Limit sets of Anosov representations

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Contents

Chapter 1

Introduction

1.1 Lie group preliminaries

We fix the Cartan subalgebra $\mathfrak{a}$ of $\mathrm{SL}(d,\mathbb{R})$:

$$\mathfrak{a} = \{\mathrm{diag}(\alpha_1,\ldots,\alpha_d) : \alpha_1 + \cdots \alpha_d = 0\}$$

and the Weyl chamber $\mathfrak{a}^+$ of $\mathrm{SL}(d,\mathbb{R})$

$$\mathfrak{a}^+ = \{\mathrm{diag}(\alpha_1,\ldots,\alpha_d) : \alpha_1 \geq \cdots \geq \alpha_d\}.$$

Denoting with $K = \mathrm{SO}(d,\mathbb{R})$, $A^+ = e^{\mathfrak{a}^+}$, we have the Cartan decomposition:

$$\begin{aligned} \mathfrak{sl}(d,\mathbb{R}) &\to K \times A^+ \times K \\ g &\mapsto (k_g, a_g, l_g) \end{aligned}$$

such that $g = k_g a_g l_g$. In particular $a_g = \mathrm{diag}(\sigma_1(g),\ldots,\sigma_d(g))$ with $\sigma_1 \geq \cdots \geq \sigma_d(g)$, where $\sigma_i(g)$ is the i-th singular value of g, i.e. eigenvalue of $g^t \cdot g$.

We will use the spaces

$$U_p(g) = \mathbb{R}u_1(g) \oplus \cdots \oplus \mathbb{R}u_p(g)$$

where $u_i(g) = k_g \cdot e_i$. One can easily show that the decomposition

$$g^{-1} \cdot U_p(g) \oplus U_{d-p}(g^{-1})$$

is orthogonal with respect to the standard inner product and that $u_p(g^{-1}) = l_g^{-1} e_{d-p+1}(g)$.

1.2 Limit set preliminaries

Definition 1.2.1. For $p \in \{2,\ldots,d\}$, $s \in \mathbb{R}$ and $g \in SL(d,\mathbb{R})$ we denote with $\Psi_s^p(g) : \mathfrak{a}^+ \to \mathbb{R}$ the functional:

$$\begin{aligned} \Psi_s^p(g) &= \alpha_{12}(a(g)) + \cdots + \alpha_{1(p-1)}(a(g)) + (s-(p-2))\alpha_{1p}(a(g)) \\ \tilde{\Psi}_s^p(g) &= \left(\frac{\sigma_2}{\sigma_1} \cdots \frac{\sigma_{p-1}}{\sigma_1}(g)\right)\left(\frac{\sigma_{p-1}}{\sigma_1}(g)\right)^{s-(p-2)} \end{aligned}$$

Remark 1.2.1. We have $\alpha_{ij}(a) = a_i - a_j, a_i(g) = \log(\sigma_i(g))$ and

$$\Psi_s^p(g) = \log \tilde{\Psi}_s^p(g)$$

and that

$$\min_{p \in [\![2,d]\!]} \left\{ \sum_{|\gamma|=T} \frac{\sigma_2}{\sigma_1} \cdots \frac{\sigma_{p-1}}{\sigma_1}(g) \left(\frac{\sigma_{p-1}}{\sigma_1}(g) \right)^{s-(p-2)} \right\} = \sum_{|\gamma|=T} e^{-\max\limits_{p \in [\![2,d]\!]} \Psi_s^p(g)}$$

Remark 1.2.2. For any $g \in \mathrm{SL}(d, \mathbb{R})$ we have that:

$$\max_{p \in [\![2,d]\!]} \Psi_s^p(g) = \Psi_s^{p_0}(g) \text{ for } s \in [p_0 - 2, p_0 - 1].$$

Indeed, a quick calculation shows that for $s \geq 0$ and $p \in [\![2, d]\!]$:

$$\Psi_s^p(g) \leq \Psi_s^p(g) \text{ if and only if } s \geq p - 1.$$

and that equality holds in the case $s = p - 1$. Thus for $s \in [p - 2, p - 1]$ we have that

$$s \geq p-2, \ldots, 1 \text{ implies that } \Psi_s^p(g) \geq \ldots \geq \Psi_s^2(g)$$
$$s \leq p, \ldots, d-1 \text{ implies that } \Psi_s^p(g) \leq \ldots \leq \Psi_s^d(g)$$

