Limit sets of Anosov representations

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Abstract

For a projective Anosov subgroup $\rho(\Gamma)$ of $\mathrm{SL}(d,\mathbb{R})$ whose limit set is a Lipschitz submanifold of $\mathbb{P}(\mathbb{R}^d)$, based on [PSW23], we show that the Hausdorff dimension of the limit set is equal to the critical exponent of the Falconer functional when the Zariski-closure of $\rho(\Gamma)$ is $\mathrm{SL}(d,\mathbb{R})$ or $\mathrm{SO}(2,d-2)$ for $d \neq 4$.

Contents

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Chapter 1

Introduction

Let $\mathbb{H}^n_\mathbb{R}$ be the real hyperbolic space and $\Gamma \leq \mathrm{SO}(1,n)$ be a discrete convex co-compact subgroup of isometries. To the action $\Gamma \curvearrowright \mathbb{H}^n_\mathbb{R}$, we associate (Definition 1.1.1) its critical exponent δ_Γ, which is a dynamical invariant expressing the exponential growth rate of the Γ-orbits in $\mathbb{H}^n_\mathbb{R}$. Another object associated (**??**) to the action is the limit set Λ_Γ, which should be thought of as the set of accumulation points of an orbit at infinity. In [Sul79], Sullivan proved that the critical exponent δ_Γ is equal to the Hausdorff dimension of the limit set Λ_Γ, providing a stunning connection between the fractal geometry of the limit set and the dynamics of the group action:

Theorem 1.0.1 ([Sul79])**.** *Let $\Gamma \leq \mathrm{PSO}(1,n)$ be convex co-compact. Then the critical exponent of Γ is equal to the Hausdorff dimension of its limit set:*

$$\dim_{\mathcal{H}}(\Lambda_\Gamma) = \delta_\Gamma.$$

To accomplish this, Sullivan generalised to all hyperbolic spaces the construction done for the hyperbolic plane by Patterson in [Pat76] of a Γ-quasi-invariant measure on the limit set. Since then, there have been numerous generalisations of this result to other settings, such as the action of other kinds of discrete groups on hyperbolic spaces (see for instance [Rob03; DOP00; Coo93]), or to other kinds of spaces (for instance [Coo93] for the case of X being a Gromov hyperbolic space).

Here, we will in large part follow [PSW23], where Pozzetti, Sambarino and Wienhard are interested in the generalisation of Sullivan's result to certain subgroups of higher rank Lie groups. In particular, they focus on Anosov representations in $\mathrm{SL}(d,\mathbb{R})$, which are defined (in Definition 1.4.2 using the characterisation of [KLP17]) as the representations of discrete, finitely generated groups into $\mathrm{SL}(d,\mathbb{R})$, with a singular value growth that is exponentially controlled by the word length of the group element. They are particularly relevant in this context because they are considered as a promising generalisation of convex co-compact subgroups of rank one Lie groups to ones of higher rank, amd their Gromov boundary is realised as a subset of the projective space through the boundary map (Theorem 1.4.3), which will be the analogue of the limit set Λ_Γ to the higher rank case. The only regularity assumption that they pose on the representations considered is that the limit set is a Lipschitz submanifold of the ambient projective space, which in particular is the case for three large classes of examples, namely maximal representations ([BIW10]), Anosov quasi-Fuchsian AdS representations ([MB12]) and $\mathbb{H}^{p,q}$-convex-cocompact representations ([DGK18]).

On second glance (more extensively described in Example 1.5.5), the critical exponent of Γ can be seen as the critical exponent (defined in Definition 1.5.4 as the radius of convergence for the respective Poincaré series) of a certain functional over the Cartan subalgebra of $\mathrm{SL}(d,\mathbb{R})$, associating to each γ, its displacement $d(o,\gamma\cdot o)$ with respect to some fixed point $o\in X$. Given this, the following modified version of the result appearing in [PSW23] should be seen as a natural generalisation of Sullivan's theorem to the higher rank case:

Theorem 3.4.1. Let $\rho:\Gamma\to\mathrm{SL}(d,\mathbb{R})$ be a Zariski-dense, projective Anosov representation in $\mathrm{SL}(d,\mathbb{R})$ such that $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold of $\mathbb{P}(\mathbb{R}^d)$. Then

$$\dim(\xi^1_\rho(\partial\Gamma)) = h_\rho(F),$$

where $h_\rho(F)$ is the critical exponent of the Falconer functional F (see Definition 1.5.1).

The above statement differs from the corresponding in [PSW23] in replacing the strong irreducibility assumption on the representation with that the Zariski-density of the subgroup $\rho(\Gamma)$ in $\mathrm{SL}(d,\mathbb{R})$, because of a gap that we noticed in the proof of the latter. Since this is a strong assumption that is not always satisfied in the three classes of examples presented above, we present the following result that covers certain cases of interest:

Theorem 4.2.1. Let $\rho:\Gamma\to\mathrm{SL}(d,\mathbb{R})$ be a projective Anosov representation such that $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold of $\mathbb{P}(\mathbb{R}^d)$. Assume moreover that $\dim\xi^1_\rho(\partial\Gamma)=d-3$, $d>4$ and $\rho(\Gamma)$ is Zariski-dense in $\mathrm{SO}(2,d-2)$. Then

$$\dim\xi^1_\rho(\partial\Gamma) = h_\rho(F) = d-3.$$

It should be noted that this result is possibly not optimal, as we hope its proof generalises (at least) to other groups of the form $\mathrm{SO}(p,q)$.

In the remaining of this chapter, we will recall the definition of the Hausdorff dimension, along with all necessary background on Lie groups, Anosov representations that will be necessary in the subsequent chapters. We will also introduce the Falconer functional (introduced in [LL23] by Ledrappier and Lessa) and the unstable Jacobian (introduced in [PSW23] by Pozzetti, Sambarino and Wienhard), which will be the functionals whose critical exponents capture the geometric information of the dimension of the limit set.

In Chapter 2, we will use a classical recipe to obtain the upper bound of the Hausdorff dimension, by finding an appropriate covering of the limit set $\xi^1_\rho(\partial\Gamma)$, for which the Hausdorff content is dominated by the Dirichlet series of the functional F. We would like to stress that the only part of the hypothesis used in that chapter is that the representation is projective Anosov, meaning that it still holds without requiring that Λ_Γ is a Lipschitz submanifold of $\mathbb{P}(\mathbb{R}^d)$ or that $\rho(\Gamma)$ is Zariski-dense.

On the other hand, to establish the upper bound of the critical exponent in Chapter 3, one uses the more elaborate machinery of Patterson–Sullivan measures as generalised into higher rank by [Qui02] and further developed in [PSW23]. For this part of the proof, the hypothesis that $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold will be introduced (in Proposition 3.3.1) to construct a Patterson–Sullivan measure μ over a space of partial flags, while the Zariski-density of $\rho(\Gamma)$ is necessary (see Example 3.3.5) to show that it is μ-irreducible (that is, the measure of the set of transverse flags to a given flag is positive, see Definition 3.3.4) and obtain estimates (in Lemma 3.3.7) that will be used to prove that the sum defining the critical exponent is finite.

Finally, in Chapter 4 we trace the gap in the proof of [PSW23] to an incorrect lemma appearing in [Lab06] and provide two counterexamples showing that it is substantial. Then we present the proof of Theorem 4.2.1, and conclude with some remarks on the generalisation of the result to other subgroups.

1.1 Limit sets and critical exponents

In this section, we recall the definition of the critical exponent of a discrete group and relate it to the volume entropy of a Riemannian space.

Definition 1.1.1. Let Γ be a discrete group of isometries of a metric space (X, d). We define the critical exponent of Γ to be the asymptotic exponential growth of its orbits, i.e. the following limit:

$$\delta_\Gamma = \limsup_{n\to\infty} \frac{\log \sharp\{\gamma \in \Gamma : d(x, \gamma x) \leq n\}}{n} =$$
$$= \inf \left\{ s \in \mathbb{R} : \sum_{\gamma \in \Gamma} e^{-sd(x, g\cdot x)} \right\} < \infty$$

for some fixed $x \in X$.

Remark 1.1.2. It is not hard to show that the critical exponent is independent of the choice of x.

Later in the thesis, we will be using certain terms to refer to lattices, which we recall below:

Definition 1.1.3. Let G be a Lie group, and $\Gamma \leq G$ a discrete subgroup.

(i) Γ is a lattice if it is co-finite, meaning that the quotient G/Γ admits a finite G-invariant measure.

(ii) Γ is a uniform lattice if it is co-compact, meaning that the quotient G/Γ is compact.

In the case where Γ is a uniform lattice, the next proposition shows that the critical exponent of Γ is in fact the same as for any other uniform lattice, and it does that by giving a rather geometric interpretation for δ_Γ. Namely, it is the exponential growth of the volume of a ball in the space $X = G/\Gamma$. Aside from offering geometric insight, this fact gives us also a way to calculate δ_Γ, as done in Section 1.7.

Lemma 1.1.4. *Let X be a complete Riemannian manifold and $\Gamma \leq \mathrm{Isom}(X)$ be discrete subgroup of isometries acting properly discontinuously and cocompactly on X. Then the critical exponent of Γ is equal to the volume entropy of X:*

$$\delta_\Gamma = \lim_{R\to\infty} \frac{\log \mathrm{Vol}(B_R(x_0))}{R}$$

for any $x_0 \in X$, where $B_R(x)$ denotes the ball of radius R around x in X.

Proof. Since Γ acts on X cocompactly, there exists a relatively compact fundamental domain K for this action. The result will follow as soon as we show that

$$\limsup_{R\to\infty} \frac{\log \mathrm{Vol}(B_{R-\mathrm{diam}\,K}(x_0))}{R} \leq \delta_\Gamma \leq \liminf_{R\to\infty} \frac{\log \mathrm{Vol}(B_{R+\mathrm{diam}\,K}(x_0))}{R}.$$

Since X is a complete Riemannian manifold and Γ is discrete, the action is properly discontinuous and hence the set $\{\gamma \in \Gamma : \gamma K \cap B_R(x_0)\} = \{\gamma_1, \cdots, \gamma_n\}$ is finite. Since K is a fundamental domain, $B_R(x_0) \subseteq \gamma_1 K \cup \cdots \cup \gamma_n K$ and hence $\mathrm{Vol}(B_R(x_0)) \leq n\,\mathrm{Vol}(K)$. But

$\{\gamma \in \Gamma : \gamma k \cap B_R(x_0) \neq 0\} \subseteq \{\gamma \in \Gamma : d(\gamma x_0, x_0) \leq R + \operatorname{diam} K\}$, which implies that $n \leq \sharp\{\gamma \in \Gamma : d(\gamma x_0, x_0) \leq R + \operatorname{diam} K\}$. Combining the two inequalities obtained we have

$$\frac{\operatorname{Vol}(B_R(x_0))}{\operatorname{Vol}(K)} \leq \sharp\{\gamma \in \Gamma : d(\gamma x_0, x_0) \leq R + \operatorname{diam} K\}$$

which gives us the left-hand-side inequality.

For the other inequality we note that when some $\gamma \in \Gamma$ satisfies $\mathrm{d}(\gamma x_0, x_0) \leq R - \operatorname{diam} K$, then $\gamma K \subseteq B_R(x_0)$, since for any $k \in K$: $d(\gamma k, x_0) \leq d(\gamma k, \gamma x_0) + d(\gamma x_0, x_0) \leq R$. Denoting with $\{\gamma_1', \cdots, \gamma_n'\} = \{\gamma \in \Gamma : d(\gamma x_0, x_0) \leq R - \operatorname{diam} K\}$, we have that $B_{R - \operatorname{diam} K}(x_0) \supseteq \gamma_1' K \cup \cdots \cup \gamma_n' K$ and hence $n \operatorname{Vol}(K) \leq \operatorname{Vol}(B_R(x_0))$, since K is a fundamental domain and thus the sets $\gamma_i K$ are disjoint with each other. Thus,

$$\sharp\{\gamma \in \Gamma : d(\gamma x_0, x_0) \leq R - \operatorname{diam} K\} \leq \frac{\operatorname{Vol}(B_R(x_0))}{\operatorname{Vol}(K)},$$

from which the right-hand-side inequality follows. □

1.2 Gromov hyperbolic groups

In this seection we recall the basic definitions and facts of Gromov hyperbolic spaces and groups which were introduced by Gromov to generalise certain metric properties of hyperbolic spaces. For more details or proofs of the following facts, we refer to [GH13].

Let (X, d) be a metric space. The Gromov product of two points $x, y \in X$ with respect to a point $z \in X$ is defined as

$$(x, y)_z = \frac{d(x, z) + d(y, z) - d(x, y)}{2}.$$

For some $\delta > 0$, we say that X is δ-hyperbolic if for all $x, y, z, w \in X$ the following relation is satisfied

$$(x, y)_z \geq \min\{(x, w)_z, (w, y)_z\} - \delta.$$

A map f defined over $[0, \infty)$ or $\mathbb{N}$ that takes values in $\mathbb{N}$ is called a geodesic ray if

$$d(f(t), f(s)) = |t - s|$$

holds for all $t, s \in [0, \infty)$ or $\mathbb{N}$ respectively. It is called a (c, C)-quasigeodesic ray if it satisfies

$$\frac{1}{C}|t - s| - c \leq d(f(t), f(s)) \leq C|t - s| + c$$

for all $t, s \in [0, \infty)$ or $\mathbb{N}$ respectively.

Supposing that X is a proper (meaning that the closure of every ball is compact), geodesic (meaning that for any $x, y \in X$, there exists a geodesic joining x and y) hyperbolic metric space, we define the Gromov boundary ∂X as the set of equivalence classes of geodesics in X under the equivalence relation

$$f \sim g \text{ if and only if } d_{\mathcal{H}}(f(\mathbb{N}), g(\mathbb{N})) < \infty,$$

where we denote with $d_{\mathcal{H}}$ the Hausdorff distance between two sets. We could consider the same equivalence relation over (c, C)-quasigeodesic rays, and the resulting space would be another model for the Gromov boundary, homeomorphic to the previous one. In any case, it is important to know that using the quotient of the topology of uniform convergence of geodesics on compact subsets, the space ∂X is metrizable and compact.

Let Γ be a finitely generated group and S be a symmetric (meaning that $s^{-1} \in S$ for any $s \in S$) generating set. The Cayley graph Γ with respect to S is the graph whose vertices are the elements of Γ and there is an edge between $g, h \in \Gamma$ if and only if $g^{-1}h \in S$. It becomes a metric space with the word metric $d_S(g,h)$ defined as the length of the shortest path in the Cayley graph connecting g and h. The group Γ is called hyperbolic if the Cayley graph is δ-hyperbolic for some $\delta > 0$, in which case the space $\hat{\Gamma} \overset{\text{def}}{=} \Gamma \cup \partial\Gamma$ is compact.

Finally, for each element $\gamma \in \Gamma$ of a hyperbolic group Γ and $c_0, c_1 > 0$, we define the (c_0, c_1)-*coarse cone* on $\partial\Gamma$ as:

$$C_\infty^{c_0,c_1}(\gamma) = \left\{ [(a_j)_{j\in\mathbb{N}}] : (a_j)_j \text{ is a } (c_0,c_1)\text{-quasi-geodesic ray, } a_0 = \gamma^{-1} \text{ and } e \in \{a_j\}_j \right\}.$$

Its definition was taken from [PSW23], and it will be useful in later chapters.

1.3 Grassmanians and flag spaces

In this section we will recall some terminology and notions related to Grassmanians and flag spaces, that will prove useful in Section 1.4 in presenting the dynamical properties of Anosov representations, as well as in Chapter 2, where we will construct an open cover for the limit set of Anosov subgroups.

For $k \in [\![1,d]\!]$, the Grassmanian $\mathcal{G}_k(\mathbb{R}^d)$ is the space of k-dimensional subspaces of $\mathbb{R}^d$. For two subspaces $V \in \mathcal{G}_k(\mathbb{R}^d), W \in \mathcal{G}_l(\mathbb{R}^d)$ of possibly different dimensions, we can define a notion of an angle as

$$\angle(V,W) \overset{\text{def}}{=} \min_{v\in V^\times, w \in W^\times} \angle(v,w) \in [0,\pi/2],$$

where $V^\times \overset{\text{def}}{=} V\backslash\{0\}$, while the angle between vectors $v, w \in \mathbb{R}^d$ is defined as

$$\angle(v,w) \overset{\text{def}}{=} \arccos\left(\frac{\langle v,w\rangle}{\langle v,v\rangle\langle w,w\rangle}\right) \in [0,\pi].$$

With these definitions, the angle between two subspaces is zero if and only if they are transverse.

For $\Theta \subseteq [\![1, d-1]\!]$ we define the space of partial flags as

$$\mathcal{F}_\Theta(\mathbb{R}^d) = \left\{(x_\theta)_{\theta\in\Theta} : x_\theta \in \mathcal{G}_i(\mathbb{R}^d) \text{ for } \theta \text{ and } x_\theta \subseteq x_{\theta'} \text{ for } \theta < \theta' \text{ in } \Theta\right\},$$

where $\mathcal{G}_r(\mathbb{R}^d)$ is the r-th Grassmanian, that is the space of r-dimensional subspaces of $\mathbb{R}^d$. To each $\Theta \subseteq [\![1, d-1]\!]$, we associate $\iota\Theta = \{d-\theta : \theta \in \Theta\}$. In this notation, two flags $(x_\theta)_\theta \in \mathcal{F}_\Theta(\mathbb{R}^d)$ and $(y_\theta)_\theta \in \mathcal{F}_{\iota\Theta}(\mathbb{R}^d)$ are transverse if $x_\theta \cap y_{d-\theta} = \{0\}$ for all $\theta \in \Theta$. For the sake of readability, we will often skip writing the braces around the indices of the flags and write $\mathcal{F}_{i,j}(\mathbb{R}^d)$ instead of $\mathcal{F}_{\{i,j\}}(\mathbb{R}^d)$.

1.4 Lie groups and Anosov representations

Let G be a real semisimple Lie group, whose Lie algebra is denoted by $\mathfrak{g}$, K is a maximal compact subgroup of G whose algebra is denoted by $\mathfrak{k}$. We let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. We denote with $\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{p}$. A non-trivial linear form $\alpha \in \mathfrak{a}^*$ is called a restricted root when the set

$$\mathfrak{g}_\alpha = \{u \in \mathfrak{g} : \mathrm{ad}_a(u) = \alpha(a)u \text{ for all } a \in \mathfrak{a}\}$$

is not zero. The set of restricted roots is denoted by Δ and the closure of any component

$$\mathfrak{a}^+ \subseteq \mathfrak{a} \backslash \bigcup_{\alpha \in \Delta} \ker \alpha$$

is called a Weyl chamber. The Weyl group $W \subseteq K$ is defined as the quotient $N_\mathfrak{a}/Z_\mathfrak{a}$, where $N_\mathfrak{a}$ and $Z_\mathfrak{a}$ are the normalizer and the centraliser of $\mathfrak{a}$ in K, i.e.

$$\begin{aligned} Z_\mathfrak{a} &= \{k \in K : \mathrm{Ad}_k(a) = a \text{ for all } a \in \mathfrak{a}\} \\ N_\mathfrak{a} &= \{k \in K : \mathrm{Ad}_k(\mathfrak{a}) \subseteq \mathfrak{a}\}. \end{aligned}$$

The following theorem is a fundemental result in the theory of semisimple Lie groups:

Theorem 1.4.1 (Cartan decomposition)**.** *Let K be maximal compact subgroup, $\mathfrak{a}$ a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{a}^+$ a Weyl chamber of $\mathfrak{a}$. Then G can be decomposed as $G = Ke^{\mathfrak{a}^+}K$, while for every $g \in G$, there exists a unique $\mu(g)$ such that $g \in Ke^{\mu(g)}K$.*

The map

$$\mu : G \to \mathfrak{a}^+$$

appearing in the theorem is called the Cartan projection.

For us, the most relevant case will be $G = \mathrm{SL}(d, \mathbb{R})$, so we will fix the maximal compact subgroup $K = \mathrm{SO}(d, \mathbb{R})$, the Cartan subalgebra

$$\mathfrak{a} = \{\mathrm{diag}(\alpha_1, \dots, \alpha_d) : \alpha_1 + \cdots + \alpha_d = 0\}$$

and its Weyl chamber

$$\mathfrak{a}^+ = \{\mathrm{diag}(\alpha_1, \dots, \alpha_d) : \alpha_1 \geq \cdots \geq \alpha_d\}.$$

The Cartan projection is then given by

$$\mu(g) = \mathrm{diag}(\mu_1(g), \dots, \mu_d(g))$$

where $\sigma_i(g) = e^{\mu_i(g)}$ is the i-th singular value of g, i.e. the square root of the eigenvalue of $g^t \cdot g$. Denoting for each $i \in [\![1, d]\!]$ with $a_i : \mathfrak{a} \to \mathbb{R}$ the projection of the i-th diagonal entry of a diagonal matrix, the restricted roots can be defined as $\mathsf{a}_{ij} = a_i - a_j \in \mathfrak{a}^*$ and the i-th fundamental weight as $\omega_{\mathsf{a}_i}(a) = a_1 + \cdots + a_i$, for $i \in [\![1, d-1]\!]$. The set of simple (restricted) roots is then $\Pi = \{\mathsf{a}_i : i \in [\![2, d]\!]\}$ and in a slight abuse of notation, we will occasionally identify a subset $\{\mathsf{a}_{i_1}, \cdots, \mathsf{a}_{i_r}\} \subseteq \Pi$ with the respective set of indices $\{i_1, \cdots, i_r\} \subseteq [\![1, d-1]\!]$. For instance, we will write $\mathcal{F}_\Theta(\mathbb{R}^d)$ for set of partial flags of dimensions $\{i \in [\![1, d-1]\!] : \mathsf{a}_i \in \Theta\}$, whenever $\Theta \subseteq \Pi$.

Before going on with the definition and the properties of Anosov representations, we will briefly discuss the way that a linear transformation in $\mathrm{SL}(d, \mathbb{R})$ acts on the euclidean space $\mathbb{R}^d$, namely by streching and contracting. Let $g \in \mathrm{SL}(d, \mathbb{R})$, and consider a Cartan decomposition $g = k_g e^{\mu(g)} l_g$. Since g acts linearly on $\mathbb{R}^d$, to understand its action on the Euclidean space it suffices to understand its action on the unit sphere $\mathbb{S}^{d-1}$, which g distorts into an ellipsoid (see Figure 1.1).

