

# Limit sets of Anosov representations

Giorgos

April 23, 2024

# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Lie group preliminaries . . . . .	2
1.2	Limit set preliminaries . . . . .	2
<b>2</b>	<b>Upper bound</b>	<b>5</b>
2.1	Proof of bound . . . . .	5
2.2	Lemmata . . . . .	6

# Chapter 1

## Introduction

### 1.1 Lie group preliminaries

We fix the Cartan subalgebra  $\mathfrak{a}$  of  $\mathrm{SL}(d, \mathbb{R})$ :

$$\mathfrak{a} = \{\mathrm{diag}(\alpha_1, \dots, \alpha_d) : \alpha_1 + \dots + \alpha_d = 0\}$$

and the Weyl chamber  $\mathfrak{a}^+$  of  $\mathrm{SL}(d, \mathbb{R})$

$$\mathfrak{a}^+ = \{\mathrm{diag}(\alpha_1, \dots, \alpha_d) : \alpha_1 \geq \dots \geq \alpha_d\}.$$

Denoting with  $K = \mathrm{SO}(d, \mathbb{R})$ ,  $A^+ = e^{\mathfrak{a}^+}$ , we have the Cartan decomposition:

$$\begin{aligned}\mathfrak{sl}(d, \mathbb{R}) &\rightarrow K \times A^+ \times K \\ g &\mapsto (k_g, a_g, l_g)\end{aligned}$$

such that  $g = k_g a_g l_g$ . In particular  $a_g = \mathrm{diag}(\sigma_1(g), \dots, \sigma_d(g))$  with  $\sigma_1 \geq \dots \geq \sigma_d(g)$ , where  $\sigma_i(g)$  is the  $i$ -th singular value of  $g$ , i.e. eigenvalue of  $g^t \cdot g$ .

We will use the spaces

$$U_p(g) = \mathbb{R}u_1(g) \oplus \dots \oplus \mathbb{R}u_p(g)$$

where  $u_i(g) = k_g \cdot e_i$ . One can easily show that the decomposition

$$g^{-1} \cdot U_p(g) \oplus U_{d-p}(g^{-1})$$

is orthogonal with respect to the standard inner product and that  $u_p(g^{-1}) = l_g^{-1} e_{d-p+1}(g)$ .

### 1.2 Limit set preliminaries

**Definition 1.2.1.** For  $p \in \{2, \dots, d\}$ ,  $s \in \mathbb{R}$  and  $g \in \mathrm{SL}(d, \mathbb{R})$  we denote with  $\tilde{\Psi}_s^p(g), \Psi_s^p(g) : \mathrm{SL}(d, \mathbb{R}) \rightarrow \mathbb{R}$  the functional:

$$\begin{aligned}\Psi_s^p(g) &= \alpha_{12}(a(g)) + \dots + \alpha_{1(p-1)}(a(g)) + (s - (p-2))\alpha_{1p}(a(g)) \\ \tilde{\Psi}_s^p(g) &= \left( \frac{\sigma_2}{\sigma_1} \dots \frac{\sigma_{p-1}}{\sigma_1}(g) \right) \left( \frac{\sigma_{p-1}}{\sigma_1}(g) \right)^{s-(p-2)}\end{aligned}$$

*Remark 1.2.1.* We have  $\alpha_{ij}(a) = a_i - a_j$ ,  $a_i(g) = \log(\sigma_i(g))$  and

$$\Psi_s^p(g) = \log \tilde{\Psi}_s^p(g)$$

and that

$$\min_{p \in \llbracket 2, d \rrbracket} \left\{ \sum_{|\gamma|=T} \frac{\sigma_2}{\sigma_1} \dots \frac{\sigma_{p-1}}{\sigma_1}(g) \left( \frac{\sigma_{p-1}}{\sigma_1}(g) \right)^{s-(p-2)} \right\} = \sum_{|\gamma|=T} e^{-\max_{p \in \llbracket 2, d \rrbracket} \Psi_s^p(g)}$$

*Remark 1.2.2.* For any  $g \in \text{SL}(d, \mathbb{R})$  we have that:

$$\max_{p \in \llbracket 2, d \rrbracket} \Psi_s^p(g) = \Psi_s^{p_0}(g) \text{ for } s \in [p_0 - 2, p_0 - 1].$$

