Notes

Giorgos

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Contents

Chapter 1

Anosov representations

1.1 Motivation

We study Anosov representations because they give a flexible generalisation of the theory of convex cocompact representations of hyperbolic groups into rank 1 Lie groups into the setting of representations of hyperbolic groups into semisimple Lie groups. They now serve as an organizing principle for the geometric and dynamical approach to the so-called Higher Teichmüller theory.

1.1.1 Convex-cocompact representations

Definition 1. Let Γ be a finitely generated discrete subgroup of $\mathrm{SO}_0(n,1)$. We say that Γ is convex-cocompact if any of the two following equivalent conditions hold:

(i) There exists a Γ-invariant convex subset $C \subseteq \mathbb{H}^n$ such that $\Gamma \backslash C$ is compact.

(ii) For some $x_0 \in \mathbb{H}^n$, the orbit map $\tau : \Gamma \to \mathbb{H}^n$ given by $\gamma \mapsto \gamma x_0$ is a quasi-isometric embedding.

A representation $\rho : \Gamma \to \mathrm{SO}_0(2,1)$ of a discrete group Γ is convex-cocompact if $\rho(\Gamma)$ is convex-cocompact.

Example 1. The holonomy representation of a hyperbolic surface group $\pi_1(S)$ into $\mathrm{SO}_0(2,1)$ is convex-cocompact.

Under certain conditions, the set of convex-cocompact representations is a union of connected components, which implies that the Teichmüller space is in fact a union of connected components. However, this is not always the case, for instance with $\hom(\mathbb{F}_2, \mathrm{SO}_0(2,1))$. A goal of Anosov representations is to generalize convex-cocompact representations. In particular, we will be interested in representations into higher rank Lie groups that satisfy the following two properties:

1. The induced orbit maps are quasi-isometric embeddings.
2. The representations should form an open subset of the representation variety.

Remark 1. For $n \geq 3$, any two discrete faithful representations convex-cocompact representations of a group Γ into $\mathrm{SO}_0(n,1)$ are conjugate, by a theorem of Mostow.

1.1.2 Anosov representations

Definition 2. Let Γ be a finitely generated group. A representation $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ is P_k Anosov if the logarithm of the k-th singular value gap is linearly controlled, i.e. there exist $C, c > 0$ such that for all $\gamma \in \Gamma$:

$$\frac{1}{C}|\gamma| - c \leq \log \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \leq C|\gamma| + c.$$

If ρ is P_k Anosov, then there exists a limit map $\xi : \partial\Gamma \to \mathcal{F}_{k,d-k}(\mathbb{R}^d)$ that is transverse and dynamics-preserving.

Example 2. Typical examples of Anosov representations include:

(i) Convex-cocompact representations of finitely generated groups into rank 1 Lie groups.

(ii) Holonomy representations of strictly convex (real) projective closed manifolds.

(iii) Hitchin representations.

1.2 Hyperbolic groups

1.2.1 The Gromov boundary

Here X will be assumed to be a proper geodesic metric space.

Definition 3. Let X be a δ-hyperbolic space. Recall that a geodesic in X is an isometric embedding $\alpha : \mathbb{R} \to X$, while a geodesic ray is an isometric embedding $\alpha : [0, \infty) \to X$. We denote with $\alpha(\infty)$ the equivalence class of α, i.e. the set of geodesic rays that stay at bounded distance from α. A generalized geodesic is a geodesic segment or a geodesic ray, i.e. an isometric embedding $\alpha : I \to X$ where $I = [0, \infty]$ or $I = [0, R]$. In the second case we consider the extension $\alpha : [0, \infty) \to X$ given by $\alpha(t) = \alpha(R)$ for $t \geq R$. We define its Gromov boundary ∂X in one of the following equivalent ways:

(i) ∂X is the set of equivalence classes of geodesic rays in X, where two rays are equivalent if they stay at bounded distance from each other.

(ii) ∂X is the set of equivalence classes of geodesic rays starting from a specific point $x_0 \in X$, where two rays are equivalent if they stay at bounded distance from each other.

Note that with the above definition, $X \cup \partial X = \{c(\infty) : c \text{ generalized geodesic ray}\}$. To topologize $X \cup \partial X$ we consider the following notion of convergence:

Definition 4. A sequence $x_n \in X \cup \partial X$ converges to $x \in \partial X$ if there exist generalized geodesic rays c_n, c such that $c(\infty) = x, c_n(\infty) = x_n$ and $c_n \to c$ uniformly on compact sets.

Proposition 1 (Visibility of ∂X)**.** Let X be a δ-hyperbolic proper geodesic space. Then for each pair of distinct points on the boundary, there exists a geodesic joining them.

Lemma 1. Let X be a geodesic δ-hyperbolic space. For $x_n, y_n \in X$ at (uniformly in n) bounded distance and $z \in \partial X$, we have that $x_n \to z$ implies $y_n \to z$.

1.2.2 Hyperbolic groups

Proposition 2 (North-South dynamics)**.** Let Γ be a δ-hyperbolic group. For every infinite order element $\gamma \in \Gamma$, there exists a quasi-isometric embedding $c_\gamma : \mathbb{R} \to \Gamma$ and $R > 0$ such that $\gamma^n(c_\gamma(t)) = c_\gamma(t+n)$ for all $t \in \mathbb{R}, n \in Z$ and $\gamma^\pm \in \partial\Gamma$ such that for all $x \in \Gamma$:

$$\lim_{n\to\infty} \gamma^n x = \gamma^+, \lim_{n\to-\infty} \gamma^n x = \gamma^-,$$

and γ fixes both γ^+ and γ^-.

1.2.3 Dynamics on the Gromov boundary

Proposition 3. Let Γ be a δ-hyperbolic group and $\gamma \in \Gamma$ an infinite order element. Then γ has exactly two fixed points $\gamma^\pm = \lim_{n\to\pm\infty} \gamma^n$ in $\partial\Gamma$, and $\gamma^n z \to \gamma^+$ for all $z \in \partial\Gamma - \{\gamma^-\}$.

1.3 Convex cocompact representations

Key takeaways:

1. The Teichmüller space can be realized as the moduli space of marked hyperbolic structures modulo homotopy, or as representations of the fundamental group into $\mathrm{PSL}(2,\mathbb{R})$ modulo conjugation.
2. Convex cocompact representaions of finitely generated groups into $\mathrm{SO}_0(n,1)$ are exactly the P_1-Anosov representations (Theorem 1) Then the image of the representation acts cocompactly on the convex hull of the limit set (Proposition 4)
3. Convex cocompact representations of a finitely generated group is stable.
4. The set of convex cocompact representations is closed for finitely generated torsion-free groups that are not virtually cyclic.

1.3.1 Teichmüller theory

Here S will be a closed surface with finitely many punctures of genus at least 2.

Definition 5. A complete orientable Riemmannian surface X is hyperbolic if it is locally isometric to the hyperbolic plane $\mathbb{H}^2$.

To every hyperbolic surface X we can associate a conjugacy class of a discrete subgroup $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ by $X \mapsto \pi_1(X)$. Recall that a marked hyperbolic structure on S is a pair (X,ϕ), where X is a hyperbolic surface and $\phi : S \to X$ is a diffeomorphism.

Definition 6 (Marked hyperbolic structures model for $\mathcal{T}(S)$)**.** A marked hyperbolic structure on S is the moduli space of marked hyperbolic structures on S:

$$\mathcal{T}(S) = \{ \text{ marked hyperbolic structures on } S\}/\sim,$$

where $(X,\phi) \sim (X',\phi')$ if there exists an isometry $f : X \to X'$ such that $\phi' = f \circ \phi$ up to isotopy.

To provide another model for $\mathcal{T}(S)$, we consider a marked hyperbolic structure (X,ϕ) on S. Since X is a hyperbolic surface, $\tilde{X}$ is isometric to the hyperbolic plane $\mathbb{H}^2$. Moreover, the fundamental group $\pi_1(X)$ acts isometrically and properly discontinuously on $\tilde{X}$. Using the marking ϕ, we identify $\pi_1(S)$ with $\pi_1(X)$, which in turn is identified with a subgroup of isometries of $\tilde{X}$ and hence with a subgroup of $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$. In this way we obtain a representation $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ that is discrete and faithful, but well-defined only up to conjugation in $\mathrm{PSL}(2,\mathbb{R})$. This motivates the following definition:

Definition 7 (Representation model for $\mathcal{T}(S)$)**.** The Teichmüller space of S is defined as the space of conjugacy classes of discrete faithful representations of $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$:

$$\mathcal{T}(S) = \hom(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\,\mathrm{PSL}(2,\mathbb{R}).$$

Recall that an isotopy is a continuous map $H : S \times [0,1] \to S$ such that for each $t \in [0,1]$, $H_t : S \to S$ is a homeomorphism.

Definition 8. The mapping class group $\mathrm{MCG}(S)$ is the group of isotopy classes of orientation-preserving homeomorphisms of S:

$$\mathrm{MCG}(S) = \{\text{orientation-preserving homeomorphisms of } S\}/\{ \text{ isotopy } \}.$$

1.3.2 Convex cocompact representations into $\mathrm{SO}_0(n,1)$

Convex cocompactness and Anosovness

Recall that a representation is almost faithful if the kernel is finite. The following proposition tells us that for $\mathrm{SO}_0(n,1)$ convex cocompactness and P_1-Anosov are equivalent and shows how we can obtain the limit map from the convex cocompactness condition.

Theorem 1. Let $\rho : \Gamma \to \mathrm{SO}_0(n,1)$ be a representation of a finitely generated group Γ. Then ρ is convex cocompact if and only if it is P_1-Anosov. In that case:

(i) Γ is Gromov hyperbolic.

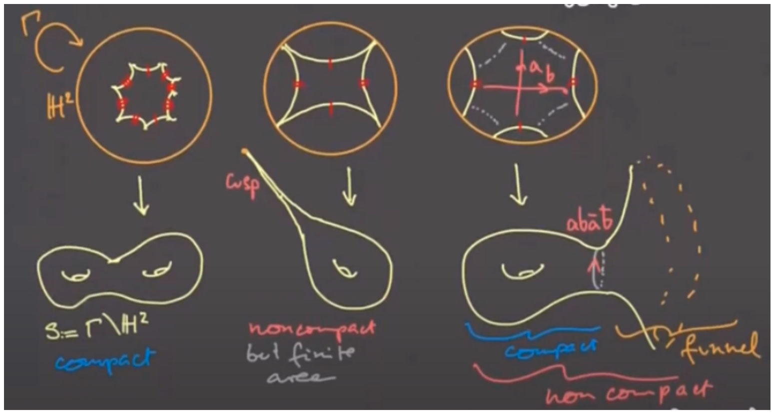


Figure 1.1: Proper actions on hyperbolic disk

(ii) ρ is discrete and almost faithful.

(iii) There exists a continuous, injective, ρ-equivariant map $\xi : \partial\Gamma \to \partial\mathbb{H}^n$ that is dynamics-preserving, i.e. we have

$$\lim_{n\to\infty} \rho(\gamma_n)x = \xi(z)$$

for sequences $\gamma_n \in \Gamma$ such that $\gamma_n \to z \in \partial\Gamma$ and points $x \in \mathbb{H}^n$. Moreover, for elements $\gamma \in \Gamma$ of infinite order, $\rho(\gamma)$ fixes $\xi(\gamma^+)$ and $\xi(\gamma^-)$.

Proof. We fix throughout the proof an element $x_0 \in \mathbb{H}^n$ and consider the orbit map corresponding to it. The hyperbolicity of Γ follows from the lemma of Milnor-Svarc. To prove that ρ is discrete, we proceed by contradiction. If this were not the case, then there would exist a sequence $\gamma_n \in \Gamma$ with no repeated elements such that $\rho(\gamma_n) \to e$. In particular, $\rho(\gamma_n)x_0$ is bounded. Since the orbit map is a quasi-isometric embedding, this implies that $|\gamma_n|$ is bounded, which is a contradiction, since this would imply that $\{\gamma_n\}$ is finite. Similarly, to show that ρ is almost faithful, we can show that the kernel is bounded, and hence finite.

For the second point, the lemma of Milnor-Svarc implies that the orbit map is a quasi-isometry, and the theory of Gromov hyperbolic spaces, we know that it must extend to a homeomorphism $\xi : \partial\Gamma \to \partial\mathbb{H}^n$ between the Gromov boundaries. Let $\gamma_n \in \Gamma$ converge to $z \in \partial\Gamma$. By Milnor-Svarc, we have that $\rho(\gamma_n)^n x_0 \to \xi(z)$ as $n \to \infty$. But for any $x \in \mathbb{H}^n$, we have that $d(\rho(\gamma_n)x_0, \rho(\gamma_n)x) = d(x_0, x)$ is bounded, and $\rho(\gamma_n)x_0 \to \xi(z)$ as $n \to infty$, so $\rho(\gamma_n)x \to \xi(z)$ by Lemma 1. In particular, for an infinite order element $\gamma \in \Gamma$, we have that $\gamma^{\pm n} \to \gamma^{\pm}$ as $n \to \infty$, so $\rho(\gamma^n)x_0 \to \xi(\gamma^{\pm})$. Since $\xi(\gamma^{\pm})$ is the limit of a sequence fixed by $\rho(\gamma)$, it will be fixed as well. □

Limit set

Definition 9. Let H be a discrete subgroup of $\mathrm{SO}_0(n,1)$. Then the limit set Λ of H is the set of accumulation points of an orbit of H that are contained in $\partial\mathbb{H}^n$:

$$\Lambda = \overline{Hx_0} - Hx_0 \subseteq \partial\mathbb{H}^n,$$

where $x_0 \in \mathbb{H}^n$ is fixed.

Remark 2. In the above definition, there have been made a few subtle assertions:

(i) The limit set $\Lambda(\rho)$ is independent of the choice of x_0. This can be seen by using Lemma 1.

(ii) The set of accumulation points in $\partial\mathbb{H}^n$ of an orbit is equal to $\overline{Hx_0} - Hx_0$. Clearly the set of accumulation points is contained in $\overline{Hx_0}$. To see that it is disjoint from Hx_0, we can use the fact that it is contained in $\partial\mathbb{H}^n$. On the other hand, every element in $\overline{Hx_0} - Hx_0$ is the limit of a sequence of discrete elements in H, so it is an accumulation point.

(iii) If one assumes that H is a discrete subgroup that acts properly discontinuously on $\mathbb{H}^n$, then the accumulation points of any orbit lie on the boundary of $\mathbb{H}^n$. To see this, assume for the sake of contradiction that there exists an accumulation point $z \in \mathbb{H}^n$ of an orbit Hx_0. Then there exists a sequence of pairwise different elements $h_n \in H$ such that $h_n x_0 \to z$. In particular $h_n x_0$ is bounded, so there exists some ball $B_R(x_0)$ that contains all $h_n x_0$ and x_0. This however implies that $h_n x_0 \in h_n B_R(x_0) \cap B_R(x_0)$ for all n, which contradicts thes proper discontinuity of the action.

Now we are ready to make the link between the geometric definition of convex cocompactness and the notion of convex cocompact representation (i.e. a representation whose orbit map is a quasi-isometric embedding).

Proposition 4. Let $\rho : \Gamma \to \mathrm{SO}_0(n,1)$ be a discrete and faithful representation. Then ρ is convex cocompact if and only if $\rho(\Gamma)$ preserves a convex subset of $\mathbb{H}^n$, on which it acts cocompactly. In that case, the convex hull of the limit set $\Lambda(\rho)$ of $\rho(\Gamma)$ is such a subset.

Stability and closedness

Two of the basic features of the space of convex cocompact representations that we look for when generalizing are their stability and closedness. To fix our notation, for a group Γ we denote with $\mathrm{CC}(\Gamma, \mathrm{SO}_0(n,1)) \subseteq \hom(\Gamma, \mathrm{SO}_0(n,1))$ the set of convex cocompact representations of Γ in $\mathrm{SO}_0(n,1)$, with $\mathrm{X}(\Gamma, \mathrm{SO}_0(n,1)) = \hom(\Gamma, \mathrm{SO}_0(n,1)) / \mathrm{SO}_0(n,1)$ the character variety of Γ in $\mathrm{SO}_0(n,1)$, and with $\hat{\mathrm{CC}}(\Gamma, \mathrm{SO}_0(n,1)) \subseteq \mathrm{X}(\Gamma, \mathrm{SO}_0(n,1))$ the set of conjugacy classes of convex cocompact representations. We will also denote with $\mathrm{DF}(\Gamma, \mathrm{SO}_0(n,1)) \subseteq \hom(\Gamma, \mathrm{SO}_0(n,1))$ the set of discrete and faithful representations and with $\hat{\mathrm{DF}}(\Gamma, \mathrm{SO}_0(n,1)) \subseteq \mathrm{X}(\Gamma, \mathrm{SO}_0(n,1))$ the set of conjugacy classes of discrete and faithful representations. Then we have:

Theorem 2. **??** Let Γ be a finitely generated group.

(i) The convex cocompact representations of Γ are open in the space of representations and in the character variety, i.e.

$$\mathrm{CC}(\Gamma, \mathrm{SO}_0(n,1)) \subseteq \hom(\Gamma, \mathrm{SO}_0(n,1)) \text{ and } \hat{\mathrm{CC}}(\Gamma, \mathrm{SO}_0(n,1)) \subseteq \mathrm{X}(\Gamma, \mathrm{SO}_0(n,1))$$

are open.

(ii) If we assume that Γ is torsion-free and not virtually cyclic, then the convex cocompact representations are closed in the space of representations and in the character variety, i.e.

$$\mathrm{CC}(\Gamma, \mathrm{SO}_0(n,1)) \subseteq \hom(\Gamma, \mathrm{SO}_0(n,1)) \text{ and } \hat{\mathrm{CC}}(\Gamma, \mathrm{SO}_0(n,1)) \subseteq \mathrm{X}(\Gamma, \mathrm{SO}_0(n,1))$$

are closed.

1.4 Various facts

Proposition 5. Let Γ be a discrete group and $\rho : \Gamma \to G$ be a P_Θ-Anosov in a semisimple Lie group G. Then the image $\rho(\Gamma)$ is discrete.

Proof. Suppose $\rho(\Gamma)$ were not discrete. Since G is Hausdorff, this is equivalent to the existence of a sequence $\rho(\gamma_n)_n$ of distinct elements of $\rho(\Gamma)$ such that $\rho(\gamma_n) \to e$. In particular, γ_n is a sequence of distinct elements of Γ. But by the continuity of the Cartan projection, we have that $\alpha(\mu(\rho(\gamma_n))) \to \alpha(\mu(e)) = 0$, for $\alpha \in \Theta$. Hence the sequence $\mu(\rho(\gamma_n))$ is bounded, which given the P_Θ-Anosov property of ρ, means that the word-lengths of $\gamma_{n n}$ are bounded. However, this implies that the sequence $\gamma_{n n}$ is finite, which is a contradiction. $\square$

Chapter 2

Lorentzian spaces

Here we recall some basic facts on Lorentzian spaces. We will introduce Lorentzian manifolds of constant sectional curvature and we will see that, as in the Riemannian case, two Lorentzian manifolds of constant sectional curvature K are locally isometric. Generally, we will focus on those with maximal isometry group, as they provide models of manifolds of constant sectional curvature: if M is a Lorentzian manifold with constant sectional curvature K and maximal isometry group, then any Lorentzian manifold with constant sectional curvature K carries a natural $(\mathrm{Isom}(M), M)$-atlas made of local isometries. Simply connected space forms have maximal isometry group, but in general there are manifolds with maximal isometry group which are not simply connected. In particular, we will focus on the case $K = -1$ and in that case it will be convenient to use models which are not simply connected.

2.1 Basic definitions

Definition 10. (i) A *Lorentzian metric* on a manifold of dimension $n+1$ is a non-degenerate symmetric 2-tensor g of signature $(n, 1)$.

(ii) A *Lorentzian manifold* is a connected manifold M equipped with a Lorentzian metric g.

(iii) In a Lorentzian manifold M we say that a non-zero vector $v \in TM$ is *spacelike, lightlike, timelike* if $g(v, v)$ is respectively positive, zero or negative. More generally, we say that a linear subspace $V \subset T_xM$ is *spacelike, lightlike, timelike* if the restriction of g_x to V is positive definite, degenerate or indefinite.

(iv) A differentiable curve is *spacelike, lightlike, timelike* if its tangent vector is spacelike (resp. lightlike, timelike) at every point. It is *causal* if the tangent vector is either timelike or lightlike.

(v) The set of lightlike vectors of T_xM is also known as the *light cone* at x.

Assuming $\dim M \geq 3$, the light cone disconnects T_xM into three regions: two convex open cones formed by timelike vectors, one opposite to the other, and the region of spacelike vectors.

Definition 11. Let M be a Lorentzian manifold.

(i) A continuous choice (in the sense of a continuous timelike vector field) of one of the two cones of time-like vectors for each point $x \in M$ is called a *time orientation* of M.

(ii) If a time-orientation of M exists, then M is said to be *time-orientable*. Timelike vectors in the same component as the time-orientation are said *future-directed*, while the rest are said *past-directed*.

(iii) Given a point x in a time-oriented Lorentzian manifold M, the *future* of x is the set $\mathrm{I}^+(x)$ of points which are connected to x by a future-directed causal curve. The *past* of x, denoted $\mathrm{I}^-(x)$, is defined similarly, for past-directed causal curves.

An *orthonormal basis* of T_xM is a basis $v_1, \dots v_{n+1}$ such that $|g(v_i, v_j)| = \delta_{ij}$, with $v_1, \dots v_n$ spacelike, and v_{n+1} timelike. As in the Riemannian setting, on a Lorentzian manifold M there is a unique linear connection ∇ which is symmetric and compatible with the Lorentzian metric g. We refer to it as the *Levi-Civita connection* of M. The Levi-Civita connection determines the Riemann curvature tensor defined by

$$R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w \ .$$

We then say that a Lorentzian manifold M has constant sectional curvature K if

$$g(R(u,v)v,u) = K\left(g(u,u)g(v,v) - g(u,v)^2\right) \tag{2.1}$$

for every pair of vectors $u, v \in T_xM$ and every $x \in M$. This definition is strictly analogous to the definition given in the Riemannian realm. However in this setting the sectional curvature can be defined only for planes in T_xM where g is non-degenerate.

Example 3. The Minkowski space $\mathbb{R}^{n,1}$ the Levi-Civita connection given by the Euclidean connection:

$$\nabla_X Y = (XY^i)\partial_i,$$

so the Riemann curvature tensor is zero, and the same is true for the sectional curvature.

Finally, we say that M is *geodesically complete* if every geodesic is defined for all times, or in other words, the exponential map is defined everywhere.

2.2 Maximal isometry groups and geodesic completeness

Constant curvature of manifolds allows us to extend isometries of tangent spaces to isometries of the whole manifold. As a result, two Lorentzian manifolds M and N of constant curvature K are locally isometric, a fact which is well-known in the Riemannian setting. More precisely, the following holds:

Lemma 2. Let M and N be Lorentzian manifolds of constant curvature K.

1. Then every linear isometry $L : T_xM \to T_yN$ extends to an isometry $f : U \to V$, where U and V are neighbourhoods of x and y respectively, and two extensions $f : U \to V$ and $f : U' \to V'$ of L coincide on $U \cap U'$.

2. If M is simply connected and N is geodesically complete, then any isometry $L : T_xM \to T_yN$ extends to a unique local isometry $f : M \to N$.

3. If M and N are both simply connected and geodesically complete, then any isometry $L : T_xM \to T_yN$ extends to a unique global isometry $f : M \to N$.

Proof. For the last statement, recall that a local isometry from a simply connected manifold to a uniquely geodesic manifold is a global isometry. □

Exactly as in the Riemannian case the proof is a simple consequence of the classical Cartan–Ambrose–Hicks Theorem. Note that this implies in particular that there is a unique simply connected geodesically complete Lorentzian manifold of constant curvature K up to isometries. For instance for $K = 0$ a model is the Minkowski space $\mathbb{R}^{n,1}$.

Another consequence of Lemma 2 is that, fixing a point $x_0 \in M$, the set of isometries of M, which we will denote by $\mathrm{Isom}(M)$, can be realized as a subset of $\mathrm{ISO}(T_{x_0}M, TM)$, namely the fiber bundle over M whose fiber over $x \in M$ is the space of linear isometries of $T_{x_0}M$ into T_xM.

It can be proved that $\mathrm{Isom}(M)$ has the structure of a Lie group with respect to composition so that the inclusion $\mathrm{Isom}(M) \hookrightarrow \mathrm{ISO}(T_{x_0}M, TM)$ is a differentiable proper embedding. It follows that the maximal dimension of $\mathrm{Isom}(M)$ is $\dim \mathrm{O}(n, 1) + n + 1 = (n + 1)(n + 2)/2$.

Definition 12. A Lorentzian manifold M has *maximal isometry group* if the action of $\mathrm{Isom}(M)$ is transitive and, for every point $x \in M$, every linear isometry $L : T_xM \to T_xM$ extends to an isometry of M.

Equivalently M has maximal isometry group if the above inclusion of $\mathrm{Isom}(M)$ into the bundle $\mathrm{ISO}(T_{x_0}M, TM)$ is a bijection. Hence, if M has maximal isometry group, then the dimension of the isometry group is maximal.

From Lemma 2, every simply connected Lorentzian manifold M has maximal isometry group if it has constant sectional curvature and is geodesically complete. The converse holds even without the simply connectedness assumption:

Lemma 3. Let M be a Lorentzian manifold.

(i) If M has a maximal isometry group then M has constant sectional curvature and is geodesically complete.

(ii) If M is simply connected, then M has maximal isometry group if and only if M has constant sectional curvature and is geodesically complete.

Chapter 3

Pseudoriemannian hyperbolic spaces

Definition 13. Let V be a real vector space of dimension $n = p + q + 1$, equipped with an inner product $\langle \cdot, \cdot \rangle$ of signature $(p, q + 1)$. We define the pseudoriemannian hyperbolic space

$$\mathbb{H}^{p,q} = \{x \in \mathbb{P}(V) : \langle x, x \rangle < 0\},$$

whose double cover is given by

$$\hat{\mathbb{H}}^{p,q} = \{x \in V : \langle x, x \rangle = -1\}$$

3.1 Second fundamental form

A reference for the material of this section is Chapter 4 of [One83]. When we say submanifold, we mean embedded (the inclusion is an emersive topological embedding).

Definition 14. Let M be a pseudoriemannian submanifold of a pseudoriemannian manifold $\tilde{M}$. Then the orthogonal decomposition of the tangent bundle TM with respect to the ambient metric defines two vector bundle homomorphisms:

$$\nabla : \Gamma(M) \times \Gamma(M) \to \Gamma(TM), \quad \mathrm{I\!I} : \Gamma(TM) \to \Gamma(NM),$$

that satisfy

$$\tilde{\nabla}_X Y = \nabla_X Y + \mathrm{I\!I}(X, Y), \text{ for } X, Y \in \Gamma(TM),$$

where $\tilde{n}abla$ is the Levi-Civita connection of $\tilde{M}$ and $\tilde{n}abla_X Y$ is evaluated for any smooth extensions of X, Y to $\tilde{M}$. The first is called the induced connection on M and the second is called the second fundamental form of M in $\tilde{M}$.

The notation ∇ is understood in the light of the following proposition.

Proposition 6. Let M be a pseudoriemannian submanifold of a pseudoriemannian manifold $\tilde{M}$. Then the induced connection ∇ is the Levi-Civita connection of M with respect to the induced metric.

The second fundamental form has a few basic properties:

Proposition 7. Let M be a pseudoriemannian submanifold of a pseudoriemannian manifold $\tilde{M}$. Then the second fundamental form II satisfies the following properties:

(i) Symmetry: $\mathrm{II}(X,Y) = \mathrm{II}(Y,X)$ for all $X, Y \in \Gamma(TM)$.

(ii) Biliniearity: II is bilinear in $C^\infty(M)$.

(iii) The value of $II(X,Y)$ at $p \in M$ for $X, Y \in \Gamma(TM)$ depends only on the values of X_p, Y_p at p.

Hence, to compute the Levi-Civita connection of a pseudoriemannian submanifold, it suffices to project the Levi-Civita connection of the ambient space onto the tangent space of the submanifold, or equivalently, compute the second fundamental form of M. Similarly, there exists an analog of the Gauss formula for vector fields over curves:

Proposition 8. Let M be a pseudoriemannian submanifold of a pseudoriemannian manifold $\tilde{M}$, $\gamma : I \to M$ a smooth curve in M and $X \in \Gamma(\gamma)$ a smooth vector field along γ (with values in TM). Then

$$\tilde{D}_t X = D_t X + \mathrm{II}(\gamma', X),$$

where $\tilde{D}_t X$ is the covariant derivative along γ in M of an extension of X on M, D_t is the covariant derivative along γ in the submanifold.

Remark 3. The Gauss formula for vector fields and curves give us two ways of interpreting the second fundamental form. If (M, g) is a pseudoriemannian submanifold of a pseudoriemannian manifold $(\tilde{M}, \tilde{g})$, then

1. The second fundamental form measures the difference between differentiation in the ambient space $\tilde{M}$ and differentiation in the submanifold M for given vector fields in M.

2. $\mathrm{II}(v,v)$ is the $\tilde{g}$-acceleration $\tilde{D}_t\tilde{\gamma}$ of the g-geodesic γ in M, with initial veolicity $v \in T_pM$.

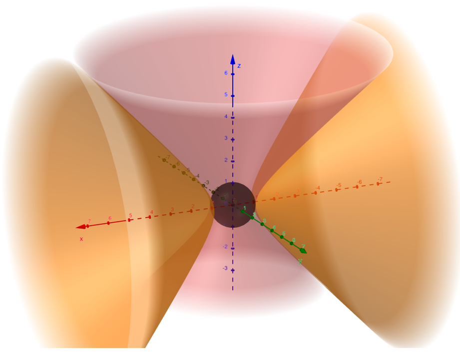
The second fundamental form can give us the curvature of a submanifold:

Proposition 9. Let M be a pseudoriemannian subbmanifold of a pseudoriemannian manifold M. Then for every $v, w \in T_pM$ that generate a nondegenerate plane, the sectional curvature of M is given by

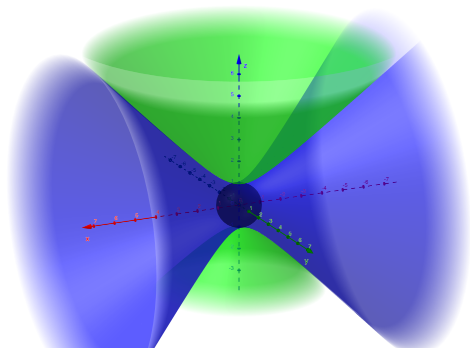
$$K(v,w) = \tilde{K}(v,w) + \frac{\langle \mathrm{II}(v,v), \mathrm{II}(w,w)\rangle - \langle \mathrm{II}(v,w), \mathrm{II}(v,w)\rangle^2}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2}.$$

3.1.1 Pseudoriemannian hypersurfaces

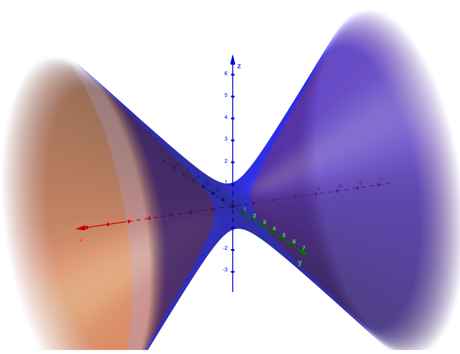
Let M be a pseudoriemannian hypersurface of a pseudoriemannian manifold $\tilde{M}$. Around every point, we can define a unit normal vector field N that is orthogonal to the tangent space of M.



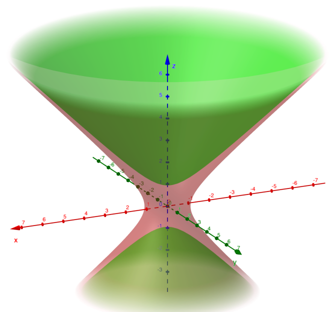
(a) Pseudospheres in different signatures



(b) Pseudohyperbolics in different signatures



(c) $\mathbb{H}^{1,1}$ and $\mathbb{S}^{0,2}$



(d) $\mathbb{H}^{2,0}$ and $\mathbb{S}^{1,1}$

Figure 3.1: Visualizations of pseudospheres and pseudohyperbolics in different signatures, $\mathbb{S}^{2,0}$: black, $\mathbb{S}^{1,1}$: red, $\mathbb{S}^{0,2}$: orange, $\mathbb{H}^{2,0}$: green, $\mathbb{H}^{1,1}$: blue, $\mathbb{H}^{0,2}$: black, Geogebra applet.

Definition 15. Let M be a pseudoriemannian hypersurface of a pseudoriemannian manifold $\tilde{M}$, and let N be a unit normal vector field. We define the shape operator $s : \Gamma(TM) \to \Gamma(TM)$ with respect to N by

$$\langle sX, Y\rangle = \langle \mathrm{I\!I}(X,Y), N\rangle, \text{ for } X, Y \in \Gamma(TM).$$

It can often by computed using the formula

$$sX = \tilde{\nabla}_X N.$$

The shape operator captures the second fundamental form of the hypersurface in the following sense:

$$\mathrm{I\!I}(X,Y) = \langle N, N\rangle\langle sX, Y\rangle N.$$

Analogously to Proposition 9, the shape operator can give us the curvature of a hypersurface:

Proposition 10. Let M be a pseudoriemannian hypersurface of a pseudoriemannian manifold M. Then for every $v, w \in T_pM$ that generate a nondegenerate plane, the sectional curvature of M is given by

$$K(v,w) = \tilde{K}(v,w) + \langle N,N\rangle \frac{\langle Sv,v\rangle\langle Sw,w\rangle - \langle Sv,w\rangle^2}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2},$$

where N is the unit normal vector field, and S is the shape operator with respect to N.

Hyperquadrics

We give the analogue of the sphere and the hyperbolic space for the pseudoriemannian case, i.e. the pseudoriemannian hypersurfaces of constant sectional curvature (see Example 4). The notation $\mathbb{S}^{p,q}, \mathbb{H}^{p,q}$ hints to the signature of the induced metric.

Definition 16. Let $0 \leq \nu \leq n$, with $n \geq 2$. We define the pseudosphere of radius $r > 0$ as

$$\mathbb{S}^n_\nu(r) = \mathbb{S}^{n-\nu,\nu}(r) = \left\{x \in \mathbb{R}^{n+1}_\nu = \mathbb{R}^{n+1-\nu,\nu} : \langle x,x\rangle = r^2\right\},$$

and the pseudohyperbolic space of radius $r > 0$ as

$$\mathbb{H}^n_\nu(r) = \mathbb{H}^{n-\nu,\nu}(r) = \left\{x \in \mathbb{R}^{n+1}_\nu = \mathbb{R}^{n-\nu,\nu+1} : \langle x,x\rangle = -r^2\right\}.$$

Example 4. The shape operator for the pseudospheres $\mathbb{S}^{p,q}$ and on the pseudohyperbolic spaces is given $Sv = -\frac{1}{r}v$. In particular, the pseudosphere $\mathbb{S}^{n-\nu,\nu}(r)$ is of constant sectional curvature $K = \frac{1}{r^2}$, and the pseudohyperbolic space $\mathbb{H}^{n-\nu,\nu}(r)$ is of constant sectional curvature $K = -\frac{1}{r^2}$.

