.1.1. Let $p \in \llbracket 2, d \rrbracket$. Then using Proposition 2.2.1, Lemma 2.2.2, and Lemma 2.2.3 we have that for $T > 0$ large enough, $\xi^1(\partial\Gamma)$ is covered by the family

$$\mathcal{U}_T = \{\rho(\gamma)B_{\alpha_1, \alpha}(\rho(\gamma)) : |\gamma| = T\},$$

and that each basin $\rho(\gamma)B_{\alpha_1, \alpha}(\rho(\gamma))$ is in turn covered by

$$2^{p-2} \cdot \frac{\sigma_p(g)^{p-2}}{\sigma_2(g) \cdots \sigma_{p-1}(g)}$$

many balls of radius

$$\sqrt{d-1} \frac{1}{\sin \alpha} \frac{\sigma_p(g)}{\sigma_1(g)}.$$

By the definition of the Hausdorff measure, for $s \geq 0$:

$$\begin{aligned} \mathcal{H}^s(\xi^1(\partial\Gamma)) &\leq \sum_{|\gamma|=T} 2^{2p+1} \cdot \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \cdots \frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \left(\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \right)^{-(p-2)} \left(\sqrt{d-1} \frac{1}{\sin \alpha} \frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \right)^s = \\ &= 2^{2p+1} \cdot \left(\frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \cdots \frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \left(\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \right)^{s-(p-2)} = \\ &= 2^{2p+1} \cdot \left(\frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} e^{-(\alpha_{12} + \dots + \alpha_{1(p-1)} + (s-(p-2))\alpha_{1p})\rho(\gamma)} \\ &= 2^{2p+1} \cdot \left(\frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} e^{-\Psi_s^p(\rho(\gamma))} \end{aligned}$$

and thus

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq 2^{2p+1} \cdot \left(\frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} e^{-\max_p \Psi_s^p(\rho(\gamma))} \lesssim \sum_{|\gamma|=T} e^{-F_s(\rho(\gamma))}.$$

To see that the above implies the upper bound, consider some $s > \dim_F(\rho)$. By the definition of the Falconer dimension, this implies that the Dirichlet series corresponding to the Falconer functional converges:

$$\sum_{\gamma \in \Gamma} e^{-F_s(\rho(\gamma))} < \infty$$

and in particular

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq \lim_{T \rightarrow \infty} e^{-F_s(\rho(\gamma))} = 0.$$

□

2.2 Lemmata

Definition 2.2.1. Let V be a finite-dimensional \mathbb{R} -vector space. We consider a decomposition

$$V = \mathbb{R}u_1 \oplus \cdots \oplus \mathbb{R}u_d$$

be a direct decomposition that is orthogonal with respect to a fixed inner-product over V . Given $\beta_2 \geq \dots \beta_d > 0$, we define an ellipsoid with axes $u_1 \oplus u_p(g)$ and lengths β_p to be the image of

$$\left\{ v = \sum_1^d v_i u_i \in V : \sum_2^d \left(\frac{v_j}{\beta_j} \right)^2 \leq 1 \right\}$$

through the projection $V \rightarrow \mathbb{P}(V)$.

The following aims to be something along the lines of [PSW23, Lemma 2.4]:

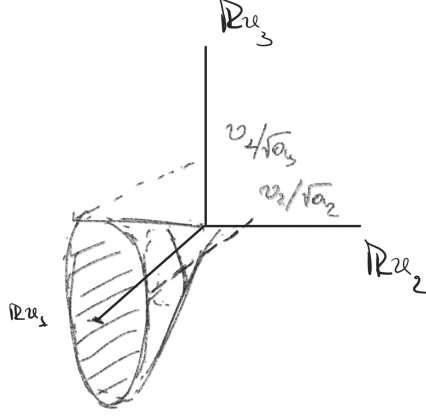


Figure 2.1: Depiction in \mathbb{R}^3 of an ellipsoid of $\mathbb{P}(\mathbb{R}^2)$

Lemma 2.2.1. Let $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. For $\alpha > 0$ small enough, there exists $L > 0$ such that for any geodesic ray $(a_j)_j$ through e we have:

$$\angle(U_1(\rho(a_i)), U_{d-1}(\rho(a_0))) > \alpha$$

when $|a_i|, |a_0| > T$.

Proof. Assume the contrary for the sake of contradiction. Then (see Figure 2.2) for each $n > 0$ there exists a geodesic ray a^n through e such that

$$|a^n_n|, |a^n_0| > n \text{ and } \angle(U_1(\rho(a^n_n)), U_1(\rho(a^n_0))) < \frac{1}{n}.$$

Due to compactness of $\Gamma \cup \partial\Gamma$ we can find some subsequence k_n and $x, y \in \partial\Gamma$ such that $a^{k_n}_{k_n} \rightarrow x$, $a^{k_n}_0 \rightarrow y$ and $x \neq y$. Since the limit map is dynamics preserving, we have that

$$\angle(\xi^1(x), \xi^{d-1}(y)) = 0,$$

which contradicts its transversality property. \square

The following is [PSW23, Proposition 3.5].

Lemma 2.2.2. Let $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$ be projective Anosov. Then for $\alpha > 0$ small enough, there exists some $T_0 > 0$ such that for all $T \geq T_0$ the family

$$\mathcal{U}_T = \{\rho(\gamma)B_{\alpha_1, \alpha}(\rho(\gamma)) : |\gamma| = T\}$$

is an open covering of $\xi^1(\partial\Gamma)$.

Proof. Let $\alpha, T > 0$ be as in the statement of Lemma 2.2.1 and $x \in \partial\Gamma$ be represented by a geodesic ray $(\gamma_j)_{j \geq 0}$ starting from e . Then $(\gamma_T^{-1}\gamma_j)_j$ is a geodesic ray starting from $(\gamma_T)^{-1}$ that passes through e , so

$$\angle(U_1(\rho(\gamma_T^{-1}\gamma_j)), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

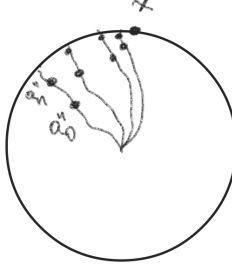


Figure 2.2: Situation in Lemma 2.2.1

as implied by Lemma 2.2.1. Taking the limit $j \rightarrow \infty$ and using the equivariance of the limit map, we obtain

$$\angle(\rho(\gamma_T^{-1})\xi^1(x), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

and thus $\xi^1(x) \in \rho(\gamma_T) \cdot B_{\alpha_1, \alpha}(\rho(\gamma_T))$. \square

The following is [PSW23, Proposition 3.8].

Proposition 2.2.1. For each $g \in \text{SL}(d, \mathbb{R})$, $\alpha > 0$, the basin of attraction $g \cdot B_{\alpha_1, \alpha}(g)$ lies in the ellipsoid with axes $u_1(g) \oplus u_p(g)$ with lengths

$$\frac{1}{\sin \alpha} \cdot \frac{\sigma_p(g)}{\sigma_1(g)}$$

Proof. Using the definition of the basin of attraction (see Figure 2.3), we have that $w = w_1 u_1(g^{-1}) + \dots + w_d u_d(g^{-1}) \in B_{\alpha_1, \alpha}(g)$ if and only if

$$w_d^2 \geq (\sin \alpha)^2 \sum_{i=1}^d w_i^2.$$

Considering now some $v = v_1 u_1(g) + \dots + v_d u_d(g) \in g \cdot B_{\alpha_1, \alpha}(g)$ we have that

$$\begin{aligned} w &= g^{-1}v = v_1 \sigma_1(g)^{-1} l_g^{-1} e_1(g) + \dots + v_d \sigma_d(g)^{-1} l_g^{-1} e_d(g) \\ &= v_1 \sigma_1(g)^{-1} u_d(g^{-1}) + \dots + v_d \sigma_d(g)^{-1} u_1(g^{-1}) \end{aligned}$$

where we used that $u_p(g^{-1}) = l_g^{-1} e_{d+1-p}$. Hence

$$\sigma_1(g)^{-2} \cdot v_1^2 \geq (\sin \alpha)^2 \sum_{i=1}^d \sigma_i(g)^{-2} v_i^2.$$

\square

The following is [PSW23, Lemma 3.7]:

Lemma 2.2.3. For any $p \in \llbracket 2, d \rrbracket$, an ellipsoid in $\mathbb{P}(\mathbb{R}^d)$ of axes lengths β_2, \dots, β_d is covered by

$$2^{p-2} \frac{\beta_2 \cdots \beta_{p-1}}{\beta_p^{p-2}}$$

many (projected) balls of radius $\sqrt{d-1} \beta_p$.

