

# Limit sets of Anosov representations

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# Chapter 1

## Introduction

**Definition 1.0.1.** Let  $\Gamma$  be a discrete group of isometries of a metric space  $(X, d)$ . We define the critical exponent of  $\Gamma$  to be the asymptotic exponential growth of its orbits, i.e. the following limit:

$$\delta_\Gamma = \limsup_{n \rightarrow \infty} \frac{\log \#\{\gamma \in \Gamma : d(x, \gamma x) \leq n\}}{n}$$

for some fixed  $x \in X$ .

*Remark 1.0.1.* It is not hard to show that the critical exponent is independent of the choice of  $x$ .

### 1.1 Lie group preliminaries

We fix the Cartan subalgebra  $\mathfrak{a}$  of  $\mathrm{SL}(d, \mathbb{R})$ :

$$\mathfrak{a} = \{\mathrm{diag}(\alpha_1, \dots, \alpha_d) : \alpha_1 + \dots + \alpha_d = 0\}$$

and the Weyl chamber  $\mathfrak{a}^+$  of  $\mathrm{SL}(d, \mathbb{R})$

$$\mathfrak{a}^+ = \{\mathrm{diag}(\alpha_1, \dots, \alpha_d) : \alpha_1 \geq \dots \geq \alpha_d\}.$$

Denoting with  $K = \mathrm{SO}(d, \mathbb{R})$ ,  $A^+ = e^{\mathfrak{a}^+}$ , we have the Cartan decomposition:

$$\begin{aligned} \mathfrak{sl}(d, \mathbb{R}) &\rightarrow K \times A^+ \times K \\ g &\mapsto (k_g, a_g, l_g) \end{aligned}$$

such that  $g = k_g a_g l_g$ . In particular  $a_g = \mathrm{diag}(\sigma_1(g), \dots, \sigma_d(g))$  with  $\sigma_1 \geq \dots \geq \sigma_d(g)$ , where  $\sigma_i(g)$  is the  $i$ -th singular value of  $g$ , i.e. eigenvalue of  $g^t \cdot g$ .

We will use the spaces

$$U_p(g) = \mathbb{R}u_1(g) \oplus \dots \oplus \mathbb{R}u_p(g)$$

where  $u_i(g) = k_g \cdot e_i$ . One can easily show that the decomposition

$$g^{-1} \cdot U_p(g) \oplus U_{d-p}(g^{-1})$$

is orthogonal with respect to the standard inner product and that  $u_p(g^{-1}) = l_g^{-1} e_{d-p+1}(g)$ .

## 1.2 Limit set preliminaries

**Definition 1.2.1.** For  $p \in \{2, \dots, d\}$ ,  $s \in \mathbb{R}$  and  $g \in SL(d, \mathbb{R})$  we denote with  $\tilde{\Psi}_s^p(g), \Psi_s^p(g) : SL(d, \mathbb{R}) \rightarrow \mathbb{R}$  the functional:

$$\begin{aligned}\Psi_s^p(g) &= \alpha_{12}(a(g)) + \dots + \alpha_{1(p-1)}(a(g)) + (s - (p-2))\alpha_{1p}(a(g)) \\ \tilde{\Psi}_s^p(g) &= \left( \frac{\sigma_2}{\sigma_1} \dots \frac{\sigma_{p-1}}{\sigma_1}(g) \right) \left( \frac{\sigma_{p-1}}{\sigma_1}(g) \right)^{s-(p-2)}\end{aligned}$$

*Remark 1.2.1.* We have  $\alpha_{ij}(a) = a_i - a_j$ ,  $a_i(g) = \log(\sigma_i(g))$  and

$$\Psi_s^p(g) = \log \tilde{\Psi}_s^p(g)$$

and that

$$\min_{p \in \llbracket 2, d \rrbracket} \left\{ \sum_{|\gamma|=T} \frac{\sigma_2}{\sigma_1} \dots \frac{\sigma_{p-1}}{\sigma_1}(g) \left( \frac{\sigma_{p-1}}{\sigma_1}(g) \right)^{s-(p-2)} \right\} = \sum_{|\gamma|=T} e^{-\max_{p \in \llbracket 2, d \rrbracket} \Psi_s^p(g)}$$

*Remark 1.2.2.* For any  $g \in SL(d, \mathbb{R})$  we have that:

$$\max_{p \in \llbracket 2, d \rrbracket} \Psi_s^p(g) = \Psi_s^{p_0}(g) \text{ for } s \in [p_0 - 2, p_0 - 1].$$

Indeed, a quick calculation shows that for  $s \geq 0$  and  $p \in \llbracket 2, d \rrbracket$ :

$$\Psi_s^p(g) \leq \Psi_s^p(g) \text{ if and only if } s \geq p - 1.$$

and that equality holds in the case  $s = p - 1$ . Thus for  $s \in [p - 2, p - 1]$  we have that

$$\begin{aligned}s \geq p - 2, \dots, 1 \text{ implies that } \Psi_s^p(g) &\geq \dots \geq \Psi_s^2(g) \\ s \leq p, \dots, d - 1 \text{ implies that } \Psi_s^p(g) &\leq \dots \leq \Psi_s^d(g)\end{aligned}$$

Another way to see this (refer to Figure 1.1) is to note that  $\Psi_s^2(g), \dots, \Psi_s^d(g)$  is a sequence of functions that are affine in  $s$ , with slopes  $\alpha_{12}(g) \leq \dots \leq \alpha_{1d}(g)$  and that they satisfy  $\Psi_1^2(g) = \Psi_2^2(g), \Psi_2^3(g) = \Psi_3^4(g) \dots, \Psi_{d-2}^{d-1}(g) = \Psi_{d-2}^d(g)$ .

The following definition comes from [LL23], in the special case of projective Anosov representations ( $P = 1$ ):

**Definition 1.2.2.** For  $s \geq 0$  we consider the Falconer functional  $F_s : SL(d, \mathbb{R}) \rightarrow \mathbb{R}$  by:

$$F_s(g) = \min \left\{ \sum_{j=2}^d s_j \alpha_{1j}(a(g)) : s_j \in (0, 1], \sum_{j=2}^d s_j = s \right\},$$

and define the Falconer dimension  $\dim_F(\rho)$  of  $\rho$  to be its critical exponent:

$$\dim_F(\rho) = \inf \left\{ s > 0 : \sum_{\gamma \in \Gamma} e^{-F_s(\rho(\gamma))} < \infty \right\}.$$

*Remark 1.2.3.* Using elementary computations one may prove that for all  $s \geq 0$ :

$$F_s(g) = \max_{p \in \llbracket 2, d \rrbracket} \Psi_s^p(g)$$

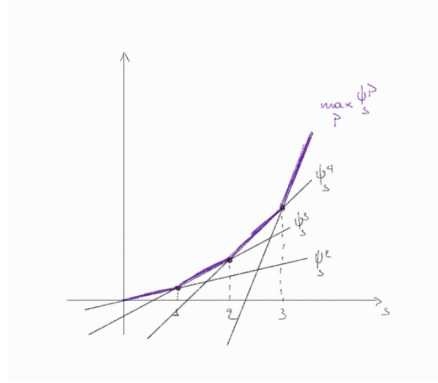


Figure 1.1: Visual illustration that  $\max_p \Psi_s^p(g) = \Psi_s^{p_0}(g)$  for  $s \in [p_0 - 2, p_0 - 1]$ .

**Definition 1.2.3.** Let  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  be a linear representation and  $p \in \llbracket 1, d - 1 \rrbracket$ . We say that  $\rho$  is  $p$ -Anosov if there exist constants  $\mu, C > 0$  such that for all  $\gamma \in \Gamma$ :

$$\frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma)) \leq C e^{-\mu|\gamma|}.$$

One can show that in that case there exist equivariant continuous maps  $\xi^p : \hat{\Gamma} \rightarrow \mathcal{G}_p(\mathbb{R}^d), \xi^{d-p} : \hat{\Gamma} \rightarrow \mathcal{G}_{d-p}(\mathbb{R}^d)$  that are transverse and restrict to

$$\xi^p(\gamma) = U_p(\rho(\gamma)), \xi^{d-p}(\gamma) = U_{d-p}(\rho(\gamma))$$

for  $\gamma \in \Gamma$ , where  $U_p(\gamma), U_{d-p}(\gamma)$  denote the flags corresponding to  $\rho(\gamma)$ .

Figure out what this exactly means

## Chapter 2

# Upper bound

### 2.1 Proof of bound

**Lemma 2.1.1** (Upper bound for dimension). *Let  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  be a projective Anosov representation. Then:*

$$\dim_H(\xi^1(\partial\Gamma)) \leq \dim_F(\rho).$$

*Remark 2.1.1.* The idea of the proof of Lemma 2.1.1 is to find a covering whose Hausdorff content is dominated by the Dirichlet series of some functional  $\Psi_s^p$ , which will in turn imply that  $\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(\Psi^p)$ . Choosing then the most "effective" cover (i.e. the one which yields the smallest Hausdorff content up to a constant) we obtain that

$$\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(\max_p \Psi^p)$$

To obtain this we first cover  $\xi^1(\partial\Gamma)$  by the bassins of attraction  $\rho(\gamma) \cdot B_{\alpha_1, \alpha}(\rho(\gamma))$  for  $\gamma \in \Gamma$  satisfying  $|\gamma| = T$ . Then we cover each bassin by an ellipsoid of axes lengths

$$\frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \dots, \frac{1}{\sin(\alpha)} \frac{\sigma_d}{\sigma_1}(\rho(\gamma)).$$

Finally we cover each ellipsoid by balls of some fixed radius  $r > 0$ . It can be shown by comparing the series appearing in the Hausdorff content of each resulting cover that the most "effective" choice of  $r$  depends only on the Hausdorff exponent  $s > 0$  and in any case will be to have  $r$  equal (up to a constant) to the the length of an axis of the ellipsoid, i.e.

$$r \in \left\{ \frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \dots, \frac{1}{\sin(\alpha)} \frac{\sigma_d}{\sigma_1}(\rho(\gamma)) \right\}$$

In particular, when  $s \in [p-2, p-1]$ , the most effective choice is  $r = \sigma_p(\rho(\gamma))/\sigma_1(\rho(\gamma))$ , whose Hausdorff content is dominated by the Dirichlet series of  $\Psi_s^p$ .