As one can suspect, the pseudospheres and the pseudohyperbolics are related to each other:

Proposition 11. The pseudospheres are anti-isometric to the pseudohyperbolic spaces. More precisely, the map

$$\begin{aligned} \sigma : \mathbb{S}^n_\nu(r) = \mathbb{S}^{n-\nu,\nu}(r) &\to \mathbb{H}^n_{n-\nu}(r) = \mathbb{H}^{\nu,n-\nu}(r) \\ (x_1, \cdots x_\nu, x_{\nu+1}, \cdots x_{n+1}) &\mapsto (x_{\nu+1}, \cdots x_{n+1}, x_1, \cdots x_\nu) \end{aligned}$$

is an anti-isometry, i.e. a diffeomorphism that satisfies $\mathrm{d}_x\sigma(u), d_x\sigma(v) = -\langle u,v\rangle$ for all $u,v \in T_x\mathbb{S}^n_\nu(r)$.

We have also a simple diffeomorphic model for the pseudospheres and pseudohyperbolics:

Proposition 12. Each $\mathbb{S}^n_\nu(r) = \mathbb{S}^{n-\nu,\nu}(r)$ is diffeomorphic to $\mathbb{R}^\nu \times \mathbb{S}^{n-\nu}$, and each $\mathbb{H}^n_\nu(r) = \mathbb{H}^{n-\nu,\nu}(r)$ is diffeomorphic to $\mathbb{S}^\nu \times \mathbb{R}^{n-\nu}$.

The following proposition gives the geodesics of the pseudospheres and the pseudohyperbolics.

Proposition 13. Let $q(\cdot,\cdot)$ be the metric defining the pseudohyperbolic space $\mathbb{H}^{n-\nu,\nu}(r)$, $p \in \mathbb{H}^{n-\nu,\nu}(r)$, and Π be a plane passing from p and the origin. Then the intersection

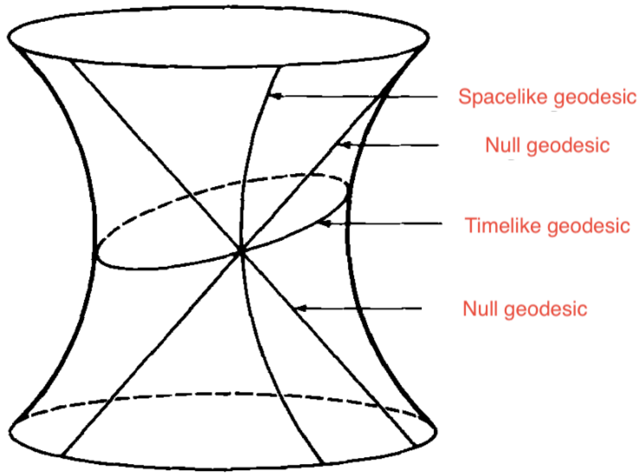


Figure 3.2: Geodesics in the $\mathbb{H}^{1,1} \subseteq R^{1,2}$

$\Pi \cap \mathbb{H}^{n-\nu,\nu}(r)$ is a geodesic in $\mathbb{H}^{n-\nu,\nu}(r)$, and we distinguish the following three cases:

(i) If $\operatorname{sgn}(q_{|\Pi}) = (0,0,2)$, then the intersection is an ellipse, and a timelike geodesic.

(ii) If $\operatorname{sgn}(q_{|\Pi}) = (0,1,1)$, then the intersection is a pair of parallel lines, and null geodesics.

(iii) If $\operatorname{sgn}(q_{|\Pi}) = (1,0,1)$, then the intersection is a hyperbola, whose branches are spacelike geodesics.

Any geodesic in $\mathbb{H}^{n-\nu,\nu}(r)$ is obtained as such, and in particular, the pseudohyperbolics are complete.

Proof. (i) Let e_1, e_2 be an orthonormal basis for q on Π, and consider the coordinates $y = (y_1, y_2)$ on $\Pi \cap \mathbb{H}^{n-\nu,\nu}$ such that $s = y^1(s)e_1 + y^2(s)e_2$ for $s \in \Pi \cap \mathbb{H}^{n-\nu,\nu}(r)$. Then

$$\begin{aligned}\Pi \cap \mathbb{H}^{n-\nu,\nu}(r) &= \left\{y \in \mathbb{R}^{n-\nu,\nu+1} : q(y,y) = -r^2\right\} \\ &= \left\{y^1 e_1 + y^2 e_2 \in \mathbb{R}^{n-\nu,\nu+1} : -(y^1)^2 - (y^2)^2 = -r^2\right\},\end{aligned}$$

which is clearly an ellipse. It can be parametrised by $\alpha(t) = r\cos(t)e_1 + r\sin(t)e_2$ for $t \in [0, 2\pi)$, for which $\alpha'(t) = -r\sin(t)\frac{\partial}{\partial y^1} + r\cos(t)\frac{\partial}{\partial y^2}$. Then $\langle \alpha'(t), \alpha'(t)\rangle = -r^2$, so α is timelike. Moreover, $\tilde{D}_t\alpha' = -P_{\alpha(t)}$ is the opposite of the position vector, so it is normal to $\mathbb{H}^{n-\nu,\nu}(r)$, and hence a geodesic.

□

3.2 Pseudoriemannian structure

3.2.1 Pseudoriemannian structure on $\mathbb{H}^{p,q}$

Since $\hat{\mathbb{H}}^{p,q}$ is the unit sphere in V with respect to the inner product chosen, it is a manfiold whose tangent space at $x \in \hat{\mathbb{H}}^{p,q}$ can be identified with the orthogonal complement of x in V:

$$T_x\hat{\mathbb{H}}^{p,q} = x^\perp = \{v \in V : \langle v, x\rangle = 0\}.$$

Since every $x \in \hat{\mathbb{H}}^{p,q}$ is a negative vector, the metric restricted on the tangent space is of signature (p,q), which makes $\hat{\mathbb{H}}^{p,q}$ a pseudoriemannian manifold of signature (p,q).

To define a pseudoriemannian structure on

Chapter 4

Linear algebra

4.1 Symplectic forms

Recall that for a bilinear form ω on a vector space V, we can define the matrix $M_B(\omega)$ of ω with respect to a basis $B = (v_1, \dots, v_n)$ of V by

$$M_B(\omega)_{ij} = \omega(v_i, v_j).$$

In that case, the form is given by

$$\omega\left(\sum_i x_i v_i, \sum_j y_j v_j\right) = \sum_{i,j} x_i y_j M_B(\omega)_{ij} = x^t M_B(\omega) y,$$

where the vectors x, y in the right hand side are represented by their coordinates in the basis B.

Definition 17. Let V be a complex or real vector space. A symplectic form $\omega : V \times V \to \mathbb{R}$ is a non-degenerate, skew-symmetric (i.e. $\omega(x,y) = -\omega(y,x)$) bilinear form. Equivalently, the associated matrix $M_B(\omega)$ is skew-symmetric and nonsingular.

Proposition 14. Every symplectic form on a finite-dimensional vector space V can be written as

$$\omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

with respect to some basis of V.

Proof. We proceed by induction on $\dim V$. If $\dim V = 2$, then we let e_1 be some non-zero vector and using non-degeneracy, we let e_2 be such that $\omega(e_1, e_2) = 1$. Then $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Supposing the statement holds for $\dim V = 2n$, we consider $\dim V = 2n+2$, and using the same arguements we can find $e_1, e_2 \in V$ such that $\omega(e_1, e_2) = 1$. In particular, $W = \operatorname{span}\{e_1, e_2\}$ is a non-degenerate subspace, so the same will be true for $W^\perp$. By the inductive hypothesis, we can find a basis of $W^\perp$ such that ω is given by

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then we can extend this basis to V by adding e_1, e_2, and after rearranging the order of the basis elements, we obtain the desired form. ∎

Definition 18. Let V be a symplectic vector space. A subspace $W \subseteq V$ is called *Lagrangian* if $W = W^{\perp}$, while a subspace $W \subseteq V$ is called *isotropic* if $W \subseteq W^{\perp}$.

Proposition 15. Let V be a symplectic vector space. Then:

(i) A subspace is lagrangian subspace if and only if it is maximally isotropic.

(ii) Every isotropic subspace is contained in a lagrangian subspace.

(iii) Symplectic vector spaces have even-dimension.

Proof.
1. Let W be a Lagrangian subspace and W' be an isotropic subspace containing W. If $W \neq W'$, then there exists $v \in W' \setminus W$. Then $v \in W^{\perp} = W$, a contradiction. Letting W be maximal isotropic, we have that $W \subseteq W^{\perp}$. But for every $v \in W^{\perp}$, we have that $\mathbb{C}v + W$ is isotropic (here we use skew-symmetry of ω to obtain that v is isotropic), so $v \in W$. That is $W = W^{\perp}$ and W is Lagrangian.

2. Let W' be isotropic and not Lagrangian. Then W is not maximal isotropic, that is, there exists $v \notin W$ such that $W + \mathbb{C}v$ is isotropic. Repeating this process, we obtain an increasing chain of isotropic subspaces containing W, which will terminate at a Lagrangian subspace.

3. Using the identity
$$\dim W + \dim W^{\perp} = \dim V$$
that holds for all subspaces W, we obtain that $2 \dim W = \dim V$ for any lagrangian subspace V.

∎

Proposition 16. Let ω, ω' be symplectic forms on V. Then there exists a subspace $W \subseteq V$ that is Lagrangian with respect to both forms.

Proof. Let $A = M_B(\omega') M_B(\omega)^{-1}$ with respect to some basis B of V. This is exactly the matrix for which
$$\omega'(v, w) = \omega(Av, w)$$
Using skew-symmetry, it is easy to see that $A^* = A$ with respect to ω' (where the $*$ denotes the adjoint with respect to ω). Then we can similarly show that it is also symmetric with respect to ω', that is:
$$\omega(Av, w) = \omega(v, Aw), \omega'(Av, w) = \omega'(v, Aw).$$

Consider the generalized eigenvalue decomposition with respect to A:
$$V = \bigoplus_{\lambda} V_{\lambda},$$
where $V_{\lambda} = \ker\left((A - \lambda I)^{k_{\lambda}}\right)$ and the sum is taken over all generalized eigenvalues of A. Moreover, the following lemma implies that the decomposition is orthogonal with respect to ω, ω'.

By counting dimensions, we see that if W_λ is a lagrangian subspace of V_λ, then $W = \bigoplus_\lambda W_\lambda$ is a lagrangian subspace of V. Hence it suffices to consider each V_λ separately. There, we may take $W_\lambda = \ker(A-\lambda I)^{k_\lambda - 1}$ and check that it is isotropic with respect to both forms. To do this, we proceed inductively on $\dim V_\lambda$. If $\dim V_\lambda = 2$, then every isotropic subspace is lagrangian, so we can take W_λ to be the span of any vector in V_λ. In particular, W_λ is lagrangian. If $\dim V_\lambda = 2n$, then the quotient space $W_\lambda^\perp / W_\lambda$ is symplectic and by the inductive hypothesis, there exist $v_1, \cdots, v_r \in W^\perp - W$ such that $W + \mathbb{C}v_1 \oplus \cdots \oplus W + \mathbb{C}v_r$ is lagrangian. Then $W \oplus \mathbb{C}v_1 + \cdots \oplus \mathbb{C}v_r$ is isotropic and by dimension counting, we see it is lagrangian as well. $\square$

Lemma 4. Let V be a vector space with bilinear form ω. Then every matrix A that is self-adjoint with respect to ω has orthogonal generalized eigenspaces.

Proof. Let $u \in V_\lambda, v \in V_\mu$ and consider polynomials $P(x) = (x-\lambda)^{k_\lambda}, Q(x) = (x-\mu)^{k_\mu}$. Then P, Q are prime with each other, so there exist polynomials U, V such that $UP + VQ = 1$. Then $\omega(u,v) = \omega((UP+VQ)(A)u, v) = \omega(VQ(A)u, v) = \omega(u, VQ(A)v) = 0$, where we use that since $P(A), Q(A)$ are polynomials of A, they are self-adjoint as well. $\square$

4.2 Proximal elements in $\mathrm{GL}(n, \mathbb{R})$

Here we will talk about basic definitions and dynamics of proximal elements in $\mathrm{GL}(n,\mathbb{R})$ and its subgroups.

Definition 19. We say that $g \in \mathrm{PGL}(d, \mathbb{R})$ is *proximal* (in $\mathbb{P}(\mathbb{R}^d)$) if it admits an attractive fixed point in $\mathbb{P}(\mathbb{R}^d)$, i.e. there exists a line $x_g^+ \in \mathbb{P}(\mathbb{R}^d)$ and a compact neighborhood $b^+ \subsetneq \mathbb{P}(\mathbb{R}^d)$ of $x + g^+$ such that $g^n x \to x$ as $n \to \infty$ uniformly for $x \in b^+$. We say that g is biproximal if both g and g^{-1} are proximal.

To better understand the dynamics of proximal elements, we will recall the Jordan decomposition of a matrix. Let $A \in \mathrm{GL}(d, \mathbb{R})$. Then A admits a Jordan canonical form $A = BJB^{-1}$, where $B \in \mathrm{GL}(d, \mathbb{R})$ and J is a block diagonal matrix with Jordan blocks $J_1, \ldots, J_k$:

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix},$$

where each J_i is either a single real entry j_i, or a (real) Jordan block of the form

$$J_i = \begin{pmatrix} C_i & 1 & & \\ & C_i & \ddots & \\ & & \ddots & 1 \\ & & & C_i \end{pmatrix},$$

with C_i being a 2×2 matrix of the form

$$C_i = \begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix},$$

and $b_i \neq 0$. In the latter case, we let j_i be one of the complex eigenvalues $j_i = \sqrt{\det C_i} e^{i \arccos(a_i)}$ of C_i.

Remark 4. Each C_i is merely a similitude (in case C_i is not a single entry) by $|j_i| = \sqrt{\det C_i}$ of a rotation by $\arccos a_i$, so J_i rotates and multiplies the plane corresponding to its first two collumns.

We call each j_i a *generalised eigenvalue* of A and the subspace E_i preserved by each J_i the *generalised eigenspace* of j_i. By changing B we may assume that $|j_1| \geq \ldots \geq |j_m|$.

We are now ready to describe the dynamics of a Jordan block, which is the same as the dynamics of A in the respective generalised eigenspace.

Lemma 5 (Dynamics of a Jordan block)**.** Let J_i be a Jordan block, and let j_i, E_i be its generalised eigenvalue and eigenspace respectively. We denote with $e_1, \cdots, e_k$ part of the standard basis that spans E_i.

(i) There exists vectors $v \in E_i$ such that

$$|J_i^n v| = |j_i|^n |v|.$$

When C_i is a real entry, this holds for $v \in \mathbb{R}v_1$, while when C_i is a 2×2 matrix, this holds for $v \in \mathbb{R}e_1 \oplus \mathbb{R}e_2$.

(ii) If J_i is upper-triangular but not a single entry, then E_i contains an actual eigenvector (namely e_1) such that for $v \in E_i$ we have $J_i^n \mathbb{R}v \to \mathbb{R}e_1$.

(iii) If J_i is not upper-triangular, then the generalized eigenspace E_i contains no eigenvalues, and there exists no attracting point in $\mathbb{P}(E_i)$.

(iv) If J_i is upper-triangular but not a single entry, then the convergence in the eigenspace E_i is not uniform.

(v) A Jordan matrix J is proximal if and only if $|j_1| > |j_2|$ and J_1 is a real entry.

Proof. While we will not give a concrete proof of this fact, we can consider examples.

(i) This follows by the fact that for v in the respective subspaces, J_i is a similitude by j_i and perhaps a rotation.

(ii) Suppose J is 2×2, so

$$J_i = \begin{pmatrix} a_i & 1 \\ 0 & a_i \end{pmatrix}.$$

Let $w \in E_2$, we can write $\mathbb{R}w = \mathbb{R}(w_1 e_1 + e_2)$. Inductively we show that

$$J_i^n \mathbb{R}w = \mathbb{R}((w_1 j + n)e_1 + je_2),$$

which clearly converges to $\mathbb{R}e_1$ as $n \to \infty$.

(iii) It is clear that $\mathbb{P}(E_i)$ contains no eigenvalues. To see that it contains no attracting fixed points, assume that J_i is 2×2. Then J_i acts on $\mathbb{R}e_1 \oplus \mathbb{R}e_2$ as a rotation and a similitude by, so any point line there does not converge.

(iv) Let

$$J = \begin{pmatrix} j & 1 \\ 0 & j \end{pmatrix} \text{ and } v_n = (n, -j).$$

Then

$$J^n = \begin{pmatrix} j & n \\ 0 & j \end{pmatrix} \text{ in } \mathrm{PGL}(2,\mathbb{R}), \text{ and } J^n \mathbb{R} v_n = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

(v) If $|j_1| > |j_2|$ and J_1 is a real entry, then for $w \in \mathbb{R}^d$, we can write $w = w_1 + \cdots + w_m$ with each $w_i \in E_i$. Then in the projective space, we have

$$J^n w = \frac{1}{|j_1|^n} J_1 w_1 + \cdots + \frac{1}{|j_1|^n} J_m w_m,$$

with

$$|\frac{1}{|j_1|^n} J^n w_i| = \left(\frac{j_i}{j_1}\right)^n |w_i| \to 0 \text{ for } i \geq 2 \text{ uniformly on compact sets.}$$

So $J^n w \to \mathbb{R}e_1$ uniformly on compact neighborhoods of $\mathbb{R}e_1$. If on the other hand $|j_1| = |j_2|$, then the same arguement shows that J^n does not converge for $w \in E_1 \oplus E_2$, while for $|j_1| > |j_2|$, not being a single entry means that either there is no attractive fixed point or that the convergence is not uniform.

□

We now consider the Jordan matrix (see Figure 4.1):

Proposition 17 (Dynamics of a Jordan matrix). Let J be a Jordan matrix with Jordan blocks $J_1, \cdots J_m$. Then J is proximal if and only if $|j_1| > |j_2|$ and J_1 is a real entry. In that case, for any $x \in \operatorname{span}\{e_2, \cdots, e_d\}^c$ we have

$$J^n x \to \mathbb{R}e_1,$$

and the convergence is uniform in compact neighborhoods of $\operatorname{span}\{e_2, \cdots, e_d\}^c$. Similarly, J is biproximal if and only if $|j_1| > |j_2|, |j_{m-1}| > |j_m|$ and J_1, J_m are real entries. In that case, for any $x \in \operatorname{span}\{e_1, \cdots, e_{d-1}\}^c$ we have

$$J^{-n} x \to \mathbb{R}e_d,$$

and the convergence is uniform in compact neighborhoods of $\operatorname{span}\{e_1, \cdots, e_{d-1}\}^c$.

Proof. Note that if $|j_1| > |j_2|$ and J_1 is a real entry, then for $w \notin \operatorname{span} e_2, \cdots, e_d$, we can write $w = w_1 + \cdots + w_m$ with each $w_i \in E_i$. Then

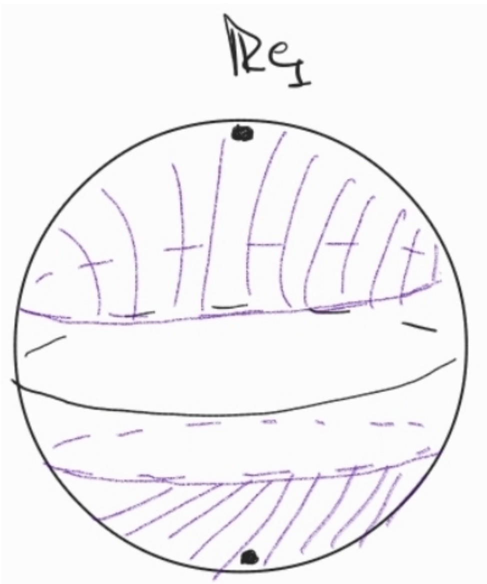
$$J^n \mathbb{R} w = \mathbb{R}(J_1^n w_1 + \cdots + J_m^n w_m) = \mathbb{R}(w_1 + \frac{J_2^n}{j_1^n} w_2 + \cdots + \frac{J_m^n}{j_1^n} w_m) \to \mathbb{R}e_1,$$

where the last convergence holds since each of the eigenvalues j_i are less than j_1.

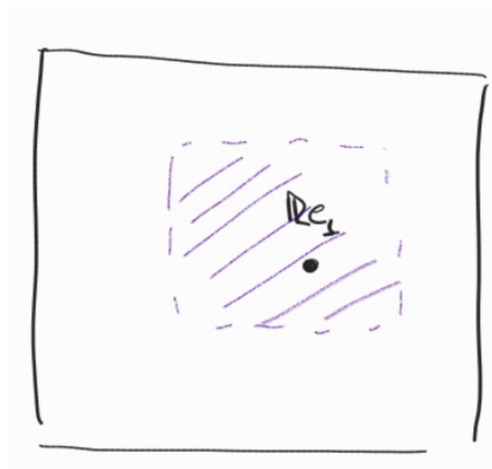
If on the other hand J is proximal, we have seen that in the remark above, that J_1 needs to be a single entry, otherwise the convergence is not uniform. On the other hand, if $|j_1| = |j_2|$, then J will rotate the plane spanned by e_2, e_3, like for instance when

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, v = (1, x, y), J^n v = \begin{pmatrix} 1 \\ \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix}$$

□



(a) In the double cover S^2 convergence is uniform when uniformly away from the equator.



(b) In the affine chart $\mathbb{R}^2 \simeq \{[1, x, y]\}$, convergence is uniform when uniformly away from the circle at infinity.

Figure 4.1: Dynamics of Jordan matrix on $\mathbb{P}(\mathbb{R}^3)$

Noting that being proximal is invariant under conjugation, we can now describe the dynamics of a proximal element in $\mathrm{GL}(d, \mathbb{R})$:

Corollary. Let $g \in \mathrm{PGL}(d, \mathbb{R})$ and denote with g any of its representatives. Then the following are equivalent:

(i) g is proximal if and only if g has a Jordan decomposition $g = BJB^{-1}$ with J being proximal.

(ii) Denoting with $\lambda_1(g), \cdots \lambda_d(g)$ the (possibly complex) eigenvalues of g in decreasing order of their modulus, g is proximal if and only if $|\lambda_1(g)| > |\lambda_2(g)|$.

and similarly for biproximal elements:

(i) g is biproximal if and only if g has a Jordan decomposition $g = BJB^{-1}$ with J being biproximal.

(ii) Denoting with $\lambda_1(g), \cdots \lambda_d(g)$ the (possibly complex) eigenvalues of g in decreasing order of their modulus, g is proximal if and only if $|\lambda_1(g)| > |\lambda_2(g)|$ and $|\lambda_{d-1}(g)| > |\lambda_d(g)|$.

If this is the case, the attracting fixed point of g is the line spanned by the eigenvector Be_1 and the convergence is uniform in compact neighborhoods of $\text{span}\{Be_2, \cdots, Be_d\}^c$.

Considering the case of $\text{PO}(p, q)$ for $p, q \geq 0$, we have that every proximal element is biproximal:

Proposition 18. In $\text{PO}(p, q)$, proximality and biproximality are equivalent.

Proof. Let $g \in \text{PO}(p, q)$ be proximal. Then the eigenvalues of g are stable under taking inverse: $\lambda(g) = \lambda(g^{-1})$. Indeed, $g \in \text{O}(p, q)$ implies that $g^t I_{p,q} g = I_{p,q}$, so for an eigenvector v of g with eigenvalue λ, we multiply $gv = \lambda v$ by $I_{p,q}$ to obtain that

$$\lambda I_{p,q} v = I_{p,q} g v = (g^t)^{-1} I_{p,q} v,$$

so $\sigma(g) \subseteq \sigma((g^t)^{-1}) = \sigma(g^{-1})$. □

4.3 Complexification of vector spaces

Definition 20. Let V be a real vector space and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of a real Lie algebra $\mathfrak{g}$.

(i) The complexification $V^{\mathbb{C}}$ of V is the complex vector space defined setwise by $V \times V$, with the addition structure of $V \oplus V$ and the scalar multiplication rule $(x + iy)(v_1, v_2) = (xv_1 - yv_2, xv_2 + yv_1)$. We write (v_1, v_2) as $v_1 + iv_2$.

(ii) If W is a complex vector space isomorphic to the complexification of V, then we say that V is a real form of W.

Recall that a map $f : V \to V$ on a complex vector space V is called *antilinear* if it it satisfies $f(u + v) = f(u) + f(v)$ and $f(\lambda v) = \overline{\lambda} f(v)$.

Definition 21. Let V be a complex vector space. A conjugation on V is an antilinear map $C : V \to V$ such that $C^2 = \text{id}$.

Example 5. If E is a real vector space and $V = E^{\mathbb{C}}$, then we can define the conjugation $C : V \to V$ by $C(v) = \overline{v}$, where $\overline{u + iv} = u - iv$.

Proposition 19. Let V be a complex vector space. Then real forms of V are in bijective correspondence with conjugations on V.

Proof. To each real form E of V, we can associate the conjugation $C(u + iv) = u - iv$, for $u, v \in E$. Conversely, for each conjugation C on V, we consider the real vector space $E = \{v \in V \mid C(v) = v\}$, which turns out to be a real form of V. □

Proposition 20. If $\mathfrak{E}$ is a representation of $\mathfrak{g}$ that is not of real type, then there exists an irreducible complex representation $V \subseteq E^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$, such that

$$E^{\mathbb{C}} = V \oplus \overline{V},$$

whre $\overline{V}$ is the conjugate representation of V, i.e. $\overline{V} = \{\overline{v} \mid v \in V\}$ and the action of $\mathfrak{g}$ on $\overline{V}$ is given by $\rho(X)(\overline{v}) = \overline{\rho(X)v}$.

Proof. Let V be a complex subspace of $E^{\mathbb{C}}$ that is invariant under $\mathfrak{g}^{\mathbb{C}}$. Then $\overline{V}$ is also invariant under $\mathfrak{g}^{\mathbb{C}}$, so the same is true for $V \cap \overline{V}$ and $V + \overline{V}$. Since these are invariant under the conjugation of $E^{\mathbb{C}}$, we have that $V + \overline{V}, V \cap \overline{V}$ admit real forms. Moreover, because the action of $\mathfrak{g}^{\mathbb{C}}$ commutes with conjugation,the real forms are invariant under the action of $\mathfrak{g}$. By irreducibility of E, we obtain that $V \oplus \overline{V} = E^{\mathbb{C}}$. $\square$

Chapter 5

Riemannian calculations

5.1 Sphere

5.1.1 Round metric on the sphere

Do the same for stereographic coordinates, see whether it is possible to obtain it in different coordinates directly through the euclidean metric, generalize to higher dimensions.

On the sphere of radious R, the round metric is defined as the metric induced by the Euclidean metric of $\mathbb{R}^{n+1}$. Recall that the spherical coordinates on $\mathbb{S}^n$ are given by

$$\begin{aligned} \phi^{-1} : (0, 2\pi) \times (0, \pi) &\to \mathbb{S}^2 \\ (\theta, \varphi) &\mapsto (R\cos\theta\sin\varphi, R\sin\theta\sin\varphi, R\cos\varphi). \end{aligned}$$

In these coordinates, the round metric is given by

$$g = R^2(\sin^2\varphi d\theta^2 + d\varphi^2).$$

Proof. We compute the frame given by the coordinate vector fields:

$$\begin{aligned} \frac{\partial}{\partial\theta} &= -R\sin\theta\sin\varphi\frac{\partial}{\partial r^1} + R\cos\theta\sin\varphi\frac{\partial}{\partial r^2} \\ \frac{\partial}{\partial\varphi} &= R\cos\theta\cos\varphi\frac{\partial}{\partial r^1} + R\sin\theta\cos\varphi\frac{\partial}{\partial r^2} - R\sin\varphi\frac{\partial}{\partial r^3}. \end{aligned}$$

Then we compute the coefficients using the Euclidean metric of $\mathbb{R}^3$:

$$\begin{aligned} g_{\theta\theta} &= \left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta} \right\rangle = R^2\sin^2\varphi \\ g_{\varphi\varphi} &= \left\langle \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi} \right\rangle = R^2 \\ g_{\theta\varphi} &= \left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi} \right\rangle = 0. \end{aligned}$$

□

5.1.2 Volume form on the sphere

We will compute the volume form of the round metric on $\mathbb{S}^2$ in spherical coordinates. We saw that in these coordinates, the round metric in matrix form is

$$g = R^2 \begin{pmatrix} \sin^2\varphi & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence the volume form is simply

$$\mathrm{Vol} = \sqrt{\det g} d\theta d\varphi = R^2 \sin\varphi \, \mathrm{d}\theta \wedge \mathrm{d}\varphi.$$

Volume entropy

5.1.3 Hyperbolic space

Coordinate representation for metric

We consider the hyperboloid model of the hyperbolic space $\mathbb{H}^n$:

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n,1} : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1, x_{n+1} > 0\},$$

with the metric induced by the Minkowski metric of $\mathbb{R}^{n,1}$.

Proposition 21 (Coordinates for projection chart)**.** Consider the chart that projects the hyperboloid onto the plane $x_3 = 0$:

$$\begin{aligned} \phi = (u, v) : \mathbb{H}^2 &\to \mathbb{R}^2 \\ (r^1, r^2, r^3) &\mapsto (r^1, r^2). \end{aligned}$$

In these coordinates, the hyperbolic metric is given by

$$g = \frac{(1+v^2)du^2 + (1+u^2)dv^2 - uvdudv}{1+u^2+v^2},$$

where $dudv$ is the symmetric product $\frac{1}{2}(du \otimes dv + dv \otimes du)$.

Proof. The parametrization we use is given by

$$\begin{aligned} \phi^{-1} : \mathbb{R}^2 &\to \mathbb{H}^2 \\ (u, v) &\mapsto (u, v, \sqrt{1+u^2+v^2}). \end{aligned}$$

We compute the frame given by the coordinate vector fields:

$$\begin{aligned} \frac{\partial}{\partial u} &= \frac{\partial}{\partial r^1} + \frac{u}{\sqrt{1+u^2+v^2}} \frac{\partial}{\partial r^3} \\ \frac{\partial}{\partial v} &= \frac{\partial}{\partial r^2} + \frac{v}{\sqrt{1+u^2+v^2}} \frac{\partial}{\partial r^3}. \end{aligned}$$

Then we compute the coefficients using the Minkowski metric of $\mathbb{R}^{2,1}$:

$$\begin{aligned} g_{uu} &= \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle = 1 - \frac{u^2}{1+u^2+v^2} = \frac{1+v^2}{1+u^2+v^2} \\ g_{vv} &= \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle = 1 - \frac{v^2}{1+u^2+v^2} = \frac{1+u^2}{1+u^2+v^2} \\ g_{uv} &= \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = -\frac{uv}{1+u^2+v^2}. \end{aligned}$$

Thus the metric is given by

$$g = \frac{(1+v^2)du^2 + (1+u^2)dv^2 - uvdudv}{1+u^2+v^2}.$$

□

Proposition 22 (Geodesic polar coordinates)**.** In geodesic polar coordinates for the hyperboloid model, the hyperbolic metric is given by

$$g = \mathrm{d}r^2 + \sinh^2(r)\,\mathrm{d}\theta^2.$$

Proof. Without loss of generality, we can assume that the point we are considering is $p = (0,0,1)$. First we calculate the exponential map at p:

$$\begin{aligned} \exp_p : T_p\mathbb{H}^2 &\to \mathbb{H}^2 \\ v_1\frac{\partial}{\partial r^1} + v_2\frac{\partial}{\partial r^2} &\mapsto \left(\sinh(\|v\|)\frac{v}{\|v\|}, \cosh(\|v\|)\right), \end{aligned}$$

where $\|v_1\partial_{r^1} + \cdots + v_3\partial_{r^3}\| = \sqrt{(v_1)^2+(v_2)^2+(v_3)^2}$ is the usual euclidean norm.

Then we identify the tangent space $T_p\mathbb{H}^2$ with $\mathbb{R}^2$, via $v_1\partial_{r^1} + v_2\partial_{r^2} \mapsto (v_1, v_2)$, and compose with the polar coordinates of $\mathbb{R}^2 - \{0\}$ so we obtain the commutative diagram

$$\begin{tikzcd} \mathbb{R}^2 \simeq T_p\mathbb{H}^2 \arrow[r, "\exp_p"] & \mathbb{H}^2 \\ \mathbb{R}_{>0} \times (0, 2\pi) \arrow[u, "\phi^{-1}"] \arrow[ur, "\phi^{-1}\circ\exp_p"'] & \end{tikzcd}$$

where ϕ is the chart of geodesic polar coordinates:

$$\begin{aligned} \phi^{-1} : \mathbb{R}_{>0} \times (0,2\pi) &\to \mathbb{H}^2 \\ (r,\theta) &\mapsto (\sinh(r)\cos(\theta), \sinh(r)\sin(\theta), \cosh(r)). \end{aligned}$$

Then we compute the frame given by the coordinate vector fields with respect to the ambient coordinates of $\mathbb{R}^{2,1}$:

$$\begin{aligned} \frac{\partial}{\partial r} &= \cosh(r)\cos(\theta)\frac{\partial}{\partial x^1} + \cosh(r)\sin(\theta)\frac{\partial}{\partial x^2} + \sinh(r)\frac{\partial}{\partial x^3} \\ \frac{\partial}{\partial \theta} &= -\sinh(r)\sin(\theta)\frac{\partial}{\partial x^1} + \sinh(r)\cos(\theta)\frac{\partial}{\partial x^2}. \end{aligned}$$

Then we compute the coefficients using the metric in the ambient coordinates given by the Minkowski metric of $\mathbb{R}^{2,1}$:

$$\begin{aligned} g_{rr} &= \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = 1 \\ g_{\theta\theta} &= \left\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right\rangle = \sinh^2(r) \\ g_{r\theta} &= \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\rangle = 0, \end{aligned}$$

and the result follows. □

We do the same for the hyperbolic n-space:

Proposition 23. Consider $\mathbb{H}^n$ in the hyperboloid model and let $p = e_{n+1}$. Then in geodesic polar coordinates centered at p, the hyperbolic metric is given by

$$g = \mathrm{d}r^2 + \sinh^2(r) g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ is the round metric on the sphere $\mathbb{S}^{n-1}$.