In other words, we claim that the image of the unit sphere through g is an ellipsoid with axes

$$u_i(g) = k_g e_i, \text{ for } i \in [\![1, d]\!]$$

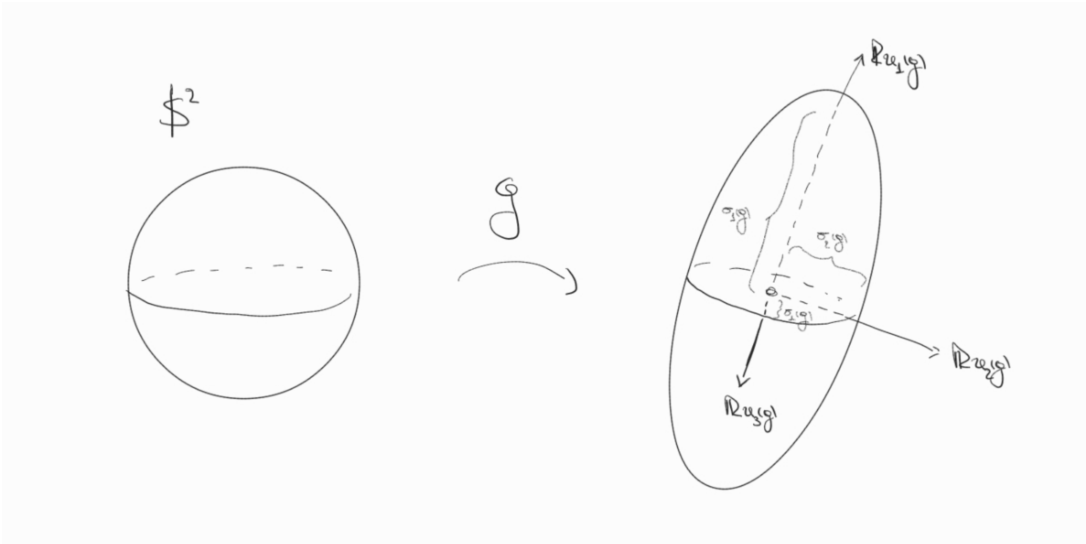


Figure 1.1: Linear transformations map spheres to elllipsoids.

and respective lengths $\sigma_i(g) = e^{\mu_i(g)}$. Indeed, if we write $w = \beta_1 u_1(g) + \cdots + \beta_d u_d(g)$, then $w \in g \cdot \mathbb{S}^{d-1}$ if and only if

$$\begin{aligned}\|g^{-1}w\|^2 &= \|\beta_1 \sigma_i(g)^{-1} l_g^{-1} + \cdots + \beta_d \sigma_d(g)^{-1}\|^2 = \\ &= \left(\frac{\beta_1}{\sigma_1(g)}\right)^2 + \cdots + \left(\frac{\beta_d}{\sigma_d(g)}\right)^2 \leq 1,\end{aligned}$$

which is precisely the defining inequality for an ellipsoid.

Recall that for a general element in $g \in \mathrm{SL}(d,\mathbb{R})$, we may not make any assumption on the singular values $\sigma_i(g)$ except from the fact that they are in non-increasing order. In particular, certain singular values may coincide, which in the context of the above discussion means that the choice of the axes $u_i(g)$ is arbitrary. In fact, when for instance $\sigma_2(g) = \sigma_3(g)$, any rotation of the orthonormal basis on the $\mathbb{R}u_i(g) \oplus \mathbb{R}u_{i+1}(g)$ plane will give the equally valid orthonormal choice of axis $u_1(g), \cos(\theta)u_1(g) + \sin(\theta)u_2(g), -\sin(\theta)u_1(g) + \cos(\theta)u_2(g), u_3(g), \cdots, u_d(g)$. This has to do with the fact that the choice of elements k_g, l_g in the Cartan decomposition of g is not unique (see Figure 1.2).

When however there is a singular value gap, i.e. $\sigma_p(g) > \sigma_{p+1}(g)$, then (noting that $\sigma_{d-p+1}(g^{-1}) < \sigma_{d-p}(g^{-1})$ as well because $\sigma_i(g^{-1}) = \sigma_{d-i+1}(g)^{-1}$) the singular planes of g and g^{-1} are well-defined:

$$\begin{aligned}U_p(g) &\stackrel{\text{def}}{=} k_g \cdot (\mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_p) \\ U_{d-p}(g^{-1}) &\stackrel{\text{def}}{=} l_g^{-1} \cdot (\mathbb{R}e_{p+1} \oplus \cdots \oplus \mathbb{R}e_d),\end{aligned}$$

and form a decomposition

$$g^{-1} \cdot U_p(g) \oplus U_{d-p}(g^{-1})$$

that is orthogonal with respect to the standard inner product on $\mathbb{R}^d$. In particular, when $g \in \mathrm{SL}(d,\mathbb{R})$ has a gap between the θ-th and $(\theta+1)$-th singular values for all $\theta \in \Theta \subseteq [\![1, d-1]\!]$,

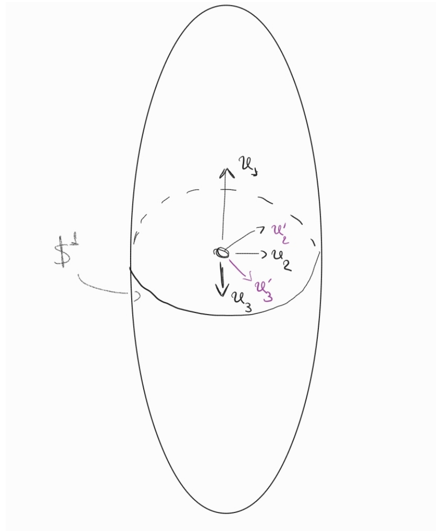


Figure 1.2: Equality between singular values results in axes ambiguity.

we define for any $\alpha > 0$ the *basin of attraction* of g as

$$B_{\Theta,\alpha}(g) \overset{\text{def}}{=} \left\{x \in \mathcal{F}_\Theta(\mathbb{R}^d) : \angle\left(x^\theta, U_{d-\theta}(g^{-1})\right) > \alpha \text{ for all } \theta \in \Theta\right\}.$$

We may think of the basin of attraction as the set of flags that are sufficiently transverse to the singular plane $\left(U_{d-\theta}(g^{-1})\right)_{\theta\in\Theta}$ of g^{-1}. To justify the name, we briefly mention informally the fact that flags contained in the basin of attraction are uniformly attracted to the attracting flag $U_\Theta(g)$ under the action of g, and refer to [BPS19][Lemma A.6]. Often, for readability purposes, we will often skip writing the curly braces around the indices comprising Θ.

We are now ready to give one of the various equivalent definitions of Anosov representations in the literature. In light of the previous discussion, one could phrase it as: "a representation in $\mathrm{SL}(d,\mathbb{R})$ is p-Anosov if the length of the p-th axis of the ellipsoid $g \cdot \mathbb{S}^{d-1}$ is exponentially larger than the length of the $(p+1)$-th axis, with the exponential rate controlled by the word length of the group element".

Definition 1.4.2. $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a linear representation of a discrete group Γ and $p \in [\![1, d-1]\!]$. We say that ρ is p-Anosov if there exist constants $\mu, C > 0$ such that for all $\gamma \in \Gamma$:

$$\frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma)) \leq Ce^{-\mu|\gamma|}.$$

When $p = 1$, we say that the representation is projective Anosov.

The following theorem shows the dynamical behaviour of Anosov representations through their action on flag spaces. In particular, a p-Anosov representation ρ will have a well-defined attracting flag $\xi^p_\rho(x)$ in $\mathcal{F}_{p,d-p}(\mathbb{R}^d)$ for every $x \in \partial\Gamma$.

Theorem 1.4.3 ([KLP17; Lab06])**.** *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a p-Anosov representation of a discrete group Γ. Then Γ is Gromov hyperbolic, and there exists an equivariant continuous map*

$\xi : \partial\Gamma \to \mathcal{F}_{p,d-p}(\mathbb{R}^d)$ *over its Gromov boundary* $\partial\Gamma$ *such that*

$$\xi^p(x) = \lim_{n\to\infty} U_p(\rho(\gamma_n))$$
$$\xi^{d-p}(x) = \lim_{n\to\infty} U_{d-p}(\rho(\gamma_n)),$$

for any $x \in \partial\Gamma$ *and any quasi-geodesic* $\gamma_n \to x$*, and it is dynamics-preserving, i.e.*

$$\rho(\gamma_n)l \to \xi^1(x)$$

for all quasi-geodesics γ_n *going from* y *to* x *and all lines* $l \in \mathbb{P}(\mathbb{R}^d)$ *that are transverse to* $\xi^{d-p}(y)$*. Moreover, it has the following transversality property on the boudnary: for every* $x, y \in \partial\Gamma$*, the flags* $\xi(x), \xi(y)$ *are transverse unless* $x = y$*.*

Again, we would like to stress that if the elements of the Anosov subgroup did not have the singular value gap, then the attracting flags U_p, U_{d-p} would not be well-defined.

The next lemma shows that in the case where $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ is projective Anosov and strongly irreducible, continuity and equivariance characterise the map $\xi^1 : \mathrm{SL}(d, \mathbb{R}) \to \mathbb{P}(\mathbb{R}^d)$.

Lemma 1.4.4. *Let* $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ *be a strongly irreducible projective Anosov representation, and denote with* $\xi_\rho : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ *its limit map. Then* ξ^1_ρ *is the unique continuous,* $\rho(\Gamma)$*-equivariant map from* $\partial\Gamma$ *to* $\mathbb{P}(\mathbb{R}^d)$*.*

Proof. Let $\eta^1 : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ be a continuous, $\rho(\Gamma)$-equivariant map. Since the action of Γ on its boundary $\partial\Gamma$ has dense orbits, it suffices to show that it agrees with ξ^1_ρ on at least one boundary point.

Suppose for the sake of contradiction that η^1 does not coincide with ξ^1 and let $z \in \partial\Gamma, y \in \partial\Gamma\backslash\{z\}$. Then for any $x \in \partial\Gamma\backslash\{y\}$ we may find some quasi-geodesic $\{\gamma_n\}_n$ such that $\gamma_n \to x, \gamma_n^{-1} \to y$ as $n \to \infty$. Then since $z \neq y$ we know that $\gamma_n z \to x$ as $n \to \infty$ and continuity of η^1 implies that $\eta^1(\gamma_n z) \to \eta^1(x)$. But equivariance of η^1 and the fact that ξ^1 is dynamics-preserving (Theorem 1.4.3) implies that $\eta(\gamma_n z) = \rho(\gamma_n)\eta(z) \to \xi^1(x)$, unless $\eta^1(z) \in \xi^{d-1}(y)$. But if in fact $\eta^1(z) \notin \xi^{d-1}(y)$, then the limits would agree, i.e. $\eta^1(x) = \xi^1(x)$ which is a contradiction. Thus, we have that $\eta^1(z) \in \xi^{d-1}(y)$ and since y was an arbitrary point of $\partial\Gamma\backslash\{z\}$, we have that

$$\eta^1(z) \in \bigcap_{y\in\partial\Gamma\backslash\{z\}} \xi^{d-1}(y) \subseteq \bigcap_{y\in\partial\Gamma\backslash\Gamma\cdot z} \xi^{d-1}(y).$$

In particular, the set appearing on the right-hand side is a ρ-invariant, non-empty proper subset of $\mathbb{R}^d$, which contradicts the strong irreducibility assumption of ρ. $\square$

1.5 Important functionals and definitions

In this section we will introduce certain functionals on the Weyl chamber, whose critical exponents (see Definition 1.5.4) are going to generalise the critical exponent of the group to the higher rank case.

Definition 1.5.1. For $p \in [\![2, \dots, d]\!]$, $s \in \mathbb{R}$ we define the functionals $\Psi_s^p : \mathfrak{a}^+ \to \mathbb{R}$, the Falconer functional $F_s : \mathfrak{a}^+ \to \mathbb{R}$, and for $p \in [\![1, d-1]\!]$ the unstable Jacobian $J_p^u : \mathfrak{a}^+ \to \mathbb{R}$ as follows:

$$\Psi_s^p = \mathsf{a}_{1,2} + \cdots + \mathsf{a}_{1,p-1} + (s - (p-2))\mathsf{a}_{1,p}$$
$$F_s(a) = \min \left\{ \sum_{j=2}^{d} s_j \alpha_{1,j}(a) : s_j \in (0,1], \sum_{j=2}^{d} s_j = s \right\}$$
$$J_p^u = (p+1)\omega_1 - \omega_{p+1} = \mathsf{a}_{1,2} + \cdots + \mathsf{a}_{1,p+1}.$$

Lemma 1.5.2. *The functionals defined above are related as follows:*

$$F_s(a) = \max_{p \in [\![2,d]\!]} \Psi_s^p(a) = \Psi_s^{p_0}(a) \textit{ for } s \in [p_0 - 2, p_0 - 1]. \textit{ for all } a \in \mathfrak{a}^+$$
$$\frac{s}{p} J_p^u \leq \Psi_{s+p}^{p+1} \textit{ for all } s > 0$$

Proof. Indeed, a quick calculation shows that for $s \geq 0, p \in [\![2, d]\!]$ and $a \in \mathfrak{a}^+$ we have that:

$$\Psi_s^{p-1}(a) \leq \Psi_s^p(a) \text{ if and only if } s \geq p-1.$$

and that equality holds in the case $s = p - 1$. Thus, for $s \in [p-2, p-1]$ we have that

$$s \geq p-2, \dots, 1 \text{ implies that } \Psi_s^p(a) \geq \dots \geq \Psi_s^2(a)$$
$$s \leq p, \dots, d-1 \text{ implies that } \Psi_s^p(a) \leq \dots \leq \Psi_s^d(a)$$

Another way to see this (refer to Figure 1.3) is to note that $\Psi_s^2(a), \cdots, \Psi_s^d(a)$ is a sequence of functions that are affine in s, with slopes $\alpha_{12}(a) \leq \cdots \leq \alpha_{1d}(a)$ and that they satisfy $\Psi_1^2(a) = \Psi_2^2(a), \Psi_2^3(a) = \Psi_3^4(a) \dots, \Psi_{d-2}^{d-1}(a) = \Psi_{d-2}^d(a)$.

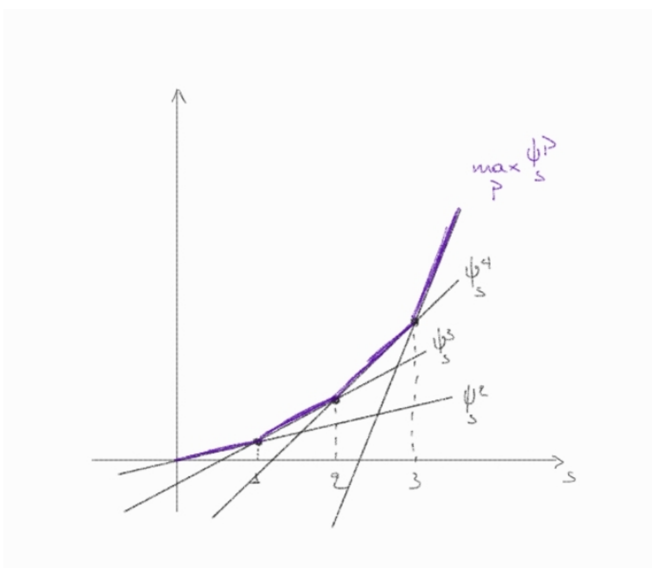


Figure 1.3: Visual illustration that $\max_p \Psi_s^p(a) = \Psi_s^{p_0}(a)$ for $s \in [p_0 - 2, p_0 - 1]$.

For the inequality, the calculations has as follows:

$$\begin{aligned}\Psi_s^{p+1} &= \mathsf{a}_{1,2} + \cdots + \mathsf{a}_{1,p+1} + (s-p)\mathsf{a}_{1,\mathsf{p+2}} \\ &= \mathsf{a}_{1,2} + \cdots + \mathsf{a}_{1,p+1} + p\left(\frac{s}{p} - 1\right)\mathsf{a}_{1,\mathsf{p+2}} \\ &\geq \mathsf{a}_{1,2} + \cdots + \mathsf{a}_{1,p+1} + \left(\frac{s}{p} - 1\right)(\mathsf{a}_{1,2} + \cdots + \mathsf{a}_{1,p+1}) \\ &= \frac{s}{p} J_p^u.\end{aligned}$$

□

Remark 1.5.3. The reason to consider the functionals Ψ_s^p and F_s will be apparent in Chapter 2, where we prove that the Hausdorff dimension of the limit set is bounded above by the critical exponent of the Falconer functional. There, we will see that the sums in the left-hand-side of the equality

$$\min_{p\in[\![2,d]\!]}\left\{\sum_{|\gamma|=T}\frac{\sigma_2}{\sigma_1}\cdots\frac{\sigma_{p-1}}{\sigma_1}(g)\left(\frac{\sigma_{p-1}}{\sigma_1}(g)\right)^{s-(p-2)}\right\} = \sum_{|\gamma|=T} e^{-F_s(a(g))}$$

correspond to the Hausdorff content of respective coverings of $\xi_\rho^1(\partial\Gamma)$ by projective ellipses of different lengths. For the use of the unstable Jacobian J_p^u we refer to Section 3.2.

As described in the begining of this section, some of the core objects that we will be working with are critical exponents and Hausdorff dimensions, which we recall below:

Definition 1.5.4. Let $\phi_s \in (a^+)^*$ be a family of non-negative functionals over the Weyl chamber indexed by and increasing in $s \in \mathbb{R}$, and $\Gamma \leq \mathrm{SL}(d,\mathbb{R})$ a discrete subgroup. We define the critical exponent of ϕ_s over Γ to be

$$h_\Gamma(\phi) = \inf\left\{s > 0 : \sum_{\gamma\in\Gamma} e^{-\phi_s(a(\gamma))} < \infty\right\}.$$

When ϕ is a functional over the Weyl chamber of $\mathrm{SL}(d,\mathbb{R})$, its critical exponent is defined as the critical exponent of the family $\{s\phi\}_{s\in\mathbb{R}}$. Similarly, we define the critical exponent of a family of functionals ϕ_s over a representation as

$$h_\Gamma(\phi) = \inf\left\{s > 0 : \sum_{\gamma\in\Gamma} e^{-\phi_s(a(\rho(\gamma)))} < \infty\right\}.$$

The following example is in a sense the link between Quint–Patterson–Sullivan theory and classical Patterson–Sullivan theory.

Example 1.5.5. For every uniform lattice $\Gamma \leq \mathrm{SO}(1,d)$, the critical exponent of Γ coincides with the critical exponent of some functional $\phi_0 \in \mathfrak{a}^+$:

$$h_\Gamma(\phi_0) = \delta_\Gamma.$$

This follows from the fact that any functional $\phi \in (a^+)^*$ over the Weyl-chamber $\mathfrak{a}^+$ of $\mathrm{SO}(d,1)$ is of the form

$$\phi(\mu(\gamma)) = c_\phi d(\gamma\cdot o, o) \text{ for some constant } c_\phi \in \mathbb{R},$$

where d is the distance on $\mathbb{H}^d$ that is invariant by some fixed maximal compact subgroup K fixing $o \in \mathbb{H}^d$.

Indeed, let $g \in \mathrm{SO}(1,d)$ and $y_0 \in \mathfrak{a}^+$. Then there exists some $r_g \in \mathbb{R}$ such that $\mu(g) = r_g \cdot y_0$, meaning that $d(g \cdot o, o) = d(e^{\mu(g)} \cdot o, o) = r_g d(e^{y_0} \cdot o, o)$. This gives the following expression for the Cartan projection of $\mathrm{SO}(1,d)$:

$$\mu(g) = \frac{d(g \cdot o, o)}{d(e^{y_0} \cdot o, o)} y_0.$$

Hence, for any $\phi \in \mathfrak{a}^*$ we have that $c_\phi = \frac{\phi(y_0)}{d(e^{y_0} \cdot o, o)}$.

Thus, we may consider the natural choice of functional $\phi_0 \in (\mathfrak{a}^+)*$ defined by $\phi_0(y_0) = d(e^{y_0} \cdot o, o)$. It is natural, in the sense that $c_\phi = 1$ and $\phi_0(\mu(g)) = d(g \cdot o, o)$

When $\phi = F$ is the Falconer functional, we obtain the following special case (for projective Anosov representations, i.e. $P = 1$) of a definition by Ledrappier and Lessa in [LL23]:

Definition 1.5.6. We define the Falconer dimension $\dim_F(\rho)$ of ρ to be the critical exponent of the Falconer functional:

$$\dim_F(\rho) = \inf \left\{ s > 0 : \sum_{\gamma \in \Gamma} e^{-F_s(\rho(\gamma))} < \infty \right\}.$$

Lastly, we recall the definition of the Hausdorff dimension:

Definition 1.5.7. Let (X, d) be a metric space and $A \subseteq X$ be a subset. We define the Hausdorff dimension of A to be

$$\dim_{\mathcal{H}}(A) = \inf \left\{ s > 0 : \mathcal{H}^s_\infty(A) = 0 \right\},$$

where $\mathcal{H}^s_\infty$ is the s-dimensional Hausdorff content of A, defined as

$$\mathcal{H}^s_\infty(A) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} \mathrm{diam}(U_i)^s : A \subseteq \bigcup_{i=1}^{\infty} U_i \right\}.$$

1.6 Hausdorff measures and densities

Let M be a submanifold of a Riemannian manifold N. Depending on the context, there are different ways to define a "natural" measure on M in a way that reflects the geometry of N, the most common of which are volume forms, densities and the Hausdorff measure.

For instance, when N is an orientable Riemannian manifold, there is a canonical volume form on N which induces a measure on N that can be restricted to M. When however neither M nor N is orientable, there exists no volume form that we can integrate. In this case, one can use a Riemannian density on N, which is a generalisation of differential forms, to obtain a measure on N and then restrict it to M. Alternatively, one could use the restriction on M of the Riemann metric on N to define the Hausdorff measure on M. The reason that we are interested in these methods is that they allow us to define a measure not only in the case where M and N are non-orientable, but also in the case where M is a Lipschitz submanifold, meaning that its tangent spaces are well-defined almost everywhere.

We begin by recalling the necessary material from the theory of densities, which can be found for instance in [LL12]. Seeing the following definition, it is apparent that densities are a generalisation of differential forms, in the sense that they are functions that obey the transformation law of differential forms. Note however that this is a strict generalisation, as densities need not be linear in any of their coordinates.

Definition 1.6.1. Let V be a vector space. A density on V is a function

$$\mu : V \times \cdots \times V \to \mathbb{R}$$

such that for every linear transformation $T : V \to V$ and every $v_1, \ldots, v_n \in V$ we have

$$\mu(Tv_1, \ldots, Tv_n) = |\det T|\mu(v_1, \ldots, v_n).$$

We say that it is a positive density on V if it is everywhere non-negative and we denote the set of densities by $\mathcal{D}(V)$.

To pass from vector spaces to manifolds, we use the notion of a density bundle, which is a direct analog of the tangent bundle. For a proof that the density bundle is indeed a line bundle, see [LL12][Proposition 16.36].

Definition 1.6.2. Let M be a smooth manifold. The density bundle is the real line bundle

$$\mathcal{D}(M) \xrightarrow{\pi} M$$

whose fiber at $x \in M$ is the space of densities $\mathcal{D}(M)_x \overset{\text{def}}{=} \mathcal{D}(T_x M)$ on the tangent space $T_x M$, and where π is the projection map $\pi_{|T_x M} = x$.

With the above definition at hand, we will henceforth call a density on M a continous section of the density bundle $\mathcal{D}(M)$, which just like every space of bundle sections, is a $C(M)$-module.

