Limit sets of Anosov representations

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July 6, 2024

Contents

Chapter 1

Introduction

Definition 1.0.1. Let Γ be a discrete group of isometries of a metric space (X, d). We define the critical exponent of Γ to be the asymptotic exponential growth of its orbits, i.e. the following limit:

$$\delta_\Gamma = \limsup_{n\to\infty} \frac{\log \sharp\{\gamma \in \Gamma : d(x, \gamma x) \leq n\}}{n}$$

for some fixed $x \in X$.

Remark 1.0.1. It is not hard to show that the critical exponent is independent of the choice of x.

1.1 Lie group preliminaries

We fix the Cartan subalgebra $\mathfrak{a}$ of $\mathrm{SL}(d, \mathbb{R})$:

$$\mathfrak{a} = \{\mathrm{diag}(\alpha_1, \ldots, \alpha_d) : \alpha_1 + \cdots \alpha_d = 0\}$$

and the Weyl chamber $\mathfrak{a}^+$ of $\mathrm{SL}(d, \mathbb{R})$

$$\mathfrak{a}^+ = \{\mathrm{diag}(\alpha_1, \ldots, \alpha_d) : \alpha_1 \geq \cdots \geq \alpha_d\}.$$

Denoting with $K = \mathrm{SO}(d, \mathbb{R})$, $A^+ = e^{\mathfrak{a}^+}$, we have the Cartan decomposition:

$$\begin{aligned} \mathfrak{sl}(d, \mathbb{R}) &\to K \times A^+ \times K \\ g &\mapsto (k_g, a_g, l_g) \end{aligned}$$

such that $g = k_g a_g l_g$. In particular $a_g = \mathrm{diag}(\sigma_1(g), \ldots, \sigma_d(g))$ with $\sigma_1 \geq \cdots \geq \sigma_d(g)$, where $\sigma_i(g)$ is the i-th singular value of g, i.e. eigenvalue of $g^t \cdot g$.

We will use the spaces

$$U_p(g) = \mathbb{R}u_1(g) \oplus \cdots \oplus \mathbb{R}u_p(g)$$

where $u_i(g) = k_g \cdot e_i$. One can easily show that the decomposition

$$g^{-1} \cdot U_p(g) \oplus U_{d-p}(g^{-1})$$

is orthogonal with respect to the standard inner product and that $u_p(g^{-1}) = l_g^{-1} e_{d-p+1}(g)$.

1.2 Limit set preliminaries

Definition 1.2.1. For $p \in \{2, \dots, d\}, s \in \mathbb{R}$ and $g \in SL(d, \mathbb{R})$ we denote with $\tilde{\Psi}^p_s(g), \Psi^p_s(g) :$ $\mathrm{SL}(d, \mathbb{R}) \to \mathbb{R}$ the functional:

$$\Psi^p_s(g) = \alpha_{12}(a(g)) + \cdots + \alpha_{1(p-1)}(a(g)) + (s - (p-2))\alpha_{1p}(a(g))$$
$$\tilde{\Psi}^p_s(g) = \left(\frac{\sigma_2}{\sigma_1} \cdots \frac{\sigma_{p-1}}{\sigma_1}(g)\right)\left(\frac{\sigma_{p-1}}{\sigma_1}(g)\right)^{s-(p-2)}$$

Remark 1.2.1. We have $\alpha_{ij}(a) = a_i - a_j, a_i(g) = \log(\sigma_i(g))$ and

$$\Psi^p_s(g) = \log \tilde{\Psi}^p_s(g)$$

and that

$$\min_{p \in [\![2,d]\!]} \left\{ \sum_{|\gamma|=T} \frac{\sigma_2}{\sigma_1} \cdots \frac{\sigma_{p-1}}{\sigma_1}(g) \left(\frac{\sigma_{p-1}}{\sigma_1}(g)\right)^{s-(p-2)} \right\} = \sum_{|\gamma|=T} e^{-\max_{p \in [\![2,d]\!]} \Psi^p_s(g)}$$

Remark 1.2.2. For any $g \in \mathrm{SL}(d, \mathbb{R})$ we have that:

$$\max_{p \in [\![2,d]\!]} \Psi^p_s(g) = \Psi^{p_0}_s(g) \text{ for } s \in [p_0 - 2, p_0 - 1].$$

Indeed, a quick calculation shows that for $s \geq 0$ and $p \in [\![2, d]\!]$:

$$\Psi^p_s(g) \leq \Psi^p_s(g) \text{ if and only if } s \geq p - 1.$$

and that equality holds in the case $s = p - 1$. Thus for $s \in [p - 2, p - 1]$ we have that

$$s \geq p - 2, \dots, 1 \text{ implies that } \Psi^p_s(g) \geq \dots \geq \Psi^2_s(g)$$
$$s \leq p, \dots, d - 1 \text{ implies that } \Psi^p_s(g) \leq \dots \leq \Psi^d_s(g)$$

Another way to see this (refer to Figure 1.1) is to note that $\Psi^2_s(g), \cdots, \Psi^d_s(g)$ is a sequence of functions that are affine in s, with slopes $\alpha_{12}(g) \leq \cdots \leq \alpha_{1d}(g)$ and that they satisfy $\Psi^2_1(g) = \Psi^2_2(g), \Psi^3_2(g) = \Psi^4_3(g) \dots, \Psi^{d-1}_{d-2}(g) = \Psi^d_{d-2}(g)$.

The following definition comes from [LL23], in the special case of projective Anosov representations ($P = 1$):

Definition 1.2.2. For $s \geq 0$ we consider the Falconer functional $F_s : \mathrm{SL}(d, \mathbb{R}) \to \mathbb{R}$ by:

$$F_s(g) = \min \left\{ \sum_{j=2}^{d} s_j \alpha_{1j}(a(g)) : s_j \in (0, 1], \sum_{j=2}^{d} s_j = s \right\},$$

and define the Falconer dimension $\dim_F(\rho)$ of ρ to be its critical exponent:

$$\dim_F(\rho) = \inf \left\{ s > 0 : \sum_{\gamma \in \Gamma} e^{-F_s(\rho(\gamma))} < \infty \right\}.$$

Remark 1.2.3. Using elementary computations one may prove that for all $s \geq 0$:

$$F_s(g) = \max_{p \in [\![2,d]\!]} \Psi^p_s(g)$$

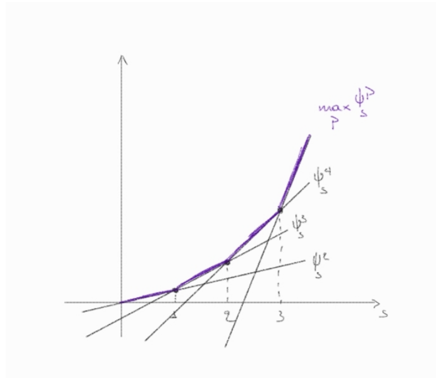


Figure 1.1: Visual illustration that $\max_p \Psi_s^p(g) = \Psi_s^{p_0}(g)$ for $s \in [p_0 - 2, p_0 - 1]$.

Definition 1.2.3. Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a linear representation and $p \in [\![1, d-1]\!]$. We say that ρ is p-Anosov if there exist constants $\mu, C > 0$ such that for all $\gamma \in \Gamma$:

$$\frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma)) \leq C e^{-\mu|\gamma|}.$$

One can show that in that case there exist equivariant continuous maps $\xi^p : \hat{\Gamma} \to \mathcal{G}_p(\mathbb{R}^d), \xi^{d-p} : \hat{\Gamma} \to \mathcal{G}_{d-p}(\mathbb{R}^d)$ that are transverse and restrict to

$$\xi^p(\gamma) = U_p(\rho(\gamma)), \xi^{d-p}(\gamma) = U_{d-p}(\rho(\gamma))$$

for $\gamma \in \Gamma$, where $U_p(\gamma), U_{d-p}(\gamma)$ denote the flags corresponding to $\rho(\gamma)$. Figure out what this exactly means

Chapter 2

Upper bound

2.1 Proof of bound

Lemma 2.1.1 (Upper bound for dimension)**.** *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. Then:*

$$\dim_H(\xi^1(\partial\Gamma)) \leq \dim_F(\rho).$$

Remark 2.1.1. The idea of the proof of Lemma 2.1.1 is to find a covering whose Hausdorff content is dominated by the Dirichlet series of some functional Ψ_s^p, which will in turn imply that $\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(\Psi^p)$. Choosing then the most "effective" cover (i.e. the one which yields the smallest Hausdorff content up to a constant) we obtain that

$$\dim_H(\xi^1(\partial\Gamma)) \leq h_\rho(\max_p \Psi^p)$$

To obtain this we first cover $\xi^1(\partial\Gamma)$ by the bassins of attraction $\rho(\gamma) \cdot B_{\alpha_1,\alpha}(\rho(\gamma))$ for $\gamma \in \Gamma$ satisfying $|\gamma| = T$. Then we cover each bassin by an ellipsoid of axes lengths

$$\frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \ldots, \frac{1}{\sin(\alpha)} \frac{\sigma_d}{\sigma_1}(\rho(\gamma)).$$

Finally we cover each ellipsoid by balls of some fixed radius $r > 0$. It can be shown by comparing the series appearing in the Hausdorff content of each resulting cover that the most "effective" choice of r depends only on the Hausdorff exponent $s > 0$ and in any case will be to have r equal (up to a constant) to the the length of an axis of the ellipsoid, i.e.

$$r \in \left\{ \frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \ldots, \frac{1}{\sin(\alpha)} \frac{\sigma_d}{\sigma_1}(\rho(\gamma)). \right\}$$

In particular, when $s \in [p-2, p-1]$, the most effective choice is $r = \sigma_p(\rho(\gamma))/\sigma_1(\rho(\gamma))$, whose Hausdorff content is dominated by the Dirichlet series of Ψ_s^p.