Proof. By Proposition 24 we have that the exponential map at p is given by

$$\begin{aligned}\exp_p : T_p\mathbb{H}^n &\to \mathbb{H}^n \\ v_1 \frac{\partial}{\partial r^1} + \cdots + v_n \frac{\partial}{\partial r^n} &\mapsto \left(\sinh(\|v\|)\frac{v}{\|v\|}, \cosh(\|v\|)\right),\end{aligned}$$

where $\|v_1 \partial_{r^1} + \cdots + v_n \partial_{r^n}\| = \sqrt{(v_1)^2 + \cdots + (v_n)^2}$ is the usual euclidean norm. We consider the polar coordinates on $\mathbb{R}^n$:

$$\begin{aligned}\mathbb{R}_{>0} \times \mathbb{S}^{n-1} &\to \mathbb{R}^n - \{0\} \\ (r, \theta) &\mapsto r\theta,\end{aligned}$$

and compose with the exponential map to obtain the chart of geodesic polar coordinates:

$$\begin{aligned}\phi : \mathbb{R}_{>0} \times \mathbb{S}^{n-1} &\to \mathbb{H}^n - \{p\} \\ (r, u) &\mapsto \exp_p(ru) = (\sinh(r)u, \cosh(r)).\end{aligned}$$

Then we compute the frame given by the coordinate vector fields with respect to the ambient coordinates of $\mathbb{R}^{n,1}$:

$$\begin{aligned}\frac{\partial}{\partial r} &= (\cosh(r)u, \sinh(r)) = \cosh(r)\sum_{i=1}^{n} u_i \frac{\partial}{\partial x^i} + \sinh(r)\frac{\partial}{\partial x^{n+1}} \\ \frac{\partial}{\partial \theta_i} &= \sinh(r)\frac{\partial}{\partial \theta^i} + 0\frac{\partial}{\partial x^{n+1}},\end{aligned}$$

where $\partial/\partial\theta^i$ are the coordinate vector fields of $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$. Then we compute the coefficients using the metric in the ambient coordinates given by the Minkowski metric of $\mathbb{R}^{n,1}$:

$$\begin{aligned}g_{rr} &= \langle(\cosh(r)u, \sinh(r))\rangle_{n,1} = \cosh^2(r)\langle u, u\rangle + \sinh^2(r) = \cosh^2(r) + \sinh^2(r) = 1, \\ g_{\theta_i\theta_j} &= \langle \sinh(r)\frac{\partial}{\partial\theta^i}, \sinh(r)\frac{\partial}{\partial\theta^j}\rangle_{n,1} = \langle \sinh(r)\frac{\partial}{\partial\theta^i}, \sinh(r)\frac{\partial}{\partial\theta^j}\rangle_{n,0} = \sinh^2(r)(g_{\mathbb{S}^{n-1}})_{ij}, \\ g_{r\theta_i} &= \langle(\cosh(r)u, \sinh(r)), \sinh(r)\frac{\partial}{\partial\theta^i}\rangle_{n,1} = 0,\end{aligned}$$

where the last equality holds since u is orthogonal to $T_u\mathbb{S}^{n-1}$. □

Note that in the above coordinates, r is the distance from the point p and θ is the angle with respect to the x^1-axis, i.e. the geodesic starting at p with initial velocity $r\left(\cos(\theta)\frac{\partial}{\partial x^1} + \sin(\theta)\frac{\partial}{\partial x^2}\right)$ is given by $t \mapsto \phi^{-1}(t, \theta)$.

Geodesics

Proposition 24 (Geodesics in the hyperboloid model of $\mathbb{H}^2$)**.** Consider $\mathbb{H}^2$ in the hyperboloid model. Then the geodesic starting at $p = (0, 0, 1)$ with initial velocity $v = v_1 \partial_{r^1} + v_2 \partial_{r^2}$ is given by

$$\gamma(t) = (\sinh(t\|v\|)\frac{v}{\|v\|}, \cosh(t\|v\|)).$$

for $t \in \mathbb{R}$, where $\|v\| = \sqrt{v_1^2 + v_2^2}$ is the euclidean norm. Similarly, for $p = (0, \cdots, 0, 1) \in \mathbb{H}^n$ and $v = v_1 \partial_{r^1} + \cdots + v_n \partial_{r^n}$, the geodesic starting at p with initial velocity v is given by

$$\gamma(t) = (\sinh(t\|v\|)\frac{v}{\|v\|}, \cosh(t\|v\|)).$$

for $t \in \mathbb{R}$, where $\|v\| = \sqrt{v_1^2 + \cdots + v_n^2}$ is the euclidean norm.

Lemma 6 (Geodesics in embedded submanifolds)**.** Let (M, g) be a embedded submanifold of $\mathbb{R}^n$ with the metric induced by the Euclidean metric of $\mathbb{R}^n$. Then $\gamma : I \to M$ is a geodesic if and only if $\gamma''(t)$ is orthogonal to $T_{\gamma(t)}M$ for all $t \in I$.

Show this for pseudoriemannian submanifolds

Proof. This is [Lee18, Corollary 5.2]. □

Volume entropy

Proposition 25. The volume entropy of the hyperbolic plane is equal to 1.

Proof. We use geodesic polar coordinates centered at a point $p = (0, 0, 1)$ in the hyperboloid model. Then the volume of a ball is given by

$$\mathrm{Vol}(B(p, R)) = \int_{r<R} \sinh(r) \, \mathrm{d}r \, \mathrm{d}\theta = 2\pi(\cosh(R) - 1) \sim \pi e^R.$$

□

We do the same for the hyperbolic n-space:

Proposition 26. The volume entropy of the hyperbolic n-space is equal to $n - 1$.

Proof. We use geodesic polar coordinates centered at a point $p = (0, \cdots, 0, 1)$ in the hyperboloid model. Then the volume of a ball is given by

$$\mathrm{Vol}(B(p, R)) = \int_{r<R} \sinh^{n-1}(r) \, \mathrm{d}r \, \mathrm{dVol}_{\mathbb{S}^{n-1}} \sim \mathrm{Vol}(\mathbb{S}^{n-1}) \frac{e^{(n-1)R}}{2^{n-1}(n-1)}.$$

Indeed, we have that $\sinh(r) \sim e^r/2$ for $r \to +\infty$, so $\sinh^{n-1}(r) \sim e^{(n-1)r}/2^{n-1}$. Thus, we obtain the result.

If we want to be more precise, we can compute the limit:

$$\lim_{R \to +\infty} \frac{1}{R} \log(\mathrm{Vol}(B(p, R)))$$

by applying l'Hôpital's rule. □

5.1.4 Projective space

Round metric in projective space:
We consider the projective space $\mathbb{P}(\mathbb{R}^{n+1})$ with the round metric, i.e. the metric induced by the round metric of the sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ through the double covering $\pi : \mathbb{S}^n \to \mathbb{P}(\mathbb{R}^{n+1})$. We use the affine chart $U_3 = \{[x : y : z] \in \mathbb{P}(\mathbb{R}^3) : z \neq 0\} \simeq \mathbb{R}^2$:

$$\begin{aligned} \phi^{-1} : \mathbb{R}^n &\to U_3 \subseteq \mathbb{P}(\mathbb{R}^3) \\ (u, v) &\mapsto [u : v : 1]. \end{aligned}$$

In these coordinates, the round metric is given by

$$g = \frac{(1 + v^2)du^2 + (1 + u^2)dv^2 - uvdudv}{(1 + u^2 + v^2)^2},$$

Proof. Note that for instance to compute g_{uu}, what we need is to compute the norm of $(\,\mathrm{d}\pi)^{-1}(\partial/\partial u)$ in the round metric of $\mathbb{S}^2$, since the metric on $\mathbb{P}(\mathbb{R}^3)$ is defined as the pull-back of the metric on $\mathbb{S}^2$ through π^{-1} (in a neighborhood where π is invertible). For this reason, we limit ourselves too the restriction $\pi^{-1} : U_3 \to \mathbb{U}_N$, where $\mathbb{U}_N$ is chart of the upper hemisphere. Then indeed,

$$g_{uu} = g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = \left\langle (\,\mathrm{d}\pi)^{-1}\left(\frac{\partial}{\partial u}\right), (\,\mathrm{d}\pi)^{-1}\left(\frac{\partial}{\partial u}\right)\right\rangle_{\mathbb{S}^2}.$$

Now we compute the images of the coordinate vector fields:

$$\begin{aligned} (\,\mathrm{d}\pi)^{-1}\left(\frac{\partial}{\partial u}\right) &= \frac{1}{(1 + u^2 + v^2)^{3/2}}\left((1 + v^2)\frac{\partial}{\partial X} - uv\frac{\partial}{\partial Y} - u\frac{\partial}{\partial Z}\right) \\ (\,\mathrm{d}\pi)^{-1}\left(\frac{\partial}{\partial v}\right) &= \frac{1}{(1 + u^2 + v^2)^{3/2}}\left(-uv\frac{\partial}{\partial X} + (1 + u^2)\frac{\partial}{\partial Y} - v\frac{\partial}{\partial Z}\right), \end{aligned}$$

where $(X, Y, Z) : U_N \to \mathbb{B}^2$ is the chart of the upper hemisphere. Then we compute the coefficients using the round metric of $\mathbb{S}^2$, which is no other than the Euclidean metric of $\mathbb{R}^3$ restricted to $\mathbb{S}^2$:

$$\begin{aligned} g_{uu} &= \left\langle (\,\mathrm{d}\pi)^{-1}\left(\frac{\partial}{\partial u}\right), (\,\mathrm{d}\pi)^{-1}\left(\frac{\partial}{\partial u}\right)\right\rangle_{\mathbb{S}^2} = \frac{1 + v^2}{(1 + u^2 + v^2)^2} \\ g_{vv} &= \left\langle (\,\mathrm{d}\pi)^{-1}\left(\frac{\partial}{\partial v}\right), (\,\mathrm{d}\pi)^{-1}\left(\frac{\partial}{\partial v}\right)\right\rangle_{\mathbb{S}^2} = \frac{1 + u^2}{(1 + u^2 + v^2)^2} \\ g_{uv} &= \left\langle (\,\mathrm{d}\pi)^{-1}\left(\frac{\partial}{\partial u}\right), (\,\mathrm{d}\pi)^{-1}\left(\frac{\partial}{\partial v}\right)\right\rangle_{\mathbb{S}^2} = -\frac{uv}{(1 + u^2 + v^2)^2}. \end{aligned}$$

Thus the metric is given by

$$g = \frac{(1 + v^2)du^2 + (1 + u^2)dv^2 - uvdudv}{(1 + u^2 + v^2)^2}.$$

$\square$

Projective space: Fubini-Study, Hilbert, round/symmetric metrics

Chapter 6

Logbook

2025-02-19

- Consider the irreducible subrepresentation of the symmetric square of the standard representation of $\mathrm{SO}(p,1)$. It takes values in $\mathrm{SO}\left(\frac{p(p-1)}{2},p\right)$:

  $$\rho = \mathrm{Sym}^2 \mathrm{std} : \mathrm{SO}(p,1) \to \mathrm{SO}\left(\frac{p(p-1)}{2},p\right)$$

  Proof. Denoting with $\omega(u,v) = u_1v_1 + \cdots + u_pv_p - u_{p+1}v_{p+1}$ the form preserved by $\mathrm{SO}(p,1)$, we can use it to identify $\mathrm{Sym}^2(\mathbb{R}^{p+1})$ with the space of symmetric forms on $\mathbb{R}^{p+1}$. Then the fact that the standard representation preserves ω is equivalent to the symmetric square preserving ω as an element of $\mathrm{Sym}^2 \mathbb{R}^{p+1}$. Moreover, the symmetric square preserves $\mathrm{Sym}^2 \omega$, so it preserves the orthogonal complement of $\mathbb{R}\omega$ with respect to $\mathrm{Sym}^2 \omega$, which we denote with $\mathrm{Sym}^2_0 \mathbb{R}^{p+1}$. Using the Weyl character formula, one can show that $\rho : \mathrm{SO}(p,1) \to \mathrm{SO}(\mathrm{Sym}^2_0 \mathbb{R}^{p+1})$ is irreducible. Also, the signature of $\mathrm{Sym}^2 \omega$ on $\mathrm{Sym}^2 \mathbb{R}^{p+1}$ is $\left(\frac{p(p-1)}{2}+1,p\right)$, and ω is positive for $\mathrm{Sym}^2 \omega$. $\square$

- Let $\Gamma \subseteq \mathrm{SO}(p,1)$ be a uniform lattice. Then $\rho : \Gamma \to \mathrm{SL}(\mathrm{Sym}^2_0(\mathbb{R}^{p+1}))$ is P_1-Anosov.

  Proof. Consider the compact subgroup $K = \mathrm{SO}(p)$ of $\mathrm{SO}(p,1)$ (block diagonal matrices with a matrix of $\mathrm{SO}(p)$ for the first block and 1 for the second block). Then $\rho(K) \subseteq K'$ for some compact subgroup K' of $\mathrm{SL}(\mathrm{Sym}^2_0 \mathbb{R}^{p+1})$. For A and A' being the diagonal matrices of $\mathrm{SO}(p,1)$ and $\mathrm{SL}(\mathrm{Sym}^2_0 \mathbb{R}^{p+1})$ respectively, we have $\rho(A) \subseteq A'$. Writing the Cartan decomposition $g = ke^{\mu(g)}l$ of some $g \in \mathrm{SO}(p,1)$, we have that $\rho(g) = \rho(k)e^{\rho_*\mu(g)}\rho(l)$, so $\mu(\rho(g)) = \rho_*(\mu(g))$. Since $\rho : \mathrm{SO}(p,1) \to \mathrm{GL}(\mathrm{Sym}^2_0 \mathbb{R}^{p+1})$ is the irreducible representation of $\mathrm{SO}(p,1)$ whose complexification is the one having the highest weight $2L_1$, we know that $\rho_*(\mu(g))$ is the diagonal matrix whose diagonal entries are the weights of ρ evaluated at $\mu(g)$. In particular, the two highest weights are $2L_1$ and $L_1 + L_2$ (the latter being equal to L_1 for $g \in \mathrm{SO}(p,1)$). Hence $\alpha_{12}(\rho_*(\mu(g))) = L_1(\mu(g))$. Since $\mathrm{SO}(p,1) = \mathrm{Isom}(H^p)$ is of rank 1, there exists a constant C such that $L_1 = d(-\cdot o, o)\mathfrak{a}^*$, where $o = e^{p+1}$ is the basepoint of H^p for which $K = \mathrm{Stab}(o)$.

  On the other hand Γ is a uniform lattice of $\mathrm{Isom}(\mathbb{H}^p)$, so the action of Γ on $\mathbb{H}^p$ is cocompact and properly discontinuous. By the lemma of Milnor-Svarc, the map $\gamma \mapsto \gamma \cdot o$ is a quasi-isometry, meaning that $\mu(\rho(g)) = Cd(\gamma \cdot o, o) \geq C|\gamma| - c$ for some $C, c > 0$. $\square$

- Calculation of the limit map $\xi : \partial\mathbb{H}^p \to \mathbb{P}(\mathbb{R}^d)$. Here we use the following description of the algebra $\mathfrak{so}(p,1) = \{x \in \mathfrak{sl}(p+1,\mathbb{R}) : x^t M + Mx = 0\}$, where

$$M = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & I_{p-1} & \vdots \\ 1 & \cdots & 0 \end{pmatrix}.$$

In matrix form:

$$\mathfrak{so}(p,1) = \left\{ \begin{pmatrix} X_{11} & X_{12} & 0 \\ X_{21} & X_{22} & -X_{12}^t \\ 0 & -X_{21}^t & -X_{11}^t \end{pmatrix} : \begin{array}{c} X_{11} \in \mathbb{R}, X_{12} \in \mathbb{R}^{1\times(p-1)}, X_{21} \in \mathbb{R}^{(p-1)\times 1}, X_{22} \in \mathbb{R}^{(p-1)\times(p-1)}, \\ X_{22}^t = -X_{22} \end{array} \right\}.$$

Fixing $x_0 = [e_{p+1}] \in \partial\mathbb{H}^p_\infty$, we have

$$\mathfrak{p}_0 = \mathrm{St}_{\mathfrak{so}(p,1)}(\mathbb{R}e_{p+1}) = \left\{ \begin{pmatrix} -t & 0 & 0 \\ X_{21} & X_{22} & 0 \\ 0 & -X_{21}^t & t \end{pmatrix} : \begin{array}{c} t \in \mathbb{R}, X_{12} \in \mathbb{R}^{1\times(p-1)}, X_{21} \in \mathbb{R}^{(p-1)\times 1}, X_{22} \in \mathbb{R}^{(p-1)\times(p-1)}, \\ X_{22}^t = -X_{22} \end{array} \right\}.$$

2025-02-21

- $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to the group of orientation-preserving isometries of the hyperbolic plane:

$$\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{SO}_0(2,1)$$

In fact, the isomorphism is given by the symmetric square of the standard representation of $\mathrm{SL}(2,\mathbb{R})$.

Proof. Consider the symmetric square $\rho = \mathrm{Sym}^2 \mathrm{std} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{GL}(\mathrm{Sym}^2(\mathbb{R}))$ of the standard representation of $\mathrm{SL}(2,\mathbb{R})$. Since the standard representation preserves the determinant, the symmetric square preserves the form $\omega = -\mathrm{Sym}^2 \det$, which can be found to be of signature $(2,1)$. We have thus a representation $\rho : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}(2,1)$. Since $\mathrm{SO}(2,\mathbb{R})$ is connected, we have that $\rho(\mathrm{SL}(2,\mathbb{R})) \subseteq \mathrm{SO}_0(2,1)$. Moreover, looking at the action of $\mathrm{SL}(2,\mathbb{R})$ on the space of symmetric 2-tensors, we see that ρ factors through $\mathrm{PSL}(2,\mathbb{R})$. Since $\mathrm{PSL}(2,\mathbb{R})$ is simple, $\rho : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{SO}_0(2,1)$ is injective. This implies that it is surjective as well, because $\mathrm{SO}_0(2,1)$ is connected (injectivity implies that the image contains a neighborhood of the identity, which will generate all of $\mathrm{SO}_0(2,1)$). $\square$

2025-02-27

- Let $\zeta_0 = e_{p+1}^2 \in \mathrm{Sym}_0 \mathbb{R}^{p,1}$. Then P_0 stabilizes $\mathbb{R}\eta_0$ when acting through $\rho : \mathrm{SO}(p,1) \to \mathrm{SO}(\mathrm{Sym}_0 \mathbb{R}^{p,1})$.

Proof. Indeed, this is equivalent to the fact that $\mathfrak{p}_0 = \mathrm{Lie}(P_0)$ stabilizes $\mathbb{R}\eta$ when acting by ρ_*, which boils down to the calculation $\rho_*(x)\eta_0 = -2e_{p+1}^2 = -2\eta_0$. $\square$

- The limit map $\xi : \partial\Gamma \to \mathbb{P}(\mathrm{Sym}_0 \mathbb{R}^{p,1})$ is given by

$$\xi(\gamma \cdot P_0) = \rho(\gamma)\eta_0,$$

under the identification $\partial\Gamma \simeq \mathrm{SO}(p,1)/P_0$, obtained by the Milnor-Svarc lemma, and the image of the differential over $\mathbb{R}\eta_0$ is

$$T_{\mathbb{R}\eta_0}\xi(\partial\Gamma) = \mathrm{d}_{P_0}\xi(T_{P_0}\partial\Gamma) = \pi\left(\rho_*(\mathfrak{so}(p,1))\eta_0\right)$$

while in the general case we have for $\mathbb{R}\eta \in \xi(\partial\Gamma)$ that

$$T_{\mathbb{R}\eta}\xi(\partial\Gamma) = \left(\rho_*(\mathfrak{so}(p,1))\eta\right)/\mathbb{R}\eta.$$

Proof. Let $\eta \in \mathrm{Sym}_0\,\mathbb{R}^{p,1}$ and $h \in \mathrm{SO}(p,1)$ be such that $\xi(hP_0) = \mathbb{R}\eta$. We know that the limit map is ρ-equivariant, meaning that the diagram

$$\begin{tikzcd} \mathrm{SO}(p,1) \arrow[r, "\rho(-)\eta"] \arrow[d] & \mathrm{Sym}_0\mathbb{R}^{p,1} \arrow[d] \\ \mathrm{SO}(p,1)/P_0 \simeq \partial\Gamma \arrow[r, "\xi"] & \mathbb{P}(\mathrm{Sym}_0\,\mathbb{R}^{p,1}) \end{tikzcd}$$

commutes, where the left vertical arrow is the map $g \mapsto gh \cdot P_0$, the right vertical arrow is the projection and $\mathrm{SO}(p,1)$ acts on $\partial\Gamma \simeq \mathrm{SO}(p,1)/P_0$ by left multiplication. Differentiating the above diagram, we obtain the following commuting diagram:

$$\begin{tikzcd} \mathfrak{so}(p,1) \arrow[r, "\rho_*(-)\eta"] \arrow[d] & T_\eta\,\mathrm{Sym}_0\,\mathbb{R}^{p,1} \arrow[r, "\simeq"] \arrow[d] & \hom(\mathbb{R}\eta, \mathrm{Sym}_0\,\mathbb{R}^{p,1}) \arrow[d, "\pi"] \\ T_{h\cdot P_0}\partial\Gamma \arrow[r, "d_{h\cdot P_0}\xi"] & T_{\mathbb{R}\eta}\mathbb{P}(\mathrm{Sym}_0\,\mathbb{R}^{p,1}) \arrow[r, "\simeq"] & \hom(\mathbb{R}\eta, \mathrm{Sym}_0\,\mathbb{R}^{p,1}/\mathbb{R}\eta) \end{tikzcd}$$

where π is the projection. □

- $\mathrm{SO}(p,1)$ acts transitively on pairs of distinct elements of $\partial\Gamma$, i.e. for every pair of discrete maximal parabolic subgroups P, P' of $\mathrm{SO}(p,1)$, there exists $g \in \mathrm{SO}(p,1)$ such that $g \cdot P = P_0$ and $g \cdot P' = P_0^t$, where $P_0 = \mathrm{Stab}_{\mathrm{SO}(p,1)}\,\mathbb{R}e_{p+1}$ and $P_0^t = \mathrm{Stab}_{\mathrm{SO}(p,1)}\,\mathbb{R}e_1$. Moreover, $g \cdot P_0 = P_0^t$ for

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{p-1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{SO}(p,1).$$

Proof. Recall that the maximal parabolic subgroups of $\mathrm{SO}(p,1)$ are exactly the stabilizers of elements in the boundary $\partial_\infty\mathbb{H}^p$ of the symmetric space. In particular, for any maximal parabolic subgroup $P = \mathrm{Stab}_{\mathrm{SO}(p,1)}\,x$ with $x \in \partial_\infty\mathbb{H}^p$, we have that $\mathrm{SO}(p,1)$ acts on maximal parabolic subgroups by conjugation: $g \cdot P \overset{\mathrm{def}}{=} gPg^{-1} = \mathrm{Stab}_{\mathrm{SO}(p,1)}(gx)$. In the case of $P_0 = \mathrm{Stab}_{\mathrm{SO}(p,1)}\,\mathbb{R}e_{p+1}$ and $P_0^t = \mathrm{Stab}_{\mathrm{SO}(p,1)}\,\mathbb{R}e_1$, we have that $g \cdot e_{p+1} = e_1$ for the element $g \in \mathrm{SO}(p,1)$ in the statement, so $gP_0g^{-1} = P_0^t$.

Let now $P = \mathrm{Stab}_{\mathrm{SO}(p,1)}\,x$ and $P' = \mathrm{Stab}_{\mathrm{SO}(p,1)}\,x'$ be two distinct maximal parabolic subgroups. Then $\langle x, x'\rangle \neq 0$, since otherwise, $\mathbb{R}x \oplus \mathbb{R}x'$ would be a two-dimensional totally isotropic subspace, which is not possible for a form of signature $(p,1)$. Thus we may assume that $\langle x, x'\rangle = 1$, i.e. the restriction of the form on $\mathbb{R}x \oplus \mathbb{R}x'$ has the following matrix form:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular $\mathbb{R}x \oplus \mathbb{R}x'$ is non-degenerate, so there exists a transformation $g \in \mathrm{SO}(p,1)$ that sends x to e_{p+1} and x' to e_1. For such a transformation, we have $g \cdot P = P_0$ and $g \cdot P' = P_0^t$. $\square$

- Consider the section

$$\begin{aligned} \zeta : \partial\Gamma &\to \mathcal{F}_{1,q}(\mathrm{Sym}_0\,\mathbb{R}^{p,1}) \\ \zeta(x) &= \big(\xi(x), \pi^{-1}(T_{\xi(x)}\xi(\partial\Gamma))\xi(x)\big) \end{aligned}$$

where we recall that $\pi^{-1}(T_{\xi(x)}\xi(\partial\Gamma))\xi(x) \subseteq \hom(\xi(x), \mathrm{Sym}_0\,\mathbb{R}^{p,1})$. Then for any two points $x, y \in \partial\Gamma$, we have that $\zeta^q(x) \cap \zeta^q(y) \neq 0$.

2025-02-08

- For $x_0 = \xi(P_0) = \eta_0 = e_1^2$, we have that

$$\zeta^q(x_0) = \rho_*(\mathfrak{so}(p,1))\eta_0 = \mathbb{R}e_2e_{p+1} \oplus \cdots \oplus \mathbb{R}e_{p+1}^2,$$

and $\rho(g)\rho_*(\mathfrak{so}(p,1))\eta_0 = \mathbb{R}e_1e_2 \oplus \cdots \oplus \mathbb{R}e_1^2$. In particular

$$\zeta^q(x_0) \cap \zeta^q(gx_0) = \zeta^q(x_0) \cap \rho(g)\zeta(x_0) = 0.$$

2025-03-12

- Let G be a group and P be a subgroup of G. If P is self-normalizing (its normalizer is itself), then G/P is in bijective correspondence with the conjugacy classes of P:

$$\begin{aligned} G/P &\simeq \{gPg^{-1} \mid g \in G\} \\ gP &\mapsto gPg^{-1} \end{aligned}$$

Recall that the normalizer of P in G is the subgroup $N_G(P)$ of elements $g \in G$ such that $gPg^{-1} = P$.

- Let G be a semisimple Lie group, and P be a minimal parabolic subgroup of G. Equivalently, $P = \mathrm{Stab}_G(x_0)$ is the stabilizer of a point $x_0 \in \partial_\infty S$ in the boundary of the symmetric space $S = G/K$. We denote with $\mathfrak{p}$ the Lie algebra of P. Then we have the following identification:

$$\begin{array}{ccccccc} \partial_\infty S & \simeq & G/P & \simeq & \{\text{ Parabolic subgroups of } P\} & \simeq & \{\text{ Parabolic subalgebras of } \mathfrak{p}\} \\ gx_0 & \leftarrow\!\shortmid & gP & \mapsto & gPg^{-1} & \mapsto & \mathrm{Ad}_g(\mathfrak{p}) \end{array}$$

In the above,we implicitly use the fact that the minimal parabolic subgroups of G are the conjugacy classes of minimal parabolic subgroups of one fixed minimal parabolic subgroup P.

- Let $G \subseteq \mathrm{PO}_0(p,q)$ be a subgroup of the identity component of projective orthogonal group, and denote with $q : \mathrm{O}(p,q) \to \mathrm{PO}(p,q)$ the quotient map. If G does not preserve a proper subspace of $\mathbb{R}^{p,q}$, then $q^{-1}(G) \cap \mathrm{SO}_0(n,1)$ does not preserve one either.

- The connected components of $\mathrm{PO}(p,q)$ are:

$$\mathrm{PO}(p,q) = \begin{cases} P(\mathrm{SO}_0(p,q)) \sqcup P(\mathrm{O}^+(p,q)) \sqcup P(\mathrm{O}^{(1,-1)}(p,q)) \sqcup P(\mathrm{O}^{(-1,+1)}(p,q)) & \text{if } p,q \text{ even} \\ P(\mathrm{SO}_0(p,q)) \sqcup P(\mathrm{O}^+(p,q)) & \text{if } p \neq q \bmod Z_2 \\ P(\mathrm{SO}_0(p,q)) \sqcup P(\mathrm{O}^{(1,-1)}(p,q)) & \text{if } p,q \text{ uneven} \end{cases},$$

  where P denotes the identification via the action of $\{\pm \mathrm{Id}\}$.

- For a semisimple Lie algebra $\mathfrak{g}$, the sum of the positive root spaces is denoted with $\mathfrak{n}^+$:

$$\mathfrak{n}^+ = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha.$$

Definition 22. Let G be a semisimple Lie group and fix a Weyl chamber $\mathfrak{a}^+$. We define the minimal parabolic subgroup of G as

$$B^+ = Z_K(\mathfrak{a}) A N^+,$$

where $A = e^{\mathfrak{a}}, N^+ = e^{\mathfrak{n}^+}$ and $Z_K(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in K. Its opposite parabolic subgroup is defined similarly and denoted with B^-. Any subgroup of G containing (up to conjugation) B^+ is called a parabolic subgroup.

- The parabolic subgroups are exactly the stabilizers of (potentially partial) flags.

- In the case of $G = \mathrm{SL}(d, \mathbb{R})$, we have

$$\begin{aligned} Z_K(\mathfrak{a}) &= \{ \text{ diagonal matrices with entries } \pm 1 \text{ with an even number of ones } \} \\ A &= \{ \text{ diagonal matrices with positive entries and determinant } 1 \} \\ N^+ &= \{ \text{ upper triangular matrices with 1 on the diagonal } \}. \end{aligned}$$

  so that

$$B^+ = \{ \text{ upper triangular matrices with determinant } 1 \},$$

  which is also the stabilizer of the full standard flag $\mathbb{R}e_1 \subseteq \cdots \subseteq \mathbb{R}e_d$. In general, denoting the stabilizer of the flag $\mathbb{R}e_1 \subseteq \cdots \subseteq \mathbb{R}e_k$ as P_k, we have that all parabolic subgroups are conjugate to P_k for some k.

- Let $\Theta \subseteq \Pi$ be a subset of simple (positive) roots. We define

$$\mathfrak{a}_\Theta = \bigcap_{\alpha \notin \Theta} \ker \alpha.$$

  Then $P_k = Z_K(\mathfrak{a}_\Theta) A N^+$.

2025-03-13

- Let $\mathcal{S}$ be the set of maximal isotropic subspaces of $\mathbb{R}^{d,d}$, topologized via $\mathrm{O}(d)$. If d is even, then:

(i) For every $V, W \in \mathcal{S}$ in different connected components:

$$gV \cap W \neq 0, \text{ for all } g \in \mathrm{SO}(d,d).$$

(ii) For any $V_0 \in \mathcal{S}$, there exist infinitely many $g \in \mathrm{SO}(d,d)$ such that

$$gV_0 \cap V_0 = 0.$$

2025-03-14

- Let X be a topological space and Γ a group acting continuously on X. If Γ acts cocompactly on X, then there exists a compact set $K \subseteq X$ that covers X under the action of Γ. If moreover X is locally compact, then the converse holds as well.

  Proof. If Γ acts cocompactly on a locally compact X, we consider for each $x \in X$ a neighborhood U_x of x and a compact subset K_x such that $U_x \subseteq K_x$. Denoting with $q : X \to X/\Gamma$ the quotient map, we have that $q(U_x)$ is an open covering of X (here we use that quotients of continuous group actions give rise to open quotient maps), so there exist $x_1, \cdots, x_n$ such that $X = \bigcup_{i=1}^n q(U_{x_i})$. Then $K = \bigcup_{i=1}^n K_{x_i}$ is a compact set that covers X under the action of Γ.

  If $\Gamma \cdot K = X$ for some compact set K, and $\{U_i\}$ is an open covering of X/Γ, then $q^{-1}(U_i)$ is an open covering of K, so it admits a finite subcovering $q^{-1}(U_{i_1}), \cdots, q^{-1}(U_{i_n})$. Then $U_{i_1}, \cdots, U_{i_n}$ is a finite subcovering of X/Γ. $\square$

- Let $\Gamma' \leq \Gamma$ be a subgroup of a group Γ. If Γ acts cocompactly on X, then Γ' acts cocompactly on X as well. Assuming moreover that Γ' is of finite index in Γ, then the converse holds as well. In particular, if $\Gamma \leq \mathrm{SO}(p,q)$ is $\mathbb{H}^{p,q}$ convex-cocompact, then any finite-index subgroup of Γ is as well.

  Proof. If Γ' acts cocompactly on X, then there exists some compact set $K \subseteq X$ that covers X under the action of Γ'. Clearly K also covers X under the action of Γ, so Γ acts cocompactly on X.

  If Γ acts cocompactly on X, then there exists a compact set $K \subseteq X$ that covers X under the action of Γ. Since Γ' is of finite index in Γ, there exists a finite set of elements $\gamma_1, \cdots, \gamma_n \in \Gamma$ such that $\Gamma = \Gamma\gamma_1 \sqcup \cdots \sqcup \Gamma\gamma_n$. Then $K' = \bigcup_{i=1}^n \gamma_i K$ is a compact set that covers X under the action of Γ'. $\square$

- The following is Corollary 1.3 from [GW12]:

  Let Γ be a discrete subgroup, Γ' be a finite-index subgroup of Γ, and $\rho : \Gamma \to G$ be a representation. Then ρ is P_1-Anosov if and only if the restriction of ρ to Γ' is P_1-Anosov.

  Proof. For details see Corollary 1.3 of [GW12]. The idea is that being Anosov is equivalent to admiting limit maps that are continuous and equivariant with certain properties. Since Γ' is of finite index, the limit sets of Γ and Γ' are the same (homeomorphic because the inclusion of Γ' into Γ is a quasi-isometry). Hence we can use the limit maps for both groups. $\square$

2025-03-15

- The realification of the standard representation of $\mathrm{SU}(1,1)$ is not irreducible.

  Proof. One can show that
  $$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\},$$
  and check that the real subspace
  $$V = \mathbb{R}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathbb{R}i \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
  is preserved by $\mathrm{SU}(1,1)$. $\square$

- $\mathrm{SU}(1,1)$ and $\mathrm{SL}(2,\mathbb{R})$ are conjugate in $\mathrm{GL}(2,\mathbb{C})$. In particular, they are isomorphic groups.