Indeed, a quick calculation shows that for  $s \geq 0$  and  $p \in \llbracket 2, d \rrbracket$ :

$$\Psi_s^p(g) \leq \Psi_s^p(g) \text{ if and only if } s \geq p - 1.$$

and that equality holds in the case  $s = p - 1$ . Thus for  $s \in [p - 2, p - 1]$  we have that

$$\begin{aligned} s \geq p - 2, \dots, 1 \text{ implies that } \Psi_s^p(g) &\geq \dots \geq \Psi_s^2(g) \\ s \leq p, \dots, d - 1 \text{ implies that } \Psi_s^p(g) &\leq \dots \leq \Psi_s^d(g) \end{aligned}$$

Another way to see this (refer to Figure 1.1) is to note that  $\Psi_s^2(g), \dots, \Psi_s^d(g)$  is a sequence of functions that are affine in  $s$ , with slopes  $\alpha_{12}(g) \leq \dots \leq \alpha_{1d}(g)$  and that they satisfy  $\Psi_1^2(g) = \Psi_2^2(g)$ ,  $\Psi_2^3(g) = \Psi_3^3(g) \dots$ ,  $\Psi_{d-2}^{d-1}(g) = \Psi_{d-2}^d(g)$ .

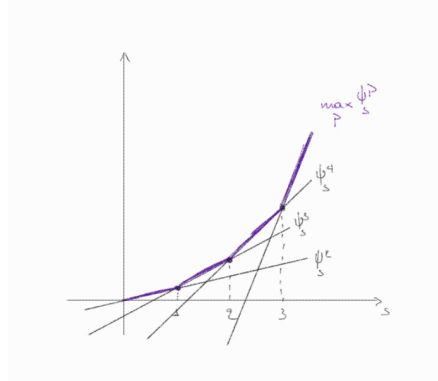


Figure 1.1: Visual illustration that  $\max_p \Psi_s^p(g) = \Psi_s^{p_0}(g)$  for  $s \in [p_0 - 2, p_0 - 1]$ .

The following definition comes from [1], in the special case of projective Anosov representations ( $P = 1$ ):

**Definition 1.2.2.** For  $s \geq 0$  we consider the Falconer functional  $F_s : \text{SL}(d, \mathbb{R}) \rightarrow \mathbb{R}$  by:

$$F_s(g) = \min \left\{ \sum_{j=2}^d s_j \alpha_{1j}(a(g)) : s_j \in (0, 1], \sum_{j=2}^d s_j = s \right\},$$

and define the Falconer dimension  $\dim_F(\rho)$  of  $\rho$  to be its critical exponent:

$$\dim_F(\rho) = \inf \left\{ s > 0 : \sum_{\gamma \in \Gamma} e^{-F_s(\rho(\gamma))} < \infty \right\}.$$

*Remark 1.2.3.* Using elementary computations one may prove that for all  $s \geq 0$ :

$$F_s(g) = \max_{p \in \llbracket 2, d \rrbracket} \Psi_s^p(g)$$

**Definition 1.2.3.** Let  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  be a linear representation and  $p \in \llbracket 1, d-1 \rrbracket$ . We say that  $\rho$  is  $p$ -Anosov if there exist constants  $\mu, C > 0$  such that for all  $\gamma \in \Gamma$ :

$$\frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma)) \leq C e^{-\mu|\gamma|}.$$

One can show that in that case there exist equivariant continuous maps  $\xi^p : \hat{\Gamma} \rightarrow \mathcal{G}_p(\mathbb{R}^d), \xi^{d-p} : \hat{\Gamma} \rightarrow \mathcal{G}_{d-p}(\mathbb{R}^d)$  that are transverse and restrict to

$$\xi^p(\gamma) = U_p(\rho(\gamma)), \xi^{d-p}(\gamma) = U_{d-p}(\rho(\gamma))$$

for  $\gamma \in \Gamma$ , where  $U_p(\gamma), U_{d-p}(\gamma)$  denote the flags corresponding to  $\rho(\gamma)$ .