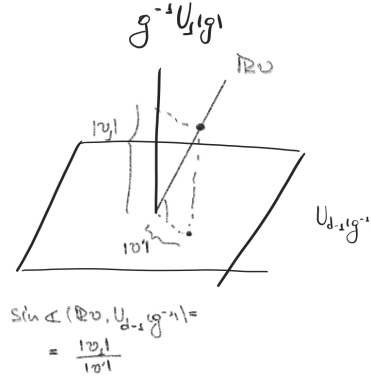


Figure 2.3: Aid for Proposition 2.2.1

Proof. We assume that E is an ellipsoid about $\mathbb{R}e_1$, so it suffice to cover its intersection $E_1 = E \cap U_1 \subseteq \mathbb{R}^{d-1}$ with the affine chart $U_1 = \{[x_1 : \dots, x_d] \in \mathbb{P}(\mathbb{R}^d) : x_1 \neq 0\}$. Clearly $E_1 \subseteq [-\beta_2, \beta_2] \times \dots \times [-\beta_d, \beta_d]$, so we proceed by covering the rectangle with side-lengths $2\beta_2, \dots, 2\beta_d$. Clearly each interval $(-\beta_j, \beta_j)$ is contained in the union of $\lceil \beta_j/\beta_p \rceil$ intervals of length $2\beta_p$, thus E_1 is contained in the union of

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil \dots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil = \left\lceil \frac{\beta_2}{\beta_p} \right\rceil \dots \left\lceil \frac{\beta_d}{\beta_p} \right\rceil$$

many squares of side-length $2\beta_p$. Since each such product is contained in a $(d-1)$ -ball of radius $\sqrt{d-1}\beta_p$ we may use at most

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil \dots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil \leq \sum_{i \in \{0,1\}^{p-2}} \prod_{j=2}^{p-1} \left(\frac{\beta_j}{\beta_p} \right)^{i_j} \leq 2^{p-2} \frac{\beta_2}{\beta_p} \dots \frac{\beta_{p-1}}{\beta_p}$$

many $(d-1)$ -balls of radius $\sqrt{d-1}\beta_p$ to cover E_1 . \square

The following can be found in [PSW23, Proposition 3.3]:

Proposition 2.2.2. *Let $\rho : \Gamma \rightarrow SL(d, \mathbb{R})$ be projective Anosov and $\alpha > 0$. Then there exist $c_0, c_1 > 0$ that depends only on α and ρ such that for all $\gamma \in \Gamma$:*

$$(\xi^1)^{-1}(B_{\alpha_1, \alpha}(\rho(\gamma))) \subseteq C_{c_0, c_1}^\infty(\gamma)$$

Proof. We begin by noting that it suffices to show this for all but finitely many $\gamma \in \Gamma$, since then we may alter the constants to satisfy the wanted inclusion for the finitely many remaining $\gamma \in \Gamma$ as well. Hence, we may assume that $|\gamma| \geq l_0$ where $l_0 > 0$ is such that

$$Ce^{-\mu l_0} < 1 \text{ and } \mathbf{a}_1(\gamma) \geq C|\gamma| - L.$$

Suppose $x \in \partial\Gamma$ such that $\xi^1(x) \in B_{\alpha_1, \alpha}(\rho(\gamma))$, and consider a geodesic ray $a_j \rightarrow x$ starting from $a_0 = e$. To prove the result, it suffices to find constants c_0, c_1 independent of γ and for which there exists a (c_0, c_1) -quasi-geodesic from γ^{-1} to x that passes through e and stays at a bounded distance from $(a_j)_{j=0}^\infty$.

Using the exponential convergence rate of $\xi^1(a_j) \rightarrow \xi^1(x)$ and the definition of $B_{\alpha_1, \alpha}(\rho(\gamma))$ we have that:

$$\begin{aligned} d(\xi^1(a_j), \xi^1(\gamma)) &\geq d(\xi^1(x), U_1(\rho(\gamma^{-1})) - d(\xi^1(a_j), \xi^1(x))) \geq \\ &\geq d(\xi^1(x), U_{d-1}(\rho(\gamma^{-1})) - d(\xi^1(a_j), \xi^1(x))) \geq \sin \alpha - Ce^{-\mu j} \end{aligned}$$

which along with the uniform continuity of $\xi^1 : \Gamma \cup \partial\Gamma \rightarrow \mathbb{P}(\mathbb{R}^d)$ this implies there exists some $\alpha' > 0$ and $L > 0$ such that for all $j \geq L$:

$$d(a_j, \gamma^{-1}) \geq \alpha'.$$

Upon considering a large L , we may also assume that $|a_L| = L > l_0$. Note that both α' and L do not depend on each γ but only on ρ and α .

Using a coarse geometric argument, we can show that for all $j \geq L$

$$d(\gamma^{-1}, a_j) > \alpha' \Rightarrow d([\gamma^{-1}, a_j], e) < \alpha''$$

for some α'' that depends only on Γ and α' , where $[a_j, \gamma^{-1}]$ denotes any geodesic segment connecting γ^{-1} and a_j . Indeed, [GH13, Lemme 2.17] states that $d([\gamma^{-1}, a_j]) \leq (\gamma_j^{-1}, a_j)_e + \delta$ where δ is the hyperbolicity constant of Γ . Thus

$$d([\gamma^{-1}, a_j]) \leq \delta + \frac{d(a_j, e) + d(\gamma^{-1}, e) + d(a_j, \gamma^{-1})}{2} \leq \delta + \frac{L + d(\gamma^{-1}, e) + \alpha'}{2}.$$

Consider the concatenation $(a'_j)_{j=L-K}^\infty$ of $[\gamma^{-1}, a_L]$ and $[a_L, x]$. To find quasi-geodesic-constants that are uniform in γ , we note that for any $c_0 \geq 1, c_1 \geq 0$:

$$c_0^{-1}|i - j| - c_1 \leq d(a'_i, a'_j) = d(a_i, a_j) \leq c_0|i - j| + c_1 \text{ when } i, j \geq L \text{ or } i, j \leq L$$

and that the upper bound follows trivially by the triangle inequality.

For the lower bound we proceed in two steps. First we bound the distance of $\gamma^{-1} = a'_{L-K}$ to a_{L+j} for $j \geq 0$:

$$\begin{aligned} d(a'_{L-K}, a'_{L+j}) &\geq \nu(|a_{L+j}| - |\gamma^{-1}|) - c'_0 - c'_1 |\log(d(U_1(\rho(a_{L+j})), U_1(\rho(\gamma^{-1}))))| \geq \\ &\geq \nu((L+j) + (K-L)) - c'_0 - c'_1 |\log(\sin a)| \geq \\ &= c_0^{-1}(j+K) - c_1 \end{aligned}$$

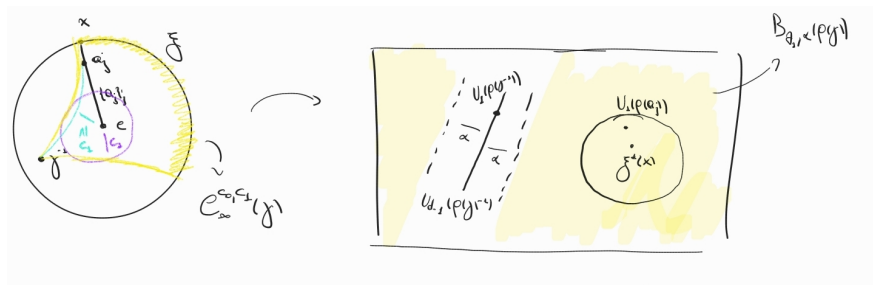
for $c_0 = \nu^{-1}, c_1 = c'_0 + c'_1 |\log(\sin a)|$. The first inequality comes from [PSW23, Lemma 3.9]. For the second inequality we estimate $|\gamma^{-1}|$ from below using the triangle inequality. We are now ready to show that the concatenation $(a'_j)_j$ is indeed a (c_0, c_1) -geodesic:

$$\begin{aligned} d(a_{L+j}, a_{L-i}) &\geq d(a_{L+j}, a_{L_K}) - d(a_{L_K}, a_{L-i}) \geq c_0^{-1}(j+K) - c_1 - (K-i) \geq \\ &\geq c_0^{-1}(j+i) - c_1. \end{aligned}$$