*Proof of Lemma 2.1.1.* Let  $p \in \llbracket 2, d \rrbracket$ . Then using Proposition 2.2.1, Lemma 2.2.2, and Lemma 2.2.3 we have that for  $T > 0$  large enough,  $\xi^1(\partial\Gamma)$  is covered by the family

$$\mathcal{U}_T = \{\rho(\gamma)B_{\alpha_1, \alpha}(\rho(\gamma)) : |\gamma| = T\},$$

and that each basin  $\rho(\gamma)B_{\alpha_1, \alpha}(\rho(\gamma))$  is in turn covered by

$$2^{p-2} \cdot \frac{\sigma_p(g)^{p-2}}{\sigma_2(g) \cdots \sigma_{p-1}(g)}$$

many balls of radius

$$\sqrt{d-1} \frac{1}{\sin \alpha} \frac{\sigma_p(g)}{\sigma_1(g)}.$$

By the definition of the Hausdorff measure, for  $s \geq 0$ :

$$\begin{aligned} \mathcal{H}^s(\xi^1(\partial\Gamma)) &\leq \sum_{|\gamma|=T} 2^{2p+1} \cdot \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \cdots \frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \left( \frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \right)^{-(p-2)} \left( \sqrt{d-1} \frac{1}{\sin \alpha} \frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \right)^s = \\ &= 2^{2p+1} \cdot \left( \frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \cdots \frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \left( \frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \right)^{s-(p-2)} = \\ &= 2^{2p+1} \cdot \left( \frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} e^{-(\alpha_{12} + \dots + \alpha_{1(p-1)} + (s-(p-2))\alpha_{1p})\rho(\gamma)} \\ &= 2^{2p+1} \cdot \left( \frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} e^{-\Psi_s^p(\rho(\gamma))} \end{aligned}$$

and thus

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq 2^{2p+1} \cdot \left( \frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} e^{-\max_p \Psi_s^p(\rho(\gamma))} \lesssim \sum_{|\gamma|=T} e^{-F_s(\rho(\gamma))}.$$

To see that the above implies the upper bound, consider some  $s > \dim_F(\rho)$ . By the definition of the Falconer dimension, this implies that the Dirichlet series corresponding to the Falconer functional converges:

$$\sum_{\gamma \in \Gamma} e^{-F_s(\rho(\gamma))} < \infty$$

and in particular

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq \lim_{T \rightarrow \infty} e^{-F_s(\rho(\gamma))} = 0.$$

□

## 2.2 Lemmata

**Definition 2.2.1.** Let  $V$  be a finite-dimensional  $\mathbb{R}$ -vector space. We consider a decomposition

$$V = \mathbb{R}u_1 \oplus \cdots \oplus \mathbb{R}u_d$$

be a direct decomposition that is orthogonal with respect to a fixed inner-product over  $V$ . Given  $\beta_2 \geq \dots \beta_d > 0$ , we define an ellipsoid with axes  $u_1 \oplus u_p(g)$  and lengths  $\beta_p$  to be the image of

$$\left\{ v = \sum_1^d v_i u_i \in V : \sum_2^d \left( \frac{v_j}{\beta_j} \right)^2 \leq 1 \right\}$$

through the projection  $V \rightarrow \mathbb{P}(V)$ .

The following aims to be something along the lines of [PSW23, Lemma 2.4]:

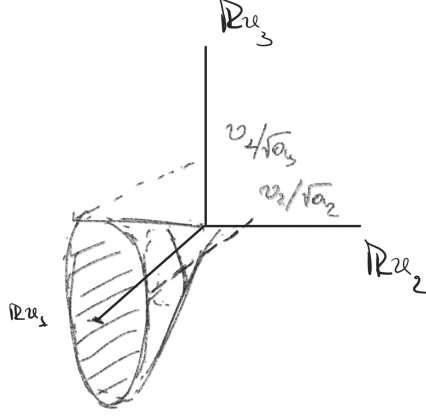


Figure 2.1: Depiction in  $\mathbb{R}^3$  of an ellipsoid of  $\mathbb{P}(\mathbb{R}^2)$

**Lemma 2.2.1.** *Let  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  be a projective Anosov representation. For  $\alpha > 0$  small enough, there exists  $L > 0$  such that for any geodesic ray  $(a_j)_j$  through  $e$  we have:*

$$\angle(U_1(\rho(a_i)), U_{d-1}(\rho(a_0))) > \alpha$$

when  $|a_i|, |a_0| > T$ .

*Proof.* Assume the contrary for the sake of contradiction. Then (see Figure 2.2) for each  $n > 0$  there exists a geodesic ray  $a^n$  through  $e$  such that

$$|a^n|, |a_0^n| > n \text{ and } \angle(U_1(\rho(a^n)), U_1(\rho(a_0^n))) < \frac{1}{n}.$$

Due to compactness of  $\Gamma \cup \partial\Gamma$  we can find some subsequence  $k_n$  and  $x, y \in \partial\Gamma$  such that  $a_{k_n}^{k_n} \rightarrow x$ ,  $a_0^{-k_n} \rightarrow y$  and  $x \neq y$ . Since the limit map is dynamics preserving, we have that

$$\angle(\xi^1(x), \xi^{d-1}(y)) = 0,$$

which contradicts its transversality property.  $\square$

The following is [PSW23, Proposition 3.5].

**Lemma 2.2.2.** *Let  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  be projective Anosov. Then for  $\alpha > 0$  small enough, there exists some  $T_0 > 0$  such that for all  $T \geq T_0$  the family*

$$\mathcal{U}_T = \{\rho(\gamma)B_{\alpha_1, \alpha}(\rho(\gamma)) : |\gamma| = T\}$$

*is an open covering of  $\xi^1(\partial\Gamma)$ .*

*Proof.* Let  $\alpha, T > 0$  be as in the statement of Lemma 2.2.1 and  $x \in \partial\Gamma$  be represented by a geodesic ray  $(\gamma_j)_{j \geq 0}$  starting from  $e$ . Then  $(\gamma_T^{-1}\gamma_j)_j$  is a geodesic ray starting from  $(\gamma_T)^{-1}$  that passes through  $e$ , so

$$\angle(U_1(\rho(\gamma_T^{-1}\gamma_j)), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$



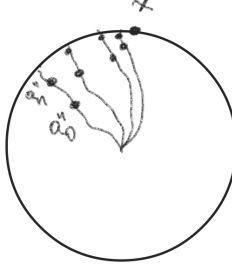


Figure 2.2: Situation in Lemma 2.2.1

as implied by Lemma 2.2.1. Taking the limit  $j \rightarrow \infty$  and using the equivariance of the limit map, we obtain

$$\angle(\rho(\gamma_T^{-1})\xi^1(x), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

and thus  $\xi^1(x) \in \rho(\gamma_T) \cdot B_{\alpha_1, \alpha}(\rho(\gamma_T))$ .  $\square$

The following is [PSW23, Proposition 3.8].

**Proposition 2.2.1.** *For each  $g \in \text{SL}(d, \mathbb{R})$ ,  $\alpha > 0$ , the basin of attraction  $g \cdot B_{\alpha_1, \alpha}(g)$  lies in the ellipsoid with axes  $u_1(g) \oplus u_p(g)$  with lengths*

$$\frac{1}{\sin \alpha} \cdot \frac{\sigma_p(g)}{\sigma_1(g)}$$

*Proof.* Using the definition of the basin of attraction (see Figure 2.3 ), we have that  $w = w_1 u_1(g^{-1}) + \dots + w_d u_d(g^{-1}) \in B_{\alpha_1, \alpha}(g)$  if and only if

$$w_d^2 \geq (\sin \alpha)^2 \sum_1^d w_i^2.$$

Considering now some  $v = v_1 u_1(g) + \dots + v_d u_d(g) \in g \cdot B_{\alpha_1, \alpha}(g)$  we have that

$$\begin{aligned} w &= g^{-1}v = v_1 \sigma_1(g)^{-1} l_g^{-1} e_1(g) + \dots + v_d \sigma_d(g)^{-1} l_g^{-1} e_d(g) \\ &= v_1 \sigma_1(g)^{-1} u_d(g^{-1}) + \dots + v_d \sigma_d(g)^{-1} u_1(g^{-1}) \end{aligned}$$

where we used that  $u_p(g^{-1}) = l_g^{-1} e_{d+1-p}$ . Hence

$$\sigma_1(g)^{-2} \cdot v_1^2 \geq (\sin \alpha)^2 \sum_1^d \sigma_i(g)^{-2} v_i^2.$$

$\square$

The following is [PSW23, Lemma 3.7]:

**Lemma 2.2.3.** *For any  $p \in \llbracket 2, d \rrbracket$ , an ellipsoid in  $\mathbb{P}(\mathbb{R}^d)$  of axes lengths  $\beta_2, \dots, \beta_d$  is covered by*

$$2^{p-2} \frac{\beta_2 \cdots \beta_{p-1}}{\beta_p^{p-2}}$$

*many (projected) balls of radius  $\sqrt{d-1} \beta_p$ .*

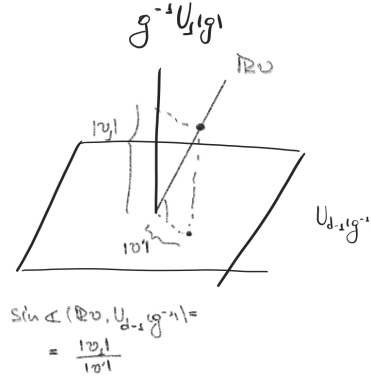


Figure 2.3: Aid for Proposition 2.2.1

*Proof.* We assume that  $E$  is an ellipsoid about  $\mathbb{R}e_1$ , so it suffice to cover its intersection  $E_1 = E \cap U_1 \subseteq \mathbb{R}^{d-1}$  with the affine chart  $U_1 = \{[x_1 : \dots, x_d] \in \mathbb{P}(\mathbb{R}^d) : x_1 \neq 0\}$ . Clearly  $E_1 \subseteq [-\beta_2, \beta_2] \times \dots \times [-\beta_d, \beta_d]$ , so we proceed by covering the rectangle with side-lengths  $2\beta_2, \dots, 2\beta_d$ . Clearly each interval  $(-\beta_j, \beta_j)$  is contained in the union of  $\lceil \beta_j/\beta_p \rceil$  intervals of length  $2\beta_p$ , thus  $E_1$  is contained in the union of

