

Limit sets of Anosov representations

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Chapter 1

Introduction

1.1 Lie group preliminaries

We fix the Cartan subalgebra \mathfrak{a} of $\mathrm{SL}(d, \mathbb{R})$:

$$\mathfrak{a} = \{\mathrm{diag}(\alpha_1, \dots, \alpha_d) : \alpha_1 + \dots + \alpha_d = 0\}$$

and the Weyl chamber \mathfrak{a}^+ of $\mathrm{SL}(d, \mathbb{R})$

$$\mathfrak{a}^+ = \{\mathrm{diag}(\alpha_1, \dots, \alpha_d) : \alpha_1 \geq \dots \geq \alpha_d\}.$$

Denoting with $K = \mathrm{SO}(d, \mathbb{R})$, $A^+ = e^{\mathfrak{a}^+}$, we have the Cartan decomposition:

$$\begin{aligned}\mathfrak{sl}(d, \mathbb{R}) &\rightarrow K \times A^+ \times K \\ g &\mapsto (k_g, a_g, l_g)\end{aligned}$$

such that $g = k_g a_g l_g$. In particular $a_g = \mathrm{diag}(\sigma_1(g), \dots, \sigma_d(g))$ with $\sigma_1 \geq \dots \geq \sigma_d(g)$, where $\sigma_i(g)$ is the i -th singular value of g , i.e. eigenvalue of $g^t \cdot g$.

We will use the spaces

$$U_p(g) = \mathbb{R}u_1(g) \oplus \dots \oplus \mathbb{R}u_p(g)$$

where $u_i(g) = k_g \cdot e_i$. One can easily show that the decomposition

$$g^{-1} \cdot U_p(g) \oplus U_{d-p}(g^{-1})$$

is orthogonal with respect to the standard inner product and that $u_p(g^{-1}) = l_g^{-1} e_{d-p+1}(g)$.

1.2 Limit set preliminaries

Definition 1.2.1. For $p \in \{2, \dots, d\}$, $s \in \mathbb{R}$ and $g \in \mathrm{SL}(d, \mathbb{R})$ we denote with $\tilde{\Psi}_s^p(g), \Psi_s^p(g) : \mathrm{SL}(d, \mathbb{R}) \rightarrow \mathbb{R}$ the functional:

$$\begin{aligned}\Psi_s^p(g) &= \alpha_{12}(a(g)) + \dots + \alpha_{1(p-1)}(a(g)) + (s - (p-2))\alpha_{1p}(a(g)) \\ \tilde{\Psi}_s^p(g) &= \left(\frac{\sigma_2}{\sigma_1} \dots \frac{\sigma_{p-1}}{\sigma_1}(g) \right) \left(\frac{\sigma_{p-1}}{\sigma_1}(g) \right)^{s-(p-2)}\end{aligned}$$

Remark 1.2.1. We have $\alpha_{ij}(a) = a_i - a_j$, $a_i(g) = \log(\sigma_i(g))$ and

$$\Psi_s^p(g) = \log \tilde{\Psi}_s^p(g)$$

and that

$$\min_{p \in \llbracket 2, d \rrbracket} \left\{ \sum_{|\gamma|=T} \frac{\sigma_2}{\sigma_1} \cdots \frac{\sigma_{p-1}}{\sigma_1}(g) \left(\frac{\sigma_{p-1}}{\sigma_1}(g) \right)^{s-(p-2)} \right\} = \sum_{|\gamma|=T} e^{-\max_{p \in \llbracket 2, d \rrbracket} \Psi_s^p(g)}$$

Remark 1.2.2. For any $g \in \mathrm{SL}(d, \mathbb{R})$ we have that:

$$\max_{p \in \llbracket 2, d \rrbracket} \Psi_s^p(g) = \Psi_s^{p_0}(g) \text{ for } s \in [p_0 - 2, p_0 - 1].$$

Indeed, a quick calculation shows that for $s \geq 0$ and $p \in \llbracket 2, d \rrbracket$:

$$\Psi_s^p(g) \leq \Psi_s^p(g) \text{ if and only if } s \geq p - 1.$$

and that equality holds in the case $s = p - 1$. Thus for $s \in [p - 2, p - 1]$ we have that

$$\begin{aligned} s \geq p - 2, \dots, 1 \text{ implies that } \Psi_s^p(g) &\geq \dots \geq \Psi_s^2(g) \\ s \leq p, \dots, d - 1 \text{ implies that } \Psi_s^p(g) &\leq \dots \leq \Psi_s^d(g) \end{aligned}$$

Another way to see this (refer to Figure 1.1) is to note that $\Psi_s^2(g), \dots, \Psi_s^d(g)$ is a sequence of functions that are affine in s , with slopes $\alpha_{12}(g) \leq \dots \leq \alpha_{1d}(g)$ and that they satisfy $\Psi_1^2(g) = \Psi_2^2(g)$, $\Psi_2^3(g) = \Psi_3^3(g) \dots$, $\Psi_{d-2}^{d-1}(g) = \Psi_{d-2}^d(g)$.

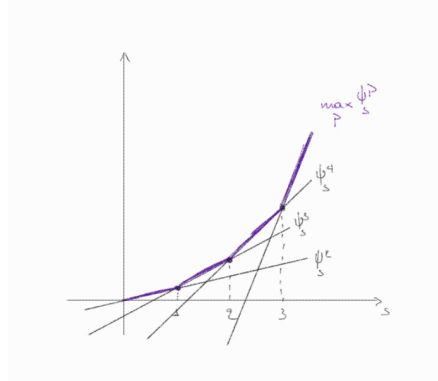


Figure 1.1: Visual illustration that $\max_p \Psi_s^p(g) = \Psi_s^{p_0}(g)$ for $s \in [p_0 - 2, p_0 - 1]$.

The following definition comes from [LL23], in the special case of projective Anosov representations ($P = 1$):

Definition 1.2.2. For $s \geq 0$ we consider the Falconer functional $F_s : \mathrm{SL}(d, \mathbb{R}) \rightarrow \mathbb{R}$ by:

$$F_s(g) = \min \left\{ \sum_{j=2}^d s_j \alpha_{1j}(a(g)) : s_j \in (0, 1], \sum_{j=2}^d s_j = s \right\},$$

and define the Falconer dimension $\dim_F(\rho)$ of ρ to be its critical exponent:

$$\dim_F(\rho) = \inf \left\{ s > 0 : \sum_{\gamma \in \Gamma} e^{-F_s(\rho(\gamma))} < \infty \right\}.$$

Remark 1.2.3. Using elementary computations one may prove that for all $s \geq 0$:

$$F_s(g) = \max_{p \in \llbracket 2, d \rrbracket} \Psi_s^p(g)$$

Definition 1.2.3. Let $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$ be a linear representation and $p \in \llbracket 1, d-1 \rrbracket$. We say that ρ is p -Anosov if there exist constants $\mu, C > 0$ such that for all $\gamma \in \Gamma$:

$$\frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma)) \leq C e^{-\mu|\gamma|}.$$

One can show that in that case there exist equivariant continuous maps $\xi^p : \hat{\Gamma} \rightarrow \mathcal{G}_p(\mathbb{R}^d), \xi^{d-p} : \hat{\Gamma} \rightarrow \mathcal{G}_{d-p}(\mathbb{R}^d)$ that are transverse and restrict to

$$\xi^p(\gamma) = U_p(\rho(\gamma)), \xi^{d-p}(\gamma) = U_{d-p}(\rho(\gamma))$$

for $\gamma \in \Gamma$, where $U_p(\gamma), U_{d-p}(\gamma)$ denote the flags corresponding to $\rho(\gamma)$.