Proof of Lemma 2.1.1. Let $p \in [\![2, d]\!]$. Then using Proposition 2.2.1, Lemma 2.2.2, and Lemma 2.2.3 we have that for $T > 0$ large enough, $\xi^1(\partial\Gamma)$ is covered by the family

$$\mathcal{U}_T = \{\rho(\gamma) B_{\alpha_1,\alpha}(\rho(\gamma)) : |\gamma| = T\},$$

and that each basin $\rho(\gamma) B_{\alpha_1,\alpha}(\rho(\gamma))$ is in turn covered by

$$2^{p-2} \cdot \frac{\sigma_p(g)^{p-2}}{\sigma_2(g) \cdots \sigma_{p-1}(g)}$$

many balls of radius

$$\sqrt{d-1}\frac{1}{\sin\alpha}\frac{\sigma_p(g)}{\sigma_1(g)}.$$

By the definition of the Hausdorff measure, for $s \geq 0$:

$$\begin{aligned}
\mathcal{H}^s(\xi^1(\partial\Gamma)) &\leq \sum_{|\gamma|=T} 2^{2p+1}\cdot \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\cdots\frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\left(\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\right)^{-(p-2)}\left(\sqrt{d-1}\frac{1}{\sin\alpha}\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\right)^s = \\
&= 2^{2p+1}\cdot\left(\frac{\sqrt{d-1}}{\sin\alpha}\right)^s\sum_{|\gamma|=T}\frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\cdots\frac{\sigma_{p-1}(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\left(\frac{\sigma_p(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\right)^{s-(p-2)} = \\
&= 2^{2p+1}\cdot\left(\frac{\sqrt{d-1}}{\sin\alpha}\right)^s\sum_{|\gamma|=T} e^{-\left(\alpha_{12}+\ldots+\alpha_{1(p-1)}+(s-(p-2))\alpha_{1p}\right)\rho(\gamma)} \\
&= 2^{2p+1}\cdot\left(\frac{\sqrt{d-1}}{\sin\alpha}\right)^s\sum_{|\gamma|=T} e^{-\Psi_s^p(\rho(\gamma))}
\end{aligned}$$

and thus

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq 2^{2p+1}\cdot\left(\frac{\sqrt{d-1}}{\sin\alpha}\right)^s\sum_{|\gamma|=T} e^{-\max_p \Psi_s^p(\rho(\gamma))} \lesssim \sum_{|\gamma|=T} e^{-F_s(\rho(\gamma))}.$$

To see that the above implies the upper bound, consider some $s > \dim_F(\rho)$. By the definition of the Falconer dimension, this implies that the Dirichlet series corresponding to the Falconer functional converges:

$$\sum_{\gamma\in\Gamma} e^{-F_s(\rho(\gamma))} < \infty$$

and in particular

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq \lim_{T\to\infty} e^{-F_s(\rho(\gamma))} = 0.$$

□

2.2 Lemmata

Definition 2.2.1. Let V be a finite-dimensional $\mathbb{R}$-vector space. We consider a decomposition

$$V = \mathbb{R}u_1 \bigoplus \cdots \bigoplus \mathbb{R}u_d$$

be a direct decomposition that is orthogonal with respect to a fixed inner-product over V. Given $\beta_2 \geq \ldots \beta_d > 0$, we define an ellipsoid with axes $u_1 \oplus u_p(g)$ and lengths β_p to be the image of

$$\left\{ v = \sum_1^d v_i u_i \in V : \sum_2^d \left(\frac{v_j}{\beta_j}\right)^2 \leq 1 \right\}$$

through the projection $V \to \mathbb{P}(V)$.

The following aims to be something along the lines of [PSW23, Lemma 2.4]:

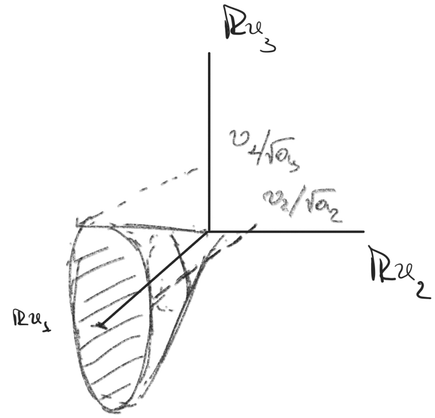


Figure 2.1: Depiction in $\mathbb{R}^3$ of an ellipsoid of $\mathbb{P}(\mathbb{R}^2)$

Lemma 2.2.1. *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. For $\alpha > 0$ small enough, there exists $L > 0$ such that for any geodesic ray $(a_j)_j$ through e we have:*

$$\angle(U_1(\rho(a_i)), U_{d-1}(\rho(a_0))) > \alpha$$

when $|a_i|, |a_0| > T$.

Proof. Assume the contrary for the shake of contradiction. Then (see Figure 2.2) for each $n > 0$ there exists a geodesic ray a^n through e such that

$$|a_n^n|, |a_0^n| > n \text{ and } \angle(U_1(\rho(a_n^n)), U_1(\rho(a_0^n))) < \frac{1}{n}.$$

Due to compactness of $\Gamma \cup \partial\Gamma$ we can find some subsequence k_n and $x, y \in \partial\Gamma$ such that $a_{k_n}^{k_n} \to x$, $a_0^{-k_n} \to y$ and $x \neq y$. Since the limit map is dynamics preserving, we have that

$$\angle(\xi^1(x), \xi^{d-1}(y)) = 0,$$

which contradicts its transversality property. □

The following is [PSW23, Proposition 3.5].

Lemma 2.2.2. *Let $\rho : \Gamma \to SL(d, \mathbb{R})$ be projective Anosov. Then for $\alpha > 0$ small enough, there exists some $T_0 > 0$ such that for all $T \geq T_0$ the family*

$$\mathcal{U}_T = \{\rho(\gamma)B_{\alpha_1,\alpha}(\rho(\gamma)) : |\gamma| = T\}$$

is an open covering of $\xi^1(\partial\Gamma)$.

Proof. Let $\alpha, T > 0$ be as in the statement of Lemma 2.2.1 and $x \in \partial\Gamma$ be represented by a geodesic ray $(\gamma_j)_{j\geq 0}$ starting from e. Then $(\gamma_T^{-1}\gamma_j)_j$ is a geodesic ray starting from $(\gamma_T)^{-1}$ that passes through e, so

$$\angle(U_1(\rho(\gamma_T^{-1}\gamma_j)), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

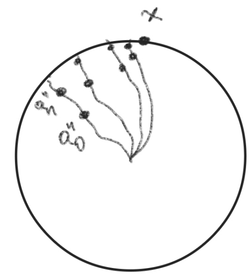


Figure 2.2: Situation in Lemma 2.2.1

as implied by Lemma 2.2.1. Taking the limit $j \to \infty$ and using the equivariance of the limit map, we obtain

$$\angle(\rho(\gamma_T^{-1})\xi^1(x), U_{d-1}(\rho(\gamma_T^{-1}))) > \alpha$$

and thus $\xi^1(x) \in \rho(\gamma_T) \cdot B_{\alpha_1,\alpha}(\rho(\gamma_T))$. $\square$

The following is [PSW23, Proposition 3.8].

Proposition 2.2.1. *For each $g \in \mathrm{SL}(d,\mathbb{R}), \alpha > 0$, the basin of attraction $g \cdot B_{\alpha_1,\alpha}(g)$ lies in the ellipsoid with axes $u_1(g) \oplus u_p(g)$ with lengths*

$$\frac{1}{\sin\alpha} \cdot \frac{\sigma_p(g)}{\sigma_1(g)}$$

Proof. Using the definition of the basin of attraction (see Figure 2.3), we have that $w = w_1 u_1(g^{-1}) + \cdots + w_d u_d(g^{-1}) \in B_{\alpha_1,\alpha}(g)$ if and only if

$$w_d^2 \geq (\sin\alpha)^2 \sum_1^d w_i^2.$$

Considering now some $v = v_1 u_1(g) + \cdots + v_d u_d(g) \in g \cdot B_{\alpha_1,\alpha}(g)$ we have that

$$\begin{aligned} w = g^{-1}v &= v_1\sigma_1(g)^{-1} l_g^{-1} e_1(g) + \cdots v_d \sigma_d(g)^{-1} l_g^{-1} e_d(g) \\ &= v_1 \sigma_1(g)^{-1} u_d(g^{-1}) + \cdots v_d \sigma_d(g)^{-1} u_1(g^{-1}) \end{aligned}$$

where we used that $u_p(g^{-1}) = l_g^{-1} e_{d+1-p}$. Hence

$$\sigma_1(g)^{-2} \cdot v_1^2 \geq (\sin a)^2 \sum_1^d \sigma_i(g)^{-2} v_i^2.$$

$\square$

The following is [PSW23, Lemma 3.7]:

Lemma 2.2.3. *For any $p \in [\![2, d]\!]$, an ellipsoid in $\mathbb{P}(\mathbb{R}^d)$ of axes lengths $\beta_2, \cdots, \beta_d$ is covered by*

$$2^{p-2} \frac{\beta_2 \cdots \beta_{p-1}}{\beta_p^{p-2}}$$

many (projected) balls of radius $\sqrt{d-1}\beta_p$.

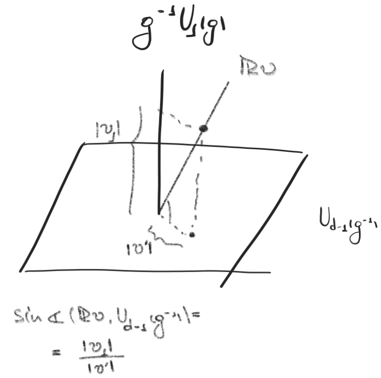


Figure 2.3: Aid for Proposition 2.2.1

Proof. We assume that E is an ellipsoid about $\mathbb{R}e_1$, so it suffice to cover its intersection $E_1 = E \cap U_1 \subseteq \mathbb{R}^{d-1}$ with the affine chart $U_1 = \{[x_1 : \ldots, x_d] \in \mathbb{P}(\mathbb{R}^d) : x_1 \neq 0\}$. Clearly $E_1 \subseteq [-\beta_2, \beta_2] \times \ldots \times [-\beta_d, \beta_d]$, so we proceed by covering the rectangle with side-lengths $2\beta_2, \ldots, 2\beta_d$. Clearly each interval $(-\beta_j, \beta_j)$ is contained in the union of $\lceil \beta_j / \beta_p \rceil$ intervals of length $2\beta_p$, thus E_1 is contained in the union of

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil \cdots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil = \left\lceil \frac{\beta_2}{\beta_p} \right\rceil \cdots \left\lceil \frac{\beta_d}{\beta_p} \right\rceil$$

many squares of side-length $2\beta_p$. Since each such product is contained in a $(d-1)$-ball of radius $\sqrt{d-1}\beta_p$ we may use at most

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil \cdots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil \leq \sum_{i \in \{0,1\}^{p-2}} \prod_{j=2}^{p-1} \left(\frac{\beta_j}{\beta_p} \right)^{i_j} \leq 2^{p-2} \frac{\beta_2}{\beta_p} \cdots \frac{\beta_{p-1}}{\beta_p}$$

many $(d-1)$-balls of radius $\sqrt{d-1}\beta_p$ to cover E_1. $\square$

The following can be found in [PSW23, Proposition 3.3]:

Proposition 2.2.2. *Let $\rho : \Gamma \to SL(d, \mathbb{R})$ be projective Anosov and $\alpha > 0$ Then there exist $c_0, c_1 > 0$ that depends only on α and ρ such that for all $\gamma \in \Gamma$:*

$$(\xi^1)^{-1}(B_{\alpha_1,\alpha}(\rho(\gamma))) \subseteq C^{\infty}_{c_0,c_1}(\gamma)$$

Proof. We begin by noting that it suffices to show this for all but finitely many $\gamma \in \Gamma$, since then we may alter the constants to satisfy the wanted inclusion for the finitely many remaining $\gamma \in \Gamma$ as well. Hence, we may assume that $|\gamma| \geq l_0$ where $l_0 > 0$ is such that

$$Ce^{-\mu l_0} < 1 \text{ and } \mathsf{a}_1(\gamma) \geq C|\gamma| - L.$$

Suppose $x \in \partial\Gamma$ such that $\xi^1(x) \in B_{\alpha_1,\alpha}(\rho(\gamma))$, and consider a geodesic ray $a_j \to x$ starting from $a_0 = e$. To prove the result, it suffices to find constants c_0, c_1 independent of γ and for which there exists a (c_0, c_1)-quasi-geodesic from γ^{-1} to x that passes through e and stays at a bounded distance from $(a_j)_{j=0}^{\infty}$.