Example 1.6.3. Let M be an orientable Riemannian manifold and denote with Vol is Riemannian volume form. Then M admits a natural positive density $|\mathrm{Vol}|$ defined by

$$|\mathrm{Vol}|(v_1, \ldots, v_n) = |\mathrm{Vol}(v_1, \ldots, v_n)|.$$

We call $|\mathrm{Vol}|$ the Riemannian density on M.

Just as in the case of differential forms, one can pull-back densities through smooth maps and obtain new ones.

Definition 1.6.4. Let $f : M \to N$ be a smooth map between smooth manifolds of the same dimension, and μ be a density on N. The pull-back of μ by f is the density $f^*\mu$ on M defined by

$$(f^*\mu)_x(v_1, \ldots, v_n) = \mu_{f(x)}(df_x v_1, \ldots, df_x v_n),$$

for $v_1, \cdots, v_n \in T_x M$.

With the above definitions at hand, we may now set out to define the integral of a density μ over a manifold M. We begin by considering the case of a density μ on $M = \mathbb{R}^d$, whose support is contained in the closure $\overline{U}$ of an open set $U \subseteq \mathbb{R}^d$. Then there exists a unique continuous function $f : \mathbb{R}^d \to \mathbb{R}$ such that $\mu = f|\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d|$ and we define

$$\int_D \mu = \int_D f \,\mathrm{d}x_1 \cdots \mathrm{d}x_d.$$

To extend this definition to any smooth manifold M, we need to show that it is invariant under diffeomorphism:

Proposition 1.6.5. *Suppose U, V are open subsets of $\mathbb{R}^n$, $f : U \to V$ is a diffeomorphism and μ is a compactly supported density on V. Then*

$$\int_U f^*\mu = \int_V \mu.$$

Letting now M be a smooth n-manifold, for a density μ that is compactly supported in a coordinate chart (U, ϕ) of M, we define:

$$\int_M \mu = \int_{\phi(U)} (\phi^{-1})^* \mu.$$

and extend this to any density μ on M by

$$\int_M \mu = \sum_i \int_M \psi_i \mu,$$

where $\{\psi_i\}$ is a partition of unity subordinate to a cover of M by coordinate charts.

1.7 Complex hyperbolic geometry

In this section we gather some material on complex hyperbolic geometry that will be useful in providing a counterexample to a lemma of [PSW23] in Section 4.1. Although we will focus on the complex hyperbolic plane, in most cases, the same facts will also hold for higher dimensional complex hyperbolic spaces. For proofs of what we are about to recall, the reader can refer to any introductory source on the matter, such as [Par03].

The complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ admits several equivalent models, according to the Hermitian form that is used to define it. For completeness, we recall that a Hermitian form over a complex vector space V is a mapping $V \times V \to \mathbb{C}$ that is linear in the first term and conjugate linear in the second. Each such form $\langle \cdot, \cdot \rangle$ is associated to a Hermitian matrix A (meaning that $A^* = A$) in the sense that $x^* A y = \langle x, y \rangle$ for all $x, y \in V$. The following example lists the Hermitian forms that we will be using:

Example 1.7.1. For $z, w \in \mathbb{C}^3$ we have the following two forms of signature $(2, 1)$:

$$\begin{aligned} \langle z, w \rangle_1 &= z_1 \overline{w_1} + z_2 \overline{w_2} - z_3 \overline{w_3} \\ \langle z, w \rangle_2 &= z_1 \overline{w_3} + z_2 \overline{w_2} + z_3 \overline{w_1} \end{aligned}$$

which correspond to the matrices

$$J_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We will denote by $\mathbb{C}^{2,1}$ the space $\mathbb{C}^3$ equipped with a $(2, 1)$-Hermitian form $\langle \cdot, \cdot \rangle$ and with $\mathrm{SU}(2, 1) \leq \mathrm{SL}(2, \mathbb{C})$ its isometries. Then the projective model for the hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ is the set of all negative complex lines in $\mathbb{C}^{2,1}$ and its boundary $\partial \mathbb{H}^2_{\mathbb{C}}$ is the set of all isotropic lines:

$$\begin{aligned} \mathbb{H}^2_{\mathbb{C}} &\stackrel{\text{def}}{=} \left\{ [z_1 : z_2 : z_3] \in \mathbb{P}(\mathbb{C}^{2,1}) : \langle z, z \rangle < 0 \right\}, \\ \partial \mathbb{H}^2_{\mathbb{C}} &= \left\{ [z_1 : z_2 : z_3] \in \mathbb{P}(\mathbb{C}^{2,1}) : \langle z, z \rangle = 0 \right\}. \end{aligned}$$

with the Bergman metric on $\mathbb{H}^2_{\mathbb{C}}$ being given by the relation

$$\cosh\left(\frac{d(z, w)}{2} \right) = \frac{\langle z, w \rangle \langle w, z \rangle}{\langle z, z \rangle \langle w, w \rangle}.$$

Note that since the aforementioned forms are of the same signature, the spaces arising from them are isometric. In particular, the volumen entropy of the Bergman metric is the same for both models. Given this, let us for the moment consider the projective model as defined from the Hermitian form $\langle\cdot,\cdot\rangle_1$. Under the identification $\mathbb{H}^2_{\mathbb{C}} \to \mathbb{C}^2, [z_1 : z_2 : z_3] \mapsto (z_1/z_3, z_2/z_3)$, the volume element is then given by

$$\mathrm{d}\,\mathrm{Vol} = \frac{16}{(1-|z_1|^2-|z_2|^2)^3}\,\mathrm{d}x_1\,\mathrm{d}y_1\,\mathrm{d}x_2\,\mathrm{d}y_2.$$

This allows us to compute the critical exponent of any uniform lattice $\Gamma \leq \mathrm{SU}(2,1)$.

Lemma 1.7.2. *Let $\Gamma \leq \mathrm{SU}(2,1)$ be a uniform lattice. Then the critical exponent of Γ with respect to the Bergman metric is*

$$\delta_\Gamma = 2.$$

Proof. Using Lemma 1.1.4, it is equivalent to showing that the volume entropy of the Bergman metric is 2:

$$\lim_{R\to\infty} \frac{\log \mathrm{Vol}\, B_R([0:0:1])}{R} = 2.$$

Indeed, under the identification $\mathbb{H}^2_{\mathbb{C}} \to C^2, [z_1 : z_2, z_3] \mapsto (z_1/z_3, z_2/z_3)$ we have the expression

$$B_R([0:0:1]) = \left\{(z_1,z_2)\in\mathbb{C}^2 : |z_1|^2+|z_2|^2 < 1 - \frac{1}{\cosh^2(R/2)}\right\}.$$

Then after switching to polar coordinates, we obtain an explicit expression for the volume of the ball $B_R([1:0:-1])$, given by

$$\begin{aligned}
\mathrm{Vol}\, B_R([[0:0:1]]) &= \int_{B_R([0:0:1])} \frac{16}{(1-|z_1|^2-|z_2|^2)^3}\,\mathrm{d}x_1\,\mathrm{d}y_1\,\mathrm{d}x_2\,\mathrm{d}y_2 \\
&= 16\int_{r_1^2+r_2^2<1-\cosh^{-2}(R/2)} \frac{1}{(1-r_1^2-r_2^2)^3}\,\mathrm{d}r_1\,\mathrm{d}r_2 \\
&= \frac{8\pi^2\tanh^4(R/2)}{(\tanh^2(R/2)-1)^2},
\end{aligned}$$

which asymptotically behaves like e^{2R}, giving us the desired result. ∎

For the rest of this section and for the next chapters, we will switch to the model of the complex hyperbolic plane defined by the Hermitian form $\langle\cdot,\cdot\rangle_2$. A transitive subgroup of its isometry group is then given by the rank 1 Lie group

$$\mathrm{SU}(2,1) \overset{\mathrm{def}}{=} \{g \in \mathrm{SL}(3,\mathbb{C}) : g^* J_2 g = J_2\},$$

with Lie algebra

$$\begin{aligned}
\mathfrak{su}(2,1) &= \{g \in \mathfrak{gl}(3,\mathbb{C}) : g^* J_2 = -J_2 g, \mathrm{tr}(g) = 0\} \\
&= \left\{\begin{pmatrix} u-is & a & it \\ b & 2is & -\bar{a} \\ ir & -\bar{b} & -u-is \end{pmatrix} : u,r,s,t\in\mathbb{R}, a,b\in\mathbb{C}\right\}.
\end{aligned}$$

The maximal compact subgroup that we will work with will be the stabiliser of $[1:0:-1]$ in $\mathrm{SU}(2,1)$ and will be denoted by K.

From the above expression of $\mathfrak{su}(2,1)$, we obtain an ordered basis of $\mathfrak{su}(2,1)$

$$\mathfrak{su}(2,1) = \mathrm{span}\left\{\begin{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & i\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, & i\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, & i\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{matrix}\right\}.$$

in which a tedious calculation yields

$$\mathrm{ad}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \mathrm{diag}(0,0,1,1,2,-1,-1,-2) \in \mathfrak{gl}\,\mathfrak{su}(2,1).$$

The Cartan subalgebra and Weyl chamber are given by

$$\mathfrak{a}^+ = \mathbb{R}\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \mathfrak{a}^+ = \mathbb{R}_{\geq 0}\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ respectively.}$$

It will be useful later to have an explicit expression of the Cartan projection

Lemma 1.7.3. *The Cartan projection $\mu : \mathfrak{su}(2,1) \to \mathfrak{a}^+$ is given by*

$$\mu(g) = \frac{d(g[1:0:-1],[1:0:-1])}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Proof. Let $g \in \mathrm{SU}(2,1)$. Since $\mathrm{rank}(\mathrm{SU}(2,1)) = 1$, there exist $r(g) \in \mathbb{R}$ and $k, l \in K$ such that

$$g = ke^{\mu(g)}l \text{ and } \mu(g) = r(g)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

To calculate $r(g)$, we note that

$$\begin{aligned} d(g[1:0:-1],[1:0:-1]) &= d\left(e^{\mu(g)}[1:0:-1],[1:0:-1]\right) \\ &= d\left([e^{r(g)}:0:-e^{-r(g)}],[1:0:-1]\right), \end{aligned}$$

and compute it through the relation defining the Bergman metric:

$$\cosh^2\left(\frac{d^2\left([e^{r(g)}:0:-e^{-r(g)}],[1:0:-1]\right)}{2}\right) = \frac{4\cosh^2 r(g)}{2\cdot 2},$$

from which we infer that $d\left([e^{r(g)}:0:-e^{-r(g)}],[1:0:-1]\right) = 2r(g)$. ∎

Moving on to parabolic subgroups, we first show how they can be used to obtain the boundary of the complex hyperbolic plane. In particular, we claim that since Γ is a uniform lattice of $\mathrm{SU}(2,1)$, we have that $\partial\Gamma$ is homeomorphic to $\mathrm{SU}(2,1)/P$, where P is a parabolic subgroup of $\mathrm{SU}(2,1)$, and it coincides with the stabiliser of some isotropic line $l \in \partial_\infty\mathbb{H}^2_{\mathbb{C}}$.

Lemma 1.7.4. *Let Γ be a uniform lattice of a Lie group* $\mathrm{SU}(2,1)$ *and P be a parabolic subgroup of* $\mathrm{SU}(2,1)$*. Then Γ is Gromov hyperbolic and its Gromov boundary $\partial\Gamma$ is homeomorphic to the quotient* $\mathrm{SU}(2,1)/P$.

Proof. The Milnor-Švarc lemma implies that for any $x_0 \in \mathbb{H}^2_{\mathbb{C}}$, the map $\Gamma \to \mathbb{H}^2_{\mathbb{C}}, \gamma \mapsto \gamma x_0$ is a quasi-isometry, implying that $\mathrm{SU}(2,1)$ is Gromov hyperbolic, because $H^2_{\mathbb{C}}$ is. Thus, the quasi-isometry extends to a homeomorphism $\partial\Gamma \to \partial H^2_{\mathbb{C}}$ of the Gromov boundaries. On the other hand, the action of $\mathrm{SU}(2,1)$ on $\partial H^2_{\mathbb{C}}$ is transitive and P, being a parabolic subgroup, equals the stabiliser of some isotropic line $l \in \partial\mathbb{H}^2_{\mathbb{C}}$, so we have that $\partial H^2_{\mathbb{C}} \simeq \mathrm{SU}(2,1)/P$. $\square$

Before moving on, we recall that all parabolic subgroups of $\mathrm{SU}(2,1)$ are conjugate to each other, so we have the following identification:

$$\begin{array}{ccccc} \mathrm{SU}(2,1)/P_0 & \leftrightarrow & \left\{ \begin{array}{c} \text{Parabolic subgroups} \\ \text{of } \mathrm{SU}(2,1) \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{c} \text{Parabolic subalgebras} \\ \text{of } \mathfrak{su}(2,1) \end{array} \right\} \\ & & & & \\ gP_0 & \leftrightarrow & gP_0g^{-1} & \leftrightarrow & \mathrm{Ad}_g(\mathfrak{p}_0) \end{array}$$

Given this remark, the following lemma tells us that $\mathrm{SU}(2,1)$ acts transitively on pairs of distinct points in the boundary $\partial_\infty\mathbb{H}^2_{\mathbb{C}}$. To prove it, we fix the parabolic subgroup of $\mathrm{SU}(2,1)$ given by $P = \mathrm{Stab}_{\mathrm{SU}(2,1)}([1:0:0])$, whose algebra is

$$\mathfrak{p} = \mathrm{Stab}_{\mathfrak{su}(2,1)}([1:0:0]) = \left\{ \begin{pmatrix} u - is & a & it \\ 0 & 2is & -\bar{a} \\ 0 & 0 & -u - is \end{pmatrix} : u, s, t \in \mathbb{R}, a \in \mathbb{C} \right\}.$$

Lemma 1.7.5. $\mathrm{Ad}_{\mathrm{SU}(2,1)}$ *acts transitively on pairs of distinct parabolic subalgebras of* $\mathfrak{su}(2,1)$*. In other words, for every distinct parabolic subalgebras $\mathfrak{p}, \mathfrak{p}' \leq \mathfrak{su}(2,1)$, there exists some $g \in \mathrm{SU}(2,1)$ such that $\mathrm{Ad}_g(\mathfrak{p}) = \mathfrak{p}_0$ and $\mathrm{Ad}_g \mathfrak{p}' = \mathfrak{p}_0^t$, where $\mathfrak{p}_0, \mathfrak{p}_0^t$ are the subalgebras of the parabolic subgroups $P_0 = \mathrm{St}_{\mathrm{SU}(2,1)}[1:0:0]$ and $P_0^t = \{g^t : g \in P_0\} = \mathrm{St}_{\mathrm{SU}(2,1)}[0:0:1]$ respectively.*

Proof. Recall that we have defined $P_0 = \mathrm{St}_{\mathrm{SU}(2,1)}[1:0:0]$. To see why $P_0^t = \mathrm{St}_{\mathrm{SU}(2,1)}[0:0:1]$, we note that $P_0^t = g_0 P_0 g_0 = g_0 P_0 g_0^{-1} = \mathrm{St}_{\mathrm{SU}(2,1)} g_0[1:0:0] = \mathrm{St}_{\mathrm{SU}(2,1)}[0:0:1]$ for

$$g_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Thus $\mathfrak{p}_0 = \mathrm{St}_{\mathfrak{su}(2,1)}[1:0:0]$ and $\mathfrak{p}_0^t = \mathrm{St}_{\mathfrak{su}(2,1)}[0:0:1]$.

Let now $l = \mathbb{R}v, l' = \mathbb{R}v' \in \partial_\infty\mathbb{H}^2_{\mathbb{C}}$ be the isotropic lines corresponding to $\mathfrak{p}, \mathfrak{p}'$ respectively, i.e. $\mathfrak{p} = \mathrm{St}_{\mathfrak{su}(2,1)} l$ and $\mathfrak{p}' \mathrm{St}_{\mathfrak{su}(2,1)} l'$. Then $\langle v, v' \rangle \neq 0$, where we denote with $\langle \cdot, \cdot \rangle$ the Hermitian form defining the complex hyperbolic plane (see Section 1.7). Indeed, if this were not the case, then $\mathrm{span}\{v, v'\}$ would be an isotropic subspace of dimension 2, which is not possible for a $(2,1)$ form. Thus we can assume that $\langle v, v' \rangle = 1$, which means that any transformation $g : \mathbb{R}e_1 \oplus \mathbb{R}e^3 \to Rl \oplus \mathbb{R}l'$ satisfying $ge_1 = l, ge_3 = l'$ preserves the form. Witt's theorem then guarantees that it can be extended to a transformation g in $\mathrm{SU}(2,1)$.

Denoting with $P = \mathrm{St}l, P' = \mathrm{St}l'$ the respective parabolic subgroups, we have that $gPg^{-1} = \mathrm{St}_{\mathrm{SU}(2,1)} gl = P_0$ and $gP'g^{-1} = \mathrm{St}_{\mathrm{SU}(2,1)} gl' = P_0^t$. $\square$

Chapter 2

Upper bound

In this section we prove the upper bound on the Hausdorff dimension, namely that for a projective Anosov representation in $\mathrm{SL}(d,\mathbb{R})$, the Hausdorff dimension of the limit set is bounded above by the Falconer dimension:

Lemma 2.2.1 (Upper bound for dimension)**.** Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a projective Anosov representation. Then:

$$\dim_{\mathcal{H}}(\xi^1(\partial\Gamma)) \leq h_\rho(F).$$

The idea of the proof is to find a covering whose Hausdorff content is dominated by the Dirichlet series of some functional Ψ^p_s, which will in turn imply that $\dim_{\mathcal{H}}(\xi^1(\partial\Gamma)) \leq h_\rho(\Psi^p)$. Choosing then the most "effective" cover (i.e. the one which yields the smallest Hausdorff content up to a constant) we obtain that

$$\dim_{\mathcal{H}}(\xi^1(\partial\Gamma)) \leq h_\rho(F) = h_\rho\left(\max_p \Psi^p\right)$$

To obtain this we first cover $\xi^1(\partial\Gamma)$ by the shifted bassins of attraction $\rho(\gamma)\cdot B_{\alpha_1,\alpha}(\rho(\gamma))$ for $\gamma \in \Gamma$ satisfying $|\gamma| = T$. Then, for each $p \in [\![2,d]\!]$, the set $\rho(\gamma)\cdot B_{\alpha_1,\alpha}(\rho(\gamma))$ is contained in an ellipsoid of axes lengths

$$\beta_2 = \frac{1}{\sin(\alpha)}\frac{\sigma_2}{\sigma_1}(\rho(\gamma)),\ldots,\beta_d = \frac{1}{\sin(\alpha)}\frac{\sigma_d}{\sigma_1}(\rho(\gamma)).$$

which in turn is covered by roughly $\beta_2\cdots\beta_{p-1}/\beta_p^{p-2}$ balls of radius $\sqrt{d-1}\beta_p$. Using this cover to bound the Hausdorff content of $\xi^1_\rho(\partial\Gamma)$, we obtain the bound

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \lesssim \sum_{|\gamma|=T} e^{-\Psi^p_s(\rho(\gamma))} \text{ for each } p \in [\![2,d]\!], \text{ so } \mathcal{H}^s(\xi^1(\partial\Gamma)) \lesssim \sum_{|\gamma|=T} e^{-F_s(\rho(\gamma))}.$$

If we assume that $s > h_\rho(F)$, then the right-hand side of the last inequality converges to zero as we let $T \to \infty$, which means that $\mathcal{H}^s(\xi^1_\rho(\partial\Gamma)) = 0$, giving us the desired bound. Finally for each $q \in [\![2,d]\!]$, we cover each ellipsoid by balls of some radius β_q.

2.1 Lemmata

To simplify the proof of the bound, we present a few preliminary facts in the form of lemmata, whose proofs constitute the content of this section. Our first goal will be to obtain an open cover for the limit set using basins of attraction of elements in the Anosov subgroup at a fixed distance from the identity. It appears as [PSW23, Proposition 3.5].

Lemma 2.1.1. *Let $\rho : \Gamma \to SL(d, \mathbb{R})$ be projective Anosov. Then for $\alpha > 0$ small enough, there exists some $T_0 > 0$ such that for all $T \geq T_0$ the family*

$$\mathcal{U}_T = \{\rho(\gamma) B_{\mathsf{a}_1,\alpha}(\rho(\gamma)) : |\gamma| = T\}$$

is an open covering of $\xi^1(\partial\Gamma)$.

To prove this, we will need the following lemma ([PSW23, Lemma 2.4]). It essentially tells us that if we evaluate an Anosov representation at two points of a geodesic passing through $e \in \Gamma$ that are far apart, then the attracting line at one point is uniformely (with respect to geodesics) far apart from the attracting hyperplane at the other point.

Lemma 2.1.2. *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. For $\alpha > 0$ small enough, there exists $T > 0$ such that for any geodesic ray $(a_j)_{j \in \mathbb{N}}$ through e we have:*

$$\angle(U_1(\rho(a_i)), U_{d-1}(\rho(a_0))) > \alpha$$

when $|a_i|, |a_0| > T$.

Proof. Assume the contrary for the sake of contradiction. Then (see Figure 2.1) for each $n > 0$, there exists a geodesic ray $(a^n)_{j \in \mathbb{N}}$ through e such that

$$|a^n_n|, |a^n_0| > n \text{ and } \angle(U_1(\rho(a^n_n)), U_{d-1}(\rho(a^n_0))) < \frac{1}{n}.$$

Due to compactness of $\Gamma \cup \partial\Gamma$ we can find some subsequence k_n and $x, y \in \partial\Gamma$ such that $a^{k_n}_{k_n} \to x$, $a^{k_n}_0 \to y$. Since there exists a geodesic joining $a^{k_n}_{k_n}, a^{k_n}_0$ passing from e, we also know that $x \neq y$. Also, the limit map being continuous (Theorem 1.4.3), we have that

$$\angle(\xi^1(x), \xi^{d-1}(y)) = 0,$$

which contradicts its transversality property. $\square$

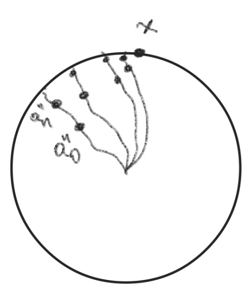


Figure 2.1: Situation in Lemma 2.1.2

We may now proceed with the proof of Lemma 2.1.1:

Proof of Lemma 2.1.1. Let $\alpha, T > 0$ be as in the statement of Lemma 2.1.2 and $x \in \partial\Gamma$ be represented by a geodesic ray $(\gamma_j)_{j \geq 0}$ starting from e. Then $(\gamma_T^{-1}\gamma_j)_j$ is a geodesic ray starting from $(\gamma_T)^{-1}$ that passes through e, so

$$\angle(U_1(\rho(\gamma_T^{-1}\gamma_j)), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

as implied by Lemma 2.1.2. Taking the limit $j \to \infty$ and using the equivariance of the limit map, we obtain

$$\angle(\rho(\gamma_T^{-1})\xi^1(x), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

and thus $\xi^1(x) \in \rho(\gamma_T) \cdot B_{\alpha_1,\alpha}(\rho(\gamma_T))$. □

The next step will be to cover each of the sets in $\mathcal{U}_T$ by ellipsoids in the projective space, which are merely the projectivisation of the usual ellipsoids in $\mathbb{R}^d$. Note that while in the case of the latter, the axes are lines (for instance $\mathbb{R}e_1, \dots, \mathbb{R}e_d$), their projective analogue should be planes (for instance $\mathbb{R}u_1 \oplus \mathbb{R}u_2, \dots \mathbb{R}u_1 \oplus \mathbb{R}u_d$).