  Proof. We use the fact that $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SU}(1,1)$ are the subsets of $\mathrm{GL}(2,\mathbb{C})$ (which acts by Möbius transformations on $\mathbb{C}^2$) that preserve the upper half-plane and the unit disk respectively. By the Riemann mapping theorem, there exists a biholomorphism $f \in \mathrm{GL}(2,\mathbb{C})$ such that $f(\mathbb{D}^2) = \mathbb{H}^2$. Then for every $g \in \mathrm{SU}(1,1)$, we have $fgf^{-1} \in \mathrm{SL}(2,\mathbb{R})$, so conjugation by f gives an isomorphism $\mathrm{SU}(1,1) \simeq \mathrm{SL}(2,\mathbb{R})$. $\square$

- $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PO}_0(2,2)$ are isomorphic groups, and are the identity component of the isometry group of the Anti-de Sitter space AdS^3.
  $$\mathrm{PO}_0(2,2) \simeq \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}).$$

  Proof. Recall that the quadric $\mathcal{H}^{2,1}$ is given by
  $$\mathcal{H}^{2,1} = \{x \in \mathbb{R}^{2,2} : q_{2,2}(x,x) = x_1^2 + x_2^2 - x_3^2 - x_4^2 = -1\},$$
  and the isometry group of $\mathcal{H}^{2,1}$ is $(2,2)$. We begin by identifying the Minkowski space $\mathbb{R}^{2,2}$ with $(\mathcal{M}_2(\mathbb{R}), -det)$: We begin in the obvious way by identifying the vector $x = (x_1, x_2, x_3, x_4)$ with the matrix
  $$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix},$$
  To see that this is an isometry, we note that the determinant is a $(2,2)$-quadratic form given by
  $$det(X,Y) = x_1 y_4 - x_2 y_3 - x_3 y_2 + x_4 y_1.$$
  Hence we have an identification $\mathrm{O}(2,2) \simeq \mathrm{Isom}(\mathcal{H}^{2,1}, -\det)$. Notice that $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ acts isometrically on $\mathcal{H}^{2,1}$ by $(A,B) \cdot X = AXB^{-1}$, giving us a homomorphism $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Isom}(\mathcal{H}^{2,1}, -\det) \simeq \mathrm{O}(2,2) \to \mathrm{PO}_(2,2)$. Since it is a Lie group homomorphism from a connected Lie group, its image is equal to $\mathrm{PO}_0(2,2)$ On the other hand, using that $\mathrm{PO}(2,2)$ is centerless, we see that the kernel is equal to $\{(\pm\mathrm{Id}, \pm\mathrm{Id})\}$, which gives us the isomorphism
  $$\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}) \simeq \left(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\right)/\{(\pm\mathrm{Id}, \pm\mathrm{Id})\} \simeq \mathrm{PO}_0(2,2).$$
  For a proof that this is the isometry group of AdS^3, see Section 3.1 of [BS20]. $\square$

2025-03-18

- A quaternionic vector is a complex vector space V equipped with a quaternionic structure, i.e. a conjugate-linear map $J : V \to V$ such that $J^2 = -\mathrm{Id}$. For instance, the quaternions $\mathbb{H}$ are a quaternionic vector space, with $J : \mathbb{H} \to \mathbb{H}$ given by $J(x) = jx$.
- We consider $\mathbb{H}^n = \mathbb{C} \oplus j\mathbb{C} = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n \oplus \mathbb{C}je_1 \oplus \cdots \oplus \mathbb{C}je_n$ as a complex vector space. Then multiplication by j gives a quaternionic structure on $\mathbb{H}^n$: $jz = \overline{z}j$ and satisfies $\overline{jz} = -jz$.

2025-03-19

- Recall that for $p, q \in \mathbb{H}^{n\times n} : (pq)^* = q^* p^*$.
- Quaternionic vector spaces:

> **Definition 23.** A quaternionic vector space is a set with an addition and a scalar multiplication to the right by quaternions. The linear maps $\hom_{\mathbb{H}}(V, W)$ between quaternionic vector spaces V, W are the maps that commute with multiplication by quaternions on the right.

In general, the space of linear maps between quaternionic vector spaces "acts on the left", just in the following two standard examples:

> **Example 6.** Let $V = \mathbb{H}$, where scalar multiplication is defined as the usual multiplication of quaternions. Then the space endhomorphisms is the space of left multiplication by quaternions: $\mathrm{End}_{\mathbb{H}}(\mathbb{H}) = \{\phi_q : q \in \mathbb{H}\}$, where $\phi_q(x) = qx$.

Note that in the above example, multiplication by quaternions to the right, is not an endomorhism, since $(vq)\lambda \neq (v\lambda)q$ for $q, \lambda \in \mathbb{H}$.

- Quaternionic hermitian forms:

> **Definition 24.** A quaternionic hermitian form on a quaternionic vector space V is an $\mathbb{H}$-conjugate symmetric map $K : V \times V \to \mathbb{H}$ that is $\mathbb{H}$-conjugate linear in the first variable and $\mathbb{H}$-linear in the second:
>
> $$K(v\lambda, w\mu) = \overline{\lambda}K(v, w)\mu, \quad K(w, v) = \overline{K(v, w)}.$$
>
> A quaternionic hermitian form is positive definite if $K(v, v) > 0$ for all $v \neq 0$. A quaternionic antihermitian form is an $\mathbb{H}$-cojugate antisymmetric map $K' : V \times V \to \mathbb{H}$ that is $\mathbb{H}$-conjugate linear in the first variable and $\mathbb{H}$-linear in the second:
>
> $$K'(v\lambda, w\mu) = \overline{\lambda}K'(v, w)\mu, \quad K'(w, v) = -\overline{K(v, w)}.$$
>
> We denote the space of quaternionic hermitian and antihermitian forms with $\mathrm{Herm}_{\mathbb{H}}(V)$ and $\mathrm{AntiHerm}_{\mathbb{H}}(V)$ respectively.

We can identify the spaces of quaternionic hermitian and antihermitian forms with subspaces of the quaternionic matrices, by fixing a base for the vector space and identifying it with the standard space $\mathbb{H}^n$.

Proposition 27. Let V be a quaternionic vector space with a basis $e_1, \cdots, e_n$. Then we have the following bijection:

$$\begin{aligned} \mathrm{Herm}_{\mathbb{H}}(V) &\simeq \left\{S \in \mathbb{H}^{n\times n} : S^* = -S\right\} \\ K &\mapsto \left(K(e_i, e_j)\right)_{ij} \\ K\left(\sum v_i e_i, \sum w_j e_j\right) = \sum_{i,j} \overline{v}_i S_{ij} w_j &\leftarrow\!\shortmid S \end{aligned}$$

and similarly for the antihermitian forms:

$$\begin{aligned} \mathrm{AntiHerm}_{\mathbb{H}}(V) &\simeq \left\{S \in \mathbb{H}^{n\times n} : S^* = -S\right\} \\ K &\mapsto \left(K(e_i, e_j)\right)_{ij} \\ K\left(\sum v_i e_i, \sum w_j e_j\right) = \sum_{i,j} \overline{v}_i S_{ij} w_j &\leftarrow\!\shortmid S \end{aligned}$$

In both cases, identifying V with $\mathbb{H}^n$ via the basis $e_1, \cdots, e_n$, we have that the sum on the left hand side is equal to $v^* S w$.

- From the above proposition it is clear that the space of quaternionic hermitian and antihermitian forms is a real vector space and not a quaternionic one.
- Hermitian $\neq$ antihermitian for quaternions: While for the field of complex numbers the hermitian forms and antihermitian forms are essentially the same (multiplying a hermitian matrix by i gives an antihermitian one), this is not the case for the quaternions. This is because the imaginary part of the complex numbers is one-dimensional, while the imaginary part of the quaternions is three-dimensional. In particular:

$$\dim_{\mathbb{R}} \mathrm{Herm}_{\mathbb{H}}(V) = 4\frac{n(n-1)}{2} + n, \quad \dim_{\mathbb{R}} \mathrm{AntiHerm}_{\mathbb{H}}(V) = 4\frac{n(n-1)}{2} + 3n.$$

- We can define the action of $\mathrm{GL}_{\mathbb{H}}(V)$ on $\mathrm{Herm}_{\mathbb{H}}(V)$ and $\mathrm{AntiHerm}_{\mathbb{H}}(V)$ by

$$(g \cdot B)(v, w) = B(g^{-1}v, g^{-1}w),$$

and we call the automorphism group of K the stabilizer of K in $\mathrm{GL}_{\mathbb{H}}(V)$:

$$\begin{aligned} \mathrm{Aut}(K) &= \{g \in \mathrm{GL}_{\mathbb{H}}(V) : K(g^{-1}u, g^{-1}v) = K(u, v)\}, \\ \mathrm{Lie}(\mathrm{Aut}(K)) &= \{X \in \mathfrak{gl}_{\mathbb{H}}(V) : K(Xu, v) + K(u, Xv) = 0\}. \end{aligned}$$

Then $\mathrm{Aut}(K)$ acts on $\mathrm{Lie}(\mathrm{Aut}(K))$ by conjugation:

$$\mathrm{Ad} : \mathrm{Aut}(K) \to \mathrm{GL}(\mathrm{Lie}(\mathrm{Aut})(K)), \quad g \mapsto \mathrm{Ad}_g,$$

where $\mathrm{Ad}_g(X) = gXg^{-1}$.

- If we fix a nondegenerate quaternionic hermitian form K, then we can identify the antihermitian forms with the Lie algebra of the automorphism group of K. Note that the same works for the other fields $\mathbb{C}, \mathbb{R}$. This is how we obtain that the adjoint representation of $\mathrm{SU}(p, 1)$ is in fact its action on the space of antihermitian forms.

Proposition 28. Let K be a nondegenerate quaternionic hermitian form on a quaternionic vector space V. We have the following isomorphism of representations:

$$\begin{aligned} \mathrm{Lie}(\mathrm{Aut}(K)) &\simeq \mathrm{AntiHerm}_{\mathbb{H}}(V) \\ X &\mapsto K_X, \end{aligned}$$

where $K_X(u, v) = K(Xu, v)$, and whose inverse is the map that assigns to a quaternionic antihermitian form ω the unique $X \in \mathrm{GL}(\mathbb{H}^n)$ such that $K_X = \omega$.

2025-03-20

- K-conjugates:

Definition 25. Let V be a quaternionic vector space and K a nondegenerate quaternionic hermitian form on V. We define the conjugation with respect to K as the quaternionic isomorphism $\mathrm{GL}(\mathbb{H}^n) \to \mathrm{GL}(\mathbb{H}^n)$ given by $g \mapsto g^{*_K}$, where $K(gu, v) = K(u, g^{*_K} v)$.

- With the above definition, the Lie algebra of automorphism group of K is the space of anti-conjugate transformations, while the space of automorphisms is given as the transformations that are "unitary" with respect to K:

$$\begin{aligned} \mathrm{Lie}(\mathrm{Aut}(K)) &= \{X \in \mathrm{End}_{\mathbb{H}}(\mathbb{H}^n) : X^{*_K} = -X\} \\ \mathrm{Aut}(K) &= \{g \in \mathrm{GL}(V) : g^{*_K} = g^{-1}\}. \end{aligned}$$

  If we fix a basis $e_1, \cdots, e_n$ for V and use it to identify $\mathrm{End}_{\mathbb{H}}(V)$ with $\mathbb{H}^{n\times n}$, then the conjugation with respect to K is given by

$$\mathbb{H}^{n\times n} \ni g \mapsto S^{-1} g^* S \in \mathbb{H}^{n\times n},$$

  where $S \in \mathrm{GL}(\mathbb{H}^n)$ is the matrix corresponding to K, i.e. $K(u, v) = u^* S v$ for $u, v \in \mathbb{H}^n \simeq V$, and $*$ denotes the standard conjugate transpose for $\mathbb{H}^n$ and $\mathbb{H}^{n\times n}$.

- Proceeding as in Proposition 28, we can identify the space of anti-conjugate transformations with the space of hermitian forms:

Proposition 29. Let K be a nondegenerate quaternionic hermitian form on a quaternionic vector space V. We have the following isomorphism of representations:

$$\begin{aligned} \{X \in \mathrm{End}_{\mathbb{H}}(V) : X^{*_K} = X\} &\simeq \mathrm{Herm}_{\mathbb{H}}(V) \\ X &\mapsto K_X, \end{aligned}$$

where $K_X(u, v) = K(Xu, v)$, the action of $\mathrm{Aut}(K)$ on $\mathrm{End}_{\mathbb{H}}(V)$ is by conjugation, and on Hermitian forms by $g \cdot K(u, v) = K(g^{-1}u, g^{-1}v)$.

- Trace operator:

Definition 26. We define the trace of a quaternionic matrix $A \in \mathbb{H}^{n\times n}$ as the sum of its diagonal entries:

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

For other fields (notably $\mathbb{C}$ and $\mathbb{R}$), the trace satisfies the commutativity property $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. In particular, $\operatorname{tr}(PAP^{-1}) = \operatorname{tr}(A)$. Because of that, letting the trace of an operator of a complex vector space be the trace of its matrix in a basis, we have that it is well-defined in the sense that it is independent of the basis.

However, this is not true for quaternions, that satisfy

$$\operatorname{tr}(BA) = \overline{\operatorname{tr}(B^*A^*)}.$$

For this reason, for a quaternionic matrix $X \in \mathbb{H}^{n\times n}$, we consider its realification $X_{\mathbb{R}} \in \mathbb{R}^{4n\times 4n}$. If we denote its trace by $\operatorname{tr}_{\mathbb{R}}(X_{\mathbb{R}})$, then we have that

$$\operatorname{tr}_{\mathbb{R}} X_{\mathbb{R}} = 4\operatorname{Re}(\operatorname{tr}_{\mathbb{H}} X).$$

Denoting with $\Phi : \mathbb{H}^{n\times n} \to \mathbb{R}^{4n\times 4n}$ be the realification of a quaternionic matrix, we can see that it is a group morphism by the following commuting diagram.

$$\begin{tikzcd}
\mathbb{H}^{n\times n} \arrow[r, hook, "\Phi"] \arrow[d, "\simeq"'] & \mathbb{R}^{4n\times 4n} \arrow[d, "\simeq"] \\
\operatorname{End}_{\mathbb{H}}(\mathbb{H}^{n\times n}) \arrow[r, hook] & \operatorname{End}_{\mathbb{R}}\mathbb{R}^{4n}
\end{tikzcd}$$

In particular, we have that $(AB)_{\mathbb{R}} = A_{\mathbb{R}}B_{\mathbb{R}}$, $(PAP^{-1})_{\mathbb{R}} = P_{\mathbb{R}}A_{\mathbb{R}}P_{\mathbb{R}}^{-1}$. Hence the real part of the trace is well-defined, i.e. invariant under conjugation by $\mathrm{GL}(\mathbb{H}^n)$, and we may define the trace of a quaternionic transformation as the real part of its trace in any basis.

Definition 27. Let V be a quaternionic vector space and $X \in \operatorname{End}_{\mathbb{H}}(V)$ be a quaternionic transformation. The trace of a quaternionic transformation $X \in \operatorname{End}_{\mathbb{H}}(V)$ is the real part of the trace of its realification:

$$\operatorname{tr}(X) = \operatorname{Re}(\operatorname{tr}(X)) = \operatorname{Re}\left(\sum_i X_{ii}\right),$$

where X_{ij} are the entries of the matrix of X in a basis of V: $Xe_j = \sum_i X_{ij}e_i$.

2025-03-23

- Inner product on quaternionic transformations:

Definition 28. Let V be a quaternionic vector space and K be a nondegenerate quaternionic hermitian form on V. We define the inner product of two quaternionic transformations $X, Y \in \operatorname{End}_{\mathbb{H}}(V)$ as the real part of the trace of their product:

$$\langle X, Y\rangle = \operatorname{Re}\operatorname{tr}(X^{*_K}Y),$$

where the trace on the right hand side can be taken in any basis of V. It is symmetric, $\mathbb{R}$-linear and non-degenerate.

- Invariance under change of basis follows from the fact that the real part of the trace is invariant under conjugation by $\mathrm{GL}(\mathbb{H}^n)$: $\operatorname{Re}\operatorname{tr}((PX^{*_K}P^{-1})(PYP^{-1})) = \operatorname{Re}\operatorname{tr}(X^{*_K}Y)$.

- Symmetry follows from the fact that every quaternionic transformation K can be written as matrix multiplication with a quaternionic matrix S: $K(u,v) = u^*Sv$, where $S^* = S$. Then the matrix form of the K-conjugate is given by $X^{*_K} = S^{-1}X^*S$, so $\operatorname{Re}\operatorname{tr}(X^{*_K}Y) = \operatorname{Re}\operatorname{tr}(S^{-1}X^*SY) = \operatorname{Re}\operatorname{tr}(X^*SYS^{-1}) = \operatorname{Re}\overline{\operatorname{tr}(X^*SYS^{-1})} = \operatorname{Re}\operatorname{tr}((S^{-1})^*Y^*S^*X) = \operatorname{Re}\operatorname{tr}(S^{-1}Y^*SX) = \operatorname{Re}\operatorname{tr}(Y^{*_K}X)$.

- $\mathbb{R}$-linearity follows from the fact that for $\lambda \in \mathbb{H}$, we have $(\lambda X)^{*_K} = X^{*_K}\overline{\lambda}$ and that $\mathbb{H}$ is the center of $\mathrm{End}_{\mathbb{H}}(V)$.

- If K is a nondegerate quaternionic hermitian form, with corresponding matrix S, we have a commutative diagram of isomorphisms of representations:

$$\begin{array}{ccc}
\{T \in \mathrm{End}_{\mathbb{H}}(V) : T^{*_K} = T\} & \longrightarrow & \mathrm{Herm}_{\mathbb{H}}(V) \\
\downarrow & & \downarrow \\
\{X \in \mathbb{H}^{n\times n} : X^*S = SX\} & \longrightarrow & \{H \in \mathbb{H}^{n\times n} : H^* = H\}
\end{array}$$

- Let $K = K_{p,q}$ be the nondegerate form on $\mathbb{H}^n$ corresponding to the matrix $I_{p,q}$, i.e. $K_{p,q}(u,v) = \sum_{i=0}^{p} \overline{u}_i v_i - \sum_{i=p+1}^{p+q} \overline{u}_i v_i$. Then

$$\begin{aligned}
&\{T \in \mathrm{End}_{\mathbb{H}}(V) : T^{*_K} = T\} \\
&= \left\{ \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix} : A \in \mathbb{H}^{p\times p}, B \in \mathbb{H}^{p\times q}, D \in \mathbb{H}^{q\times q}, A^* = A, D^* = D \right\} \\
&= \mathfrak{k} \overset{\perp}{\oplus} \mathfrak{p},
\end{aligned}$$

  where $\mathfrak{k}$ and $\mathfrak{p}$ are given

$$\begin{aligned}
\mathfrak{k} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in \mathbb{H}^{p\times p}, D \in \mathbb{H}^{q\times q}, A^* = A, D^* = D \right\}, \\
\mathfrak{p} &= \left\{ \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} : B \in \mathbb{H}^{p\times q} \right\}.
\end{aligned}$$

- $\mathfrak{k}$ is positive definite, $\mathfrak{p}$ is negative definite and they are orthogonal to each other. Given that $\dim \mathfrak{k} = 4\left(\frac{p(p-1)}{2} + \frac{q(q-1)}{2} + p + q\right)$ and $\dim \mathfrak{p} = 4pq$, we have that

$$\mathrm{Sp}(p,q) = \mathrm{Aut}(K_{p,q}) \subseteq \mathrm{SO}\left(4\left(\frac{p(p-1)}{2} + \frac{q(q-1)}{2}\right) + p + q, 4pq\right).$$

- Under the isomorphism of representations from Chapter 6, $K_{p,q} \in \mathrm{Herm}_{\mathbb{H}}(\mathbb{H}^n)$ corresponds to the identity transformation $\mathrm{Id} \in \mathrm{End}_{\mathbb{H}}(\mathbb{H}^n)$ and the identity matrix $I_n \in \mathbb{H}^{n\times n}$. Clearly, the latter is invariant under the action of $\mathrm{Aut}(K_{p,q}) = \mathrm{Sp}(p,q) : g^{-1}I_n g = I_n$.

- Since $I_n \in \mathfrak{k}$, we have that $\langle I_n, I_n \rangle > 0$, so in fact $\mathrm{Sp}(p,q)$ preserves its orthogonal complement $I_n^\perp$, and:

$$\mathrm{Sp}(p,q) = \mathrm{Aut}(K_{p,q}) \subseteq \mathrm{SO}\left(4\left(\frac{p(p-1)}{2} + \frac{q(q-1)}{2}\right) - 1, 4pq\right).$$

- We have

$$\mathfrak{sp}(p,q) = \left\{ \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} : A \in \mathbb{H}^{p\times p}, B \in \mathbb{H}^{p\times q}, D \in \mathbb{H}^{q\times q}, A^* = -A, D^* = -D \right\},$$

where we use the form $K_{p,q}$ to define $\mathrm{Sp}(p,q)$.

2025-03-24

- Identification between $\mathcal{H}^n$ and $\mathbb{C}^{2n}$: Identifying $\mathcal{H} \simeq \mathbb{C} \oplus j\mathbb{C}$, we have the following identification between $\mathcal{H}^n$ and $\mathbb{C}^{2n}$:

$$\mathcal{H}^n \simeq \mathbb{C}^{2n}$$
$$\begin{pmatrix} a_1 + jb_1 \\ \vdots \\ a_n + jb_n \end{pmatrix} \mapsto \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

- Identification between $\mathcal{H}^{n\times n} \simeq \mathbb{C}^{2n\times 2n}$: Keeping up with the above definition, we have the following:

$$\mathcal{H}^{n\times n} \simeq \mathbb{C}^{2n\times 2n}$$
$$\begin{pmatrix} z_{11} + jw_{11} & \cdots & z_{1n} + jw_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} + jw_{n1} & \cdots & z_{nn} + jw_{nn} \end{pmatrix} \mapsto \begin{pmatrix} z_{11} & \cdots & z_{1n} & -\overline{w}_{11} & \cdots & -\overline{w}_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} & -\overline{w}_{n1} & \cdots & -\overline{w}_{nn} \\ w_{11} & \cdots & w_{1n} & \overline{z}_{11} & \cdots & \overline{z}_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} & \overline{z}_{n1} & \cdots & \overline{z}_{nn} \end{pmatrix}$$
$$Z + jW \mapsto \begin{pmatrix} Z & -\overline{W} \\ W & \overline{Z} \end{pmatrix}.$$

- Calculation of $\mathfrak{sp}(p,1)$, for $n = p+1$:

$$\begin{aligned}\mathfrak{sp}(p,1) &= \{X \in \mathfrak{gl}(n,\mathcal{H}) : -JX^*J = X\} \\ &== \left\{ \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & -X_{12}^* \\ X_{31} & -X_{21}^* & -\overline{X}_{11} \end{pmatrix} : \begin{array}{c} X_{11}, X_{13}, X_{31} \in \mathcal{H}, X_{12} \in \mathcal{H}^{1\times p}, X_{21} \in \mathcal{H}^{p\times 1}, X_{22} \in \mathcal{H}^{p\times p}, \\ X_{13} = -\overline{X}_{13}, X_{22} = -X_{22}^*, X_{31} = -\overline{X}_{31} \end{array} \right\}.\end{aligned}$$

- The Cartan involution is given by $\theta(X) = -X^*$, so

$$\begin{aligned}\mathfrak{p} &= \{X \in \mathfrak{sp}(p,1) : X^* = X\} = \\ &= \left\{ \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^* & 0 & -X_{12}^* \\ X_{13}^* & -X_{12} & -X_{11} \end{pmatrix} : \begin{array}{c} X_{11} \in \mathbb{R}, X_{13} \in \mathcal{H}, X_{12} \in \mathcal{H}^{1\times p}, \\ X_{13} = -\overline{X}_{13} \end{array} \right\}.\end{aligned}$$

- Cartan subalgebra of $\mathfrak{sp}(p,1)$:

$$\mathfrak{a} = \mathbb{R}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \simeq \mathbb{R}\mathrm{diag}(1,0,\cdots,0,-1,1,0,\cdots,0,-1),$$

where on the right hand side the we have a the corresponding complex matrix under the indentification $\mathcal{H}^{n\times n} \simeq \mathbb{C}^{2n\times 2n}$.

- From the above, it is clear that the first and second largest eigenvalue of the adjoint representation are equal, so the adjoint representation of $\mathfrak{sp}(p,1)$ is not proximal. Considering that the latter is isomorphic to the action of $\mathfrak{sp}(p,1)$ on the space of antihermitian forms $\mathrm{AntiHerm}_{\mathbb{H}}(\mathcal{H}^{p+1})$, we see that it would not give us a proximal representation of $\mathrm{Sp}(p,1)$. This is why we consider hermitian forms.

- Identifying $\mathrm{Herm}_{\mathcal{H}}(\mathcal{H}^{p+1})$ with the subset of $\mathcal{H}^{p+1\times p+1}$ given by $A^{*_K} = J^{-1}AJ = A$, we have that

$$\begin{aligned}&\mathrm{Herm}_{\mathcal{H}}(\mathcal{H}^{p+1}) \\ &= \left\{ \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{12}^* \\ A_{31} & A_{21}^* & \overline{A}_{11} \end{pmatrix} : \begin{array}{c} A_{11} \in \mathcal{H}, A_{13}, A_{31} \in \mathbb{R}, A_{12} \in \mathcal{H}^{1\times p}, A_{22} \in \mathcal{H}^{p\times p}, \\ A_{22} = A_{22}^* \end{array} \right\}.\end{aligned}$$

- Let $H_0 = \mathrm{diag}(1,0,-1) \in a$, $A \in \mathrm{Herm}_{\mathcal{H}}(\mathcal{H}^{p+1})$. Then

$$\rho_*(H_0) = [H_0, A] = \begin{pmatrix} 0 & A_{12} & 2A_{13} \\ -A_{21} & 0 & A_{12} \\ -2A_{31} & -A_{21}^* & 0 \end{pmatrix},$$

where $\rho : \mathrm{Sp}(p,1) \to \mathrm{GL}_{\mathcal{H}}(\mathcal{H}^{p+1})$. Hence the two largest eigenvalues of $\rho_*(H_0)$ are $2L_1, L_1$, where $L_1(H_0) = 1$.

2025-03-25

- Let Γ be a discrete subgroup of the isometry group of a symmetric space X. If Γ acts properly discontinuously on X, then the set of accumulation points of Γ lies in the boundary $\partial_\infty X$ of X (see Item (iii)).

- Another way of looking at the action of $\mathrm{Sp}(p,q)$ on $\mathrm{Herm}_{\mathcal{H}}(\mathcal{H}^{p+q})$. Fix a nondegenerate quaternionic hermitian form K, and denote with S the matrix that transforms the standard inner product on $\mathcal{H}^n$ to K, i.e. $K(u,v) = \langle Su, v\rangle_0$. Then the following to involutions of $\mathfrak{sl}(n,\mathcal{H})$ commute:

$$\theta(X) = -X^*, \quad \sigma(X) = X^{*_K} = S^{-1}X^*S.$$

Each of these two involutions gives a decomposition of the Lie algebra:

$$\mathfrak{sl}(n, \mathcal{H}) = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q},$$

where k, h are the eigenspaces of θ, σ with eigenvalue 1, and p, q are the eigenspaces with eigenvalue -1. By their definition, we can see that $\mathfrak{k}$ is the Lie algebra $\mathfrak{sp}(n)$ of the compact symplectic group . If K is of signature (p, q), then $\mathfrak{h}$ is the Lie algebra $\mathfrak{sp}(p, q)$, which can be identified with the Antihermitian quaternionic forms of $\mathcal{H}^{p+q}$. Similarly $\mathfrak{q}$ can be identified with the Hermitian quaternionic forms of $\mathcal{H}^{p+q}$. Since the two involutions commute, they preserve each others eigenspaces and we have the decomposition:

$$\mathfrak{sl}(n, \mathcal{H}) = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}).$$

Then the representation considered as the action of $\mathrm{Sp}(p, q)$ on Hermitian forms is the same as the restriction of the adjoint representation of $\mathrm{SL}(n, \mathcal{H})$ to $\mathrm{Sp}(p, q)$ and the Hermitian forms:

$$\rho = \mathrm{Ad}\,|_{\mathrm{Sp}(p,q)} : \mathrm{Sp}(p, q) \to \mathrm{GL}(\mathfrak{q}).$$

Now, we know that the adjoint representation of $\mathrm{Sp}(p, q)$ preserves the Killing form of $\mathfrak{sl}(n, \mathcal{H})$, which is nondegenerate and usually given by $(X, Y) \mapsto \mathrm{Re}\,\mathrm{tr}(XY)$. Thus it will preserve the inner product

$$\langle X, Y\rangle = -\mathrm{Re}\,\mathrm{tr}(XY), \quad X, Y \in \mathfrak{sl}(n, \mathcal{H}).$$

Since θ is a Cartan involution, $\mathfrak{k}$ is positive definite, $\mathfrak{p}$ is negative definite, and they are orthogonal to each other with respect to this product. The same properties carry on to ρ, being a restriction:

$$\rho : \mathrm{Sp}(p, q) \to \mathrm{SO}((\mathfrak{q} \cap \mathfrak{k}) \overset{\perp}{\oplus} (\mathfrak{q} \cap \mathfrak{p}), \langle\cdot, \cdot\rangle).$$

Following the definitions:

$$\mathfrak{k} \cap \mathfrak{q} = \{T \in \mathfrak{sl}(n, \mathcal{H}) : T^* = T = T^{*_K}\} \simeq \left\{ \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} : B \in \mathcal{H}^{p\times q} \right\}$$

$$\mathfrak{p} \cap \mathfrak{q} = \{T \in \mathfrak{sl}(n, \mathcal{H}) : T^* = -T, T = T^{*_K}\} \simeq \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : \begin{matrix} A \in \mathcal{H}^{p\times p}, D \in \mathcal{H}^{q\times q}, \\ A^* = A, D^* = D, \mathrm{tr}(A) = -\mathrm{tr}(D) \end{matrix} \right\}$$

We can now calculate the dimsensions and see that

$$\rho : \mathrm{Sp}(p, q) \to \mathrm{SO}\left(4pq, 4\left(\frac{p(p-1)}{2} + \frac{q(q-1)}{2}\right) + p + q - 1\right)$$

2025-03-26

- Consider $\mathrm{Sp}(p, 1) = \left\{g \in \mathrm{SL}(p + 1, \mathcal{H}) : g^{*_{I_{p,1}}} = g^{-1}\right\} \simeq \left\{ \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} : A^* = -A, D^* = -D \right\}$, where we use the identification of $\mathrm{SL}(p+1, \mathcal{H})$ as a subset of $\mathcal{H}^{p+1\times p+1}$ via the standard basis. Then we have that the maximal compact subgroup of $\mathrm{Sp}(p, 1)$ is

$$\begin{aligned} \mathrm{Sp}(p, 1) \cap \mathrm{Sp}(p + 1) &= \left\{g \in \mathrm{Sp}(p, 1) : g^* = g^{-1}\right\} \\ &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A^* = A^{-1}, |D|^2 = 1 \right\} \simeq \mathrm{Sp}(p) \times \mathrm{Sp}(1). \end{aligned}$$

$\mathrm{Sp}(p,1) \cap \mathrm{Sp}(p+1)$, which has Lie algebra:

$$\begin{aligned}\mathfrak{k} &= \mathfrak{sp}(p,1) \cap \mathfrak{sp}(p+1) \\ &= \{X \in \mathfrak{sp}(p,1) : X^* = -X\} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A^* = -A, D^* = -D \right\}.\end{aligned}$$

On the other hand, the tangent space of the symmetric space $\mathrm{Sp}(p,1)/K$ is identified with the Lie algebra

$$\mathfrak{p} = \{X \in \mathfrak{sp}(p,1) : X^* = X\} = \left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} : B \in \mathcal{H}^{p\times 1} \right\}.$$

- Using the form $I_{p,1}$, we can identify the space of quaternionic hermitian forms with a subspace of endomorphisms of $\mathcal{H}^{p+1}$, which can be again identified with a matrix subspace of $\mathcal{H}^{p+1\times p+1}$ using the standard basis:

$$\mathrm{Herm}_{\mathcal{H}}(\mathcal{H}^{p+1}) = \{X \in \mathfrak{gl}(p+1,\mathcal{H}) : X^{*_{I_{p,1}}} = X\} = \left\{ \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix} : A^* = A, D^* = D \right\},$$

which is equipped with the inner product $\langle X, Y\rangle = -\mathrm{Re}\,\mathrm{tr}(XY)$.

- In particular, we fix the following form, that satisfies $\langle x_0, x_0\rangle = -1$:

$$x_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- In the following, we will consider the pseudohyperbolic space

$$\mathbb{H}^{4p,2p(p-1)+p} = \left\{ l \in \mathbb{P}(\mathrm{Herm}_{\mathcal{H}}(\mathcal{H}^{p+1})) : \langle l, l\rangle < 0 \right\},$$

with double cover

$$\hat{\mathbb{H}}^{4p,2p(p-1)+p} = \left\{ X \in \mathrm{Herm}_{\mathcal{H}}(\mathcal{H}^{p+1}) : \langle X, X\rangle = -1 \right\}.$$

- We would like to find a map

$$f : \mathrm{Sp}(p,1)/K \to \mathbb{H}^{4p,2p(p-1)+p}$$

that is ρ-equivariant and it pulls back the metric of $\mathbb{H}^{4p,2p(p-1)+p}$ to a positive definite metric on $\mathrm{Sp}(p,1)/K$.

- It suffices for us to find a ρ-equivariant map $\tilde{f} : \mathrm{Sp}(p,1) \to \hat{\mathbb{H}}^{4p,2p(p-1)+p}$ to the double cover with the same properties.

- ρ-equivariance is equivalent to finding some $v \in \hat{\mathbb{H}}^{4p,2p(p-1)+p}$ that is invariant under the action of K through ρ, since then we can define $\tilde{f}(gK) = \rho(g)\cdot v$.

- This is the case for $v = x_0$, since:

$$\rho(k)\cdot x_0 = k x_0 k^{-1} = x_0, \text{ for all } k \in \mathrm{Sp}(p)\times \mathrm{Sp}(1),$$

so $\rho(k)\cdot \mathbb{R}x_0 = \mathbb{R}x_0$.

- We calculate the derivative of f over the tangent space $\mathrm{d}_K(\mathrm{Sp}(p,1)/K) \simeq \mathfrak{sp}(p,1)/\mathfrak{k} \simeq \mathfrak{p}$:

$$d_K f(X+\mathfrak{k}) = \rho_*(X)\cdot \mathbb{R}x_0 = \mathbb{R}[X, x_0] = \mathbb{R}\begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}, \text{ for } X = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \in \mathfrak{p}.$$

In particular $\langle d_K f(X+\mathfrak{k}), d_K f(X+\mathfrak{k})\rangle = 2\mathrm{Re}\,\mathrm{tr}(BB^*) > 0$.

2025-03-27

- Characterisation of properness.

> **Lemma 7.** Let $f : X \to Y$ be a continuous map between topological spaces. Then f is proper if and only if it maps sequences that escape to infinity to sequences that escape to infinity.

2025-03-31

- The adjoint action of $\mathrm{Sp}(p, q)$ on $\mathrm{Herm}^0_{\mathcal{H}}(\mathcal{H}^{p+q})$ is proximal and takes values in $\mathrm{SO}(p', q')$:

$$\rho : \mathrm{Sp}(p, q) \to \mathrm{SO}\left(4pq, 4\left(\frac{p(p-1)}{2} + \frac{q(q-1)}{2}\right) + p + q - 1\right)$$

2025-04-04

- There exist no groups of Beyrer-Kassel in $\mathrm{SL}(d, \mathbb{R})$

> **Lemma 8.** Let $\rho : \mathrm{SL}(d, \mathbb{R}) \to \mathrm{SO}(p, q)$ be irreducible with $p > q$. Then there exists no $\Gamma \leq \mathrm{SL}(d, \mathbb{R})$ discrete such that $\rho_{|\Gamma}$ is $\mathbb{H}^{p,q}$-convex cocompact of maximal dimension.

Proof. Assume for the sake of contradiction the contrary, and denote with V the vector space on which ρ acts irreducibly. Since ρ is irreducible, we know that it is a subrepresentation of

$$\bigotimes_{i=1}^{d-1} \mathrm{Sym}^{a_i} \bigwedge^i \mathrm{std} : \mathrm{SL}(d, \mathbb{R}) \to \mathrm{GL}\left(\bigotimes_{i=1}^{d-1} \mathrm{Sym}^{a_i} \bigwedge^i \mathbb{R}^d\right)$$

Since it is self-dual, we know that $a_i = a_{d-i}$ for $i = 1, \ldots, d-1$.

