

Limit sets of Anosov representations

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Chapter 1

Introduction

1.1 Lie group preliminaries

We fix the Cartan subalgebra \mathfrak{a} of $\mathrm{SL}(d, \mathbb{R})$:

$$\mathfrak{a} = \{\mathrm{diag}(\alpha_1, \dots, \alpha_d) : \alpha_1 + \dots + \alpha_d = 0\}$$

and the Weyl chamber \mathfrak{a}^+ of $\mathrm{SL}(d, \mathbb{R})$

$$\mathfrak{a}^+ = \{\mathrm{diag}(\alpha_1, \dots, \alpha_d) : \alpha_1 \geq \dots \geq \alpha_d\}.$$

Denoting with $K = \mathrm{SO}(d, \mathbb{R})$, $A^+ = e^{\mathfrak{a}^+}$, we have the Cartan decomposition:

$$\begin{aligned} \mathfrak{sl}(d, \mathbb{R}) &\rightarrow K \times A^+ \times K \\ g &\mapsto (k_g, a_g, l_g) \end{aligned}$$

such that $g = k_g a_g l_g$. In particular $a_g = \mathrm{diag}(\sigma_1(g), \dots, \sigma_d(g))$ with $\sigma_1 \geq \dots \geq \sigma_d(g)$, where $\sigma_i(g)$ is the i -th singular value of g , i.e. eigenvalue of $g^t \cdot g$.

We will use the spaces

$$U_p(g) = \mathbb{R}u_1(g) \oplus \dots \oplus \mathbb{R}u_p(g)$$

where $u_i(g) = k_g \cdot e_i$. One can easily show that the decomposition

$$g^{-1} \cdot U_p(g) \oplus U_{d-p}(g^{-1})$$

is orthogonal with respect to the standard inner product and that $u_p(g^{-1}) = l_g^{-1} e_{d-p+1}(g)$.

1.2 Limit set preliminaries

Definition 1.2.1. For $p \in \{2, \dots, d\}$, $s \in \mathbb{R}$ and $g \in \mathrm{SL}(d, \mathbb{R})$ we denote with $\Psi_s^p(g) : \mathfrak{a}^+ \rightarrow \mathbb{R}$ the functional:

$$\begin{aligned} \Psi_s^p(g) &= \alpha_{12}(a(g)) + \dots + \alpha_{1(p-1)}(a(g)) + (s - (p-2))\alpha_{1p}(a(g)) \\ \tilde{\Psi}_s^p(g) &= \left(\frac{\sigma_2}{\sigma_1} \dots \frac{\sigma_{p-1}}{\sigma_1}(g) \right) \left(\frac{\sigma_{p-1}}{\sigma_1}(g) \right)^{s-(p-2)} \end{aligned}$$

Remark 1.2.1. We have $\alpha_{ij}(a) = a_i - a_j$, $a_i(g) = \log(\sigma_i(g))$ and

$$\Psi_s^p(g) = \log \tilde{\Psi}_s^p(g)$$

and that

$$\min_{p \in \llbracket 2, d \rrbracket} \left\{ \sum_{|\gamma|=T} \frac{\sigma_2}{\sigma_1} \cdots \frac{\sigma_{p-1}}{\sigma_1}(g) \left(\frac{\sigma_{p-1}}{\sigma_1}(g) \right)^{s-(p-2)} \right\} = \sum_{|\gamma|=T} e^{-\max_{p \in \llbracket 2, d \rrbracket} \Psi_s^p(g)}$$

The following definition comes from [1], in the special case of projective Anosov representations ($P = 1$):

Definition 1.2.2. For $s \geq 0$ we consider the Falconer functional $F_s : \mathrm{SL}(d, \mathbb{R}) \rightarrow \mathbb{R}$ by:

$$F_s(g) = \min \left\{ \sum_{j=2}^d s_j \alpha_{1j}(a(g)) : s_j \in (0, 1], \sum_{j=2}^d s_j = s \right\}$$

Remark 1.2.2. Using elementary computations one may prove that for all $s \geq 0$:

$$F_s(g) = \min_{p \in \llbracket 2, d \rrbracket} \Psi_s^p(g)$$

Definition 1.2.3. Let $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$ be a linear representation and $p \in \llbracket 1, d-1 \rrbracket$. We say that ρ is p -Anosov if there exist constants $\mu, C > 0$ such that for all $\gamma \in \Gamma$:

$$\frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma)) \leq C e^{-\mu|\gamma|}.$$

One can show that in that case there exist equivariant continuous maps $\xi^p : \hat{\Gamma} \rightarrow \mathcal{G}_p(\mathbb{R}^d)$, $\xi^{d-p} : \hat{\Gamma} \rightarrow \mathcal{G}_{d-p}(\mathbb{R}^d)$ that are transverse and restrict to

$$\xi^p(\gamma) = U_p(\rho(\gamma)), \xi^{d-p}(\gamma) = U_{d-p}(\rho(\gamma))$$

for $\gamma \in \Gamma$, where $U_p(\gamma), U_{d-p}(\gamma)$ denote the flags corresponding to $\rho(\gamma)$.