Figure out  
what this  
exactly  
means

# Chapter 2

## Upper bound

### 2.1 Proof of bound

**Lemma 2.1.1** (Upper bound for dimension). *Let  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  be a projective Anosov representation. Then:*

$$\dim_H(\xi^1(\partial\Gamma)) \leq \inf \left\{ s : \sum_{|\gamma|=T} e^{-F_s(\rho(\gamma))} < \infty \right\}.$$

*Remark 2.1.1.* The idea of the proof of Lemma 2.1.1 is to find a covering whose Hausdorff content is dominated by the Dirichlet series of some functional  $\Psi_s^p$ , which will in turn imply that  $\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(\Psi^p)$ . Choosing then the most "effective" cover (i.e. the one which yields the smallest Hausdorff content up to a constant) we obtain that

$$\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(\max_p \Psi^p)$$

To obtain this we first cover  $\xi^1(\partial\Gamma)$  by the bassins of attraction  $\rho(\gamma) \cdot B_{\alpha_1, \alpha}(\rho(\gamma))$  for  $\gamma \in \Gamma$  satisfying  $|\gamma| = T$ . Then we cover each bassin by an ellipsoid of axes lengths

$$\frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \dots, \frac{1}{\sin(\alpha)} \frac{\sigma_d}{\sigma_1}(\rho(\gamma)).$$

Finally we cover each ellipsoid by balls of some fixed radius  $r > 0$ . It can be shown by comparing the series appearing in the Hausdorff content of each resulting cover that the most "effective" choice of  $r$  depends only on the Hausdorff exponent  $s > 0$  and in any case will be to have  $r$  equal (up to a constant) to the the length of an axis of the ellipsoid, i.e.

$$r \in \left\{ \frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \dots, \frac{1}{\sin(\alpha)} \frac{\sigma_d}{\sigma_1}(\rho(\gamma)) \right\}$$

In particular, when  $s \in [p-2, p-1]$ , the most effective choice is  $r = \sigma_p(\rho(\gamma))/\sigma_1(\rho(\gamma))$ , whose Hausdorff content is dominated by the Dirichlet series of  $\Psi_s^p$ .

*Proof of Lemma 2.1.1.* Let  $p \in \llbracket 2, d \rrbracket$ . Then using Proposition 2.2.1, Lemma 2.2.2, and Lemma 2.2.3 we have that for  $T > 0$  large enough,  $\xi^1(\partial\Gamma)$  is covered by the family

$$\mathcal{U}_T = \{\rho(\gamma)B_{\alpha_1, \alpha}(\rho(\gamma)) : |\gamma| = T\},$$

and that each basin  $\rho(\gamma)B_{\alpha_1, \alpha}(\rho(\gamma))$  is in turn covered by

$$2^{p-2} \cdot \frac{\sigma_p(g)^{p-2}}{\sigma_2(g) \cdots \sigma_{p-1}(g)}$$

many balls of radius

$$\sqrt{d-1} \frac{1}{\sin \alpha} \frac{\sigma_p(g)}{\sigma_1(g)}.$$

By the definition of the Hausdorff measure, for  $s \geq 0$ :

$$\begin{aligned} \mathcal{H}^s(\xi^1(\partial\Gamma)) &\leq \sum_{|\gamma|=T} 2^{2p+1} \cdot \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \cdots \frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \left( \frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \right)^{-(p-2)} \left( \sqrt{d-1} \frac{1}{\sin \alpha} \frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \right)^s = \\ &= 2^{2p+1} \cdot \left( \frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \cdots \frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \left( \frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \right)^{s-(p-2)} = \\ &= 2^{2p+1} \cdot \left( \frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} e^{-(\alpha_{12} + \dots + \alpha_{1(p-1)} + (s-(p-2))\alpha_{1p})\rho(\gamma)} \\ &= 2^{2p+1} \cdot \left( \frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} e^{-\Psi_s^p(\rho(\gamma))} \end{aligned}$$

and thus

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq 2^{2p+1} \cdot \left( \frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} e^{-\max_p \Psi_s^p(\rho(\gamma))} \lesssim \sum_{|\gamma|=T} e^{-F_s(\rho(\gamma))}.$$