Definition 2.1.3. Let V be a finite-dimensional $\mathbb{R}$-vector space. We consider a decomposition

$$V = \mathbb{R}u_1 \bigoplus \cdots \bigoplus \mathbb{R}u_d$$

that is orthogonal with respect to a fixed inner-product over V. Given $\beta_2 \geq \dots \beta_d > 0$, we define an ellipsoid with axes $(u_1 \oplus u_p)_{p \in [\![2,d]\!]}$ and lengths $(\beta_p)_{p\in[\![2,d]\!]}$ to be the image of

$$\left\{ v = \sum_1^d v_i u_i \in V : \sum_2^d \left(\frac{v_j}{\beta_j}\right)^2 \leq v_1^2 \right\}$$

through the projection $V \to \mathbb{P}(V)$.

When we want to distinguish between the projective ellipsoid of the above definition and the usual ellipsoid in $\mathbb{R}^d$, we will refer to the latter as a euclidean ellipsoid. When there is no ambiguity, we will simply refer to the projective ellipsoid as an ellipsoid.

The following is [PSW23, Proposition 3.8] and tells us that each set of the covering $\mathcal{U}_T$ from Lemma 2.1.1 is contained in a projective ellipsoid, whose axes correspond to the axes of the euclidean ellipsoid $g \cdot \mathbb{S}^{d-1}$, and whose axes depend on the singular values of g. Recalling the discussion preceeding Definition 1.4.2, it should not come as a surprise since the bassin of attraction is a set of flags that are streched and contracted at a uniform degree under the action of g.

Proposition 2.1.4. *Let $g \in \mathrm{SL}(d,\mathbb{R}), \alpha > 0$, fix a Cartan decomposition $g = k_g e^{\mu(g)} l_g$ and consider the corresponding choice of axes $u_i(g) = k_g e_i, u_i(g^{-1}) = l_g^{-1} e_{d-i+1}$ for $i \in [\![1,d]\!]$, along with the respective flags $U_i(g), U_i(g^{-1})$. Denoting with*

$$B_{\mathsf{a}_1,\alpha}(g) = \left\{x \in \mathbb{P}(\mathbb{R}^d) : \angle(x, U_{d-1}(g^{-1})) > \alpha\right\}$$

the bassin of attraction of g, the set $g \cdot B_{\mathsf{a}_1,\alpha}(g)$ lies in the ellipsoid with axes $(u_1(g) \oplus u_p(g))_{p\in[\![2,d]\!]}$ of lengths

$$\frac{1}{\sin\alpha} \cdot \frac{\sigma_p(g)}{\sigma_1(g)}$$

Proof. Using the definition of the basin of attraction (see Figure 2.3), for $w \in \mathbb{R}^d$, we have (up to considering $-w$ instead of w) that:

$$\angle(\mathbb{R}w, U_{d-1}(g^{-1})) = \angle(w, w') = \frac{|w_d|}{|w|},$$

when

$$\begin{aligned} w &= w_1 u_1(g^{-1}) + \cdots + w_d u_d(g^{-1}) \\ &= w_1 l_g^{-1} e_c(g) + \cdots + w_d l_g^{-1} e_1(g) \end{aligned}$$

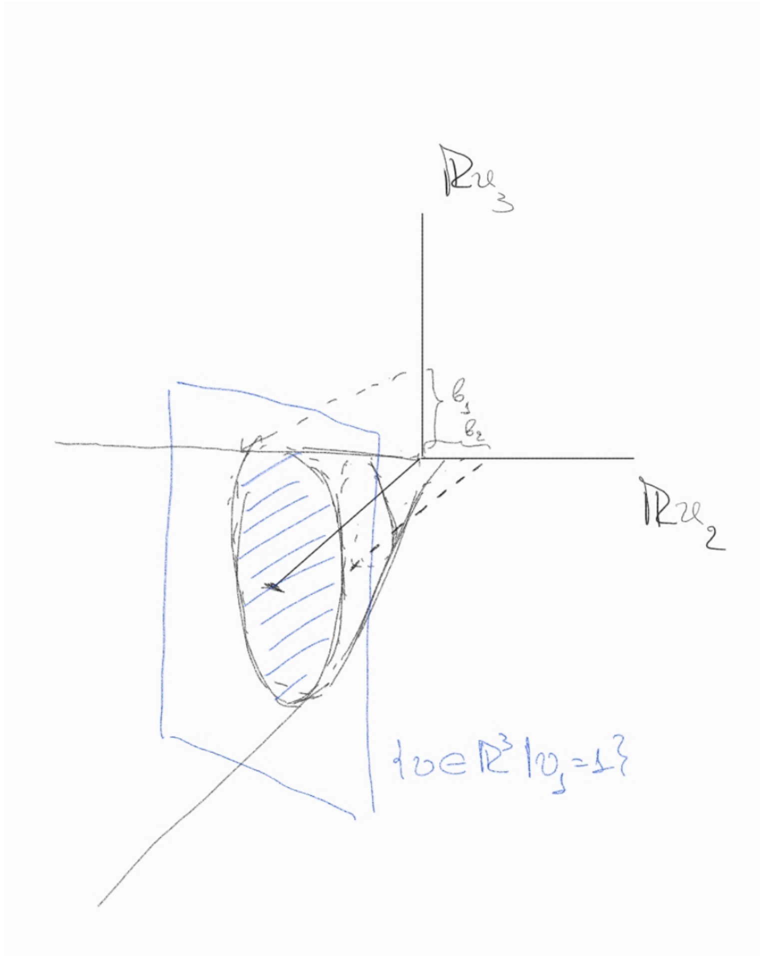


Figure 2.2: Depiction in $\mathbb{R}^3$ of an ellipsoid of $\mathbb{P}(\mathbb{R}^2)$

and

$$\begin{aligned} w' &= w - w_d u_d(g^{-1}) = \\ &= w_1 u_1(g^{-1}) + \cdots + w_{d-1} u_{d-1}(g^{-1}) \\ &= w_d l_g^{-1} e_1(g) + \cdots + w_1 l_g^{-1} e_{d-1}(g). \end{aligned}$$

Then $\mathbb{R}w \in B_{\alpha_1,\alpha}(g)$ if and only if

$$w_d^2 \geq (\sin\alpha)^2 \sum_1^d w_i^2.$$

Considering now some $v = v_1 u_1(g) + \cdots + v_d u_d(g)$ such that $\mathbb{R}v \in g \cdot B_{\alpha_1,\alpha}(g)$ we have that

$$\begin{aligned} w = g^{-1} v &= v_1 \sigma_1(g)^{-1} l_g^{-1} e_1(g) + \cdots v_d \sigma_d(g)^{-1} l_g^{-1} e_d(g) \\ &= v_1 \sigma_1(g)^{-1} u_d(g^{-1}) + \cdots v_d \sigma_d(g)^{-1} u_1(g^{-1}) \end{aligned}$$

where we used that $u_p(g^{-1}) = l_g^{-1} e_{d+1-p}$. Hence

$$\sigma_1(g)^{-2} \cdot v_1^2 \geq (\sin a)^2 \sum_1^d \sigma_i(g)^{-2} v_i^2.$$

Since the only condition on v is that the line through v lies in $g \cdot B_{\alpha_1,\alpha}(g)$, we may choose v such that $v_1 = \sigma_1(g)$. Indeed, the only case that this would not be possible is when

$$v \in \mathbb{R}(u_2(g) \oplus \cdots u_d(g)) = gU_{d-1}(g^{-1}),$$

which is not possible since this would imply that $g^{-1}v \in U_{d-1}(g^{-1})$. Thus, the inequality above becomes precisely the condition for $\mathbb{R}w$ to lie in the projective ellipsoid. $\square$

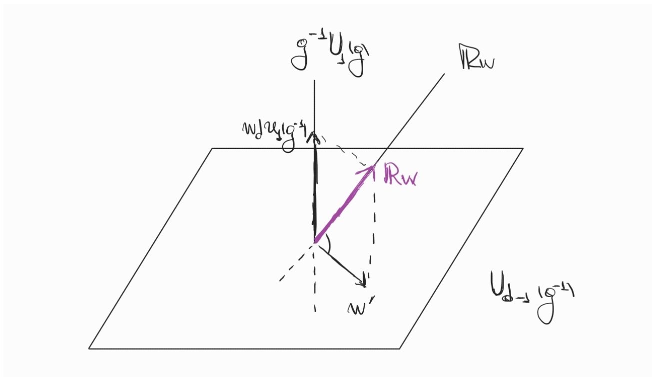


Figure 2.3: Aid for Proposition 2.1.4

The following is [PSW23, Lemma 3.7]:

Lemma 2.1.5. *For any $p \in [\![2, d]\!]$, an ellipsoid in $\mathbb{P}(\mathbb{R}^d)$ of axes lengths $\beta_2 \geq \cdots \geq \beta_d$ is covered by*

$$2^{p-2} \frac{\beta_2 \cdots \beta_{p-1}}{\beta_p^{p-2}}$$

many (projected) balls of radius $\sqrt{d-1}\beta_p$.

Proof. We assume that E is an ellipsoid about $\mathbb{R}e_1$, so it suffices to cover its intersection $E_1 = E \cap U_1 \subseteq \mathbb{R}^{d-1}$ with the affine chart $U_1 = \{[x_1 : \ldots, x_d] \in \mathbb{P}(\mathbb{R}^d) : x_1 \neq 0\}$. Clearly $E_1 \subseteq [-\beta_2, \beta_2] \times \ldots \times [-\beta_d, \beta_d]$, so we proceed by covering the rectangle with side-lengths $2\beta_2, \ldots, 2\beta_d$. Furthermore each interval $(-\beta_j, \beta_j)$ is contained in the union of $\lceil \beta_j/\beta_p \rceil$ intervals of length $2\beta_p$, thus E_1 is contained in the union of

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil \cdots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil = \left\lceil \frac{\beta_2}{\beta_p} \right\rceil \cdots \left\lceil \frac{\beta_d}{\beta_p} \right\rceil$$

many squares of side-length $2\beta_p$. Since each such product is contained in a $(d-1)$-ball of radius $\sqrt{d-1}\beta_p$ we may use at most

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil \cdots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil \leq \sum_{i \in \{0,1\}^{p-2}} \prod_{j=2}^{p-1} \left(\frac{\beta_j}{\beta_p} \right)^{i_j} \leq 2^{p-2} \frac{\beta_2}{\beta_p} \cdots \frac{\beta_{p-1}}{\beta_p}$$

many $(d-1)$-balls of radius $\sqrt{d-1}\beta_p$ to cover E_1. $\square$

The following can be found in [PSW23, Proposition 3.3]. It will not be used until Chapter 3, but we include it here to stress the fact that (similarly to all results of this chapter and in contrast to the next one) it holds for any projective Anosov representation, without any more assumptions (like Zariski-density of the subgroup or regularity of the limit set).

Proposition 2.1.6. *Let $\rho : \Gamma \to SL(d, \mathbb{R})$ be projective Anosov and $\alpha > 0$ Then there exist $c_0, c_1 > 0$ that depends only on α and ρ such that for all $\gamma \in \Gamma$:*

$$(\xi^1)^{-1}(B_{\alpha_1,\alpha}(\rho(\gamma))) \subseteq C^{\infty}_{c_0,c_1}(\gamma)$$

Proof. We begin by noting that it suffices to show this for all but finitely many $\gamma \in \Gamma$, since then we may alter the constants to satisfy the wanted inclusion for the finitely many remaining $\gamma \in \Gamma$ as well. Hence, we may assume that $|\gamma| \geq l_0$ where $l_0 > 0$ is such that

$$Ce^{-\mu l_0} < 1 \text{ and } \mathsf{a}_1(\gamma) \geq C|\gamma| - L.$$

Suppose $x \in \partial\Gamma$ such that $\xi^1(x) \in B_{\alpha_1,\alpha}(\rho(\gamma))$, and consider a geodesic ray $a_j \to x$ starting from $a_0 = e$. To prove the result, it suffices to find constants c_0, c_1 independent of γ and for which there exists a (c_0, c_1)-quasi-geodesic from γ^{-1} to x that passes through e and stays at a bounded distance from $(a_j)_{j=0}^{\infty}$.

Using the exponential convergence rate of $\xi^1(a_j) \to \xi^1(x)$ and the definition of $B_{\alpha_1,\alpha}(\rho(\gamma))$ we have that:

$$\begin{aligned} d(\xi^1(a_j), \xi^1(\gamma)) &\geq d(\xi^1(x), U_1(\rho(\gamma^{-1})) - d(\xi^1(a_j), \xi^1(x))) \geq \\ &\geq d(\xi^1(x), U_{d-1}(\rho(\gamma^{-1})) - d(\xi^1(a_j), \xi^1(x))) \geq \sin\alpha - Ce^{-\mu j} \end{aligned}$$

which along with the uniform continuity of $\xi^1 : \Gamma \cup \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ implies that there exists some $\alpha' > 0$ and $L > 0$ such that for all $j \geq L$:

$$d(a_j, \gamma^{-1}) \geq \alpha'.$$

Upon considering a large L, we may also assume that $|a_L| = L > l_0$. Note that both α' and L do not depend on each γ but only on ρ and α.

Using a coarse geometric arguement, we can show that for all $j \geq L$

$$d(\gamma^{-1}, a_j) > \alpha' \Rightarrow d_{\mathcal{H}}([\gamma^{-1}, a_j], e) < \alpha''$$

for some α'' that depends only on Γ and α', where $[a_j, \gamma^{-1}]$ denotes any geodesic segment connecting γ^{-1} and a_j, and $d_{\mathcal{H}}$ denotes the Hausdorff distance between sets. Indeed, [GH13, Lemme 2.17] states that $d([\gamma^{-1}, a_j]) \leq (\gamma_j^{-1}, a_j)_e + \delta$ where δ is the hyperbolicity constant of Γ. Thus

$$d([\gamma^{-1}, a_j]) \leq \delta + \frac{d(a_j, e) + d(\gamma^{-1}, e) + d(a_j, \gamma^{-1})}{2} \leq \delta + \frac{L + d(\gamma^{-1}, e) + \alpha'}{2}.$$

Consider the concatenation $(a'_j)_{j=L-K}^{\infty}$ of $[\gamma^{-1}, a_L]$ and $[a_L, x]$. To find quasi-geodesic-constants that are uniform in γ, we note that for any $c_0 \geq 1, c_1 \geq 0$:

$$c_0^{-1}|i - j| - c_1 \leq d(a'_i, a'_j) = d(a_i, a_j) \leq c_0|i - j| + c_1 \text{ when } i, j \geq L \text{ or } i, j \leq L$$

and that the upper bound follows trivially by the triangle inequality.

For the lower bound we proceed in two steps. First we bound the distance of $\gamma^{-1} = a'_{L-K}$ to a_{L+j} for $j \geq 0$:

$$
\begin{aligned}
d(a'_{L-K}, a'_{L+j}) &\geq \nu(|a_{L+j}| - |\gamma^{-1}|) - c'_0 - c'_1 |\log(d(U_1(\rho(a_{L+j})), U_1(\rho(\gamma^{-1}))))| \geq \\
&\geq \nu((L+j) + (K-L)) - c'_0 - c'_1 |\log(\sin a)| \geq \\
&= c_0^{-1}(j+K) - c_1
\end{aligned}
$$

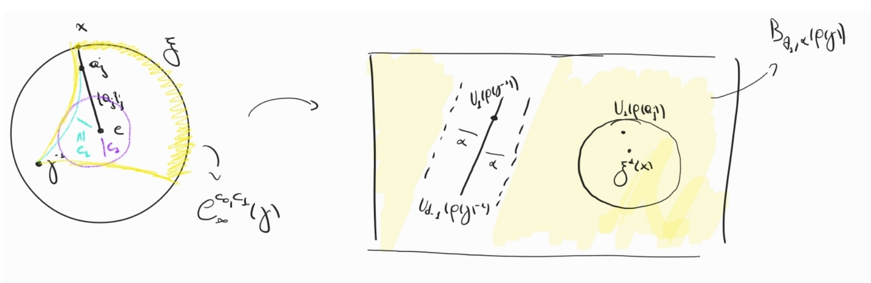
for $c_0 = \nu^{-1}, c_1 = c'_0 + c'_1|\log(\sin \alpha)|$. The first inequality comes from [PSW23, Lemma 3.9]. For the second inequality we estimate $|\gamma^{-1}|$ from below using the triangle inequality. We are now ready to show that the concatenation $(a'_j)_j$ is indeed a $(c_0,\, c_1)$-geodesic:

$$
\begin{aligned}
d(a_{L+j}, a_{L-i}) &\geq d(a_{L+j}, a_{L_K}) - d(a_{L_K}, a_{L-i}) \geq c_0^{-1}(j+K) - c_1 - (K-i) \geq \\
&\geq c_0^{-1}(j+i) - c_1.
\end{aligned}
$$

Note however that (a'_j) does not necessarily lie in $C_\infty^{c_0,c_1}$ since it may not pass through e. For this reason we some $L-K \leq i_0 \leq L$ such that $|a_{i_0}| < \alpha''$, the existence of which is guaranteed by the fact that $d([\gamma^{-1}, a_L], \epsilon) < \alpha''$. We then consider alter (a'_j) at i_0 so that it passes through e to obtain

$$
a''_j = \begin{cases} a_j & \text{for } j \neq i_0 \\ e & \text{for } j = i_0 \end{cases}
$$

which is a $(c_0, c_1 + \alpha'')$-quasigeodesic passing from e and converging to x. $\square$



2.2 Proof of bound

We are now ready to prove the upper bound of the Hausdorff dimension of the limit set by formalizing the proof strategy outlined in the beginning of this chapter.

Lemma 2.2.1 (Upper bound for dimension)**.** *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. Then:*

$$
\dim_{\mathcal{H}}(\xi^1(\partial\Gamma)) \leq h_\rho(F).
$$

Proof of Lemma 2.2.1. Let $p \in [\![2, d]\!]$. Then using Proposition 2.1.4, Lemma 2.1.1, and Lemma 2.1.5 we have that for $T > 0$ large enough and $\alpha > 0$ small enough, $\xi^1(\partial\Gamma)$ is covered by the family

$$
\mathcal{U}_T = \{\rho(\gamma) B_{\alpha_1, \alpha}(\rho(\gamma)) : |\gamma| = T\},
$$

and that each basin $\rho(\gamma)B_{\alpha_1,\alpha}(\rho(\gamma))$ is in turn covered by

$$2^{p-2} \cdot \frac{\sigma_2(g)\cdots\sigma_{p-1}(g)}{\sigma_p(g)^{p-2}}$$

many balls of radius

$$\sqrt{d-1}\frac{1}{\sin\alpha}\frac{\sigma_p(g)}{\sigma_1(g)}.$$

By the definition of the Hausdorff measure, for such a choice of constants $T, \alpha > 0$ and $s \geq 0$:

$$\begin{aligned}
\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq \sum_{|\gamma|=T} 2^{p-2} \cdot \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\cdots\frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\left(\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\right)^{-(p-2)}\left(\sqrt{d-1}\frac{1}{\sin\alpha}\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\right)^s &= \\
= 2^{p-2}\cdot\left(\frac{\sqrt{d-1}}{\sin\alpha}\right)^s \sum_{|\gamma|=T}\frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\cdots\frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\left(\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\right)^{s-(p-2)} &= \\
= 2^{p-2}\cdot\left(\frac{\sqrt{d-1}}{\sin\alpha}\right)^s \sum_{|\gamma|=T} e^{-\left(\alpha_{12}+\ldots+\alpha_{1(p-1)}+(s-(p-2))\alpha_{1p}\right)\rho(\gamma)} \\
= 2^{p-2}\cdot\left(\frac{\sqrt{d-1}}{\sin\alpha}\right)^s \sum_{|\gamma|=T} e^{-\Psi_s^p(\rho(\gamma))}
\end{aligned}$$

and thus

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq 2^{p-2}\cdot\left(\frac{\sqrt{d-1}}{\sin\alpha}\right)^s \sum_{|\gamma|=T} e^{-\max_p \Psi_s^p(\rho(\gamma))} \lesssim \sum_{|\gamma|=T} e^{-F_s(\rho(\gamma))}.$$

To see that the above implies the upper bound, consider some $s > h_\rho(F)$. By the definition of the Falconer dimension, this implies that the Dirichlet series corresponding to the Falconer functional converges:

$$\sum_{\gamma\in\Gamma} e^{-F_s(\rho(\gamma))} < \infty$$

and in particular

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq \lim_{T\to\infty}\sum_{|\gamma|=T} e^{-F_s(\rho(\gamma))} = 0.$$

Hence $s \geq \dim_{\mathcal{H}}(\xi^1(\partial\Gamma))$. $\square$

Looking again at the proof above, one could wonder whether the cover of the ellipsoid with axes $\beta_2, \cdots, \beta_d$ by balls of radius $\sqrt{d-1}\beta_p$ is optimal, in the sense that it gives the smallest Hausdorff content. To show that this is the case, consider the cover by balls of radius $r > 0$. This yields bounds of the form $\mathcal{H}^s(\xi^1_\rho(\partial\Gamma)) \lesssim \Phi_{r,s}$, where

$$\Phi_{r,s} = \begin{cases} \sum_{|\gamma|=T} r^s, \text{ for } r > \beta_2 \\ \sum_{|\gamma|=T} \frac{\beta_2\cdots\beta_{p-1}}{r^{p-2}} r^s, \text{ for } \beta_{p-1} < r < \beta_p, p \in [\![3, d]\!] \\ \vdots \\ \sum_{|\gamma|=T} \frac{\beta_2\cdots\beta_d}{r^{d-1}} r^s, \text{ for } 0 < r < \beta_d. \end{cases}$$

The opimality of $r = \beta_p$ can be seen that for $s \in [0,1], \Phi_{r,s}$ is increasing in $[\beta_2, \infty]$, and decreasing in $[0, \beta_2]$, so it is minimized as a function of r for $r = \beta_2$, and thus gives the optimal Hausdroff content when $s \in [0,1]$. Proceeding similarly, we see that the analogue holds for each of the other intervals as well.

Chapter 3

Lower bound

In this chapter we will be bounding the Hausdorff dimension of the limit set from below by the Falconer dimension. Combining with the reverse inequality obtained in Chapter 2, this will yield the main result of the thesis:

Theorem 3.4.1. Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a Zariski-dense, projective Anosov representation such that $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold of $\mathbb{P}(\mathbb{R}^d)$ of dimension d_ρ. Then the dimension of the limit set $\xi^1_\rho(\partial\Gamma)$ equals the Falconer dimension of ρ:

$$\dim \xi^1_\rho(\partial\Gamma) = h_\rho(F)$$

where $F_s(a) = \max\{\Psi^p_s(a) : p \in [\![2, d]\!]\}$ is the Falconer functional.