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil \dots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil = \left\lceil \frac{\beta_2}{\beta_p} \right\rceil \dots \left\lceil \frac{\beta_d}{\beta_p} \right\rceil$$

many squares of side-length  $2\beta_p$ . Since each such product is contained in a  $(d-1)$ -ball of radius  $\sqrt{d-1}\beta_p$  we may use at most

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil \dots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil \leq \sum_{i \in \{0,1\}^{p-2}} \prod_{j=2}^{p-1} \left( \frac{\beta_j}{\beta_p} \right)^{i_j} \leq 2^{p-2} \frac{\beta_2}{\beta_p} \dots \frac{\beta_{p-1}}{\beta_p}$$

many  $(d-1)$ -balls of radius  $\sqrt{d-1}\beta_p$  to cover  $E_1$ .  $\square$

The following can be found in [PSW23, Proposition 3.3]:

**Proposition 2.2.2.** *Let  $\rho : \Gamma \rightarrow SL(d, \mathbb{R})$  be projective Anosov and  $\alpha > 0$ . Then there exist  $c_0, c_1 > 0$  that depends only on  $\alpha$  and  $\rho$  such that for all  $\gamma \in \Gamma$ :*

$$(\xi^1)^{-1}(B_{\alpha_1, \alpha}(\rho(\gamma))) \subseteq C_{c_0, c_1}^\infty(\gamma)$$

*Proof.* We begin by noting that it suffices to show this for all but finitely many  $\gamma \in \Gamma$ , since then we may alter the constants to satisfy the wanted inclusion for the finitely many remaining  $\gamma \in \Gamma$  as well. Hence, we may assume that  $|\gamma| \geq l_0$  where  $l_0 > 0$  is such that

$$Ce^{-\mu l_0} < 1 \text{ and } \mathbf{a}_1(\gamma) \geq C|\gamma| - L.$$

Suppose  $x \in \partial\Gamma$  such that  $\xi^1(x) \in B_{\alpha_1, \alpha}(\rho(\gamma))$ , and consider a geodesic ray  $a_j \rightarrow x$  starting from  $a_0 = e$ . To prove the result, it suffices to find constants  $c_0, c_1$  independent of  $\gamma$  and for which there exists a  $(c_0, c_1)$ -quasi-geodesic from  $\gamma^{-1}$  to  $x$  that passes through  $e$  and stays at a bounded distance from  $(a_j)_{j=0}^\infty$ .

Using the exponential convergence rate of  $\xi^1(a_j) \rightarrow \xi^1(x)$  and the definition of  $B_{\alpha_1, \alpha}(\rho(\gamma))$  we have that:

$$\begin{aligned} d(\xi^1(a_j), \xi^1(\gamma)) &\geq d(\xi^1(x), U_1(\rho(\gamma^{-1})) - d(\xi^1(a_j), \xi^1(x))) \geq \\ &\geq d(\xi^1(x), U_{d-1}(\rho(\gamma^{-1})) - d(\xi^1(a_j), \xi^1(x))) \geq \sin \alpha - Ce^{-\mu j} \end{aligned}$$

which along with the uniform continuity of  $\xi^1 : \Gamma \cup \partial\Gamma \rightarrow \mathbb{P}(\mathbb{R}^d)$  this implies there exists some  $\alpha' > 0$  and  $L > 0$  such that for all  $j \geq L$ :

$$d(a_j, \gamma^{-1}) \geq \alpha'.$$

Upon considering a large  $L$ , we may also assume that  $|a_L| = L > l_0$ . Note that both  $\alpha'$  and  $L$  do not depend on each  $\gamma$  but only on  $\rho$  and  $\alpha$ .

Using a coarse geometric argument, we can show that for all  $j \geq L$

$$d(\gamma^{-1}, a_j) > \alpha' \Rightarrow d([\gamma^{-1}, a_j], e) < \alpha''$$

for some  $\alpha''$  that depends only on  $\Gamma$  and  $\alpha'$ , where  $[a_j, \gamma^{-1}]$  denotes any geodesic segment connecting  $\gamma^{-1}$  and  $a_j$ . Indeed, [GH13, Lemme 2.17] states that  $d([\gamma^{-1}, a_j]) \leq (\gamma_j^{-1}, a_j)_e + \delta$  where  $\delta$  is the hyperbolicity constant of  $\Gamma$ . Thus

$$d([\gamma^{-1}, a_j]) \leq \delta + \frac{d(a_j, e) + d(\gamma^{-1}, e) + d(a_j, \gamma^{-1})}{2} \leq \delta + \frac{L + d(\gamma^{-1}, e) + \alpha'}{2}.$$

Consider the concatenation  $(a'_j)_{j=L-K}^\infty$  of  $[\gamma^{-1}, a_L]$  and  $[a_L, x]$ . To find quasi-geodesic-constants that are uniform in  $\gamma$ , we note that for any  $c_0 \geq 1, c_1 \geq 0$ :

$$c_0^{-1}|i - j| - c_1 \leq d(a'_i, a'_j) = d(a_i, a_j) \leq c_0|i - j| + c_1 \text{ when } i, j \geq L \text{ or } i, j \leq L$$

and that the upper bound follows trivially by the triangle inequality.

For the lower bound we proceed in two steps. First we bound the distance of  $\gamma^{-1} = a'_{L-K}$  to  $a_{L+j}$  for  $j \geq 0$ :

$$\begin{aligned} d(a'_{L-K}, a'_{L+j}) &\geq \nu(|a_{L+j}| - |\gamma^{-1}|) - c'_0 - c'_1 |\log(d(U_1(\rho(a_{L+j})), U_1(\rho(\gamma^{-1}))))| \geq \\ &\geq \nu((L+j) + (K-L)) - c'_0 - c'_1 |\log(\sin a)| \geq \\ &= c_0^{-1}(j+K) - c_1 \end{aligned}$$

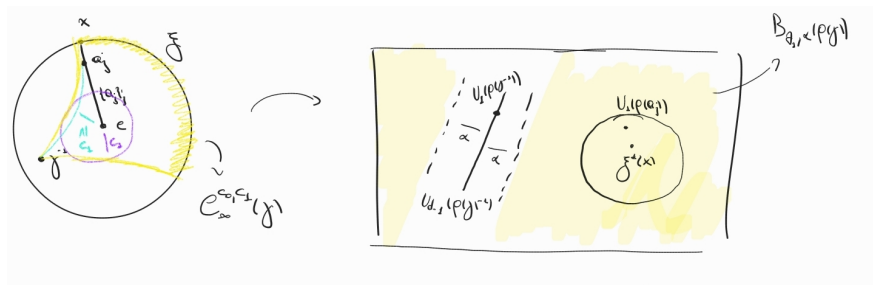
for  $c_0 = \nu^{-1}, c_1 = c'_0 + c'_1 |\log(\sin a)|$ . The first inequality comes from [PSW23, Lemma 3.9]. For the second inequality we estimate  $|\gamma^{-1}|$  from below using the triangle inequality. We are now ready to show that the concatenation  $(a'_j)_j$  is indeed a  $(c_0, c_1)$ -geodesic:

$$\begin{aligned} d(a_{L+j}, a_{L-i}) &\geq d(a_{L+j}, a_{L_K}) - d(a_{L_K}, a_{L-i}) \geq c_0^{-1}(j+K) - c_1 - (K-i) \geq \\ &\geq c_0^{-1}(j+i) - c_1. \end{aligned}$$

Note however that  $(a'_j)$  does not necessarily lie in  $C_\infty^{c_0, c_1}$  since it may not pass through  $e$ . For this reason we some  $L-K \leq i_0 \leq L$  such that  $|a_{i_0}| < \alpha''$ , the existence of which is guaranteed by the fact that  $d([\gamma^{-1}, a_L], e) < \alpha''$ . We then consider alter  $(a'_j)$  at  $i_0$  so that it passes through  $e$  to obtain

$$a''_j = \begin{cases} a_j & \text{for } j \neq i_0 \\ e & \text{for } j = i_0 \end{cases}$$

which is a  $(c_0, c_1 + \alpha'')$ -quasigeodesic passing from  $e$  and converging to  $x$ .  $\square$



# Chapter 3

## Lower bound

### 3.1 Busemann cocycle and Patterson-Sullivan measures

We denote with  $\Pi$  the set of simple positive roots, and for  $\Theta \subseteq \Pi$  we consider the Levi-Anosov subspace of  $\mathfrak{a}$

$$\mathfrak{a}_\Theta = \bigcap_{\alpha \notin \Theta} \ker \alpha,$$

which in particular admits  $\{\omega_{\alpha_i} : \alpha_i \in \Theta\}$  as a basis.

**Definition 3.1.1.** Let  $\Theta \subseteq \Pi$ . We define the Busemann cocycle

$$b : \mathrm{PSL}(d, \mathbb{R}) \times \mathcal{F}_\Theta \rightarrow \mathfrak{a}_\Theta$$

as the unique element  $b(g, kP_\Theta) \in \mathfrak{a}_\Theta$  such that

$$gk \in Ke^{b(g, kP_\Theta)}N.$$

where  $N = \{n \in \mathrm{SL}(d, \mathbb{R}) : n_{ij} = 0 \text{ for } i > j, n_{ii} = 1 \text{ for all } i\}$  is the unipotent group of upper subgroup of upper triangular matrices with 1s on the diagonal, and  $P_\Theta$  is the parabolic subgroup of  $\mathrm{PSL}(d, \mathbb{R})$  corresponding to  $\Theta$ , i.e.  $\mathcal{F}_\Theta = \mathrm{PSL}(d, \mathbb{R})/P_\Theta$ .

**Lemma 3.1.1.** For  $y \in \mathcal{F}_\Theta(\mathbb{R}^d)$  we have

$$\omega_{\alpha_i}(b_\Theta(g, x)) = \log \frac{\|gv_1 \wedge \cdots \wedge gv_i\|}{\|v_1 \wedge \cdots \wedge v_i\|} \text{ for all } \alpha_i \in \Theta$$

for any basis  $v_1, \dots, v_i$  of  $x^i \in \mathcal{G}_i(\mathbb{R}^d)$ , where  $\|\cdot\|$  denotes the norm on  $\bigwedge^i \mathbb{R}^d$  induced by the euclidean inner product on  $\mathbb{R}^d$ , i.e.  $\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle := \det(\langle v_i, w_j \rangle)$ .