We denote with $\mathfrak{a}_0$ the Cartan subalgebra of $\mathfrak{sl}(d, \mathbb{R})$ consisting of traceless real diagonal matrices. Let $\mathcal{P}_+$ be the set of positive weights of ρ. Fix some i for which $a_i > 0$ and let $\mathcal{P}^i_+$ be the set of weights of $\mathrm{Sym}^{a_i} \bigwedge^i \mathbb{R}^d$.

The weights of $\mathrm{Sym}^{a_i} \bigwedge^i \mathbb{R}^d \otimes \mathrm{Sym}^{a_{d-i}} \bigwedge^{d-i} \mathbb{R}^d$ are the differences of the weights of $\mathrm{Sym}^{a_i} \bigwedge^i \mathbb{R}^d$. □

2025-04-05

- Real representations in $\mathfrak{so}(p, q)$ come from complex representations in $\mathfrak{so}(p+q, \mathbb{C})$.

> **Lemma 9.** Let $\mathfrak{g}_0$ be a real simple Lie algebra that complexifies to $\mathfrak{g}$ and $\rho_0 : \mathfrak{g}_0 \to \mathfrak{gl}(V_0)$ be an irreducible real representation of $\mathfrak{g}_0$. Then there exists an irreducible complex subrepresentation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of the complexification $\rho_0(\mathbb{C}) : \mathfrak{g} \to \mathfrak{gl}(V_0(\mathbb{C}))$ such that ρ_0 is either $\left(\rho_{|\mathfrak{g}_0}\right)_{\mathbb{R}} : \mathfrak{g}_0 \to \mathfrak{gl}(V_{\mathbb{R}})$, or a subrepresentation of $\left(\rho_{|\mathfrak{g}_0}\right)_{\mathbb{R}}$.

If moreover $\rho_0(\mathfrak{g}_0) \subseteq \mathfrak{so}(p,q)$, then $\rho(\mathfrak{g}) \subseteq \mathfrak{so}(p+q,\mathbb{C})$.

Proof. We know that either the complexification $(\rho_0)(\mathbb{C}) : \mathfrak{g} \to \mathfrak{gl}(V_0(\mathbb{C}))$ is irreducible, or there exists a complex structure $J : V_0 \to V_0$ on V_0, and an irreducible complex representation $\rho' : \mathfrak{g} \to \mathfrak{gl}(V_0, J)$ such that $\rho_0 = \left(\rho'_{|\mathfrak{g}_0}\right)_{\mathbb{R}}$. In the first case, letting $\rho = \rho_0(\mathbb{C})$, we have that ρ_0 is the subrepresentation of $(\rho_{|\mathfrak{g}_0})_{\mathbb{R}}$ in the real form V_0 of $V_0(\mathbb{C})$. Note that in this case, ρ is exactly the complexification of ρ_0. In the second case, we can take $\rho = \rho'$, and we have that ρ_0 is $(\rho_{|\mathfrak{g}_0})_{\mathbb{R}}$. Moreover, we have that $\rho^{\mathbb{C}} = ((\rho'_{|\mathfrak{g}_0})_{\mathbb{R}})^{\mathbb{C}} = \rho' + \overline{\rho}'$. In particular, $\rho'_{|\mathfrak{g}_0}$ is a subrepresentation of $\rho_0^{\mathbb{C}} : \mathfrak{g}_0 \to \mathfrak{gl}(V_0(\mathbb{C}))$, and hence ρ' is a subrepresentation of $\rho_0(\mathbb{C})$.

Assuming now that ρ_0 preserves a form ω, we can consider ω as a complex form on $V_0(\mathbb{C})$, and we have that the complexification $\rho_0(\mathbb{C})$ preserves this complex form. In particular, ρ will also preserve it, since it is a subrepresentation of $\rho_0(\mathbb{C})$. □

- Bound on positive weights

Lemma 10. Let $\mathfrak{g}_0$ be a real simple Lie algebra that complexifies to $\mathfrak{g}$. We denote with $\mathfrak{a}$ the Cartan subalgebra of $\mathfrak{g}$ and define

$$\mathfrak{a}_0 = \{H \in \mathfrak{a} : \alpha(H) \in \mathbb{R} \text{ for all } \alpha \in \Delta\}.$$

We assume that $\mathfrak{g}_0 \cap \mathfrak{a} \subseteq \mathfrak{a}_0$. We consider a connected component D of the set

$$\mathfrak{a} \cap \mathfrak{g}_0 - \bigcup_{\substack{\alpha \in \mathcal{P}(V) \\ \alpha_{|\mathfrak{a} \cap \mathfrak{g}_0} \neq 0}} \ker\alpha,$$

and consider the set of weights that are positive on D:

$$\mathcal{P}_{++} = \left\{\alpha \in \mathcal{P}(V) : \alpha_{|D} > 0\right\}.$$

Let $\rho : \mathfrak{g}_0 \to \mathfrak{so}(p,q) \subseteq \mathfrak{gl}(V_0)$ be an irreducible real representation of $\mathfrak{g}_0$. Then there exists a complex irreducible representation $\rho : \mathfrak{g} \to \mathfrak{so}(p+q,\mathbb{C}) \subseteq \mathfrak{gl}(V)$ such that ρ_0 is a (possibly nonstrict) subrepresentation of its realification $\left(\rho_{|\mathfrak{g}_0}\right)_{\mathbb{R}}$.
Then

$$p, q \geq \sharp\mathcal{P}_{++}$$

Proof. The existence of the representation $\rho : \mathfrak{g} \to \mathfrak{so}(p+q,\mathbb{C})$ follows from Lemma 9. For every positive weight $\alpha \in \mathcal{P}_{+}$, we denote with V_α the corresponding weight space. As in Lemma 9, we consider two cases for the representation ρ_0:

The first case is when the complexification $\rho = \rho_0(\mathbb{C}) : \mathfrak{g}_0 \to \mathfrak{gl}(V_0(\mathbb{C}))$ is irreducible. We denote with $\mathrm{Re}, \mathrm{Im} : V_0(\mathbb{C}) = V_0 \oplus iV_0 \to V_0$ the projections that satisfy $v = \mathrm{Re}v + i\mathrm{Im}v$ for $v \in V_0(\mathbb{C})$. Then we have that

$$\bigoplus_{\alpha \in \mathcal{P}_{++}} \mathrm{Re}V_\alpha$$

is a totally isotropic subspace of V_0.

We begin by showing

$$\mathrm{Re}V_\alpha \subseteq V_\alpha \cap V_0.$$

For this, it suffices that $\rho(H_0)v_\alpha = \langle \alpha, H_0\rangle v_\alpha$ for $H_0 \in \mathfrak{a}_0$, because $\mathfrak{a}_0$ is a real form of $\mathfrak{a}$. Let $H \in \mathfrak{a}_0$ and $v_\alpha \in V_\alpha$. Then $\mathrm{Re}v_\alpha \in V_\alpha$, since

$$\rho(H)\mathrm{Re}v_\alpha = \rho(H)\left(\frac{v_\alpha + \sigma v_\alpha}{2}\right) = \frac{1}{2}\left(\langle \alpha, H\rangle + \langle \overline{\alpha}, H\rangle\right) v_\alpha = \mathrm{Re}v_\alpha$$

On the other hand we have that $\mathrm{Im}V_\alpha = \mathrm{Re}V_\alpha$. Thus we have that $V_\alpha = (\mathrm{Re}V_\alpha)^{\mathbb{C}}$. In particular, since $\mathrm{Re}V_\alpha \subseteq V_\alpha$, and the weight spaces are linearly independent, the real parts $\mathrm{Re}V_\alpha$ are linearly independent as well and their real dimension is the same as the complex dimension of the weight spaces:

$$\dim_{\mathbb{R}} \bigoplus_{\alpha \in \mathcal{P}_{++}} \mathrm{Re}V_\alpha = \dim_{\mathbb{C}} \bigoplus_{\alpha \in \mathcal{P}_{++}} V_\alpha \geq \sharp\mathcal{P}_{++}.$$

To see that they are isotropic, consider To see that they are isotropic, take for instance $v_\alpha \in V_\alpha, v_\beta \in V_\beta$ be two weight vectors for $\alpha, \beta \in \mathcal{P}_{++}$. Fix some $H_0 \in D$, in which case $\langle \alpha, H_0\rangle, \langle \beta, H_0\rangle > 0$ and Then we have that

$$e^{\langle \alpha+\beta, H_0\rangle}\langle \mathrm{Re}v_\alpha, \mathrm{Re}v_\beta\rangle = \langle e^{\rho_0(H_0)}\mathrm{Re}v_\alpha, e^{\rho_0(H_0)}\mathrm{Re}v_\beta\rangle = \langle \mathrm{Re}v_\alpha, \mathrm{Re}v_\beta\rangle.$$

But $\langle \alpha + \beta, H_0\rangle > 0$, so $e^{\langle \alpha+\beta, H_0\rangle} > 1$, and $\langle \mathrm{Re}v_\alpha, \mathrm{Re}v_\beta\rangle = 0$.

The second case is when $\rho(\mathbb{C})$ is not irreducible, and we have that there exists a complex structure $J : V_0 \to V_0$ which commutes with ρ_0 and renders its extension $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V_0, J)$ to $\mathfrak{g}$ an irreudible complex representation. Denoting with $(V_0, J) = \oplus_\alpha V_\alpha$ the weight decomposition of ρ, we have that the (real) vector space

$$\bigoplus_{\alpha \in \mathcal{P}_{++}} (V_\alpha)_{\mathbb{R}}$$

is a totally isotropic subspace of V_0. Indeed, the same arguements as before show that it is isotropic. In that case, the bound on the dimension is given by

$$\dim_{\mathbb{R}} \bigoplus_{\alpha \in \mathcal{P}_{++}} (V_\alpha)_{\mathbb{R}} = 2\dim_{\mathbb{C}} \bigoplus_{\alpha \in \mathcal{P}_{++}} V_\alpha \geq 2\sharp\mathcal{P}_{++}.$$

□

2025-04-06

> **Lemma 11.** Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be an irreducible complex representation of a real simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{a} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Then for every weight vector $v_\alpha \in V_\alpha$, there exists some $H \in \mathfrak{a}$ such that $\langle \alpha, H\rangle > 0$.

- *Proof.* Let $\alpha \in \mathcal{P}(V)$ be a weight and v_α be a weight vector. Clearly there exists some $H \in \mathfrak{a}$ such that $\langle \alpha, H\rangle \neq 0$. Then there exists some $\theta \in [0, 2\pi)$ such that $e^{i\theta}\langle \alpha, H\rangle = \langle \alpha, e^{i\theta}H\rangle > 0$. □

2025-04-07

- The weights of a complex representation of a complex semisimple Lie algebra are real valued on the split Cartan subalgebra.

> **Lemma 12.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{a}$ be a Cartan subalgebra. We define the subalgebra $\mathfrak{a}_0$ as the set of elements of $\mathfrak{a}$ that will be real valued on the roots:
> $$\mathfrak{a}_0 = \{H \in \mathfrak{a} : \alpha(H) \in \mathbb{R} \text{ for all } \alpha \in \Delta\}.$$
> Then the weights of any complex representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ are real valued on $\mathfrak{a}_0$.

Proof. Denoting with B the Killing form of $\mathfrak{g}$, we define for each root α a corresponding element $H_\alpha \in \mathfrak{a}$ through the isomorphism $\mathfrak{a} \simeq \mathfrak{a}^*$ given by the Killing form (possible since B is nondegenerate on $\mathfrak{h} \times \mathfrak{h}$).

Let $X_\alpha, X_{-\alpha}$ be root vectors corresponding to α and $-\alpha$, scaled so that $B(X_\alpha, X_{-\alpha}) = 1$. Then we have that
$$H_\alpha = [X_\alpha, X_{-\alpha}].$$
Indeed, we have $B([X_\alpha, X_{-\alpha}], H) = -B(X_{-\alpha}, [X_\alpha, H]) = +B(X_{-\alpha}, \langle \alpha, H\rangle X_\alpha) = \langle \alpha, H\rangle$, for $H \in \mathfrak{a}$. Renormalizing
$$H'_\alpha = \frac{2}{\langle \alpha, H_\alpha\rangle} H_\alpha, \quad X'_\alpha = \frac{2}{\langle \alpha, H_\alpha\rangle} X_\alpha, \quad X_{-\alpha'} = X_{-\alpha},$$
we have that
$$\begin{aligned}
[H'_\alpha, X'_\alpha] &= 2X'_\alpha, & [h, e] &= 2e,\\
[H'_\alpha, X_{-\alpha'}] &= -2X_{-\alpha'}, & [h, f] &= -2f,\\
[X'_\alpha, X_{-\alpha'}] &= H'_\alpha, & [e, f] &= h.
\end{aligned}$$
Thus we have that the subalgebra $\mathfrak{sl}_\alpha(2, \mathbb{C}) = \mathbb{C}\{H_\alpha, X_\alpha, X_{-\alpha}\}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, via the map taking $H'_\alpha, X'_\alpha, X_{-\alpha'}$ to H, e, f. Restricting ρ to $\mathfrak{sl}_\alpha(2, \mathbb{C})$, we obtain a representation ρ_α of $\mathfrak{sl}(2, \mathbb{C})$, for which it is known that the eigenvalues of $\rho(H'_\alpha)$ are integers. In particular, we have that the eigenvalues of $\rho(H_\alpha)$ are real. Since for any weight $\beta \in \mathcal{P}$, with a weight vector $v_\beta \in V_\beta$, we have $\rho(H_\alpha)v_\beta = \langle \beta, H_\alpha\rangle v_\beta$, we have that $\langle \beta, H_\alpha\rangle$ is real. But
$$\mathfrak{a}_0 = \oplus_{\alpha \in \Delta} \mathbb{R} H_\alpha,$$
so β will be real valued on $\mathfrak{a}_0$. □

- Weight decomposition for real representations:

> **Lemma 13.** Let $\mathfrak{g}_0$ be a real simple Lie algebra that complexifies to $\mathfrak{g}$. We denote with $\mathfrak{a}$ the Cartan subalgebra of $\mathfrak{g}$ and define
> $$\mathfrak{a}_0 = \{H \in \mathfrak{a} : \alpha(H) \in \mathbb{R} \text{ for all } \alpha \in \Delta\}.$$
> We assume that $\mathfrak{g}_0 \cap \mathfrak{a} \subseteq \mathfrak{a}_0$.
>
> Let $\rho : \mathfrak{g}_0 \to \mathfrak{so}(p, q) \subseteq \mathfrak{gl}(V_0)$ be an irreducible real representation of $\mathfrak{g}_0$. Then there exists a complex irreducible representation $\rho : \mathfrak{g} \to \mathfrak{so}(p+q, \mathbb{C}) \subseteq \mathfrak{gl}(V)$ such that ρ_0 is a (possibly nonstrict) subrepresentation of its realification $\left(\rho_{|\mathfrak{g}_0}\right)_{\mathbb{R}}$. If we denote with $\mathcal{P}$ the weights of ρ and V_α its weight spaces, then we have the following weight

decomposition for ρ_0:

$$V_0 = \bigoplus_{\alpha \in \mathcal{P}} V_\alpha \cap V_0,$$

in the sense that for all $H \in \mathfrak{a} \cap \mathfrak{g}_0$, we have $\rho_0(H) = \langle \alpha, H \rangle \mathrm{Id}$ over $V_\alpha \cap V_0$.

Proof. We distinguish two cases. The first case is when the complexification $\rho = \rho_0(\mathbb{C}) : \mathfrak{g}_0 \to \mathfrak{gl}(V_0(\mathbb{C}))$ is irreducible. In Lemma 10 we saw that $\mathrm{Re} V_\alpha = V_\alpha \cap V_0$. In particular $\mathrm{Re} V_\alpha \subseteq V_\alpha$, so $\mathrm{Re} V_\alpha$ is a weight space of ρ_0. To see that V_0 is the sum of the $\mathrm{Re} V_\alpha$, we consider the weight decopmosition of some $v = \sum_\alpha c_\alpha v_\alpha \in V_0$ and apply the projection Re to get that $v = \sum_\alpha c_\alpha \mathrm{Re} v_\alpha$.

In the second case where $\rho(\mathbb{C})$ is not irreducible, we have that there exists a complex structure $J : V_0 \to V_0$ which commutes with ρ_0 and renders its extension $\rho : \mathfrak{g} \to \mathfrak{gl}(V_0, J)$ to $\mathfrak{g}$ an irreudible complex representation. Then the weight decomposition of ρ_0 is the same as the one of ρ. □

2025-04-10

- Spacelike cocompact groups

Definition 29. A representation $\rho : \Gamma \to \mathrm{SO}(p, q)$ of a discrete group Γ is spacelike cocompact if it is injective and $\rho(\Gamma)$ acts properly discontinuously and cocompactly on a surface-like submanifold $\tilde{M}$ of $\mathbb{H}^{p,q}$ with dimension $\dim \tilde{M} = p$. In that case $\Gamma = \pi_1(M)$, where $M = \tilde{M}/\rho(\Gamma)$.

- Criteria for $\mathbb{H}^{p,q}$-convex cocompactness

Lemma 14. Let $\rho : \Gamma \to \mathrm{SO}(p, q)$ be a spacelike cocompact representation of a discrete group Γ. The following are equivalent:

(i) Γ is Gromov hyperbolic.

(ii) ρ is $\mathbb{H}^{p,q}$-convex cocompact.

(iii) ρ is $\mathbb{H}^{p,q}$-convex cocompact of maximal dimension.

(iv) ρ is Anosov.

(v) $\partial_\infty \tilde{M}$ contains no segments.

In that case, if $q = 1$, then $\tilde{M}$ can be taken to be convex.

- Proper embeddings in $\mathbb{H}^{p,q}$ (see Seppi, Smith, Toulisse).

Lemma 15. Let $f : M^p \to \mathbb{H}^{p,q}$ be a spacelike immersion of a manifold M^p into $\mathbb{H}^{p,q}$, i.e. $f^* g_{p,q}$ is a Riemannian metric on M^p. If one of the following two conditions holds:

(i) $f^* g_{p,q}$ is complete

(ii) f is proper

then $M \simeq \mathbb{D}^p$ is diffeomorphic to a disk, and f is a proper embedding.

- Criterion for spacelike cocompact.

> **Lemma 16.** Let G be a real semisimple Lie group with maximal compact subgroup K and $\Gamma \leq G$ a discrete subgroup acting cocompactly on G/K, admitting an injective representaiton $\rho : \Gamma \to \mathrm{SO}(p,q)$. If there exists a ρ-equivariant spacelike immersion $f : G/K \to \mathbb{H}^{p,q}$, then ρ is spacelike cocompact.

Proof. Denote with $\tilde{M} = f(G/K)$ the image of G/K under f. By equivariance, f factors through the quotient $\Gamma\backslash G/K$, as in the following commutative diagram:

$$\begin{tikzcd} G/K \arrow[r, "f"] \arrow[d] & \tilde{M} \arrow[d] \\ \Gamma\backslash G/K \arrow[r, dashed] & M \end{tikzcd}$$

Since Γ acts cocompactly, $\Gamma\backslash G/K$ is compact, so f is proper. By the previous lemma, we have that f is a proper embedding, so $\tilde{M}$ is a spacelike submanifold. To see that $\rho(\Gamma)$ acts properly discontinuously, we argue by absurd and consider an injective sequence $\{g_n\} \subseteq \Gamma$ such that $\rho(\gamma_n)K \cap K \neq \emptyset$, for some $K \subseteq \tilde{M}$ compact. The set $K' = f^{-1}(K)$ is compact, because f is proper, and there exist $k'_n \in K'$ such that $f(k'_n) = k_n$. Then $f(\gamma_n k'_n) = \rho(\gamma_n)k_n \in K$, so the sequence γ_n satisfies also that $\gamma_n K' \cap K' \neq \emptyset$, so γ_n is not properly discontinuous, which is absurd. Finally, $M = f(\Gamma\backslash G/K)$ is compact, because $\Gamma\backslash G/K$ is compact. □

2025-05-06

> **Proposition 30.** Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a proximal irreducible representation of a discrete subgroup Γ of some semisimple Lie group G. Then ρ is P_1-Anosov if and only if Γ is Θ-Anosov with $\Theta = \{\alpha \in \Pi : \langle \alpha, \lambda \rangle \neq 0\}$.
>
> In particular, if ρ is P_1-Anosov, then Γ is P_i-Anosov for some i.

- *Proof.* Let $\lambda_1, \cdots, \lambda_m$ be the weights of ρ with λ_1 being the highest weight. Then for every $H \in \mathfrak{a}^+$, we have that $\lambda_1(H) \geq \lambda_i(H)$ for $i = 1, \cdots, m$. Fix some $v \in \mathfrak{a}^+$. Then we have that for $\alpha \in \Pi$

$$\lambda_1 - \alpha \text{ is a weight of } \rho \text{ if and only if } \langle \alpha, \lambda_1 \rangle \neq 0.$$

To see this, we let $s_\alpha \in \mathcal{W}$ be the reflection with respect to $\ker \alpha$. Then $s_\alpha(\lambda_1)$ is a weight of ρ and is among the extremal points on the set of weights of ρ. Moreover, there exists $k \geq 0$ such that $\lambda_1, \lambda_1 - \alpha, \cdots, \lambda_1 - k\alpha$ are all weights of ρ and $k = 0$ if and only if $\langle \alpha, \lambda_1 \rangle = 0$, i.e. λ lies on a wall of the Weyl chamber of α. Thus $\langle \alpha, \lambda_1 \rangle \neq 0$ if and only if $\lambda_1 - \alpha$ is a weight of ρ.

Before continuing, we note that we can pick the Cartan subalgebras $\mathfrak{a} \subseteq \mathfrak{g}$, $\mathfrak{h} \subseteq \mathfrak{sl}(d,\mathbb{R})$ such that $\rho_*(\mathfrak{a}^+) = \mathfrak{h}^+$, where $\mathfrak{g}$ is the Lie algebra of G. Then, using the Cartan decompositions of G and $\mathrm{SL}(d,\mathbb{R})$ we can show that

$$\mu(\rho(\gamma)) = \rho_*(\gamma)$$

Recall that ρ is P_1-Anosov if and only if there exist $C, c > 0$ such that for every $\gamma \in \Gamma$, $\alpha_{12}(\mu(\rho(\gamma))) \geq C|\gamma| - c$. However, we know that $\alpha_{12}(\mu(\rho(\gamma))) = \alpha_{12}(\rho_*(\mu(\gamma)))$ is the difference between the first two largest eigenvalues of $\rho_*(\mu(\gamma)) \in \mathfrak{h}^+$. Clearly, the first largest eigenvalue of $\rho_*(\mu(\gamma))$ is given by the highest weight $\lambda_1(\rho_*(\mu(\gamma)))$. Since the representation ρ is assumed to be proximal, the second largest eigenvalue is given by another weight of ρ. Hence we have that:

$$\begin{aligned}\alpha_{12}(\mu(\rho(\gamma))) &= \alpha_{12}(\rho_*(\mu(\gamma))) \\ &= \lambda_1(\mu(\rho(\gamma))) - \max_{\alpha\in\Theta}(\lambda_1 - \alpha)(\mu(\rho(\gamma))) \\ &= \min_{\alpha\in\Theta} \alpha(\mu(\rho(\gamma))),\end{aligned}$$

where $\Theta = \{\alpha \in \Pi : \langle \alpha, \lambda_1 \rangle \neq 0\}$. Here we make use of the fact that any other weight of ρ is obtained by substracting a positive integral combination of the simple roots, so the weight with the second largest value in $\mu(\rho(\gamma))$ is the one that is obtained by substracting a simple root $\alpha \in \Theta$.

Clearly, this quantity grows linearly with respect to the length of γ, if and only if $\alpha(\mu(\rho(\gamma)))$ does for all $\alpha \in \Theta$, which is equivalent to ρ being Θ-Anosov. □

2025-05-12

- Let $\mathfrak{g}$ be a real semisimple Lie algebra and fix a Cartan subalgebra $\mathfrak{a}$. Then for any two roots $\alpha, \beta \in \Delta$, we have that

$$n_{\alpha\beta} = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}.$$

Proof. Consider the subalgebra $\mathfrak{sl}_\alpha = \mathbb{C}\{H_\alpha, X_\alpha, X_{-\alpha}\} \simeq \mathfrak{sl}(2, \mathbb{C})$. Then the the subspace

$$W = \oplus_{k\in\mathbb{Z}} V_{\beta+k\alpha}$$

is a representation of $\mathfrak{sl}_\alpha$, where we consider $V = \mathfrak{g}$ as the adjoint representation of $\mathfrak{g}$ In particular, up to substituting β by some $\beta+k\alpha$, we have that $W = V_\beta \oplus \cdots \oplus V_{\beta+m\alpha}$, where m is a positive integer. But being a representation of $\mathfrak{sl}_\alpha$, we have that the eigenvalues of $\mathrm{Ad}(H_\alpha)$ is a string of integers that is symmetric arround zero. But these are exactly $\beta(H_\alpha), \beta(H_\alpha) + 2, \cdots, \beta(H_\alpha) + 2m$, so $2(\beta, \alpha)/(\alpha, \alpha) = \beta(H_\alpha) = -m \in \mathbb{Z}$. □

2025-05-13

- The irreducible representation of $2n + 1, \mathbb{C})$ with highest weight $2L_1$ is the kernel of the representation morphism:

$$\phi : \mathrm{Sym}^2(\mathbb{C}^{2n+1}) \to \mathbb{C}, \quad vw \mapsto \langle v, w \rangle$$

Proof. We have that $(\mathrm{Sym}^2(\mathbb{C}^{2n+1}))_{2L_1} = e_1^2$, which lies in the kernel of ϕ (here we do not use the standard form on $\mathbb{C}^{2n+1}$), so the kernel of ϕ contains the irreducible representation Γ_{2L_1} with highest weight $2L_1$. On the other hand, due to dimensional reasons, $\mathrm{Sym}^2(\mathbb{C}^{2n+1}) = \ker\phi \oplus \mathbb{C}$, where $\mathbb{C}$ is the trivial representation, and $\Gamma_{2L_1} \subseteq \ker\phi$. Since $\mathrm{Sym}^2(\mathbb{C}^{2n+1})$ has weights $0, \pm 2L_i, \pm L_i$ for $i = 1, \cdots, n$ and $2(\pm L_i \pm L_j)$ for $i \neq j$, with all but 0 being simple, we have that the kernel of ϕ is either Γ_{2L_1} or Γ_{2L_1} along with trivial factors. However, since $\mathrm{Sym}^2(\mathbb{C}^{2n+1}) \simeq \mathrm{Sym}^2((\mathbb{C}^{2n+1})^*)$, we have that invariant bilinear forms of the standard representation are equivalent to trivial summands of $\mathrm{Sym}^2(\mathbb{C}^{2n+1})$. Since every irreducible representation admits at most one (up to scalar multiplication) invariant bilinear form, we have that $\mathrm{Sym}^2(\mathbb{C}^{2n+1})$ contains exactly one trivial summand. Hence $\Gamma_{2L_1} = \ker\phi$. □

2025-06-05: Patterson-Sullivan in rank 1

Assumption 1. Throughout, X will be a complete, connected Riemannian manifold with sectional curvature bounded below by a negative constant.

Definition 30. We define the Busemann function $b : \overline{X} \times X \times X \to \mathbb{R}$ as follows:

$$b_z(x, y) = d(x, z) - d(y, z), b_\xi(x, y) = \lim_{X \ni z \to \xi} b_z(x, y).$$

For each $x \in X$, we define the Gromov product as seen from x by

$$(\xi|\eta)_x = \frac{1}{2}\left(b_\xi(x, y) + b_\eta(x, y)\right),$$

where $y \in X$ is a point on the geodesic joining ξ and η. We may now equip the boundary $\partial_\infty X$ with the distance $d_x : \partial X \times \partial X \to \mathbb{R}$ as seen from x:

$$d_x(\xi, \eta) = e^{-(\xi|\eta)_x}.$$

It turns out that for any two points $x, y \in X$, the respective distances on the boundary are conformally equivalent:

Lemma 17. Let $x \in X, \xi, \eta \in \partial_\infty X$ and $g \in \mathrm{Isom}(X)$.

(i) For $y \in X$:
$$d_y(\xi, \eta) = e^{\frac{1}{2}(b_\xi(x,y)+b_\eta(x,y))} d_x(\xi, \eta).$$

(ii) $d_{gx}(\xi, \eta) = e^{d(x,gx)} d_x(\xi, \eta)$.

Proof. (i) One can show that for $z \in X$ a point on the geodesic joining ξ and η, we have

$$\begin{aligned}(\xi|\eta)_y &= (\xi|\eta)_x + \frac{1}{2}\left(b_\xi(x, z) - b_\eta(x, z)\right) + \frac{1}{2}\left(b_\xi(y, z) - b_\eta(y, z)\right) \\ &= (\xi|\eta)_x + \frac{1}{2}\left(b_\xi(x, y) + b_\eta(x, y)\right),\end{aligned}$$

where we used that $b_\xi(y, z) - b_\xi(x, z) = -b_\xi(x, y)$.

(ii) First we show that $d_{gx}(\xi, \eta) \leq e^{d(x,gx)} d_x(\xi, \eta)$ by using the previous point and that $b_\xi(gx, x) \leq d(gx, x)$. Using this relation we establish the reverse inequality: $d_x(\xi, \eta) = d_{g^{-1}gx}(\xi, \eta) \leq e^{d(gx,g^{-1}gx)} d_{gx}(\xi, \eta) = e^{d(x,gx)} d_{gx}(\xi, \eta)$.

□

We begin by recalling the definition of a conformal density:

Definition 31. Let Γ be a nonelementary discrete group of isometries of X and $\beta \in \mathbb{R}$. A Γ-conformal density is a family of Radon measures $\{\nu_x\}_{x \in X}$ on ∂X such that

(i) It is Γ-equivariant, i.e. for all $g \in \Gamma$ and $x \in X$, we have that $\nu_{gx} = g_* \nu_x$.

(ii) For all $x, y \in X$, we have that ν_x, ν_y are equivalent with density

$$\frac{d\nu_y}{d\nu_x}(\xi) = e^{-\beta b_\xi(y,x)}.$$

Since the distances on the boundary are conformally equivalent, the corresponding Hausdorff measures $\mathcal{H}_x$ are equivalent as well. In particular, the Hausdorff dimension of ∂X is well-defined.

Lemma 18. Denote with $\mathcal{H}_x^s$ to be the s-dimensional Hausdorff measure on ∂X that is induced by the distance d_x, for $x \in X$:

$$H_x^s(A) = \lim_{\delta \to 0} \inf \left\{ \sum_i \operatorname{diam}(U_i)^s : A \subseteq \bigcup_i U_i, \operatorname{diam}(U_i) < \delta \right\}.$$

Then the Hausdorff measures $\mathcal{H}_x$ are equivalent under the action of $\operatorname{Isom}(X)$ by

$$\mathcal{H}_y^s(A) = e^{\frac{s}{2}(b_\xi(x,y) + b_\eta(x,y))} \mathcal{H}_x^s(A),$$

and in particular, for $g \in \operatorname{Isom}(X)$, we have that

$$\mathcal{H}_{gx}^s(A) = e^{sd(x,gx)} \mathcal{H}_x^s(A),$$

In particular, the Hausdorff dimension of ∂X is the same for every distance, and we will refer to it as the Hausdorff dimension of ∂X.

2025-06-20

- Let $\mathfrak{g}_0 = \mathfrak{su}(2,1)$ and $\mathfrak{h}_0 = \mathbb{R} \cdot \operatorname{diag}(1, 0, -1)$ be a Cartan subalgebra for $\mathfrak{g}_0$ and $\mathfrak{a}_0 = \{\operatorname{diag}(a, b, -a-b) : a, b \in \mathbb{R}\}$. Then the functionals of $\mathfrak{a}_0^*$ that vanish when restricted to $\mathfrak{h}_0$ are in duality with the orthogonal complement $\mathfrak{h}_0^\perp$ with respect to the Killing form, here given by

$$\mathfrak{h}_0^\perp = \{\operatorname{diag}(a, -2a, a) : a \in \mathbb{R}\} \simeq \mathbb{R}(\alpha_1 - \alpha_2) = \mathbb{R}(L_2 + \mathbb{R}(L_1 + L_2 + L_3)).$$

- The only spacelike cocompact representations of $\operatorname{SL}(d, \mathbb{R})$ with $p = \dim(G/K)$ are the symmetric cube

$$\rho : \operatorname{SL}(2, \mathbb{R}) \to \operatorname{Sym}^3(\mathbb{R}^2) \to \operatorname{SO}(3, 2),$$

and the symmetric square of the wedge square of the standard representation of $\operatorname{SL}(4, \mathbb{R})$, i.e.

$$\rho : \operatorname{SL}(4, \mathbb{R}) \to \operatorname{SO}(\operatorname{Sym}^2 \wedge^2 \mathbb{R}^4) \simeq \operatorname{SO}(12, 9).$$

Proof. To see this, recall that for $G = \operatorname{SL}(d, \mathbb{R})$, we have

$$\dim(G/K) = \frac{d(d+1)}{2} - 1.$$

We know that if $\rho : G \to \mathrm{SO}(p,q)$ gives rise to a spacelike cocompact representation, then the number of positive weights is at most $\min\{p,q\} = p$. Since in our case this is equal to $\dim(G/K)$, we need to show that $p > d(d+1)/2 - 1$. Indeed, the fundamental weights of self-dual representations of $\mathrm{SL}(d,\mathbb{R})$ are given by $\omega_i + \omega_{d-i}$ for $i < d/2$ and $\omega_{d/2}$ for d even. Assuming that $2 \leq i \neq d/2$, half of the orbit of $\Lambda = \omega_i + \omega_{d-i}$ under the action of the Weyl group $\mathcal{W}$ has cardinality

$$\frac{1}{2} \cdot \frac{d!}{i!(d-2i)!} \geq \frac{1}{2} \cdot \frac{d!}{2!(d-4)!} > \frac{d(d+1)}{2} - 1, \text{ for } d \geq 5.$$

On the other hand, if $i = d/2$, then half of the orbit of $\Lambda = \omega_{d/2}$ has cardinality

$$\frac{1}{2}(2k)!/(k!) > \frac{2k(2k+1)}{2} - 1, \text{ for } k \geq 3$$

Hence it suffices to check the case $d = 2,3,4,\ i = 1$ and $(i,d) = (1,2),(2,4)$. The case $i = 1, d \geq 3$ corresponds to the adjoint representation

$$\mathrm{Ad} : \mathrm{SL}(d,\mathbb{R}) \to \mathrm{SO}\left(\frac{d(d+1)}{2} - 1, \frac{d(d-1)}{2}\right),$$

which is not spacelike cocompact becase there exists no element of the Lie algebra $\mathfrak{sl}(d,\mathbb{R})$ that is commutes with all elements of $K = \mathrm{SO}(d)$. On the other hand, since it admits the zero weight, its symmetric powers have at least $\binom{d}{2}$ positive weights, which is larger than $\dim(G/K)$ for $d \geq 3$. This handles also the case of $d = 3$.