To see that the above implies the upper bound, consider some  $s > \dim_F(\rho)$ . By the definition of the Falconer dimension, this implies that the Dirichlet series corresponding to the Falconer functional converges:

$$\sum_{\gamma \in \Gamma} e^{-F_s(\rho(\gamma))} < \infty$$

and in particular

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq \lim_{T \rightarrow \infty} e^{-F_s(\rho(\gamma))} = 0.$$

□

## 2.2 Lemmata

**Definition 2.2.1.** Let  $V$  be a finite-dimensional  $\mathbb{R}$ -vector space. We consider a decomposition

$$V = \mathbb{R}u_1 \oplus \cdots \oplus \mathbb{R}u_d$$

be a direct decomposition that is orthogonal with respect to a fixed inner-product over  $V$ . Given  $\beta_2 \geq \dots \beta_d > 0$ , we define an ellipsoid with axes  $u_1 \oplus u_p(g)$  and lengths  $\beta_p$  to be the image of

$$\left\{ v = \sum_1^d v_i u_i \in V : \sum_2^d \left( \frac{v_j}{\beta_j} \right)^2 \leq 1 \right\}$$

through the projection  $V \rightarrow \mathbb{P}(V)$ .

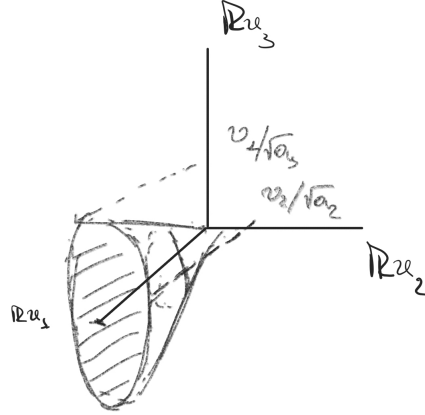


Figure 2.1: Depiction in  $\mathbb{R}^3$  of an ellipsoid of  $\mathbb{P}(\mathbb{R}^2)$

The following aims to be something along the lines of [2, Lemma 2.4]:

**Lemma 2.2.1.** *Let  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$  be a projective Anosov representation. For  $\alpha > 0$  small enough, there exists  $L > 0$  such that for any geodesic ray  $(a_j)_j$  through  $e$  we have:*

$$\angle(U_1(\rho(a_i)), U_{d-1}(\rho(a_0))) > \alpha$$

when  $|a_i|, |a_0| > T$ .

*Proof.* Assume the contrary for the sake of contradiction. Then (see Figure 2.2) for each  $n > 0$  there exists a geodesic ray  $a^n$  through  $e$  such that

$$|a_n^n|, |a_0^n| > n \text{ and } \angle(U_1(\rho(a_n^n)), U_1(\rho(a_0^n))) < \frac{1}{n}.$$

Due to compactness of  $\partial\Gamma$  we may assume (up to a subsequence) that  $a^n \rightarrow x$  in  $\partial\Gamma$  for some  $x \in \partial\Gamma$ . Then  $a_n^n, a_0^n \rightarrow x$  in  $\hat{\Gamma}$  which implies

$$\angle(\xi^1(x), \xi^{d-1}(x)) = 0$$

using the fact that the limit maps  $\xi^1, \xi^{d-1}$  are continuous, which contradicts their transversality.  $\square$

The following is [2, Proposition 3.5].

**Lemma 2.2.2.** *Let  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$  be projective Anosov. Then for  $\alpha > 0$  small enough, there exists some  $T_0 > 0$  such that for all  $T \geq T_0$  the family*

$$\mathcal{U}_T = \{\rho(\gamma)B_{\alpha_1, \alpha}(\rho(\gamma)) : |\gamma| = T\}$$

*is an open covering of  $\xi^1(\partial\Gamma)$ .*

Not sure if this is true.