Note however that (a'_j) does not necessarily lie in $C_\infty^{c_0, c_1}$ since it may not pass through e . For this reason we some $L-K \leq i_0 \leq L$ such that $|a_{i_0}| < \alpha''$, the existence of which is guaranteed by the fact that $d([\gamma^{-1}, a_L], e) < \alpha''$. We then consider alter (a'_j) at i_0 so that it passes through e to obtain

$$a''_j = \begin{cases} a_j & \text{for } j \neq i_0 \\ e & \text{for } j = i_0 \end{cases}$$

which is a $(c_0, c_1 + \alpha'')$ -quasigeodesic passing from e and converging to x . \square



Chapter 3

Lower bound

3.1 Busemann cocycle and Patterson-Sullivan measures

We denote with Π the set of simple positive roots, and for $\Theta \subseteq \Pi$ we consider the Levi-Anosov subspace of \mathfrak{a}

$$\mathfrak{a}_\Theta = \bigcap_{\alpha \notin \Theta} \ker \alpha,$$

which in particular admits $\{\omega_{\alpha_i} : \alpha_i \in \Theta\}$ as a basis.

Definition 3.1.1. Let $\Theta \subseteq \Pi$. We define the Busemann cocycle

$$b : \mathrm{PSL}(d, \mathbb{R}) \times \mathcal{F}_\Theta \rightarrow \mathfrak{a}_\Theta$$

as the unique element $b(g, kP_\Theta) \in \mathfrak{a}_\Theta$ such that

$$gk \in Ke^{b(g, kP_\Theta)} N.$$

where $N = \{n \in \mathrm{SL}(d, \mathbb{R}) : n_{ij} = 0 \text{ for } i > j, n_{ii} = 1 \text{ for all } i\}$ is the unipotent group of upper subgroup of upper triangular matrices with 1s on the diagonal, and P_Θ is the parabolic subgroup of $\mathrm{PSL}(d, \mathbb{R})$ corresponding to Θ , i.e. $\mathcal{F}_\Theta = \mathrm{PSL}(d, \mathbb{R})/P_\Theta$.

Lemma 3.1.1. For $y \in \mathcal{F}_\Theta(\mathbb{R}^d)$ we have

$$\omega_{\alpha_i}(b_\Theta(g, x)) = \log \frac{\|gv_1 \wedge \cdots gv_i\|}{\|v_1 \wedge \cdots v_i\|} \text{ for all } \alpha_i \in \Theta$$

for any basis v_1, \dots, v_i of $x^i \in \mathcal{G}_i(\mathbb{R}^d)$, where $\|\cdot\|$ denotes the norm on $\bigwedge^i \mathbb{R}^d$ induced by the euclidean inner product on \mathbb{R}^d , i.e. $\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle := \det(\langle v_i, w_j \rangle)$.

Definition 3.1.2. We define

$$\Lambda^k : \mathrm{SL} d, \mathbb{R} \rightarrow \mathrm{SL} \Lambda^k \mathbb{R}^d, \quad \Lambda^k : \mathcal{G}_k(\mathbb{R}^d) \rightarrow \mathbb{P}(\Lambda^k \mathbb{R}^d)$$

as

$$\Lambda^k(g)(v_1 \wedge \cdots \wedge v_k) = gv_1 \wedge \cdots \wedge gv_k \Lambda^k(\mathbb{R}v_1 \oplus \cdots \mathbb{R}v_k) = v_1 \wedge \cdots \wedge v_k$$

Definition 3.1.3. For a discrete subgroup $\Gamma < \mathrm{PSL}(d, \mathbb{R})$, $\phi \in (\alpha_\Theta)^*$, a (Γ, ϕ) -Patterson Sullivan measure on \mathcal{F}_Θ is a finite Radon measure μ such that for every $\gamma \in \Gamma$

$$\frac{d\gamma_*\mu}{d\mu}(x) = e^{-\phi(b_\Theta(g^{-1}, x))}, \text{ for all } x \in \mathcal{F}_\Theta(\mathbb{R}^d).$$

Lemma 3.1.2. Let $\alpha > 0, \Theta \subseteq \Pi$. There exists $K = K(\alpha) > 0$ such that for each $g \in \mathrm{SL}(d, \mathbb{R})$, $\mathbf{a}_i \in \Theta, \mathbf{y} \in \mathbf{B}_{\Theta, \alpha}(\mathbf{g}), \phi \in \mathfrak{a}_\Theta$

$$|\phi(a(g) - b(g, \mathbf{y}))| \leq K.$$

Proof. We begin by noting that it suffices to consider the case where $\phi = \mathbf{a}_k$ for $\mathbf{a}_k \in \Theta$, since $\{\omega_i\}_{\mathbf{a}_i \in \Theta}$ is a basis for \mathfrak{a}_Θ^* .

We first consider the case where $k = 1$. We recall that the first component of the Cartan projection coincides with the spectral norm of g , i.e.

$$a_1(g) = \log \sup_{v \neq 0} \frac{\|gv\|}{\|v\|} = \log \|gk_2^{-1}e_1\|$$

where $g = k_1 e^{a(g)} k_2$ is the Cartan decomposition of g . Let $v = v_1 k_2^{-1} e_1 + \dots + v_d k_2^{-1} e_d \in \mathbb{R}^d$ be such that $\|v\| = 1$ and $y = \mathbb{R}v$, we have

$$\begin{aligned} |\omega_1(a(g) - b(g, y))| &= |\log \|gk_2^{-1}e_1\| - \log \|gv\|| = \\ &= |\log |e^{a_1(g)}| - \log \|e^{a_1(g)}v_1 k_1 e_1 + \dots + e^{a_d(g)}v_d k_1 e_d\|| = \\ &= \left| \log \left\| v_1 k_1 e_1 + e^{-a_{12}(g)} v_2 k_1 e_2 + \dots + e^{-a_{1d}(g)} v_d k_1 e_d \right\| \right| \leq \\ &\leq |\log |v_1|| = |\log \sin(\angle(v, U_{d-1}(g^{-1})))| \leq |\log \sin \alpha|. \end{aligned}$$

For the case where $\Theta = \{\mathbf{a}_k\}$, we consider $\alpha' > 0$ such that

$$\Lambda^k(B_{\mathbf{a}_k, \alpha}(g)) \subseteq B_{\mathbf{a}_k, \alpha'}(\Lambda^k g)$$

Then using the case $k = 1$ we have that

$$|\omega_k(a(g) - b(g, y))| = |\omega_1(a(\Lambda^k g) - b(\Lambda^k g, \Lambda^k y))| \leq |\log \sin \alpha'|.$$

□

3.2 Proof strategy

Denoting with $d_\Gamma = \dim_H \xi_\rho^1(\partial\Gamma)$ the Hausdorff dimension of the limit set, we shall outline how to obtain its lower bound by the Falconer dimension

$$d_\Gamma \geq h_\rho(F).$$

First we recall that $F_s(a) = \max\{\Psi_s^p(a) : p \in [2, d]\}$ and in particular $h_\rho(F) \leq h_\rho(\Psi^{d_\Gamma+1})$. Thus the lower bound will follow once we have shown that

$$d_\Gamma \geq h_\rho(\Psi^{d_\Gamma+1}).$$

Noting that $(s+1)J_{d_\Gamma^s} \leq \Psi_{s+d_\Gamma}^{d_\Gamma+1}$, the above bound will follow as soon as we have shown that

$$h_\rho(J_{d_\Gamma}) \leq 1. \tag{LB}$$

To obtain inequality (LB), one uses the method of Patterson-Sullivan-Quint, which may be summed up in

There exists a (ϕ, ρ) -Patterson-Sullivan measure on $\mathcal{F}_\Theta(\mathbb{R}^d) \Rightarrow h_\rho(\phi) \leq 1$,

where $\phi \in \mathfrak{a}_\Theta$ and $\Theta \subseteq \Pi$. The property that we will need of our measure is that there exists a collection of open sets $U_\gamma \in \Gamma$ such that

$$\mu(U_\gamma) \sim e^{-J_{d_\Gamma}^u(a(\rho(\gamma)))} \text{ and } M := \sup_{n \in \mathbb{N}} \max \left\{ \sharp A : A \subseteq \Gamma_n, \bigcap_{\gamma \in A} U_\gamma \neq \emptyset \right\} < \infty \quad (\text{MP})$$

where $\Gamma_n = \{\gamma \in \Gamma : |\gamma| = n\}$. For the proof of the existence of a $(J_{d_\Gamma}^u, \rho)$ -Patterson-Sullivan measure that satisfies the property (MP) we refer to Section 3.3, noting that the Zariski-density assumption is necessary only for the equivalence appearing on the left hand side of Equation (MP). Assuming that for the time being, below we outline the Patterson-Sullivan-Quint method of obtaining Equation (LB).