Using the exponential convergence rate of $\xi^1(a_j) \to \xi^1(x)$ and the definition of $B_{\alpha_1,\alpha}(\rho(\gamma))$ we have that:

$$d(\xi^1(a_j), \xi^1(\gamma)) \geq d(\xi^1(x), U_1(\rho(\gamma^{-1})) - d(\xi^1(a_j), \xi^1(x))) \geq \\ \geq d(\xi^1(x), U_{d-1}(\rho(\gamma^{-1})) - d(\xi^1(a_j), \xi^1(x))) \geq \sin\alpha - Ce^{-\mu j}$$

which along with the uniform continuity of $\xi^1 : \Gamma \cup \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ this implies there exists some $\alpha' > 0$ and $L > 0$ such that for all $j \geq L$:

$$d(a_j, \gamma^{-1}) \geq \alpha'.$$

Upon considering a large L, we may also assume that $|a_L| = L > l_0$. Note that both α' and L do not depend on each γ but only on ρ and α.

Using a coarse geometric arguement, we can show that for all $j \geq L$

$$d(\gamma^{-1}, a_j) > \alpha' \Rightarrow d([\gamma^{-1}, a_j], e) < \alpha''$$

for some α'' that depends only on Γ and α', where $[a_j, \gamma^{-1}]$ denotes any geodesic segment connecting γ^{-1} and a_j. Indeed, [GH13, Lemme 2.17] states that $d([\gamma^{-1}, a_j]) \leq (\gamma_j^{-1}, a_j)_e + \delta$ where δ is the hyperbolicity constant of Γ. Thus

$$d([\gamma^{-1}, a_j]) \leq \delta + \frac{d(a_j, e) + d(\gamma^{-1}, e) + d(a_j, \gamma^{-1})}{2} \leq \delta + \frac{L + d(\gamma^{-1}, e) + \alpha'}{2}.$$

Consider the concatenation $(a'_j)_{j=L-K}^{\infty}$ of $[\gamma^{-1}, a_L]$ and $[a_L, x]$. To find quasi-geodesic-constants that are uniform in γ, we note that for any $c_0 \geq 1, c_1 \geq 0$:

$$c_0^{-1}|i-j| - c_1 \leq d(a'_i, a'_j) = d(a_i, a_j) \leq c_0|i-j| + c_1 \text{ when } i, j \geq L \text{ or } i, j \leq L$$

and that the upper bound follows trivially by the triangle inequality.

For the lower bound we proceed in two steps. First we bound the distance of $\gamma^{-1} = a'_{L-K}$ to a_{L+j} for $j \geq 0$:

$$d(a'_{L-K}, a'_{L+j}) \geq \nu(|a_{L+j}| - |\gamma^{-1}|) - c'_0 - c'_1|\log(d(U_1(\rho(a_{L+j})), U_1(\rho(\gamma^{-1}))))| \geq \\ \geq \nu((L+j) + (K-L)) - c'_0 - c'_1|\log(\sin a)| \geq \\ = c_0^{-1}(j+K) - c_1$$

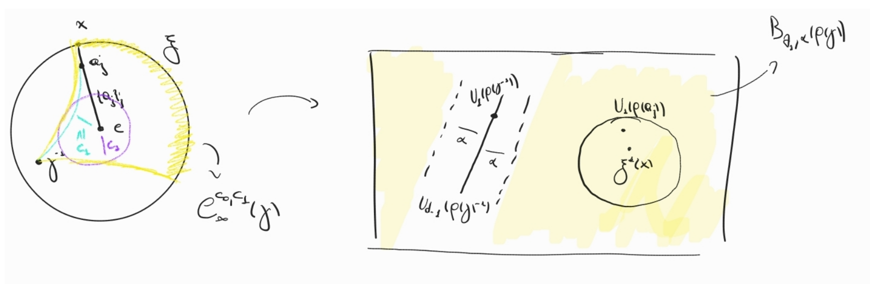
for $c_0 = \nu^{-1}, c_1 = c'_0 + c'_1|\log(\sin\alpha)|$. The first inequality comes from [PSW23, Lemma 3.9]. For the second inequality we estimate $|\gamma^{-1}|$ from below using the triangle inequality. We are now ready to show that the concatenation $(a'_j)_j$ is indeed a (c_0, c_1)-geodesic:

$$d(a_{L+j}, a_{L-i}) \geq d(a_{L+j}, a_{L_K}) - d(a_{L_K}, a_{L-i}) \geq c_0^{-1}(j+K) - c_1 - (K-i) \geq \\ \geq c_0^{-1}(j+i) - c_1.$$

Note however that (a'_j) does not necessarily lie in $C_\infty^{c_0,c_1}$ since it may not pass through e. For this reason we some $L - K \leq i_0 \leq L$ such that $|a_{i_0}| < \alpha''$, the existence of which is guaranteed by the fact that $d([\gamma^{-1}, a_L], \epsilon) < \alpha''$. We then consider alter (a'_j) at i_0 so that it passes through e to obtain

$$a''_j = \begin{cases} a_j & \text{for } j \neq i_0 \\ e & \text{for } j = i_0 \end{cases}$$

which is a $(c_0, c_1 + \alpha'')$-quasigeodesic passing from e and converging to x. □



Chapter 3

Lower bound

3.1 Busemann cocyle and Patterson-Sullivan measures

We denote with Π the set of simple positive roots, and for $\Theta \subseteq \Pi$ we consider the Levi-Anosov subspace of $\mathfrak{a}$

$$\mathfrak{a}_\Theta = \bigcap_{\alpha \notin \Theta} \ker \alpha,$$

which in particular admits $\{\omega_{\alpha_i} : \alpha_i \in \Theta\}$ as a basis.

Definition 3.1.1. Let $\Theta \subseteq \Pi$. We define the Busemann cocycle

$$b : \mathrm{PSL}(d, \mathbb{R}) \times \mathcal{F}_\Theta \to \mathfrak{a}_\Theta$$

as the unique element $b(g, kP_\Theta) \in \mathfrak{a}_\Theta$ such that

$$gk \in Ke^{b(g,kP_\Theta)}N.$$

where $N = \{n \in \mathrm{SL}(d, \mathbb{R}) : n_{ij} = 0 \text{ for } i > j, n_{ii} = 1 \text{ for all } i\}$ is the unipotent group of upper subgroup of upper triangular matrices with 1s on the diagonal, and P_Θ is the parabolic subgroup of $\mathrm{PSL}(d, \mathbb{R})$ corresponding to Θ, i.e. $\mathcal{F}_\Theta = \mathrm{PSL}(d, \mathbb{R})/P_\Theta$.

Lemma 3.1.1. *For $y \in \mathcal{F}_\Theta(\mathbb{R}^d)$ we have*

$$\omega_{\alpha_i}(b_\Theta(g, x)) = \log \frac{\|gv_1 \wedge \cdots gv_i\|}{\|v_1 \wedge \cdots v_i\|} \text{ for all } \alpha_i \in \Theta$$

for any basis $v_1, \ldots, v_i$ of $x^i \in \mathcal{G}_i(\mathbb{R}^d)$, where $\|\cdot\|$ denotes the norm on $\bigwedge^i \mathbb{R}^d$ induced by the euclidean inner product on $\mathbb{R}^d$, i.e. $\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k\rangle := \det(\langle v_i, w_j\rangle)$.

Definition 3.1.2. We define

$$\Lambda^k : \mathrm{SL}(d, \mathbb{R}) \to \mathrm{SL}(\Lambda^k \mathbb{R}^d), \quad \Lambda^k : \mathcal{G}_k(\mathbb{R}^d) \to \mathbb{P}(\Lambda^k \mathbb{R}^d)$$

as

$$\Lambda^k(g)(v_1 \wedge \cdots \wedge v_k) = gv_1 \wedge \cdots \wedge gv_k \Lambda^k(\mathbb{R}v_1 \oplus \cdots \mathbb{R}v_k) = v_1 \wedge \cdots \wedge v_k$$

Lemma 3.1.2. *Let $g \in \mathrm{SL}(d, \mathbb{R}), \alpha > 0$ and $k \in [\![2, d]\!]$.*

(i) $\omega_1(a(\Lambda^k g)) = \omega_k(a(g))$ *and* $\omega_1(b(\Lambda^k g, \Lambda^k y)) = \omega_k(b(g,y))$.

(ii) There exists some $\alpha' > 0$ *independent of* g *such that* $\Lambda^k B_{\mathsf{a}_k,\alpha}(\Lambda^k g) \subseteq B_{\mathsf{a}_1,\alpha'}(\Lambda^k g)$.

Proof. (i) Follows from the definitions of the fundamental weights and the Cartan projection.

(ii) Let $g = k_g e^{a(g)} l_g$ be the Cartan decomposition of g. Then using the definitions of the respective subspaces:

$$U_{d-k}(g^{-1}l_g^{-1}) = \mathbb{R}e_{k+1} \oplus \cdots \oplus \mathbb{R}e_d$$
$$x_0 := U_{d-1}(\Lambda^k g^{-1} l_g^{-1}) = \bigoplus_{\substack{i_1<\cdots<i_k \\ (i_1,\cdots,i_k)\neq(1,\cdots,k)}} \mathbb{R}e_{i_1} \oplus \cdots \oplus \mathbb{R}e_{i_k}$$

The first equality implies that

$$y \in B_{\mathsf{a}_k,\alpha}(g) \Leftrightarrow l_g y \in B_{\mathsf{a}_k,\alpha}(g l_g^{-1}) = B_{\mathsf{a}_k,\alpha}(\mathrm{Id}),$$

so for every $y \in B_{\mathsf{a}_k,\alpha}(g)$ we have that

$$l_g y = l U_k(\mathrm{Id}) \text{ for some } l \in L$$

where

$$L = \{ l \in \mathrm{SO}(d,\mathbb{R}) : l U_k(\mathrm{Id}) \in B_{\mathsf{a}_k,\alpha}(\mathrm{Id}) \}.$$

Note that L is compact, being a closed subset of a compact group. Moreover, the fact that $\Lambda^k(y) \notin \mathrm{U}_{d-1}(\Lambda^k g^{-1})$ implies that

$$0 < \angle(\Lambda^k(y), \mathrm{U}_{d-1}(\Lambda^k g^{-1})) = \angle(\Lambda^k(l_g y), \mathrm{U}_{d-1}(\Lambda^k g^{-1} l_g^{-1})) = \angle(\Lambda^k(l)\,\mathrm{U}_k(\mathrm{Id}), x_0)$$

The right-hand side is in the image of the compact set L under a continuous map, so it is bounded below by a positive number $\alpha' > 0$.

(iii) Follows from the definition of the Cartan projection and the Busemann cocycle.