Figure out
what this
exactly
means

Chapter 2

Upper bound

2.1 Proof of bound

Lemma 2.1.1 (Upper bound for dimension). *Let $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. Then:*

$$\dim_H(\xi^1(\partial\Gamma)) \leq \dim_F(\rho).$$

Remark 2.1.1. The idea of the proof of Lemma 2.1.1 is to find a covering whose Hausdorff content is dominated by the Dirichlet series of some functional Ψ_s^p , which will in turn imply that $\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(\Psi^p)$. Choosing then the most "effective" cover (i.e. the one which yields the smallest Hausdorff content up to a constant) we obtain that

$$\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(\max_p \Psi^p)$$

To obtain this we first cover $\xi^1(\partial\Gamma)$ by the bassins of attraction $\rho(\gamma) \cdot B_{\alpha_1, \alpha}(\rho(\gamma))$ for $\gamma \in \Gamma$ satisfying $|\gamma| = T$. Then we cover each bassin by an ellipsoid of axes lengths

$$\frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \dots, \frac{1}{\sin(\alpha)} \frac{\sigma_d}{\sigma_1}(\rho(\gamma)).$$

Finally we cover each ellipsoid by balls of some fixed radius $r > 0$. It can be shown by comparing the series appearing in the Hausdorff content of each resulting cover that the most "effective" choice of r depends only on the Hausdorff exponent $s > 0$ and in any case will be to have r equal (up to a constant) to the the length of an axis of the ellipsoid, i.e.

$$r \in \left\{ \frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \dots, \frac{1}{\sin(\alpha)} \frac{\sigma_d}{\sigma_1}(\rho(\gamma)) \right\}$$

In particular, when $s \in [p-2, p-1]$, the most effective choice is $r = \sigma_p(\rho(\gamma))/\sigma_1(\rho(\gamma))$, whose Hausdorff content is dominated by the Dirichlet series of Ψ_s^p .

Proof of Lemma 2.1.1. Let $p \in \llbracket 2, d \rrbracket$. Then using Proposition 2.2.1, Lemma 2.2.2, and Lemma 2.2.3 we have that for $T > 0$ large enough, $\xi^1(\partial\Gamma)$ is covered by the family

$$\mathcal{U}_T = \{\rho(\gamma)B_{\alpha_1, \alpha}(\rho(\gamma)) : |\gamma| = T\},$$

and that each basin $\rho(\gamma)B_{\alpha_1, \alpha}(\rho(\gamma))$ is in turn covered by

$$2^{p-2} \cdot \frac{\sigma_p(g)^{p-2}}{\sigma_2(g) \cdots \sigma_{p-1}(g)}$$

many balls of radius

$$\sqrt{d-1} \frac{1}{\sin \alpha} \frac{\sigma_p(g)}{\sigma_1(g)}.$$

By the definition of the Hausdorff measure, for $s \geq 0$:

$$\begin{aligned} \mathcal{H}^s(\xi^1(\partial\Gamma)) &\leq \sum_{|\gamma|=T} 2^{2p+1} \cdot \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \cdots \frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \left(\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \right)^{-(p-2)} \left(\sqrt{d-1} \frac{1}{\sin \alpha} \frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \right)^s = \\ &= 2^{2p+1} \cdot \left(\frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \cdots \frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \left(\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \right)^{s-(p-2)} = \\ &= 2^{2p+1} \cdot \left(\frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} e^{-(\alpha_{12} + \dots + \alpha_{1(p-1)} + (s-(p-2))\alpha_{1p})\rho(\gamma)} \\ &= 2^{2p+1} \cdot \left(\frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} e^{-\Psi_s^p(\rho(\gamma))} \end{aligned}$$

and thus

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq 2^{2p+1} \cdot \left(\frac{\sqrt{d-1}}{\sin \alpha} \right)^s \sum_{|\gamma|=T} e^{-\max_p \Psi_s^p(\rho(\gamma))} \lesssim \sum_{|\gamma|=T} e^{-F_s(\rho(\gamma))}.$$

To see that the above implies the upper bound, consider some $s > \dim_F(\rho)$. By the definition of the Falconer dimension, this implies that the Dirichlet series corresponding to the Falconer functional converges:

$$\sum_{\gamma \in \Gamma} e^{-F_s(\rho(\gamma))} < \infty$$

and in particular

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq \lim_{T \rightarrow \infty} e^{-F_s(\rho(\gamma))} = 0.$$

□

2.2 Lemmata

Definition 2.2.1. Let V be a finite-dimensional \mathbb{R} -vector space. We consider a decomposition

$$V = \mathbb{R}u_1 \oplus \cdots \oplus \mathbb{R}u_d$$

be a direct decomposition that is orthogonal with respect to a fixed inner-product over V . Given $\beta_2 \geq \dots \beta_d > 0$, we define an ellipsoid with axes $u_1 \oplus u_p(g)$ and lengths β_p to be the image of

$$\left\{ v = \sum_1^d v_i u_i \in V : \sum_2^d \left(\frac{v_j}{\beta_j} \right)^2 \leq 1 \right\}$$

through the projection $V \rightarrow \mathbb{P}(V)$.

The following aims to be something along the lines of [PSW23, Lemma 2.4]:

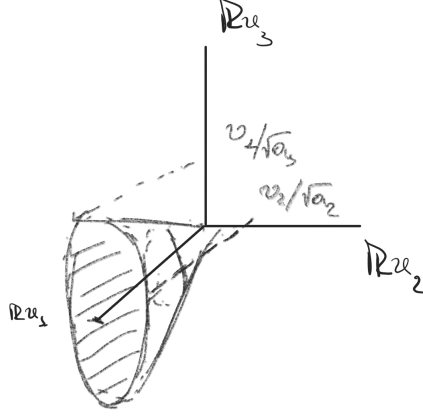


Figure 2.1: Depiction in \mathbb{R}^3 of an ellipsoid of $\mathbb{P}(\mathbb{R}^2)$

Lemma 2.2.1. *Let $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$ be a projective Anosov representation. For $\alpha > 0$ small enough, there exists $L > 0$ such that for any geodesic ray $(a_j)_j$ through e we have:*

$$\angle(U_1(\rho(a_i)), U_{d-1}(\rho(a_0))) > \alpha$$

when $|a_i|, |a_0| > T$.

Proof. Assume the contrary for the sake of contradiction. Then (see Figure 2.2) for each $n > 0$ there exists a geodesic ray a^n through e such that

$$|a_n^n|, |a_0^n| > n \text{ and } \angle(U_1(\rho(a_n^n)), U_1(\rho(a_0^n))) < \frac{1}{n}.$$

Due to compactness of $\partial\Gamma$ we may assume (up to a subsequence) that $a^n \rightarrow x$ in $\partial\Gamma$ for some $x \in \partial\Gamma$. Then $a_n^n, a_0^n \rightarrow x$ in $\hat{\Gamma}$ which implies

$$\angle(\xi^1(x), \xi^{d-1}(x)) = 0$$

using the fact that the limit maps ξ^1, ξ^{d-1} are continuous, which contradicts their transversality. \square

The following is [PSW23, Proposition 3.5].

Lemma 2.2.2. *Let $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$ be projective Anosov. Then for $\alpha > 0$ small enough, there exists some $T_0 > 0$ such that for all $T \geq T_0$ the family*

$$\mathcal{U}_T = \{\rho(\gamma)B_{\alpha_1, \alpha}(\rho(\gamma)) : |\gamma| = T\}$$

is an open covering of $\xi^1(\partial\Gamma)$.

Proof. Let $\alpha, T > 0$ be as in the statement of Lemma 2.2.1 and $x \in \partial\Gamma$ be represented by a geodesic ray $(\gamma_j)_{j \geq 0}$ starting from e . Then $(\gamma_T^{-1}\gamma_j)_j$ is a geodesic ray starting from $(\gamma_T)^{-1}$ that passes through e , so

$$\angle(U_1(\rho(\gamma_T^{-1}\gamma_j)), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

Not sure if this is true.