**Definition 3.1.2.** We define

$$\Lambda^k : \mathrm{SL}(d, \mathbb{R}) \rightarrow \mathrm{SL}(\bigwedge^k \mathbb{R}^d), \quad \Lambda^k : \mathcal{G}_k(\mathbb{R}^d) \rightarrow \mathbb{P}(\bigwedge^k \mathbb{R}^d)$$

as

$$\Lambda^k(g)(v_1 \wedge \cdots \wedge v_k) = gv_1 \wedge \cdots \wedge gv_k \Lambda^k(\mathbb{R}v_1 \oplus \cdots \oplus \mathbb{R}v_k) = v_1 \wedge \cdots \wedge v_k$$

**Lemma 3.1.2.** Let  $g \in \mathrm{SL}(d, \mathbb{R})$ ,  $\alpha > 0$  and  $k \in \llbracket 2, d \rrbracket$ .

(i)  $\omega_1(a(\Lambda^k g)) = \omega_k(a(g))$  and  $\omega_1(b(\Lambda^k g, \Lambda^k y)) = \omega_k(b(g, y))$ .

(ii) There exists some  $\alpha' > 0$  independent of  $g$  such that  $\Lambda^k B_{\mathfrak{a}_k, \alpha}(\Lambda^k g) \subseteq B_{\mathfrak{a}_1, \alpha'}(\Lambda^k g)$ .

*Proof.* (i) Follows from the definitions of the fundamental weights and the Cartan projection.

(ii) Let  $g = k_g e^{a(g)} l_g$  be the Cartan decomposition of  $g$ . Then using the definitions of the respective subspaces:

$$\begin{aligned} U_{d-k}(g^{-1} l_g^{-1}) &= \mathbb{R} e_{k+1} \oplus \cdots \oplus \mathbb{R} e_d \\ x_0 := U_{d-1}(\Lambda^k g^{-1} l_g^{-1}) &= \bigoplus_{\substack{i_1 < \cdots < i_k \\ (i_1, \dots, i_k) \neq (1, \dots, k)}} \mathbb{R} e_{i_1} \oplus \cdots \oplus \mathbb{R} e_{i_k} \end{aligned}$$

The first equality implies that

$$y \in B_{\mathfrak{a}_k, \alpha}(g) \Leftrightarrow l_g y \in B_{\mathfrak{a}_k, \alpha}(g l_g^{-1}) = B_{\mathfrak{a}_k, \alpha}(\text{Id}),$$

so for every  $y \in B_{\mathfrak{a}_k, \alpha}(g)$  we have that

$$l_g y = l U_k(\text{Id}) \text{ for some } l \in L$$

where

$$L = \{l \in \text{SO}(d, \mathbb{R}) : l U_k(\text{Id}) \in B_{\mathfrak{a}_k, \alpha}(\text{Id})\}.$$

Note that  $L$  is compact, being a closed subset of a compact group. Moreover, the fact that  $\Lambda^k(y) \notin U_{d-1}(\Lambda^k g^{-1})$  implies that

$$0 < \angle(\Lambda^k(y), U_{d-1}(\Lambda^k g^{-1})) = \angle(\Lambda^k(l_g y), U_{d-1}(\Lambda^k g^{-1} l_g^{-1})) = \angle(\Lambda^k(l) U_k(\text{Id}), x_0)$$

The right-hand side is in the image of the compact set  $L$  under a continuous map, so it is bounded below by a positive number  $\alpha' > 0$ .

(iii) Follows from the definition of the Cartan projection and the Busemann cocycle.  $\square$

**Definition 3.1.3.** For a discrete subgroup  $\Gamma < \text{PSL}(d, \mathbb{R})$ ,  $\phi \in (\alpha_\Theta)^*$ , a  $(\Gamma, \phi)$ -Patterson Sullivan measure on  $\mathcal{F}_\Theta$  is a finite Radon measure  $\mu$  such that for every  $\gamma \in \Gamma$

$$\frac{d\gamma_* \mu}{d\mu}(x) = e^{-\phi(b_\Theta(g^{-1}, x))}, \text{ for all } x \in \mathcal{F}_\Theta(\mathbb{R}^d).$$

**Lemma 3.1.3.** Let  $\alpha > 0, \Theta \subseteq \Pi$ . There exists  $K = K(\alpha) > 0$  such that for each  $g \in \text{SL}(d, \mathbb{R})$ ,  $\mathfrak{a}_i \in \Theta$ ,  $\mathfrak{y} \in B_{\Theta, \alpha}(g)$ ,  $\phi \in \mathfrak{a}_\Theta$

$$|\phi(a(g) - b(g, y))| \leq K.$$

*Proof.* We begin by noting that it suffices to consider the case where  $\phi = \mathfrak{a}_k$  for  $\mathfrak{a}_k \in \Theta$ , since  $\{\omega_i\}_{\mathfrak{a}_i \in \Theta}$  is a basis for  $\mathfrak{a}_\Theta^*$ .

We first consider the case where  $k = 1$ . We recall that the first component of the Cartan projection coincides with the spectral norm of  $g$ , i.e.

$$a_1(g) = \log \sup_{v \neq 0} \frac{\|gv\|}{\|v\|} = \log \|gk_2^{-1} e_1\|$$

where  $g = k_1 e^{a(g)} k_2$  is the Cartan decomposition of  $g$ . Let  $v = v_1 k_2^{-1} e_1 + \cdots + v_d k_2^{-1} e_d \in \mathbb{R}^d$  be such that  $\|v\| = 1$  and  $y = \mathbb{R}v$ , we have

$$\begin{aligned} |\omega_1(a(g) - b(g, y))| &= |\log \|g k_2^{-1} e_1\| - \log \|g v\|| = \\ &= |\log |e^{a_1(g)}| - \log \|e^{a_1(g)} v_1 k_1 e_1 + \cdots + e^{a_d(g)} v_d k_1 e_d\|| = \\ &= \left| \log \left\| v_1 k_1 e_1 + e^{-a_{12}(g)} v_2 k_1 e_2 + \cdots + e^{-a_{1d}(g)} v_d k_1 e_d \right\| \right| \leq \\ &\leq |\log |v_1|| = |\log \sin(\angle(v, U_{d-1}(g^{-1})))| \leq |\log \sin \alpha|. \end{aligned}$$

For the case where  $\Theta = \{\mathbf{a}_k\}$ , we consider  $\alpha' > 0$  such that

$$\Lambda^k(B_{\mathbf{a}_k, \alpha}(g)) \subseteq B_{\mathbf{a}_k, \alpha'}(\Lambda^k g)$$

Then using the case  $k = 1$  we have that

$$|\omega_k(a(g) - b(g, y))| = |\omega_1(a(\Lambda^k g) - b(\Lambda^k g, \Lambda^k y))| \leq |\log \sin \alpha'|.$$

□

## 3.2 Proof strategy

Denoting with  $d_\Gamma = \dim_H \xi_\rho^1(\partial\Gamma)$  the Hausdorff dimension of the limit set, we shall outline how to obtain its lower bound by the Falconer dimension

$$d_\Gamma \geq h_\rho(F).$$

First we recall that  $F_s(a) = \max\{\Psi_s^p(a) : p \in \llbracket 2, d \rrbracket\}$  and in particular  $h_\rho(F) \leq h_\rho(\Psi^{d_\Gamma+1})$ . Thus the lower bound will follow once we have shown that

$$d_\Gamma \geq h_\rho(\Psi^{d_\Gamma+1}).$$

Noting that  $(s+1)J_{d_\Gamma}^u \leq \Psi_{s+d_\Gamma}^{d_\Gamma+1}$ , the above bound will follow as soon as we have shown that

$$h_\rho(J_{d_\Gamma}) \leq 1. \tag{LB}$$

To obtain inequality (LB), one uses the method of Patterson-Sullivan-Quint, which may be summed up in

There exists a  $(\phi, \rho)$ -Patterson-Sullivan measure on  $\mathcal{F}_\Theta(\mathbb{R}^d) \Rightarrow h_\rho(\phi) \leq 1$ ,

where  $\phi \in \mathfrak{a}_\Theta$  and  $\Theta \subseteq \Pi$ . The property that we will need of our measure is that there exists a collection of open sets  $U_\gamma \in \Gamma$  such that

$$\mu(U_\gamma) \sim e^{-J_{d_\Gamma}^u(a(\rho(\gamma)))} \text{ and } M := \sup_{n \in \mathbb{N}} \max \left\{ \sharp A : A \subseteq \Gamma_n, \bigcap_{\gamma \in A} U_\gamma \neq \emptyset \right\} < \infty \tag{MP}$$

where  $\Gamma_n = \{\gamma \in \Gamma : |\gamma| = n\}$ . For the proof of the existence of a  $(J_{d_\Gamma}^u, \rho)$ -Patterson-Sullivan measure that satisfies the property (MP) we refer to Section 3.3, noting that the Zariski-density assumption is necessary only for the equivalence appearing on the left hand side of

Equation (MP). Assuming that for the time being, below we outline the Patterson-Sullivan-Quint method of obtaining Equation (LB).

Indeed, we first obtain the uniform in  $n$  bound:

$$\sum_{\gamma \in \Gamma_n} e^{-J_{d_\Gamma}^u(a(\rho(\gamma)))} \lesssim \sum_{\gamma \in \Gamma_n} \mu(U_\gamma) \leq \frac{1}{M} \mu(\mathcal{F}_\Theta(\mathbb{R}^d)) < \infty$$

along with the bound implied by the Anosov property of  $\rho$ :

$$J_{d_\Gamma}(a(\rho(\gamma))) \geq \mathbf{a}_{12}(a(\rho(\gamma))) \geq C|\gamma| - b$$

to conclude that

$$\sum_{\gamma \in \Gamma} e^{-(s+1)J_{d_\Gamma}^u(a(\rho(\gamma)))} = \sum_{n \geq 0} \sum_{\gamma \in \Gamma_n} e^{-sJ_{d_\Gamma}^u(a(\rho(\gamma)))} e^{J_{d_\Gamma}^u(a(\rho(\gamma)))} \lesssim \sum_{n \geq 0} e^{-s(Cn-b)} < \infty$$

which holds for any  $s > 0$ , and thus Equation (LB) holds.