Another way to see this (refer to Figure 1.1) is to note that $\Psi_s^2(g), \cdots, \Psi_s^d(g)$ is a sequence of functions that are affine in s, with slopes $\alpha_{12}(g) \leq \cdots \leq \alpha_{1d}(g)$ and that they satisfy $\Psi_1^2(g) = \Psi_2^2(g), \Psi_2^3(g) = \Psi_3^4(g) \ldots, \Psi_{d-2}^{d-1}(g) = \Psi_{d-2}^d(g)$.

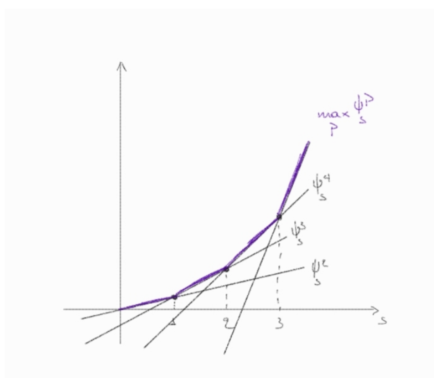


Figure 1.1: Visual illustration that $\max_p \Psi_s^p(g) = \Psi_s^{p_0}(g)$ for $s \in [p_0 - 2, p_0 - 1]$.

The following definition comes from [1], in the special case of projective Anosov representations ($P = 1$):

Definition 1.2.2. For $s \geq 0$ we consider the Falconer functional $F_s : \mathrm{SL}(d, \mathbb{R}) \to \mathbb{R}$ by:

$$F_s(g) = \min \left\{ \sum_{j=2}^{d} s_j \alpha_{1j}(a(g)) : s_j \in (0, 1], \sum_{j=2}^{d} s_j = s \right\}$$

Remark 1.2.3. Using elementary computations one may prove that for all $s \geq 0$:

$$F_s(g) = \min_{p \in [\![2,d]\!]} \Psi_s^p(g)$$

Definition 1.2.3. Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a linear representation and $p \in [\![1, d-1]\!]$. We say that ρ is p-Anosov if there exist constants $\mu, C > 0$ such that for all $\gamma \in \Gamma$:

$$\frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma)) \leq C e^{-\mu|\gamma|}.$$

One can show that in that case there exist equivariant continuous maps $\xi^p : \hat{\Gamma} \to \mathcal{G}_p(\mathbb{R}^d), \xi^{d-p} : \hat{\Gamma} \to \mathcal{G}_{d-p}(\mathbb{R}^d)$ that are transverse and restrict to

$$\xi^p(\gamma) = U_p(\rho(\gamma)), \xi^{d-p}(\gamma) = U_{d-p}(\rho(\gamma))$$

for $\gamma \in \Gamma$, where $U_p(\gamma), U_{d-p}(\gamma)$ denote the flags corresponding to $\rho(\gamma)$.

Figure out what this exactly means

Chapter 2

Upper bound

Lemma 2.0.1 (Upper bound for dimension). *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. Then:*

$$\dim_H(\xi^1(\partial\Gamma)) \leq \inf \left\{ s : \sum_{|\gamma|=T} e^{-F_s(\rho(\gamma))} < \infty \right\}.$$

Remark 2.0.1. The idea of the proof of Lemma 2.0.1 is to find a covering whose Hausdorff content is dominated by the Dirichlet series of some functional Ψ^p_s, which will in turn imply that $\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(\Psi^p)$. Choosing then the most "effective" cover (i.e. the one which yields the smallest Hausdorff content up to a constant) we obtain that

$$\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(\max_p \Psi^p)$$

To obtain this we first cover $\xi^1(\partial\Gamma)$ by the bassins of attraction $\rho(\gamma) \cdot B_{\alpha_1,\alpha}(\rho(\gamma))$ for $\gamma \in \Gamma$ satisfying $|\gamma| = T$. Then we cover each bassin by an ellipsoid of axes lengths

$$\frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \ldots, \frac{1}{\sin(\alpha)} \frac{\sigma_d}{\sigma_1}(\rho(\gamma)).$$