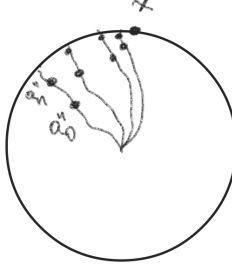


Figure 2.2: Situation in Lemma 2.2.1

as implied by Lemma 2.2.1. Taking the limit $j \rightarrow \infty$ and using the equivariance of the limit map, we obtain

$$\angle(\rho(\gamma_T^{-1})\xi^1(x), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

and thus $\xi^1(x) \in \rho(\gamma_T) \cdot B_{\alpha_1, \alpha}(\rho(\gamma_T))$. \square

The following is [PSW23, Proposition 3.8].

Proposition 2.2.1. *For each $g \in \text{SL}(d, \mathbb{R})$, $\alpha > 0$, the basin of attraction $g \cdot B_{\alpha_1, \alpha}(g)$ lies in the ellipsoid with axes $u_1(g) \oplus u_p(g)$ with lengths*

$$\frac{1}{\sin \alpha} \cdot \frac{\sigma_p(g)}{\sigma_1(g)}$$

Proof. Using the definition of the basin of attraction (see Figure 2.3), we have that $w = w_1 u_1(g^{-1}) + \dots + w_d u_d(g^{-1}) \in B_{\alpha_1, \alpha}(g)$ if and only if

$$w_d^2 \geq (\sin \alpha)^2 \sum_1^d w_i^2.$$

Considering now some $v = v_1 u_1(g) + \dots + v_d u_d(g) \in g \cdot B_{\alpha_1, \alpha}(g)$ we have that

$$\begin{aligned} w &= g^{-1}v = v_1 \sigma_1(g)^{-1} l_g^{-1} e_1(g) + \dots + v_d \sigma_d(g)^{-1} l_g^{-1} e_d(g) \\ &= v_1 \sigma_1(g)^{-1} u_d(g^{-1}) + \dots + v_d \sigma_d(g)^{-1} u_1(g^{-1}) \end{aligned}$$

where we used that $u_p(g^{-1}) = l_g^{-1} e_{d+1-p}$. Hence

$$\sigma_1(g)^{-2} \cdot v_1^2 \geq (\sin \alpha)^2 \sum_1^d \sigma_i(g)^{-2} v_i^2.$$

\square

The following is [PSW23, Lemma 3.7]:

Lemma 2.2.3. *For any $p \in \llbracket 2, d \rrbracket$, an ellipsoid in $\mathbb{P}(\mathbb{R}^d)$ of axes lengths β_2, \dots, β_d is covered by*

$$2^{p-2} \frac{\beta_2 \cdots \beta_{p-1}}{\beta_p^{p-2}}$$

many (projected) balls of radius $\sqrt{d-1} \beta_p$.

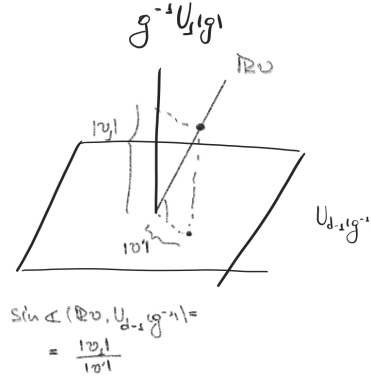


Figure 2.3: Aid for Proposition 2.2.1

Proof. We assume that E is an ellipsoid about $\mathbb{R}e_1$, so it suffice to cover its intersection $E_1 = E \cap U_1 \subseteq \mathbb{R}^{d-1}$ with the affine chart $U_1 = \{[x_1 : \dots, x_d] \in \mathbb{P}(\mathbb{R}^d) : x_1 \neq 0\}$. Clearly $E_1 \subseteq [-\beta_2, \beta_2] \times \dots \times [-\beta_d, \beta_d]$, so we proceed by covering the rectangle with side-lengths $2\beta_2, \dots, 2\beta_d$. Clearly each interval $(-\beta_j, \beta_j)$ is contained in the union of $\lceil \beta_j/\beta_p \rceil$ intervals of length $2\beta_p$, thus E_1 is contained in the union of

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil \dots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil = \left\lceil \frac{\beta_2}{\beta_p} \right\rceil \dots \left\lceil \frac{\beta_d}{\beta_p} \right\rceil$$

many squares of side-length $2\beta_p$. Since each such product is contained in a $(d-1)$ -ball of radius $\sqrt{d-1}\beta_p$ we may use at most

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil \dots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil \leq \sum_{i \in \{0,1\}^{p-2}} \prod_{j=2}^{p-1} \left(\frac{\beta_j}{\beta_p} \right)^{i_j} \leq 2^{p-2} \frac{\beta_2}{\beta_p} \dots \frac{\beta_{p-1}}{\beta_p}$$

many $(d-1)$ -balls of radius $\sqrt{d-1}\beta_p$ to cover E_1 . □

The following can be found in [PSW23, Proposition 3.3]:

Proposition 2.2.2. *Let $\rho : \Gamma \rightarrow SL(d, \mathbb{R})$ be projective Anosov and $\alpha > 0$. Then there exist $c_0, c_1 > 0$ that depends only on α and ρ such that for all $\gamma \in \Gamma$:*

$$(\xi^1)^{-1}(B_{\alpha_1, \alpha}(\rho(\gamma))) \subseteq C_{c_0, c_1}^\infty(\gamma)$$

Proof. We begin by noting that it suffices to show this for all but finitely many $\gamma \in \Gamma$, since then we may alter the constants to satisfy the wanted inclusion also for the finitely many remaining $\gamma \in \Gamma$. Given this, we shall assume that $|\gamma| \geq l_0$ where $l_0 > 0$ is such that $Ce^{-\mu l_0} < 1$ and $C, \mu > 0$ are the constants appearing in the definition of the Anosov property of ρ .

Suppose $x \in \partial\Gamma$ such that $\xi^1(x) \in B_{\alpha_1, \alpha}(\rho(\gamma))$, and consider a geodesic ray $a_j \rightarrow x$ starting from $a_0 = e$. To prove the result, it suffices to find constants c_0, c_1 independent of γ and a (c_0, c_1) -quasi-geodesic from γ^{-1} to x that passes through e and stays at a bounded distance from $(a_j)_{j=0}^\infty$.

Using [PSW23, Proposition 2.5] we have that $d(\xi^1(a_j), U_{d-1}(\rho(\gamma^{-1}))) \leq Ce^{-\mu j}$, so there exists some $L > 0$ that depends only on α such that for all $j \geq L : U_1(\rho(a_j)) \in B_{\alpha_1, \alpha}(\rho(\gamma))$ and in particular

$$d(\xi^1(a_j), \gamma^{-1}) = d(U_1(\rho(a_j)), U_1(\rho(\gamma^{-1}))) \geq d(U_1(\rho(a_j)), U_{d-1}(\rho(\gamma^{-1}))) > \sin \alpha.$$

Along with the uniform continuity of $\xi^1 : \Gamma \cup \partial\Gamma \rightarrow \mathbb{P}(\mathbb{R}^d)$ this implies there exists some $\alpha' > 0$ and $L > 0$ such that for all $j \geq L$:

$$d(a_j, \gamma^{-1}) \geq \alpha'.$$

Upon considering a large L , we may also assume that $|a_L| = L > l_0$. Note that both α' and L do not depend on each γ but only on ρ and α .

Using some geometric group theory, we can show that for all $j \geq L$

$$d(\gamma^{-1}, a_j) > \alpha' \Rightarrow d([\gamma^{-1}, a_j], e) < \alpha''$$

for some α'' that depends only on Γ and α' , where $[a_j, \gamma^{-1}]$ denotes the geodesic segment connecting γ^{-1} and a_j .

Consider the concatenation $(a'_j)_{j=L-K}^\infty$ of $[\gamma^{-1}, a_L]$ and $[a_L, x]$. To find quasi-geodesic-constants that are uniform in γ , we note that for any $c_0 \geq 1, c_1 \geq 0$:

$$c_0^{-1}|i-j| - c_1 \leq d(a'_i, a'_j) = d(a_i, a_j) \leq d(a_i)c_0|i-j| + c_1 \text{ when } i, j \geq L \text{ or } i, j \leq L$$

and that the upper bound follows trivially by the triangle inequality.