### 3.3 Existence of Patterson-Sullivan measure

**Definition 3.3.1.** For  $p \in \llbracket 2, d \rrbracket$ , we denote the  $p$ -th unstable Jacobian  $J_p^u \in \mathfrak{a}^*$  by

$$J_p^u = (p+1)\omega_{\mathfrak{a}_1} - \omega_{\mathfrak{a}_{p+1}} = \mathbf{a}_{12} + \cdots + \mathbf{a}_{1(p+1)}.$$

**Definition 3.3.2.** Let  $V \in \mathcal{G}_{p+1}\mathbb{R}^d$  and  $l \in \mathbb{P}(V)$ . Using the canonical identification  $T_l\mathbb{P}(V) \simeq \text{hom}(l, V/l)$ , we define the density  $|\Omega_{l,V}|$  on  $\bigwedge^p T_l\mathbb{P}(V)$  by

$$|\Omega_{l,V}|(\phi_1, \dots, \phi_p) = \frac{\|v \wedge \tilde{\phi}_1(v) \wedge \cdots \wedge \tilde{\phi}_p(v)\|}{\|v\|^{p+1}}$$

for any  $v \in l - \{0\}$ , where  $\tilde{\phi}_1, \dots, \tilde{\phi}_p \in \text{hom}(l, V)$  are such that  $\phi_i = \tilde{\phi}_i + \text{hom}(l, l)$  and  $\|\cdot\|$  denotes the norm on  $\bigwedge^{p+1}\mathbb{R}^d$  induced by the euclidean inner product.

The following is [PSW23, Proposition 6.4]:

**Proposition 3.3.1.** *Assume that  $\xi_\rho^1(\partial\Gamma)$  is a Lipschitz submanifold of dimension  $d_\Gamma$ . Then there exists a  $(\rho(\Gamma), J_{d_\Gamma}^u)$ -Patterson-Sullivan measure on  $\mathcal{F}_{1,d_\Gamma+1}$ .*

*Proof.* By Rademacher's theorem,  $\xi_\rho^1(\partial\Gamma)$  has a well-defined Lebesgue measure class, and Lebesgue-almost every  $\xi_\rho^1(x) \in \xi_\rho^1(\partial\Gamma)$  admits a well-defined tangent space  $T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma)$ . Considering such a  $\xi_\rho^1(x)$  we let

$$\pi : \text{hom}(\xi_\rho^1(x), \mathbb{R}^d) \rightarrow \text{hom}(\xi_\rho^1(x), \mathbb{R}^d/\xi_\rho^1(x)) \simeq T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma),$$

and

$$x^{d_\Gamma+1} = \pi^{-1}(T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma))\xi_\rho^1(x) \in \mathcal{G}_{d_\Gamma+1}(\mathbb{R}^d),$$

for which one can show that

$$T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma) \simeq \text{hom}(\xi_\rho^1(x), \mathbb{R}^d/\xi_\rho^1(x)) \simeq \text{hom}(\xi_\rho^1(x), x^{d_\Gamma+1}/\xi_\rho^1(x)).$$

In this notation, we shall define (Lebesgue-almost everywhere) the map

$$\zeta_\rho : \xi_\rho^1(\partial\Gamma) \rightarrow \mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d), \quad \zeta_\rho(\xi_\rho^1(x)) = (\xi_\rho^1(x), x^{d_\Gamma+1}).$$



We now define the non-negative density on  $\xi_\rho^1(\partial\Gamma)$

$$\mu_{\xi_\rho^1(x)} = |\Omega_{\zeta_\rho(\xi_\rho^1(x))}|$$

which satisfies

$$\frac{d\gamma_*\mu}{d\mu}(\xi) = \frac{d(\rho(\gamma)^{-1})^*\mu}{d\mu}(\xi) = e^{-J_{d_\Gamma+1}^u(b_\Theta(\rho(\gamma)^{-1}, \zeta(x)))},$$

where the term on the left-hand side involves measures, the term between the two equalities involves densities, and  $\Theta = \{1, d_\Gamma + 1\}$ . Indeed, for  $\phi_1, \dots, \phi_{d_\Gamma} \in T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma)$ :

$$\begin{aligned} & (\rho(\gamma)^*\mu)_{\xi_\rho^1(x)}(\phi_1, \dots, \phi_{d_\Gamma}) \\ &= \mu_{\rho(\gamma)\xi_\rho^1(x)}(\rho(\gamma)\phi_1\rho(\gamma)^{-1}, \dots, \rho(\gamma)\phi_{d_\Gamma}\rho(\gamma)^{-1}) = \\ &= \frac{\|\rho(\gamma)\xi_\rho^1(x) \wedge \rho(\gamma)\phi_1(\xi_\rho^1(x)) \wedge \dots \wedge \rho(\gamma)\phi_{d_\Gamma}(\xi_\rho^1(x))\|}{\|\rho(\gamma)\xi_\rho^1(x)\|^{d_\Gamma+1}} = \\ &= \frac{\|\rho(\gamma)\xi_\rho^1(x) \wedge \rho(\gamma)\phi_1(\xi_\rho^1(x)) \wedge \dots \wedge \rho(\gamma)\phi_{d_\Gamma}(\xi_\rho^1(x))\|}{\|\xi_\rho^1(x) \wedge \phi_1(\xi_\rho^1(x)) \wedge \dots \wedge \phi_{d_\Gamma}(\xi_\rho^1(x))\|} \cdot \frac{\|\xi_\rho^1(x) \wedge \phi_1(\xi_\rho^1(x)) \wedge \dots \wedge \phi_{d_\Gamma}(\xi_\rho^1(x))\|}{\|\rho(\gamma)\xi_\rho^1(x)\|^{d_\Gamma+1}} \cdot \frac{\|\rho(\gamma)\xi_\rho^1(x)\|^{d_\Gamma+1}}{\|\xi_\rho^1(x)\|^{d_\Gamma+1}} = \\ &= e^{\omega_{d_\Gamma}(b_\Theta(\rho(\gamma), \zeta_\rho(\xi_\rho^1(x))))} \cdot \mu_{\xi_\rho^1(x)}(\phi_1, \dots, \phi_{d_\Gamma}) \cdot e^{-(p+1)\omega_1(b_\Theta(\rho(\gamma), \zeta_\rho(\xi_\rho^1(x))))} = \\ &= e^{-J_{d_\Gamma}^u(b_\Theta(\rho(\gamma)^{-1}, \zeta(\xi_\rho^1(x))))} \mu_{\xi_\rho^1(x)}(\phi_1, \dots, \phi_{d_\Gamma}). \end{aligned}$$

Finally, we let  $\nu = \zeta_{\rho*}\mu$ , which is the wanted Patterson-Sullivan measure on  $\mathcal{F}_{1, d_\Gamma+1}(\mathbb{R}^d)$ , since for  $f \in C_c(\mathcal{F}_{1, d_\Gamma+1}(\mathbb{R}^d))$ :

$$\begin{aligned} \int_{\mathcal{F}_{1, d_\Gamma+1}(\mathbb{R}^d)} f d(\gamma_*\zeta_{\rho*}\mu) &= \int_{\xi_\rho^1(\partial\Gamma)} f \circ \gamma \circ \zeta_\rho d\mu = \int_{\xi_\rho^1(\partial\Gamma)} f \circ \zeta_\rho \circ \gamma d\mu = \\ &= \int_{\xi_\rho^1(\partial\Gamma)} f \circ \zeta_\rho(\xi) e^{-J_{d_\Gamma}^u(b_\Theta(\rho(\gamma)^{-1}, \zeta(\xi_\rho^1(x))))} d\mu(\xi_\rho^1(x)) = \\ &= \int_{\mathcal{F}_{1, d_\Gamma+1}(\mathbb{R}^d)} f(y) e^{-J_{d_\Gamma}^u(b_\Theta(\rho(\gamma)^{-1}, y))} d(\zeta_{\rho*}\mu)(y) \end{aligned}$$

□

Before giving the next definition, we recall that the annihilator annihilator of an element  $y \in \mathcal{F}_F i\Theta(\mathbb{R}^d)$  is the set of partial flags that are not transverse to  $y$ , that is:

$$\text{Ann}(y) = \{x \in \mathcal{F}_\Theta(\mathbb{R}^d) : x^\theta \cap y^{d-\theta} \neq \emptyset \text{ for some } \theta \in \Theta\}.$$

**Definition 3.3.3.** Let  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$  be a linear representation,  $\Theta \subseteq \Pi$  and  $\mu$  a measure over  $\mathcal{F}_\Theta(\mathbb{R}^d)$ . We say that  $\rho$  is  $\mu$ -irreducible there is no element in  $\mathcal{F}_{i\Theta}(\mathbb{R}^d)$ , whose annihilator is of full measure, i.e. for all  $y \in \mathcal{F}_{i\Theta}(\mathbb{R}^d)$ :

$$\mu(\text{Ann}(y)) < \mu(\mathcal{F}_\Theta(\mathbb{R}^d)).$$

**Example 3.3.1.** If  $\rho(\Gamma)$  is Zariski-dense in  $\text{SL}(d, \mathbb{R})$ , then  $\rho$  is  $\mu$ -irreducible for any  $\rho$ -quasi-equivariant measure  $\mu$ , and in particular for any  $(\rho(\Gamma), \phi)$ -Patterson-Sullivan measure.

*Remark 3.3.1.* The reason that we introduce the concept of  $\mu$ -irreducibility is that for any  $\mu$ -irreducible representation  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$ , there exist  $\alpha, \kappa > 0$  such that  $\mu(B_{\Theta, \alpha}(\rho(\gamma))) \geq \kappa$  for all  $\gamma \in \Gamma$ .