Indeed, we first obtain the uniform in n bound:

$$\sum_{\gamma \in \Gamma_n} e^{-J_{d_\Gamma}^u(a(\rho(\gamma)))} \lesssim \sum_{\gamma \in \Gamma_n} \mu(U_\gamma) \leq \frac{1}{M} \mu(\mathcal{F}_\Theta(\mathbb{R}^d)) < \infty$$

along with the bound implied by the Anosov property of ρ :

$$J_{d_\Gamma}(a(\rho(\gamma))) \geq \mathfrak{a}_{12}(a(\rho(\gamma))) \geq C|\gamma| - b$$

to conclude that

$$\sum_{\gamma \in \Gamma} e^{-(s+1)J_{d_\Gamma}^u(a(\rho(\gamma)))} = \sum_{n \geq 0} \sum_{\gamma \in \Gamma_n} e^{-sJ_{d_\Gamma}^u(a(\rho(\gamma)))} e^{J_{d_\Gamma}^u(a(\rho(\gamma)))} \lesssim \sum_{n \geq 0} e^{-s(Cn-b)} < \infty$$

which holds for any $s > 0$, and thus Equation (LB) holds.

3.3 Existence of Patterson-Sullivan measure

Definition 3.3.1. For $p \in \llbracket 2, d \rrbracket$, we denote the p -th unstable Jacobian $J_p^u \in \mathfrak{a}^*$ by

$$J_p^u = (p+1)\omega_{\mathfrak{a}_1} - \omega_{\mathfrak{a}_{p+1}} = \mathfrak{a}_{12} + \cdots + \mathfrak{a}_{1(p+1)}.$$

Definition 3.3.2. Let $V \in \mathcal{G}_{p+1}\mathbb{R}^d$ and $l \in \mathbb{P}(V)$. Using the canonical identification $T_l\mathbb{P}(V) \simeq \text{hom}(l, V/l)$, we define the density $|\Omega_{l,V}|$ on $\bigwedge^p T_l\mathbb{P}(V)$ by

$$|\Omega_{l,V}|(\phi_1, \dots, \phi_p) = \frac{\|v \wedge \tilde{\phi}_1(v) \wedge \cdots \wedge \tilde{\phi}_p(v)\|}{\|v\|^{p+1}}$$

for any $v \in l - \{0\}$, where $\tilde{\phi}_1, \dots, \tilde{\phi}_p \in \text{hom}(l, V)$ are such that $\phi_i = \tilde{\phi}_i + \text{hom}(l, l)$ and $\|\cdot\|$ denotes the norm on $\bigwedge^{p+1}\mathbb{R}^d$ induced by the euclidean inner product.

The following is [PSW23, Proposition 6.4]:

Proposition 3.3.1. Assume that $\xi_\rho^1(\partial\Gamma)$ is a Lipschitz submanifold of dimension d_Γ . Then there exists a $(\rho(\Gamma), J_{d_\Gamma}^u)$ -Patterson-Sullivan measure on $\mathcal{F}_{1, d_\Gamma+1}$.

Proof. By Rademacher's theorem, $\xi_\rho^1(\partial\Gamma)$ has a well-defined Lebesgue measure class, and Lebesgue-almost every $\xi_\rho^1(x) \in \xi_\rho^1(\partial\Gamma)$ admits a well-defined tangent space $T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma)$. Considering such a $\xi_\rho^1(x)$ we let

$$\pi : \text{hom}(\xi_\rho^1(x), \mathbb{R}^d) \rightarrow \text{hom}(\xi_\rho^1(x), \mathbb{R}^d / \xi_\rho^1(x)) \simeq T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma),$$

and

$$x^{d_\Gamma+1} = \pi^{-1}(T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma))\xi_\rho^1(x) \in \mathcal{G}_{d_\Gamma+1}(\mathbb{R}^d),$$

for which one can show that

$$T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma) \simeq \text{hom}(\xi_\rho^1(x), \mathbb{R}^d/\xi_\rho^1(x)) \simeq \text{hom}(\xi_\rho^1(x), x^{d_\Gamma+1}/\xi_\rho^1(x)).$$

In this notation, we shall define (Lebesgue-almost everywhere) the map

$$\zeta_\rho : \xi_\rho^1(\partial\Gamma) \rightarrow \mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d), \quad \zeta_\rho(\xi_\rho^1(x)) = (\xi_\rho^1(x), x^{d_\Gamma+1}).$$

We now define the non-negative density on $\xi_\rho^1(\partial\Gamma)$

$$\mu_{\xi_\rho^1(x)} = |\Omega_{\zeta_\rho(\xi_\rho^1(x))}|$$

which satisfies

$$\frac{d\gamma_*\mu}{d\mu}(\xi) = \frac{d(\rho(\gamma)^{-1})^*\mu}{d\mu}(\xi) = e^{-J_{d_\Gamma+1}^u(b_\Theta(\rho(\gamma)^{-1}, \zeta(x)))},$$

where the term on the left-hand side involves measures, the term between the two equalities involves densities, and $\Theta = \{1, d_\Gamma + 1\}$. Indeed, for $\phi_1, \dots, \phi_{d_\Gamma} \in T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma)$:

$$\begin{aligned} & (\rho(\gamma)^*\mu)_{\xi_\rho^1(x)}(\phi_1, \dots, \phi_{d_\Gamma}) \\ &= \mu_{\rho(\gamma)\xi_\rho^1(x)}(\rho(\gamma)\phi_1\rho(\gamma)^{-1}, \dots, \rho(\gamma)\phi_{d_\Gamma}\rho(\gamma)^{-1}) = \\ &= \frac{\|\rho(\gamma)\xi_\rho^1(x) \wedge \rho(\gamma)\phi_1(\xi_\rho^1(x)) \wedge \dots \wedge \rho(\gamma)\phi_{d_\Gamma}(\xi_\rho^1(x))\|}{\|\rho(\gamma)\xi_\rho^1(x)\|^{d_\Gamma+1}} = \\ &= \frac{\|\rho(\gamma)\xi_\rho^1(x) \wedge \rho(\gamma)\phi_1(\xi_\rho^1(x)) \wedge \dots \wedge \rho(\gamma)\phi_{d_\Gamma}(\xi_\rho^1(x))\|}{\|\xi_\rho^1(x) \wedge \phi_1(\xi_\rho^1(x)) \wedge \dots \wedge \phi_{d_\Gamma}(\xi_\rho^1(x))\|} \cdot \frac{\|\xi_\rho^1(x) \wedge \phi_1(\xi_\rho^1(x)) \wedge \dots \wedge \phi_{d_\Gamma}(\xi_\rho^1(x))\|}{\|\rho(\gamma)\xi_\rho^1(x)\|^{d_\Gamma+1}} \cdot \frac{\|\rho(\gamma)\xi_\rho^1(x)\|^{d_\Gamma+1}}{\|\xi_\rho^1(x)\|^{d_\Gamma+1}} = \\ &= e^{\omega_{d_\Gamma}(b_\Theta(\rho(\gamma), \zeta_\rho(\xi_\rho^1(x))))} \cdot \mu_{\xi_\rho^1(x)}(\phi_1, \dots, \phi_{d_\Gamma}) \cdot e^{-(p+1)\omega_1(b_\Theta(\rho(\gamma), \zeta_\rho(\xi_\rho^1(x))))} = \\ &= e^{-J_{d_\Gamma}^u(b_\Theta(\rho(\gamma)^{-1}, \zeta(\xi_\rho^1(x))))} \mu_{\xi_\rho^1(x)}(\phi_1, \dots, \phi_{d_\Gamma}). \end{aligned}$$