For the lower bound we proceed in two steps. First we bound the distance of $\gamma^{-1} = a'_{L-K}$ to a_{L+j} for $j \geq 0$:

$$\begin{aligned} d(a'_{L-K}, a'_{L+j}) &\geq \nu(|a_{L+j}| - |\gamma^{-1}|) - c'_0 - c'_1 |\log(d(U_1(\rho(a_{L+j})), U_1(\rho(\gamma^{-1}))))| \geq \\ &\geq \nu((L+j) + (K-L)) - c'_0 - c'_1 |\log(\sin a)| \geq \\ &= c_0^{-1}(j+K) - c_1 \end{aligned}$$

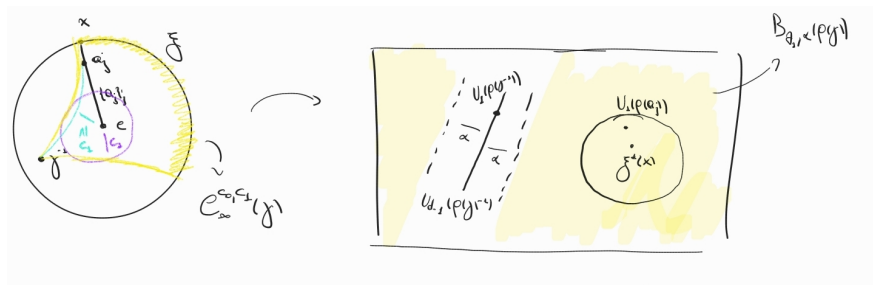
for $c_0 = \nu^{-1}, c_1 = c'_0 + c'_1 |\log(\sin \alpha)|$. The first inequality comes from [PSW23, Lemma 3.9]. For the second inequality we estimate $|\gamma^{-1}|$ from below using the triangle inequality. We are now ready to show that the concatenation $(a'_j)_j$ is indeed a (c_0, c_1) -geodesic:

$$\begin{aligned} d(a_{L+j}, a_{L-i}) &\geq d(a_{L+j}, a_{L-K}) - d(a_{L-K}, a_{L-i}) \geq c_0^{-1}(j+K) - c_1 - (K-i) \geq \\ &\geq c_0^{-1}(j+i) - c_1. \end{aligned}$$

Note however that (a'_j) does not necessarily lie in $C_\infty^{c_0, c_1}$ since it may not pass through e . For this reason we some $L-K \leq i_0 \leq L$ such that $|a_{i_0}| < \alpha''$, the existence of which is guaranteed by the fact that $d([\gamma^{-1}, a_L], e) < \alpha''$. We then consider alter (a'_j) at i_0 so that it passes through e to obtain

$$a''_j = \begin{cases} a_j & \text{for } j \neq i_0 \\ e & \text{for } j = i_0 \end{cases}$$

which is a $(c_0, c_1 + \alpha'')$ -quasigeodesic passing from e and converging to x . \square



Chapter 3

Lower bound

We denote with Π the set of simple positive roots, and for $\Theta \subseteq \Pi$ we consider the Levi-Anosov subspace of \mathfrak{a}

$$\mathfrak{a}_\Theta = \bigcap_{\alpha \notin \Theta} \ker \alpha,$$

which in particular admits $\{\omega_{\alpha_i} : \alpha_i \in \Theta\}$ as a basis. Finally, we shall consider the Busemann cocycle

$$b_\Theta : \mathrm{PSL}(d, \mathbb{R}) \times \mathcal{F}_\Theta \rightarrow \mathfrak{a}_\Theta$$

which might as well be defined as

$$\omega_{\alpha_i}(b_\Theta(g, x)) = \log \frac{\|gv_1 \wedge \cdots \wedge gv_i\|}{\|v_1 \wedge \cdots \wedge v_i\|} \text{ for all } \alpha_i \in \Theta$$

for any basis v_1, \dots, v_i of $x^i \in \mathcal{G}_i(\mathbb{R}^d)$, where $\|\cdot\|$ denotes the norm on $\bigwedge^i \mathbb{R}^d$ induced by the euclidean inner product on \mathbb{R}^d , i.e. $\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle := \det(\langle v_i, w_j \rangle)$.

Definition 3.0.1. For a discrete subgroup $\Gamma < \mathrm{PSL}(d, \mathbb{R})$, $\phi \in (\alpha_\Theta)^*$, a (Γ, ϕ) -Patterson Sullivan measure on \mathcal{F}_Θ is a finite Radon measure μ such that for every $\gamma \in \Gamma$

$$\frac{d\gamma_*\mu}{d\mu}(x) = e^{-\phi(b_\Theta(g^{-1}, x))}, \text{ for all } x \in \mathcal{F}_\Theta(\mathbb{R}^d).$$

Lemma 3.0.1. Let $\alpha > 0, \Theta \subseteq \Pi$. There exists $K = K(\alpha) > 0$ such that for each $g \in \mathrm{SL}(d, \mathbb{R})$, $\mathfrak{a}_i \in \Theta$, $y \in B_{\Theta, \alpha}(g)$

$$|\omega_i(a(g) - b(g, y))| \leq K.$$

Recalling that $\{\omega_i\}_{\mathfrak{a}_i \in \Theta}$ is a basis for \mathfrak{a}_Θ , the above implies in particular that for each $\phi \in \mathfrak{a}_\Theta^*$ there exists $K = K(\alpha, \phi) > 0$ such that for all $g \in \mathrm{SL}(d, \mathbb{R})$, $y \in B_{\Theta, \alpha}(g)$

$$|\phi(a(g) - b(g, y))| \leq K.$$

How to prove this?

3.1 Proof strategy

Denoting with $d_\Gamma = \dim_H \xi_\rho^1(\partial\Gamma)$ the Hausdorff dimension of the limit set, we shall outline how to obtain its lower bound by the Falconer dimension

$$d_\Gamma \geq h_\rho(F).$$

First we recall that $F_s(a) = \max\{\Psi_s^p(a) : p \in \llbracket 2, d \rrbracket\}$ and in particular $h_\rho(F) \leq h_\rho(\Psi^{d_\Gamma+1})$. Thus the lower bound will follow once we have shown that

$$d_\Gamma \geq h_\rho(\Psi^{d_\Gamma+1}).$$

Noting that $(s+1)J_{d_\Gamma}^u \geq \Psi_{s+d_\Gamma}^{d_\Gamma+1}$, the above bound will follow as soon as we have shown that

$$h_\rho(J_{d_\Gamma}) \geq 1. \quad (\text{LB})$$

To obtain Equation (LB), one uses the method of Patterson-Sullivan-Quint, which may be summed up in

There exists a (ϕ, ρ) -Patterson-Sullivan measure on $\mathcal{F}_\Theta(\mathbb{R}^d) \Rightarrow h_\rho(\phi) \geq 1$,

where $\phi \in \mathfrak{a}_\Theta$ and $\Theta \subseteq \Pi$. The property that we will need of our measure is that there exists a collection of open sets $U_\gamma \in \Gamma$ such that

$$\mu(U_\gamma) \sim e^{-J_{d_\Gamma}^u(a(\rho(\gamma)))} \text{ and } M := \sup_{n \in \mathbb{N}} \max \left\{ \#A : A \subseteq \Gamma_n, \bigcap_{\gamma \in A} U_\gamma \neq \emptyset \right\} < \infty \quad (\text{MP})$$

where $\Gamma_n = \{\gamma \in \Gamma : |\gamma| = n\}$. The existence of a $(J_{d_\Gamma}^u, \rho)$ -Patterson-Sullivan measure that satisfies the property (MP) will be proved in Section 3.2. Assuming that for the time being, below we outline the Patterson-Sullivan-Quint method of obtaining Equation (LB).