Finally we cover each ellipsoid by balls of some fixed radius $r > 0$. It can be shown by comparing the series appearing in the Hausdorff content of each resulting cover that the most "effective" choice of r depends only on the Hausdorff exponent $s > 0$ and in any case will be to have r equal (up to a constant) to the the length of an axis of the ellipsoid, i.e.

$$r \in \left\{ \frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \ldots, \frac{1}{\sin(\alpha)} \frac{\sigma_d}{\sigma_1}(\rho(\gamma)). \right\}$$

In particular, when $s \in [p-2, p-1]$, the most effective choice is $r = \sigma_p(\rho(\gamma))/\sigma_1(\rho(\gamma))$, whose Hausdorff content is dominated by the Dirichlet series of Ψ^p_s.

Proof of Lemma 2.0.1. Let $p \in [\![2, d]\!]$. Then using Proposition 2.1.1, Lemma 2.1.2, and Lemma 2.1.3 we have that for $T > 0$ large enough, $\xi^1(\partial\Gamma)$ is covered by the family

$$\mathcal{U}_T = \{\rho(\gamma) B_{\alpha_1,\alpha}(\rho(\gamma)) : |\gamma| = T\},$$

and that each basin $\rho(\gamma)B_{\alpha_1,\alpha}(\rho(\gamma))$ is in turn covered by

$$2^{2p+1} \cdot \frac{\sigma_p(g)^{p-2}}{\sigma_2(g)\cdots\sigma_{p-1}(g)}$$

many balls of radius

$$\sqrt{d}\frac{1}{\sin\alpha}\frac{\sigma_p(g)}{\sigma_1(g)}.$$

By the definition of the Hausdorff measure, for $s \geq 0$:

$$\begin{aligned}
\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq \sum_{|\gamma|=T} 2^{2p+1} \cdot \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\cdots\frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\left(\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\right)^{-(p-2)}\left(\sqrt{d}\frac{1}{\sin\alpha}\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\right)^s = \\
= 2^{2p+1} \cdot \left(\frac{\sqrt{d}}{\sin\alpha}\right)^s \sum_{|\gamma|=T} \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\cdots\frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\left(\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\right)^{s-(p-2)} = \\
= 2^{2p+1} \cdot \left(\frac{\sqrt{d}}{\sin\alpha}\right)^s \sum_{|\gamma|=T} e^{-\left(\alpha_{12}+\ldots+\alpha_{1(p-1)}+(s-(p-2))\alpha_{1p}\right)\rho(\gamma)} \\
= 2^{2p+1} \cdot \left(\frac{\sqrt{d}}{\sin\alpha}\right)^s \sum_{|\gamma|=T} e^{-\Psi_s^p(\rho(\gamma))}
\end{aligned}$$

and thus

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq 2^{2p+1} \cdot \left(\frac{\sqrt{d}}{\sin\alpha}\right)^s \sum_{|\gamma|=T} e^{-\max_p \Psi_s^p(\rho(\gamma))}$$

□

Will this not imply that $\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(max_p\Psi^p)$ which may be strictly smaller than $h_\rho(min_p\Psi^p) = h_\rho(F)$?

2.1 Lemmata

Definition 2.1.1. Let V be a finite-dimensional $\mathbb{R}$-vector space. We consider a decomposition

$$V = \mathbb{R}u_1 \bigoplus \cdots \bigoplus \mathbb{R}u_d$$

be a direct decomposition that is orthogonal with respect to a fixed inner-product over V. Given $\beta_2 \geq \ldots \beta_d > 0$, we define an ellipsoid with axes $u_1 \oplus u_p(g)$ and lengths β_p to be the image of

$$\left\{v = \sum_1^d v_i u_i \in V : \sum_2^d \left(\frac{v_j}{\beta_j}\right)^2 \leq 1\right\}$$

through the projection $V \to \mathbb{P}(V)$.

The following aims to be something along the lines of [2, Lemma 2.4]:

Lemma 2.1.1. *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. For $\alpha > 0$ small enough, there exists $L > 0$ such that for any geodesic ray $(a_j)_j$ through e we have:*

$$\angle(U_1(\rho(a_i)), U_1(\rho(a_0))) > \alpha$$

when $|a_i|, |a_0| > T$.