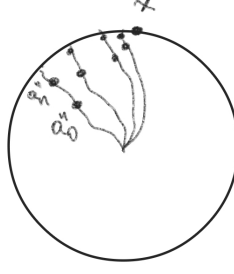


Figure 2.2: Situation in Lemma 2.2.1

*Proof.* Let  $\alpha, T > 0$  be as in the statement of Lemma 2.2.1 and  $x \in \partial\Gamma$  be represented by a geodesic ray  $(\gamma_j)_{j \geq 0}$  starting from  $e$ . Then  $(\gamma_T^{-1} \gamma_j)_j$  is a geodesic ray starting from  $(\gamma_T)^{-1}$  that passes through  $e$ , so

$$\angle(U_1(\rho(\gamma_T^{-1} \gamma_j)), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

as implied by Lemma 2.2.1. Taking the limit  $j \rightarrow \infty$  and using the equivariance of the limit map, we obtain

$$\angle(\rho(\gamma_T^{-1})\xi^1(x), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

and thus  $\xi^1(x) \in \rho(\gamma_T) \cdot B_{\alpha_1, \alpha}(\rho(\gamma_T))$ .  $\square$

The following is [2, Proposition 3.8].

**Proposition 2.2.1.** *For each  $g \in \mathrm{SL}(d, \mathbb{R})$ ,  $\alpha > 0$ , the basin of attraction  $g \cdot B_{\alpha_1, \alpha}(g)$  lies in the ellipsoid with axes  $u_1(g) \oplus u_p(g)$  with lengths*

$$\frac{1}{\sin \alpha} \cdot \frac{\sigma_p(g)}{\sigma_1(g)}$$

*Proof.* Using the definition of the basin of attraction (see Figure 2.3 ), we have that  $w = w_1 u_1(g^{-1}) + \dots + w_d u_d(g^{-1}) \in B_{\alpha_1, \alpha}(g)$  if and only if

$$w_d^2 \geq (\sin \alpha)^2 \sum_1^d w_i^2.$$

Considering now some  $v = v_1 u_1(g) + \dots + v_d u_d(g) \in g \cdot B_{\alpha_1, \alpha}(g)$  we have that

$$\begin{aligned} w &= g^{-1} v = v_1 \sigma_1(g)^{-1} l_g^{-1} e_1(g) + \dots + v_d \sigma_d(g)^{-1} l_g^{-1} e_d(g) \\ &= v_1 \sigma_1(g)^{-1} u_d(g^{-1}) + \dots + v_d \sigma_d(g)^{-1} u_1(g^{-1}) \end{aligned}$$

where we used that  $u_p(g^{-1}) = l_g^{-1} e_{d+1-p}$ . Hence

$$\sigma_1(g)^{-2} \cdot v_1^2 \geq (\sin \alpha)^2 \sum_1^d \sigma_i(g)^{-2} v_i^2.$$

$\square$

The following is [2, Lemma 3.7]:

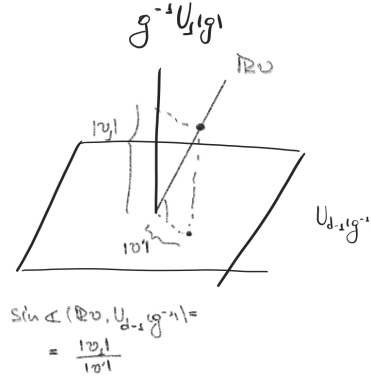


Figure 2.3: Aid for Proposition 2.2.1

**Lemma 2.2.3.** *For any  $p \in \llbracket 2, d \rrbracket$ , an ellipsoid in  $\mathbb{P}(\mathbb{R}^d)$  of axes lengths  $\beta_2, \dots, \beta_d$  is covered by*

$$2^{p-2} \frac{\beta_2 \cdots \beta_{p-1}}{\beta_p^{p-2}}$$

*many (projected) balls of radius  $\sqrt{d-1}\beta_p$ .*

*Proof.* We assume that  $E$  is an ellipsoid about  $\mathbb{R}e_1$ , so it suffice to cover its intersection  $E_1 = E \cap U_1 \subseteq \mathbb{R}^{d-1}$  with the affine chart  $U_1 = \{[x_1 : \dots, x_d] \in \mathbb{P}(\mathbb{R}^d) : x_1 \neq 0\}$ . Clearly  $E_1 \subseteq [-\beta_2, \beta_2] \times \dots \times [-\beta_d, \beta_d]$ , so we proceed by covering the rectangle with side-lengths  $2\beta_2, \dots, 2\beta_d$ . Clearly each interval  $(-\beta_j, \beta_j)$  is contained in the union of  $\lceil \beta_j/\beta_p \rceil$  intervals of length  $2\beta_p$ , thus  $E_1$  is contained in the union of