Indeed, we first obtain the uniform in n bound:

$$\sum_{\gamma \in \Gamma_n} e^{-J_{d_\Gamma}^u(a(\rho(\gamma)))} \lesssim \sum_{\gamma \in \Gamma_n} \mu(\rho(U_\gamma)) \leq \frac{1}{M} \mu(\mathcal{F}_\Theta(\mathbb{R}^d)) < \infty$$

along with the bound implied by the Anosov property of ρ :

$$J_{d_\Gamma}(a(\rho(\gamma))) \geq \mathfrak{a}_{12}(a(\rho(\gamma))) \geq C|\gamma| - b$$

to conclude that

$$\sum_{\gamma \in \Gamma} e^{-(s+1)J_{d_\Gamma}^u(a(\rho(\gamma)))} = \sum_{n \geq 0} \sum_{\gamma \in \Gamma_n} e^{-sJ_{d_\Gamma}^u(a(\rho(\gamma)))} e^{J_{d_\Gamma}^u(a(\rho(\gamma)))} \lesssim \sum_{n \geq 0} e^{-s(Cn-b)} < \infty$$

which holds for any $s > 0$, and thus Equation (LB) holds.

3.2 Existence of Patterson-Sullivan measure

Definition 3.2.1. Let $V \in \mathcal{G}_{p+1}\mathbb{R}^d$ and $l \in \mathbb{P}(V)$. Using the canonical identification $T_l\mathbb{P}(V) \simeq \text{hom}(l, V/l)$, we define the density $|\Omega_{l,V}|$ on $\bigwedge^p T_l\mathbb{P}(V)$ by

$$|\Omega_{l,V}|(\phi_1, \dots, \phi_p) = \frac{\|v \wedge \tilde{\phi}_1(v) \wedge \dots \wedge \tilde{\phi}_p(v)\|}{\|v\|^{p+1}}$$

for any $v \in l - \{0\}$, where $\tilde{\phi}_1, \dots, \tilde{\phi}_p \in \text{hom}(l, V)$ are such that $\phi_i = \tilde{\phi}_i + \text{hom}(l, l)$ and $\|\cdot\|$ denotes the norm on $\bigwedge^{p+1} \mathbb{R}^d$ induced by the euclidean inner product.

The following is [PSW23, Proposition 6.4]:

Proposition 3.2.1. *Assume that $\xi_\rho^1(\partial\Gamma)$ is a Lipschitz submanifold of dimension d_Γ . Then there exists a $(\rho(\Gamma), J_{d_\Gamma}^u)$ -Patterson-Sullivan measure on $\mathcal{F}_{1,d_\Gamma+1}$.*

Proof. By Rademacher's theorem, $\xi_\rho^1(\partial\Gamma)$ has a well-defined Lebesgue measure class, and Lebesgue-almost every $\xi_\rho^1(x) \in \xi_\rho^1(\partial\Gamma)$ admits a well-defined tangent space $T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma)$. Considering such a $\xi_\rho^1(x)$ we let

$$\pi : \text{hom}(\xi_\rho^1(x), \mathbb{R}^d) \rightarrow \text{hom}(\xi_\rho^1(x), \mathbb{R}^d / \xi_\rho^1(x)) \simeq T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma),$$

and

$$x^{d_\Gamma+1} = \pi^{-1}(T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma))\xi_\rho^1(x) \in \mathcal{G}_{d_\Gamma+1}(\mathbb{R}^d),$$

for which one can show that

$$T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma) \simeq \text{hom}(\xi_\rho^1(x), \mathbb{R}^d / \xi_\rho^1(x)) \simeq \text{hom}(\xi_\rho^1(x), x^{d_\Gamma+1} / \xi_\rho^1(x)).$$

In this notation, we shall define (Lebesgue-almost everywhere) the map

$$\zeta_\rho : \xi_\rho^1(\partial\Gamma) \rightarrow \mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d), \quad \zeta_\rho(\xi_\rho^1(x)) = (\xi_\rho^1(x), x^{d_\Gamma+1}).$$

We now define the non-negative density on $\xi_\rho^1(\partial\Gamma)$

$$\mu_{\xi_\rho^1(x)} = |\Omega_{\zeta_\rho(\xi_\rho^1(x))}|$$

which satisfies

$$\frac{d\gamma_*\mu}{d\mu}(\xi) = \frac{d(\rho(\gamma)^{-1})^*\mu}{d\mu}(\xi) = e^{-J_{d_\Gamma+1}^u(b_\Theta(\rho(\gamma)^{-1}, \zeta(x)))},$$

where the term on the left-hand side involves measures, the term between the two equalities involves densities, and $\Theta = \{1, d_\Gamma + 1\}$. Indeed, for $\phi_1, \dots, \phi_{d_\Gamma} \in T_{\xi_\rho^1(x)}\xi_\rho^1(\partial\Gamma)$:

$$\begin{aligned} & (\rho(\gamma)^*\mu)_{\xi_\rho^1(x)}(\phi_1, \dots, \phi_{d_\Gamma}) \\ &= \mu_{\rho(\gamma)\xi_\rho^1(x)}(\rho(\gamma)\phi_1\rho(\gamma)^{-1}, \dots, \rho(\gamma)\phi_{d_\Gamma}\rho(\gamma)^{-1}) = \\ &= \frac{\|\rho(\gamma)\xi_\rho^1(x) \wedge \rho(\gamma)\phi_1(\xi_\rho^1(x)) \wedge \dots \wedge \rho(\gamma)\phi_{d_\Gamma}(\xi_\rho^1(x))\|}{\|\rho(\gamma)\xi_\rho^1(x)\|^{d_\Gamma+1}} = \\ &= \frac{\|\rho(\gamma)\xi_\rho^1(x) \wedge \rho(\gamma)\phi_1(\xi_\rho^1(x)) \wedge \dots \wedge \rho(\gamma)\phi_{d_\Gamma}(\xi_\rho^1(x))\|}{\|\xi_\rho^1(x) \wedge \phi_1(\xi_\rho^1(x)) \wedge \dots \wedge \phi_{d_\Gamma}(\xi_\rho^1(x))\|} \cdot \frac{\|\xi_\rho^1(x) \wedge \phi_1(\xi_\rho^1(x)) \wedge \dots \wedge \phi_{d_\Gamma}(\xi_\rho^1(x))\|}{\|\rho(\gamma)\xi_\rho^1(x)\|^{d_\Gamma+1}} \cdot \frac{\|\rho(\gamma)\xi_\rho^1(x)\|^{d_\Gamma+1}}{\|\xi_\rho^1(x)\|^{d_\Gamma+1}} = \\ &= e^{\omega_{d_\Gamma}(b_\Theta(\rho(\gamma), \zeta_\rho(\xi_\rho^1(x))))} \cdot \mu_{\xi_\rho^1(x)}(\phi_1, \dots, \phi_{d_\Gamma}) \cdot e^{-(p+1)\omega_1(b_\Theta(\rho(\gamma), \zeta_\rho(\xi_\rho^1(x))))} = \\ &= e^{-J_{d_\Gamma}^u(b_\Theta(\rho(\gamma)^{-1}, \zeta(\xi_\rho^1(x))))} \mu_{\xi_\rho^1(x)}(\phi_1, \dots, \phi_{d_\Gamma}). \end{aligned}$$

Finally, we let $\nu = \zeta_{\rho*}\mu$, which is the wanted Patterson-Sullivan measure on $\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d)$, since for $f \in C_c(\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d))$:

$$\begin{aligned} \int_{\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d)} f d(\gamma_*\zeta_{\rho*}\mu) &= \int_{\xi_\rho^1(\partial\Gamma)} f \circ \gamma \circ \zeta_\rho d\mu = \int_{\xi_\rho^1(\partial\Gamma)} f \circ \zeta_\rho \circ \gamma d\mu = \\ &= \int_{\xi_\rho^1(\partial\Gamma)} f \circ \zeta_\rho(\xi) e^{-J_{d_\Gamma}^u(b_\Theta(\rho(\gamma)^{-1}, \zeta(\xi_\rho^1(x))))} d\mu(\xi_\rho^1(x)) = \\ &= \int_{\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d)} f(y) e^{-J_{d_\Gamma}^u(b_\Theta(\rho(\gamma)^{-1}, y))} d(\zeta_{\rho*}\mu)(y) \end{aligned}$$