As can be seen from the statement, the lower bound on the dimension will need two more assumptions on the representation: its limit set needs to be a Lipschitz submanifold of the projective space and the image needs to be Zariski-dense in $\mathrm{SL}(d, \mathbb{R})$. While the first assumption appears in [PSW23] as well, the second one does not, and replaces the assumption that the representation is strongly irreducible. The reason for this change is that in [PSW23] the authors suggest that strong irreducibility of the representation implies that Patterson–Sullivan measures do not admit annihilators of full measure. However, this is not true, as we show in Chapter 4, where we also discuss further ways to circumvent this issue.

In Section 3.1 we give the necessary definitions due to [Qui02] that generalise the Busemann cocycle and Patterson–Sullivan measures to Lie groups of higher rank and flag spaces. Using these, we outline the strategy for the proof of Theorem 3.4.1 in Section 3.2, which is given in full in Section 3.4. In Section 3.3, we show that there exists a Patterson–Sullivan measure with the properties that we need.

3.1 Busemann cocyle and Patterson–Sullivan measures

We denote with Π the set of simple positive roots, and for $\Theta \subseteq \Pi$ we consider the Levi-Anosov subspace of $\mathfrak{a}$

$$\mathfrak{a}_\Theta = \bigcap_{\mathsf{a}_i \notin \Theta} \ker \mathsf{a}_i,$$

which in particular admits $\{\omega_{\mathsf{a}_i} : i \in \Theta\}$ as a basis.

Definition 3.1.1. Let $\Theta \subseteq \Pi$. We define the Busemann cocycle

$$b_\Pi : \mathrm{PSL}(d, \mathbb{R}) \times \mathcal{F}_\Pi \to \mathfrak{a}$$

as the unique element $b_\Pi(g, kP) \in \mathfrak{a}$ such that

$$gk \in Ke^{b_\Pi(g,kP)}N,$$

where $N = \{n \in \mathrm{SL}(d, \mathbb{R}) : n_{ij} = 0 \text{ for } i > j, n_{ii} = 1 \text{ for all } i\}$ is the unipotent group of upper subgroup of upper triangular matrices with 1s on the diagonal, and P is the minimal parabolic subgroup of $\mathrm{PSL}(d, \mathbb{R})$ given by the set of upper triangular matrices.

However we will not be directly working with the Busemann cocycle itself, but rather with a cocycle define on $\mathrm{PSL}(d, \mathbb{R}) \times \mathcal{F}_\Theta \to \mathfrak{a}_\Theta$. To construct it, we will use a projection p_Θ, whose definition is well-posed due to the following lemma:

Lemma 3.1.2. *Let W_Θ the subgroup of the Weyl group that fixes all elements of the Levi-Anosov subspace:*

$$W_\Theta = \bigcap_{\mathsf{a}_i \in \mathfrak{a}_\Theta} \mathrm{Stab}_W(\mathsf{a}_i).$$

Then there exists a unique projection $p_\Theta : \mathfrak{a} \to \mathfrak{a}_\Theta$ that is invariant under W_Θ.

Proof. Recall that a linear projection over some vector space V is a linear mapping $p : \mathbb{R}^d \to V$ such that $p^2 = p$, or equivalently the decomposition of V into ordered pairs (W, Z) of complementary subspaces, in which case $Z = \ker p, W = \mathrm{Im} p$. So it suffices to show that every W_Θ-invariant projection should satisfy $Z = \ker p$ and $\mathfrak{a}_\Theta = \mathrm{Im} p$, where

$$Z = \mathrm{span}\left\{e_i - e_{i+1} : i \notin \Theta\right\},$$

and $e_1, \cdots, e_d$ are the standard basis of $\mathbb{R}^d$. Clearly $\dim Z = d - 1 - \sharp\Theta$ and $Z + \mathfrak{a}_\Theta = \mathfrak{a}$.

Using the definition of $\mathfrak{a}_\Theta$, we can calculate its dimension: $\dim \mathfrak{a}_\Theta = \sharp\Theta$. Recalling that W are the matrices that permute the coordinates of $\mathbb{R}^d$, we see that W_Θ is the subgroup of W that is generated by the permutations of the coordinates of the simple roots that are not in Θ:

$$W_\Theta = \langle\{(i, i + 1) - \text{permutation matrix} : i \notin \Theta\}\rangle$$

where the $(i, i+1)$-permutation matrix is the matrix that swaps the i-th and $(i+1)$-th coordinates of $\mathbb{R}^d$. Then every W_Θ-invariant projection p must satisfy $\mathrm{Im}(w-\mathrm{id}) \subseteq \ker p$ for $w \in W_\Theta$. Letting w be the $(i, i + 1)$ permutation matrix, we see that $e_i - e_{i+1} \in \ker p$ for all i such that $i \notin \Theta$, and thus $Z \leq \ker p$. Counting dimensions, this implies that $Z = \ker p$. □

Lemma 3.1.3 ([Qui02], Lemme 6.1)**.** *Let $p_\Theta : \mathfrak{a} \to \mathfrak{a}_\Theta$ be the unique projection that is invariant under W_Θ. Then the map $p_\Theta \circ b_\Pi$ factors through a unique map $b_\Theta : \mathrm{PSL}(d, \mathbb{R}) \times \mathcal{F}_\Theta \to \mathfrak{a}_\Theta$.*

Moreover, this map is a cocycle:

$$b_\Theta(gh, x) = b_\Theta(g, hx) + b_\Theta(h, x).$$

Lemma 3.1.4. *For $y \in \mathcal{F}_\Theta(\mathbb{R}^d)$ we have*

$$\omega_{\alpha_i}(b_\Theta(g, x)) = \log \frac{\|gv_1 \wedge \cdots gv_i\|}{\|v_1 \wedge \cdots v_i\|} \text{ for all } i \in \Theta$$

for any basis $v_1, \ldots, v_i$ of $x^i \in \mathcal{G}_i(\mathbb{R}^d)$, where $\|\cdot\|$ denotes the norm on $\bigwedge^i \mathbb{R}^d$ induced by the euclidean inner product on $\mathbb{R}^d$, i.e. $\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k\rangle := \det(\langle v_i, w_j\rangle)$.

Proof. We begin by showing the wanted property for the case $\Theta = \Pi$. We let $g \in \mathrm{SL}(d, \mathbb{R})$, $y \in \mathcal{F}_\Pi(\mathbb{R}^d)$ and consider some $k \in K$ such that $y^i = k \cdot (\mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_i)$. Then $gk = k' e^{b_\Pi(g,y)} n$ for some $k' \in K, n \in N$, and

$$\begin{aligned}
\|gy^i\| &= \|e^{b_\Pi(g,y)} n_1 \wedge \cdots \wedge e^{b_\Pi(g,y)} n_1\| \\
&= \|e^{a_1(b_\Pi(g,y))} e_1 \wedge \left(e^{a_1(b_\Pi(g,y))} n_{1,2} e_1 + e^{a_2(b_\Pi(g,y))} e_2\right) \wedge \cdots \| \\
&= \|e^{a_1(b_\Pi(g,y))} e_1 \wedge e^{a_2(b_\Pi(g,y))} e_2 \wedge \cdots \wedge e^{a_i(b_\Pi(g,y))} e_i\| = e^{\omega_{\mathsf{a}_i}(b_\Pi(g,y))},
\end{aligned}$$

where we denoted by $n_{i,j}$ the (i,j)-entry of n and with $n_i = (n_{i,1}, \cdots, n_{i,d})^t$ its i-th column.

Returning to the general case $\Theta \subseteq \Pi$, we note that p_Θ does not change the value of the weights corresponding to the roots in Θ, i.e. $\omega_{\mathsf{a}_i} \circ p_\Theta = \omega_{\mathsf{a}_i}$ for $\mathsf{a}_i \in \Theta$, making the following diagram commute:

$$\begin{tikzcd}
\mathrm{PSL}(d, \mathbb{R}) \times \mathcal{F}_\Theta(\mathbb{R}^d) \arrow[r, "b_\Pi"] \arrow[d, hook] & \mathfrak{a} \arrow[r, "\omega_{\mathsf{a}_i}"] \arrow[d, "p_\Theta"] & \mathbb{R} \\
\mathrm{PSL}(d, \mathbb{R}) \times \mathcal{F}_\Theta(\mathbb{R}^d) \arrow[r, "b_\Theta"] & \mathfrak{a}_\Theta \arrow[ur, "\omega_{\mathsf{a}_i}"'] &
\end{tikzcd}$$

Since p_Θ is a projection, this follows from the fact that $\ker p_\Theta \subseteq \ker \omega_{\mathsf{a}_i}$.

Letting now $y \in \mathcal{F}_\Theta(\mathbb{R}^d)$, we can consider any of its lifts $y' \in \mathcal{F}_\Pi(\mathbb{R}^d)$. Then

$$\frac{\|gv_1 \wedge \cdots \wedge gv_i\|}{\|v_1 \wedge \cdots \wedge v_i\|} = \omega_{\mathsf{a}_i}(b_\Pi(g, y')) = \omega_{\mathsf{a}_i}(p_\Theta b_\Pi(g, y')) = \omega_{\mathsf{a}_i}(b_\Theta(g, y)),$$

for $y^i = \mathbb{R}v_1 \oplus \cdots \oplus \mathbb{R}v_i$, which is the desired result. □

Definition 3.1.5. We define

$$\Lambda^k : \mathrm{SL}(d, \mathbb{R}) \to \mathrm{SL}(\Lambda^k \mathbb{R}^d), \quad \Lambda^k : \mathcal{G}_k(\mathbb{R}^d) \to \mathbb{P}(\Lambda^k \mathbb{R}^d)$$

as

$$\Lambda^k(g)(v_1 \wedge \cdots \wedge v_k) = gv_1 \wedge \cdots \wedge gv_k, \quad \Lambda^k(\mathbb{R}v_1 \oplus \cdots \mathbb{R}v_k) = v_1 \wedge \cdots \wedge v_k$$

Lemma 3.1.6. *Let $g \in \mathrm{SL}(d, \mathbb{R}), \alpha > 0$ and $k \in [\![2, d]\!]$.*

(i) $\omega_{\mathsf{a}_1}(a(\Lambda^k g)) = \omega_{\mathsf{a}_k}(a(g))$ and $\omega_{\mathsf{a}_1}(b(\Lambda^k g, \Lambda^k y)) = \omega_{\mathsf{a}_k}(b(g, y))$.

(ii) There exists some $\alpha' > 0$ independent of g such that $\Lambda^k B_{\mathsf{a}_k,\alpha}(\Lambda^k g) \subseteq B_{\mathsf{a}_1,\alpha'}(\Lambda^k g)$.

Proof. (i) Follows from the definitions of the fundamental weights and the Cartan projection.

(ii) Let $g = k_g e^{a(g)} l_g$ be the Cartan decomposition of g. Then using the definitions of the respective subspaces:

$$\begin{aligned}
U_{d-k}(g^{-1} l_g^{-1}) &= \mathbb{R}e_{k+1} \oplus \cdots \oplus \mathbb{R}e_d \\
x_0 := U_{d-1}(\Lambda^k g^{-1} l_g^{-1}) &= \bigoplus_{\substack{i_1 < \cdots < i_k \\ (i_1, \cdots, i_k) \neq (1, \cdots, k)}} \mathbb{R}e_{i_1} \oplus \cdots \oplus \mathbb{R}e_{i_k}
\end{aligned}$$

The first equality implies that

$$y \in B_{\mathsf{a}_k,\alpha}(g) \Leftrightarrow l_g y \in B_{\mathsf{a}_k,\alpha}(g l_g^{-1}) = B_{\mathsf{a}_k,\alpha}(\mathrm{Id}),$$

so for every $y \in B_{\mathsf{a}_k,\alpha}(g)$ we have that

$$l_g y = lU_k(\mathrm{Id}) \text{ for some } l \in L$$

where

$$L = \{l \in \mathrm{SO}(d,\mathbb{R}) : lU_k(\mathrm{Id}) \in B_{\mathsf{a}_k,\alpha}(\mathrm{Id})\}.$$

Note that L is compact, being a closed subset of a compact group. Moreover, the fact that $\Lambda^k(y) \notin \mathrm{U}_{d-1}(\Lambda^k g^{-1})$ implies that

$$0 < \angle(\Lambda^k(y), \mathrm{U}_{d-1}(\Lambda^k g^{-1})) = \angle(\Lambda^k(l_g y), \mathrm{U}_{d-1}(\Lambda^k g^{-1} l_g^{-1})) = \angle(\Lambda^k(l)\,\mathrm{U}_k(\mathrm{Id}), x_0)$$

The right-hand side is in the image of the compact set L under a continuous map, so it is bounded below by a positive number $\alpha' > 0$.

(iii) Follows from the definition of the Cartan projection and the Busemann cocycle.

□

Definition 3.1.7. For a discrete subgroup $H < \mathrm{PSL}(d,\mathbb{R}), \phi \in (\alpha_\Theta)^*$, an (H,ϕ)-Patterson–Sullivan measure on $\mathcal{F}_\Theta$ is a finite Radon measure μ such that for every $h \in H$

$$\frac{\mathrm{d}h_*\mu}{\mathrm{d}\mu}(x) = e^{-\phi(b_\Theta(h^{-1},x))}, \text{ for all } x \in \mathcal{F}_\Theta(\mathbb{R}^d).$$

Lemma 3.1.8. *Let $\alpha > 0, \Theta \subseteq \Pi$. There exists $K = K(\alpha) > 0$ such that for each $g \in \mathrm{SL}(d,\mathbb{R}), \mathsf{a_i} \in \Theta, \mathsf{y} \in \mathsf{B}_{\Theta,\alpha}(\mathsf{g}), \phi \in \mathfrak{a}_\Theta$*

$$|\phi(a(g) - b(g,y))| \leq K.$$

Proof. We begin by noting that it suffices to consider the case where $\phi = \mathsf{a}_k$ for $\mathsf{a}_k \in \Theta$, since $\{\omega_{\mathsf{a}_i}\}_{\mathsf{a}_i \in \Theta}$ is a basis for $\mathfrak{a}^*_\Theta$.

Considering the case where $k = 1$, we recall that the first component of the Cartan projection coincides with the spectral norm of g, i.e.

$$a_1(g) = \log \sup_{v \neq 0} \frac{\|gv\|}{\|v\|} = \log \|g k_2^{-1} e_1\|$$

where $g = k_1 e^{a(g)} k_2$ is the Cartan decomposition of g. Let $v = v_1 k_2^{-1} e_1 + \cdots + v_d k_2^{-1} e_d \in \mathbb{R}^d$ be such that $\|v\| = 1$ and $y = \mathbb{R}v$, we have

$$\begin{aligned}
|\omega_{\mathsf{a}_1}(a(g) - b(g,y))| &= |\log \|g k_2^{-1} e_1\| - \log \|gv\|| = \\
&= |\log |e^{a_1(g)}| - \log \|e^{a_1(g)} v_1 k_1 e_1 + \cdots + e^{a_d(g)} v_d k_1 e_d\|| = \\
&= \left|\log \left\| v_1 k_1 e_1 + e^{-\mathsf{a}_{12}(g)} v_2 k_1 e_2 + \cdots + e^{-\mathsf{a}_{1d}(g)} v_d k_1 e_d \right\|\right| \leq \\
&\leq |\log |v_1|| = \left|\log \sin(\angle(v, U_{d-1}(g^{-1})))\right| \leq |\log \sin \alpha|.
\end{aligned}$$

For the case where $\Theta = \{\mathsf{a}_k\}$, we consider $\alpha' > 0$ such that

$$\Lambda^k(B_{\mathsf{a}_k,\alpha}(g)) \subseteq B_{\mathsf{a}_k,\alpha'}(\Lambda^k g)$$

Then using the case $k = 1$ we have that

$$|\omega_{\mathsf{a}_k}(a(g) - b(g,y))| = |\omega_{\mathsf{a}_1}(a(\Lambda^k g) - b(\Lambda^k g, \Lambda^k y))| \leq |\log \sin \alpha'|.$$

□

3.2 Strategy of the proof

Denoting with $d_\rho = \dim_{\mathcal{H}} \xi^1_\rho(\partial\Gamma)$ the Hausdorff dimension of the limit set, we shall outline how to obtain its lower bound by the Falconer dimension

$$d_\rho \geq h_\rho(F).$$

First we recall that $F_s(a) = \max\{\Psi^p_s(a) : p \in [\![2,d]\!]\}$, so the lower bound will follow once we have shown that

$$d_\rho \geq h_\rho(\Psi^{d_\rho+1}),$$

since $h_\rho(F) \leq h_\rho(\Psi^{d_\rho+1})$.

But in Lemma 1.5.2, we showed that $\frac{s}{d_\rho} J^u_{d_\rho} \leq \Psi^{d_\rho+1}_{s+d_\rho}$, so the above bound will follow as soon as we have shown that

$$h_\rho(J_{d_\rho}) \leq 1. \tag{LB}$$

which will be obtained using the method of Patterson–Sullivan–Quint. To apply the latter, apart from showing the existence of a $(J^u_{d_\rho}, \rho)$-Patterson–Sullivan measure μ, we will also need to show that there exists a collection of open sets $\{U_\gamma\}_\gamma \in \Gamma$ such that

$$\mu(U_\gamma) \sim e^{-J^u_{d_\rho}(a(\rho(\gamma)))} \text{ and } M \overset{\text{def}}{=} \sup_{n\in\mathbb{N}} \max\left\{\sharp A : A \subseteq \Gamma_n, \bigcap_{\gamma\in A} U_\gamma \neq \emptyset\right\} < \infty \tag{MP}$$

where $\Gamma_n = \{\gamma \in \Gamma : |\gamma| = n\}$. For the proof of the existence of a $(J^u_{d_\rho}, \rho)$-Patterson–Sullivan measure that satisfies the property (MP) we refer to Section 3.3, noting that the Zariski-density assumption is necessary only for the equivalence appearing on the left hand side of Equation (MP). Assuming that for the time being, below we outline the Patterson–Sullivan-Quint method of obtaining Equation (LB).

Indeed, we first obtain the bound uniform in n:

$$\sum_{\gamma\in\Gamma_n} e^{-J^u_{d_\rho}(a(\rho(\gamma)))} \lesssim \sum_{\gamma\in\Gamma_n} \mu(U_\gamma) \leq \frac{1}{M}\mu(\mathcal{F}_\Theta(\mathbb{R}^d)) < \infty$$

along with the bound implied by the Anosov property of ρ:

$$J_{d_\rho}(a(\rho(\gamma))) \geq \mathsf{a}_{12}(a(\rho(\gamma))) \geq C|\gamma| - b$$

to conclude that

$$\sum_{\gamma\in\Gamma} e^{-(s+1)J^u_{d_\rho}(a(\rho(\gamma)))} = \sum_{n\geq 0}\sum_{\gamma\in\Gamma_n} e^{-sJ^u_{d_\rho}(a(\rho(\gamma)))} e^{-J^u_{d_\rho}(a(\rho(\gamma)))} \lesssim \sum_{n\geq 0} e^{-s(Cn-b)} < \infty$$

which holds for any $s > 0$, and thus Equation (LB) holds.

3.3 Existence of Patterson–Sullivan measure

Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a projective Anosov representation such that $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold of dimension d_ρ. A naive approach would be to attempt to construct a Patterson–Sullivan measure over the limit set $\xi^1_\rho(\partial\Gamma)$. However, complications can arise in the Lipschitz regularity case, because the limit set does not admit a well-defined tangent space at every point.

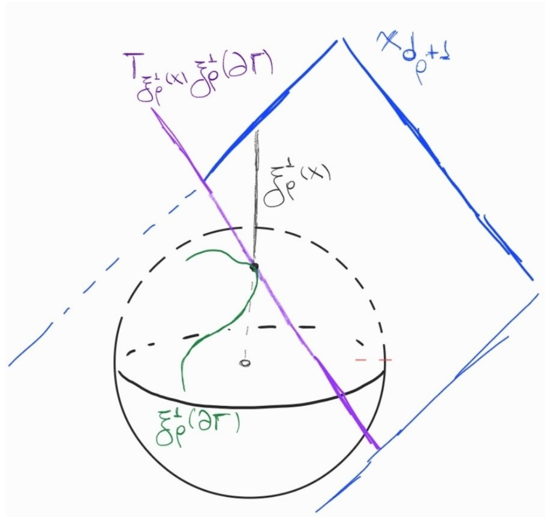


Figure 3.1: Tangent space to the limit set.

To overcome this difficulty, proceed as follows. To almost every $\xi \in \xi^1_\rho(\partial\Gamma)$, we have a well-defined tangent space $T_\xi \xi^1_\rho(\partial\Gamma)$. The latter, being a subspace of the tangent space $T_\xi(\mathbb{P}(\mathbb{R}^d))$ to the projective space $\mathbb{P}(\mathbb{R}^d)$, can be seen as the projectivisation of a subspace of $\mathbb{R}^d$. In this way we can associate to almost every $\xi \in \xi^1_\rho(\partial\Gamma)$ an element of the flag space $\mathcal{F}_{1,d_\rho+1}(\mathbb{R}^d)$, thus defining a measurable section ζ_ρ of $\pi_{\mathsf{a}_1} : \mathcal{F}_{1,d_\rho+1}(\mathbb{R}^d) \to \mathbb{P}(\mathbb{R}^d)$. Over each point $\zeta_\rho(\xi)$ we can define a density $|\Omega_{\zeta_\rho(\xi)}|$ over $T_\xi \xi^1_\rho(\partial\Gamma)$, and thus obtain an almost everywhere defined density over the limit set $\xi^1_\rho(\partial\Gamma)$:

$$\xi \mapsto \left|\Omega_{\zeta_\rho(\xi)}\right|.$$

The following is [PSW23, Proposition 6.4]:

Proposition 3.3.1. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a projective Anosov representation, and assume that $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold of dimension d_ρ. Then there exists a $(\rho(\Gamma), J^u_{d_\rho})$-Patterson–Sullivan measure on $\mathcal{F}_{1,d_\rho+1}(\mathbb{R}^d)$, which is supported on the image of a ρ-equivariant measurable section*

$$\zeta_\rho : \xi^1_\rho(\partial\Gamma) \to \mathcal{F}_{1,d_\rho+1}(\mathbb{R}^d)$$

of the projection $\mathrm{pr}_1 : \mathcal{F}_{1,d_\rho+1}(\mathbb{R}^d) \to \mathbb{P}(\mathbb{R}^d)$.