Figure out what this exactly means

Chapter 2

Upper bound

Lemma 2.0.1 (Upper bound for dimension). *Let $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. Then:*

$$\dim_H(\xi^1(\partial\Gamma)) \leq \inf \left\{ s : \sum_{|\gamma|=T} e^{-F_s(\rho(\gamma))} < \infty \right\}.$$

Definition 2.0.1. Let V be a finite-dimensional \mathbb{R} -vector space. We consider a decomposition

$$V = \mathbb{R}u_1 \oplus \dots \oplus \mathbb{R}u_d$$

be a direct decomposition that is orthogonal with respect to a fixed inner-product over V . Given $\beta_2 \geq \dots \beta_d > 0$, we define an ellipsoid with axes $u_1 \oplus u_p(g)$ and lengths β_p to be the image of

$$\left\{ v = \sum_1^d v_i u_i \in V : \sum_2^d \left(\frac{v_j}{\beta_j} \right)^2 \leq 1 \right\}$$

through the projection $V \rightarrow \mathbb{P}(V)$.

The following aims to be something along the lines of [2, Lemma 2.4]:

Lemma 2.0.2. *Let $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. For $\alpha > 0$ small enough, there exists $L > 0$ such that for any geodesic ray $(a_j)_j$ through e we have:*

$$\angle(U_1(\rho(a_i)), U_1(\rho(a_0))) > \alpha$$

when $|a_i|, |a_0| > T$.

Proof. Assume the contrary for the sake of contradiction. Then (see Figure 2.1) for each $n > 0$ there exists a geodesic ray a^n through e such that

$$|a_n^n|, |a_0^n| > n \text{ and } \angle(U_1(\rho(a_n^n)), U_1(\rho(a_0^n))) < \frac{1}{n}.$$

Due to compactness of $\partial\Gamma$ we may assume (up to a subsequence) that $a^n \rightarrow x$ in $\partial\Gamma$ for some $x \in \partial\Gamma$. Then $\underline{a_n^n}, \underline{a_0^n} \rightarrow x$ in $\hat{\Gamma}$ which implies

$$\angle(\xi^1(x), \xi^{d-1}(x)) = 0$$

using the fact that the limit maps ξ^1, ξ^{d-1} are continuous, which contradicts their transversality. \square

Not sure if this is true.

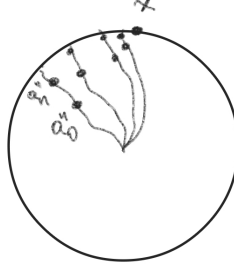


Figure 2.1: Situation in Lemma 2.0.2

The following is [2, Proposition 3.5].

Lemma 2.0.3. *Let $\rho : \Gamma \rightarrow SL(d, \mathbb{R})$ be projective Anosov. Then for $\alpha > 0$ small enough, there exists some $T_0 > 0$ such that for all $T \geq T_0$ the family*

$$\mathcal{U}_T = \{\rho(\gamma)B_{\alpha_1, \alpha}(\rho(\gamma)) : |\gamma| = T\}$$

is an open covering of $\xi^1(\partial\Gamma)$.

Proof. Let $\alpha, T > 0$ be as in the statement of Lemma 2.0.2 and $x \in \partial\Gamma$ be represented by a geodesic ray $(\gamma_j)_{j \geq 0}$ starting from e . Then $(\gamma_T^{-1}\gamma_j)_j$ is a geodesic ray starting from $(\gamma_T)^{-1}$ that passes through e , so

$$\angle(U_1(\rho(\gamma_T^{-1}\gamma_j)), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

as implied by Lemma 2.0.2. Taking the limit $j \rightarrow \infty$ and using the equivariance of the limit map, we obtain

$$\angle(\rho(\gamma_T^{-1})\xi^1(x), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

and thus $\xi^1(x) \in \rho(\gamma_T) \cdot B_{\alpha_1, \alpha}(\rho(\gamma_T))$. □

The following is [2, Proposition 3.8].