□

Before giving the next definition, we recall that the annihilator of an element $y \in \mathcal{F}_\Theta(\mathbb{R}^d)$ is the set of partial flags that are not transverse to y , that is:

$$\text{Ann}(y) = \{x \in \mathcal{F}_\Theta(\mathbb{R}^d) : x^\theta \cap y^{d-\theta} \neq 0 \text{ for some } \theta \in \Theta\}.$$

Definition 3.2.2. Let $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$ be a linear representation, $\Theta \subseteq \Pi$ and μ a measure over $\mathcal{F}_\Theta(\mathbb{R}^d)$. We say that ρ is μ -irreducible there is no element in $\mathcal{F}_\Theta(\mathbb{R}^d)$, whose annihilator is of full measure, i.e. for all $y \in \mathcal{F}_\Theta(\mathbb{R}^d)$:

$$\mu(\text{Ann}(y)) < \mu(\mathcal{F}_\Theta(\mathbb{R}^d)).$$

Example 3.2.1. If $\rho(\Gamma)$ is Zariski-dense in $\text{SL}(d, \mathbb{R})$, then ρ is μ -irreducible for any ρ -quasi-invariant measure μ , and in particular for any $(\rho(\Gamma), \phi)$ -Patterson-Sullivan measure.

Remark 3.2.1. The reason that we introduce the concept of μ -irreducibility is that for any μ -irreducible representation $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$, there exist $\alpha, \kappa > 0$ such that $\mu(B_{\Theta, \alpha}(\rho(\gamma))) \geq \kappa$ for all $\gamma \in \Gamma$.

Indeed, if this were not the case, then there would exist a sequence $\alpha_n \searrow 0$ and $\gamma_n \in \Gamma$ such that

$$\mu(B_{\Theta, \alpha_n}(\rho(\gamma_n))) \leq \frac{1}{n}.$$

Due to the compactness of $\mathcal{F}_\Theta(\mathbb{R}^d)$, up to considering a subsequence, we may assume that the repelling flags or $\rho(\gamma_n)$ converge to some $\xi \in \mathcal{F}_\Theta(\mathbb{R}^d)$:

$$(U_{d-i}(\rho(\gamma_n)^{-1}))_{a_i \in \Theta} \rightarrow \xi$$

In that case, the complements $B_{\Theta, \alpha_n}^c(\rho(\gamma_n))$ will converge to the annihilator of ξ , in the sense:

$$\limsup_n B_{\Theta, \alpha_n}^c(\rho(\gamma_n)) \subseteq \text{Ann}(\xi).$$

Indeed, let $y \in \limsup_n B_{\Theta, \alpha_n}^c(\rho(\gamma_n))$ and consider a subsequence k_n such that $y \in B_{\Theta, \alpha_n}^c(\rho(\gamma_{k_n}))$. By the very definition of $B_{\Theta, \alpha_n}^c(\rho(\gamma_n))$, there exists some p such that up to considering a subsequence of k_n ,

$$\angle(y^p, U_{d-p}(\rho(\gamma_n)^{-1})) \leq \alpha_n$$

holds. Taking the limit as $n \rightarrow \infty$, we have that $y^p \cap \xi^{d-p} \neq 0$ and hence $y \in \text{Ann}(\xi)$.

Using a measure-theoretic argument we conclude that $\text{Ann}(\xi)$ is of full measure, which contradicts the μ -irreducibility of ρ :

$$\mu(\text{Ann}(\xi)) \geq \mu(\limsup_n B_{\Theta, \alpha_n}^c(\rho(\gamma_{k_n}))) \geq \limsup_n \mu(B_{\Theta, \alpha_n}^c(\rho(\gamma_{k_n}))) = \mu(\mathcal{F}_\Theta(\mathbb{R}^d)).$$

Lemma 3.2.1. Let $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$ be a representation and μ^ϕ be a $(\rho(\Gamma), \phi)$ -Patterson-Sullivan measure. If $\rho(\Gamma)$ is μ -irreducible, then there exists some $\alpha_0 > 0$, such that for any $\alpha \in (0, \alpha_0)$, there's some $k = k(\alpha) > 0$ for which

$$\frac{1}{k} e^{-\phi(a(\rho(\gamma)))} \leq \mu^\phi(\rho(\gamma) B_{\Theta, \alpha}(\rho(\gamma))) \leq k e^{-\phi(a(\rho(\gamma)))}$$

for all $\gamma \in \Gamma$.

Proof. Let $\alpha_0, k > 0$ be as in the remark preceeding the statement of the lemma. As noted in Lemma 3.0.1, there exists some $K = K(\alpha_0, \phi) > 0$ such that for any $\alpha \in (0, \alpha_0)$ and $y \in B_{\Theta, \alpha}(\rho(\gamma))$:

$$|\phi(a(\rho(\gamma))) - b(\rho(\gamma), y)| \leq K,$$

from which we obtain the upper bound

$$\begin{aligned} \mu^\phi(\rho(\gamma)B_{\Theta, \alpha}(\rho(\gamma))) &= (\rho(\gamma^{-1})_*\mu^\phi)(B_{\Theta, \alpha}(\rho(\gamma))) = \int_{\mathcal{F}_\Theta(\mathbb{R}^d)} e^{-\phi(b(\rho(\gamma), y))} d\mu^\phi(y) \leq \\ &\leq e^{-K} \mu^\phi(\mathcal{F}_\Theta(\mathbb{R}^d)) e^{-\phi(a(\rho(\gamma)))}. \end{aligned}$$

Similarly we obtain the lower bound

□

Appendix A

Tangent space to the Grassmanian

Let V be a d -dimensional real vector space. We denote with $\mathcal{G}_k(V)$ the Grassmanian of k -dimensional subspaces of V . Our first objective is to find a convenient way to express its tangent space.

Proposition A.0.1. *We have the following canonical identification:*

$$\begin{aligned} \text{hom}(W, V/W) &\simeq T_W \mathcal{G}_k(V) \\ \phi &\mapsto \left. \frac{d}{dt} \right|_{t=0} \Gamma(t\phi) \end{aligned}$$

where $\Gamma(\phi) = (Id + \phi)(W)$ is the graph of ϕ .

Proof. We will consider the map

$$F : \text{Injhom}(W, V) \rightarrow \mathcal{G}_k(V), \quad \phi \mapsto \phi(W).$$

whose derivative is given by:

$$d_I F(\phi) = F \left(\left. \frac{d}{dt} \right|_{t=0} (I + t\phi) \right) = \left. \frac{d}{dt} \right|_{t=0} (I + t\phi(W)) = \left. \frac{d}{dt} \right|_{t=0} \Gamma(t\phi).$$

The result will follow as soon as we have shown that $d_I F$ is surjective and that $\ker d_I F = \text{hom}(W, W)$.

To show that it is surjective, we consider a $(d-k)$ -dimensional subspace $W' \in \mathcal{G}_{d-k}(V)$ that is complementary to W , i.e. $V = W \oplus W'$. Denoting with $U_{W'} = \{Z \in \mathcal{G}_k(V) : Z \cap W' = 0\}$, we recall the corresponding chart:

$$\begin{aligned} \Psi : \text{hom}(W, W') &\rightarrow U_{W'} \\ \phi &\mapsto \Gamma(\phi). \end{aligned}$$

Surjectivity of $d_I F$ now follows by the fact that

$$d_I F(\phi) = \left. \frac{d}{dt} \right|_{t=0} \Gamma(t\phi) = d_0 \Psi(\phi).$$

To show that $\ker d_I F = \text{hom}(W, W)$, we first note that clearly $\ker d_I F \supseteq \text{hom}(W, W)$. Equality then follows by the fact that $\dim \text{hom}(W, W) = \dim \ker d_I F$, which is a direct consequence of the surjectivity. \square

Note that another way to prove the above identification through the fact that the Grassmanian is a homogeneous space of $\mathrm{GL}(d, \mathbb{R})$, giving us the diffeomorphism

$$\begin{aligned} \mathrm{GL}(V)/\mathrm{St}_{\mathrm{GL}(V)}W &\rightarrow \mathcal{G}_k(V) \\ [g] &\mapsto gW, \end{aligned}$$

where $\mathrm{St}_{\mathrm{GL}(V)}W = \{g \in \mathrm{GL}(V) : gW = W\}$ is the stabilizer of W . Thus an expression for the tangent space at W may be obtained by differentiating the map above at the identity coset:

$$\mathrm{hom}(W, V/W) \simeq \mathrm{hom}(V, V)/\mathrm{hom}(W, W) \simeq T_W \mathcal{G}_k(V).$$

In particular the above gives us a direct proof that the kernel of the map constructed in the previous proof is indeed $\mathrm{hom}(W, W)$.