For the case $d = 4$, we note that $(i,d) = (2,4)$ corresponds to the isomorphism $\mathrm{SL}(4,\mathbb{R}) \to \mathrm{SO}(3,3)$, which is not spacelike cocompact. On the other hand, the symmetric square of the exterior square of the standard representation takes values in $\mathrm{SO}(\mathrm{Sym}^2(\wedge^2\mathbb{R}^4)) \simeq \mathrm{SO}(12,9)$, and all its weights are simple, so it is irreducible. To see that it is spacelike cocompact, we note that the wedge product of the standard representation $\wedge^2\mathbb{R}^4$ is a special isomorphism of $\mathrm{SL}(4,\mathbb{R})$ and $\mathrm{SO}(3,3)$. Thus the symmetric square $\wedge^2\mathbb{R}^4$ is the same representation as the symmetric square of the standard representation $\mathrm{Sym}^2(\mathbb{R}^{3,3})$ of $\mathrm{SO}(3,3)$. Since $\mathrm{Sym}^2(\mathbb{R}^6)$ is the isomorphic to the space of symmetric bilinear forms on $\mathbb{R}^6$, we know that the action of $\mathrm{SO}(3,3)$ is in fact the adjoint action of $\mathrm{SO}(3,3)$ on the space of symmetric bilinear forms on $\mathbb{R}^6$, which we have shown to be spacelike cocompact.

Finally, for the case $d = 2$, we have that $\mathrm{Sym}^2(\mathbb{R}^2)$ has signature $(2,1)$, so it is not spacelike cocompact. On the other hand, $\mathrm{Sym}^4(\mathbb{R}^2)$ has signature $(3,2)$, so it could be spacelike cocompact. Seeing this representation in another way, we have that $\mathrm{Sym}^3(\mathbb{R}^2) \simeq \mathrm{Sp}(4,\mathbb{R})$, but composing with the exceptional isomorphism $\mathrm{Sp}(4,\mathbb{R}) \to \mathrm{SO}(3,2)$, we obtain a representation $\rho : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}(3,2)$. This is a Hitchin representation (and thus maximal), which is known to be spacelike cocompact, since maximal representations are spacelike cocompact. $\square$

2025-07-02 - Classification of real subgroups of Lie group

Let G be a Lie group. Then the classification of simple subgroups of G is the same as classifying the representations $\rho : H \to G$ where H is a simple Lie group, in the sense that the following correspondence is <u>NOT</u> a bijection:

$$\{\text{Simple subgroups of } G\} \leftrightarrow \{\rho : H \to G : H \text{ simple}, \rho \text{ not trivial}\}$$
$$H \mapsto H \hookrightarrow G,$$
$$\rho(H) \leftarrow\!\shortmid \rho : H \to G.$$

Here, simplicity of G and nontriviality imply that ρ is injective, so $\rho(H) \simeq H$.

- LRL = Id.
- Right to left direction is surjective, since we can take the injections.
- RLR $\neq$ Id since $\rho \mapsto \rho(H) \mapsto i_{\rho(H)} \neq H$. We could try fixing this by considering a map

$$\{\text{Simple subgroups of } G\} / \simeq \leftrightarrow \{\rho : H \to G : H \text{ simple}, \rho \text{ not trivial}\} / \sim$$

  where $\rho : H \to G$ is equivalent to $\rho' : H' \to G$ if there exists an isomorphism $\phi : H \to H'$ such that $\rho' \simeq \rho \circ \phi$ as representations. But then, the left to right direction is not well defined, since for $\phi : H \to H'$ an isomorphism, we have that $\phi \circ i_H \not\simeq i_{H'}$ in general. On the other hand, if we consider a map

$$\{\text{Simple subgroups of } G\} \leftrightarrow \{\rho : H \to G : H \text{ simple}, \rho \text{ not trivial}\} / \sim$$

  it is not well defined either because for $\rho \sim \rho'$, we only have $\rho(H) \simeq H \simeq H' \simeq \rho(H')$ and not equality.

2025-08-07

- Irreducible representations of $\mathrm{PSL}(2, \mathbb{R})$:
  It is known that the irreducible representations of $\mathrm{SL}(2, \mathbb{R})$ are exactly the symmetric products of the standard representation. Seen differently, $\rho_d : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(\mathrm{Sym}^{d-1} \mathbb{R}^2)$ acts on the space of homogeneous polynomials of degree $d-1$ in two variables as

$$\rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) x^i y^j = (ax + cy)^i (by + dy)^j,$$

  for $i + j = d - 1$. Clearly this representation factors factors through a projective representation $\mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(\mathbb{R}^d)$, which is irreducible, since the projection $\mathrm{SL}(i, \mathbb{R}) \to \mathrm{PSL}(i, \mathbb{R})$ is a covering map and the differential is an irreducible representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$.

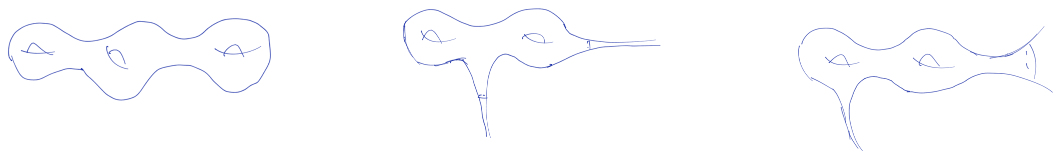
- Quasifuchsian representation
  Let Γ be a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Then a representation $\rho : \Gamma \to \mathrm{PSL}(d, \mathbb{R})$ is a quasifuchsian representation if it is the restriction of the irreducible representation $\mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(d, \mathbb{R})$ to Γ.

- Hitchin representations
  A representation of a hyperbolic surface group $\pi_1(S) \leq \mathrm{SL}(2, \mathbb{R})$ is Hitchin if it is in the same connected compoment of $\hom(\pi_1(S), \mathrm{SL}(2, \mathbb{R}))$ as the Fuchsian representation of $\pi_1(S)$.

- Discrete subgroups of $\mathrm{SL}(2, \mathbb{R})$
  Let $\Gamma \leq \mathrm{SL}(2, \mathbb{R})$ be a discrete torsion-free subgroup. Then we have the following cases (see Figure 6.1):

  (i) Γ is a uniform lattice: Then $\mathbb{H}^2/\Gamma$ is a closed hyperbolic surface.

  (ii) Γ is a nonuniform lattice: Then $\mathbb{H}^2/\Gamma$ is a noncompact hyperbolic surface, which contains cusps (volume stays finite).



(a) Quotient with uniform lattice

(b) Quotient with nonuniform lattice

(c) Quotient with nonlattice

Figure 6.1: Quotients with discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$

(iii) Γ is not a lattice: Then $\mathbb{H}^2/\Gamma$ is a noncompact hyperbolic surface, which contains funnels and possibly cusps.

- Irreducibility assumption for $\mathbb{H}^{p,q}$-convex-cocompact and spacelike cocompact representations

> **Lemma 19.** Let $\rho : \Gamma \to \mathrm{SO}(p,q)$. Under certain conditions on ρ, exactly one of the three occurs:
>
> (i) ρ is irreducible.
>
> (ii) ρ is reducible, with $p \neq q$.
> In this case, $\rho(G) \subseteq \mathrm{S}(\mathrm{O}(p,q') \times \mathrm{O}(q-q'))$, and the projection of ρ onto $\mathrm{SO}(p,q')$ satisfies some of the certain conditions.
>
> (iii) ρ is reducible, with $p = q$.
> In this case, $\rho(G)$ preserves two isotropic transverse isotropic subspaces of dimension p.

The conditions could include that ρ is $\mathbb{H}^{p,q}$-convex cocompact or spacelike cocompact.

- If $\rho : \Gamma \to \mathrm{SO}(p,q)$ is reducible, $\mathbb{H}^{p,q}$-convex cocompact, and preserves a non-degenerate subspace $V \subseteq \mathbb{R}^{p,q}$, then $\rho(\Gamma) \subseteq \mathrm{S}(\mathrm{O}(p,q') \times \mathrm{O}(q-q'))$.
The idea is to note that since ρ is $\mathbb{H}^{p,q}$-convex cocompact, it is Anosov, with limit set Λ_ρ being negative. Since the limit set Λ_ρ is contained in $\partial\mathbb{H}^{p,q}$, we have that $W = \mathrm{span}_{\mathbb{R}}(\pi^{-1}(\Lambda_\rho))$ is non-degenerate (since it is generated by negative lines), where $\pi : \mathbb{R}^{p,q} \to \mathbb{H}^{p,q}$. Thus (by semisimplicity of the representation) there exists a non-degenerate subspace $V \subseteq W$ such that $\rho(\Gamma)$ preserves V. Letting $q - q' = \dim W$, we have that $\mathrm{sgn}(W) = (0, q-q')$ and $\mathrm{sgn}(V) = (p, q-q')$, implying that $\rho(\Gamma) \subseteq \mathrm{S}(\mathrm{O}(p,q') \times \mathrm{O}(q-q'))$. Moreover, when we consider the projection onto V, the limit set does not change and it is negative, and it still satisfies the Anosov property, so it is still $\mathbb{H}^{p,q'-1}$-convex cocompact.

Not used non-degeneracy of preserved subspace

2025-07-10

- The span of the limit set for an Anosov representation is irreducible.

> **Lemma 20.** Let Γ be a non-elementary hyperbolic group and $\rho : \Gamma \to G$ be a Θ-Anosov representation into a linear group G. Then the span $\mathrm{span}(\Lambda_\rho)$ of the limit set Λ_ρ is an invariant subspace of $\rho(\Gamma)$, on which $\rho(\Gamma)$ acts irreducibly.

Proof. Let $\xi^+(x) \in \Lambda_\rho$ be a point in the limit set, where $x \in \partial_\infty\Gamma$. Then by the equivariance of the limit map, we have that $\rho(\gamma)\xi^+(x) = \xi^+(\gamma x) \in \Lambda_\rho$ for all $\gamma \in \Gamma$, so Λ_ρ is invariant under the action of $\rho(\Gamma)$.

Do we really need that $\mathrm{vcd}(\Gamma) = p$ and that ρ is $\mathbb{H}^{p,q}$-convex cocompact?

To see that $\rho(\Gamma)$ acts irreducibly on $V = \mathrm{span}(\Lambda_\rho)$, we assume the contrary for the sake of contradiction. Let $W \subsetneq V$ be an invariant subspace of $\rho(\Gamma)$. Consider some $\xi^+(\gamma^+) \in \Lambda_\rho$ for $x \in \partial_\infty\Gamma$, and let $\gamma^- \in \Gamma - \{\gamma^+\}$. Then there exists a geodesic γ_n such that $\lim_{n\to\pm\infty}\gamma_n = \gamma^\pm$. If $w \in W - \xi^-(\gamma^-)$, we have that

Maybe we just need (and should somehow use) that Λ_ρ is connected?

$$\rho(\gamma_n)w \to \xi^+(\gamma^+) \in W,$$

meaning that $\xi^+(\gamma^+) \in W$. This will imply that $\mathrm{span}(\Lambda_\rho) \subseteq W$, unless every element $w \in W$ is contained in $\xi^-(\gamma^-)$, for all $\gamma^- \in \Gamma - \{\gamma^+\}$. In other words, it suffices to exclude the case that

$$W \subseteq \bigcap_{\gamma^- \in \Gamma - \{\gamma^+\}} \xi^-(\gamma^-) = (\xi(\partial_\infty\Gamma - x))^\perp,$$

where the last equality follows from the fact that $\xi^-(y) = \xi^+(y)^\perp$ for all $y \in \partial_\infty\Gamma$.

We claim that the element on the right-hand side is trivial, i.e. $W = \{0\}$. Indeed, for each two distinct $y, z \in \partial_\infty\Gamma$, we have that $\xi^-(y) \cap \xi^-(z)$ is a subspace of codimension at most one in the hyperplane $\xi^-(y)$. This follows from the fact that $\xi^-(y) \neq \xi^-(z)$ for $y \neq z$, since $\xi^+(y) \in \xi^-(y)$ and $\xi^+(y) \cap \xi^-(z) = \{0\}$. The same argument applies to the intersection of a finite number of hyperplanes $\xi^-(y_1), \cdots, \xi^-(y_n)$, where $y_i \in \partial_\infty\Gamma$ are distinct. Since Γ is non-elementary, $\partial_\infty\Gamma$ is infinite and due to finite-dimensionality, we have that the intersection of all hyperplanes $\xi^-(y)$ for $y \in \partial_\infty\Gamma$ is trivial, i.e. $W = \{0\}$. □

- A claim for $\mathbb{H}^{p,q}$-convex cocompact representations of maximal dimension:

  Could have something to do with the j-crowns in [BK23]

  Claim 1. Let $\rho : \Gamma \to \mathrm{SO}(p,q) \to \mathbb{H}^{p,q}$ be a $\mathbb{H}^{p,q}$-convex cocompact representation of maximal dimension. Then there exist points $x_1, \cdots, x_{p+1}$ in Λ_ρ that span a subspace of signature $(p,1)$, i.e.
  $$\mathrm{span}\{x_1, \cdots, x_{p+1}\} \simeq \mathbb{R}^{p,1}.$$

- If the above claim is true, then we can deduce the following: Reducible $\mathbb{H}^{p,q}$-convex cocompact representations of maximal dimension restrict to a subrepresentation that is $\mathbb{H}^{p,q'}$-convex cocompact of maximal dimension.

Lemma 21. Let $\rho : \Gamma \to \mathrm{SO}(p,q) \to \mathbb{H}^{p,q}$ be a $\mathbb{H}^{p,q}$-convex cocompact representation of maximal dimension, where Γ is a non-elementary hyperbolic group. If rho is reducible, then $\rho(G) \subseteq \mathrm{S}(\mathrm{O}(p,q') \times \mathrm{O}(q-q'))$, and the restriction of ρ to the subrepresentation $\rho' : \Gamma \to \mathrm{SO}(p,q')$ is $\mathbb{H}^{p,q'}$-convex cocompact of maximal dimension.

Proof. Let $V = \mathrm{span}(\Lambda_\rho)$ be the span of the limit set Λ_ρ. We know that V is an irreducible subrepresentation of ρ, and since $\rho(\Gamma)$ acts cocompactly on $\mathrm{Conv}(\Lambda_\rho)$, we have that $\rho : \Gamma \to \mathrm{SO}(p',q')$ is $\mathbb{H}^{p',q'}$-convex cocompact, where (p',q') is the signature of V.

To show that $p' = p$, let $x_1, \cdots, x_{p+1} \in \Lambda_\rho$ be such that $W = \mathrm{span}\{x_1, \cdots, x_{p+1}\} \simeq \mathbb{R}^{p,1}$. Then we have that $W \subseteq V$, so $p' \leq p$. □

2025-07-11

- There exist no irreducible $\mathbb{H}^{p,q}$-convex cocompact representations of maximal dimension from $\mathrm{Sp}(2n,\mathbb{R})$, except for the $\mathbb{H}^{p,q}$-convex cocompact subgroups of $\mathrm{Sp}(4,\mathbb{R}) \simeq \mathrm{SO}(3,2)$ of maximal dimension.

Lemma 22. Let $\rho : \mathrm{Sp}(2n,\mathbb{R}) \to \mathrm{SO}(p,q)$ be an irreducible representation such that there exists some discrete subgroup $\Gamma \leq G$ for which $\rho_{|\Gamma}$ is an $\mathbb{H}^{p,q+1}$-convex cocompact representation of maximal dimension. Then $n = 2, \mathrm{vcd}(\Gamma) = 2$ and ρ is the special Lie group isomorphism

$$\rho : \mathrm{Sp}(4,\mathbb{R}) \to \mathrm{SO}(\wedge^2\mathbb{R}^4) \simeq \mathrm{SO}(2,3).$$

Proof. Assume that $\rho : \mathrm{Sp}(2n,\mathbb{R}) \to \mathrm{SO}(p,q)$ is an irreducible representation such that there exists some discrete subgroup $\Gamma \leq G$ for which $\rho_{|\Gamma}$ is an $\mathbb{H}^{p,q+1}$-convex cocompact representation of maximal dimension. Then ρ is P_1-Anosov, and Γ is P_Θ-Anosov in $\mathrm{Sp}(2n,\mathbb{R})$, for

$$\Theta = \{\alpha \in \Delta(\mathrm{Sp}(2n,\mathbb{R})) : \langle \alpha, \Lambda \rangle \neq 0\},$$

where Λ is the highest weight of ρ. In particular, Γ is P_i-Anosov in $\mathrm{Sp}(2n,\mathbb{R})$ for some $1 \leq i \leq n$.

Moreover, since Γ is P_i-Anosov in $\mathrm{Sp}(2n,\mathbb{R})$, we have that the limit set $\partial_\infty\Gamma$ of Γ embeds into the flag variety $\mathcal{F}_\Theta = \mathrm{Sp}(2n,\mathbb{R})/P_\Theta$ of $\mathrm{Sp}(2n,\mathbb{R})$, which in turn embeds into each flag variety $\mathcal{F}_j = \mathrm{Sp}(2n,\mathbb{R})/P_j$ for each $\alpha_j \in \Theta$. Since the Lie algebra of every maximal parabolic subgroup P_j of $\mathrm{Sp}(2n,\mathbb{R})$ is given as the intersection of $^{(}2n,\mathbb{R})$ with the stepwise upper triangular matrices (where the step is in the j-th collumn), we have that the maximal parabolic subgroup with the minimum dimension is P_1. Equivalently, the flag variety $\mathcal{F}_1$ has the smallest dimension among all flag varieties $\mathcal{F}_j$. In other words:

$$\dim(\mathcal{F}_\Theta) \leq \dim(\mathcal{F}_1) = n - 1.$$

Using the equality of Bestvina-Mess, we have that $\mathrm{vcd}(\Gamma) = \dim(\partial_\infty\Gamma) + 1 \leq \dim(\mathcal{F}_\Theta) + 1 \leq n$. Combining with the fact that the positive weights of ρ are at most $\min\{p,q\}$, we have that

$$\mathcal{P}_{>0}(\rho) \leq p = \mathrm{vcd}(\Gamma) \leq n \leq \dim(\mathcal{F}_1) + 1.$$

Since ρ is an irreducible representation of $\mathrm{Sp}(2n,\mathbb{R})$, we know that it is a subrepresentation of the tensor product of symmetric powers of exterior powers of the standard representation $\mathbb{R}^{2n}$:

$$\rho : \mathrm{Sp}(2n,\mathbb{R}) \to \bigotimes_{i=1}^{n} \mathrm{Sym}^{k_i}(\wedge^i\mathbb{R}^{2n}),$$

where $k_i \geq 0$ are integers. If a symmetric power of the exterior power $\wedge^i\mathbb{R}^{2n}$ appears in the product, then the highest weight of ρ is given by $\Lambda = L_1 \oplus + \cdots + L_i$, which has an orbit of cardinality $2^i \cdot n!/i!$ through the Weyl group. Since each such representation is self-dual, half of these weights are positive, meaning that the number of positive weights is at least $2^{i-1} \cdot n(n-1)\cdots(n-i+1)$.

Clearly, for $n \geq 3$, these are more than n positive weights, so we must have $n = 2$ and thus $i = 1, 2$. If $i = 1$, then the standard representation appears in the product. Since it is symplectic it needs to appear either along with another exterior power that carries a symplectic structure, or to appear as a symmetric power of the standard representation. Since the only exterior powers that appear as factors are $\mathbb{R}^{2n}$ and $\wedge^2\mathbb{R}^{2n}$, the first option is not feasible. On the other hand, neither is the second option, since any symmetric power of the standard representation will have more than n positive weights.

Hence we must have $i = 2$, with highest weight $\Lambda(\mathrm{Sym}^k(\wedge^2\mathbb{R}^{2n})) = 2k(L_1 + L_2)$. The orbit of the highest weight has at least two weights: $2k(L_1 \pm L_2)$. If $k > 1$, then there exist also positive weights that are not in the orbit of the highest weight, so we must have $k = 1$. Said differently, the only possibility is the special Lie group isomorphism of $\mathrm{Sp}(4,\mathbb{R})$ in the orthogonal complement $(\mathbb{R}\omega)^\perp \subseteq \wedge^2\mathbb{R}^4$, where ω is the symplectic form on $\mathbb{R}^4$ preserved by $\mathrm{Sp}(4,\mathbb{R})$.

$$\wedge^2(\mathrm{std}) : \mathrm{Sp}(4,\mathbb{R}) \to \mathrm{SO}(2,3).$$

□

2025-07-15

- Most irreducible representation from $\mathrm{SO}(n+1,n)$ to some $\mathrm{SO}(p,q)$ that restricts to a convex-cocompact representation of maximal dimension are the standard representation.

Lemma 23. Let $\rho : \mathrm{SO}(n+1,n) \to \mathrm{SO}(p,q)$ be an irreducible representation such that there exists some discrete subgroup $\Gamma \leq G$ for which $\rho_{|\Gamma}$ is an $\mathbb{H}^{p,q+1}$-convex cocompact representation of maximal dimension. Then ρ is either the standard representation $\mathrm{SO}(n+1,n) \to \mathrm{SO}(n+1,n)$, or it is the adjoint representation of $\mathrm{SO}(2,1)$ or $\mathrm{SO}(3,2)$:

$$\begin{aligned} \mathrm{Ad} : \Gamma \leq \mathrm{SO}(2,1) \to \mathrm{SO}(1,2), \text{ with } \mathrm{vcd}(\Gamma) = 1 \\ \mathrm{Ad} : \Gamma \leq \mathrm{SO}(3,2) \to \mathrm{SO}(4,6), \text{ with } \mathrm{vcd}(\Gamma) = 4. \end{aligned}$$

Proof. Assume that $\rho : \mathrm{SO}(n+1,n) \to \mathrm{SO}(p,q)$ is an irreducible representation such that there exists some discrete subgroup $\Gamma \leq G$ for which $\rho_{|\Gamma}$ is an $\mathbb{H}^{p,q+1}$-convex cocompact representation of maximal dimension. Then ρ is P_1-Anosov, and Γ is P_Θ-Anosov in $\mathrm{SO}(n+1,n)$, for some $\Theta \subseteq \{1,\dots,n\}$. In particular, $\partial_\infty\Gamma$ embeds into the flag variety $\mathcal{F}_\Theta = \mathrm{SO}(n+1,n)/P_\Theta$ of $\mathrm{SO}(n+1,n)$, which in turn embeds into each flag variety $\mathcal{F}_j = \mathrm{SO}(n+1,n)/P_j$ for each $\alpha_j \in \Theta$. Since the maximal parabolic subgroups of $\mathrm{SO}(n+1,n)$ are the stabilizers of isotropic k-subspaces for $k = 1,\dots,n$, whose matrix form is given by the stepwise upper triangular matrices, we have that the maximal parabolic subgroup with the minimum dimension is P_1, or equivalently, the flag variety $\mathcal{F}_1$ has the smallest dimension among all flag varieties $\mathcal{F}_j$. Using the result of Bestvina-Mess, we have that

$$\mathrm{vcd}(\Gamma) = \dim(\partial_\infty\Gamma) + 1 \leq \dim(\mathcal{F}_{\{\mathtt{a}_1\}}) + 1 \leq 2n + 1.$$

Combining with the fact that the positive weights of ρ are at most $\min\{p,q\}$, we have that

$$\sharp\mathcal{P}_{>0}(\rho) \leq p = \mathrm{vcd}(\Gamma) \leq 2n + 1.$$

Since ρ is an irreducible representation of $\mathrm{SO}(n+1,n)$, we know that it is a subrepresentation of the tensor product of symmetric powers of exterior powers of the standard representation $\mathbb{R}^{n+1,n}$:

$$\rho : \mathrm{SO}(n+1,n) \to \bigotimes_{i=1}^{n} \mathrm{Sym}^{k_i}(\wedge^i \mathbb{R}^{n+1,n}),$$

where $k_i \geq 0$ are integers.

It will be useful to note that the weights of each exterior (or symmetric) power are also weights of every higher exterior (or symmetric) power, since 0 is a weight of the standard representation. This implies that exterior powers higher than 3 do not appear in the product, since $\wedge^3\mathbb{R}^{n+1,n}$ has at least

$$\binom{n}{3} + 2 \cdot \binom{n}{2} + n$$

positive weights (as sums $L_i + L_j + L_k, L_i + L_j, L_i - L_j, L_i$ for $1 \leq i < j < k \leq n$), which is larger than $2n+1$ for $n \geq 3$.

On the other hand, we note that the exterior square $\wedge^2\mathbb{R}^{n+1,n}$ is the adjoint representation which can be found to land in $\mathrm{SO}(n^2, n(2n+1))$, which means that it can't appear in the product for $n \geq 3$.

Symmetric powers of the standard representation $\mathrm{Sym}^k(\mathbb{R}^{n+1,n})$ can not appear either for $k \geq 2$, since its weights are given by $\pm 2L_i, \pm L_i, \pm L_i \pm L_j$ for $1 \leq i < j \leq n$, so the number of positive ones exceeds $2n+1$. Similarly we exclude the case of products involving $\mathbb{R}^{n+1,n}$ and $\wedge^2\mathbb{R}^{n+1,n}$. $\square$

2025-09-04

- Let Γ be a non-elementary hyperbolic group and $\rho : \Gamma \to G$ be a Θ-Anosov representation into a $\mathrm{SL}(d,\mathbb{R})$. Then for $\alpha_k \in \Theta$, there exists no subspace in $\mathcal{G}_k(R^d)$ that is in all the subspaces $\xi^{d-k}(x)$ for $x \in \partial_\infty\Gamma$.

  Proof. What we want to show is that

  $$\cap_{x\in\partial_\infty\Gamma}\xi^{d-k}(x) = \{0\}.$$

  Let $x \neq y \in \partial_\infty\Gamma$. Then $\xi^{d-k}(x) \cap \xi^{d-k}(y)$ is a subspace of codimension at least one in $\xi^{d-k}(x)$, since $\xi^{d-k}(x) \neq \xi^{d-k}(y)$. The same arguement implies that in each intersection, the dimension decreases by at least one. Since Γ is non-elementary, $\partial_\infty\Gamma$ is infinite, so the intersection must be trivial. $\square$

- Let Γ be a non-elementary hyperbolic group and $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be an $\mathbb{H}^{p,q}$-convex cocompact representation. If $\Omega \subseteq \mathbb{R}^{p,q}$ is a properly convex domain preserved by $\rho(\Gamma)$, then the limit set Λ_ρ is contained in the closure $\overline{\mathbb{P}(\Omega)}$ of the projectivization $\mathbb{P}(\Omega)$ of Ω.

  Proof. Since the action of Γ on $\xi^1(\partial_\infty\Gamma)$ is minimal, i.e. every orbit is dense, it suffices to show that there exists some $x \in \partial_\infty\Gamma$ such that $\xi^1(x) \in \overline{\mathbb{P}(\Omega)}$. $\square$

- Let $\rho : \Gamma \to \mathrm{SO}(p, q+1)$ be an $\mathbb{H}^{p,q}$-convex cocompact representation, and $\Omega \subseteq \mathbb{R}^{p,q+1}$ be a properly convex domain preserved by $\rho(\Gamma)$. Then the lift of the limit set $\pi^{-1}(\Lambda_\rho)$ under the double covering $\pi : \mathbb{S}^{p-1} \times \mathbb{S}^q \to \partial\mathbb{H}^{p,q}$ has two connected components, each of which is a copy of $pi^{-1}(\Lambda_\rho)$, in the sense that when restricted to each, π is a homeomorphism onto Λ_ρ.

Proof. First we show that $\pi^{-1}(\Lambda_\rho)$ is not connected, i.e. it has at least two connected components. Assume the contrary for the sake of contradiction. Then for $x \in \pi^{-1}(\Lambda_\rho)$, we have that $-x \in \pi^{-1}(\Lambda_\rho)$ as well. Since Ω is properly convex, there exists a hyperplane H that does not intersect $\bar{\Omega}$. Let $\gamma : I \to \pi^{-1}(\Lambda_\rho)$ be a path from x to $-x$. Then γ must intersect H at some point $\gamma(t_0)$, which is a contradiction since $\pi(\gamma(t_0)) \in \Lambda_\rho \subseteq \overline{\mathbb{P}(\Omega)}$. In particular, we have shown that each component of $\pi^{-1}(\Lambda_\rho)$ contains at most one lift of each point in Λ_ρ.

To show that it has at most two components, it suffices to show that the restriction of π to each component is surjective, since π is a double covering. Let $\tilde{x} \in \pi^{-1}(\Lambda_\rho)$ be a point in some component, that projects to $\pi(\tilde{x}) = x \in \Lambda_\rho$. Then for any $y \in \Lambda_\rho$, there exists a path γ in Λ_ρ from x to y. Then it lifts to a path $\tilde{\gamma}$ in $\pi^{-1}(\Lambda_\rho)$ from $\tilde{x}$ to some lift of y, which will be contained in the same component as $\tilde{x}$. Thus the restriction of π to the component is surjective. $\square$

2025-09-08 – meeting – Limit set regularity rigidity

Why deformations are interesting

Short answer: they give us representations of discrete groups that do not come from a representation of a Lie group.

Let G be a real semisimple Lie group. In general, we are interested in lattices of G, because roughly they give us manifolds or orbifolds. However, lattices are rigid in higher rank Lie groups, in the sense that there only a few of them. This is because under the conditions of the Margulis rigidity theorem, any representation of a lattice $\Gamma \leq G$ into another Lie group H can be extended to a representation of G into H. However, there exist finitely many representations of G into H, which can be studied individually.

For this reason, we are led to consider discrete subgroups of G that admit deformations (whose image is not a discrete group and) which do not extend to G. If we take for instance $Z \leq G$ a rank-1 Lie subgroup of G, then we can take $\Gamma \leq Z$ a lattice, and $\rho : Z \to G$ a representation for which $\rho_{|\Gamma}$ is Anosov. Then it is known that the deformations of $\rho_{|\Gamma}$ are Anosov as well. In this way, we obtain an infinite family of interesting representations ρ_t which do not extend to G.

Rigidity with respect to regularity

Let $\Gamma \leq G$ be a discrete subgroup of the semisimple Lie group G and $\rho : \Gamma \to H$ be a representation into another semisimple Lie group H, that is assumed to be Anosov. One would like to know when this representation is "banal", in the sense that it comes from a representation of G into H. One way to do this is to look at the regularity of the limit set Λ_ρ.

In the "banal" case, the limit set Λ_ρ is very regular.

AdS-quasifuchsian representations

In the case of AdS-quasifuchsian representations, one has the result of [GM18]. We recall:

Definition 32. Let $\Gamma \leq \mathrm{SO}(n, 2)$ be a discrete subgroup. Then:

1. Γ is AdS^{n+1}-quasifuchsian if it is $\mathbb{H}^{n,1}$-convex cocompact of maximal dimension, i.e.

it acts properly discontinuously and cocompactly on some properly convex closed subset of $\mathbb{H}^{n,1}$ and its limit set $\Lambda_\Gamma \subseteq \partial\mathbb{H}^{n,1}$ is homeomorphic to $\mathbb{S}^{n-1}$.

2. Γ is AdS^{n+1}-Fuchsian if it is AdS^{n+1}-quasifuchsian and preserves a totally geodesic copy of $\mathbb{H}^n \subseteq \mathbb{H}^{n,1}$.

One knows that the limit set of an AdS^{n+1}-Fuchsian representation is (smooth and hence) $\mathcal{C}^1$. The reason for that is that one can show that it will act cocompactly on the totally geodesic copy of $\mathbb{H}^n$ that it preserves, so the boundary will be a regular sphere of dimension $n-1$. The result of [GM18] states that $\mathcal{C}^1$-regularity of the limit set is rigid, in the sense that it characterizes the AdS^{n+1}-Fuchsian representations among quasifuchsian ones:

Theorem 3 (Theorem 1.3 of [GM18]). Let Γ be an AdS^{n+1}-quasifuchsian group. Then its limit set Λ_Γ is a $\mathcal{C}^1$ submanifold of $\partial\mathrm{AdS}^{n+1}$ if and only if Γ is AdS^{n+1}-Fuchsian.

Hence in a sense, regularity of the limit set is only possible for examples that come from the rank-1 case (i.e. $\mathrm{SO}(n,1)$).

Convex-cocompact representations

Here similarly we have that $\mathcal{C}^1$-regularity characterises strict convexity:

Theorem 4 (Benoist - Convex divisibles I). Let Ω be a divisible convex subset of $\mathbb{P}(\mathbb{R}^d)$ divided by Γ. Then the following are equivalent:

1. The metric space (Ω, d_Ω) is Gromov-hyperbolic.
2. The convex set Ω is strictly convex.
3. The boundary $\partial\Omega$ is $\mathcal{C}^1$.
4. The group Γ is Gromov-hyperbolic.

On the other hand we have that $\mathcal{C}^2$-regularity is even more rigid, since it characterises ellipsoids among strictly convex sets:

Theorem 5 (Benoist ?). Let Ω be a strictly convex divisible convex subset of $\mathbb{P}(\mathbb{R}^d)$ divided by Γ. Then $\partial\Omega$ is $\mathcal{C}^2$ if and only if Ω is an ellipsoid.

A few conjectures

In the spirit of the above, we can make the following conjectures:

Conjecture 1. Let $\Gamma \leq \mathrm{SO}(p,q)$ be $\mathbb{H}^{p,q}$-convex cocompact of maximal dimension. Then its limit set Λ_Γ is a $\mathcal{C}^2$ submanifold of $\partial\mathbb{H}^{p,q}$ if and only if Γ is a lattice of a Lie subgroup G of rank 1 of $\mathrm{SO}(p,q)$.

Moreover, the groups G that appear can be classified into some families of representations.

Conjecture 2. If $q \leq p$ or $p \geq 3$, then Conjecture 1 holds for $\mathcal{C}^1$-regularity.

Note however that $\mathcal{C}^1$-regularity is not always attained. For instance, if $\rho : \pi_1(S) \to \mathrm{SO}(2,3)$ is a Hitchin representation, then its limit set is a $\mathcal{C}^1$-submanifold and not $\mathcal{C}^2$ unless it is Hitchin Fuchsian (?).

2025-09-15 – comité de suivi – geodesic flow and Anosov representations

This was a crash course by Nicolas and Daniel.
Unless otherwise stated, M will be a compact manifold with strictly negative curvature.

Geodesic flow basics

Then we can define the geodesic flow $\phi_t : T^1M \to T^1M$ on the unit tangent bundle T^1M of M.

Note that the unit tangent bundle of the universal cover $\widetilde{M}$ of M can be identified by a pair of points in the boundary at infinity $\partial_\infty \widetilde{M}$ and a real number:

$$T^1\widetilde{M} \simeq \partial_\infty \widetilde{M} \times \mathbb{R}.$$

The boundaries at infinity designate a geodesic in $\widetilde{M}$, and the real number designates a point on that geodesic.

Definition 33. A closed orbit of the geodesic flow ϕ_t is a set of the form

$$\{\phi_t(v) : t \in \mathbb{R}\},$$

where $v \in T^1M$ is such that $\phi_T(v) = v$ for some $T > 0$.