□

Definition 3.1.3. For a discrete subgroup $\Gamma < \mathrm{PSL}(d,\mathbb{R}), \phi \in (\alpha_\Theta)^*$, a (Γ,ϕ)-Patterson Sullivan measure on $\mathcal{F}_\Theta$ is a finite Radon measure μ such that for every $\gamma \in \Gamma$

$$\frac{\mathrm{d}\gamma_*\mu}{\mathrm{d}\mu}(x) = e^{-\phi(b_\Theta(g^{-1},x))}, \text{ for all } x \in \mathcal{F}_\Theta(\mathbb{R}^d).$$

Lemma 3.1.3. *Let* $\alpha > 0, \Theta \subseteq \Pi$. *There exists* $K = K(\alpha) > 0$ *such that for each* $g \in \mathrm{SL}(d,\mathbb{R}), \mathsf{a_i} \in \Theta, \mathsf{y} \in \mathsf{B}_{\Theta,\alpha}(\mathsf{g}), \phi \in \mathfrak{a}_\Theta$

$$|\phi(a(g) - b(g,y))| \leq K.$$

Proof. We begin by noting that it suffices to consider the case where $\phi = \mathsf{a}_k$ for $\mathsf{a_k} \in \Theta$, since $\{\omega_i\}_{\mathsf{a_i}\in\Theta}$ is a basis for $\mathfrak{a}_\Theta^*$.

We first consider the case where $k = 1$. We recall that the first component of the Cartan projection coincides with the spectral norm of g, i.e.

$$a_1(g) = \log \sup_{v\neq 0} \frac{\|gv\|}{\|v\|} = \log \|g k_2^{-1} e_1\|$$

where $g = k_1 e^{a(g)} k_2$ is the Cartan decomposition of g. Let $v = v_1 k_2^{-1} e_1 + \cdots + v_d k_2^{-1} e_d \in \mathbb{R}^d$ be such that $\|v\| = 1$ and $y = \mathbb{R}v$, we have

$$\begin{aligned}
|\omega_1(a(g) - b(g,y))| &= |\log \|g k_2^{-1} e_1\| - \log \|gv\|| = \\
&= |\log |e^{a_1(g)}| - \log \|e^{a_1(g)} v_1 k_1 e_1 + \cdots + e^{a_d(g)} v_d k_1 e_d\|| = \\
&= \left|\log \left\| v_1 k_1 e_1 + e^{-\mathsf{a}_{12}(g)} v_2 k_1 e_2 + \cdots + e^{-\mathsf{a}_{1d}(g)} v_d k_1 e_d \right\|\right| \leq \\
&\leq |\log |v_1|| = \left|\log \sin(\angle(v, U_{d-1}(g^{-1})))\right| \leq |\log \sin \alpha| \,.
\end{aligned}$$

For the case where $\Theta = \{\mathsf{a}_k\}$, we consider $\alpha' > 0$ such that

$$\Lambda^k(B_{\mathsf{a}_k,\alpha}(g)) \subseteq B_{\mathsf{a}_k,\alpha'}(\Lambda^k g)$$

Then using the case $k = 1$ we have that

$$|\omega_k(a(g) - b(g,y))| = |\omega_1(a(\Lambda^k g) - b(\Lambda^k g, \Lambda^k y))| \leq |\log \sin \alpha'| \,.$$

$\square$

3.2 Proof strategy

Denoting with $d_\Gamma = \dim_H \xi^1_\rho(\partial\Gamma)$ the Hausdorff dimension of the limit set, we shall outline how to obtain its lower bound by the Falconer dimension

$$d_\Gamma \geq h_\rho(F).$$

First we recall that $F_s(a) = \max\{\Psi^p_s(a) : p \in [\![2, d]\!]\}$ and in particular $h_\rho(F) \leq h_\rho(\Psi^{d_\Gamma+1})$. Thus the lower bound will follow once we have shown that

$$d_\Gamma \geq h_\rho(\Psi^{d_\Gamma+1}).$$

Noting that $\frac{s}{d_\Gamma} J^u_{d_\Gamma} \leq \Psi^{d_\Gamma+1}_{s+d_\Gamma}$, the above bound will follow as soon as we have shown that

$$h_\rho(J_{d_\Gamma}) \leq 1. \tag{LB}$$

To obtain inequality (LB), one uses the method of Patterson-Sullivan-Quint, which may be summed up in

$$\text{There exists a } (\phi,\rho)\text{-Patterson-Sullivan measure on } \mathcal{F}_\Theta(\mathbb{R}^d) \Rightarrow h_\rho(\phi) \leq 1,$$

where $\phi \in \mathfrak{a}_\Theta$ and $\Theta \subseteq \Pi$. The property that we will need of our measure is that there exists a collection of open sets $U_{\gamma_\gamma} \in \Gamma$ such that

$$\mu(U_\gamma) \sim e^{-J^u_{d_\Gamma}(a(\rho(\gamma)))} \text{ and } M := \sup_{n\in\mathbb{N}} \max \left\{ \sharp A : A \subseteq \Gamma_n, \bigcap_{\gamma\in A} U_\gamma \neq \emptyset \right\} < \infty \tag{MP}$$

where $\Gamma_n = \{\gamma \in \Gamma : |\gamma| = n\}$. For the proof of the existence of a $(J^u_{d_\Gamma}, \rho)$-Patterson-Sullivan measure that satisfies the property (MP) we refer to Section 3.3, noting that the Zariski-density assumption is necessary only for the equivalence appearing on the left hand side of

Equation (MP). Assuming that for the time being, below we outline the Patterson-Sullivan-Quint method of obtaining Equation (LB).

Indeed, we first obtain the uniform in n bound:

$$\sum_{\gamma\in\Gamma_n} e^{-J^u_{d_\Gamma}(a(\rho(\gamma)))} \lesssim \sum_{\gamma\in\Gamma_n} \mu(U_\gamma) \leq \frac{1}{M}\mu(\mathcal{F}_\Theta(\mathbb{R}^d)) < \infty$$

along with the bound implied by the Anosov property of ρ:

$$J_{d_\Gamma}(a(\rho(\gamma))) \geq \mathsf{a}_{12}(a(\rho(\gamma))) \geq C|\gamma| - b$$

to conclude that

$$\sum_{\gamma\in\Gamma} e^{-(s+1)J^u_{d_\Gamma}(a(\rho(\gamma)))} = \sum_{n\geq 0}\sum_{\gamma\in\Gamma_n} e^{-sJ^u_{d_\Gamma}(a(\rho(\gamma)))}e^{-J^u_{d_\Gamma}(a(\rho(\gamma)))} \lesssim \sum_{n\geq 0} e^{-s(Cn-b)} < \infty$$

which holds for any $s > 0$, and thus Equation (LB) holds.

3.3 Existence of Patterson-Sullivan measure

Definition 3.3.1. For $p \in [\![2, d]\!]$, we denote the p-th unstable Jacobian $J^u_p \in \mathfrak{a}^*$ by

$$J^u_p = (p+1)\omega_{\mathsf{a}_1} - \omega_{\mathsf{a}_{p+1}} = \mathsf{a}_{12} + \cdots + \mathsf{a}_{1(p+1)}.$$

Definition 3.3.2. Let $V \in \mathcal{G}_{p+1}\mathbb{R}^d$ and $l \in \mathbb{P}(V)$. Using the canonical identification $T_l\mathbb{P}(V) \simeq \hom(l, V/l)$, we define the density $|\Omega_{l,V}|$ on $\bigwedge^p T_l\mathbb{P}(V)$ by

$$|\Omega_{l,V}|(\phi_1, \ldots, \phi_p) = \frac{\|v \wedge \tilde{\phi}_1(v) \wedge \cdots \wedge \tilde{\phi}_p(v)\|}{\|v\|^{p+1}}$$

for any $v \in l - \{0\}$, where $\tilde{\phi}_1, \ldots \tilde{\phi}_p \in \hom(l, V)$ are such that $\phi_i = \tilde{\phi}_i + \hom(l, l)$ and $\|\cdot\|$ denotes the norm on $\bigwedge^{p+1}\mathbb{R}^d$ induced by the euclidean inner product.

The following is [PSW23, Proposition 6.4]:

Proposition 3.3.1. *Assume that $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold of dimension d_Γ. Then there exists a $(\rho(\Gamma), J^u_{d_\Gamma})$-Patterson-Sullivan measure on $\mathcal{F}_{1,d_\Gamma+1}$.*

Proof. By Rademacher's theorem, $\xi^1_\rho(\partial\Gamma)$ has a well-defined Lebesgue measure class, and Lebesgue-almost every $\xi^1_\rho(x) \in \xi^1_\rho(\partial\Gamma)$ admits a well-defined tangent space $T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma)$. Considering such a $\xi^1_\rho(x)$ we let

$$\pi : \hom(\xi^1_\rho(x), \mathbb{R}^d) \to \hom(\xi^1_\rho(x), \mathbb{R}^d/\xi^1_\rho(x)) \simeq T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma),$$

and

$$x^{d_\Gamma+1} = \pi^{-1}(T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma))\xi^1_\rho(x) \in \mathcal{G}_{d_\Gamma+1}(\mathbb{R}^d),$$

for which one can show that

$$T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma) \simeq \hom(\xi^1_\rho(x), \mathbb{R}^d/\xi^1_\rho(x)) \simeq \hom(\xi^1_\rho(x), x^{d_\Gamma+1}/\xi^1_\rho(x)).$$

In this notation, we shall define (Lebesgue-almost eeverywhere) the map

$$\zeta_\rho : \xi^1_\rho(\partial\Gamma) \to \mathcal{F}_{1,d_\Gamma+1}(R^d), \quad \zeta_\rho(\xi^1_\rho(x)) = (\xi^1_\rho(x), x^{d_\Gamma+1}).$$