Our second objective is to identify subspaces of $T_l \mathbb{P}(V)$ with subspaces of V , by considering the first as projectivisation of the second. More concretely, we shall consider the space

$$\mathcal{P} = \{(l, P) : l \in \mathbb{P}(V), P \in \mathcal{G}_k(T_l \mathbb{P}(V))\}$$

as a homogeneous space of $\mathrm{SL}(V)$, where the action is given by

$$g \cdot (l, P) = (gl, d_l g(P) = g\pi^{-1}(P)g^{-1} + \mathrm{hom}(gl, gl)).$$

where we use the identification of $T_l \mathbb{P}(V)$ with $\mathrm{hom}(l, V/l)$ as above and denote with $\pi : \mathrm{hom}(l, V) \rightarrow \mathrm{hom}(l, V/l)$ the canonical projection. For the sake of completeness, we outline the calculation of the differential:

$$\begin{aligned} \mathrm{hom}(l, V/l) &\rightarrow T_l \mathbb{P}(V) && \rightarrow T_{gl} \mathbb{P}(V) && \rightarrow \mathrm{hom}(gl, V/gl) \\ \phi &\mapsto \left. \frac{d}{dt} \right|_{t=0} g(I + t\tilde{\phi})(l) && \mapsto \left. \frac{d}{dt} \right|_{t=0} (I + tg\tilde{\phi}g^{-1})(gl) && \mapsto g\tilde{\phi}g^{-1} + \mathrm{hom}(gl, gl) \end{aligned}$$

where $\phi \in \mathrm{hom}(l, V/l)$, $\tilde{\phi} \in \mathrm{hom}(l, V)$ such that $\tilde{\phi} + \mathrm{hom}(l, l) = \phi$.

We are now ready to express the needed identification:

Proposition A.0.2. *We have the following $\mathrm{SL}(V)$ equivariant identification:*

$$\begin{aligned} \mathcal{P} &\rightarrow \mathcal{F}_{1,k+1}(V) \\ (l, P) &\mapsto (l, \pi^{-1}(P)l) \\ (l, \mathrm{hom}(l, Q/l)) &\mapsto (l, Q). \end{aligned}$$

where $\pi : \mathrm{hom}(l, V) \rightarrow \mathrm{hom}(l, V/l)$ is the canonical projection.

Proof. We begin by showing that the left-to-right direction of the map is well-defined. For this, we first need to check that for $(l, P) \in \mathcal{P}$, we have that $\dim \pi^{-1}(P)l = k + 1$. Indeed, we have that $\dim \pi^{-1}(P) = k + 1$ as implied by the rank-nullity theorem for $\pi : \pi^{-1}(P) \rightarrow P$. The result then follows by the fact that $\pi^{-1}(P)l = T_1(l) \oplus \dots \oplus T_{k+1}(l)$ for any base T_1, \dots, T_{k+1} of $\pi^{-1}(P)$. The second thing to check is that $l \leq \pi^{-1}(P)l$, which holds since $\ker \pi = \mathrm{hom}(l, l) \leq \pi^{-1}(P)$.

To see that the two directions above are inverse to each other, we begin by examining the right-to-left-to-right composition:

$$(l, Q) \mapsto (l, \pi(\mathrm{hom}(l, Q))) \mapsto (l, \pi^{-1}\pi(\mathrm{hom}(l, Q))) = (l, \mathrm{hom}(l, Q)l) = (l, Q).$$

and for the left-to-right-to-left composition

$$(l, P) \mapsto (l, \pi^{-1}(P)l) \mapsto (l, \mathrm{hom}(l, \pi^{-1}(P)l)),$$

so it suffices to show that $\text{hom}(l, \pi^{-1}(P)l/l) = P$. Indeed, for $\pi^{-1}(P) = \mathbb{R}T_1 \oplus \cdots \oplus \mathbb{R}T_k$, we have that

$$\begin{aligned} \text{hom}(l, \pi^{-1}(P)l/l) &= \text{hom}(l, \pi^{-1}(P)l) / \text{hom}(l, l) = (\oplus_i \text{hom}(l, T_i(l)) / \text{hom}(l, l)) = \\ &= (\oplus_i \mathbb{R}T_i) / \text{hom}(l, l) = \pi^{-1}(P) / \text{hom}(l, l) = P. \end{aligned}$$

For the equivariance, the calculations has as follows:

$$\begin{array}{ccc} (l, P) & \xrightarrow{\quad\quad\quad} & (l, \pi^{-1}(P)l) \\ \downarrow g & & \downarrow g \\ (gl, g\pi^{-1}(P)g^{-1} + \text{hom}(gl, gl)) & \xrightarrow{\quad\quad\quad} & (gl, (g\pi^{-1}(P)g^{-1})(gl)) = (gl, g\pi^{-1}(P)l) \end{array} \quad \square$$

Appendix B

Irreducible actions problem

The matter of this chapter has to do with an obstruction, found in the proof of this lemma:

Lemma B.0.1 (Lemma 6.8 in [PSW23]). *Let Γ be a hyperbolic group and $\eta : \Gamma \rightarrow \mathrm{PGL}(d, \mathbb{R})$ be a strongly irreducible projective Anosov representation such that $\xi_\eta(\partial\Gamma)$ is homeomorphic to S^{d-1} , and which admits a measurable η -equivariant section $\zeta : \partial\Gamma \rightarrow \mathcal{F}_{\{\mathbf{a}_1, \mathbf{a}_{d+1}\}}(\mathbb{R}^d)$. Then η is μ -irreducible for any $(\eta(\Gamma), \phi)$ -Patterson-Sullivan measure μ on $\mathcal{F}_{\{\mathbf{a}_1, \mathbf{a}_{d+1}\}}(\mathbb{R}^d)$.*

For convenience, we recall that a linear representation $\rho : \Gamma \rightarrow \mathrm{GL}(d, \mathbb{R})$ is strongly irreducible if there is no proper $\rho(\Gamma)$ -invariant subspace of \mathbb{R}^d , and it is μ -irreducible if there is no element in $\mathcal{F}_{i\Theta}(\mathbb{R}^d)$, whose annihilator is of full measure.

In what follows, we will show that the above lemma is false, by providing a counterexample. Let Γ be a uniform lattice of $\mathrm{SU}(2, 1)$ (i.e. acts convex cocompactly), and $\eta : \Gamma \rightarrow \mathrm{SL}(\mathfrak{su}(2, 1))$ be the restriction of the adjoint representation, i.e. $\eta(\gamma) = \mathrm{Ad}_\gamma$ for all $\gamma \in \Gamma$.

For convenience, we recall the definition of a uniform lattice:

Definition B.0.1. Let G be a locally compact group. A uniform lattice is a discrete subgroup $\Gamma \leq G$ that is co-compact, i.e. G/Γ is compact.

Remark B.0.1. When $G = \mathrm{Isom}(X)$ is the isometry group of a complete Riemannian manifold X , and Γ is a uniform lattice of G , then it acts properly discontinuously and cocompactly on X .

We begin by showing proving the Anosov property of η .