Indeed, if this were not the case, then there would exist a sequence  $\alpha_n \searrow 0$  and  $\gamma_n \in \Gamma$  such that

$$\mu(B_{\theta, \alpha}(\rho(\gamma))) \leq \frac{1}{n}.$$

Due to the compactness of  $\mathcal{F}_\Theta(\mathbb{R}^d)$ , up to considering a subsequence, we may assume that the repelling flags or  $\rho(\gamma_n)$  converge to some  $\xi \in \mathcal{F}_\Theta(\mathbb{R}^d)$ :

$$(U_{d-i}(\rho(\gamma_n)^{-1}))_{a_i \in \Theta} \rightarrow \xi$$

In that case, the complements  $B_{\Theta, \alpha_n}^c(\rho(\gamma_n))$  will converge to the annihilator of  $\xi$ , in the sense:

$$\limsup_n B_{\Theta, \alpha_n}^c(\rho(\gamma_n)) \subseteq \text{Ann}(\xi).$$

Indeed, let  $y \in \limsup_n B_{\Theta, \alpha_n}^c(\rho(\gamma_n))$  and consider a subsequence  $k_n$  such that  $y \in B_{\Theta, \alpha_n}^c(\rho(\gamma_{k_n}))$ . By the very definition of  $B_{\Theta, \alpha_n}^c(\rho(\gamma_n))$ , there exists some  $p$  such that up to considering a subsequence of  $k_n$ ,

$$\angle(y^p, U_{d-p}(\rho(\gamma_n)^{-1})) \leq \alpha_n$$

holds. Taking the limit as  $n \rightarrow \infty$ , we have that  $y^p \cap \xi^{d-p} \neq 0$  and hence  $y \in \text{Ann}(\xi)$ .

Using a measure-theoretic argument we conclude that  $\text{Ann}(\xi)$  is of full measure, which contradicts the  $\mu$ -irreducibility of  $\rho$ :

$$\mu(\text{Ann}(\xi)) \geq \mu(\limsup_n B_{\Theta, \alpha_n}^c(\rho(\gamma_{k_n}))) \geq \limsup_n \mu(B_{\Theta, \alpha_n}^c(\rho(\gamma_{k_n}))) = \mu(\mathcal{F}_\Theta(\mathbb{R}^d)).$$

**Lemma 3.3.1.** *Let  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$  be a representation and  $\mu^\phi$  be a  $(\rho(\Gamma), \phi)$ -Patterson-Sullivan measure. If  $\rho(\Gamma)$  is  $\mu$ -irreducible, then there exists some  $\alpha_0 > 0$ , such that for any  $\alpha \in (0, \alpha_0)$ , there's some  $k = k(\alpha) > 0$  for which*

$$\frac{1}{k} e^{-\phi(a(\rho(\gamma)))} \leq \mu^\phi(\rho(\gamma) B_{\Theta, \alpha}(\rho(\gamma))) \leq k e^{-\phi(a(\rho(\gamma)))}$$

for all  $\gamma \in \Gamma$ .

*Proof.* Let  $\alpha_0, k > 0$  be as in the remark preceeding the statement of the lemma. As noted in Lemma 3.1.3, there exists some  $K = K(\alpha_0, \phi) > 0$  such that for any  $\alpha \in (0, \alpha_0)$  and  $y \in B_{\Theta, \alpha}(\rho(\gamma))$ :

$$|\phi(a(\rho(\gamma))) - b(\rho(\gamma), y)| \leq K,$$

from which we obtain the upper bound

$$\begin{aligned} \mu^\phi(\rho(\gamma) B_{\Theta, \alpha}(\rho(\gamma))) &= (\rho(\gamma^{-1})_* \mu^\phi)(B_{\Theta, \alpha}(\rho(\gamma))) = \int_{\mathcal{F}_\Theta(\mathbb{R}^d)} e^{-\phi(b(\rho(\gamma), y))} d\mu^\phi(y) \leq \\ &\leq e^{-K} \mu^\phi(\mathcal{F}_\Theta(\mathbb{R}^d)) e^{-\phi(a(\rho(\gamma)))}. \end{aligned}$$

Similarly we obtain the lower bound

□

## Appendix A

# Tangent space to the Grassmanian

Let  $V$  be a  $d$ -dimensional real vector space. We denote with  $\mathcal{G}_k(V)$  the Grassmanian of  $k$ -dimensional subspaces of  $V$ . Our first objective is to find a convenient way to express its tangent space.

**Proposition A.0.1.** *We have the following canonical identification:*

$$\begin{aligned} \text{hom}(W, V/W) &\simeq T_W \mathcal{G}_k(V) \\ \phi &\mapsto \left. \frac{d}{dt} \right|_{t=0} \Gamma(t\phi) \end{aligned}$$

where  $\Gamma(\phi) = (Id + \phi)(W)$  is the graph of  $\phi$ .

*Proof.* We will consider the map

$$F : \text{Injhom}(W, V) \rightarrow \mathcal{G}_k(V), \quad \phi \mapsto \phi(W).$$

whose derivative is given by:

$$d_I F(\phi) = F \left( \left. \frac{d}{dt} \right|_{t=0} (I + t\phi) \right) = \left. \frac{d}{dt} \right|_{t=0} (I + t\phi(W)) = \left. \frac{d}{dt} \right|_{t=0} \Gamma(t\phi).$$

The result will follow as soon as we have shown that  $d_I F$  is surjective and that  $\ker d_I F = \text{hom}(W, W)$ .

To show that it is surjective, we consider a  $(d-k)$ -dimensional subspace  $W' \in \mathcal{G}_{d-k}(V)$  that is complementary to  $W$ , i.e.  $V = W \oplus W'$ . Denoting with  $U_{W'} = \{Z \in \mathcal{G}_k(V) : Z \cap W' = 0\}$ , we recall the corresponding chart:

$$\begin{aligned} \Psi : \text{hom}(W, W') &\rightarrow U_{W'} \\ \phi &\mapsto \Gamma(\phi). \end{aligned}$$

Surjectivity of  $d_I F$  now follows by the fact that

$$d_I F(\phi) = \left. \frac{d}{dt} \right|_{t=0} \Gamma(t\phi) = d_0 \Psi(\phi).$$

To show that  $\ker d_I F = \text{hom}(W, W)$ , we first note that clearly  $\ker d_I F \supseteq \text{hom}(W, W)$ . Equality then follows by the fact that  $\dim \text{hom}(W, W) = \dim \ker d_I F$ , which is a direct consequence of the surjectivity.  $\square$

Note that another way to prove the above identification throught the fact that the Grassmanian is a homogeneous space of  $\mathrm{GL}(d, \mathbb{R})$ , giving us the diffeomorphism

$$\begin{aligned} \mathrm{GL}(V)/\mathrm{St}_{\mathrm{GL}(V)}W &\rightarrow \mathcal{G}_k(V) \\ [g] &\mapsto gW, \end{aligned}$$

where  $\mathrm{St}_{\mathrm{GL}(V)}W = \{g \in \mathrm{GL}(V) : gW = W\}$  is the stabilizer of  $W$ . Thus an expression for the tangent space at  $W$  may be obtained by differentiating the map above at the identity coset:

$$\mathrm{hom}(W, V/W) \simeq \mathrm{hom}(V, V)/\mathrm{hom}(W, W) \simeq T_W \mathcal{G}_k(V).$$

In particular the above gives us a direct proof that the kernel of the map constructed in the previous proof is indeed  $\mathrm{hom}(W, W)$ .

Our second objective is to identify subspaces of  $T_l \mathbb{P}(V)$  with subspaces of  $V$ , by considering the first as projectivisation of the second. More concretely, we shall consider the space

$$\mathcal{P} = \{(l, P) : l \in \mathbb{P}(V), P \in \mathcal{G}_k(T_l \mathbb{P}(V))\}$$

as a homogenous space of  $\mathrm{SL}(V)$ , where the action is given by

$$g \cdot (l, P) = (gl, d_l g(P) = g\pi^{-1}(P)g^{-1} + \mathrm{hom}(gl, gl)).$$

where we use the identification of  $T_l \mathbb{P}(V)$  with  $\mathrm{hom}(l, V/l)$  as above and denote with  $\pi : \mathrm{hom}(l, V) \rightarrow \mathrm{hom}(l, V/l)$  the canonical projection. For the sake of completeness, we outline the calculation of the differential:

$$\begin{aligned} \mathrm{hom}(l, V/l) &\rightarrow T_l \mathbb{P}(V) && \rightarrow T_{gl} \mathbb{P}(V) && \rightarrow \mathrm{hom}(gl, V/gl) \\ \phi &\mapsto \left. \frac{d}{dt} \right|_{t=0} g(I + t\tilde{\phi})(l) && \mapsto \left. \frac{d}{dt} \right|_{t=0} (I + tg\tilde{\phi}g^{-1})(gl) && \mapsto g\tilde{\phi}g^{-1} + \mathrm{hom}(gl, gl) \end{aligned}$$

where  $\phi \in \mathrm{hom}(l, V/l)$ ,  $\tilde{\phi} \in \mathrm{hom}(l, V)$  such that  $\tilde{\phi} + \mathrm{hom}(l, l) = \phi$ .

We are now ready to express the needed identification:

**Proposition A.0.2.** *We have the following  $\mathrm{SL}(V)$  equivariant identification:*

$$\begin{aligned} \mathcal{P} &\rightarrow \mathcal{F}_{1, k+1}(V) \\ (l, P) &\mapsto (l, \pi^{-1}(P)l) \\ (l, \mathrm{hom}(l, Q/l)) &\mapsto (l, Q). \end{aligned}$$

where  $\pi : \mathrm{hom}(l, V) \rightarrow \mathrm{hom}(l, V/l)$  is the canonical projection.

*Proof.* We begin by showing that the left-to-right direction of the map is well-defined. For this, we first need to check that for  $(l, P) \in \mathcal{P}$ , we have that  $\dim \pi^{-1}(P)l = k + 1$ . Indeed, we have that  $\dim \pi^{-1}(P) = k + 1$  as implied by the rank-nullity theorem for  $\pi : \pi^{-1}(P) \rightarrow P$ . The result then follows by the fact that  $\pi^{-1}(P)l = T_1(l) \oplus \dots \oplus T_{k+1}(l)$  for any base  $T_1, \dots, T_{k+1}$  of  $\pi^{-1}(P)$ . The second thing to check is that  $l \leq \pi^{-1}(P)l$ , which holds since  $\ker \pi = \mathrm{hom}(l, l) \leq \pi^{-1}(P)$ .