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil \cdots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil = \left\lceil \frac{\beta_2}{\beta_p} \right\rceil \cdots \left\lceil \frac{\beta_d}{\beta_p} \right\rceil$$

many squares of side-length  $2\beta_p$ . Since each such product is contained in a  $(d-1)$ -ball of radius  $\sqrt{d-1}\beta_p$  we may use at most

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil \cdots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil \leq \sum_{i \in \{0,1\}^{p-2}} \prod_{j=2}^{p-1} \left( \frac{\beta_j}{\beta_p} \right)^{i_j} \leq 2^{p-2} \frac{\beta_2}{\beta_p} \cdots \frac{\beta_{p-1}}{\beta_p}$$

many  $(d-1)$ -balls of radius  $\sqrt{d-1}\beta_p$  to cover  $E_1$ . □

The following can be found in [2, Proposition 3.3]:

**Proposition 2.2.2.** *Let  $\rho : \Gamma \rightarrow SL(d, \mathbb{R})$  be projective Anosov and  $\alpha > 0$ . Then there exist  $c_0, c_1 > 0$  that depends only on  $\alpha$  and  $\rho$  such that for all  $\gamma \in \Gamma$ :*

$$(\xi^1)^{-1}(B_{\alpha_1, \alpha}(\rho(\gamma))) \subseteq C_{c_0, c_1}^\infty(\gamma)$$

*Proof.* We begin by noting that it suffices to show this for all but finitely many  $\gamma \in \Gamma$ , since then we may alter the constants to satisfy the wanted inclusion also for the finitely many remaining  $\gamma \in \Gamma$ . Given this, we shall assume that  $|\gamma| \geq l_0$  where  $l_0 > 0$  is such that  $Ce^{-\mu l_0} < 1$  and  $C, \mu > 0$  are the constants appearing in the definition of the Anosov property of  $\rho$ .

Suppose  $x \in \partial\Gamma$  such that  $\xi^1(x) \in B_{\alpha_1, \alpha}(\rho(\gamma))$ , and consider a geodesic ray  $a_j \rightarrow x$  starting from  $a_0 = e$ . To prove the result, it suffices to find constants  $c_0, c_1$  independent of  $\gamma$  and a  $(c_0, c_1)$ -quasi-geodesic from  $\gamma^{-1}$  to  $x$  that passes through  $e$  and stays at a bounded distance from  $(a_j)_{j=0}^\infty$ .

Using [2, Proposition 2.5] we have that  $d(\xi^1(a_j), U_{d-1}(\rho(\gamma^{-1}))) \leq Ce^{-\mu j}$ , so there exists some  $L > 0$  that depends only on  $\alpha$  such that for all  $j \geq L : U_1(\rho(a_j)) \in B_{\alpha_1, \alpha}(\rho(\gamma))$  and in particular

$$d(\xi^1(a_j), \gamma^{-1}) = d(U_1(\rho(a_j)), U_1(\rho(\gamma^{-1}))) \geq d(U_1(\rho(a_j)), U_{d-1}(\rho(\gamma^{-1}))) > \sin \alpha.$$

Along with the uniform continuity of  $\xi^1 : \Gamma \cup \partial\Gamma \rightarrow \mathbb{P}(\mathbb{R}^d)$  this implies there exists some  $\alpha' > 0$  and  $L > 0$  such that for all  $j \geq L$ :

$$d(a_j, \gamma^{-1}) \geq \alpha'.$$

Upon considering a large  $L$ , we may also assume that  $|a_L| = L > l_0$ . Note that both  $\alpha'$  and  $L$  do not depend on each  $\gamma$  but only on  $\rho$  and  $\alpha$ .

Using some geometric group theory, we can show that for all  $j \geq L$

$$d(\gamma^{-1}, a_j) > \alpha' \Rightarrow d([\gamma^{-1}, a_j], e) < \alpha''$$

for some  $\alpha''$  that depends only on  $\Gamma$  and  $\alpha'$ , where  $[a_j, \gamma^{-1}]$  denotes the geodesic segment connecting  $\gamma^{-1}$  and  $a_j$ .