Clearly, closed geodesics in M correspond to closed orbits of the geodesic flow, which in turn correspond to conjugacy classes of primitive elements in $\pi_1(M)$, which are known to be in bijection with free homotopy classes of closed curves in M. In particular, a reparametrization does not influence the closed orbits of the flow (it only changes their period).

Definition 34. Let $f : M \to \mathbb{R}_{>0}$ be a positive function. Then there exists a flow $\phi_t^f : T^1M \to T^1M$ such that the generator of ϕ_t^f is f times the generator of ϕ_t, i.e.

$$\left.\frac{d}{dt}\right|_{t=0} \phi_t^f(v) = f(v) \cdot \left.\frac{d}{dt}\right|_{t=0} \phi_t(v).$$

We call ϕ_t^f the reparametrization of ϕ_t by f.

Note that if $T > 0$ is the period of a closed orbit starting at $v \in T^1M$, then the period T' of the closed orbit starting at v for the reparametrized flow ϕ_t^f is given by

$$T' = \int_0^T f(\phi_t(v))dt.$$

Moreover, if μ is a ϕ_t-invariant probability measure on T^1M, then $\mu^f = f\,\mathrm{d}\mu$ is a ϕ_t^f-invariant probability measure on T^1M. Why?

Definition 35 (Exponential mixing)**.** Let ψ_t be a flow on a metric space X and μ an invariant measure. Then we say that ψ_t mixes exponentially with respect to μ if for all α-Hölder functions $f, g : X \to \mathbb{R}$:

$$\int_X f(\psi_t(x))g(x)d\mu(x) \to \int_X f d\mu \int_X g d\mu$$

The classical tools to show exponential mixing are based on the regularity of the stable and unstable leaves of the flow.

Definition 36. Let ψ_t be a flow on a metric space X. The strongly and weakly stable leaves of ψ_t at $x \in X$ are given by

$$W^{ss}(x) = \{y \in X : d(\psi_t(x), \psi_t(y)) \to 0 \text{ as } t \to +\infty\}$$
$$W^{ws}(x) = \{y \in X : d(\psi_t(x), \psi_t(y)) \text{ is bounded for } t > 0\}.$$

Similarly, the strongly and weakly unstable leaves of ψ_t at $x \in X$ are given by

$$W^{su}(x) = \{y \in X : d(\psi_t(x), \psi_t(y)) \to 0 \text{ as } t \to -\infty\}$$
$$W^{wu}(x) = \{y \in X : d(\psi_t(x), \psi_t(y)) \text{ is bounded for } t < 0\}.$$

Under certain conditions, these define folitations of a certain regularity. For instance, in the geodesic flow of the hyperbolic space, the strongly stable leaves are the horocycles based at the forward endpoint of the geodesic in the boundary at infinity. The weakly stable leaves are the sets of geodesics that share the same forward endpoint.

Given an endpoint $p \in \partial_\infty \mathbb{H}^n$ we can define the Busemann function based at p, which measures the distances of points as seen from the point p at infinity. In particular, for $x, y \in T^1\mathbb{H}^n$ pointing towards p (in which case we have $y \in W^{ws}(x)$), we define

$$B_p(x, y) = \lim_{t\to\infty} d(\phi_t(x), \phi_t(y)).$$

Geodesic flow and Anosov representations – Sambarino's flow

Let $\rho : \pi_1(M) \to G$ be a Θ-Anosov representation into a semisimple Lie group G. The for every functional $\alpha \in \mathfrak{a}^*$ that is positive on the Weyl chamber $\mathfrak{a}^+$ and that is of the form

$$\alpha = \sum_{\theta\in\Theta} \alpha_\theta \omega_\theta,$$

where ω_θ are the fundamental weights of G, we can construct a reparametrization (depending on the choice of α and the representation) ϕ^f of the geodesic flow ϕ_t on T^1M such that

$$\mathcal{L}_f(\gamma) = \alpha(\lambda(\rho(\gamma))),$$

where $\mathcal{L}_f(\gamma)$ is the length of the geodesic corresponding to $\gamma \in \pi_1(M)$ for the reparametrized flow ϕ_t^f. Note that although the reparametrized flow is not necessarily a geodesic flow for some metric, it still has the same closed orbits as the original geodesic flow, which are in bijection with conjugacy classes of primitive elements in $\pi_1(M)$. Using this we can obtain estimates for the number of closed orbits of the flow ϕ_t^f of a certain bounded length:

$$\sharp\{[\gamma] \in [\pi_1(M)] : \mathcal{L}_f(\gamma) \le R\} \sim \frac{e^{\delta_1(\rho)R}}{\delta_1(\rho)R},$$

In the case of projective Anosov representations into $\mathrm{SL}(d,\mathbb{R})$, we have (probably due to Daniel) that there exists some $\epsilon > 0$ such that

$$\sharp\{[\gamma] \in [\pi_1(M)] : \alpha(\lambda(\rho(\gamma))) \leq R\} \sim \frac{e^{\delta_1(\rho)R}}{\delta_1(\rho)R} \cdot (1 + O(e^{-\epsilon R})).$$

For this, we need the fact that the reparametrized flow ϕ_t^f mixes exponentially with respect to the Borel-Margulis measure. Since its leaves are only Hölder, the classical tools do not apply. For this, we embed it into a larger smooth flow for which the classical tools apply, and then we restrict back to our flow.

Theorem 6 (DMS)**.** Let $\rho : \pi_1(M) \to \mathrm{SL}(d,\mathbb{R})$ be a projective Anosov representation. Then there exists a reparametrization ϕ_t^ρ of the geodesic flow ϕ_t on T^1M for which $\mathcal{L}_{\phi^\rho}(\gamma) = \lambda_1(\rho(\gamma))$ for all $\gamma \in \pi_1(M)$.

There also exists a smooth flow ψ_t on a smooth manifold N, smooth foliations $\mathcal{W}_\psi^{ss}, \mathcal{W}_\psi^{su}$ of N and a Hölder embedding $F : T^1M \to N$ that conjugates ϕ_t^ρ with the restriction of ψ_t to $F(T^1M)$, such that $F(\mathcal{W}_{\phi^\rho}^{ss}) = \mathcal{W}_\psi^{ss} \cap F(T^1M)$ and $F(\mathcal{W}_{\phi^\rho}^{su}) = \mathcal{W}_\psi^{su} \cap F(T^1M)$.

Note that the flow ϕ_t^ρ is different than the one constructed by Sambarino.

Example 7. Let M be a compact hyperbolic surface and $\rho : \pi_1(M) \to \mathrm{SO}(3,1) \leq \mathrm{SL}(4,\mathbb{R})$ be a quasi-Fuchsian representation. Then $N = T^1\mathbb{H}^3/\rho(\pi_1(M)) \simeq T^1(M \times \mathbb{R})$.

Ehresmann - Thurston principle

For a $(G, G/H)$-structure on M we can define the holonomy representation $\rho : \pi_1(M) \to G$ and the developing map $\mathrm{dev} : \widetilde{M} \to G/H$. Then we have the following:

Theorem 7 (Ehresmann - Thurston principle)**.** If ρ' is sufficiently close to ρ, then there exists a $(G, G/H)$-structure on M with holonomy ρ'.

2025-09-16

Proposition 31. The adjoint representation of $\mathrm{GL}(n,\mathbb{R})$ lies into $\mathrm{SL}(\mathfrak{gl}(n,\mathbb{R}))$.

Proof. Recall that for a vector space V we have the identification

$$\begin{aligned} V \otimes V^* &\simeq \mathrm{End}(V) \\ v \otimes \varphi &\mapsto (w \mapsto \varphi(w)v) \\ E_{ij} &\leftarrow\!\shortmid e_i \otimes e^j. \end{aligned}$$

Then we can see left and right multiplication by g (acting on $\mathfrak{gl}(n,V)$) as endomorphisms of $V \otimes V^*$:

$$\begin{aligned} L_g &= g \otimes \mathrm{id}_{V^*} \\ R_g &= \mathrm{id}_V \otimes g^t. \end{aligned}$$

Then the adjoint representation is given by $\mathrm{Ad}(g) = L_g \circ R_{g^{-1}} = g \otimes (g^{-1})^t$. Knowing that $\det(A \otimes B) = (\det A)^{\dim B}(\det B)^{\dim A}$, we have that $\det(\mathrm{Ad}(g)) = 1$ for all $g \in \mathrm{GL}(n,\mathbb{R})$. $\square$

2025-09-21

- The limit set of a uniform lattice of $\mathrm{SO}(n,1)$ is the full boundary $\partial\mathbb{H}^n \simeq \mathbb{S}^{n-1}$. Recall that the limit set is given as $\Lambda_\Gamma = \overline{\Gamma \cdot x} \cap \partial\mathbb{H}^n$ for some $x \in \mathbb{H}^n$, and $\partial\mathbb{H}^n$ is the Gromov boundary of $\mathbb{H}^n$.

> **Proposition 32.** Let $\Gamma \leq \mathrm{SO}(n,1)$ be a uniform lattice. Then its limit set is the full boundary $\partial\mathbb{H}^n$.

Proof. Since the action is cocompact, there exists a compact set $K \subseteq \mathbb{H}^n$ such that $\Gamma \cdot K = \mathbb{H}^n$. Without loss of generality, we can assume that $o \in K$. Let $\xi \in \partial\mathbb{H}^n$, and let $r : \mathbb{N} \to \mathbb{H}^n$ be a geodesic ray such that $r(\infty) = \xi$. Then for every n, there exists $\gamma_n \in \Gamma$ such that $\gamma_n^{-1} r_n \in K$. In particular, $d(\gamma_n^{-1} r_n, o) = d(r_n, \gamma_n o) \leq \mathrm{diam}(K)$ is bounded. By the Fellow Traveller property, this implies that $\gamma_n o \to \xi$ as $n \to \infty$. ☐

2025-09-23

- Labourie's lemma is correct in dimension 1 and when the group G is very large.

> **Proposition 33.** Let G be a subgroup of $\mathrm{SL}(d,\mathbb{R})$, such that
>
> $$\dim G \geq d-1.$$
>
> Then for every $l \in \mathbb{P}(\mathbb{R}^d), W \in \mathcal{G}_{d-k}(\mathbb{R}^d)$, there exists $g \in G$ such that $gl \oplus W = \mathbb{R}^d$.

Proof. We know the orbit of a Lie group is an immersed submanifold of $\mathbb{R}^d$. In our case, $G \cdot l$ is an immersed submanifold of $\mathbb{P}(\mathbb{R}^d)$ of dimension at least $d-1$. Since the inclusion map is an immersion, its differential is injective. But since the dimension of the domain and codomain are equal, it is an isomorphism, meaning that it is a diffeomorphism onto its image.

Thus $G \cdot l$ contains an open neighborhood of l in $\mathbb{P}(\mathbb{R}^d)$ and no projective hyperplane can contain an open set, so there exists $g \in G$ such that $gl \oplus W = \mathbb{R}^d$. ☐

- Transversality in Grassmanian coordinates:

> **Proposition 34.** Let $P \in \mathcal{G}_k(\mathbb{R}^d)$ and $Q \in \mathcal{G}_{d-k}(\mathbb{R}^d)$ be two complementary subspaces. We define U_Q and U_P as the transverse subspaces to Q and P respectively:
>
> $$U_Q = \{W \in \mathcal{G}_k(\mathbb{R}^d) : W \oplus Q = \mathbb{R}^d\}$$
> $$U_P = \{W \in \mathcal{G}_{d-k}(\mathbb{R}^d) : W \oplus P = \mathbb{R}^d\}.$$
>
> Then we have the following parametrizations of the Grassmanians:
>
> $$\mathcal{L}(Q,P) \to U_Q \subseteq \mathcal{G}_k(\mathbb{R}^d)$$
> $$X \mapsto \mathrm{graph}(X)$$
> $$\mathcal{L}(P,Q) \to U_P \subseteq \mathcal{G}_{d-k}(\mathbb{R}^d)$$
> $$Y \mapsto \mathrm{graph}(Y),$$

where $\mathcal{L}(A, B)$ is the space of linear maps from A to B.

In these coordinates, the transversality condition of a flag $(\mathrm{graph}(X), \mathrm{graph}(Y)) \in U_Q \times U_P \subseteq \mathcal{G}_k(\mathbb{R}^d) \times \mathcal{G}_{d-k}(\mathbb{R}^d)$ is:

$$\mathrm{graph}(X) \pitchfork \mathrm{graph}(Y) \iff \mathrm{Id} - YX \in \mathrm{GL}(Q).$$

Fix a basis of $\mathbb{R}^d$ adapted to the decomposition $\mathbb{R}^d = P \oplus Q$. In these coordinates for $\mathrm{GL}(d, \mathbb{R})$ we have that the transversality condition under the action of $g \in \mathrm{GL}(d, \mathbb{R})$ is given by

$$g \cdot \mathrm{graph}(X) \pitchfork \mathrm{graph}(Y) \iff (CX + D) - Y(AX + B) \in \mathrm{GL}(Q),$$

where

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in the chosen basis, i.e. $A : P \to P, B : Q \to P, C : P \to Q, D : Q \to Q$, given by $\mathrm{pr}_P g_{|P}, \mathrm{pr}_P g_{|Q}, \mathrm{pr}_Q g_{|P}, \mathrm{pr}_Q g_{|Q}$ respectively.

This allows us either to disprove Labourie's property by finding X, Y such that for all A, B, C, D the above is not invertible, or to prove it for flags in $U_Q \times U_P$ by finding A, B, C, D for all X, Y such that the above is invertible.

- It feels like all irreducible representations of $\mathrm{SL}(2, \mathbb{R})$ satisfy Labourie's property for $k = 1$. We have that

$$\mathrm{span}\left(\rho(\mathrm{SL}(2, \mathbb{R}))e_1^i\right) = \mathbb{R}^d$$

2025-09-28

1. Let $\mathfrak{g}$ be a simple Lie algebra, whose Cartan involution is θ. If σ is another involution of $\mathfrak{g}$ that commutes with θ. Then the restriction of the adjoint representation $\mathrm{ad}_{\mathfrak{g}^\sigma}$ on the fixed point set preserves the antifixed point set $\mathfrak{g}^{-\sigma}$. This is not always irreducible:

Example 8. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, $\sigma(X) = I_{p,q} X I_{p,q}$. Then

$$\mathfrak{g}^\sigma = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in \mathfrak{sl}(p, \mathbb{R}), D \in \mathfrak{sl}(q, \mathbb{R}) \right\}, \mathfrak{g}^{-\sigma} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} : B \in M_{p,q}(\mathbb{R}), C \in M_{q,p}(\mathbb{R}) \right\}.$$

Clearly $\mathrm{ad}_{\mathfrak{g}^\sigma}$ is not irreducible on $\mathfrak{g}^{-\sigma}$, and admits two invariant subspaces.

Regarding the above situation, it is possibly correct that if the representation is not irreducible, then it admits two invariant subspaces, each of which is isotropic for the Killing form and which are permuted by the Cartan involution.

2025-09-29

1. Limit sets of uniform lattice and convex-cocompact subgroups.

   Let G be a rank 1 semisimple Lie group, and $\phi : G \to H$ be a Lie group morphism, where H is a semisimple Lie group. Then there exists a parabolic subgroup P_H of H such that

$\phi_{|\Gamma}$ is P_H-Anosov for every uniform lattice Γ of G. In fact, for every uniform lattice Γ of G, the limit set is the same, and it is a smooth embedded submanifold of G/P_H.

In fact, denoting with P_G the unique parabolic proper subgroup of G, there exists an equivariant smooth embedding $\xi_G : G/P_G \to H/P_H$, and the limit set of $\phi_{|\Gamma}$ is $\xi_G(G/P_G)$.

If Γ is convex-cocompact, then it has a limit set $\Lambda_\Gamma \subseteq \partial_\infty(G/K) = G/P_G$ that can be identified with the Gromov boundary of Γ via the Milnor-Svarc lemma, and the limit set of $\phi_{|\Gamma}$ is $\xi_G(\Lambda_\Gamma)$.

This can be found in Guichard-Wienhard 2012, Proposition 4.7.

2025-09-30 - Random walks on groups

Definition of a random walk on a group

Let G be a locally compact group and μ be a probability measure on G. To every such pair we can associate a random walk on G with increments distributed according to μ as follows. First we define the space of (forward) increments to be the space $G^{\mathbb{N}}$ endowed with the product measure $\mu^{\otimes\mathbb{N}}$, which is essentially the space of sequences of elements of G chosen independently according to the law μ. The multiplication of the first n increments gives us a random walk on G, whose increments are distributed according to μ:

$$Z : G^{\mathbb{N}} \times \mathbb{N} \to G, \quad Z_n\left((g_i)_{i\in\mathbb{N}}\right) = g_1 g_2 \cdots g_n.$$

Motivation of boundary theory and stationary measures

Let G be a locally compact group, μ be a probability measure on G and X be a compact metric space on which G acts continuously. We denote with Z_n the random walk on G corresponding to μ. We are interested in the following question:

Question 1. What is the asymptotic behaviour of the orbits of Z_n on X, i.e. what can we say about the sequence $Z_n \cdot x$ for $x \in X$?

For this, boundary theory defines a boundary space Y equiped with a measure ν that is invariant under the dynamical system of the random walk. More concretely we ask for ν to be a μ-stationary measure:

$$\mu * \nu = \nu,$$

where the convolution is defined as follows:

Definition 37. Let G be a group acting on a space X. If μ is a probability measure on G and ν is a probability measure on X, then the convolution $\mu * \nu$ is the probability measure on X defined by

$$\mu * \nu = \int_G g_* \nu d\mu(g), \text{ i.e. } \int_X f d(\mu * \nu) = \int_G \int_X f(g \cdot x) d\nu(x) d\mu(g)$$

Another way to interpret stationarity is that if we consider a random walk on Y with increments distributed according to μ and starting distribution ν, then the distribution of the position at any time n is still ν.

The reasons to look for such a measure and space are the following:

1. Such a measure exists: for instance when the space X is compact.

2. By a martingale argument we have a convergence result: there exists a family of measures $\{\nu_\omega\}_{\omega \in G^{\mathbb{N}}}$ such that in the weak-$*$ topology
$$Z_n(\omega)_*\nu \to \nu_\omega.$$
Moreover, we can recover ν by integrating the ν_ω:
$$\nu = \int_{G^{\mathbb{N}}} \nu_\omega d\mu^{\otimes \mathbb{N}}(\omega).$$
3. Often, one can show that the measures ν_ω are Dirac masses, i.e. there exists a measurable map $\xi : G^{\mathbb{N}} \to X$ such that $\nu_\omega = \delta_{\xi(\omega)}$. This is in fact a contraction property for the dynamical system on the boundary Y.

To illustrate the last point above, we ask for the following two conditions:

1. ν_ω is a Dirac mass for almost every $\omega \in G^{\mathbb{N}}$.
2. The support of ν is "large" in Y.

Then the support of ν is contracted by the random walk around ω, in the sense that for a suitable subset $O \subseteq Y$ that contains $\xi(\omega)$,
$$Z_n(\omega)_*\nu(O) = \nu(Z_n(\omega)^{-1}O) \to 1.$$

Moreover, provided that the support of μ is large, we can show that the random walk on X converges to $\xi(\omega)$:
$$Z_n(\omega) \cdot x \to \xi(\omega) \text{ for many } x \in X.$$
To see this, assume that X is the hyperbolic disc and Y its boundary. Then for a point $x \in X$, we assume that there exist points $\xi_1, \xi_1' \in \operatorname{supp}(\nu) \subseteq Y$ such that the geodesic ray from ξ_1 to ξ_1' passes through x. Similarly, we assume that there exist points $\xi_n, \xi_n' \in \operatorname{supp}(\nu) \subseteq Y$ such that the geodesic ray from ξ_n to ξ_n' passes through $Z_n(\omega) \cdot x$. Then the above discussion tells us that $\xi_n, \xi_n' \to \xi(\omega)$, so the geodesic ray from ξ_n to ξ_n' converges to the geodesic ray from $\xi(\omega)$ to itself, which is just the point $\xi(\omega)$.

A toy example

Let $G = \mathrm{GL}(d, \mathbb{R})$ and $\mu = \delta_A$ for some diagonal matrix $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$ with $\lambda_1 > \lambda_2 > \cdots > \lambda_d > 0$. Then the random walk is given by $Z_n = A^n$. Indeed, the measure $\mu^{\otimes \mathbb{N}}$ is the Dirac mass on the sequence $(A, A, A, \ldots)$.

2025-10-01

Boundary theory basics

Assumption 2. Let G be a second countable locally compact group and μ be a probability measure on G.

> **Definition 38.** Let G be a group acting continuously on a topological space Y, equipped with the Borel σ-algebra and a probability measure ν. Then
>
> (i) The action of G on Y is *non-singular* if every element of G preserves the measure class of ν.

(ii) The convolution $\mu * \nu$ is defined as the measure on Y given by

$$\mu * \nu = \int_G g_* \nu d\mu(g),$$

i.e. for every measurable function $f : Y \to \mathbb{R}$ we have

$$\int_Y f d(\mu * \nu) = \int_G \int_Y f(g \cdot y) d\nu(y) d\mu(g).$$

We denote with $\mu^{*n} = \mu * \cdots * \mu$ (n times) the n-th convolution power of μ, where G is considered with the action on itself by left multiplication.

(iii) The measure ν is *μ-stationary* if $\mu * \nu = \nu$. In this case we say that the pair (Y, ν) is a *(G, μ)-space*.

(iv) The measure $\mu \in \mathrm{Prob}(G)$ is *admissible* if the support of μ generates G as a semigroup and there exists $n \in \mathbb{N}$ such that μ^{*n} and the Haar measure on G are non-singular.

Cocycles and Gregs

Definition 39. Let G be a group, $\mathbb{T} \in \{\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{Z}, \mathbb{N}\}$ be time (or any other group), and $\theta : \mathbb{T} \times X \to X$ be a flow (or an action of $\mathbb{T}$) on X. A *cocycle* on X over θ is a map $w : \mathbb{T} \times X \to G$ such that for all $s, t \in \mathbb{T}$ and $x \in X$ we have

$$w(s + t, x) = w(s, \theta_t(x)) w(t, x).$$

Remark 5 (Cocycle condition)**.** To motivate the cocycle condition, consider a group (time) $\mathbb{T}$ acting on a space X and a group G. Then we can define a lift of the action (flow) of $\mathbb{T}$ on X to an action of $\mathbb{T}$ on $X \times G$ by

$$t \cdot (x, g) = (\theta_t(x), w(t, x)g).$$

Then the cocycle condition is equivalent to the fact that this defines a flow. Thus

$$\{\text{cocycles } \mathbb{T} \times X \to G\} \leftrightarrow \{ \text{ actions of } \mathbb{T} \text{ on } X \times G \text{ that lift } \mathbb{T} \curvearrowright X\}.$$

Consider now a group (time) $\mathbb{T}$ acting on a space X and a group G acting on a space Y. Then for every cocycle $w : \mathbb{T} \times X \to G$ we can define a lift of the action (flow) of $\mathbb{T}$ on X to an action (flow) of $\mathbb{T}$ on $X \times Y$ by

$$t \cdot (x, y) = (\theta_t(x), w(t, x) \cdot y).$$

In this case, the cocycle condition guarantees that this defines a flow on $X \times Y$ (but being a flow on $X \times Y$ does not guarantee that the cocycle condition holds, unless the action of G on Y is faithful).

Example 9 (Linear cocycles)**.** For an example of the second construction above, one can consider a space X with a flow ϕ_t, the space $Y = \mathbb{R}^d$ and the group $G = \mathrm{GL}(d, \mathbb{R})$ acting on it. Then the cocycle $w : \mathbb{T} \times X \to G$ defines a flow on the trivial vector bundle $X \times \mathbb{R}^d$ over X, which covers the flow ϕ_t on X. This is called a *linear cocycle* over the flow ϕ_t in [BPS19].

Remark 6. In the case of discrete time $\mathbb{T} \in \{\mathbb{Z}, \mathbb{N}\}$, the flow is determined by the map $T = \theta_1 : X \to X$, and the cocycle is determined by the map $w = w_1 : X \to G$ by the relations:

$$\begin{aligned} w(n, x) &= w(T^{n-1}(x)) \cdots w(T(x))w(x) \text{ for } n > 0, \\ w(0, x) &= e, \\ w(-n, x) &= w(T^{-n}(x))^{-1} \cdots w(T^{-1}(x))^{-1} \text{ for } n > 0. \end{aligned} \tag{6.1}$$

We call the map $w : \mathbb{T} \times X \to G$ the *generator* of the cocycle.

On the other hand, given a map $w : X \to G$, and a map $T : X \to X$, the relations in Equation (6.1) define a cocycle on X over the flow generated by T (which is over $\mathbb{Z}$ when T is invertible and over $\mathbb{N}$ otherwise).

If the flow of the cocycle is over a probability space, then the cocycle is called a greg:

Definition 40. Let $(X, \mathcal{X}, m)$ be a probability space, $T : X \to X$ a measurable measure-preserving transformation. Then we say that $(X, \mathcal{X}, m, T, w, G)$ is a *group random element generator*, where $w : \mathbb{T} \times X \to G$ is the discrete-time cocycle corresponding to the map $w : X \to G$ over the flow generated by $T : X \to X$.

(Directed) Lyapunov exponents of cocycles

Definition 41. Let $w : \mathbb{N} \times X \to \mathrm{GL}(d, \mathbb{R})$ be a cocycle with values in $\mathrm{GL}(d, \mathbb{R})$. We define the *upper Lyapunov exponent* of $(x, v) \in X \times \mathbb{R}^d$ with respect to w as

$$\bar{\chi}^+(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log \|w(n, x)v\| \in [-\infty, +\infty].$$

If the limit

$$\chi^+(x, v) = \lim_{n\to\infty} \frac{1}{n} \log \|w(n, x)v\|$$

exists, we call it the *Lyapunov exponent* of (x, v) with respect to w.

The proof of the following is [KKH95, Lemma S.2.6]

Lemma 24 (Properties of Lyapunov exponents)**.** Let $w : \mathbb{N} \times X \to \mathrm{GL}(d, \mathbb{R})$ be a cocycle with values in $\mathrm{GL}(d, \mathbb{R})$. Then

(i) $\bar{\chi}^+(x, v)$ is homogeneous in v, i.e. $\bar{\chi}^+(x, tv) = \bar{\chi}^+(x, v)$ for every $t \in \mathbb{R} \setminus \{0\}$.

(ii) $\bar{\chi}^+(x, v_1 + v_2) \leq \max\{\bar{\chi}^+(x, v_1), \bar{\chi}^+(x, v_2)\}$ for every $v_1, v_2 \in \mathbb{R}^d$. If in particular, $\bar{\chi}^+(x, v_1) < \bar{\chi}^+(x, v_2)$ then $\bar{\chi}^+(x, v_1 + v_2) = \bar{\chi}^+(x, v_2)$.

We can now define the Lyapunov exponents of a cocycle at a point, as the discontinuity points of the function $\chi \mapsto \dim E_\chi(x)$, where $E_\chi(x) = \{v \in \mathbb{R}^d : \bar{\chi}^+(x, v) \leq \chi\}$.

Definition 42. Let $w : \mathbb{N} \times X \to \mathrm{GL}(d, \mathbb{R})$ be a cocycle with values in $\mathrm{GL}(d, \mathbb{R})$. If the Lyapunov exponents of w at some point x are all finite, then there exist

$$\chi_1(x) < \chi_2(x) < \cdots < \chi_{k(x)}(x) \in \mathbb{R},$$
$$\{0\} \subsetneq E_1(x) \subsetneq E_2(x) \subsetneq \cdots \subsetneq E_{k(x)}(x) = \mathbb{R}^d,$$

where we define the vector spaces $E_\chi(x) = \{v \in \mathbb{R}^d : \bar{\chi}^+(x, v) \leq \chi\}$, and we ask that for $v \in E_i(x) \setminus E_{i-1}(x)$ we have $\bar{\chi}^+(x, v) = \chi_i(x)$.

The numbers $\chi_i(x)$ are called the *upper Lyapunov exponents* of w at x, and the spaces $E_i(x)$ are called the *filtration* at x associated to the cocycle w. The number

$$\lambda_i(x) = \dim E_i(x) - \dim E_{i-1}(x), i = 2, \ldots, k(x), \quad \lambda_1(x) = \dim E_1(x)$$

is called the *multiplicity* of the Lyapunov exponent $\chi_i(x)$, and the set

$$\mathrm{Sp}_x w = \{(\chi_i(x), \lambda_i(x)) : i = 1, \ldots, k(x)\}$$

is called the *spectrum* of w at x.

Remark 7 (Well-definiteness of exponents and filtrations)**.** First note that if all Lyapunov exponents at x are finite, then the Lyapunov exponents at x are bounded uniformly in $v \in \mathbb{R}^d \setminus \{0\}$, since for any base $\{e_1, \ldots, e_d\}$ of $\mathbb{R}^d$ we have $\bar{\chi}^+(x, v) \leq \max_{i=1,\ldots,d} \bar{\chi}^+(x, e_i)$ for every $v \in \mathbb{R}^d \setminus \{0\}$. Thus the plot of $\dim E_\chi$ shown in Figure 6.2 shows that the filtration is finite. Moreover, it gives a reason for asking that $\bar{\chi}^+(x, v) = \chi_i(x)$ for $v \in E_i(x) \setminus E_{i-1}(x)$, since otherwise, one could pick χ

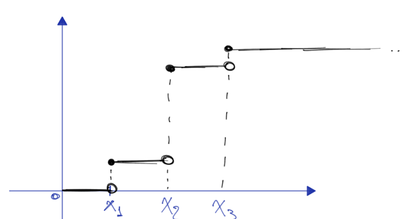


Figure 6.2: Filtration of Lyapunov exponents: the plot of $\dim E_\chi$ as a function of χ is a non-decreasing step function.

If we restrict ourselves to linear gregs, i.e. gregs with cocycles $w : \mathbb{T} \times X \to \mathrm{GL}(d, \mathbb{R})$ taking values in $\mathrm{GL}(d, \mathbb{R})$, then the pointwise version of Osedelec's multiplicative ergodic theorem tells us that the directed Lyapunov exponents exist for almost every point. Its statement (and some more results) can be found as [KKH95, Theorem S.2.9]:

Theorem 8 (Osedelec's pointwise multiplicative theorem). Let $(X, \mathcal{X}, m, T, w, \mathrm{GL}(d, \mathbb{R}))$ be a $\mathbb{Z}$-time greg, and assume that $(X, \mathcal{X}, m)$ is a Lebesgue space. If

$$\log^+ \|w(x)^{\pm 1}\| \in L^1(X, m),$$

then there exists a co-null set $Y \subseteq X$ such that for every $x \in Y$:

(i) There exists a decomposition

$$\mathbb{R}^d = \bigoplus_{i=1}^{k(x)} H_i(x),$$

that is invariant under the action of $\mathbb{Z}$ on $X \times \mathbb{R}^d$.

(ii) The Lyapunov exponents over x exist, are finite, invariant under T, the same in all directions of $H_i(x) \setminus \{0\}$. In particular,

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|w(T^n(x))\| = \pm \chi_i(x)$$

uniformly in $v \in H_i(x) \setminus \{0\}$.

Osedelec's theorem and equivalent definitions of Lyapunov exponents

What is the relationship between directed and nondirected exponents?

Using Kingman's subadditive ergodic theorem, one can give for ergodic gregs a definition of Lyapunov exponents that does not depend on direction v and the point x. The statement of the following can be found as [DK+16, Theorem 4.3], or [BF25, Theorem 5.17]

Theorem 9 (Kingman's subadditive ergodic theorem). Let $(X, \mathcal{X}, m, T, w, \mathrm{GL}(d, \mathbb{R}))$ be a $\mathbb{Z}$-time greg, with $(X, \mathcal{X}, m, T)$ ergodic. If $f_n : X \to [-\infty, +\infty)$ is a sequence of measurable functions such that

$$f_{n+m} \leq f_n + f_m \circ T^n,$$

and $f_1^+ \in L^1(X, m)$, then for m-almost every $x \in X$ we have

$$\lim_{n \to \infty} \frac{1}{n} f_n(x) = \inf_{n \in \mathbb{N}} \frac{1}{n} \int_X f_n dm$$

where the limit exists and is finite.

Definition 43 ((Nondirected) Lyapunov exponents for ergodic gregs).
Let $(X, \mathcal{X}, m, T, w, \mathrm{GL}(d, \mathbb{R}))$ be a $\mathbb{N}$-time greg, with $(X, \mathcal{X}, m, T)$ ergodic. If

$$\log^+ \|w(x)\| \in L^1(X, m),$$

then the limits

$$\chi_1 = \lim_{n\to\infty} \frac{1}{n} \int_X \log \|w(n,x)\| dm(x),$$
$$\chi_1 + \cdots + \chi_i = \lim_{n\to\infty} \frac{1}{n} \int_X \log \| \wedge^i w(n,x)\| dm(x), \quad i = 2, \dots, d,$$

exist, are finite, and are called the *Lyapunov exponents* of w. Moreover,

(i) The limits above can be replaced with $\inf_{n\in\mathbb{N}}$.

(ii) The Lyapunov exponents χ_i can be computed as

$$\chi_1 = \lim_{n\to\infty} \frac{1}{n} \log \|w(n,x)\|,$$
$$\chi_1 + \cdots + \chi_i = \lim_{n\to\infty} \frac{1}{n} \log \| \wedge^i w(n,x)\|, \quad i = 1, \dots, d.$$

(iii) The Lyapunov exponents χ_i can also be computed as

$$\chi_1 = \lim_{n\to\infty} \frac{1}{n} \log \sigma_1(w(n,x)),$$
$$\chi_i = \lim_{n\to\infty} \frac{1}{n} \log \sigma_i(w(n,x)), \quad i = 1, \dots, d,$$

where $\sigma_1(g) \geq \sigma_2(g) \geq \cdots \geq \sigma_d(g)$ are the singular values of $g \in \mathrm{GL}(d,\mathbb{R})$.

Proof of equivalence. (i) Follows from Theorem 9 applied to the functions $f_n(x) = \log \| \wedge^i w(n,x)\|$. To see that the cocycles $\wedge^i w(n,x)$ will give a function in $L^1(X,m)$, note that $\| \wedge^i w(x)\| \leq \|w(x)\|^i$ for every $x \in X$.

(ii) The equalities follow again from Theorem 9 applied to the functions

$$f_n(x) = \log \|w(n,x)\| \text{ and } f_n(x) = \log \| \wedge^i w(n,x)\|.$$

(iii) The first equality follows from the previous point and the fact that $\sigma_1(g) = \|g\|$ for every $g \in \mathrm{GL}(d,\mathbb{R})$, where the norm is the operator norm induced by the Euclidean norm on $\mathbb{R}^d$. For the rest, note that

$$\sigma_i(g) = \frac{\| \wedge^i g\|}{\| \wedge^{i-1} g\|} \text{ for } i = 2, \dots, d,$$

so $\log \sigma_i(g) = \log \| \wedge^i g\| - \log \| \wedge^{i-1} g\|$ for every $g \in \mathrm{GL}(d,\mathbb{R})$, giving us that the result inductively.