Finally, we let $\nu = \zeta_{\rho*}\mu$, which is the wanted Patterson-Sullivan measure on $\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d)$, since for $f \in C_c(\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d))$:

$$\begin{aligned} \int_{\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d)} f d(\gamma_*\zeta_{\rho*}\mu) &= \int_{\xi_\rho^1(\partial\Gamma)} f \circ \gamma \circ \zeta_\rho d\mu = \int_{\xi_\rho^1(\partial\Gamma)} f \circ \zeta_\rho \circ \gamma d\mu = \\ &= \int_{\xi_\rho^1(\partial\Gamma)} f \circ \zeta_\rho(\xi) e^{-J_{d_\Gamma}^u(b_\Theta(\rho(\gamma)^{-1}, \zeta(\xi_\rho^1(x))))} d\mu(\xi_\rho^1(x)) = \\ &= \int_{\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d)} f(y) e^{-J_{d_\Gamma}^u(b_\Theta(\rho(\gamma)^{-1}, y))} d(\zeta_{\rho*}\mu)(y) \end{aligned}$$

□

Before giving the next definition, we recall that the annihilator of an element $y \in \mathcal{F}_F i\Theta(\mathbb{R}^d)$ is the set of partial flags that are not transverse to y , that is:

$$\text{Ann}(y) = \{x \in \mathcal{F}_\Theta(\mathbb{R}^d) : x^\theta \cap y^{d-\theta} \neq \emptyset \text{ for some } \theta \in \Theta\}.$$

Definition 3.3.3. Let $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$ be a linear representation, $\Theta \subseteq \Pi$ and μ a measure over $\mathcal{F}_\Theta(\mathbb{R}^d)$. We say that ρ is μ -irreducible there is no element in $\mathcal{F}_\Theta(\mathbb{R}^d)$, whose annihilator is of full measure, i.e. for all $y \in \mathcal{F}_\Theta(\mathbb{R}^d)$:

$$\mu(\mathrm{Ann}(y)) < \mu(\mathcal{F}_\Theta(\mathbb{R}^d)).$$

Example 3.3.1. If $\rho(\Gamma)$ is Zariski-dense in $\mathrm{SL}(d, \mathbb{R})$, then ρ is μ -irreducible for any ρ -quasi-equivariant measure μ , and in particular for any $(\rho(\Gamma), \phi)$ -Patterson-Sullivan measure.

Remark 3.3.1. The reason that we introduce the concept of μ -irreducibility is that for any μ -irreducible representation $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$, there exist $\alpha, \kappa > 0$ such that $\mu(B_{\Theta, \alpha}(\rho(\gamma))) \geq \kappa$ for all $\gamma \in \Gamma$.

Indeed, if this were not the case, then there would exist a sequence $\alpha_n \searrow 0$ and $\gamma_n \in \Gamma$ such that

$$\mu(B_{\Theta, \alpha_n}(\rho(\gamma_n))) \leq \frac{1}{n}.$$

Due to the compactness of $\mathcal{F}_\Theta(\mathbb{R}^d)$, up to considering a subsequence, we may assume that the repelling flags or $\rho(\gamma_n)$ converge to some $\xi \in \mathcal{F}_\Theta(\mathbb{R}^d)$:

$$(U_{d-i}(\rho(\gamma_n)^{-1}))_{\mathbf{a}_i \in \Theta} \rightarrow \xi$$

In that case, the complements $B_{\Theta, \alpha_n}^c(\rho(\gamma_n))$ will converge to the annihilator of ξ , in the sense:

$$\limsup_n B_{\Theta, \alpha_n}^c(\rho(\gamma_n)) \subseteq \mathrm{Ann}(\xi).$$

Indeed, let $y \in \limsup_n B_{\Theta, \alpha_n}^c(\rho(\gamma_n))$ and consider a subsequence k_n such that $y \in B_{\Theta, \alpha_n}^c(\rho(\gamma_{k_n}))$. By the very definition of $B_{\Theta, \alpha_n}^c(\rho(\gamma_n))$, there exists some p such that up to considering a subsequence of k_n ,

$$\angle(y^p, U_{d-p}(\rho(\gamma_n)^{-1})) \leq \alpha_n$$

holds. Taking the limit as $n \rightarrow \infty$, we have that $y^p \cap \xi^{d-p} \neq 0$ and hence $y \in \mathrm{Ann}(\xi)$.

Using a measure-theoretic argument we conclude that $\mathrm{Ann}(\xi)$ is of full measure, which contradicts the μ -irreducibility of ρ :

$$\mu(\mathrm{Ann}(\xi)) \geq \mu(\limsup_n B_{\Theta, \alpha_n}^c(\rho(\gamma_{k_n}))) \geq \limsup_n \mu(B_{\Theta, \alpha_n}^c(\rho(\gamma_{k_n}))) = \mu(\mathcal{F}_\Theta(\mathbb{R}^d)).$$

Lemma 3.3.1. Let $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$ be a representation and μ^ϕ be a $(\rho(\Gamma), \phi)$ -Patterson-Sullivan measure. If $\rho(\Gamma)$ is μ -irreducible, then there exists some $\alpha_0 > 0$, such that for any $\alpha \in (0, \alpha_0)$, there's some $k = k(\alpha) > 0$ for which

$$\frac{1}{k} e^{-\phi(a(\rho(\gamma)))} \leq \mu^\phi(\rho(\gamma) B_{\Theta, \alpha}(\rho(\gamma))) \leq k e^{-\phi(a(\rho(\gamma)))}$$

for all $\gamma \in \Gamma$.

Proof. Let $\alpha_0, k > 0$ be as in the remark preceding the statement of the lemma. As noted in Lemma 3.1.2, there exists some $K = K(\alpha_0, \phi) > 0$ such that for any $\alpha \in (0, \alpha_0)$ and $y \in B_{\Theta, \alpha}(\rho(\gamma))$:

$$|\phi(a(\rho(\gamma))) - b(\rho(\gamma), y)| \leq K,$$

from which we obtain the upper bound

$$\begin{aligned} \mu^\phi(\rho(\gamma) B_{\Theta, \alpha}(\rho(\gamma))) &= (\rho(\gamma^{-1})_* \mu^\phi)(B_{\Theta, \alpha}(\rho(\gamma))) = \int_{\mathcal{F}_\Theta(\mathbb{R}^d)} e^{-\phi(b(\rho(\gamma), y))} d\mu^\phi(y) \leq \\ &\leq e^{-K} \mu^\phi(\mathcal{F}_\Theta(\mathbb{R}^d)) e^{-\phi(a(\rho(\gamma)))}. \end{aligned}$$

Similarly we obtain the lower bound □

Appendix A

Tangent space to the Grassmanian

Let V be a d -dimensional real vector space. We denote with $\mathcal{G}_k(V)$ the Grassmanian of k -dimensional subspaces of V . Our first objective is to find a convenient way to express its tangent space.

Proposition A.0.1. *We have the following canonical identification:*

$$\begin{aligned} \text{hom}(W, V/W) &\simeq T_W \mathcal{G}_k(V) \\ \phi &\mapsto \left. \frac{d}{dt} \right|_{t=0} \Gamma(t\phi) \end{aligned}$$

where $\Gamma(\phi) = (Id + \phi)(W)$ is the graph of ϕ .

Proof. We will consider the map

$$F : \text{Injhom}(W, V) \rightarrow \mathcal{G}_k(V), \quad \phi \mapsto \phi(W).$$

whose derivative is given by:

$$d_I F(\phi) = F \left(\left. \frac{d}{dt} \right|_{t=0} (I + t\phi) \right) = \left. \frac{d}{dt} \right|_{t=0} (I + t\phi(W)) = \left. \frac{d}{dt} \right|_{t=0} \Gamma(t\phi).$$

The result will follow as soon as we have shown that $d_I F$ is surjective and that $\ker d_I F = \text{hom}(W, W)$.