Proposition 2.0.1. *For each $g \in SL(d, \mathbb{R})$, $\alpha > 0$, the basin of attraction $g \cdot B_{\alpha_1, \alpha}(g)$ lies in the ellipsoid with axes $u_1(g) \oplus u_p(g)$ with lengths*

$$\frac{1}{\sin \alpha} \cdot \frac{\sigma_p(g)}{\sigma_1(g)}$$

Proof. Using the definition of the basin of attraction (see Figure 2.2), we have that $w = w_1 u_1(g^{-1}) + \dots + w_d u_d(g^{-1}) \in B_{\alpha_1, \alpha}(g)$ if and only if

$$w_d^2 \geq (\sin \alpha)^2 \sum_{i=1}^d w_i^2.$$

Considering now some $v = v_1 u_1(g) + \dots + v_d u_d(g) \in g \cdot B_{\alpha_1, \alpha}(g)$ we have that

$$\begin{aligned} w &= g^{-1}v = v_1 \sigma_1(g)^{-1} l_g^{-1} e_1(g) + \dots + v_d \sigma_d(g)^{-1} l_g^{-1} e_d(g) \\ &= v_1 \sigma_1(g)^{-1} u_d(g^{-1}) + \dots + v_d \sigma_d(g)^{-1} u_1(g^{-1}) \end{aligned}$$

Using some geometric group theory, we can show that for all $j \geq L$

$$d(\gamma^{-1}, a_j) > \alpha' \Rightarrow d([\gamma^{-1}, a_j], e) < \alpha''$$

for some α'' that depends only on Γ and α' , where $[a_j, \gamma^{-1}]$ denotes the geodesic segment connecting γ^{-1} and a_j .

Consider the concatenation $(a'_j)_{j=L-K}^\infty$ of $[\gamma^{-1}, a_L]$ and $[a_L, x]$. To find quasi-geodesic-constants that are uniform in γ , we note that for any $c_0 \geq 1, c_1 \geq 0$:

$$c_0^{-1}|i-j| - c_1 \leq d(a'_i, a'_j) = d(a_i, a_j) \leq d(a_i)c_0|i-j| + c_1 \text{ when } i, j \geq L \text{ or } i, j \leq L$$

and that the upper bound follows trivially by the triangle inequality.

For the lower bound we proceed in two steps. First we bound the distance of $\gamma^{-1} = a'_{L-K}$ to a_{L+j} for $j \geq 0$:

$$\begin{aligned} d(a'_{L-K}, a'_{L+j}) &\geq \nu(|a_{L+j}| - |\gamma^{-1}|) - c'_0 - c'_1 |\log(d(U_1(\rho(a_{L+j})), U_1(\rho(\gamma^{-1}))))| \geq \\ &\geq \nu((L+j) + (K-L)) - c'_0 - c'_1 |\log(\sin a)| \geq \\ &= c_0^{-1}(j+K) - c_1 \end{aligned}$$

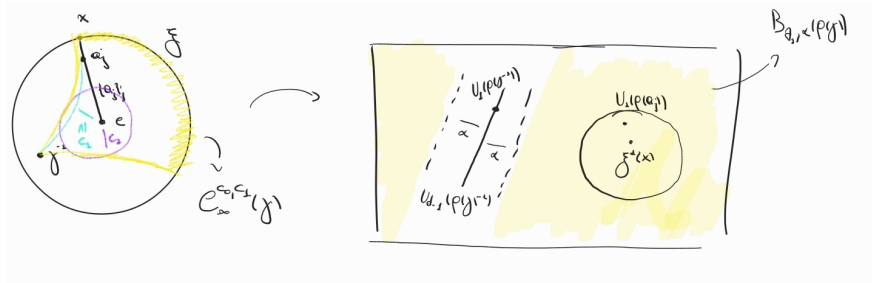
for $c_0 = \nu^{-1}, c_1 = c'_0 + c'_1 |\log(\sin \alpha)|$. The first inequality comes from [2, Lemma 3.9]. For the second inequality we estimate $|\gamma^{-1}|$ from below using the triangle inequality. We are now ready to show that the concatenation $(a'_j)_j$ is indeed a (c_0, c_1) -geodesic:

$$\begin{aligned} d(a_{L+j}, a_{L-i}) &\geq d(a_{L+j}, a_{L_K}) - d(a_{L_K}, a_{L-i}) \geq c_0^{-1}(j+K) - c_1 - (K-i) \geq \\ &\geq c_0^{-1}(j+i) - c_1. \end{aligned}$$

Note however that (a'_j) does not necessarily lie in $C_{\infty}^{c_0, c_1}$ since it may not pass through e . For this reason we some $L-K \leq i_0 \leq L$ such that $|a_{i_0}| < \alpha''$, the existence of which is guaranteed by the fact that $d([\gamma^{-1}, a_L], e) < \alpha''$. We then consider alter (a'_j) at i_0 so that it passes through e to obtain

$$a''_j = \begin{cases} a_j & \text{for } j \neq i_0 \\ e & \text{for } j = i_0 \end{cases}$$

which is a $(c_0, c_1 + \alpha'')$ -quasigeodesic passing from e and converging to x . □



Bibliography

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- [2] Beatrice Pozzetti, Andrés Sambarino, and Anna Wienhard. “Anosov representations with Lipschitz limit set”. In: *Geometry & Topology* 27.8 (Nov. 2023). arXiv:1910.06627 [math], pp. 3303–3360. issn: 1364-0380, 1465-3060 (cit. on pp. 4–7).