Proof. By Rademacher's theorem, $\xi^1_\rho(\partial\Gamma)$ has a well-defined Lebesgue measure class, and Lebesgue-almost every $\xi^1_\rho(x) \in \xi^1_\rho(\partial\Gamma)$ admits a well-defined tangent space $T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma)$. Considering such a $\xi^1_\rho(x)$ we let

$$\pi : \hom(\xi^1_\rho(x), \mathbb{R}^d) \to \hom(\xi^1_\rho(x), \mathbb{R}^d/\xi^1_\rho(x)) \simeq T_{\xi^1_\rho(x)}\mathbb{P}(\mathbb{R}^d),$$

and

$$x^{d_\rho+1} = \pi^{-1}(T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma))\xi^1_\rho(x) \in \mathcal{G}_{d_\rho+1}(\mathbb{R}^d).$$

The calculation of the dimension follows from the rank-nullity theorem on π restricted to $\pi^{-1}(T_{\xi^1_\rho(x)}\xi^1_\rho(\partial_\Gamma))$ and the fact that $\xi^1_\rho(\partial\Gamma) \leq \pi^{-1}(T_{\xi^1_\rho(x)}\xi^1_\rho(\partial_\Gamma))$. Then Proposition A.0.3 implies that

$$T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma) = \hom(\xi^1_\rho(x), x^{d_\rho+1}/\xi^1_\rho(x)),$$

In this notation, we shall define (Lebesgue-almost everywhere) the map

$$\zeta_\rho : \xi^1_\rho(\partial\Gamma) \to \mathcal{F}_{1,d_\rho+1}(\mathbb{R}^d), \quad \zeta_\rho(\xi^1_\rho(x)) = (\xi^1_\rho(x), x^{d_\rho+1}).$$

which is a section in the sense that the projection $\mathrm{pr}_1 : \mathcal{F}_{1,d_\rho+1} \to \mathbb{P}(\mathbb{R}^d)$ to the first coordinate satisfies $\mathrm{pr}_1 \circ \zeta_\rho = \mathrm{Id}$. Moreover, it is $\rho(\Gamma)$ equivariant, since for $\gamma \in \Gamma$ we have

$$\begin{aligned}\rho(\gamma)\zeta_\rho(\xi^1_\rho(x)) &= \Big(\rho(\gamma)\xi^1_\rho(x), \rho(\gamma)\pi^{-1}(T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma))\Big) = \\ &= \Big(\rho(\gamma)\xi^1_\rho(x), \rho(\gamma)\pi^{-1}(T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma))\rho(\gamma)^{-1}\rho(\gamma)\xi^1_\rho(x)\Big) = \\ &= \Big(\rho(\gamma)\xi^1_\rho(x), \pi^{-1}(\rho(\gamma)T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma))\Big) = \\ &= \Big(\rho(\gamma)\xi^1_\rho(x), \pi^{-1}(T_{\rho(\gamma)\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma))\Big) = \\ &= \zeta_\rho(\rho(\gamma)\xi^1_\rho(x)) = \zeta_\rho(\xi^1_\rho(\gamma x)).\end{aligned}$$

We now define the non-negative density on $\xi^1_\rho(\partial\Gamma)$

$$\mu_{\xi^1_\rho(x)} = |\Omega_{\zeta_\rho(\xi^1_\rho(x))}|,$$

where $\left|\Omega_{\xi^1_\rho(x), x^{d_\rho+1}}\right| : \left(T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma)\right)^{d_\rho} \to \mathbb{R}_{\geq 0}$ is a density over $T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma)$ defined as

$$|\Omega_{\xi^1_\rho(x), x^{d_\rho+1}}|(\phi_1, \dots, \phi_{d_\rho}) = \frac{\|v \wedge \tilde{\phi}_1(v) \wedge \dots \wedge \tilde{\phi}_{d_\rho}(v)\|}{\|v\|^{d_\rho+1}}$$

for any $v \in \xi^1_\rho(x) - \{0\}$, $\phi_1, \cdots, \phi_{d_\rho} \in T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma) = \hom(\xi^1_\rho(x), x^{d_\rho+1}/\xi^1_\rho(x))$ and $\tilde{\phi}_1, \dots \tilde{\phi}_{d_\rho} \in \hom(\xi^1_\rho(x), x^{d_\rho+1})$ such that $\phi_i = \tilde{\phi}_i + \hom(\xi^1_\rho(x), \xi^1_\rho(x))$. Here $\|\cdot\|$ denotes the norm on $\bigwedge^{d_\rho+1}\mathbb{R}^d$ induced by the standard euclidean inner product.

This density satisfies

$$\frac{\mathrm{d}\rho(\gamma)_*\mu}{\mathrm{d}\mu}(\xi) = \frac{\mathrm{d}(\rho(\gamma)^{-1})^*\mu}{\mathrm{d}\mu}(\xi) = e^{-J^u_{d_\rho+1}(b_\Theta(\rho(\gamma)^{-1}, \zeta_\rho(\xi)))},$$

where the term on the left-hand side involves measures, the term between the two equalities involves densities, and $\Theta = \{1, d_\rho+1\}$. The first equality holds for any density and its respective measure, while for the second one we consider $\phi_1, \dots, \phi_{d_\rho} \in T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma)$ and calculate:

$$\begin{aligned}&(\rho(\gamma)^*\mu)_{\xi^1_\rho(x)}(\phi_1, \dots, \phi_{d_\rho}) \\ &= \mu_{\rho(\gamma)\xi^1_\rho(x)}(\rho(\gamma)\phi_1\rho(\gamma)^{-1}, \dots, \rho(\gamma)\phi_{d_\rho}\rho(\gamma)^{-1}) \\ &= \frac{\|\rho(\gamma)\xi^1_\rho(x) \wedge \rho(\gamma)\phi_1(\xi^1_\rho(x)) \wedge \dots \wedge \rho(\gamma)\phi_{d_\rho}(\xi^1_\rho(x))\|}{\|\rho(\gamma)\xi^1_\rho(x)\|^{d_\rho+1}} \\ &= \frac{\|\rho(\gamma)\xi^1_\rho(x) \wedge \rho(\gamma)\phi_1(\xi^1_\rho(x)) \wedge \dots \wedge \rho(\gamma)\phi_{d_\rho}(\xi^1_\rho(x))\|}{\|\xi^1_\rho(x) \wedge \phi_1(\xi^1_\rho(x)) \wedge \dots \wedge \phi_{d_\rho}(\xi^1_\rho(x))\|} \\ &\quad \cdot \frac{\|\xi^1_\rho(x) \wedge \phi_1(\xi^1_\rho(x)) \wedge \dots \wedge \phi_{d_\rho}(\xi^1_\rho(x))\|}{\|\rho(\gamma)\xi^1_\rho(x)\|^{d_\rho+1}} \cdot \frac{\|\rho(\gamma)\xi^1_\rho(x)\|^{d_\rho+1}}{\|\xi^1_\rho(x)\|^{d_\rho+1}} \\ &= e^{\omega_{d_\rho}(b_\Theta(\rho(\gamma), \zeta_\rho(\xi^1_\rho(x))))} \cdot \mu_{\xi^1_\rho(x)}(\phi_1, \dots, \phi_{d_\rho}) \\ &\quad \cdot e^{-(d_\rho+1)\omega_1(b_\Theta(\rho(\gamma), \zeta_\rho(\xi^1_\rho(x))))} \\ &= e^{-J^u_{d_\rho}(b_\Theta(\rho(\gamma)^{-1}, \zeta_\rho(\xi^1_\rho(x))))}\mu_{\xi^1_\rho(x)}(\phi_1, \dots, \phi_{d_\rho}).\end{aligned}$$

Finally, we let $\nu = \zeta_{\rho*}\mu$, which is the wanted Patterson–Sullivan measure on $\mathcal{F}_{1,d_\rho+1}(\mathbb{R}^d)$, since for $f \in C_c(\mathcal{F}_{1,d_\rho+1}(\mathbb{R}^d))$:

$$\begin{aligned}\int_{\mathcal{F}_{1,d_\rho+1}(\mathbb{R}^d)} f\,\mathrm{d}(\rho(\gamma)_*\zeta_{\rho*}\mu) &= \int_{\xi^1_\rho(\partial\Gamma)} f\circ\rho(\gamma)\circ\zeta_\rho\,\mathrm{d}\mu = \int_{\xi^1_\rho(\partial\Gamma)} f\circ\zeta_\rho\circ\rho(\gamma)\,\mathrm{d}\mu = \\ &= \int_{\xi^1_\rho(\partial\Gamma)} f\circ\zeta_\rho(\xi^1_\rho(x))e^{-J^u_{d_\rho}(b_\Theta(\rho(\gamma)^{-1},\zeta_\rho(\xi^1_\rho(x)))}\,\mathrm{d}\mu(\xi^1_\rho(x)) = \\ &= \int_{\mathcal{F}_{1,d_\rho+1}(\mathbb{R}^d)} f(y)e^{-J^u_{d_\rho}(b_\Theta(\rho(\gamma)^{-1},y)}\,\mathrm{d}(\zeta_{\rho*}\mu)(y)\end{aligned}$$

□

Our next goal is to find a collection of sets $(U_\gamma)_{\gamma\in\Gamma}$ such that $e^{-J^u_{d_\rho}(a(\rho(\gamma)))} \lesssim \mu(U_\gamma)$ and whose sets do not intersect a lot. In the next lemma we define this collection and show its intersection property. It should be regarded as an analog of Lemma 2.1.1 and Proposition 2.1.6 to arbitrary flag varieties, and relies only on the Anosov property of ρ, and the fact that ζ_ρ is a section of $\pi_{\mathsf{a}_1} : \mathcal{F}_{1,d_{\rho,\Gamma}+1}(\mathbb{R}^d) \to \mathbb{P}(\mathbb{R}^d)$.

Lemma 3.3.2. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be projective Anosov and $\Theta = \{\mathsf{a}_1, \mathsf{a}_{d_\rho}\} \subseteq \Pi$. Then for $\alpha > 0$ small enough, there exists some $C, T_0 > 0$ such that for all $T \geq T_0$ the family*

$$\mathcal{U}_T = \{\rho(\gamma)B_{\Theta,\alpha}(\rho(\gamma)) : |\gamma| = T\}$$

is a collection of open subsets of $\mathcal{F}_\Theta(\mathbb{R}^d)$ such that every $\zeta_\rho(\xi^1_\rho(x))$ is contained in at most C many sets in $\mathcal{U}_T$.

Proof. Suppose $\zeta_\rho(\xi^1_\rho(x)) \in \rho(\gamma)B_{\Theta,\alpha}(\rho(\gamma)) \cap \rho(\eta)B_{\Theta,\alpha}(\rho(\eta))$ for some $\gamma, \eta \in \Gamma_T$. Then $\xi^1_\rho(x) \in \rho(\gamma)B_{\mathsf{a}_1,\alpha}(\rho(\gamma)) \cap \rho(\eta)B_{\mathsf{a}_1,\alpha}(\rho(\eta))$. But using Proposition 2.1.6 we have that

$$\begin{aligned} x &\in (\xi^1_\rho)^{-1}(\rho(\gamma)B_{\mathsf{a}_1,\alpha}(\rho(\gamma)) \cap \rho(\eta)B_{\mathsf{a}_1,\alpha}(\rho(\eta))) = \\ &= \gamma(\xi^1_\rho)^{-1}(B_{\mathsf{a}_1,\alpha}(\rho(\gamma))) \cap \eta(\xi^1_\rho)^{-1}(B_{\mathsf{a}_1,\alpha}(\rho(\eta))) \subseteq \\ &\subseteq \gamma C_{c_0,c_1}(\gamma) \cap \eta C_{c_0,c_1}(\eta).\end{aligned}$$

Thus x is represented by (c_0, c_1)-quasi-geodesic rays $(a_j)_0^\infty, (b_j)_0^\infty$, that start from e and pass from γ and η respectively. By Morse's lemma, we know that there exists some geodesic ray starting from e and some $A > 0$ depending only on c_0, c_1 and the hyperbolicity constant of Γ such that the Hausdorff distance of the geodesic ray to each of the quasi-geodesics is bounded by A. Let ϵ_0, ϵ_1 be two points on the geodesic ray such that $\mathrm{d}(\gamma, \epsilon_0), \mathrm{d}(\eta, \epsilon_1) \leq A$. Then we have that

$$\begin{aligned}\mathrm{d}(\gamma,\eta) &\leq \mathrm{d}(\gamma,\epsilon_0) + \mathrm{d}(\epsilon_0,\epsilon_1) + \mathrm{d}(\epsilon_1,\eta) \leq 2A + ||\epsilon_0| - |\epsilon_1|| \leq \\ &\leq 2A + ||\epsilon_0| - |\gamma|| + ||\gamma| - |\eta|| + ||\epsilon_1| - |\eta|| \leq 4A.\end{aligned}$$

In particular, any γ' such that $\zeta_\rho(\xi^1_\rho(\gamma')) \in \xi^1_\rho(\rho(\gamma')B_{\Theta,\alpha}(\rho(\gamma')))$, will lie in a ball of radius $4A$ around γ. Since Γ is finitely generated, there exists some $C > 0$ such that the ball of radius $4A$ around γ contains at most C elements of Γ. □

To show the equivalence $e^{-J^u_{d_\rho}(a(\rho(\gamma)))} \sim \mu(\rho(\gamma)B_{\Theta,\alpha}(\rho(\gamma)))$, aside from the equivariance property of μ, we will also need for it to "see the whole space of flags $\mathcal{F}_\Theta(\mathbb{R}^d)$" and not just part of it. To make this more concrete, we introduce the following two definitions from [PSW23].

Definition 3.3.3. Let $\Theta \subseteq \Pi$. We define the annihilator of an element $y \in \mathcal{F}_{i\Theta}(\mathbb{R}^d)$ is the set of partial flags in $\mathcal{F}_\Theta(\mathbb{R}^d)$ that are not transverse to y, that is:

$$\mathrm{Ann}(y) = \left\{x \in \mathcal{F}_\Theta(\mathbb{R}^d) : x^\theta \cap y^{d-\theta} \neq 0 \text{ for some } \theta \in \Theta\right\}.$$

Definition 3.3.4. Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a linear representation, $\Theta \subseteq \Pi$ and μ a $(\rho(\Gamma), \phi)$-Patterson–Sullivan measure over $\mathcal{F}_\Theta(\mathbb{R}^d)$. We say that ρ is μ-irreducible if there is no element in $\mathcal{F}_{i\Theta}(\mathbb{R}^d)$, whose annihilator is of full measure, i.e. for all $y \in \mathcal{F}_{i\Theta}(\mathbb{R}^d)$:

$$\mu(\mathrm{Ann}(y)) < \mu(\mathcal{F}_\Theta(\mathbb{R}^d)).$$

The following example will be also the case that is of interest to us in this chapter.

Example 3.3.5. Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a linear representation and $\Theta \subseteq \Pi$. If $\rho(\Gamma)$ is Zariski-dense in $\mathrm{SL}(d, \mathbb{R})$, then ρ is μ-irreducible for any $\rho(\Gamma)$-quasi-invariant measure μ on $\mathcal{F}_\Theta(\mathbb{R}^d)$, and in particular for any $(\rho(\Gamma), \phi)$-Patterson–Sullivan measure.

Proof. Since the anhiliator $\mathrm{Ann}(y)$ of a flag $y \in \mathcal{F}_{\iota\Theta}$ is a proper subvariety of the flag variety $\mathcal{F}_\Theta(\mathbb{R}^d)$, the result will follow as soon as we show that $\operatorname{supp}\mu$ is Zariski-dense. Indeed, quasi-invariance implies that for all $\rho(\gamma) \in \rho(\Gamma)$

$$\operatorname{supp}\mu = \operatorname{supp}\rho(\gamma)_*\mu = \rho(\gamma)\operatorname{supp}\mu,$$

and thus the support of μ is $\rho(\Gamma)$-invariant: $\rho(\Gamma)\operatorname{supp}\mu = \operatorname{supp}\mu$. Taking the Zariski-closure of both sets, we see that the Zariski-closure of $\operatorname{supp}\mu$ is in fact $\mathrm{SL}(d, \mathbb{R})$-invariant and hence contains the whole space $\mathcal{F}_\Theta(\mathbb{R}^d)$ because the action of $\mathrm{SL}(d, \mathbb{R})$ on the latter is transitive. $\square$

The following proposition shows the reason that we introduce the concept of μ-irredubility. It allows us to bound the measure of bassins of attraction uniformly from below.

Proposition 3.3.6. *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a linear representation, $\Theta \subseteq \Pi$ and μ a $(\rho(\Gamma), \phi)$-Patterson–Sullivan measure over $\mathcal{F}_\Theta(\mathbb{R}^d)$. If ρ is μ-irreducible, then there exist $\alpha, \kappa > 0$ such that $\mu(B_{\Theta,\alpha}(\rho(\gamma))) \geq k$ for all $\gamma \in \Gamma$.*

Proof. If this were not the case, then there would exist a sequence $\alpha_n \searrow 0$ and $\gamma_n \in \Gamma$ such that

$$\mu(B_{\Theta,\alpha_n}(\rho(\gamma))) \leq \frac{1}{n}.$$

Due to the compactness of $\mathcal{F}_\Theta(\mathbb{R}^d)$, up to considering a subsequence, we may assume that the repelling flags or $\rho(\gamma_n)$ converge to some $\xi \in \mathcal{F}_\Theta(\mathbb{R}^d)$:

$$(U_{d-i}(\rho(\gamma_n)^{-1}))_{\mathsf{a}_i \in \Theta} \to \xi$$

In that case, the complements $B^c_{\Theta,\alpha_n}(\rho(\gamma_n))$ will converge to the annihilator of ξ, in the sense:

$$\limsup_n B^c_{\Theta,\alpha_n}(\rho(\gamma_n)) \subseteq \mathrm{Ann}(\xi).$$

Indeed, let $y \in \limsup_n B^c_{\Theta,\alpha_n}(\rho(\gamma_n))$ and consider a subsequence k_n such that $y \in B^c_{\Theta,\alpha_n}(\rho(\gamma_{k_n}))$. By the very definition of $B_{\Theta,\alpha_n}(\rho(\gamma_n))$, there exists some p such that up to considering a subsequence of k_n,

$$\angle(y^p, U_{d-p}(\rho(\gamma_n)^{-1})) \leq \alpha_n$$

holds. Taking the limit as $n \to \infty$, we have that $y^p \cap \xi^{d-p} \neq 0$ and hence $y \in \mathrm{Ann}(\xi)$.

Using a measure-theoretic arguement we conclude that $\mathrm{Ann}(\xi)$ is of full measure

$$\mu(\mathrm{Ann}(\xi)) \geq \mu(\limsup_n B^c_{\Theta,\alpha_n}(\rho(\gamma_{k_n}))) \geq \limsup_n \mu(B^c_{\Theta,\alpha_n}(\rho(\gamma_{k_n}))) = \mu(\mathcal{F}_\Theta(\mathbb{R}^d)),$$

which contradicts the μ-irreducibility of ρ. $\square$

We are now ready to show the equivalence $\mu(\rho(\gamma)B_{\Theta,\alpha}(\rho(\gamma))) \simeq e^{-J^u_{d_\rho}(a(\rho(\gamma)))}$ when the representation μ-irreducible.

Lemma 3.3.7. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a representation and μ^ϕ be a $(\rho(\Gamma),\phi)$-Patterson–Sullivan measure. If $\rho(\Gamma)$ is μ-irreducible, then there exists some $\alpha_0 > 0$, such that for any $\alpha \in (0,\alpha_0)$, there is some $k = k(\alpha) > 0$ for which*

$$\frac{1}{k}e^{-\phi(a(\rho(\gamma)))} \leq \mu^\phi(\rho(\gamma)B_{\Theta,\alpha}(\rho(\gamma))) \leq ke^{-\phi(a(\rho(\gamma)))}$$

for all $\gamma \in \Gamma$.

Proof. Let $\alpha_0, k > 0$ be as in the remark preceeding the statement of the lemma. As noted in Lemma 3.1.8, there exists some $K = K(\alpha_0,\phi) > 0$ such that for any $\alpha \in (0,\alpha_0)$ and $y \in B_{\Theta,\alpha}(\rho(\gamma))$:

$$|\phi(a(\rho(\gamma)) - b(\rho(\gamma),y))| \leq K,$$

from which we obtain the upper bound

$$\mu^\phi(\rho(\gamma)B_{\Theta,\alpha}(\rho(\gamma))) = (\rho(\gamma^{-1})_*\mu^\phi)(B_{\Theta,\alpha}(\rho(\gamma))) = \int_{\mathcal{F}_\Theta(\mathbb{R}^d)} e^{-\phi(b(\rho(\gamma),y))}d\mu^\phi(y) \leq$$
$$\leq e^{-K}\mu^\phi(\mathcal{F}_\Theta(\mathbb{R}^d))e^{-\phi(a(\rho(\gamma)))}.$$

Similarly we obtain the lower bound. □

3.4 Proof of the main theorem

In this section we shall prove the main theorem, which we restate for the reader's convenience:

Theorem 3.4.1. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a Zariski-dense, projective Anosov representation such that $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold of $\mathbb{P}(\mathbb{R}^d)$ of dimension d_ρ. Then the dimension of the limit set $\xi^1_\rho(\partial\Gamma)$ equals the Falconer dimension of ρ:*

$$\dim \xi^1_\rho(\partial\Gamma) = h_\rho(F)$$

where $F_s(a) = \max\{\Psi^p_s(a) : p \in [\![2,d]\!]\}$ is the Falconer functional.

Proof. We have already seen in Lemma 2.2.1 that $d_\rho \leq h_\rho(F)$. For the opposite inequality, we merely need to piece together the results of the previous sections as outlined in Section 3.2. There we have seen that $h_\rho(F) \leq h_\rho(\Psi^{d_\rho+2})$ since $F_s \geq \Psi^{d_\rho+2}_s$, so may as well show that $h_\rho(\Psi^{d_\rho+2}) \leq d_\rho$, i.e. that

$$\sum_{\gamma\in\Gamma} e^{-\Psi^{d_\rho+2}_s(\rho(\gamma))} < \infty$$

for all $s \geq d_\rho$. This will follow as soon as we have shown that $h_\rho(J_{d_\rho}) \leq 1$, since $\Psi^{d_\rho+1}_s \geq \frac{s}{d_\rho}J^u_{d_\rho}$, as we saw in Lemma 1.5.2.