To show that it is surjective, we consider a $(d-k)$ -dimensional subspace $W' \in \mathcal{G}_{d-k}(V)$ that is complementary to W , i.e. $V = W \oplus W'$. Denoting with $U_{W'} = \{Z \in \mathcal{G}_k(V) : Z \cap W' = 0\}$, we recall the corresponding chart:

$$\begin{aligned} \Psi : \text{hom}(W, W') &\rightarrow U_{W'} \\ \phi &\mapsto \Gamma(\phi). \end{aligned}$$

Surjectivity of $d_I F$ now follows by the fact that

$$d_I F(\phi) = \left. \frac{d}{dt} \right|_{t=0} \Gamma(t\phi) = d_0 \Psi(\phi).$$

To show that $\ker d_I F = \text{hom}(W, W)$, we first note that clearly $\ker d_I F \supseteq \text{hom}(W, W)$. Equality then follows by the fact that $\dim \text{hom}(W, W) = \dim \ker d_I F$, which is a direct consequence of the surjectivity. \square

Note that another way to prove the above identification throught the fact that the Grassmanian is a homogeneous space of $\mathrm{GL}(d, \mathbb{R})$, giving us the diffeomorphism

$$\begin{aligned} \mathrm{GL}(V)/\mathrm{St}_{\mathrm{GL}(V)}W &\rightarrow \mathcal{G}_k(V) \\ [g] &\mapsto gW, \end{aligned}$$

where $\mathrm{St}_{\mathrm{GL}(V)}W = \{g \in \mathrm{GL}(V) : gW = W\}$ is the stabilizer of W . Thus an expression for the tangent space at W may be obtained by differentiating the map above at the identity coset:

$$\mathrm{hom}(W, V/W) \simeq \mathrm{hom}(V, V)/\mathrm{hom}(W, W) \simeq T_W \mathcal{G}_k(V).$$

In particular the above gives us a direct proof that the kernel of the map constructed in the previous proof is indeed $\mathrm{hom}(W, W)$.

Our second objective is to identify subspaces of $T_l \mathbb{P}(V)$ with subspaces of V , by considering the first as projectivisation of the second. More concretely, we shall consider the space

$$\mathcal{P} = \{(l, P) : l \in \mathbb{P}(V), P \in \mathcal{G}_k(T_l \mathbb{P}(V))\}$$

as a homogenous space of $\mathrm{SL}(V)$, where the action is given by

$$g \cdot (l, P) = (gl, d_l g(P) = g\pi^{-1}(P)g^{-1} + \mathrm{hom}(gl, gl)).$$

where we use the identification of $T_l \mathbb{P}(V)$ with $\mathrm{hom}(l, V/l)$ as above and denote with $\pi : \mathrm{hom}(l, V) \rightarrow \mathrm{hom}(l, V/l)$ the canonical projection. For the sake of completeness, we outline the calculation of the differential:

$$\begin{aligned} \mathrm{hom}(l, V/l) &\rightarrow T_l \mathbb{P}(V) && \rightarrow T_{gl} \mathbb{P}(V) && \rightarrow \mathrm{hom}(gl, V/gl) \\ \phi &\mapsto \left. \frac{d}{dt} \right|_{t=0} g(I + t\tilde{\phi})(l) && \mapsto \left. \frac{d}{dt} \right|_{t=0} (I + tg\tilde{\phi}g^{-1})(gl) && \mapsto g\tilde{\phi}g^{-1} + \mathrm{hom}(gl, gl) \end{aligned}$$

where $\phi \in \mathrm{hom}(l, V/l)$, $\tilde{\phi} \in \mathrm{hom}(l, V)$ such that $\tilde{\phi} + \mathrm{hom}(l, l) = \phi$.

We are now ready to express the needed identification:

Proposition A.0.2. *We have the following $\mathrm{SL}(V)$ equivariant identification:*

$$\begin{aligned} \mathcal{P} &\rightarrow \mathcal{F}_{1, k+1}(V) \\ (l, P) &\mapsto (l, \pi^{-1}(P)l) \\ (l, \mathrm{hom}(l, Q/l)) &\mapsto (l, Q). \end{aligned}$$

where $\pi : \mathrm{hom}(l, V) \rightarrow \mathrm{hom}(l, V/l)$ is the canonical projection.

Proof. We begin by showing that the left-to-right direction of the map is well-defined. For this, we first need to check that for $(l, P) \in \mathcal{P}$, we have that $\dim \pi^{-1}(P)l = k + 1$. Indeed, we have that $\dim \pi^{-1}(P) = k + 1$ as implied by the rank-nullity theorem for $\pi : \pi^{-1}(P) \rightarrow P$. The result then follows by the fact that $\pi^{-1}(P)l = T_1(l) \oplus \dots \oplus T_{k+1}(l)$ for any base T_1, \dots, T_{k+1} of $\pi^{-1}(P)$. The second thing to check is that $l \leq \pi^{-1}(P)l$, which holds since $\ker \pi = \mathrm{hom}(l, l) \leq \pi^{-1}(P)$.

To see that the two directions above are inverse to each other, we begin by examining the right-to-left-to-right composition:

$$(l, Q) \mapsto (l, \pi(\mathrm{hom}(l, Q))) \mapsto (l, \pi^{-1}\pi(\mathrm{hom}(l, Q))) = (l, \mathrm{hom}(l, Q)l) = (l, Q).$$

and for the left-to-right-to-left composition

$$(l, P) \mapsto (l, \pi^{-1}(P)l) \mapsto (l, \mathrm{hom}(l, \pi^{-1}(P)l)),$$

so it suffices to show that $\text{hom}(l, \pi^{-1}(P)l/l) = P$. Indeed, for $\pi^{-1}(P) = \mathbb{R}T_1 \oplus \cdots \oplus \mathbb{R}T_k$, we have that

$$\begin{aligned} \text{hom}(l, \pi^{-1}(P)l/l) &= \text{hom}(l, \pi^{-1}(P)l) / \text{hom}(l, l) = (\oplus_i \text{hom}(l, T_i(l)) / \text{hom}(l, l)) = \\ &= (\oplus_i \mathbb{R}T_i) / \text{hom}(l, l) = \pi^{-1}(P) / \text{hom}(l, l) = P. \end{aligned}$$

For the equivariance, the calculations has as follows:

$$\begin{array}{ccc} (l, P) & \xrightarrow{\quad\quad\quad} & (l, \pi^{-1}(P)l) \\ \downarrow g & & \downarrow g \\ (gl, g\pi^{-1}(P)g^{-1} + \text{hom}(gl, gl)) & \xrightarrow{\quad\quad\quad} & (gl, (g\pi^{-1}(P)g^{-1})(gl)) = (gl, g\pi^{-1}(P)l) \end{array} \quad \square$$

Appendix B

Irreducible actions problem

The matter of this chapter has to do with an obstruction, found in the proof of this lemma:

Lemma B.0.1 (Lemma 6.8 in [PSW23]). Let Γ be a hyperbolic group and $\eta : \Gamma \rightarrow \mathrm{PGL}(d, \mathbb{R})$ be a strongly irreducible projective Anosov representation such that $\xi_\eta(\partial\Gamma)$ is homeomorphic to S^{d-1} , and which admits a measurable η -equivariant section $\zeta : \partial\Gamma \rightarrow \mathcal{F}_{\{\mathbf{a}_1, \mathbf{a}_{d-1}+1\}}(\mathbb{R}^d)$. Then η is μ -irreducible for any $(\eta(\Gamma), \phi)$ -Patterson-Sullivan measure μ on $\mathcal{F}_{\{\mathbf{a}_1, \mathbf{a}_{d-1}+1\}}(\mathbb{R}^d)$.

For convenience, we recall that a linear representation $\rho : \Gamma \rightarrow \mathrm{GL}(d, \mathbb{R})$ is strongly irreducible if there is no proper $\rho(\Gamma)$ -invariant subspace of \mathbb{R}^d , and it is μ -irreducible if there is no element in $\mathcal{F}_{i\Theta}(\mathbb{R}^d)$, whose annihilator is of full measure.

In what follows, we will show that the above lemma is false, by providing a counterexample. Let Γ be a uniform lattice of $\mathrm{SU}(2, 1)$ (i.e. acts convex cocompactly), and $\eta : \Gamma \rightarrow \mathrm{SL}(\mathfrak{su}(2, 1))$ be the restriction of the adjoint representation, i.e. $\eta(\gamma) = \mathrm{Ad}_\gamma$ for all $\gamma \in \Gamma$.

