Notes

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Contents

Chapter 1

Lorentzian spaces

Here we recall some basic facts on Lorentzian spaces. We will introduce Lorentzian manifolds of constant sectional curvature and we will see that, as in the Riemannian case, two Lorentzian manifolds of constant sectional curvature K are locally isometric. Generally, we will focus on those with maximal isometry group, as they provide models of manifolds of constant sectional curvature: if M is a Lorentzian manifold with constant sectional curvature K and maximal isometry group, then any Lorentzian manifold with constant sectional curvature K carries a natural $(\mathrm{Isom}(M), M)$-atlas made of local isometries. Simply connected space forms have maximal isometry group, but in general there are manifolds with maximal isometry group which are not simply connected. In particular, we will focus on the case $K = -1$ and in that case it will be convenient to use models which are not simply connected.

1.1 Basic definitions

Definition 1.1.1. (i) A *Lorentzian metric* on a manifold of dimension $n + 1$ is a non-degenerate symmetric 2-tensor g of signature $(n, 1)$.

(ii) A *Lorentzian manifold* is a connected manifold M equipped with a Lorentzian metric g.

(iii) In a Lorentzian manifold M we say that a non-zero vector $v \in TM$ is *spacelike, lightlike, timelike* if $g(v, v)$ is respectively positive, zero or negative. More generally, we say that a linear subspace $V \subset T_xM$ is *spacelike, lightlike, timelike* if the restriction of g_x to V is positive definite, degenerate or indefinite.

(iv) A differentiable curve is *spacelike, lightlike, timelike* if its tangent vector is spacelike (resp. lightlike, timelike) at every point. It is *causal* if the tangent vector is either timelike or lightlike.

(v) The set of lightlike vectors of T_xM is also known as the *light cone* at x.

Assuming $\dim M \geq 3$, the light cone disconnects T_xM into three regions: two convex open cones formed by timelike vectors, one opposite to the other, and the region of spacelike vectors.

Definition 1.1.2. Let M be a Lorentzian manifold.

(i) A continuous choice (in the sense of a continuous timelike vector field) of one of the two cones of time-like vectors for each point $x \in M$ is called a *time orientation* of M.

(ii) If a time-orientation of M exists, then M is said to be *time-orientable*. Timelike vectors in the same component as the time-orientation are said *future-directed*, while the rest are said *past-directed*.

(iii) Given a point x in a time-oriented Lorentzian manifold M, the *future* of x is the set $\mathrm{I}^+(x)$ of points which are connected to x by a future-directed causal curve. The *past* of x, denoted $\mathrm{I}^-(x)$, is defined similarly, for past-directed causal curves.

An *orthonormal basis* of T_xM is a basis $v_1, \dots v_{n+1}$ such that $|g(v_i, v_j)| = \delta_{ij}$, with $v_1, \dots v_n$ spacelike, and v_{n+1} timelike. As in the Riemannian setting, on a Lorentzian manifold M there is a unique linear connection ∇ which is symmetric and compatible with the Lorentzian metric g. We refer to it as the *Levi-Civita connection* of M. The Levi-Civita connection determines the Riemann curvature tensor defined by

$$R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w \ .$$

We then say that a Lorentzian manifold M has constant sectional curvature K if

$$g(R(u,v)v,u) = K\left(g(u,u)g(v,v) - g(u,v)^2\right) \tag{1.1}$$

for every pair of vectors $u, v \in T_xM$ and every $x \in M$. This definition is strictly analogous to the definition given in the Riemannian realm. However in this setting the sectional curvature can be defined only for planes in T_xM where g is non-degenerate.

Example 1.1.1. The Minkowski space $\mathbb{R}^{n,1}$ the Levi-Civita connection given by the Euclidean connection:

$$\nabla_X Y = (XY^i)\partial_i,$$

so the Riemann curvature tensor is zero, and the same is true for the sectional curvature.

Finally, we say that M is *geodesically complete* if every geodesic is defined for all times, or in other words, the exponential map is defined everywhere.