Using the Anosov property of ρ we have that

$$J_{d_\rho}(a(\rho(\gamma))) \geq \mathsf{a}_{12}(a(\rho(\gamma))) \geq C|\gamma| - b$$

for certain $C, b > 0$ which, when we break up the sum defining the critical exponent into the sum over the sets $\Gamma_T = \{\gamma \in \Gamma : |\gamma| = T\}$, give us:

$$\begin{aligned}\sum_{\gamma\in\Gamma} e^{-(s+1)J^u_{d_\rho}(a(\rho(\gamma)))} &= \sum_{T\geq 0}\sum_{\gamma\in\Gamma_T} e^{-sJ^u_{d_\rho}(a(\rho(\gamma)))} e^{-J^u_{d_\rho}(a(\rho(\gamma)))} = \\ &= \sum_{T\geq 0} e^{-sJ^u_{d_\rho}(a(\rho(\gamma)))} \sum_{\gamma\in\Gamma_T} e^{-J^u_{d_\rho}(a(\rho(\gamma)))} \leq \\ &\leq \sum_{T\geq 0} e^{-s(CT-b)} \sum_{\gamma\in\Gamma_T} e^{-J^u_{d_\rho}(a(\rho(\gamma)))}\end{aligned}$$

To obtain a bound on the inner sums that is uniform in T, we recall Proposition 3.3.1. There we saw that $\xi^1_\rho(\partial\Gamma)$ being a Lipschitz submanifold of $\mathbb{P}(\mathbb{R}^d)$ implies the existence of a $(\rho(\Gamma), J^u_{d_\rho})$-Patterson–Sullivan measure μ on $\zeta^1_\rho(\xi^1(\partial\Gamma)) \subseteq \mathcal{F}_{1,d_\rho+1}(\mathbb{R}^d)$. By Lemma 3.3.2 we have that for $\alpha > 0$ small enough, there exists some $M, T_0 > 0$ such that for all $T \geq T_0$ the family

$$\mathcal{U}_T = \{\rho(\gamma)B_{\alpha,\Theta}(\rho(\gamma)) : |\gamma| = T\}$$

is an open covering of $\zeta_\rho(\xi^1_\rho(\partial\Gamma))$ for which

$$\max\left\{\sharp A : A \subseteq \Gamma_T, \bigcap_{\gamma\in A}\rho(\gamma)B_{\alpha,\Theta}(\rho(\gamma)) \neq \emptyset\right\} \leq M.$$

But μ is $\rho(\Gamma)$-quasi-invariant which along with the Zariski-density of $\rho(\Gamma)$ implies that ρ is μ-irreducible, as we have seen in Example 3.3.5. Hence the bound in Lemma 3.3.7 applies and we have that

$$\begin{aligned}\sum_{\gamma\in\Gamma_T} e^{-J^u_{d_\rho}(a(\rho(\gamma)))} &\leq \sum_{\gamma\in\Gamma_T}\mu\left(\rho(\gamma)B_{\alpha,\Theta}(\rho(\gamma))\right) \leq \\ &\leq \frac{1}{M}\mu\left(\mathcal{F}_{1,d_\rho+1}(\mathbb{R}^d)\right) < \infty.\end{aligned}$$

□

Chapter 4

Dropping the Zariski-density assumption

Since many interesting examples of Anosov subgroups are not Zariski-dense, one could certainly argue that the Zariski-density assumption in Theorem 3.4.1 is not cheap. The reason that it is needed is that otherwise one can't guarantee that it is μ-irreducible for a Patterson–Sullivan measure μ. In fact, Zariski-density is not assumed and takes the place of strong irredubility in [PSW23], where the authors suggest the latter implies μ-irredubility through the following lemma which as we will see below is false.

Lemma 4.0.1 (Lemma 6.8 in [PSW23])**.** *Let Γ be a hyperbolic group and $\eta : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a strongly irreducible projective Anosov representation such that $\xi_\eta(\partial\Gamma)$ is homeomorphic to $\mathbb{S}^{d_\rho}$, and which admits a measurable η-equivariant section $\zeta : \partial\Gamma \to \mathcal{F}_{\{\mathsf{a}_1, \mathsf{a}_{d_\rho+1}\}}(\mathbb{R}^d)$. Then η is μ-irreducible for any $(\eta(\Gamma), \phi)$-Patterson–Sullivan measure μ on $\zeta(\partial\Gamma) \subseteq \mathcal{F}_{\{\mathsf{a}_1, \mathsf{a}_{d_\rho+1}\}}(\mathbb{R}^d)$.*

For convenience, we recall that a linear representation $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is strongly irreducible if there is no proper $\rho(\Gamma)$-invariant subspace of $\mathbb{R}^d$, and it is μ-irreducible if there is no element in $\mathcal{F}_{i\Theta}(\mathbb{R}^d)$, whose annihilator is of full measure.

More specifically, in the proof of the preceeding lemma, the authors argue by contradiction that if the result is not true, then there exists subspaces $W_0 \in \mathcal{G}_{d_\rho+1}$ and $V \in \mathcal{G}_{d-d_\rho-1}$ such that

$$\eta(\gamma)V \cap W_0 \neq 0$$

for all $\gamma \in \Gamma$. Indeed, if this were not the case, then there exist $(W_0, P_0) \in \mathcal{F}_{\mathsf{a}_{d-d_\rho-1}, \mathsf{a}_{d-1}}$ such that $\mathrm{Ann}(W_0, P_0)$ is of full μ measure. In fact, η-equivariance of ζ implies that one can choose W_0, P_0 such that $P_0 \in \xi_\eta^{d-1}(\partial\Gamma)$, meaning that for μ-almost every $(\xi_\eta^1(x), \zeta(x)^{d_\rho+1}) \in \zeta(\partial\Gamma)$, we have that $\zeta(x)^{d_\rho+1} \cap W_0 \neq 0$. Again using the η-equivariance of ζ, we have that for all $\gamma \in \Gamma$, the set $\left\{\zeta(x) \in \zeta(\partial\Gamma) : \zeta(x)^{d_\rho+1} \cap \gamma W_0 \neq 0\right\}$ is of full measure, which implies that their intersection is non-empty, and we can take $V = \zeta(x)^p$ for any $x \in \partial\Gamma$ such that $\zeta(x)$ lies in this intersection.

Finally to conlude, the authors assert that this contradicts strong irreducibility by using [Lab06, Proposition 10.3]:

Proposition 4.0.2 (Labourie)**.** *Let $G \leq \mathrm{SL}(d, \mathbb{R})$ be an algebraic group. If $C \in \mathcal{G}_k(\mathbb{R}^d), B \in \mathcal{G}_{d-k}(\mathbb{R}^d)$ such that*

$$gC \cap B \neq 0 \text{ for all } g \in G, \tag{4.1}$$

then the connected component of G containing the identity element is not irreducible.

However, Proposition 4.0.2 and consequently Lemma 4.0.1 are false, leaving room for further investigation. In particular, below we present a direct counterexample to Proposition 4.0.2, followed by a construction which shows that Equation (4.1) does not contradict strong irreducibility, and where the representation η is not μ-irreducible, thus disproving Lemma 4.0.1. Nevertheless, if we proceed to calculate the dimension of the respective limit set and the critical exponent of the Falconer functional, we notice that they coincide and hence do not provide a counterexample to the main result presented in [PSW23], which suggests that the result may still hold (at least for certain special cases).

4.1 Two counterexamples

We begin by providing a counterexample to Proposition 4.0.2. For this, we use the space $\mathbb{R}^{d,d}$, which we recall to be $\mathbb{R}^{2d}$ equipped with the standard bilinear form of signature (d,d): $\langle x,y\rangle = x_1y_1 + \cdots + x_dy_d - x_{d+1}y_{d+1} - \cdots x_{2d}y_{2d}$.

Example 4.1.1. Let $S \subseteq \mathcal{G}_d(\mathbb{R}^{2d})$ to be the set of maximal isotropic subspaces of $\mathbb{R}^{d,d}$. If d is odd, then for any subspaces V, W in the same component of S, we have

$$gV \cap W \neq 0 \text{ for all } g \in \mathrm{SO}(d,d).$$

Proof. We begin by parametrizing the set S by $\mathrm{O}(d)$. For completeness we note that this parametrization is independent of the parity of d.

$$\begin{array}{lcl} S & \leftrightarrow & \mathrm{O}(d) \\ V & \mapsto & \mathrm{pr}_2 \circ (\mathrm{pr}_{1|V})^{-1} \\ \Gamma(\phi) & \mapsfrom & \phi \end{array}$$

where we denote with $\mathrm{pr}_1, \mathrm{pr}_2 : \mathbb{R}^{d,d} \to \mathbb{R}^d$ the projections to the first and last d coordinates, and $\Gamma(\phi) = \{(x,\phi(x)) : x \in \mathbb{R}^d\}$ the graph of $\phi \in \mathrm{O}(d)$. The left-to-right direction of this correspondence is well-defined because all $V \in S$ are d-dimensional and transverse to $\mathbb{R}^d \oplus 0$ and $0 \oplus \mathbb{R}^d$.

For the next step, we will need the assumption that d is odd and we will show that any pair of spaces V, W in the same connected component of S are non-transverse. Indeed, note that $\phi, \psi \in \mathrm{O}(d) : \Gamma(\phi) \cap \Gamma(\psi) \neq 0$ if and only if $\psi \circ \phi^{-1}$ admits 1 as an eigenvalue. The latter is true when $\det \psi = \det \phi$ or equivalently when ϕ, ψ are in the same component of $\mathrm{O}(d)$.

Given this, the non-transversality of gV and W for every $g \in \mathrm{SO}(d,d)$ will follow as soon as we show that the action of $\mathrm{SO}(d,d)$ preserves the components of S. Since it is a semisimple group, the Cartan projection implies that it contracts to its maximal compact subgroup $S(\mathrm{O}(d) \times \mathrm{O}(d))$, which preserves the components of S because its action is given by $(g,h)\Gamma(\phi) = \Gamma(h\phi g^{-1})$. Hence $\mathrm{SO}(d,d)$ preserves the components of S as well. $\square$

After providing a counterexample to Proposition 4.0.2 as above, one could object that this does not necessarily imply that the result of Lemma 4.0.1 is false. For this, one would need to provide another counterexample for Lemma 4.0.1 as well, which is the goal of the next pages and is summarised in Proposition 4.1.2. Before giving the statement of the latter, we recall that a uniform lattice Γ of a locally compact group G is a discrete subgroup of G that is co-compact, i.e. G/Γ is compact.

Since we will be working with $\mathrm{SU}(2,1)$ and $\mathrm{SL}(su(2,1))$,we will use different symbols for their Cartan projections to avoid confusion. We will denote with $\mu : \mathrm{SU}(2,1) \to \mathfrak{a}^+$ and $a : \mathrm{SL}(\mathfrak{su}(2,1)) \to \mathfrak{a}^+$ the projections of $\mathrm{SU}(2,1)$ and $\mathrm{SL}(\mathfrak{su}(2,1))$ respectively.

Proposition 4.1.2. *Let $\Gamma \leq \mathrm{SU}(2,1)$ be a uniform lattice and $\eta : \Gamma \to \mathrm{SL}(\mathfrak{su}(2,1))$ be the restriction of the adjoint representation. Then*

(i) η is strongly irreducible,

(ii) η is projective Anosov

(iii) η admits a continuous (Γ, η)-equivariant section:

$$\begin{aligned}\zeta : \partial\Gamma &\to \mathcal{F}_{\{1,4\}}(\mathfrak{su}(2,1)) \simeq \mathcal{P} \\ x &\mapsto (\xi^1(x), T_{\xi^1(x)}\xi^1(\partial\Gamma)) \simeq (\xi^1(x), (\,\mathrm{d}_{\xi^1(x)}p)^{-1}(T_{\xi^1(x)}\xi^1(\partial\Gamma))\xi^1(x)).\end{aligned}$$

where $d_{\xi^1(x)}p : \hom(\xi^1(x), \mathfrak{su}(2,1)) \to \hom(\xi^1(x), \mathfrak{su}(2,1)/\xi^1(x))$ is the canonical projection.

(iv) For all $x, y \in \partial\Gamma : \zeta^4(x) \cap \zeta^4(y) \neq 0$.

(v) For any $y_0 \in \mathrm{SU}(2,1)/P_0$ and $W_0 \in \mathcal{G}_7(\mathbb{R}^4)$ that contains $\zeta(y_0)^4$, we have that $\mathrm{Ann}(\zeta(y_0)^4, W_0) \supseteq \zeta(\mathrm{SU}(2,1)/P_0)$ and is in particular of full μ-measure, for any $(\eta(\Gamma), \phi)$-Patterson–Sullivan measure μ supported over $\zeta(\partial\Gamma)$.

(vi)

$$h_\rho(F) = \dim_{\mathcal{H}}(\xi^1_\rho(\partial\Gamma)) = 3.$$

For the sake of readability, we have broken up the proof of Proposition 4.1.2 into multiple propositions which are then combined.

Remark 4.1.3. When $G = \mathrm{Isom}(X)$ is the isometry group of a complete Riemannian manifold X, and Γ is a uniform lattice of G,then it acts properly discontinuously and cocompactly on X.

Before diving in and proving the Anosov property of η, we will first give an expression for the Cartan projection of $\mathrm{SL}(\mathfrak{su}(2,1))$ over $\eta(\Gamma) \subseteq \mathrm{Ad}_{\mathrm{SU}(2,1)}$ that will be used in a few occasions later on.

Lemma 4.1.4. *Let Γ be a uniform lattice of $\mathrm{SU}(2,1)$, and $\eta : \Gamma \to \mathrm{SL}(\mathfrak{su}(2,1))$ be the restriction of the adjoint representation: $\eta(\gamma) = \mathrm{Ad}_\gamma$ for all $\gamma \in \Gamma$. Denoting the Cartan projection with $a : \mathrm{SL}(\mathfrak{su}(2,1)) \to \mathfrak{a}^+$, it is given on $\mathrm{Ad}(\mathrm{SU}(2,1))$ by*

$$a(\mathrm{Ad}_g) = \frac{d\,(g[1:0:-1], [1:0:-1])}{2}(0,0,1,1,2,-1,-1,-2),$$

where $d(\cdot,\cdot)$ is the Bergam metric on $\mathbb{H}^2_{\mathbb{C}}$.

Proof. Let $g \in \mathrm{SU}(2,1)$. Denoting with $\mu : \mathrm{SU}(2,1) \to a^+$ the Cartan projection of $\mathrm{SU}(2,1)$, we have that

$$g = k_1 e^{\mu(g)} k_2,$$

for some k_1, k_2 in the maximal compact subgroup K of $\mathrm{SU}(2,1)$. Letting $k_1' = Ad_{k_1}, k_2' = Ad_{k_2}$ and $K' \leq \mathrm{SL}(\mathfrak{su}(2,1))$ be a maximal comapct subgroup containing them, we have that:

$$\mathrm{Ad}_g = k_1' \,\mathrm{Ad}_{\exp(\mu(g))}\, k_2' = k_1' \exp(\mathrm{ad}_{\mu(g)}) k_2'.$$

Plugging in the expression $\mu(g) = \frac{d(g[1:0:-1],[1:0:-1])}{2}\,\mathrm{diag}(2,0,-1)$ from Lemma 1.7.3, we obtain that

$$
\begin{aligned}
a(g) = \operatorname{ad}\mu(g) &= \frac{d\left(g[1:0:-1],[1:0:-1]\right)}{2}\operatorname{ad}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\\
&= \frac{d\left(g[1:0:-1],[1:0:-1]\right)}{2}\operatorname{diag}(0,0,1,1,2,-1,-1,-2).
\end{aligned}
$$

in the basis of $\mathfrak{su}(2,1)$ that is presented in Section 1.7. □

We begin by proving the Anosov property of η.

Proposition 4.1.5. *Let Γ be a uniform lattice of $\mathrm{SU}(2,1)$, and $\eta : \Gamma \to \mathrm{SL}(\mathfrak{su}(2,1))$ be the restriction of the adjoint representation: $\eta(\gamma) = \mathrm{Ad}_\gamma$ for all $\gamma \in \Gamma$. Then η is projective Anosov.*

Proof. Let $\gamma \in \Gamma$. Using the expression in Lemma 4.1.4, we have that

$$
\mathsf{a}_1(\operatorname{ad}\mu(\gamma)) = \frac{d\left(\gamma[1:0:-1],[1:0:-1]\right)}{2},
$$

where a_1 denotes here the first fundamental root of $\mathfrak{su}(2,1)$.

But by the definition of a uniform lattice, we have that Γ acts properly discontinuously and cocompactly, so the mapping $\Gamma \to \mathbb{H}^2_{\mathbb{C}}$ given by $\gamma \mapsto \gamma[1:0:-1]$ is a quasi-isometric embedding. In particular $d\left(\gamma[1:0:-1],[1:0:-1]\right) \geq C|\gamma| - c$ for some constants $c, C > 0$ and for all γ. Using this estimate in the expression of a_1 we have:

$$
\mathsf{a}_1(a(\eta(\gamma))) \geq \frac{C}{2}|\gamma| - \frac{c}{2},
$$

where $a : \mathrm{SL}(\mathfrak{su}(2,1)) \to \left\{(a_1,\cdots,a_8) \in \mathbb{R}^8 : a_1 + \cdots a_8 = 0\right\}$ is the Cartan projection for $\mathfrak{su}(2,1)$. Hence η is projective Anosov. □

To calculate the projective part of the limit map, we make use of Lemma 1.4.4, according to which there exists a unique $\mathrm{SU}(2,1)$-equivariant map from $\partial\Gamma$ to $\mathbb{P}(\mathfrak{su}(2,1)) \simeq \mathbb{P}(\mathbb{R}^8)$. Hence it suffices to find an $\mathrm{SU}(2,1)$-equivariant continuous map from $\partial\Gamma$ to $\mathbb{P}(\mathbb{R}^d)$, which is constructed in the following proposition. Recall that the necessary material on $\mathrm{SU}(2,1)$ and its parabolic groups can be found in Section 1.7.

Proposition 4.1.6. *Let Γ be a uniform lattice of $\mathrm{SU}(2,1)$, and $\eta : \Gamma \to \mathrm{SL}(\mathfrak{su}(2,1))$ be the restriction of the adjoint representation: $\eta(\gamma) = \mathrm{Ad}_\gamma$ for all $\gamma \in \Gamma$. The projective part of the limit map of η is given by*

$$
\xi_\eta : \partial\Gamma = \mathrm{SU}(2,1)/P_0 \to \mathbb{P}(\mathfrak{su}(2,1)), \quad \xi_\eta(gP_0) = \mathbb{R}\,\mathrm{Ad}_\gamma x_0.
$$

where $P_0 = \mathrm{St}_{\mathrm{SU}(2,1)}[1:0:0]$ and

$$
x_0 = i\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix} \in \mathfrak{su}(2,1).
$$

Its derivative satisfies:

$$
d_x\xi(T_x\,\mathrm{SU}(2,1)/P_0) = \pi(\operatorname{ad}_{\xi^1(x)}\mathfrak{su}(2,1))
$$

where $\pi : \hom(\xi^1(x),\mathfrak{su}(2,1)) \to \hom(\xi^1(x),\mathfrak{su}(2,1)/\xi^1(x))$ is the canonical projection.

Proof. Since by Lemma 1.4.4 the limit map is the unique continuous ρ-equivariant map from the boundary of Γ to the projective space, it suffices to show that there exists an η-equivariant map $\xi^1 : \mathrm{SU}(2,1)/P_0 \to \mathbb{P}(\mathfrak{su}(2,1))$, since it will then restrict to the limit map on $\partial\Gamma$.

We consider the parabolic subgroup $P_0 = \mathrm{St}_{\mathrm{SU}(2,1)}[1:0:0]$ of $\mathrm{SU}(2,1)$. Then its Lie algebra is given by:

$$\mathfrak{p}_0 = \mathrm{St}_{\mathfrak{su}(2,1)}[1:0:0] = \left\{ \begin{pmatrix} u-is & a & it \\ 0 & 2is & -\bar{a} \\ 0 & 0 & -u-is \end{pmatrix} : a \in \mathbb{C}, u, s, t \in \mathbb{R} \right\}.$$

Since for $\mathbb{R}x \in \mathbb{P}(\mathfrak{su}(2,1))$ we have that P_0 fixes $\mathbb{R}x$ if and only if $\mathfrak{p}_0$ fixes $\mathbb{R}x$. But a quick calculation shows that the only element of $\mathfrak{su}(2,1)$ fixed by $\mathfrak{p}_0$ is x_0.

For the calculation of the image of the differential at the identity coset P, we differentiate the commutative diagram:

$$\begin{array}{ccc} \mathrm{SU}(2,1) & \xrightarrow{\mathrm{Ad}.\, x_0} & \mathfrak{su}(2,1) \\ \downarrow & & \downarrow \\ \mathrm{SU}(2,1)/P_0 & \xrightarrow{\xi^1} & \mathbb{P}(\mathfrak{su}(2,1)) \end{array} \quad \text{to get} \quad \begin{array}{ccc} \mathfrak{su}(2,1) & \xrightarrow{\mathrm{ad}.\, x_0} & \mathfrak{su}(2,1) \\ \downarrow & & \downarrow \pi \\ \mathfrak{su}(2,1)/\mathfrak{p}_0 & \xrightarrow{\mathrm{d}_P \xi^1} & T_{\xi^1(P)}\mathbb{P}(\mathfrak{su}(2,1)) \end{array}$$

In the general case we use the equivariance of the limit map

$$\begin{aligned} \mathrm{d}_{gP}\xi^1(T_{gP}\,\mathrm{SU}(2,1)/P_0) &= d_{gP}\xi^1 d_P g(T_P\,\mathrm{SU}(2,1)/P_0) = d_{\xi^1(P)} g d_P \xi^1(T_P\,\mathrm{SU}(2,1)/P_0) = \\ &= d_{\xi^1(P)} g \pi(\mathrm{ad}_{\xi^1(P)}\,\mathfrak{su}(2,1)) = \\ &= \pi(Ad_g(\mathrm{ad}_{\xi^1(P)}\,\mathfrak{su}(2,1))) = \pi(\mathrm{ad}_{Ad_g \xi^1(P)}\,\mathfrak{su}(2,1)) = \\ &= \pi(\mathrm{ad}_{\xi^1(gP)}\,\mathfrak{su}(2,1)). \end{aligned}$$

□

We are now ready to fill in the gaps, and provide a proof of Proposition 4.1.2.

Proof of Proposition 4.1.2. (i) Follows from the fact that $\mathrm{SU}(2,1)$ is a simple Lie group.

(ii) Shown in Proposition 4.1.5.

(iii) Follows from the fact that ξ^1 is $\mathrm{SU}(2,1)$-equivariant and the equivariant identification of $\mathcal{F}_{\{1,4\}}(\mathfrak{su}(2,1)) \simeq \mathcal{P}$.

(iv) Letting $g \in \mathrm{SU}(2,1)$ be as in Lemma 1.7.5, we have that $\mathrm{Ad}_g(\mathfrak{p}_0) = \mathfrak{p}$ and $\mathrm{Ad}_g(\mathfrak{p}_0^t) = \mathfrak{p}'$. Thus

$$\begin{aligned} Ad_g(\zeta(x)^4 \cap \zeta(y)^4) = \mathrm{Ad}_g\,\zeta(x)^4 \cap \mathrm{Ad}_g\,\zeta(y)^4 = \zeta(gx)^4 \cap \zeta(gy)^4 = \zeta(\mathfrak{p}_0)^4 \cap \zeta(\mathfrak{p}_0^t)^4 = \\ = \pi\left(\mathbb{R}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right) \neq \emptyset. \end{aligned}$$

For the last equality, we use Proposition 4.1.6 and the fact that $\mathfrak{p}_0^t = \mathrm{Ad}_g\,\mathfrak{p}_0$ for

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

to conclude that

$$\zeta(\mathfrak{p}_0) = \zeta(P_0) = \pi\left(\left\{\begin{pmatrix} u & a & it \\ 0 & 0 & -\bar{a} \\ 0 & 0 & -u \end{pmatrix} : a \in \mathbb{C}, u, t \in \mathbb{R}\right\}\right),$$

and

$$\zeta(\mathfrak{p}_0^t) = \zeta(gP_0) = \mathrm{Ad}_g\,\zeta(P_0) = \pi\left(\left\{\begin{pmatrix} u & 0 & 0 \\ a & 0 & 0 \\ it & -\bar{a} & -u \end{pmatrix} : a \in \mathbb{C}, u, t \in \mathbb{R}\right\}\right).$$

(v) This follows from the preceeding point of the proposition.