For convenience, we recall the definition of a uniform lattice:

Definition B.0.1. Let G be a locally compact group. A uniform lattice is a discrete subgroup $\Gamma \leq G$ that is co-compact, i.e. G/Γ is compact.

Remark B.0.1. When $G = \mathrm{Isom}(X)$ is the isometry group of a complete Riemannian manifold X , and Γ is a uniform lattice of G , then it acts properly discontinuously and cocompactly on X .

We begin by showing proving the Anosov property of η .

Proposition B.0.1. η is projective Anosov.

Proof. Let $\gamma \in \Gamma$. Since $\gamma \in \mathrm{SU}(2, 1)$, we have that

$$\gamma = k_1 \exp(r(\gamma)x_0) k_2$$

for x_0 a fixed non-zero in the Weyl-chamber $\mathbb{R}x_0$ of $\mathfrak{su}(2, 1)$, $r(\gamma) \in \mathbb{R}$ and $k_1, k_2 \in \mathrm{U}(2)$. Then by the definition of a uniform lattice, we have that Γ acts properly discontinuously and cocompactly, which means that the inclusion $\Gamma \hookrightarrow \mathrm{SU}(2, 1)$ is projective Anosov (since $\mathrm{SU}(2, 1)$ is of rank 1). Thus there exist constants $L \geq 1, b \geq 0$ such that for all $\gamma \in \Gamma$:

$$r(\gamma) \geq \mathbf{a}_1(x_0)^{-1}(L|\gamma| - b) = L'|\gamma| - b'.$$

Note that $\mathbf{a}_1(x_0) > 0$ since x_0 is in the interior of the Weyl-chamber $\mathbb{R}x_0$.

Letting $k'_1 = \text{Ad}_{k_1}, k'_2 = \text{Ad}_{k_2}$ and $K' \leq \text{SL}(\mathfrak{su}(2, 1))$ be a maximal compact subgroup containing them, we have that:

$$\eta(\gamma) = \text{Ad}_\gamma = k'_1 \text{Ad}_{\exp(r(\gamma)x_0)} k'_2 = k'_1 \exp(r(\gamma) \text{ad}_{x_0}) k'_2.$$

Thus

$$\mathbf{a}_1(\mu(\eta(\gamma))) = r(\gamma) \mathbf{a}_1(\text{ad}_{x_0}) \geq (L'|\gamma| - b') \mathbf{a}_1(x_0)$$

which is Anosov because $\mathbf{a}_1(\text{ad}_{x_0}) > 0$, as can be seen by concrete calculations. \square

Before giving an expression for the projective part of the limit map of η , we make a few observations regarding Gromov boundary of Γ . In particular, we claim that since Γ is a uniform lattice of $\text{SU}(2, 1)$, we have that $\partial\Gamma$ is homeomorphic to $\text{SU}(2, 1)/P$, where P is a parabolic subgroup of $\text{SU}(2, 1)$, and it coincides with the stabilizer of some isotropic line $l \in \partial_\infty \mathbb{H}_{\mathbb{C}}^2$.

Indeed, for a uniform lattice Γ of the isometry group G of a homogenous G -space X , the Milnor-Švarc lemma implies that for any $x_0 \in X$, the map $\Gamma \rightarrow X, \gamma \mapsto \gamma x_0$ is a quasi-isometry. In our case $G = \text{SU}(2, 1)$ and $X = \mathbb{H}_{\mathbb{C}}^2$ is a hyperbolic metric space, so the quasi-isometry extends to a homeomorphism $\partial\Gamma \rightarrow \partial\mathbb{H}_{\mathbb{C}}^2$ of the Gromov-boundaries. On the other hand, the action of $\text{SU}(2, 1)$ on $\partial_\infty \mathbb{H}_{\mathbb{C}}^2$ is transitive, so we have that $\partial\mathbb{H}_{\mathbb{C}}^2 \simeq \text{SU}(2, 1)/P$ where P is the stabilizer of a point in $\partial\mathbb{H}_{\mathbb{C}}^2$. In fact, we have that P is a parabolic subgroup of $\text{SU}(2, 1)$. The combination of the above, along with the fact that the geometric and the Gromov boundaries agree in the case of $\mathbb{H}_{\mathbb{C}}^2$, we deduce that $\partial\Gamma \simeq \text{SU}(2, 1)/P$.

To calculate the projective part of the limit map, we shall show that there exists a unique $\text{SU}(2, 1)$ -equivariant map from $\partial\Gamma$ to $\mathbb{P}(\mathbb{R}^d)$. The uniqueness follows from the following characterisation of limit maps:

Add proofs of these.

Lemma B.0.2. *Let $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$ be a strongly irreducible projective Anosov representation, and denote with $\xi_\rho : \partial\Gamma \rightarrow \mathbb{P}(\mathbb{R}^d)$ its limit map. Then ξ_ρ^1 is the unique continuous, $\rho(\Gamma)$ -equivariant map from $\partial\Gamma$ to $\mathbb{P}(\mathbb{R}^d)$.*

Proof. Let $\eta^1 : \partial\Gamma \rightarrow \mathbb{P}(\mathbb{R}^d)$ be a continuous, $\rho(\Gamma)$ -equivariant map. Since the action of Γ on its boundary $\partial\Gamma$ has dense orbits, it suffices to show that it agrees with ξ_ρ^1 on at least one boundary point.

Suppose for the sake of contradiction that η^1 does not coincide with ξ^1 and let $z \in \partial\Gamma, y \in \partial\Gamma \setminus \{z\}$. Then for any $x \in \partial\Gamma \setminus \{y\}$ we may find some quasi-geodesic $\{\gamma_n\}_n$ such that $\gamma_n \rightarrow x, \gamma_{-n} \rightarrow y$ as $n \rightarrow \infty$. Then since $z \neq y$ we know that $\gamma_n z \rightarrow z$ as $n \rightarrow \infty$ and continuity of η^1 implies that $\eta^1(\gamma_n z) \rightarrow \eta^1(z)$. But equivariance of η^1 and the fact that ξ^1 is dynamics-preserving implies that $\eta(\gamma_n z) = \rho(\gamma_n) \eta(z) \rightarrow \xi^1(x)$, unless $\eta^1(z) \in \xi^{d-1}(y)$. But if in fact $\eta^1(z) \notin \xi^{d-1}(y)$, then the limits would agree, i.e. $\eta^1(x) = \xi^1(x)$ which is a contradiction. Thus we have that $\eta^1(z) \in \xi^{d-1}(y)$ and since y was an arbitrary point of $\partial\Gamma \setminus \{z\}$, we have that

$$\eta^1(z) \in \bigcap_{y \in \partial\Gamma \setminus \{z\}} \xi^{d-1}(y) \subseteq \bigcap_{y \in \partial\Gamma \setminus \Gamma \cdot z} \xi^{d-1}(y).$$

In particular, the set appearing on the right hand side is ρ -equivariant, non-empty proper subset of \mathbb{R}^d , which contradicts the strong irreducibility assumption of ρ . \square

Given the lemma above, it suffices to find an $\text{SU}(2, 1)$ -equivariant map from $\partial\Gamma$ to $\mathbb{P}(\mathbb{R}^d)$.

Proposition B.0.2. *The projective part of the limit map of η is given by*

$$\xi_\eta : \partial\Gamma = \text{SU}(2, 1)/P_0 \rightarrow \mathbb{P}(\mathfrak{su}(2, 1)), \quad \xi_\eta(gP_0) = \mathbb{R} \text{Ad}_\gamma x_0.$$

where $P_0 = \text{St}_{\text{SU}(2,1)}[1 : 0 : 0]$ and

$$x_0 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{su}(2, 1).$$

Its derivative satisfies:

$$d_x \xi(T_x \text{SU}(2, 1)/P_0) = \pi(\text{ad}_{\xi^1(x)} \mathfrak{su}(2, 1))$$

where $\pi : \text{hom}(\xi^1(x), \mathfrak{su}(2, 1)) \rightarrow \text{hom}(\xi^1(x), \mathfrak{su}(2, 1)/\xi^1(x))$ is the canonical projection.