We now define the non-negative density on $\xi^1_\rho(\partial\Gamma)$

$$\mu_{\xi^1_\rho(x)} = |\Omega_{\zeta_\rho(\xi^1_\rho(x))}|$$

which satisfies

$$\frac{\mathrm{d}\gamma_*\mu}{\mathrm{d}\mu}(\xi) = \frac{\mathrm{d}(\rho(\gamma)^{-1})^*\mu}{\mathrm{d}\mu}(\xi) = e^{-J^u_{d_\Gamma+1}(b_\Theta(\rho(\gamma)^{-1},\zeta(x))))},$$

where the term on the left-hand side involves measures, the term between the two equalities involves densities, and $\Theta = \{1, d_\Gamma + 1\}$. Indeed, for $\phi_1, \dots, \phi_{d_\Gamma} \in T_{\xi^1_\rho(x)}\xi^1_\rho(\partial\Gamma)$:

$$\begin{aligned}
&(\rho(\gamma)^*\mu)_{\xi^1_\rho(x)}(\phi_1, \dots, \phi_{d_\Gamma}) \\
&= \mu_{\rho(\gamma)\xi^1_\rho(x)}(\rho(\gamma)\phi_1\rho(\gamma)^{-1}, \dots, \rho(\gamma)\phi_{d_\Gamma}\rho(\gamma)^{-1}) \\
&= \frac{\|\rho(\gamma)\xi^1_\rho(x) \wedge \rho(\gamma)\phi_1(\xi^1_\rho(x)) \wedge \cdots \wedge \rho(\gamma)\phi_{d_\Gamma}(\xi^1_\rho(x))\|}{\|\rho(\gamma)\xi^1_\rho(x)\|^{d_\Gamma+1}} \\
&= \frac{\|\rho(\gamma)\xi^1_\rho(x) \wedge \rho(\gamma)\phi_1(\xi^1_\rho(x)) \wedge \cdots \wedge \rho(\gamma)\phi_{d_\Gamma}(\xi^1_\rho(x))\|}{\|\xi^1_\rho(x) \wedge \phi_1(\xi^1_\rho(x)) \wedge \cdots \wedge \phi_{d_\Gamma}(\xi^1_\rho(x))\|} \\
&\quad \cdot \frac{\|\xi^1_\rho(x) \wedge \phi_1(\xi^1_\rho(x)) \wedge \cdots \wedge \phi_{d_\Gamma}(\xi^1_\rho(x))\|}{\|\rho(\gamma)\xi^1_\rho(x)\|^{d_\Gamma+1}} \cdot \frac{\|\rho(\gamma)\xi^1_\rho(x)\|^{d_\Gamma+1}}{\|\xi^1_\rho(x)\|^{d_\Gamma+1}} \\
&= e^{\omega_{d_\Gamma}(b_\Theta(\rho(\gamma),\zeta_\rho(\xi^1_\rho(x))))} \cdot \mu_{\xi^1_\rho(x)}(\phi_1, \dots, \phi_{d_\Gamma}) \\
&\quad \cdot e^{-(p+1)\omega_1(b_\Theta(\rho(\gamma),\zeta_\rho(\xi^1_\rho(x))))} \\
&= e^{-J^u_{d_\Gamma}(b_\Theta(\rho(\gamma)^{-1},\zeta(\xi^1_\rho(x)))}\mu_{\xi^1_\rho(x)}(\phi_1, \dots, \phi_{d_\Gamma}).
\end{aligned}$$

Finally, we let $\nu = \zeta_{\rho*}\mu$, which is the wanted Patterson-Sullivan measure on $\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d)$, since for $f \in C_c(\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d))$:

$$\begin{aligned}
\int_{\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d)} f \,\mathrm{d}(\gamma_*\zeta_{\rho*}\mu) &= \int_{\xi^1_\rho(\partial\Gamma)} f \circ \gamma \circ \zeta_\rho \,\mathrm{d}\mu = \int_{\xi^1_\rho(\partial\Gamma)} f \circ \zeta_\rho \circ \gamma \,\mathrm{d}\mu = \\
&= \int_{\xi^1_\rho(\partial\Gamma)} f \circ \zeta_\rho(\xi) e^{-J^u_{d_\Gamma}(b_\Theta(\rho(\gamma)^{-1},\zeta(\xi^1_\rho(x)))} \,\mathrm{d}\mu(\xi^1_\rho(x)) = \\
&= \int_{\mathcal{F}_{1,d_\Gamma+1}(\mathbb{R}^d)} f(y) e^{-J^u_{d_\Gamma}(b_\Theta(\rho(\gamma)^{-1},y)} \,\mathrm{d}(\zeta_{\rho*}\mu)(y)
\end{aligned}$$

□

The next lemma is should regarded as an analog of Lemma 2.2.2 and Proposition 2.2.2 to arbitrary flag varieties, and relies only on the Anosov property of ρ, and the fact that ζ_ρ is a section of $\pi_{\mathsf{a}_1} : \mathcal{F}_{1,d_{\rho,\Gamma}+1}(\mathbb{R}^d) \to \mathbb{P}(\mathbb{R}^d)$.

Lemma 3.3.1. *Let $\rho : \Gamma \to SL(d, \mathbb{R})$ be projective Anosov and $\Theta = \{\mathsf{a}_1, \mathsf{a}_{d_{\rho,\Gamma}}\} \subseteq \Pi$. Then for $\alpha > 0$ small enough, there exists some $C, T_0 > 0$ such that for all $T \geq T_0$ the family*

$$\mathcal{U}_T = \{\rho(\gamma)B_{\Theta,\alpha}(\rho(\gamma)) : |\gamma| = T\}$$

is an collection of open subsets of $\mathcal{F}_\Theta(\mathbb{R}^d)$ such that every $\zeta_\rho(\xi^1_\rho(x))$ is contained in at most C many sets in $\mathcal{U}_T$.

Proof. Suppose $\zeta_\rho(\xi^1_\rho(x)) \in \rho(\gamma)B_{\Theta,\alpha}(\rho(\gamma)) \cap \rho(\eta)B_{\Theta,\alpha}(\rho(\eta))$ for some $\gamma, \eta \in \Gamma_T$. Then $\xi^1_\rho(x) \in \rho(\gamma)B_{\mathsf{a}_1,\alpha}(\rho(\gamma)) \cap \rho(\eta)B_{\mathsf{a}_1,\alpha}(\rho(\eta))$. But using Proposition 2.2.2 we have that

$$\begin{aligned} x \in & (\xi^1_\rho)^{-1}(\rho(\gamma)B_{\mathsf{a}_1,\alpha}(\rho(\gamma)) \cap \rho(\eta)B_{\mathsf{a}_1,\alpha}(\rho(\eta))) = \\ & = \gamma(\xi^1_\rho)^{-1}(B_{\mathsf{a}_1,\alpha}(\rho(\gamma))) \cap \eta(\xi^{-1}_\rho)(B_{\mathsf{a}_1,\alpha}(\rho(\eta))) \subseteq \\ & \subseteq \gamma C_{c_0,c_1}(\gamma) \cap \gamma C_{c_0,c_1}(\eta). \end{aligned}$$

Thus x is represented by (c_0, c_1)-quasi-geodesic rays $(a_j)_0^\infty, (b_j)_0^\infty$, that start from e and pass from γ and η respectively. By Morse's lemma, we know that there exists some geodesic ray starting from e and some $A > 0$ depending only on c_0, c_1 and the hyperbolicity constant of Γ such that the Hausdorff distance of the geodesic ray to each of the quasi-geodesics is bounded by A. Let ϵ_0, ϵ_1 be two points on the geodesic ray such that $|\,\mathrm{d}(\gamma, \epsilon_0), \mathrm{d}(\eta, \epsilon_1) \leq A$. Then we have that

$$\begin{aligned} \mathrm{d}(\gamma, \eta) \leq & \ \mathrm{d}(\gamma, \epsilon_0) + \mathrm{d}(\epsilon_0, \epsilon_1) + \mathrm{d}(\epsilon_1, \eta) \leq 2A + ||\epsilon_0| - |\epsilon_1|| \leq \\ & \leq 2A + ||\epsilon_0| - |\gamma|| + ||\gamma| - |\eta|| + ||\epsilon_1| - |\eta|| \leq 4A. \end{aligned}$$

In particular, any γ' such that $\zeta_\rho(\xi^1_\rho(\gamma')) \in \xi^1_\rho(\rho(\gamma')B_{\Theta,\alpha}(\rho(\gamma')))$, will lie in a ball of radius $4A$ around γ. Since Γ is finitely generated, there exists some $C > 0$ such that the ball of radius $4A$ around γ contains at most C elements of Γ. □

Before giving the next definition, we recall that the annihilator annihilator of an element $y \in \mathcal{F}_F i\Theta(\mathbb{R}^d)$ is the set of partial flags that are not transverse to y, that is:

$$\mathrm{Ann}(y) = \left\{x \in \mathcal{F}_\Theta(\mathbb{R}^d) : x^\theta \cap y^{d-\theta} \neq 0 \text{ for some } \theta \in \Theta\right\}.$$

Definition 3.3.3. Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a linear representation, $\Theta \subseteq \Pi$ and μ a measure over $\mathcal{F}_\Theta(\mathbb{R}^d)$. We say that ρ is μ-irreducible there is no element in $\mathcal{F}_{i\Theta}(\mathbb{R}^d)$, whose annihilator is of full measure, i.e. for all $y \in \mathcal{F}_{i\Theta}(\mathbb{R}^d)$:

$$\mu(\mathrm{Ann}(y)) < \mu(\mathcal{F}_\Theta(\mathbb{R}^d)).$$

Example 3.3.1. If $\rho(\Gamma)$ is Zariski-dense in $\mathrm{SL}(d, \mathbb{R})$, then ρ is μ-irreducible for any ρ-quasi-equivariant measure μ, and in particular for any $(\rho(\Gamma), \phi)$-Patterson-Sullivan measure.

Remark 3.3.1. The reason that we introduce the concept of μ-irreducibility is that for any μ-irreducible representation $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$, there exist $\alpha, \kappa > 0$ such that $\mu(B_{\Theta,\alpha}(\rho(\gamma))) \geq k$ for all $\gamma \in \Gamma$.

Indeed, if this were not the case, then there would exists a sequence $\alpha_n \searrow 0$ and $\gamma_n \in \Gamma$ such that

$$\mu(B_{\theta,\alpha}(\rho(\gamma))) \leq \frac{1}{n}.$$

Due to the compactness of $\mathcal{F}_\Theta(\mathbb{R}^d)$, up to considering a subsequence, we may assume that the reppeling flags or $\rho(\gamma_n)$ converge to some $\xi \in \mathcal{F}_\Theta(\mathbb{R}^d)$:

$$(U_{d-i}(\rho(\gamma_n)^{-1}))_{\mathsf{a}_i \in \Theta} \to \xi$$

In that case, the complements $B^c_{\Theta,\alpha_n}(\rho(\gamma_n))$ will converge to the annihilator of ξ, in the sense:

$$\limsup_n B^c_{\Theta,\alpha_n}(\rho(\gamma_n)) \subseteq \mathrm{Ann}(\xi).$$

Indeed, let $y \in \limsup_n B^c_{\Theta,\alpha_n}(\rho(\gamma_n))$ and consider a subsequence k_n such that $y \in B^c_{\Theta,\alpha_n}(\rho(\gamma_{k_n}))$. By the very definition of $B_{\Theta,\alpha_n}(\rho(\gamma_n))$, there exists some p such that up to considering a subsequence of k_n,

$$\angle(y^p, U_{d-p}(\rho(\gamma_n)^{-1})) \leq \alpha_n$$

holds. Taking the limit as $n \to \infty$, we have that $y^p \cap \xi^{d-p} \neq 0$ and hence $y \in \mathrm{Ann}(\xi)$.

Using a measure-theoretic arguement we conclude that $\mathrm{Ann}(\xi)$ is of full measure, which contradicts the μ-irreducibility of ρ:

$$\mu(\mathrm{Ann}(\xi)) \geq \mu(\limsup_n B^c_{\Theta,\alpha_n}(\rho(\gamma_{k_n}))) \geq \limsup_n \mu(B^c_{\Theta,\alpha_n}(\rho(\gamma_{k_n}))) = \mu(\mathcal{F}_\Theta(\mathbb{R}^d)).$$

Lemma 3.3.2. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a representation and μ^ϕ be a $(\rho(\Gamma),\phi)$-Patterson-Sullivan measure. If $\rho(\Gamma)$ is μ-irreducible, then there exists some $\alpha_0 > 0$, such that for any $\alpha \in (0,\alpha_0)$, there's some $k = k(\alpha) > 0$ for which*

$$\frac{1}{k} e^{-\phi(a(\rho(\gamma)))} \leq \mu^\phi(\rho(\gamma) B_{\Theta,\alpha}(\rho(\gamma))) \leq k e^{-\phi(a(\rho(\gamma)))}$$

for all $\gamma \in \Gamma$.