(vi) Since $\Gamma \leq \mathrm{SU}(2,1)$ is a uniform lattice, we know that $\partial\Gamma \simeq \partial_\infty H^2_{\mathbb{C}} \simeq \mathbb{S}^3$, hence $d_\Gamma = 3$. Thus, by the arguements presented in Chapter 2, it suffices to show that $h_\eta(J_3^u) = 1$. Using the definition of the unstable Jacobian and Lemma 4.1.4:

$$\begin{aligned} J_3^u(a(\eta(\gamma))) &= \frac{d\,(g[1:0:-1],[1:0:-1])}{2} J_3^u((0,0,1,1,2,-1,-1,-2)) \\ &= \frac{d\,(g[1:0:-1],[1:0:-1])}{2}(\mathsf{a}_1+\mathsf{a}_2+\mathsf{a}_3+\mathsf{a}_4)(a(0,0,1,1,2,-1,-1,-2)) \\ &= \frac{d\,(g[1:0:-1],[1:0:-1])}{2}((2-1)+(2-1)+(2-0)) \\ &= 2d\,(g[1:0:-1],[1:0:-1])\,, \end{aligned}$$

which implies that the critical exponent of J_3^u is equal to half of the critical exponent δ_Γ of Γ, i.e.

$$h_\eta(J_3^u) = \frac{\delta_\Gamma}{2} := \lim_{R\to\infty} \frac{\sharp\{\gamma : d(\gamma[1:0:-1],[1:0:-1]) \leq R\}}{R}.$$

From Lemma 1.1.4, we know that the critical exponent of Γ is equal to the volume entropy of the metric d, which can be explicitly computed:

$$\delta_\Gamma = \lim_{R\to\infty} \frac{\log \mathrm{Vol}\,(B_R\,([1:0:-1]))}{R} = 2.$$

□

4.2 Zariski-density, but not in $\mathrm{SL}(d,\mathbb{R})$

As can be seen in the Proposition 4.1.2, the above construction provides a counterexample to Lemma 4.0.1 and to Proposition 4.1.2, but our calculation shows that it verifies the result of the main theorem. For this reason, it is natural to look for special cases, where Equation (4.1) constitutes indeed a contradiction. In particular, we have found that this is indeed true if we add the assumption that $\eta(\Gamma)$ is Zariski-dense in $\mathrm{SO}(2,p)$ for some p different from 2, which covers a particularly interesting source of examples of discrete subgroups of $\mathrm{SO}(2,p)$, which are closely connected with the construction of globally hyperbolic maximally compact Cauchy manifolds (see for instance [MST23]). This provides a modified version of the main theorem:

Theorem 4.2.1. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a projective Anosov representation such that $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold of $\mathbb{P}(\mathbb{R}^d)$. Assume moreover that $\dim \xi^1_\rho(\partial\Gamma) = d-3, d > 4$ and $\rho(\Gamma)$ is Zariski-dense in $\mathrm{SO}(2,d-2)$. Then*

$$\dim \xi^1_\rho(\partial\Gamma) = h_\rho(F).$$

The proof of the above theorem will in part be based on reducing it to the analogous result for the case of $\mathrm{SO}(1,d-1)$, which we will first state and prove.

Lemma 4.2.2. *The complement of every hyperplane $W \subseteq \mathbb{R}^d$ contains vectors of all signatures. In other words, for any $V \in P(\mathbb{R}^d), W \in \mathcal{G}_{d-1}(\mathbb{R}^d)$ there exists some $V' \in \mathcal{G}_1(\mathbb{R}^d)$ such that*

$$\mathrm{sgn}\omega_{|\mathrm{V}} = \mathrm{sgn}\omega_{|\mathrm{V'}} \quad \textit{and} \quad V' \cap W = 0,$$

where ω is the standard form on $\mathbb{R}^{1,d-1}$.

Proof. What we will actually show, is that the complement W^c contains two isotropic vectors, infinitely many positive vectors and infinitely many negative vectors.

Let $w \in W$ be a non-zero vector. Such a vector exists because otherwise W would be an isotropic subspace of dimension $d-1 \geq 2$ which is not possible for a form of signature $(1, d-1)$. Let $u \in W^c$ be a vector of the opposite sign to w, which exists since otherwise ω would be either positive or negative on $\mathbb{R}^d$. Then the affine line $L = \{u_t := u + tw : t \in \mathbb{R}\}$ does not intersect W and contains vectors of all signatures. Indeed, assume without loss of generality that $\omega(u,u) > 0 > \omega(w,w)$. Then

$$\omega(u_t, u_t) = \omega(w,w)t^2 + 2\omega(u,w)t + \omega(u,u).$$

The discriminant of the above quadratic is $4\omega(u,w)^2 - 4\omega(u,u)\omega(w,w)$, which is positive since u and w have opposite signs. Let $t_1 < t_2$ be the two roots of the above quadratic. Then

$$\omega(u_t,u_t) \begin{cases} > 0 & \text{if } t \in (t_1,t_2) \neq \emptyset \\ = 0 & \text{if } t = t_1 \text{ or } t = t_2 \\ < 0 & \text{if } t \notin [t_1,t_2] \end{cases}. \tag{4.2}$$

□

To demonstrate the use of the lemma above, we will show how it can be used to obtain the desired result in the rank 1 case. This is however nothing new, because it follows from the classical Patterson–Sullivan theory.

Lemma 4.2.3. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a projective Anosov representation such that $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold of $\mathbb{P}(\mathbb{R}^d)$. Assume moreover that $\dim \xi^1_\rho(\partial\Gamma) = d-2, d \geq 3$, and $\rho(\Gamma)$ is Zariski-dense in $\mathrm{SO}(1,d-1)$. Then*

$$\dim \xi^1_\rho(\partial\Gamma) = h_\rho(F).$$

Proof. The proof that we will present is analogous to the proof of Theorem 3.4.1, with the only difference that μ-irreducibility does not follow trivially from Zariski-density. Nevertheless, we still follow the strategy of [PSW23] and argue by contradiction. Up to swapping V and W we can assume that $\dim V = 1$ and $\dim W = d-1$. Then the arguements following the statement of Lemma 4.0.1 show that it suffices to show that for any $V \in \mathcal{G}_1(\mathbb{R}^d), W \in \mathcal{G}_{d-1}(\mathbb{R}^d)$ there exists some $g \in \mathrm{SO}(1,d-1)$ such that

$$gV \cap W = 0.$$

Using Witt's theorem, we see that this is equivalent to showing that for any $V \in \mathcal{G}_1(\mathbb{R}^d), W \in \mathcal{G}_{d-1}(\mathbb{R}^d)$ there exists some $V' \in \mathcal{G}_1(\mathbb{R}^d)$ such that

$$\mathrm{sgn}\omega_{|\mathrm{V}} = \mathrm{sgn}\omega_{|\mathrm{V'}} \quad \text{and} \quad V' \cap W = 0,$$

which is precisely the statement of Lemma 4.2.2. □

To reduce the proof of Theorem 4.2.1 to the situation of Lemma 4.2.2, we will use the following lemma:

Lemma 4.2.4. *Let ω be a non-degenerate symmetric bilinear form on $\mathbb{R}^d$ and v a null vector for ω. Then ω induces the form $\tilde{\omega}$ on $v^\perp/\mathbb{R}v$ given by*

$$\tilde{\omega}(x+\mathbb{R}v, y+\mathbb{R}v) = \omega(x,y),$$

and which has signature $\operatorname{sgn}(\tilde{\omega}) = \operatorname{sgn}(\omega) - (1,1)$.

Proof. Let $e_1 = u$. We claim that there exists some isotropic vector $e_2 \in \mathbb{R}^d$ such that $\omega(e_1, e_2) = 1$. To find one, consider some $f \in \mathbb{R}^d$ such that $\omega(e_1, f) = 1$, which is exists since the form is non-degenerate. Then we can take $e_2 = f - \omega(e_1, f)e_1$.

Over $\operatorname{span}\{e_1, e_2\}$, ω is a $(1,1)$ form given by

$$\omega_{|\operatorname{span}\{e_1,e_2\}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in the basis $\{e_1, e_2\}$. In particular, ω is non-denegerate on $\operatorname{span}\{e_1, e_2\}$, so it will be non-degenerate on $\operatorname{span}\{e_1, e_2\}^\perp$ as well, with $\operatorname{sgn}(\omega) = \operatorname{sgn}(\omega_{|\operatorname{span}\{e_1,e_2\}^\perp}) - (1,1)$. Letting $e_3, \cdots, e_d$ be a basis for $\operatorname{span}\{e_1, e_2\}^\perp$, we have that $e_1^\perp = \operatorname{span}\{e_1, e_3, \cdots, e_d\}$, so:

$$\omega = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \cdots & * \end{pmatrix} \text{ in the basis } \{e_1, \cdots, e_d\} \text{ and } \omega_{|e_1^\perp} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}$$

in the bases $\{e_1, \cdots, e_d\}$ and $\{e_1, e_3, \cdots, e_d\}$. Thus $\operatorname{sgn}(\omega_{|e_1^\perp}) = \operatorname{sgn}(\omega) - (1,0,1) + (0,1,0)$. Denoting with $q : e_1^\perp \to e_1^\perp/\mathbb{R}e_1$ the canonical projection, we have that $\tilde{\omega} = q^*\omega$, so $\operatorname{sgn}(\tilde{\omega}) = \operatorname{sgn}(\omega_{|\operatorname{span}\{e_1,e_2\}}) = \operatorname{sgn}(\omega) - (1,1)$. □

We are now ready to provide a proof for Theorem 4.2.1:

Proof of Theorem 4.2.1. The same arguements as in the proof of Lemma 4.2.3 show that it suffices to show that for any $V \in \mathcal{G}_2(\mathbb{R}^d), W \in \mathcal{G}_{d-2}(\mathbb{R}^d)$ there exists some $V' \in \mathcal{G}_2(\mathbb{R}^d)$ that is transverse to W and has the same signature as V.

For the case where V is non-degenerate, we can perturb V to a non-degenerate hyperplane V' that is transverse to W without changing its signature.

The case where V is degenerate is more subtle. Assuming that $\dim(V \cap V^\perp) \geq 1$, we first show that there exists some zero vector $v \in W^c$ such that $W \not\subseteq v^\perp$. Indeed, the same arguements as in the Lemma 4.2.3 show that there exist $w \in W, u \in W^c$ such that $\omega(w,w) > 0 > \omega(u,u)$ and that there exist $t_1 < t_2$ such that Equation (4.2) holds for the affine line $L = \{u_t := u + tw : t \in \mathbb{R}\} \subseteq W^c$. In particular u_{t_1}, u_{t_2} are not contained in W. Moreover, at least one of u_{t_1}, u_{t_2} is not orthogonal to W, since otherwise $w = (t_2-t_1)^{-1}(u_{t_2}-u_{t_1}) \in \langle u_{t_1}, u_{t_2} \rangle$ would be orthogonal to W, which is absurd because w was assumed to be non-isotropic.

Let $p : v^\perp \to v^\perp/\mathbb{R}v$ be the canonical projection. We claim that for $W' = p(W \cap v^\perp)$,

$$W' \text{ is a hyperplane of } v^\perp/\mathbb{R}v \text{ and } \omega \text{ induces a form of signature } (1, d-3) \text{ on } v^\perp/\mathbb{R}v.$$

The signature of the induced form follows from Lemma 4.2.4. To calculate the dimension of W', note that $\dim p(W \cap v^\perp) = \dim(W \cap v^\perp)$ since $\ker p \cap W = \mathbb{R}v \cap W = 0$, by the fact that v is not contained in W. On the other hand $\dim(W \cap v^\perp) = \dim(\ker W \ni x \mapsto (v, x)) = \dim W - 1 = d - 3$, while $\dim(v^\perp/\mathbb{R}v) = \dim v^\perp - 1 = (d-1) - 1 = d - 2$.

For any $x \in v^\perp$ such that $p(x)$ is transverse to W', we have that $V' = \operatorname{span}\{v, x\}$ is a plane that is transverse to W. To show transversality, we let $r, s \in \mathbb{R}$ such that $rv + sx \in W$. Then $p(rv + sx) = sp(x) \in W'$, which implies that $s = 0$ by the choice of x. But then $rv \in W$ which means that $r = 0$ by the choice of v. In fact, the same arguements show that x and v are linearly independent and hence V is a plane.

Finally, we show that by picking the apropriate x, we can ensure that V' has the same signature as V. Indeed, $\dim V = 2$, so $\operatorname{sgn}(\omega_{|V}) \in \{(1,1,0), (0,2,0), (0,1,1)\}$. Letting x be a positive, negative or zero vector respectively, we have that

$$\omega_{|V'} \in \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, 0, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\} \text{ in the basis } \{v, x\}$$

respectively. Thus Witt's theorem gives us the desired result. $\square$

Chapter 5

Conclusion

Anosov representations have been a subject of interest ever since their introduction in [Lab06], and are now considered as a natural genralisation of convex-cocompact representations into Lie groups of rank higher than 1. Their limit set, aside from being a natural object to study in its own right, has been shown to have applications in such diverse fields as anti-de Sitter geometry, being important in the characterisation of globally hyperbolic maximally compact Cauchy manifolds.

In this thesis, based upon the work of [PSW23], we have explored different situations where that the Falconer functional of an Anosov subgroup is equal to the Hausdorff dimension of its limit set, provided that the latter is a Lipschitz submanifold of the projective space. In Chapter 2 we have done this under the assumption that the Anosov subgroup is Zariski-dense in $\mathrm{SL}(d,\mathbb{R})$ and in Chapter 4 we have shown that this is not necessary, provided that the Anosov subgroup is Zariski-dense in $\mathrm{SO}(2,p)$ for some $p \neq 2$.

However, we have yet no reason to believe that this last result is optimal, and it would be interesting to investigate whether it can be generalised for instance to the case where the group is Anosov and Zariski dense in $\mathrm{SO}(p,q)$, or even to the non-Anosov case.

Appendix A

Tangent space to the Grassmanian

Let V be a d-dimensional real vector space. We denote with $\mathcal{G}_k(V)$ the Grassmanian of k-dimensional subspaces of V. Our first objective is to find a convenient way to express its tangent space.

Proposition A.0.1. *We have the following canonical identification:*

$$\begin{aligned} \hom(W, V/W) &\simeq T_W\mathcal{G}_k(V) \\ \phi &\mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Gamma(t\phi) \end{aligned}$$

where $\Gamma(\phi) = (Id + \phi)(W)$ *is the graph of* ϕ.

Proof. We will consider the map

$$F : \mathrm{Injhom}(W, V) \to \mathcal{G}_k(V), \quad \phi \mapsto \phi(W).$$

whose derivative is given by:

$$d_I F(\phi) = F\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(I + t\phi)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(I + t\phi(W)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Gamma(t\phi).$$

The result will follow as soon as we have shown that $d_I F$ is surjective and that $\ker d_I F = \hom(W, W)$.

To show that it is surjective, we consider a $(d-k)$-dimensional subspace $W' \in \mathcal{G}_{d-k}(V)$ that is complementary to W, i.e. $V = W \oplus W'$. Denoting with $U_{W'} = \{Z \in \mathcal{G}_k(V) : Z \cap W' = 0\}$, we recall the corresponding chart:

$$\begin{aligned} \Psi : \hom(W, W') &\to U_{W'} \\ \phi &\mapsto \Gamma(\phi). \end{aligned}$$

Surjectivity of $d_I F$ now follows by the fact that

$$d_I F(\phi) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Gamma(t\phi) = d_0\Psi(\phi).$$

To show that $\ker d_I F = \hom(W, W)$, we first note that clearly $\ker\, \mathrm{d}_I F \supseteq \hom(W, W)$. Equality then follows by the fact that $\dim \hom(W, W) = \dim \ker\, \mathrm{d}_I F$, which is a direct consequence of the surjectivity. $\square$

Note that another way to prove the above identification throught the fact that the Grassmanian is a homogeneous space of $\mathrm{GL}(d, \mathbb{R})$, giving us the diffeomorphism

$$\begin{aligned} \mathrm{GL}(V)/\mathrm{St}_{GL(V)}W &\to \mathcal{G}_k(V) \\ [g] &\mapsto gW, \end{aligned}$$

where $\mathrm{St}_{GL(V)}W = \{g \in \mathrm{GL}(V) : gW = W\}$ is the stabiliser of W. Thus an expression for the tangent space at W may be obtained by differentiating the map above at the identity coset:

$$\hom(W, V/W) \simeq \hom(V, V)/\hom(W, W) \simeq T_W\mathcal{G}_k(V).$$

In particular the above gives us a direct proof that the kernel of the map constructed in the previous proof is indeed $\hom(W, W)$.

Our second objective is to identify subspaces of $T_l\mathbb{P}(V)$ with subspaces of V, by considering the first as projectivisation of the second. More concretely, we shall consider the space

$$\mathcal{P} = \{(l, P) : l \in \mathbb{P}(V), P \in \mathcal{G}_k(T_l\mathbb{P}(V))\}$$

as a homogenous space of $\mathrm{SL}(V)$, where the action is given by

$$g \cdot (l, P) = (gl,\, \mathrm{d}_l g(P) = g\pi^{-1}(P)g^{-1} + \hom(gl, gl)).$$

where we use the identification of $T_l\mathbb{P}(V)$ with $\hom(l, V/l)$ as above and denote with $\pi : \hom(l, V) \to \hom(l, V/l)$ the canonical projection. For the sake of completeness, we outline the calculation of the differential:

$$\begin{aligned} \hom(l, V/l) \to T_l\mathbb{P}(V) \qquad &\to T_{gl}\mathbb{P}(V) \qquad &\to \hom(gl, V/gl) \\ \phi \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g(I + t\tilde{\phi})(l) \qquad &\mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(I + tg\tilde{\phi}g^{-1})(gl) \qquad &\mapsto g\tilde{\phi}g^{-1} + \hom(gl, gl) \end{aligned}$$

where $\phi \in \hom(l, V/l), \tilde{\phi} \in \hom(l, V)$ such that $\tilde{\phi} + \hom(l, l) = \phi$.

Another subtle point is that conjugation by g helps us pass from $\mathcal{G}_k(T_x\mathbb{P}(\mathbb{V}))$ to $\mathcal{G}_k(T_{gx}\mathbb{P}(V))$:

Lemma A.0.2. *Let $g \in \mathrm{SL}(V)$. Then for every $x \in \mathbb{P}(V), P \in \mathcal{G}_k(T_x\mathbb{P}(V))$, we have:*

$$\pi^{-1}(gPg^{-1}) = g\pi^{-1}(P)g^{-1},$$

where π denotes the projection $\hom(gx, \mathbb{R}^d) \to T_{gx}\mathbb{P}(\mathbb{R}^d)$ on the left hand side of the equality, and the projection $\hom(x, \mathbb{R}^d) \to T_x\mathbb{P}(\mathbb{R}^d)$ on the right hand side.

Proof. We denote with $C_g : \hom(x, \mathbb{V}) \to \hom(gx, \mathbb{V})$ the isomorphism by conjugation $C_g(T) = gTg^{-1}$. Since the following diagram commutes

$$\begin{array}{ccc} \hom(x, V) & \xrightarrow{C_g} & \hom(gx, V) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \hom(x, V/x) & \xrightarrow{\tilde{C}_g} & \hom(gx, V/gx) \end{array}$$

C_g induces an isomorphism $\tilde{C}_g$ between $T_x(\mathbb{P}(V))$ and $T_{gx}(\mathbb{P}(V))$. We now have

$$\pi^{-1}(gPg^{-1}) = \pi^{-1}\left(\tilde{C}_g(P)\right) = (\tilde{C}_g^{-1}\pi)^{-1}(P) = (\pi C_g^{-1})^{-1}(P) = C_g\left(\pi^{-1}(P)\right),$$

which is a rephrasing of the desired equality. $\square$

We are now ready to express the needed identification:

Proposition A.0.3. *We have the following* $\mathrm{SL}(V)$ *equivariant identification:*

$$\begin{aligned} \mathcal{P} &\to \mathcal{F}_{1,k+1}(V) \\ (l,P) &\mapsto (l,\pi^{-1}(P)l) \\ (l,\hom(l,Q/l)) &\leftarrow\!\shortmid (l,Q). \end{aligned}$$

where $\pi : \hom(l,V) \to \hom(l,V/l)$ *is the canonical projection.*

Proof. We begin by showing that the left-to-right direction of the map is well-defined. For this, we first need to check that for $(l,P) \in \mathcal{P}$, we have that $\dim \pi^{-1}(P)l = k+1$. Indeed, we have that $\dim \pi^{-1}(P) = k+1$ as implied by the rank-nullity theorem for $\pi : \pi^{-1}(P) \to P$. The result then follows by the fact that $\pi^{-1}(P)l = T_1(l) \oplus \cdots \oplus T_{k+1}(l)$ for any base $T_1, \ldots, T_{k+1}$ of $\pi^{-1}(P)$. The second thing to check is that $l \leq \pi^{-1}(P)l$, which holds since $\ker \pi = \hom(l,l) \leq \pi^{-1}(P)$.

To see that the two directions above are inverse to each other, we begin by examining the right-to-left-to-right composition:

$$(l,Q) \mapsto (l,\pi(\hom(l,Q))) \mapsto (l,\pi^{-1}\pi(\hom(l,Q))l) = (l,\hom(l,Q)l) = (l,Q),$$

where we did the calculation $\pi^{-1}\pi(\hom(l,Q)l) = \hom(l,Q) + \ker\pi = \hom(l,Q)$ in the second to last inequality. For the left-to-right-to-left composition we have

$$(l,P) \mapsto (l,\pi^{-1}(P)l) \mapsto (l,\hom(l,\pi^{-1}(P)l/l)),$$

so it suffices to show that $\hom(l,\pi^{-1}(P)l/l) = P$. Indeed, for $\pi^{-1}(P) = \mathbb{R}T_1 \oplus \cdots \oplus \mathbb{R}T_k$, we have that

$$\begin{aligned} \hom(l,\pi^{-1}(P)l/l) &= \hom(l,\pi^{-1}(P)l)/\hom(l,l) = (\oplus_i \hom(l,T_i(l))/\hom(l,l)) = \\ &= (\oplus_i \mathbb{R}T_i)/\hom(l,l) = \pi^{-1}(P)/\hom(l,l) = P. \end{aligned}$$

For the equivariance, the calculations has as follows:

$$\begin{array}{ccc} (l,P) & \longmapsto & (l,\pi^{-1}(P)l) \\ \downarrow g & & \downarrow g \\ (gl, g\pi^{-1}(P)g^{-1} + \hom(gl,gl)) & \longmapsto & (gl,(g\pi^{-1}(P)g^{-1})(gl)) = (gl, g\pi^{-1}(P)l) \end{array}$$

$\square$

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