□

Equivalent cocycles

Definition 44 (Tempered maps). Let $(X, \mathcal{X}, m, T, w, \mathrm{GL}(d,\mathbb{R}))$ be a greg. A measurable map $C : X \to \mathrm{GL}(d,\mathbb{R})$ is called *tempered* with respect to T if for m-almost every $x \in X$

we have

$$\lim_{n\to\infty} \frac{1}{n} \log \|C(T^n(x))^{\pm 1}\| = 0.$$

The following is [KKH95, Lemma S.2.4]:

Lemma 25 (Criterion for temperedness)**.** Let $(X, \mathcal{X}, m, T, w, \mathrm{GL}(d, \mathbb{R}))$ be a greg on $\mathbb{R}$. If $C : X \to \mathrm{GL}(d, \mathbb{R})$ is such that

$$\max\{\log \|C(x)\|, \log \|C(x)^{-1}\|\} \in L^1(X, m),$$

then C is tempered with respect to T.

Proof. Use Birkhoff's ergodic theorem on the functions $\log \|C(x)\|$ and $\log \|C(x)^{-1}\|$. □

Definition 45 (Equivalent cocycles)**.** Consider two gregs $(A, \mathrm{GL}(d, \mathbb{R}))$ and $(B, \mathrm{GL}(d, \mathbb{R}))$ over the same flow and probability space $(X, \mathcal{X}, m, T)$. We say that the cocycles A and B are *equivalent* if there exists a tempered map $C : X \to \mathrm{GL}(d, \mathbb{R})$ such that for m-almost every $x \in X$ we have

$$A(x) = C(T(x))^{-1} B(x) C(x)^{-1}.$$

A cocycle is called *rigid* if it is equivalent to a cocycle with generator independent from x, i.e. $B(n, x) = B^n$ for some $B \in \mathrm{GL}(d, \mathbb{R})$ and every $n \in \mathbb{N}, x \in X$.

Lyapunov exponents and random products

The goal of this section is to motivate the study of gregs and their Lyapunov exponents, by showing how they arise naturally when studying random products of matrices.

Consider a discrete-time ergodic dynamical system $(X, \mathcal{X}, m, T)$ and a measurable map $\alpha : X \to \mathbb{R}$. To understand this system, a fundamental question is to study the asymptotic behaviour of

$$\alpha(T^{n-1}(x)) + \cdots + \alpha(x) \text{ as } n \to \infty.$$

The answer comes from Birkhoff's ergodic theorem, which states that for m-almost every $x \in X$ we have

$$\alpha(T^{n-1}(x)) + \cdots + \alpha(x) \sim n \int_X \alpha(x) dm(x).$$

A natural generalization is to consider an ergodic dynamical system $(X, \mathcal{X}, m, T)$ and a measurable map $w : X \to \mathbb{G}L(d, \mathbb{R})$, and study the asymptotic behaviour of the product

$$w(T^{n-1}(x)) \cdots w(x) \text{ as } n \to \infty.$$

Note that the product is exactly $w(n, x)$, where $w(n, x)$ is the cocycle generated by w, so our first observation is that we arrive at the study of gregs. On the other hand, Osedelet's multiplicative ergodic theorem tells us that for m-almost every $x \in X$ there the Lyapunov exponents $\chi_1(x) > \cdots > \chi_k(x)$ and a decomposition $\mathbb{R}^d = E_1(x) \oplus \cdots \oplus E_k(x)$ such that for every $v \in E_i(x) \setminus \{0\}$ we have

$$\|w(n, x)v\| \sim e^{n\chi_i(x)},$$

while Kingman's subadditive ergodic theorem tells us that for m-almost every $x \in X$

$$\log \|w(n, x)\| \sim e^{n\chi_1(x)} \int_X \log \|w(x)\| dm(x).$$

2025-10-03

Proof of $2\delta_{1,n} \leq \delta_{1,2}$

Note that critical exponents of functionals are defined for positive functionals on the Weyl chamber because we want that for $s > \delta_\Gamma(\varphi)$ the series converges.

Lemma 26 (Properties of critical exponents)**.** Let Γ be a discrete subgroup of $\mathrm{SL}(d, \mathbb{R})$. Then for functionals $\varphi, \psi : \mathfrak{a} \to \mathbb{R}$ that are positive on the Weyl chamber $\mathfrak{a}^+$ we have

(i) $\delta_\Gamma(\lambda\varphi) = \frac{1}{\lambda}\delta_\Gamma(\varphi)$ for every $\lambda > 0$.

(ii) If $\varphi \leq \psi$ on $\mathfrak{a}^+$, then $\delta_\Gamma(\varphi) \geq \delta_\Gamma(\psi)$.

We want to show that

Proposition 35. Let Γ be a discrete subgroup of $\mathrm{SL}(d, \mathbb{R})$. Then

$$2\delta_{1,n}(\Gamma) \leq \delta_{1,2}(\Gamma),$$

where $\delta_{1,n}(\Gamma) = \delta_\Gamma(\alpha_{1,n})$ and $\delta_{1,2}(\Gamma) = \delta_\Gamma(\alpha_{1,2})$, and $\alpha_{i,j}(\mathrm{diag}(a_1, \dots, a_d)) = a_i - a_j$.

By the above lemma, it would be enough to show that $\alpha_{1,n} \geq \frac{1}{2}\alpha_{1,2}$ on the Weyl chamber $\mathfrak{a}^+$, but this is not true. A counter example in $\mathrm{SL}(3, \mathbb{R})$ is given by the matrix $\mathrm{diag}(3, -2, -1)$, for which $\alpha_{1,3} = 4$ and $\alpha_{1,2} = 5$.

We will use the fact that for the opposition involution $\iota : \mathfrak{a} \to \mathfrak{a}$ in $\mathfrak{sl}(d, \mathbb{R})$ we have

$$\alpha_{i,j}(H) = \alpha_{d-j+1,d-i+1}(\iota(H)).$$

We recall its general definition:

Definition 46 (Opposition involution)**.** Consider a choice of a Cartan subalgebra $\mathfrak{a}$ and a Weyl chamber $\mathfrak{a}^+$ of a semisimple Lie algebra $\mathfrak{g}$. There exists a unique element of the Weyl group w_0 that sends $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. The *opposition involution* is the map $\iota : \mathfrak{a} \to \mathfrak{a}$ defined by $\iota(H) = -w_0(H)$.

We also recall that the Cartan decomposition is unique up to the action of the Weyl group. This is [KK96, Proposition 7.39].

Proposition 36 (Uniqueness of Cartan decomposition)**.** Let G be a reductive group. Then

$$KaK = Ka'K \iff a' = w \cdot a \text{ for some } w \in W,$$

where W is the Weyl group of G and $a, a' \in A$.

Proof of Proposition 35. Note that the proposition above tells us that for every $g \in \mathrm{SL}(d, \mathbb{R})$:

$$\alpha_{n-1,n}(\mu(g)) = \alpha_{1,2}(\mu(g^{-1})),$$

since for $k, l \in K$ we have $g = k\exp(\mu(g))l$ and

$$g^{-1} = l^{-1}\exp(-\mu(g))k^{-1} = l^{-1} - w_0(\mu(g))k^{-1} = l^{-1}\exp(\iota(\mu(g)))k^{-1}$$

, so $\mu(g^{-1}) = \iota(\mu(g))$ which negates and inverses the entries of $\mu(g)$.

Since $\alpha_{1,n} = \alpha_{1,2} + \alpha_{2,3} + \cdots + \alpha_{n-1,n} \geq \alpha_{1,2} + \alpha_{n-1,n}$, Hölder's inequality gives us that for $1/p + 1/q = 1$ and $s > 0$:

$$\sum_{\gamma\in\Gamma} e^{-s\alpha_{1,n}(\mu(\gamma))} \leq \sum_{\gamma\in\Gamma} e^{-s(\alpha_{1,2}(\mu(\gamma))+\alpha_{n-1,n}(\mu(\gamma)))} \leq$$

$$leq \left(\sum_{\gamma\in\Gamma} e^{-s\alpha_{1,2}(\mu(\gamma))}\right)^{1/p} \left(\sum_{\gamma\in\Gamma} e^{-s\alpha_{n-1,n}(\mu(\gamma))}\right)^{1/q}$$

Note that the sum of the right-hand side converges if

$$s > \max\{\frac{1}{p}\delta_{1,2}, \left(1-\frac{1}{p}\right)\delta_{n-1,n}\}$$

which is optimal (i.e. as small as possible) when $p^{-1} = \delta_{n-1,n}/(\delta_{1,2}+\delta_{n-1,n})$. Thus the sum on the left-hand side converges if

$$s > \frac{\delta_{1,2}\delta_{n-1,n}}{\delta_{1,2}+\delta_{n-1,n}}.$$

meaning that

$$\delta_{1,n} \leq \frac{\delta_{1,2}\delta_{n-1,n}}{\delta_{1,2}+\delta_{n-1,n}}.$$

However, the fact that $\alpha_{1,2}(\mu(g)) = \alpha_{n-1,n}(\mu(g^{-1}))$ implies that $\delta_{1,2} = \delta_{n-1,n}$, so the result follows. □

Geodesic flow on groups

We would like to explain the definition of the geodesic flow on a hyperbolic group Γ. To motivate it, we start with the case of the geodesic flow on a simply connected negatively curved manifold M, then pass to the case of a group Γ acting properly discontinuously on a negatively curved manifold, and finally to the case of a hyperbolic group Γ.

Geodesic flow on a simply connected negatively curved manifold

Why do we not allow zero curvature?

Let $\tilde{M}$ be a simply connected, complete, negatively curved manifold. Then by the Cartan-Hadamard theorem, $\tilde{M}$ is diffeomorphic to a ball $\mathbb{B}^n$ and the exponential map $\exp_x : T_x\tilde{M} \to \tilde{M}$ is a diffeomorphism for every $x \in \tilde{M}$. Using $\exp_x$ we can identify the Gromov boundary $\partial_\infty\tilde{M}$ with the unit tangent sphere $\mathbb{S}^{n-1} \simeq T_x^1\tilde{M}$ at x. So we can think of $\tilde{M} \cup \partial_\infty\tilde{M}$ as the closed ball $\overline{\mathbb{B}^n}$, just like in the case of Poincaré-disc model of the hyperbolic space $\mathbb{H}^n$.

We have the following fiber bundle over the space $\partial_\infty\tilde{M}^{(2)}$ of distinct pairs of boundary points, with fiber isomorphic to $\mathbb{R}$:

$$\begin{tikzcd} T^1\tilde{M} \arrow[d, "\pi"] & \mathbb{R} \arrow[l, two heads] \\ \partial_\infty\tilde{M}^{(2)} & \end{tikzcd}$$

where

$$\pi : T^1\tilde{M} \to \partial_\infty \tilde{M}^{(2)}, \quad v \mapsto \left(\lim_{t\to+\infty} \phi_t(v), \lim_{t\to-\infty} \phi_t(v) \right),$$

and $\phi_t : T^1\tilde{M} \to T^1\tilde{M}$ is the geodesic flow on $\tilde{M}$. The fiber over a pair of distinct boundary points $(\xi^-, \xi^+) \in \partial_\infty \tilde{M}^{(2)}$ are the unit tangent vectors pointing towards ξ^+ that are tangent to the geodesic connecting η and ξ^+.

This is a principal bundle. The reals $\mathbb{R}$ should be thought rather as the affine line, since the choice of a base point 0 is not canonical.

To trivialize the bundle, we consider a section $s : \partial_\infty \tilde{M}^{(2)} \to T^1\tilde{M}$ which gives us a base point $s(\xi^-, \xi^+)$ in each fiber $\pi^{-1}(\xi^-, \xi^+)$. Then we can identify $T^1\tilde{M}$ with $\partial_\infty \tilde{M}^{(2)} \times \mathbb{R}$ via the map

$$\begin{aligned} \partial_\infty \tilde{M}^{(2)} \times \mathbb{R} &\to T^1\tilde{M}, \\ (\xi^-, \xi^+, t) &\mapsto \phi_t(s(\xi^-, \xi^+)). \end{aligned}$$

Thus, given a section $s : \partial_\infty \tilde{M}^{(2)} \to T^1\tilde{M}$, we associate to each $v \in T^1\tilde{M}$ the triple (ξ^-, ξ^+, t), where $\xi^\pm$ are the endpoints of the (oriented) geodesic tangent to v and t is the signed distance between $s(\xi^-, \xi^+)$ and v along the geodesic.

Geodesic flow on a negatively curved manifold
Or a group acting properly discontinuously on a negatively curved simply connected manifold

Consider now a complete, negatively curved manifold M. Then M is isometric to the quotient $\Gamma\backslash\tilde{M}$, where $\tilde{M}$ is the universal cover of M and $\Gamma = \pi_1(M)$ acts on $\tilde{M}$ by isometries.

Note that this is equivalent to considering a torsion-free group Γ acting by isometries, properly discontinuously on a simply connected, complete, negatively curved manifold $\tilde{M}$. This is the setting that we will work on. From an analog of the Milnor-Svarc lemma for non-cocompact actions, we know that the orbit maps $\Gamma \to \tilde{M}$ are quasi-isometric embeddings. Extending these to topological embeddings of the boundaries gives us an embedding of the boundaries $\partial_\infty\Gamma \subseteq \partial_\infty\tilde{M}$, which does not depend on the choice of the orbit.

Our goal is to define a properly discontinuous action of Γ on $T^1\tilde{M}$ that commutes with the geodesic flow $\phi_t : T^1\tilde{M} \to T^1\tilde{M}$. Then we will have defined a geodesic flow on $M = \Gamma\backslash\tilde{M}$ as the quotient flow $\Gamma\backslash T^1\tilde{M}$. We will use the (arbitrary) choice of a section $s : \partial_\infty\tilde{M}^{(2)} \to T^1\tilde{M}$ to obtain a trivialization $\Phi : \tilde{M}^{(2)} \times \mathbb{R} \to T^1\tilde{M}$. We define the action of γ on $\partial_\infty\tilde{M}^{(2)} \times \mathbb{R}$ as

$$\begin{aligned} \Phi^{-1}\gamma\Phi : \partial_\infty\tilde{M}^{(2)} \times \mathbb{R} &\to \partial_\infty\tilde{M}^{(2)} \times \mathbb{R}, \\ (\xi^-, \xi^+, t) &\mapsto (\gamma\xi^-, \gamma\xi^+, t + c(\gamma, \xi^-, \xi^+)), \end{aligned}$$

where $c : \Gamma \times \partial_\infty\tilde{M}^{(2)} \to \mathbb{R}$. Since this clearly lifts the action of Γ on $\tilde{M}$, we know from the theory of cocycles that $\Phi^{-1}\gamma\Phi$ is an action if and only if c is a cocycle. To see this, one can use the definition of Φ, or the fact that γ acts by isometries.

> **Remark 8** (Action and arbitrariness of parametrisation). Note that while the action of Γ on $T^1\tilde{M}$ is canonical, the action on $\partial_\infty\tilde{M}^{(2)} \times \mathbb{R}$ depends on the choice of the section $s : \partial_\infty\tilde{M}^{(2)} \to T^1\tilde{M}$. To see this consider another section $s' : \partial_\infty\tilde{M}^{(2)} \to T^1\tilde{M}$, which we denote with $s'(x) = (x, \bar{s}'(x))$, and let
>
> $$f : \partial_\infty\tilde{M}^{(2)} \to \mathbb{R}, \quad f(x) = \bar{s}'(x) - \bar{s}(x).$$

Then using the definition of Φ one can check that $\phi_t\Phi(x,t) = \Phi T_{f(x)}(x,t)$, where $T_s : \mathbb{R} \to \mathbb{R}$ is the translation $T_s(t) = t + s$, and that $\Phi'(x,t) = \phi_{f(x)}\Phi(x,t)$. Using this one can calculate that

$$\Phi'^{-1}\gamma\Phi'(x,t) = (\gamma x, t + c(\gamma, x) + f(x) - f(\gamma x)).$$

Hence $c'(\gamma, x) = c(\gamma, x) + f(x) - f(\gamma x)$, i.e. c and c' are cohomologous cocycles.

In this sense, the parametrised action of Γ on $\partial_\infty \tilde{M}^{(2)} \times \mathbb{R}$ is well-defined up to cocycle cohomology class.

The following lemma shows that the action of Γ on $T^1\tilde{M}$ is properly discontinuous.

Lemma 27. If a group Γ acts properly discontinuously on a manifold N, then it acts properly discontinuously on T^1N.

Moreover, since Γ acts by isometries, the action on $T^1\tilde{M}$ commutes with the geodesic flow $\phi_t : T^1\tilde{M} \to T^1\tilde{M}$, which thus descends to a flow on $\Gamma\backslash T^1\tilde{M} = T^1M$. Denoting with $\pi : T^1\tilde{M} \to \partial_\infty\tilde{M}^{(2)}$ the projection, one can show that

Fact 1. The set

$$\Gamma\backslash\partial_\infty\Gamma^{(2)} \times \mathbb{R} \simeq \Gamma\backslash\pi^{-1}(\partial_\infty\Gamma^{(2)}) \subseteq T^1M$$

is compact and coincides with the non-wandering set of the geodesic flow on T^1M.

In the case where Γ is cocompact, this is the whole T^1M, since $\partial_\infty\Gamma = \partial_\infty\tilde{M}$. Then things are a bit easier:

Exercise 1. Let Γ be a torsion-free discrete group acting cocompactly by isometries on a complete negatively curved manifold $\tilde{M}$. Then Γ acts properly discontinuously and cocompactly on T^1M.

Solution. We will denote with $\pi : T^1M \to M$ the bundle projection. It is easy to see that the quotient map $T^1M \to T^1M/\Gamma$ lifts the quotient map $M \to M/\Gamma$, making the following diagram commute:

$$\begin{tikzcd} T^1M \arrow[r] \arrow[d, "\pi"] & T^1M/\Gamma \arrow[d, "\bar{\pi}"] \\ M \arrow[r] & M/\Gamma \end{tikzcd}$$

To prove proper discontinuity, let $K \subseteq T^1M$ be compact and $\gamma_n \in \Gamma$ be a sequence in Γ such that $\gamma_nK \cap K \neq \emptyset$. Then $\pi(\gamma_nK) \cap \pi(K) \neq \emptyset$, so $\{\gamma_n\}$ is finite since Γ acts properly discontinuously on M.

For cocompactness, let $K \subseteq M$ be a compact set such that $\Gamma K = M$ and $v_n \in T^1M$ be a sequence. Then we can find $\gamma_n \in \Gamma$ such that $\gamma_n\pi(v_n) = \pi(\gamma_nv_n) \in \pi(K)$, which up to a subsequence will converge to a point $p \in \pi(K)$. Using an orthonormal frame around p, we can trivialize a neighborhood of p in T^1M as $\Phi : \pi^{-1}(U) \to U \times \mathbb{S}^{n-1}$. Then $\Phi(\gamma_nv_n)$ will converge to a point in $\{p\} \times \mathbb{S}^{n-1}$, so $[\gamma_nv_n]$ will converge to a point in $T^1(M)/\Gamma$.

Alternatively, one can show that T^1M/Γ is homeomorphic to $T^1(M/\Gamma)$, which is compact since M/Γ is compact. $\square$

Geodesic flow on a hyperbolic group

Let Γ be a hyperbolic group acting properly discontinuously and cocompactly on a hyperbolic space (this is the case for instance when X is the Cayley graph of Γ with respect to a finite generating set). A consequence of the Milnor-Svarc lemma is that $\partial_\infty \Gamma \simeq \partial_\infty X$.

> **Remark 9** (Analogies to negatively curved manifolds)**.** Here X is an analog of the universal cover of a negatively curved manifold, and Γ is an analog of the fundamental group acting by isometries on X.

The analog of the unit tangent bundle is defined as the set of parametrized unit-speed geodesics

$$T^1X = \{v : \mathbb{R} \to X \mid v \text{ is an isometry}\},$$

which comes with the analog of the bundle projection and the geodesic flow

$$\pi : T^1X \to X, \quad v \mapsto (v(-\infty), v(+\infty)),$$
$$\phi_t : T^1X \to T^1X, \quad v \mapsto (s \mapsto v(s+t)).$$

One can show that the action of Γ on X induces a properly discontinuous and cocompact action on T^1X that commutes with the geodesic flow. As before, we can consider the fiber bundle

$$\begin{tikzcd}
T^1X \arrow[d, "\rho"] & & \mathbb{R} \arrow[ll] \\
\partial_\infty X^{(2)} \simeq \partial_\infty \Gamma^{(2)}
\end{tikzcd}$$

where the projection is defined as before

$$\rho : T^1X \to \partial_\infty X^{(2)}, \quad v \mapsto \left(\lim_{t\to+\infty} \phi_t(v), \lim_{t\to-\infty} \phi_t(v) \right),$$

and for a section $s : \partial_\infty \Gamma^{(2)} \to T^1X$ we have a trivialization $\Phi : \partial_\infty \Gamma^{(2)} \times \mathbb{R} \to T^1X$. Then one can show

> **Fact 2.** There exists a cocycle $c : \Gamma \times \partial_\infty X^{(2)} \to \mathbb{R}$ such that the action of Γ on T^1X is properly discontinuous, cocompact and commutes with the geodesic flow.

2024-10-07

A reference for the discussion below is [Tho19].

Let S be a closed surface of genus $g \geq 2$. Then by the classical Teichmüller theory, the Teichmüller space $\mathcal{T}(S)$ of S has at least two models:

$$\left\{ \begin{array}{l} \text{Discrete faithful} \\ \text{representations} \\ \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R}) \end{array} \right\} // \, \mathrm{PSL}(2,\mathbb{R}) \simeq \left\{ \begin{array}{l} \text{Hyperbolic markings} \\ \text{on } S \end{array} \right\} / \mathrm{Diff}_0(S).$$

Thus for a given representation $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$, we can consider the associated hyperbolic surface $\Sigma = \rho(\pi_1(S))\backslash\mathbb{H}^2$ which carries a class of hyperbolic metrics.

Consider a representation $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ and the associated hyperbolic metric on $S \simeq \rho(\pi_1(S))\backslash\mathbb{H}^2$. Then we can consider the geodesic flow $\phi_t : T^1S \to T^1S$ on the unit tangent bundle of S, construct a bundle $E_\rho = T^1\mathbb{H}^2 \times \mathbb{R}^2/\pi_1(S)$ over T^1S and lift the geodesic flow to a linear flow $\psi_t : E_\rho \to E_\rho$ on E_ρ, à la [BPS19]. In this way, the choice of a parametrization of the geodesic flow on S (i.e. a choice of a hyperbolic metric) is compatible with the choice of a representation $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ used to construct the bundle E_ρ and the linear flow $\psi_t : E_\rho \to E_\rho$.

However, one could consider a different parametrization of the geodesic flow on S induced by a different hyperbolic metric. It is a fact that the geodesic flows on S induced by two different hyperbolic metrics are orbit equivalent, i.e. there exists a homeomorphism $h : T^1S \to T^1S$ that sends orbits of one flow to orbits of the other. Using the same parametrization, we could then lift the geodesic flow to a linear flow on E_ρ. The new flow $\psi'_t : E_\rho \to E_\rho$ is dominated if and only if the original flow $\psi_t : E_\rho \to E_\rho$ is dominated, which makes sense given the fact that ρ being Anosov does not depend on the choice of a hyperbolic metric on S.

Thus there is no reason to look for compatibility in the choice of a hyperbolic metric on S and a representation $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$. More on that can be found in

To see how the orbital equivalence of two geodesic flows on S works, consider two hyperbolic metrics g, g' on S and the associated geodesic flows $\phi_t : T^1S \to T^1S$ and $\phi'_t : T^1S \to T^1S$. Let $j : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ and $j' : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ be the two representations associated to g and g' respectively. To recall how to construct these representations, we consider the universal cover $\tilde{S}$ of S. Then one can lift the metrics g and g' to metrics on $\tilde{S}$, for which the fundamental group $\pi_1(S)$ acts by isometries. Since $\tilde{S}$ with each of these metrics is a simply connected, complete, Riemannian manifold of constant curvature -1, we can obtain two isometries $f, f' : \tilde{S} \to \mathbb{H}^2$. Then we can define for $f(\tilde{x}) \in \mathbb{H}^2$ and $\gamma \in \pi_1(S)$:

$$j(\gamma) \cdot f(\tilde{x}) = f(\gamma \cdot \tilde{x}),$$

and similarly for j'.

In particular, the maps f, f' are equivariant, so the map $f' \circ f^{-1} : \mathbb{H}^2 \to \mathbb{H}^2$ is a j-j'-equivariant isometry. This isometry induces a homeomorphism $\partial_\infty(f' \circ f^{-1}) : \partial_\infty\mathbb{H}^2 \to \partial_\infty\mathbb{H}^2$ that is also j-j'-equivariant. We consider the (j, j')-equivariant homeomorphism $\tilde{h} : \partial_\infty{\mathbb{H}^2}^{(3)} \to \partial_\infty{\mathbb{H}^2}^{(3)}$. Identifying $T^1\mathbb{H}^2$ with $\partial_\infty{\mathbb{H}^2}^{(3)}$ using the orthogonal projection of the third point on the geodesic defined by the first two points, we obtain a j-j'-equivariant homeomorphism $\tilde{h} : T^1\mathbb{H}^2 \to T^1\mathbb{H}^2$ that descends to a homeomorphism $h : T^1S \to T^1S$. One can check that h sends orbits of ϕ_t to orbits of ϕ'_t, giving us the desired orbit equivalence (see Figure 6.3).

One should note that the homeomorphism $h : T^1S \to T^1S$ is not induced from a diffeomorphism of S. Thus we have proved that

Proposition 37. All geodesic flows on a closed surface S of genus $g \geq 2$ are orbit equivalent.

2024-10-9

The critical exponent of a subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ is not the same as its critical exponent, seen as a subgroup of $\mathrm{SO}_0(2,1)$.

Proposition 38. Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$, and denote with $\rho : \mathrm{SL}(2,\mathbb{R}) \to$

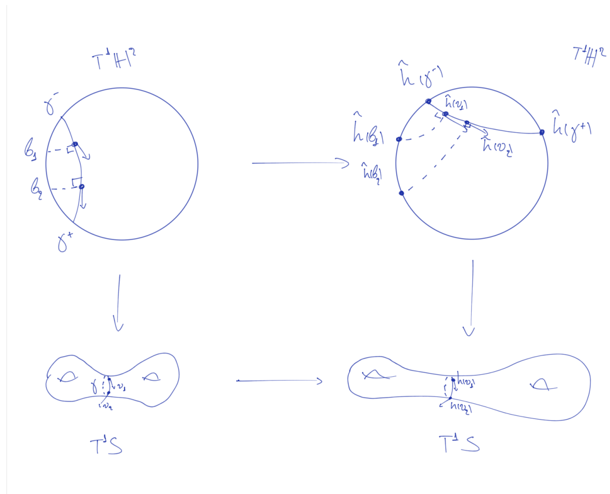


Figure 6.3: Orbit equivalence between geodesic flows associated to different hyperbolic metrics on S.

$\mathrm{SO}_0(2,1)$ the irreducible isomorphism. Then

$$\delta_{1,2}(\Gamma) = 2\delta(\rho(\Gamma)).$$

In particular, the Cartan projections $\mu : \mathrm{SL}(2,\mathbb{R}) \to \mathfrak{a}^+$ and $\mu' : \mathrm{SO}_0(2,1) \to \mathfrak{a}'^+$ of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SO}_0(2,1)$ are related via

$$\mu'(\rho(g)) = (2\mu_1(g), 0, -2\mu_1(g)).$$

Proof. The equality for the critical exponents follows from the equality of the Cartan projections, so it suffices to prove the latter.

Recall that the symmetric square of the standard representation of $\mathrm{SL}(2,\mathbb{R})$ is an isomorphism $\rho : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}_0(\mathrm{Sym}^2\mathbb{R}^2, B)$, where $B = \mathrm{Sym}^2(\det)$ is a bilinear form of signature $(2,1)$. In particular, in the base e_1^2, e_1e_2, e_2^2 of $\mathrm{Sym}^2\mathbb{R}^2$ we have

$$B = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

We can calculate that in the same basis for $\mathrm{Sym}^2\mathbb{R}^2$ and for the standard basis of $\mathbb{R}^2$ we have

$$\rho_*\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) = \begin{pmatrix} 2a & b & 0 \\ 2c & 0 & 2b \\ 0 & c & -2a \end{pmatrix}.$$

Thus we have that

$$\rho_*(\mu(g)) = \begin{pmatrix} 2\mu_1(g) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2\mu_1(g) \end{pmatrix}.$$

Writing the Cartan decomposition $g = k\exp(\mu(g))l$ for $k, l \in \mathrm{SO}(2)$, we have that $\rho(g) = \rho(k)\exp(\rho_*(\mu(g)))\rho(l)$ is a Cartan decomposition of $\rho(g)$, since $\rho(\mathrm{SO}(2))$ is a maximal compact subgroup of $\mathrm{SO}_0(2,1)$. Thus $\mu'(\rho(g)) = (2\mu_1(g), 0, -2\mu_1(g))$. $\square$

Proposition 39. Let $\Gamma \subseteq \mathrm{SL}(2,\mathbb{R})$ be a uniform lattice, and consider the irreducible representation $\rho_{\mathrm{irr}} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R})$ and the reducible representation $\rho_{\mathrm{red}} : \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R})$. Both restrictions on Γ are projective Anosov, and we have

$$\begin{aligned} 2\delta_{1,3}(\rho_{\mathrm{irr}}(\Gamma)) &= \delta_{1,2}(\rho_{\mathrm{irr}}(\Gamma)) = 1, \\ \delta_{1,3}(\rho_{\mathrm{red}}(\Gamma)) &= \frac{1}{2}\delta_{1,2}(\rho_{\mathrm{red}}(\Gamma)) = 1. \end{aligned}$$

Before the next example, we will need the following theorem:

Theorem 10 (Borel density theorem)**.** Let G be a semisimple real algebraic group and Γ be a lattice in G. Then Γ is Zariski-dense in G.

Proposition 40. Let Γ be a lattice of $\mathrm{SL}(2,\mathbb{R})$ and $u : \Gamma \to \mathbb{R}^2$. We define $\rho_u : \Gamma \to \mathrm{SL}(3,\mathbb{R})$ as

$$\rho_u(\gamma) = \begin{pmatrix} \gamma & u(\gamma) \\ 0 & 1 \end{pmatrix}.$$

Then

1. ρ_u is a representation if and only if u is a cocycle:

$$u(\gamma\eta) = u(\gamma) + \gamma u(\eta).$$

2. ρ_u is projective Anosov.

3. The limit sets in the projective space are independent of u and are all equal to the projective line $\mathbb{P}(\mathbb{R}^2 \times \{0\})$.

4. The dual limit set $\xi_u^* : \partial_\infty\Gamma \to \mathbb{P}((\mathbb{R}^3)^*)$ is a projective line if and only if u is a cobordism.

Proof. 1. This is a straightforward calculation.

2. First note that ρ_0 is projective Anosov, so ρ_{tu} is projective Anosov for t small enough, by the fact that the set of Anosov representations is open in the representation variety. Moreover, ρ_u is conjugate to ρ_{tu}.

3. First note that $W = \mathbb{R}^2 \times \{0\}$ is $\rho_u(\Gamma)$-invariant. This implies that the limit set $\Lambda_{\rho_u(\Gamma)}$ is contained in $\mathbb{P}(W)$. Indeed, one reason is that the projective limit set can be defined as the smallest closed $\rho_u(\Gamma)$-invariant subset of $\mathbb{P}(\mathbb{R}^3)$. Alternatively, for any $w \in W$, we can find some $\gamma_- \in \partial_\infty\Gamma$ such that $w \notin \xi_u^*(\gamma_-)$, because $\partial_\infty\Gamma$ is infinite and ξ_u^* is injective. Then $\lim_{n\to\infty} \rho_u(\gamma^n)\mathbb{R}w = \xi_u(\gamma_+) \in \mathbb{P}(W)$, where γ_+ is the attracting fixed point of γ.

   To see the reverse inclusion, note first that since Γ is a lattice of $\mathrm{SL}(2,\mathbb{R})$, its limit set is the whole $\mathbb{P}(\mathbb{R}^2)$. It is thus reasonable to expect that $\Lambda_{\rho_u(\Gamma)} = \mathbb{P}(W)$, given that the action of $\rho_u(\Gamma)$ on W is the same as the action of Γ on $\mathbb{R}^2$. Indeed, for $\mathbb{R}(x,0) \in \mathbb{P}(W)$, we can find a sequence $\gamma_n \in \Gamma$ such that $\lim_{n\to\infty} \gamma_n\mathbb{R}x = \mathbb{R}x'$. Then $\lim_{n\to\infty} \rho_u(\gamma_n)\mathbb{R}(x,0) = \mathbb{R}(\gamma_n x, 0) = \mathbb{R}x'$, so $\mathbb{P}(W) \subseteq \Lambda_{\rho_u(\Gamma)}$.

4. First note that ρ_u is conjugated to ρ_0 if and only if u is a cobordism, so if u is a cobordism then ξ_u^* is a projective line. If ξ_u^* is a projective line $\mathbb{P}(W^*)$, then W^* is $\rho_u(\Gamma)$-invariant, so the annihilator $W = (W^*)^\perp$ is also $\rho_u(\Gamma)$-invariant and $\dim W = 1$.

   We will show that ρ_u does not preserve any line if u is not a cobordism, which will conclude the proof. Assuming the contrary, writing $(w,t) \in W$, for every γ there exists $\lambda(\gamma)$ such that
$$\rho_u(\gamma)(w,t) = (\gamma w + tu(\gamma), t) = (\lambda(\gamma)w, \lambda(\gamma)t).$$
   If $t = 0$, this implies that $\gamma w = \lambda(\gamma)w$, so $\mathbb{R}w$ is Γ-invariant. But $\Gamma \subseteq \mathrm{SL}(2,\mathbb{R})$ is a lattice, so by the Borel density theorem it is Zariski-dense, and thus does not preserve any line. Thus $t \neq 0$. The calculation above shows that $\lambda(\gamma) = 1$ for every γ, so $u(\gamma) = w - \gamma w$, i.e. u is a cobordism.

□

Theorem 11 (Equivalence of notions of entropy)**.** Let $\pi_1(M)$ be the fundamental group of a closed negatively curved manifold M. Then the following notions of entropy coincide:

- the topological entropy of the geodesic flow on T^1M;
- the volume entropy of M (i.e. the exponential growth rate of the volume of balls in the universal cover of M);
- the critical exponent of $\pi_1(M)$ seen as a discrete subgroup of the isometry group of the universal cover of M:
$$\delta(\pi_1(M)) = \limsup_{R\to\infty} \frac{1}{R} \log \#\{\gamma \in \pi_1(M) : d(\tilde{x}, \gamma\tilde{x}) \leq R\},$$
  where $\tilde{x}$ is a point in the universal cover of M.
- The critical exponent of the length spectrum of M:
$$\delta_{\text{length}}(\pi_1(M)) = \limsup_{R\to\infty} \frac{1}{R} \log \#\{[\gamma] \in [\pi_1(M)] : \ell([\gamma]) \leq R\},$$
  where $[\pi_1(M)]$ is the set of conjugacy classes of $\pi_1(M)$ and $\ell([\gamma])$ is the length of the closed geodesic in the free homotopy class associated to $[\gamma]$.