Proof. Let $\alpha_0, k > 0$ be as in the remark preceeding the statement of the lemma. As noted in Lemma 3.1.3, there exists some $K = K(\alpha_0,\phi) > 0$ such that for any $\alpha \in (0,\alpha_0)$ and $y \in B_{\Theta,\alpha}(\rho(\gamma))$:

$$|\phi(a(\rho(\gamma)) - b(\rho(\gamma), y))| \leq K,$$

from which we obtain the upper bound

$$\mu^\phi(\rho(\gamma) B_{\Theta,\alpha}(\rho(\gamma))) = (\rho(\gamma^{-1})_* \mu^\phi)(B_{\Theta,\alpha}(\rho(\gamma))) = \int_{\mathcal{F}_\Theta(\mathbb{R}^d)} e^{-\phi(b(\rho(\gamma),y))} d\mu^\phi(y) \leq$$
$$\leq e^{-K} \mu^\phi(\mathcal{F}_\Theta(\mathbb{R}^d)) e^{-\phi(a(\rho(\gamma)))}.$$

Similarly we obtain the lower bound. □

3.4 Proof of the main theorem

In this section we shall prove the main theorem, which we restate for the reader's convenience:

Theorem 1. *Let $\Gamma < \mathrm{PSL}(d,\mathbb{R})$ be a discrete subgroup, $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a Zariski-dense, projective Anosov representation such that $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold of $\mathbb{P}(\mathbb{R}^d)$. Denoting with $d_{\rho,\Gamma}$ the dimension of $\xi^1_\rho(\partial\Gamma)$, we have that*

$$d_{\rho,\Gamma} = \dim_F(\rho)$$

where $\dim_F(\rho) = h_\rho(F)$ is the Falconer dimension, and $F_s(a) = \max\{\Psi^p_s(a) : p \in [\![2,d]\!]\}$ is the Falconer functional.

Proof. We have already seen in Lemma 2.1.1 that $d_{\rho,\Gamma} \leq \dim_F(\rho)$. For the opposite inequality, we merely need to piece together the results of the previous sections as outlined in Section 3.2. There we have seen that $h_\rho(F) \leq h_\rho(\Psi^{d_{\rho,\Gamma}+2})$ since $F_s \geq \Psi_s^{d_{\rho,\Gamma}+2}$, so may as well show that $h_\rho(\Psi^{d_{\rho,\Gamma}+2}) \leq d_{\rho,\Gamma}$, i.e. that

$$\sum_{\gamma \in \Gamma} e^{-\Psi_s^{d_{\rho,\Gamma}+2}(\rho(\gamma))} < \infty$$

for all $s \geq d_{\rho,\Gamma}$. This will follow as soon as we have shown that $h_\rho(J_{d_{\rho,\Gamma}}) \leq 1$, since

$$\begin{aligned}\Psi_s^{d_{\rho,\Gamma}+1} \circ a &= \mathsf{a}_{12} + \cdots + \mathsf{a}_{1(d_{\rho,\Gamma}+1)} + (s - d_{\rho,\Gamma})\mathsf{a}_{1(\mathsf{d}_{\rho,\Gamma}+2)} = \\ &= \mathsf{a}_{12} + \cdots + \mathsf{a}_{1(d_{\rho,\Gamma}+1)} + d_{\rho,\Gamma}\left(\frac{s}{d_{\rho,\Gamma}} - 1\right)\mathsf{a}_{1(\mathsf{d}_{\rho,\Gamma}+2)} \geq \\ &\geq \mathsf{a}_{12} + \cdots + \mathsf{a}_{1(d_{\rho,\Gamma}+1)} + \left(\frac{s}{d_{\rho,\Gamma}} - 1\right)\left(\mathsf{a}_{12} + \cdots + \mathsf{a}_{1(d_{\rho,\Gamma}+1)}\right) = \\ &= \frac{s}{d_{\rho,\Gamma}} J^u_{d_{\rho,\Gamma}}.\end{aligned}$$

Using the Anosov property of ρ we have that

$$J_{d_\Gamma}(a(\rho(\gamma))) \geq \mathsf{a}_{12}(a(\rho(\gamma))) \geq C|\gamma| - b$$

for certain $C, b > 0$ which when we break up the sum defining the critical exponent into the sum over the sets $\Gamma_T = \{\gamma \in \Gamma : |\gamma| = T\}$ gives us:

$$\begin{aligned}\sum_{\gamma\in\Gamma} e^{-(s+1)J^u_{d_{\rho,\Gamma}}(a(\rho(\gamma)))} &= \sum_{T\geq 0}\sum_{\gamma\in\Gamma_T} e^{-sJ^u_{d_{\rho,\Gamma}}(a(\rho(\gamma)))} e^{-J^u_{d_{\rho,\Gamma}}(a(\rho(\gamma)))} = \\ &= \sum_{T\geq 0} e^{-sJ^u_{d_{\rho,\Gamma}}(a(\rho(\gamma)))} \sum_{\gamma\in\Gamma_T} e^{-J^u_{d_{\rho,\Gamma}}(a(\rho(\gamma)))} \leq \\ &\leq \sum_{T\geq 0} e^{-s(CT-b)} \sum_{\gamma\in\Gamma_T} e^{-J^u_{d_{\rho,\Gamma}}(a(\rho(\gamma)))}\end{aligned}$$

To obtain a bound on the inner sums that is uniform in T, we recall Proposition 3.3.1. There we saw that $\xi^1_\rho(\partial\Gamma)$ being a Lipschitz submanifold of $\mathbb{P}(\mathbb{R}^d)$ implies the existence of a $(\rho(\Gamma), J^u_{d_{\rho,\Gamma}})$-Patterson-Sullivan measure μ on $\zeta^1_\rho(\xi^1(\partial\Gamma)) \subseteq \mathcal{F}_{1,d_{\rho,\Gamma}+1}(\mathbb{R}^d)$. By Lemma 3.3.1 we have that for $\alpha > 0$ small enough, there exists some $M, T_0 > 0$ such that for all $T \geq T_0$ the family

$$\mathcal{U}_T = \{\rho(\gamma)B_{\alpha,\Theta}(\rho(\gamma)) : |\gamma| = T\}$$

is an open covering of $\zeta_\rho(\xi^1_\rho(\partial\Gamma))$ for which

$$\max\left\{\sharp A : A \subseteq \Gamma_T, \bigcap_{\gamma\in A}\rho(\gamma)B_{\alpha,\Theta}(\rho(\gamma)) \neq \emptyset\right\} \leq M.$$

But μ is in particualar ρ-quasi-equivariant which along with the Zariski-density of $\rho(\Gamma)$ implies that ρ is μ-irreducible, as we have seen in Example 3.3.1. Hence the bound in Lemma 3.3.2 applies and we have that

$$\begin{aligned}\sum_{\gamma\in\Gamma_T} e^{-J^u_{d_{\rho,\Gamma}}(a(\rho(\gamma)))} &\leq \sum_{\gamma\in\Gamma_T}\mu\left(\rho(\gamma)B_{\alpha,\Theta}(\rho(\gamma))\right) \leq \\ &\leq \frac{1}{M}\mu\left(\mathcal{F}_{1,d_{\rho,\Gamma}+1}(\mathbb{R}^d)\right) < \infty.\end{aligned}$$

□

Appendix A

Tangent space to the Grassmanian

Let V be a d-dimensional real vector space. We denote with $\mathcal{G}_k(V)$ the Grassmanian of k-dimensional subspaces of V. Our first objective is to find a convenient way to express its tangent space.

Proposition A.0.1. *We have the following canonical identification:*

$$\begin{aligned} \hom(W, V/W) &\simeq T_W \mathcal{G}_k(V) \\ \phi &\mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Gamma(t\phi) \end{aligned}$$

where $\Gamma(\phi) = (Id + \phi)(W)$ *is the graph of* ϕ.

Proof. We will consider the map

$$F : \mathrm{Injhom}(W, V) \to \mathcal{G}_k(V), \quad \phi \mapsto \phi(W).$$

whose derivative is given by:

$$d_I F(\phi) = F\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(I + t\phi)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(I + t\phi(W)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Gamma(t\phi).$$

The result will follow as soon as we have shown that $d_I F$ is surjective and that $\ker d_I F = \hom(W, W)$.

To show that it is surjective, we consider a $(d-k)$-dimensional subspace $W' \in \mathcal{G}_{d-k}(V)$ that is complementary to W, i.e. $V = W \oplus W'$. Denoting with $U_{W'} = \{Z \in \mathcal{G}_k(V) : Z \cap W' = 0\}$, we recall the corresponding chart:

$$\begin{aligned} \Psi : \hom(W, W') &\to U_{W'} \\ \phi &\mapsto \Gamma(\phi). \end{aligned}$$

Surjectivity of $d_I F$ now follows by the fact that

$$d_I F(\phi) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Gamma(t\phi) = d_0\Psi(\phi).$$

To show that $\ker d_I F = \hom(W, W)$, we first note that clearly $\ker \mathrm{d}_I F \supseteq \hom(W, W)$. Equality then follows by the fact that $\dim \hom(W, W) = \dim \ker \mathrm{d}_I F$, which is a direct consequence of the surjectivity. $\square$

Note that another way to prove the above identification throught the fact that the Grassmanian is a homogeneous space of $\mathrm{GL}(d,\mathbb{R})$, giving us the diffeomorphism

$$\begin{aligned}\mathrm{GL}(V)/\mathrm{St}_{GL(V)}W &\to \mathcal{G}_k(V)\\ [g] &\mapsto gW,\end{aligned}$$

where $\mathrm{St}_{GL(V)}W = \{g \in \mathrm{GL}(V) : gW = W\}$ is the stabilizer of W. Thus an expression for the tangent space at W may be obtained by differentiating the map above at the identity coset:

$$\hom(W, V/W) \simeq \hom(V, V)/\hom(W, W) \simeq T_W\mathcal{G}_k(V).$$

In particular the above gives us a direct proof that the kernel of the map constructed in the previous proof is indeed $\hom(W, W)$.