Consider the concatenation  $(a'_j)_{j=L-K}^\infty$  of  $[\gamma^{-1}, a_L]$  and  $[a_L, x]$ . To find quasi-geodesic-constants that are uniform in  $\gamma$ , we note that for any  $c_0 \geq 1, c_1 \geq 0$ :

$$c_0^{-1}|i-j| - c_1 \leq d(a'_i, a'_j) = d(a_i, a_j) \leq d(a_i, c_0|i-j| + c_1) \text{ when } i, j \geq L \text{ or } i, j \leq L$$

and that the upper bound follows trivially by the triangle inequality.

For the lower bound we proceed in two steps. First we bound the distance of  $\gamma^{-1} = a'_{L-K}$  to  $a_{L+j}$  for  $j \geq 0$ :

$$\begin{aligned} d(a'_{L-K}, a'_{L+j}) &\geq \nu(|a_{L+j}| - |\gamma^{-1}|) - c'_0 - c'_1 |\log(d(U_1(\rho(a_{L+j})), U_1(\rho(\gamma^{-1}))))| \geq \\ &\geq \nu((L+j) + (K-L)) - c'_0 - c'_1 |\log(\sin a)| \geq \\ &= c_0^{-1}(j+K) - c_1 \end{aligned}$$

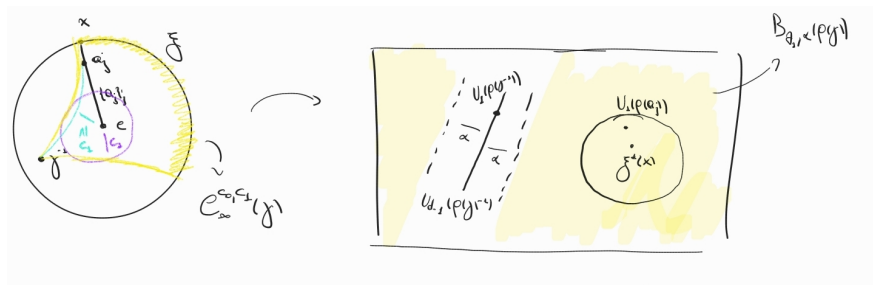
for  $c_0 = \nu^{-1}, c_1 = c'_0 + c'_1 |\log(\sin \alpha)|$ . The first inequality comes from [2, Lemma 3.9]. For the second inequality we estimate  $|\gamma^{-1}|$  from below using the triangle inequality. We are now ready to show that the concatenation  $(a'_j)_j$  is indeed a  $(c_0, c_1)$ -geodesic:

$$\begin{aligned} d(a_{L+j}, a_{L-i}) &\geq d(a_{L+j}, a_{L-K}) - d(a_{L-K}, a_{L-i}) \geq c_0^{-1}(j+K) - c_1 - (K-i) \geq \\ &\geq c_0^{-1}(j+i) - c_1. \end{aligned}$$

Note however that  $(a'_j)$  does not necessarily lie in  $C_{\infty}^{c_0, c_1}$  since it may not pass through  $e$ . For this reason we find some  $L-K \leq i_0 \leq L$  such that  $|a_{i_0}| < \alpha''$ , the existence of which is guaranteed by the fact that  $d([\gamma^{-1}, a_L], e) < \alpha''$ . We then consider alter  $(a'_j)$  at  $i_0$  so that it passes through  $e$  to obtain

$$a''_j = \begin{cases} a_j & \text{for } j \neq i_0 \\ e & \text{for } j = i_0 \end{cases}$$

which is a  $(c_0, c_1 + \alpha'')$ -quasigeodesic passing from  $e$  and converging to  $x$ .  $\square$



# Bibliography

- [1] François Ledrappier and Pablo Lessa. “Dimension gap and variational principle for Anosov representations”. In: (Dec. 2023). arXiv:2310.13465 [math] version: 3 (cit. on p. 3).
- [2] Beatrice Pozzetti, Andrés Sambarino, and Anna Wienhard. “Anosov representations with Lipschitz limit set”. In: *Geometry & Topology* 27.8 (Nov. 2023). arXiv:1910.06627 [math], pp. 3303–3360. issn: 1364-0380, 1465-3060 (cit. on pp. 7–10).