Proposition B.0.1. *η is projective Anosov.*

Proof. Let $\gamma \in \Gamma$. Since $\gamma \in \mathrm{SU}(2, 1)$, we have that

$$\gamma = k_1 \exp(r(\gamma)x_0) k_2$$

for x_0 a fixed non-zero in the Weyl-chamber $\mathbb{R}x_0$ of $\mathfrak{su}(2, 1)$, $r(\gamma) \in \mathbb{R}$ and $k_1, k_2 \in \mathrm{U}(2)$. Then by the definition of a uniform lattice, we have that Γ acts properly discontinuously and cocompactly, which means that the inclusion $\Gamma \hookrightarrow \mathrm{SU}(2, 1)$ is projective Anosov (since $\mathrm{SU}(2, 1)$ is of rank 1). Thus there exist constants $L \geq 1, b \geq 0$ such that for all $\gamma \in \Gamma$:

$$r(\gamma) \geq \mathbf{a}_1(x_0)^{-1}(L|\gamma| - b) = L'|\gamma| - b'.$$

Note that $\mathbf{a}_1(x_0) > 0$ since x_0 is in the interior of the Weyl-chamber $\mathbb{R}x_0$.

Letting $k'_1 = Ad_{k_1}, k'_2 = Ad_{k_2}$ and $K' \leq \mathrm{SL}(\mathfrak{su}(2, 1))$ be a maximal comapct subgroup containing them, we have that:

$$\eta(\gamma) = \mathrm{Ad}_\gamma = k'_1 \mathrm{Ad}_{\exp(r(\gamma)x_0)} k'_2 = k'_1 \exp(r(\gamma) \mathrm{ad}_{x_0}) k'_2.$$

Thus

$$\mathbf{a}_1(\mu(\eta(\gamma))) = r(\gamma) \mathbf{a}_1(\mathrm{ad}_{x_0}) \geq (L'|\gamma| - b') \mathbf{a}_1(x_0)$$

which is Anosov because $\mathbf{a}_1(\mathrm{ad}_{x_0}) > 0$, as can be seen by concrete calculations. \square

Before giving an expression for the projective part of the limit map of η , we make a few observations regarding Gromov boundary of Γ . In particular, we claim that since Γ is a uniform lattice os $\mathrm{SU}(2, 1)$, we have that $\partial\Gamma$ is homeomorphic to $\mathrm{SU}(2, 1)/P$, where P is a parabolic subgroup of $\mathrm{SU}(2, 1)$, and it coincides with the stabilizer of some isotropic line $l \in \partial_\infty \mathbb{H}_{\mathbb{C}}^2$.

Indeed, for a uniform lattice Γ of the isometry group G of a homogenous G -space X , the Milnor-Švarc lemma implies that for any $x_0 \in X$, the map $\Gamma \rightarrow X, \gamma \mapsto \gamma x_0$ is a quasi-isometry. In our case $G = \mathrm{SU}(2, 1)$ and $X = \mathbb{H}_{\mathbb{C}}^2$ is a hyperbolic metric space, so it the quasi-isometry extends to a homeomorphism $\partial\Gamma \rightarrow \partial H_{\mathbb{C}}^2$ of the Gromov-boundaries. On the other hand, the action of $\mathrm{SU}(2, 1)$ on $\partial_\infty \mathbb{H}_{\mathbb{C}}^2$ is transitive, so we have that $\partial \mathbb{H}_{\mathbb{C}}^2 \simeq \mathrm{SU}(2, 1)/P$ where P is the stabilizer of a point in $\partial \mathbb{H}_{\mathbb{C}}^2$. In fact, we have that P is a parabolic subgroup of $\mathrm{SU}(2, 1)$. The combination of the above, along with the fact that the geometric and the Gromov boundaries agree in the case of $\mathbb{H}_{\mathbb{C}}^2$, we deduce that $\partial\Gamma \simeq \mathrm{SU}(2, 1)/P$.

We now proceed to calculate the projective part of the limit map of η .

Add proofs of these.

Proposition B.0.2. *The projective part of the limit map of η is given by*

$$\xi_\eta : \partial\Gamma = \mathrm{SU}(2, 1)/P_0 \rightarrow \mathbb{P}(\mathfrak{su}(2, 1)), \quad \xi_\eta(gP_0) = \mathbb{R} \mathrm{Ad}_\gamma x_0.$$

where $P_0 = \mathrm{St}_{\mathrm{SU}(2, 1)}[1 : 0 : 0]$ and

$$x_0 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{su}(2, 1).$$

Proof. Since the limit map of an Anosov representation is equivariant, it suffices to show that there exists a unique η -equivariant map $\xi^1 : \mathrm{SU}(2, 1)/P_0 \rightarrow \mathbb{P}(\mathfrak{su}(2, 1))$.

We consider the parabolic subgroup $P_0 = \mathrm{St}_{\mathrm{SU}(2, 1)}[1 : 0 : 0]$ of $\mathrm{SU}(2, 1)$. Then its Lie algebra is given by:

$$\mathfrak{p}_0 = \mathrm{St}_{\mathfrak{su}(2, 1)}[1 : 0 : 0] = \left\{ \begin{pmatrix} u - is & a & it \\ 0 & 2is & -\bar{a} \\ 0 & 0 & -u - is \end{pmatrix} : a \in \mathbb{C}, u, s, t \in \mathbb{R} \right\}.$$

Since for $\mathbb{R}x \in \mathbb{P}(\mathfrak{su}(2, 1))$ we have that P_0 fixes $\mathbb{R}x$ if and only if \mathfrak{p}_0 fixes $\mathbb{R}x$. But a quick calculation shows that the only element of $\mathfrak{su}(2, 1)$ fixed by \mathfrak{p}_0 is x_0 . \square

Recall that all parabolic subgroups of $\mathrm{SU}(2, 1)$ are conjugate to each other, so we have the following identification:

$$\begin{aligned} \mathrm{SU}(2, 1)/P_0 &\leftrightarrow \{ \text{Parabolic subgroups of } \mathrm{SU}(2, 1) \} &\leftrightarrow \{ \text{Parabolic subalgebras of } \mathfrak{su}(2, 1) \} \\ gP_0 &\leftrightarrow gP_0g^{-1} &\leftrightarrow \mathrm{Ad}_g(\mathfrak{p}_0) \end{aligned}$$

This means that it is equivariant with respect to the Ad-joint action of Γ . However we are only proving uniqueness of an $\mathrm{SU}(2, 1)$ -equivariant map.

Lemma B.0.2. *Let $\mathfrak{p}, \mathfrak{p}' \leq \mathfrak{su}(2, 1)$ be two distinct parabolic subalgebras. Then there exists some $g \in \mathrm{SU}(2, 1)$ such that $\mathrm{Ad}_g(\mathfrak{p}) = \mathfrak{p}_0$ and $\mathrm{Ad}_g \mathfrak{p}' = \mathfrak{p}_0^t$.*

The following proposition implies that the falsehood of the lemma in the beginning of this chapter.

Proposition B.0.3. *Let $\Gamma \leq \mathrm{SU}(2, 1)$ be a uniform lattice and $\eta : \Gamma \rightarrow \mathrm{SL}(\mathfrak{su}(2, 1))$ be the restriction of the adjoint representation. Then*

- (i) η is strongly irreducible,
- (ii) η is projective Anosov
- (iii) η admits a measurable η -equivariant section:

$$\begin{aligned} \zeta : \partial\Gamma &\rightarrow \mathcal{F}_{\{1, d_\Gamma+1\}}(\mathfrak{su}(2, 1)) \simeq \mathcal{P} \\ x &\mapsto (\xi^1(x), T_{\xi^1(x)}\xi^1(\partial\Gamma)) \simeq (\xi^1(x), (d_{\xi^1(x)}p)^{-1}(T_{\xi^1(x)}\xi^1(\partial\Gamma))\xi^1(x)). \end{aligned}$$

- (iv) For all $x, y \in \partial\Gamma : \zeta(x), \zeta(y)$ are not transverse.

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