Proof. Since by Lemma B.0.2 the limit map is the unique continuous ρ -equivariant from the boundary of Γ to the projective space, it suffices to show that there exists an η -equivariant map $\xi^1 : \text{SU}(2, 1)/P_0 \rightarrow \mathbb{P}(\mathfrak{su}(2, 1))$, since it will then restrict to the limit map on $\partial\Gamma$.

We consider the parabolic subgroup $P_0 = \text{St}_{\text{SU}(2,1)}[1 : 0 : 0]$ of $\text{SU}(2, 1)$. Then its Lie algebra is given by:

$$\mathfrak{p}_0 = \text{St}_{\mathfrak{su}(2,1)}[1 : 0 : 0] = \left\{ \begin{pmatrix} u - is & a & it \\ 0 & 2is & -\bar{a} \\ 0 & 0 & -u - is \end{pmatrix} : a \in \mathbb{C}, u, s, t \in \mathbb{R} \right\}.$$

Since for $\mathbb{R}x \in \mathbb{P}(\mathfrak{su}(2, 1))$ we have that P_0 fixes $\mathbb{R}x$ if and only if \mathfrak{p}_0 fixes $\mathbb{R}x$. But a quick calculation shows that the only element of $\mathfrak{su}(2, 1)$ fixed by \mathfrak{p}_0 is x_0 .

For the calculation of the image of the differential at the identity coset P , we differentiate the commutative diagram:

$$\begin{array}{ccc} \text{SU}(2, 1) & \xrightarrow{\text{Ad. } x_0} & \mathfrak{su}(2, 1) \\ \downarrow & & \downarrow \\ \text{SU}(2, 1)/P_0 & \xrightarrow{\xi^1} & \mathbb{P}(\mathfrak{su}(2, 1)) \end{array} \quad \text{to get} \quad \begin{array}{ccc} \mathfrak{su}(2, 1) & \xrightarrow{\text{ad. } x_0} & \mathfrak{su}(2, 1) \\ \downarrow & & \downarrow \pi \\ \mathfrak{su}(2, 1)/\mathfrak{p}_0 & \xrightarrow{d_P \xi^1} & T_{\xi^1(P)} \mathbb{P}(\mathfrak{su}(2, 1)) \end{array}$$

In the general case we use the equivariance of the limit map

$$\begin{aligned} d_{gP} \xi^1(T_{gP} \text{SU}(2, 1)/P_0) &= d_{gP} \xi^1 d_P g(T_P \text{SU}(2, 1)/P_0) = d_{\xi^1(P)} g d_P \xi^1(T_P \text{SU}(2, 1)/P_0) = \\ &= d_{\xi^1(P)} g \pi(\text{ad}_{\xi^1(P)} \mathfrak{su}(2, 1)) = \\ &= \pi(\text{Ad}_g(\text{ad}_{\xi^1(P)} \mathfrak{su}(2, 1))) = \pi(\text{ad}_{\text{Ad}_g \xi^1(P)} \mathfrak{su}(2, 1)) = \\ &= \pi(\text{ad}_{\xi^1(gP)} \mathfrak{su}(2, 1)). \end{aligned}$$

□

Recall that all parabolic subgroups of $\text{SU}(2, 1)$ are conjugate to each other, so we have the following identification:

$$\begin{aligned} \text{SU}(2, 1)/P_0 &\leftrightarrow \{ \text{Parabolic subgroups of } \text{SU}(2, 1) \} \leftrightarrow \{ \text{Parabolic subalgebras of } \mathfrak{su}(2, 1) \} \\ gP_0 &\leftrightarrow gP_0 g^{-1} \leftrightarrow \text{Ad}_g(\mathfrak{p}_0) \end{aligned}$$

Lemma B.0.3. Let $\mathfrak{p}, \mathfrak{p}' \leq \mathfrak{su}(2, 1)$ be two distinct parabolic subalgebras. Then there exists some $g \in \text{SU}(2, 1)$ such that $\text{Ad}_g(\mathfrak{p}) = \mathfrak{p}_0$ and $\text{Ad}_g \mathfrak{p}' = \mathfrak{p}_0^t$.

The following proposition implies that the falsehood of the lemma in the beginning of this chapter.

Proposition B.0.3. Let $\Gamma \leq \mathrm{SU}(2, 1)$ be a uniform lattice and $\eta : \Gamma \rightarrow \mathrm{SL}(\mathfrak{su}(2, 1))$ be the restriction of the adjoint representation. Then

- (i) η is strongly irreducible,
- (ii) η is projective Anosov
- (iii) η admits a measurable η -equivariant section:

$$\begin{aligned} \zeta : \partial\Gamma &\rightarrow \mathcal{F}_{\{1,4\}}(\mathfrak{su}(2, 1)) \simeq \mathcal{P} \\ x &\mapsto (\xi^1(x), T_{\xi^1(x)}\xi^1(\partial\Gamma)) \simeq (\xi^1(x), (d_{\xi^1(x)}p)^{-1}(T_{\xi^1(x)}\xi^1(\partial\Gamma))\xi^1(x)). \end{aligned}$$

where $d_{\xi^1(x)}p : \mathrm{hom}(\xi^1(x), \mathfrak{su}(2, 1)) \rightarrow \mathrm{hom}(\xi^1(x), \mathfrak{su}(2, 1)/\xi^1(x))$ is the canonical projection.

- (iv) For all $x, y \in \partial\Gamma : \zeta(x)^4 \cap \zeta(y)^4 \neq \emptyset$.
- (v) For any $y_0 \in \mathrm{SU}(2, 1)/P_0$ and $W_0 \in \mathcal{G}_7(\mathbb{R}^4)$ that contains $\zeta(y_0)^4$, we have that $\mathrm{Ann}(\zeta(y_0)^4, W_0) \supseteq \zeta(\mathrm{SU}(2, 1)/P_0)$ and is in particular of full μ -measure, for any $(\eta(\Gamma), \phi)$ -Patterson-Sullivan measure μ supported over $\zeta(\partial\Gamma)$.

Proof. (i) Follows from the fact that $\mathrm{SU}(2, 1)$ is a simple Lie group.

(ii) Shown in Proposition B.0.1.

(iii) Follows from the fact that ξ^1 is $\mathrm{SU}(2, 1)$ -equivariant and the equivariant identification of $\mathcal{F}_{\{1,4\}}(\mathfrak{su}(2, 1)) \simeq \mathcal{P}$.

(iv) Letting $g \in \mathrm{SU}(2, 1)$ be as in Lemma B.0.3, we have that $\mathrm{Ad}_g(\mathfrak{p}_0) = \mathfrak{p}$ and $\mathrm{Ad}_g(\mathfrak{p}_0^t) = \mathfrak{p}'$. Thus $\zeta(x)^4 \cap \zeta(y)^4 \neq \emptyset$ if and only if

$$\begin{aligned} \emptyset \neq \mathrm{Ad}_g(\zeta(x)^4 \cap \zeta(y)^4) &= \mathrm{Ad}_g \zeta(x)^4 \cap \mathrm{Ad}_g \zeta(y)^4 = \zeta(gx)^4 \cap \zeta(gy)^4 = \zeta(\mathfrak{p}_0)^4 \cap \zeta(\mathfrak{p}_0^t)^4 = \\ &= \pi \left(\mathbb{R} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right). \end{aligned}$$

For the last equality, we use Proposition B.0.2 and the fact that $\mathfrak{p}_0^t = \mathrm{Ad}_g \mathfrak{p}_0$ for

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

to conclude that

$$\zeta(\mathfrak{p}_0) = \zeta(P_0) = \pi \left(\left\{ \begin{pmatrix} u & a & it \\ 0 & 0 & -\bar{a} \\ 0 & 0 & -u \end{pmatrix} : a \in \mathbb{C}, u, t \in \mathbb{R} \right\} \right),$$

and

$$\zeta(\mathfrak{p}_0^t) = \zeta(gP_0) = \mathrm{Ad}_g \zeta(P_0) = \pi \left(\left\{ \begin{pmatrix} u & 0 & 0 \\ a & 0 & 0 \\ it & -\bar{a} & 0 \end{pmatrix} : a \in \mathbb{C}, u, t \in \mathbb{R} \right\} \right).$$

□

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