To see that the two directions above are inverse to each other, we begin by examining the right-to-left-to-right composition:

$$(l, Q) \mapsto (l, \pi(\mathrm{hom}(l, Q))) \mapsto (l, \pi^{-1}\pi(\mathrm{hom}(l, Q))) = (l, \mathrm{hom}(l, Q)l) = (l, Q).$$

and for the left-to-right-to-left composition

$$(l, P) \mapsto (l, \pi^{-1}(P)l) \mapsto (l, \mathrm{hom}(l, \pi^{-1}(P)l)),$$

so it suffices to show that  $\text{hom}(l, \pi^{-1}(P)l/l) = P$ . Indeed, for  $\pi^{-1}(P) = \mathbb{R}T_1 \oplus \cdots \oplus \mathbb{R}T_k$ , we have that

$$\begin{aligned} \text{hom}(l, \pi^{-1}(P)l/l) &= \text{hom}(l, \pi^{-1}(P)l) / \text{hom}(l, l) = (\oplus_i \text{hom}(l, T_i(l)) / \text{hom}(l, l)) = \\ &= (\oplus_i \mathbb{R}T_i) / \text{hom}(l, l) = \pi^{-1}(P) / \text{hom}(l, l) = P. \end{aligned}$$

For the equivariance, the calculations has as follows:

$$\begin{array}{ccc} (l, P) & \xrightarrow{\quad\quad\quad} & (l, \pi^{-1}(P)l) \\ \downarrow g & & \downarrow g \\ (gl, g\pi^{-1}(P)g^{-1} + \text{hom}(gl, gl)) & \xrightarrow{\quad\quad\quad} & (gl, (g\pi^{-1}(P)g^{-1})(gl)) = (gl, g\pi^{-1}(P)l) \end{array} \quad \square$$

## Appendix B

# Irreducible actions problem

The matter of this chapter has to do with an obstruction, found in the proof of this lemma:

**Lemma B.0.1** (Lemma 6.8 in [PSW23]). *Let  $\Gamma$  be a hyperbolic group and  $\eta : \Gamma \rightarrow \mathrm{PGL}(d, \mathbb{R})$  be a strongly irreducible projective Anosov representation such that  $\xi_\eta(\partial\Gamma)$  is homeomorphic to  $S^{d-1}$ , and which admits a measurable  $\eta$ -equivariant section  $\zeta : \partial\Gamma \rightarrow \mathcal{F}_{\{a_1, a_{d+1}\}}(\mathbb{R}^d)$ . Then  $\eta$  is  $\mu$ -irreducible for any  $(\eta(\Gamma), \phi)$ -Patterson-Sullivan measure  $\mu$  on  $\mathcal{F}_{\{a_1, a_{d+1}\}}(\mathbb{R}^d)$ .*

For convenience, we recall that a linear representation  $\rho : \Gamma \rightarrow \mathrm{GL}(d, \mathbb{R})$  is strongly irreducible if there is no proper  $\rho(\Gamma)$ -invariant subspace of  $\mathbb{R}^d$ , and it is  $\mu$ -irreducible if there is no element in  $\mathcal{F}_{i\Theta}(\mathbb{R}^d)$ , whose annihilator is of full measure.

In what follows, we will show that the above lemma is false, by providing a counterexample. Let  $\Gamma$  be a uniform lattice of  $\mathrm{SU}(2, 1)$  (i.e. acts convex cocompactly), and  $\eta : \Gamma \rightarrow \mathrm{SL}(\mathfrak{su}(2, 1))$  be the restriction of the adjoint representation, i.e.  $\eta(\gamma) = \mathrm{Ad}_\gamma$  for all  $\gamma \in \Gamma$ .

For convenience, we recall the definition of a uniform lattice:

**Definition B.0.1.** Let  $G$  be a locally compact group. A uniform lattice is a discrete subgroup  $\Gamma \leq G$  that is co-compact, i.e.  $G/\Gamma$  is compact.

*Remark B.0.1.* When  $G = \mathrm{Isom}(X)$  is the isometry group of a complete Riemannian manifold  $X$ , and  $\Gamma$  is a uniform lattice of  $G$ , then it acts properly discontinuously and cocompactly on  $X$ .

We begin by showing proving the Anosov property of  $\eta$ .

**Proposition B.0.1.**  *$\eta$  is projective Anosov.*

*Proof.* Let  $\gamma \in \Gamma$ . Since  $\gamma \in \mathrm{SU}(2, 1)$ , we have that

$$\gamma = k_1 \exp(r(\gamma)x_0) k_2$$

for  $x_0$  a fixed non-zero in the Weyl-chamber  $\mathbb{R}x_0$  of  $\mathfrak{su}(2, 1)$ ,  $r(\gamma) \in \mathbb{R}$  and  $k_1, k_2 \in \mathrm{U}(2)$ . Then by the definition of a uniform lattice, we have that  $\Gamma$  acts properly discontinuously and cocompactly, which means that the inclusion  $\Gamma \hookrightarrow \mathrm{SU}(2, 1)$  is projective Anosov (since  $\mathrm{SU}(2, 1)$  is of rank 1). Thus there exist constants  $L \geq 1, b \geq 0$  such that for all  $\gamma \in \Gamma$ :

$$r(\gamma) \geq a_1(x_0)^{-1}(L|\gamma| - b) = L'|\gamma| - b'.$$

Note that  $a_1(x_0) > 0$  since  $x_0$  is in the interior of the Weyl-chamber  $\mathbb{R}x_0$ .

Letting  $k'_1 = \text{Ad}_{k_1}, k'_2 = \text{Ad}_{k_2}$  and  $K' \leq \text{SL}(\mathfrak{su}(2, 1))$  be a maximal compact subgroup containing them, we have that:

$$\eta(\gamma) = \text{Ad}_\gamma = k'_1 \text{Ad}_{\exp(r(\gamma)x_0)} k'_2 = k'_1 \exp(r(\gamma) \text{ad}_{x_0}) k'_2.$$

Thus

$$\mathfrak{a}_1(\mu(\eta(\gamma))) = r(\gamma) \mathfrak{a}_1(\text{ad}_{x_0}) \geq (L'|\gamma| - b') \mathfrak{a}_1(x_0)$$

which is Anosov because  $\mathfrak{a}_1(\text{ad}_{x_0}) > 0$ , as can be seen by concrete calculations.  $\square$

Before giving an expression for the projective part of the limit map of  $\eta$ , we make a few observations regarding Gromov boundary of  $\Gamma$ . In particular, we claim that since  $\Gamma$  is a uniform lattice of  $\text{SU}(2, 1)$ , we have that  $\partial\Gamma$  is homeomorphic to  $\text{SU}(2, 1)/P$ , where  $P$  is a parabolic subgroup of  $\text{SU}(2, 1)$ , and it coincides with the stabilizer of some isotropic line  $l \in \partial_\infty \mathbb{H}_{\mathbb{C}}^2$ .

Indeed, for a uniform lattice  $\Gamma$  of the isometry group  $G$  of a homogenous  $G$ -space  $X$ , the Milnor-Švarc lemma implies that for any  $x_0 \in X$ , the map  $\Gamma \rightarrow X, \gamma \mapsto \gamma x_0$  is a quasi-isometry. In our case  $G = \text{SU}(2, 1)$  and  $X = \mathbb{H}_{\mathbb{C}}^2$  is a hyperbolic metric space, so the quasi-isometry extends to a homeomorphism  $\partial\Gamma \rightarrow \partial\mathbb{H}_{\mathbb{C}}^2$  of the Gromov-boundaries. On the other hand, the action of  $\text{SU}(2, 1)$  on  $\partial_\infty \mathbb{H}_{\mathbb{C}}^2$  is transitive, so we have that  $\partial\mathbb{H}_{\mathbb{C}}^2 \simeq \text{SU}(2, 1)/P$  where  $P$  is the stabilizer of a point in  $\partial\mathbb{H}_{\mathbb{C}}^2$ . In fact, we have that  $P$  is a parabolic subgroup of  $\text{SU}(2, 1)$ . The combination of the above, along with the fact that the geometric and the Gromov boundaries agree in the case of  $\mathbb{H}_{\mathbb{C}}^2$ , we deduce that  $\partial\Gamma \simeq \text{SU}(2, 1)/P$ .

To calculate the projective part of the limit map, we shall show that there exists a unique  $\text{SU}(2, 1)$ -equivariant map from  $\partial\Gamma$  to  $\mathbb{P}(\mathbb{R}^d)$ . The uniqueness follows from the following characterisation of limit maps:

Add proofs of these.

**Lemma B.0.2.** *Let  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$  be a strongly irreducible projective Anosov representation, and denote with  $\xi_\rho : \partial\Gamma \rightarrow \mathbb{P}(\mathbb{R}^d)$  its limit map. Then  $\xi_\rho^1$  is the unique continuous,  $\rho(\Gamma)$ -equivariant map from  $\partial\Gamma$  to  $\mathbb{P}(\mathbb{R}^d)$ .*

*Proof.* Let  $\eta^1 : \partial\Gamma \rightarrow \mathbb{P}(\mathbb{R}^d)$  be a continuous,  $\rho(\Gamma)$ -equivariant map. Since the action of  $\Gamma$  on its boundary  $\partial\Gamma$  has dense orbits, it suffices to show that it agrees with  $\xi_\rho^1$  on at least one boundary point.

Suppose for the sake of contradiction that  $\eta^1$  does not coincide with  $\xi_\rho^1$  and let  $z \in \partial\Gamma, y \in \partial\Gamma \setminus \{z\}$ . Then for any  $x \in \partial\Gamma \setminus \{y\}$  we may find some quasi-geodesic  $\{\gamma_n\}_n$  such that  $\gamma_n \rightarrow x, \gamma_{-n} \rightarrow y$  as  $n \rightarrow \infty$ . Then since  $z \neq y$  we know that  $\gamma_n z \rightarrow z$  as  $n \rightarrow \infty$  and continuity of  $\eta^1$  implies that  $\eta^1(\gamma_n z) \rightarrow \eta^1(z)$ . But equivariance of  $\eta^1$  and the fact that  $\xi^1$  is dynamics-preserving implies that  $\eta(\gamma_n z) = \rho(\gamma_n) \eta(z) \rightarrow \xi^1(x)$ , unless  $\eta^1(z) \in \xi^{d-1}(y)$ . But if in fact  $\eta^1(z) \notin \xi^{d-1}(y)$ , then the limits would agree, i.e.  $\eta^1(x) = \xi^1(x)$  which is a contradiction. Thus we have that  $\eta^1(z) \in \xi^{d-1}(y)$  and since  $y$  was an arbitrary point of  $\partial\Gamma \setminus \{z\}$ , we have that

$$\eta^1(z) \in \bigcap_{y \in \partial\Gamma \setminus \{z\}} \xi^{d-1}(y) \subseteq \bigcap_{y \in \partial\Gamma \setminus \Gamma \cdot z} \xi^{d-1}(y).$$

In particular, the set appearing on the right hand side is  $\rho$ -equivariant, non-empty proper subset of  $\mathbb{R}^d$ , which contradicts the strong irreducibility assumption of  $\rho$ .  $\square$

Given the lemma above, it suffices to find an  $\text{SU}(2, 1)$ -equivariant map from  $\partial\Gamma$  to  $\mathbb{P}(\mathbb{R}^d)$ .