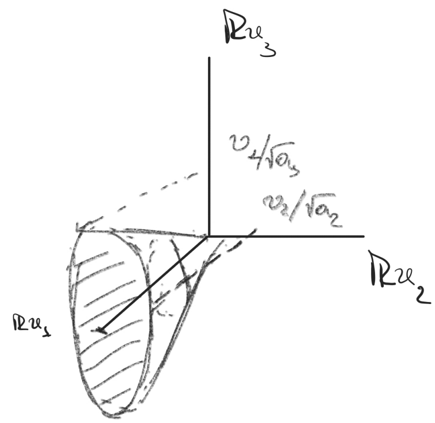


Figure 2.1: Depiction in $\mathbb{R}^3$ of an ellipsoid of $\mathbb{P}(\mathbb{R}^2)$

Proof. Assume the contrary for the shake of contradiction. Then (see Figure 2.2) for each $n > 0$ there exists a geodesic ray a^n through e such that

$$|a_n^n|, |a_0^n| > n \text{ and } \angle(U_1(\rho(a_n^n)), U_1(\rho(a_0^n))) < \frac{1}{n}.$$

Due to compactness of $\partial\Gamma$ we may assume (up to a subsequence) that $a^n \to x$ in $\partial\Gamma$ for some $x \in \partial\Gamma$. Then $a_n^n, a_0^n \to x$ in $\hat{\Gamma}$ which implies

Not sure if this is true.

$$\angle(\xi^1(x), \xi^{d-1}(x)) = 0$$

using the fact that the limit maps ξ^1, ξ^{d-1} are continuous, which contradicts their tranversality. □

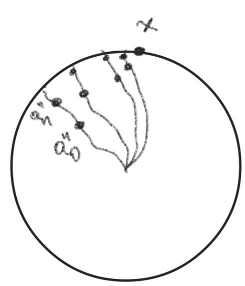


Figure 2.2: Situation in Lemma 2.1.1

The following is [2, Proposition 3.5].

Lemma 2.1.2. *Let $\rho : \Gamma \to SL(d, \mathbb{R})$ be projective Anosov. Then for $\alpha > 0$ small enough, there exists some $T_0 > 0$ such that for all $T \geq T_0$ the family*

$$\mathcal{U}_T = \{\rho(\gamma) B_{\alpha_1,\alpha}(\rho(\gamma)) : |\gamma| = T\}$$

is an open covering of $\xi^1(\partial\Gamma)$.

Proof. Let $\alpha, T > 0$ be as in the statement of Lemma 2.1.1 and $x \in \partial\Gamma$ be represented by a geodesic ray $(\gamma_j)_{j\geq 0}$ starting from e. Then $(\gamma_T^{-1}\gamma_j)_j$ is a geodesic ray starting from $(\gamma_T)^{-1}$ that passes through e, so

$$\angle(U_1(\rho(\gamma_T^{-1}\gamma_j)), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

as implied by Lemma 2.1.1. Taking the limit $j \to \infty$ and using the equivariance of the limit map, we obtain

$$\angle(\rho(\gamma_T^{-1})\xi^1(x), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

and thus $\xi^1(x) \in \rho(\gamma_T) \cdot B_{\alpha_1,\alpha}(\rho(\gamma_T))$. □

The following is [2, Proposition 3.8].

Proposition 2.1.1. *For each* $g \in \mathrm{SL}(d, \mathbb{R}), \alpha > 0$, *the basin of attraction* $g \cdot B_{\alpha_1,\alpha}(g)$ *lies in the ellipsoid with axes* $u_1(g) \oplus u_p(g)$ *with lengths*

$$\frac{1}{\sin\alpha} \cdot \frac{\sigma_p(g)}{\sigma_1(g)}$$

Proof. Using the definition of the basin of attraction (see Figure 2.3), we have that $w = w_1u_1(g^{-1}) + \cdots + w_du_d(g^{-1}) \in B_{\alpha_1,\alpha}(g)$ if and only if

$$w_d^2 \geq (\sin\alpha)^2 \sum_1^d w_i^2.$$

Considering now some $v = v_1u_1(g) + \cdots + v_du_d(g) \in g \cdot B_{\alpha_1,\alpha}(g)$ we have that

$$\begin{aligned} w = g^{-1}v &= v_1\sigma_1(g)^{-1}l_g^{-1}e_1(g) + \cdots v_d\sigma_d(g)^{-1}l_g^{-1}e_d(g) \\ &= v_1\sigma_1(g)^{-1}u_d(g^{-1}) + \cdots v_d\sigma_d(g)^{-1}u_1(g^{-1}) \end{aligned}$$

where we used that $u_p(g^{-1}) = l_g^{-1}e_{d+1-p}$. Hence

$$\sigma_1(g)^{-2} \cdot v_1^2 \geq (\sin a)^2 \sum_1^d \sigma_i(g)^{-2}v_i^2.$$

□

The following is [2, Lemma 3.7]:

Lemma 2.1.3. *For any* $p \in [\![2, d]\!]$, *an ellipsoid in* $\mathbb{P}(\mathbb{R}^d)$ *of axes lengths* $\beta_2, \cdots, \beta_d$ *is covered by*

$$2^{2p+1}\frac{\beta_2 \cdots \beta_{p-1}}{\beta_p^{p-2}}$$

many balls of radius $\sqrt{d}\beta_p$.

The following can be found in [2, Proposition 3.3]:

Proposition 2.1.2. *Let* $\rho : \Gamma \to SL(d, \mathbb{R})$ *be projective Anosov and* $\alpha > 0$ *Then there exist* $c_0, c_1 > 0$ *that depends only on* α *and* ρ *such that for all* $\gamma \in \Gamma$*:*

$$(\xi^1)^{-1}(B_{\alpha_1,\alpha}(\rho(\gamma))) \subseteq C^\infty_{c_0,c_1}(\gamma)$$

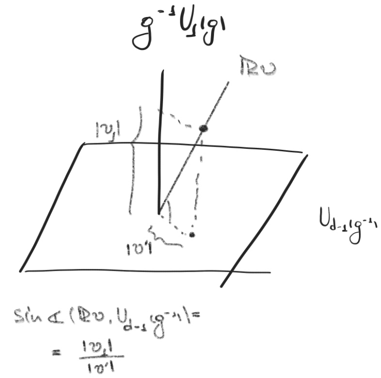


Figure 2.3: Aid for Proposition 2.1.1

Proof. We begin by noting that it suffices to show this for all but finitely many $\gamma \in \Gamma$, since then we may alter the constants to satisfy the wanted inclusion also for the finitely many remaining $\gamma \in \Gamma$. Given this, we shall assume that $|\gamma| \geq l_0$ where $l_0 > 0$ is such that $Ce^{-\mu l_0} < 1$ and $C, \mu > 0$ are the constants appearing in the definition of the Anosov property of ρ..

Suppose $x \in \partial\Gamma$ such that $\xi^1(x) \in B_{\alpha_1,\alpha}(\rho(\gamma))$, and consider a geodesic ray $a_j \to x$ starting from $a_0 = e$. To prove the result, it suffices to find constants c_0, c_1 independent of γ and a (c_0, c_1)-quasi-geodesic from γ^{-1} to x that passes through e and stays at a bounded distance from $(a_j)_{j=0}^{\infty}$

Using [2, Proposition 2.5] we have that $d(\xi^1(a_j), U_{d-1}(\rho(\gamma^{-1}))) \leq Ce^{-\mu j}$, so there exists some $L > 0$ that depends only on α such that for all $j \geq L : U_1(\rho(a_j)) \in B_{\alpha_1,\alpha}(\rho(\gamma))$ and in particular

$$d(\xi^1(a_j), \gamma^{-1}) = d(U_1(\rho(a_j)), U_1(\rho(\gamma^{-1}))) \geq d(U_1(\rho(a_j)), U_{d-1}(\rho(\gamma^{-1}))) > \sin\alpha.$$

Along with the uniform continuity of $\xi^1 : \Gamma \cup \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ this implies there exists some $\alpha' > 0$ and $L > 0$ such that for all $j \geq L$:

$$d(a_j, \gamma^{-1}) \geq \alpha'.$$

Upon considering a large L, we may also assume that $|a_L| = L > l_0$. Note that both α' and L do not depend on each γ but only on ρ and α.