Our second objective is to identify subspaces of $T_l\mathbb{P}(V)$ with subspaces of V, by considering the first as projectivisation of the second. More concretely, we shall consider the space

$$\mathcal{P} = \{(l, P) : l \in \mathbb{P}(V), P \in \mathcal{G}_k(T_l\mathbb{P}(V))\}$$

as a homogenous space of $\mathrm{SL}(V)$, where the action is given by

$$g \cdot (l, P) = (gl,\, \mathrm{d}_l g(P) = g\pi^{-1}(P)g^{-1} + \hom(gl, gl)).$$

where we use the identification of $T_l\mathbb{P}(V)$ with $\hom(l, V/l)$ as above and denote with $\pi : \hom(l, V) \to \hom(l, V/l)$ the canonical projection. For the sake of completeness, we outline the calculation of the differential:

$$\begin{aligned}\hom(l, V/l) \to T_l\mathbb{P}(V) &\to T_{gl}\mathbb{P}(V) &&\to \hom(gl, V/gl)\\ \phi \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g(I + t\tilde{\phi})(l) &\mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(I + tg\tilde{\phi}g^{-1})(gl) &&\mapsto g\tilde{\phi}g^{-1} + \hom(gl, gl)\end{aligned}$$

where $\phi \in \hom(l, V/l), \tilde{\phi} \in \hom(l, V)$ such that $\tilde{\phi} + \hom(l, l) = \phi$.

We are now ready to express the needed identification:

Proposition A.0.2. *We have the following* $\mathrm{SL}(V)$ *equivariant identification:*

$$\begin{aligned}\mathcal{P} &\to \mathcal{F}_{1,k+1}(V)\\ (l, P) &\mapsto (l, \pi^{-1}(P)l)\\ (l, \hom(l, Q/l)) &\mapsfrom (l, Q).\end{aligned}$$

where $\pi : \hom(l, V) \to \hom(l, V/l)$ *is the canonical projection.*

Proof. We begin by showing that the left-to-right direction of the map is well-defined. For this, we first need to check that for $(l, P) \in \mathcal{P}$, we have that $\dim \pi^{-1}(P)l = k + 1$. Indeed, we have that $\dim \pi^{-1}(P) = k+1$ as implied by the rank-nullity theorem for $\pi : \pi^{-1}(P) \to P$. The result then follows by the fact that $\pi^{-1}(P)l = T_1(l) \oplus \cdots \oplus T_{k+1}(l)$ for any base $T_1, \ldots, T_{k+1}$ of $\pi^{-1}(P)$. The second thing to check is that $l \leq \pi^{-1}(P)l$, which holds since $\ker \pi = \hom(l, l) \leq \pi^{-1}(P)$.

To see that the two directions above are inverse to each other, we begin by examining the right-to-left-to-right composition:

$$(l, Q) \mapsto (l, \pi(\hom(l, Q))) \mapsto (l, \pi^{-1}\pi(\hom(l, Q))) = (l, \hom(l, Q)l) = (l, Q).$$

and for the left-to-right-to-left composition

$$(l, P) \mapsto (l, \pi^{-1}(P)l) \mapsto (l, \hom(l, \pi^{-1}(P)l)),$$

so it suffices to show that $\hom(l, \pi^{-1}(P)l/l) = P$. Indeed, for $\pi^{-1}(P) = \mathbb{R}T_1 \oplus \cdots \oplus \mathbb{R}T_k$, we have that

$$\begin{aligned}\hom(l, \pi^{-1}(P)l/l) &= \hom(l, \pi^{-1}(P)l)/\hom(l,l) = (\oplus_i \hom(l, T_i(l))/\hom(l,l)) = \\ &= (\oplus_i \mathbb{R}T_i)\,/\hom(l,l) = \pi^{-1}(P)/\hom(l,l) = P.\end{aligned}$$

For the equivariance, the calculations has as follows:

$$\begin{tikzcd} (l, P) \arrow[r, mapsto] \arrow[d, mapsto, "g"] & (l, \pi^{-1}(P)l) \arrow[d, mapsto, "g"] \\ (gl, g\pi^{-1}(P)g^{-1} + \hom(gl, gl)) \arrow[r, mapsto] & (gl, (g\pi^{-1}(P)g^{-1})(gl)) = (gl, g\pi^{-1}(P)l) \end{tikzcd}$$

$\square$

Appendix B

Irreducible actions problem

The matter of this chapter has to do with an obstruction, found in the proof of this lemma:

Lemma B.0.1 (Lemma 6.8 in [PSW23])**.** *Let Γ be a hyperbolic group and $\eta : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a strongly irreducible projective Anosov representation such that $\xi_\eta(\partial\Gamma)$ is homeomorphic to S^{d_Γ}, and which admits a measurable η-equivariant section $\zeta : \partial\Gamma \to \mathcal{F}_{\{\mathsf{a}_1, \mathsf{a}_{d_\Gamma+1}\}}(\mathbb{R}^d)$. Then η is μ-irreducible for any $(\eta(\Gamma), \phi)$-Patterson-Sullivan measure μ on $\mathcal{F}_{\{\mathsf{a}_1, \mathsf{a}_{d_\Gamma+1}\}}(\mathbb{R}^d)$.*

For convenience, we recall that a linear representation $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is strongly irreducible if there is no proper $\rho(\Gamma)$-invariant subspace of $\mathbb{R}^d$, and it is μ-irreducible if there is no element in $\mathcal{F}_{i\Theta}(\mathbb{R}^d)$, whose annihilator is of full measure.

In what follows, we will show that the above lemma is false, by providing a counterexample. Let Γ be a uniform lattice of $\mathrm{SU}(2,1)$ (i.e. acts convex cocompactly), and $\eta : \Gamma \to \mathrm{SL}(\mathfrak{su}(2,1))$ be the restriction of the adjoint representation, i.e. $\eta(\gamma) = \mathrm{Ad}_\gamma$ for all $\gamma \in \Gamma$.

For convenience, we recall the definition of a uniform lattice:

Definition B.0.1. Let G be a locally compact group. A uniform lattice is a discrete subgroup $\Gamma \leq G$ that is co-compact, i.e. G/Γ is compact.

Remark B.0.1. When $G = \mathrm{Isom}(X)$ is the isometry group of a complete Riemannian manifold X, and Γ is a uniform lattice of G,then it acts properly discontinuously and cocompactly on X.

We begin by showing proving the Anosov property of η.

Proposition B.0.1. *η is projective Anosov.*

Proof. Let $\gamma \in \Gamma$. Since $\gamma \in \mathrm{SU}(2,1)$, we have that

$$\gamma = k_1 \exp\left(r(\gamma) x_0\right) k_2$$

for x_0 a fixed non-zero in the Weyl-chamber $\mathbb{R}x_0$ of $\mathfrak{su}(2,1), r(\gamma) \in \mathbb{R}$ and $k_1, k_2 \in \mathrm{U}(2)$. Then by the definition of a uniform lattice, we have that Γ acts properly discontinuously and cocompactly, which means that the inclusion $\Gamma \hookrightarrow \mathrm{SU}(2,1)$ is projective Anosov (since $\mathrm{SU}(2,1)$ is of rank 1). Thus there exist constants $L \geq 1, b \geq 0$ such that for all $\gamma \in \Gamma$:

$$r(\gamma) \geq \mathsf{a}_1(x_0)^{-1}(L|\gamma| - b) = L'|\gamma| - b'.$$

Note that $\mathsf{a}_1(x_0) > 0$ since x_0 is in the interior of the Weyl-chamber $\mathbb{R}x_0$.

Letting $k_1' = Ad_{k_1}, k_2' = Ad_{k_2}$ and $K' \leq \mathrm{SL}(\mathfrak{su}(2,1))$ be a maximal comapct subgroup containing them, we have that:

$$\eta(\gamma) = \mathrm{Ad}_\gamma = k_1' \,\mathrm{Ad}_{\exp(r(\gamma)x_0)}\, k_2' = k_1' \exp(r(\gamma)\,\mathrm{ad}_{x_0})k_2'.$$

Thus

$$\mathsf{a}_1(\mu(\eta(\gamma))) = r(\gamma)\mathsf{a}_1(\mathrm{ad}_{x_0}) \geq (L'|\gamma| - b')\mathsf{a}_1(x_0)$$

which is Anosov because $\mathsf{a}_1(\mathrm{ad}_{x_0}) > 0$, as can be seen by concrete calculations. □

Before giving an expression for the projective part of the limit map of η, we make a few observations regarding Gromov boundary of Γ. In particular, we claim that since Γ is a uniform lattice os $\mathrm{SU}(2,1)$, we have that $\partial\Gamma$ is homeomorphic to $\mathrm{SU}(2,1)/P$, where P is a parabolic subgroup of $\mathrm{SU}(2,1)$, and it coincides with the stabilizer of some isotropic line $l \in \partial_\infty \mathbb{H}^2_{\mathbb{C}}$.

Indeed, for a uniform lattice Γ of the isometry group G of a homogenous G-space X, the Milnor-Švarc lemma implies that for any $x_0 \in X$, the map $\Gamma \to X, \gamma \mapsto \gamma x_0$ is a quasi-isometry. In our case $G = \mathrm{SU}(2,1)$ and $X = \mathbb{H}^2_{\mathbb{C}}$ is a hyperbolic metric space, so it the quasi-isometry extends to a homeomorphism $\partial\Gamma \to \partial H^2_{\mathbb{C}}$ of the Gromov-boundaries. On the other hand, the action of $\mathrm{SU}(2,1)$ on $\partial_\infty \mathbb{H}^2_{\mathbb{C}}$ is transitive, so we have that $\partial\mathbb{H}^2_{\mathbb{C}} \simeq \mathrm{SU}(2,1)/P$ where P is the stabilizer of a point in $\partial\mathbb{H}^2_{\mathbb{C}}$. In fact, we have that P is a parabolic subgroup of $SU(2,1)$. The combination of the above, along with the fact that the geometric and the Gromov boundaries agree in the case of $\mathbb{H}^2_{\mathbb{C}}$, we deduce that $\partial\Gamma \simeq \mathrm{SU}(2,1)/P$.

Add proofs of these.

To calculate the projective part of the limit map, we shall show that there exists a unique $\mathrm{SU}(2,1)$-equivariant map from $\partial\Gamma$ to $\mathbb{P}(\mathbb{R}^d)$. The uniqueness follows from the following characterisation of limit maps:

Lemma B.0.2. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a strongly irreducible projective Anosov representation, and denote with $\xi_\rho : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ its limit map. Then ξ^1_ρ is the unique continuous, $\rho(\Gamma)$-equivariant map from $\partial\Gamma$ to $\mathbb{P}(\mathbb{R}^d)$.*

Proof. Let $\eta^1 : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ be a continuous, $\rho(\Gamma)$-equivariant map. Since the action of Γ on its boundary $\partial\Gamma$ has dense orbits, it suffices to show that it agrees with ξ^1_ρ on at least one boundary point.