**Proposition B.0.2.** *The projective part of the limit map of  $\eta$  is given by*

$$\xi_\eta : \partial\Gamma = \text{SU}(2, 1)/P_0 \rightarrow \mathbb{P}(\mathfrak{su}(2, 1)), \quad \xi_\eta(gP_0) = \mathbb{R} \text{Ad}_\gamma x_0.$$

where  $P_0 = \text{St}_{\text{SU}(2,1)}[1 : 0 : 0]$  and

$$x_0 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{su}(2, 1).$$

Its derivative satisfies:

$$d_x \xi(T_x \text{SU}(2, 1)/P_0) = \pi(\text{ad}_{\xi^1(x)} \mathfrak{su}(2, 1))$$

where  $\pi : \text{hom}(\xi^1(x), \mathfrak{su}(2, 1)) \rightarrow \text{hom}(\xi^1(x), \mathfrak{su}(2, 1)/\xi^1(x))$  is the canonical projection.

*Proof.* Since by Lemma B.0.2 the limit map is the unique continuous  $\rho$ -equivariant from the boundary of  $\Gamma$  to the projective space, it suffices to show that there exists an  $\eta$ -equivariant map  $\xi^1 : \text{SU}(2, 1)/P_0 \rightarrow \mathbb{P}(\mathfrak{su}(2, 1))$ , since it will then restrict to the limit map on  $\partial\Gamma$ .

We consider the parabolic subgroup  $P_0 = \text{St}_{\text{SU}(2,1)}[1 : 0 : 0]$  of  $\text{SU}(2, 1)$ . Then its Lie algebra is given by:

$$\mathfrak{p}_0 = \text{St}_{\mathfrak{su}(2,1)}[1 : 0 : 0] = \left\{ \begin{pmatrix} u - is & a & it \\ 0 & 2is & -\bar{a} \\ 0 & 0 & -u - is \end{pmatrix} : a \in \mathbb{C}, u, s, t \in \mathbb{R} \right\}.$$

Since for  $\mathbb{R}x \in \mathbb{P}(\mathfrak{su}(2, 1))$  we have that  $P_0$  fixes  $\mathbb{R}x$  if and only if  $\mathfrak{p}_0$  fixes  $\mathbb{R}x$ . But a quick calculation shows that the only element of  $\mathfrak{su}(2, 1)$  fixed by  $\mathfrak{p}_0$  is  $x_0$ .

For the calculation of the image of the differential at the identity coset  $P$ , we differentiate the commutative diagram:

$$\begin{array}{ccc} \text{SU}(2, 1) & \xrightarrow{\text{Ad. } x_0} & \mathfrak{su}(2, 1) \\ \downarrow & & \downarrow \\ \text{SU}(2, 1)/P_0 & \xrightarrow{\xi^1} & \mathbb{P}(\mathfrak{su}(2, 1)) \end{array} \quad \text{to get} \quad \begin{array}{ccc} \mathfrak{su}(2, 1) & \xrightarrow{\text{ad. } x_0} & \mathfrak{su}(2, 1) \\ \downarrow & & \downarrow \pi \\ \mathfrak{su}(2, 1)/\mathfrak{p}_0 & \xrightarrow{d_P \xi^1} & T_{\xi^1(P)} \mathbb{P}(\mathfrak{su}(2, 1)) \end{array}$$

In the general case we use the equivariance of the limit map

$$\begin{aligned} d_{gP} \xi^1(T_{gP} \text{SU}(2, 1)/P_0) &= d_{gP} \xi^1 d_P g(T_P \text{SU}(2, 1)/P_0) = d_{\xi^1(P)} g d_P \xi^1(T_P \text{SU}(2, 1)/P_0) = \\ &= d_{\xi^1(P)} g \pi(\text{ad}_{\xi^1(P)} \mathfrak{su}(2, 1)) = \\ &= \pi(\text{Ad}_g(\text{ad}_{\xi^1(P)} \mathfrak{su}(2, 1))) = \pi(\text{ad}_{\text{Ad}_g \xi^1(P)} \mathfrak{su}(2, 1)) = \\ &= \pi(\text{ad}_{\xi^1(gP)} \mathfrak{su}(2, 1)). \end{aligned}$$

□

Recall that all parabolic subgroups of  $\text{SU}(2, 1)$  are conjugate to each other, so we have the following identification:

$$\begin{aligned} \text{SU}(2, 1)/P_0 &\leftrightarrow \{ \text{Parabolic subgroups of } \text{SU}(2, 1) \} \leftrightarrow \{ \text{Parabolic subalgebras of } \mathfrak{su}(2, 1) \} \\ gP_0 &\leftrightarrow gP_0 g^{-1} \leftrightarrow \text{Ad}_g(\mathfrak{p}_0) \end{aligned}$$

**Lemma B.0.3.** *Let  $\mathfrak{p}, \mathfrak{p}' \leq \mathfrak{su}(2, 1)$  be two distinct parabolic subalgebras. Then there exists some  $g \in \text{SU}(2, 1)$  such that  $\text{Ad}_g(\mathfrak{p}) = \mathfrak{p}_0$  and  $\text{Ad}_g \mathfrak{p}' = \mathfrak{p}_0^t$ .*

The following proposition implies that the falsehood of the lemma in the beginning of this chapter.



**Proposition B.0.3.** *Let  $\Gamma \leq \mathrm{SU}(2, 1)$  be a uniform lattice and  $\eta : \Gamma \rightarrow \mathrm{SL}(\mathfrak{su}(2, 1))$  be the restriction of the adjoint representation. Then*

- (i)  $\eta$  is strongly irreducible,
- (ii)  $\eta$  is projective Anosov
- (iii)  $\eta$  admits a measurable  $\eta$ -equivariant section:

$$\begin{aligned} \zeta : \partial\Gamma &\rightarrow \mathcal{F}_{\{1,4\}}(\mathfrak{su}(2, 1)) \simeq \mathcal{P} \\ x &\mapsto (\xi^1(x), T_{\xi^1(x)}\xi^1(\partial\Gamma)) \simeq (\xi^1(x), (d_{\xi^1(x)}p)^{-1}(T_{\xi^1(x)}\xi^1(\partial\Gamma))\xi^1(x)). \end{aligned}$$

where  $d_{\xi^1(x)}p : \mathrm{hom}(\xi^1(x), \mathfrak{su}(2, 1)) \rightarrow \mathrm{hom}(\xi^1(x), \mathfrak{su}(2, 1)/\xi^1(x))$  is the canonical projection.

- (iv) For all  $x, y \in \partial\Gamma : \zeta(x)^4 \cap \zeta(y)^4 \neq \emptyset$ .
- (v) For any  $y_0 \in \mathrm{SU}(2, 1)/P_0$  and  $W_0 \in \mathcal{G}_7(\mathbb{R}^4)$  that contains  $\zeta(y_0)^4$ , we have that  $\mathrm{Ann}(\zeta(y_0)^4, W_0) \supseteq \zeta(\mathrm{SU}(2, 1)/P_0)$  and is in particular of full  $\mu$ -measure, for any  $(\eta(\Gamma), \phi)$ -Patterson-Sullivan measure  $\mu$  supported over  $\zeta(\partial\Gamma)$ .

*Proof.* (i) Follows from the fact that  $\mathrm{SU}(2, 1)$  is a simple Lie group.

(ii) Shown in Proposition B.0.1.

(iii) Follows from the fact that  $\xi^1$  is  $\mathrm{SU}(2, 1)$ -equivariant and the equivariant identification of  $\mathcal{F}_{\{1,4\}}(\mathfrak{su}(2, 1)) \simeq \mathcal{P}$ .

(iv) Letting  $g \in \mathrm{SU}(2, 1)$  be as in Lemma B.0.3, we have that  $\mathrm{Ad}_g(\mathfrak{p}_0) = \mathfrak{p}$  and  $\mathrm{Ad}_g(\mathfrak{p}_0^t) = \mathfrak{p}'$ . Thus  $\zeta(x)^4 \cap \zeta(y)^4 \neq \emptyset$  if and only if

$$\begin{aligned} \emptyset \neq \mathrm{Ad}_g(\zeta(x)^4 \cap \zeta(y)^4) &= \mathrm{Ad}_g \zeta(x)^4 \cap \mathrm{Ad}_g \zeta(y)^4 = \zeta(gx)^4 \cap \zeta(gy)^4 = \zeta(\mathfrak{p}_0)^4 \cap \zeta(\mathfrak{p}_0^t)^4 = \\ &= \pi \left( \mathbb{R} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right). \end{aligned}$$

For the last equality, we use Proposition B.0.2 and the fact that  $\mathfrak{p}_0^t = \mathrm{Ad}_g \mathfrak{p}_0$  for

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

to conclude that

$$\zeta(\mathfrak{p}_0) = \zeta(P_0) = \pi \left( \left\{ \begin{pmatrix} u & a & it \\ 0 & 0 & -\bar{a} \\ 0 & 0 & -u \end{pmatrix} : a \in \mathbb{C}, u, t \in \mathbb{R} \right\} \right),$$

and

$$\zeta(\mathfrak{p}_0^t) = \zeta(gP_0) = \mathrm{Ad}_g \zeta(P_0) = \pi \left( \left\{ \begin{pmatrix} u & 0 & 0 \\ a & 0 & 0 \\ it & -\bar{a} & 0 \end{pmatrix} : a \in \mathbb{C}, u, t \in \mathbb{R} \right\} \right).$$

□

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