Using some geometric group theory, we can show that for all $j \geq L$

$$d(\gamma^{-1}, a_j) > \alpha' \Rightarrow d([\gamma^{-1}, a_j], e) < \alpha''$$

for some α'' that depends only on Γ and α', where $[a_j, \gamma^{-1}]$ denotes the geodesic segment connecting γ^{-1} and a_j.

Consider the concatenation $(a'_j)_{j=L-K}^{\infty}$ of $[\gamma^{-1}, a_L]$ and $[a_L, x]$. To find quasi-geodesic-constants that are uniform in γ, we note that for any $c_0 \geq 1, c_1 \geq 0$:

$$c_0^{-1}|i-j| - c_1 \leq d(a'_i, a'_j) = d(a_i, a_j) \leq d(a_i)c_0|i-j| + c_1 \text{ when } i,j \geq L \text{ or } i,j \leq L$$

and that the upper bound follows trivially by the triangle inequality.

For the lower bound we proceed in two steps. First we bound the distance of $\gamma^{-1} = a'_{L-K}$

to a_{L+j} for $j \geq 0$:

$$\begin{aligned}
d(a'_{L-K}, a'_{L+j}) &\geq \nu(|a_{L+j}| - |\gamma^{-1}|) - c'_0 - c'_1|\log(d(U_1(\rho(a_{L+j})), U_1(\rho(\gamma^{-1}))))| \geq \\
&\geq \nu((L+j) + (K-L)) - c'_0 - c'_1|\log(\sin a)| \geq \\
&= c_0^{-1}(j+K) - c_1
\end{aligned}$$

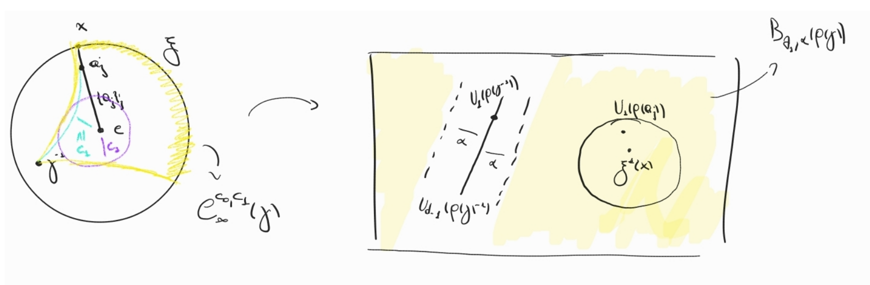
for $c_0 = \nu^{-1}, c_1 = c'_0 + c'_1|\log(\sin\alpha)|$. The first inequality comes from [2, Lemma 3.9]. For the second inequality we estimate $|\gamma^{-1}|$ from below using the triangle inequality. We are now ready to show that the concatenation $(a'_j)_j$ is indeed a $(c_0,\, c_1)$-geodesic:

$$\begin{aligned}
d(a_{L+j}, a_{L-i}) &\geq d(a_{L+j}, a_{L_K}) - d(a_{L_K}, a_{L-i}) \geq c_0^{-1}(j+K) - c_1 - (K-i) \geq \\
&\geq c_0^{-1}(j+i) - c_1.
\end{aligned}$$

Note however that (a'_j) does not necessarily lie in $C_\infty^{c_0,c_1}$ since it may not pass through e. For this reason we some $L-K \leq i_0 \leq L$ such that $|a_{i_0}| < \alpha''$, the existence of which is guaranteed by the fact that $d([\gamma^{-1}, a_L], \epsilon) < \alpha''$. We then consider alter (a'_j) at i_0 so that it passes through e to obtain

$$a''_j = \begin{cases} a_j & \text{for } j \neq i_0 \\ e & \text{for } j = i_0 \end{cases}$$

which is a $(c_0, c_1 + \alpha'')$-quasigeodesic passing from e and converging to x. $\square$



Bibliography

[1] François Ledrappier and Pablo Lessa. “Dimension gap and variational principle for Anosov representations”. In: (Dec. 2023). arXiv:2310.13465 [math] version: 3 (cit. on p. 3).

[2] Beatrice Pozzetti, Andrés Sambarino, and Anna Wienhard. “Anosov representations with Lipschitz limit set”. In: *Geometry & Topology* 27.8 (Nov. 2023). arXiv:1910.06627 [math], pp. 3303–3360. ISSN: 1364-0380, 1465-3060 (cit. on pp. 6–10).