Suppose for the shake of contradiction that η^1 does not coincide with ξ^1 and let $z \in \partial\Gamma, y \in \partial\Gamma\backslash\{z\}$. Then for any $x \in \partial\Gamma\backslash\{y\}$ we may find some quasi-geodesic $\{\gamma_n\}_n$ such that $\gamma_n \to x, \gamma_{-n} \to y$ as $n \to \infty$. Then since $z \neq y$ we know that $\gamma_n z \to z$ as $n \to \infty$ and continuity of η^1 implies that $\eta^1(\gamma_n z) \to \eta^1(z)$. But equivariance of η^1 and the fact that ξ^1 is dynamics-preserving implies that $\eta(\gamma_n z) = \rho(\gamma_n)\eta(z) \to \xi^1(x)$, unless $\eta^1(z) \in \xi^{d-1}(y)$. But if in fact $\eta^1(z) \notin \xi^{d-1}(y)$, then the limits would agree, i.e. $\eta^1(x) = \xi^1(x)$ which is a contradiction. Thus we have that $\eta^1(z) \in \xi^{d-1}(y)$ and since y was an arbitrary points of $\partial\Gamma\backslash\{y\}$, we have that

$$\eta^1(z) \in \bigcap_{y\in\partial\Gamma\backslash\{z\}} \xi^{d-1}(y) \subseteq \bigcap_{y\in\partial\Gamma\backslash\Gamma\cdot z} \xi^{d-1}(y).$$

In particular, the set appearing on the right hand side is ρ-equivariant, non-empty proper subset of $\mathbb{R}^d$, which contradicts the strong irreducibility assumption of ρ. □

Given the lemma above, it suffices to find an $\mathrm{SU}(2,1)$-equivariant map from $\partial\Gamma$ to $\mathbb{P}(\mathbb{R}^d)$.

Proposition B.0.2. *The projective part of the limit map of η is given by*

$$\xi_\eta : \partial\Gamma = \mathrm{SU}(2,1)/P_0 \to \mathbb{P}(\mathfrak{su}(2,1)), \quad \xi_\eta(gP_0) = \mathbb{R}\,\mathrm{Ad}_\gamma\, x_0.$$

where $P_0 = \mathrm{St}_{\mathrm{SU}(2,1)}[1:0:0]$ *and*

$$x_0 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{su}(2,1).$$

Its derivative satisfies:

$$d_x\xi(T_x\,\mathrm{SU}(2,1)/P_0) = \pi(\mathrm{ad}_{\xi^1(x)}\,\mathfrak{su}(2,1))$$

where $\pi : \hom(\xi^1(x), \mathfrak{su}(2,1)) \to \hom(\xi^1(x), \mathfrak{su}(2,1)/\xi^1(x))$ *is the canonical projection.*

Proof. Since by Lemma B.0.2 the limit map is the unique continuous ρ-equivariant from the boundary of Γ to the projective space, it suffices to show that there exists an η-equivariant map $\xi^1 : \mathrm{SU}(2,1)/P_0 \to \mathbb{P}(\mathfrak{su}(2,1))$, since it will then restrict to the limit map on $\partial\Gamma$.

We consider the parabolic subgroup $P_0 = \mathrm{St}_{\mathrm{SU}(2,1)}[1:0:0]$ of $\mathrm{SU}(2,1)$. Then its Lie algebra is given by:

$$\mathfrak{p}_0 = \mathrm{St}_{\mathfrak{su}(2,1)}[1:0:0] = \left\{ \begin{pmatrix} u-is & a & it \\ 0 & 2is & -\bar{a} \\ 0 & 0 & -u-is \end{pmatrix} : a \in \mathbb{C}, u,s,t \in \mathbb{R} \right\}.$$

Since for $\mathbb{R}x \in \mathbb{P}(\mathfrak{su}(2,1))$ we have that P_0 fixes $\mathbb{R}x$ if and only if $\mathfrak{p}_0$ fixes $\mathbb{R}x$. But a quick calculation shows that the only element of $\mathfrak{su}(2,1)$ fixed by $\mathfrak{p}_0$ is x_0.

For the calculation of the image of the differential at the identity coset P, we differentiate the commutative diagram:

$$\begin{array}{ccc} \mathrm{SU}(2,1) & \xrightarrow{\mathrm{Ad}.\,x_0} & \mathfrak{su}(2,1) \\ \downarrow & & \downarrow \\ \mathrm{SU}(2,1)/P_0 & \xrightarrow{\xi^1} & \mathbb{P}(\mathfrak{su}(2,1)) \end{array} \quad \text{to get} \quad \begin{array}{ccc} \mathfrak{su}(2,1) & \xrightarrow{\mathrm{ad}.\,x_0} & \mathfrak{su}(2,1) \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathfrak{su}(2,1)/\mathfrak{p}_0 & \xrightarrow{\mathrm{d}_P\xi^1} & T_{\xi^1(P)}\mathbb{P}(\mathfrak{su}(2,1)) \end{array}$$

In the general case we use the equivariance of the limit map

$$\begin{aligned} \mathrm{d}_{gP}\xi^1(T_{gP}\,\mathrm{SU}(2,1)/P_0) &= d_{gP}\xi^1 d_P g(T_P\,\mathrm{SU}(2,1)/P_0) = d_{\xi^1(P)}g d_P\xi^1(T_P\,\mathrm{SU}(2,1)/P_0) = \\ &= d_{\xi^1(P)}g\pi(\mathrm{ad}_{\xi^1(P)}\,\mathfrak{su}(2,1)) = \\ &= \pi(Ad_g(\mathrm{ad}_{\xi^1(P)}\,\mathfrak{su}(2,1))) = \pi(\mathrm{ad}_{Ad_g\xi^1(P)}\,\mathfrak{su}(2,1)) = \\ &= \pi(\mathrm{ad}_{\xi^1(gP)}\,\mathfrak{su}(2,1)). \end{aligned}$$

□

Recall that all parabolic subgroups of $\mathrm{SU}(2,1)$ are conjugate to each other, so we have the following identification:

$$\begin{aligned} \mathrm{SU}(2,1)/P_0 &\leftrightarrow \{\text{ Parabolic subgroups of } \mathrm{SU}(2,1)\} &&\leftrightarrow \{\text{ Parabolic subalgebras of } \mathfrak{su}(2,1)\} \\ gP_0 &\leftrightarrow gP_0g^{-1} &&\leftrightarrow \mathrm{Ad}_g(\mathfrak{p}_0) \end{aligned}$$

Lemma B.0.3. *Let* $\mathfrak{p}, \mathfrak{p}' \leq \mathfrak{su}(2,1)$ *be two distinct parabolic subalgebras. Then there exists some* $g \in \mathrm{SU}(2,1)$ *such that* $\mathrm{Ad}_g(\mathfrak{p}) = \mathfrak{p}_0$ *and* $\mathrm{Ad}_g\,\mathfrak{p}' = \mathfrak{p}_0^t$.

The following proposition implies that the falsehood of the lemma in the beginning of this chapter.

Proposition B.0.3. *Let $\Gamma \leq \mathrm{SU}(2,1)$ be a uniform lattice and $\eta : \Gamma \to \mathrm{SL}(\mathfrak{su}(2,1))$ be the restriction of the adjoint representation. Then*

(i) η is strongly irreducible,

(ii) η is projective Anosov

(iii) η admits a measurable η-equivariant section:

$$\begin{aligned}\zeta : \partial\Gamma &\to \mathcal{F}_{\{1,4\}}(\mathfrak{su}(2,1)) \simeq \mathcal{P}\\ x &\mapsto (\xi^1(x), T_{\xi^1(x)}\xi^1(\partial\Gamma)) \simeq (\xi^1(x), (\,\mathrm{d}_{\xi^1(x)}p)^{-1}(T_{\xi^1(x)}\xi^1(\partial\Gamma))\xi^1(x)).\end{aligned}$$

where $d_{\xi^1(x)}p : \hom(\xi^1(x), \mathfrak{su}(2,1)) \to \hom(\xi^1(x), \mathfrak{su}(2,1)/\xi^1(x))$ is the canonical projection.

(iv) For all $x, y \in \partial\Gamma : \zeta(x)^4 \cap \zeta(y)^4 \neq 0$.

(v) For any $y_0 \in \mathrm{SU}(2,1)/P_0$ and $W_0 \in \mathcal{G}_7(\mathbb{R}^4)$ that contains $\zeta(y_0)^4$, we have that $\mathrm{Ann}(\zeta(y_0)^4, W_0) \supseteq \zeta(\mathrm{SU}(2,1)/P_0)$ and is in particular of full μ-measure, for any $(\eta(\Gamma), \phi)$-Patterson-Sullivan measure μ supported over $\zeta(\partial\Gamma)$.

Proof. (i) Follows from the fact that $\mathrm{SU}(2,1)$ is a simple Lie group.

(ii) Shown in Proposition B.0.1.

(iii) Follows from the fact that ξ^1 is $\mathrm{SU}(2,1)$-equivariant and the equivariant identification of $\mathcal{F}_{\{1,4\}}(\mathfrak{su}(2,1)) \simeq \mathcal{P}$.

(iv) Letting $g \in \mathrm{SU}(2,1)$ be as in Lemma B.0.3, we have that $\mathrm{Ad}_g(\mathfrak{p}_0) = \mathfrak{p}$ and $\mathrm{Ad}_g(\mathfrak{p}_0^t) = \mathfrak{p}'$. Thus $\zeta(x)^4 \cap \zeta(y)^4 \neq \emptyset$ if and only if

$$\begin{aligned}\emptyset \neq Ad_g(\zeta(x)^4 \cap \zeta(y)^4) = \mathrm{Ad}_g\, \zeta(x)^4 \cap \mathrm{Ad}_g\, \zeta(y)^4 = \zeta(gx)^4 \cap \zeta(gy)^4 = \zeta(\mathfrak{p}_0)^4 \cap \zeta(\mathfrak{p}_0^t)^4 =\\ = \pi\left(\mathbb{R}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\right).\end{aligned}$$

For the last equality, we use Proposition B.0.2 and the fact that $\mathfrak{p}_0^t = \mathrm{Ad}_g\, \mathfrak{p}_0$ for

$$g = \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix},$$

to conclude that

$$\zeta(\mathfrak{p}_0) = \zeta(P_0) = \pi\left(\left\{\begin{pmatrix}u & a & it\\ 0 & 0 & -\bar{a}\\ 0 & 0 & -u\end{pmatrix} : a \in \mathbb{C}, u, t \in \mathbb{R}\right\}\right),$$

and

$$\zeta(\mathfrak{p}_0^t) = \zeta(gP_0) = \mathrm{Ad}_g\, \zeta(P_0) = \pi\left(\left\{\begin{pmatrix}u & 0 & 0\\ a & 0 & 0\\ it & -\bar{a} & 0\end{pmatrix} : a \in \mathbb{C}, u, t \in \mathbb{R}\right\}\right).$$

□

Bibliography

[LL23] François Ledrappier and Pablo Lessa. "Dimension gap and variational principle for Anosov representations". In: (Dec. 2023). arXiv:2310.13465 [math] version: 3 (cit. on p. 3).

[PSW23] Beatrice Pozzetti, Andrés Sambarino, and Anna Wienhard. "Anosov representations with Lipschitz limit set". In: *Geometry & Topology* 27.8 (Nov. 2023). arXiv:1910.06627 [math], pp. 3303–3360. ISSN: 1364-0380, 1465-3060 (cit. on pp. 6–10, 15, 23).

[GH13] Etienne Ghys and Pierre da la Harpe. *Sur les groupes hyperboliques d'apres Mikhael Gromov.* Vol. 83. Springer Science & Business Media, 2013 (cit. on p. 10).