1.2 Maximal isometry groups and geodesic completeness

Constant curvature of manifolds allows us to extend isometries of tangent spaces to isometries of the whole manifold. As a result, two Lorentzian manifolds M and N of constant curvature K are locally isometric, a fact which is well-known in the Riemannian setting. More precisely, the following holds:

Lemma 1.2.1. *Let M and N be Lorentzian manifolds of constant curvature K.*

1. *Then every linear isometry $L : T_xM \to T_yN$ extends to an isometry $f : U \to V$, where U and V are neighbourhoods of x and y respectively, and two extensions $f : U \to V$ and $f : U' \to V'$ of L coincide on $U \cap U'$.*

2. *If M is simply connected and N is geodesically complete, then any isometry $L : T_xM \to T_yN$ extends to a unique local isometry $f : M \to N$.*

3. *If M and N are both simply connected and geodesically complete, then any isometry $L : T_xM \to T_yN$ extends to a unique global isometry $f : M \to N$.*

Proof. For the last statement, recall that a local isometry from a simply connected manifold to a uniquely geodesic manifold is a global isometry. $\square$

Exactly as in the Riemannian case the proof is a simple consequence of the classical Cartan–Ambrose–Hicks Theorem. Note that this implies in particular that there is a unique simply connected geodesically complete Lorentzian manifold of constant curvature K up to isometries. For instance for $K = 0$ a model is the Minkowski space $\mathbb{R}^{n,1}$.

Another consequence of Lemma 1.2.1 is that, fixing a point $x_0 \in M$, the set of isometries of M, which we will denote by $\mathrm{Isom}(M)$, can be realized as a subset of $\mathrm{ISO}(T_{x_0}M, TM)$, namely the fiber bundle over M whose fiber over $x \in M$ is the space of linear isometries of $T_{x_0}M$ into T_xM.

It can be proved that $\mathrm{Isom}(M)$ has the structure of a Lie group with respect to composition so that the inclusion $\mathrm{Isom}(M) \hookrightarrow \mathrm{ISO}(T_{x_0}M, TM)$ is a differentiable proper embedding. It follows that the maximal dimension of $\mathrm{Isom}(M)$ is $\dim \mathrm{O}(n, 1) + n + 1 = (n + 1)(n + 2)/2$.

Definition 1.2.1. A Lorentzian manifold M has *maximal isometry group* if the action of $\mathrm{Isom}(M)$ is transitive and, for every point $x \in M$, every linear isometry $L : T_xM \to T_xM$ extends to an isometry of M.

Equivalently M has maximal isometry group if the above inclusion of $\mathrm{Isom}(M)$ into the bundle $\mathrm{ISO}(T_{x_0}M, TM)$ is a bijection. Hence, if M has maximal isometry group, then the dimension of the isometry group is maximal.

From Lemma 1.2.1, every simply connected Lorentzian manifold M has maximal isometry group if it has constant sectional curvature and is geodesically complete. The converse holds even without the simply connectedness assumption:

Lemma 1.2.2. *Let M be a Lorentzian manifold.*

(i) If M has a maximal isometry group then M has constant sectional curvature and is geodesically complete.

(ii) If M is simply connected, then M has maximal isometry group if and only if M has constant sectional curvature and is geodesically complete.

Chapter 2

Linear algebra

2.1 Symplectic forms

Recall that for a bilinear form ω on a vector space V, we can define the matrix $M_B(\omega)$ of ω with respect to a basis $B = (v_1, \ldots, v_n)$ of V by

$$M_B(\omega)_{ij} = \omega(v_i, v_j).$$

In that case, the form is given by

$$\omega\left(\sum_i x_i v_i, \sum_j y_j v_j\right) = \sum_{i,j} x_i y_j M_B(\omega)_{ij} = x^t M_B(\omega) y,$$

where the vectors x, y in the right hand side are represented by their coordinates in the basis B.

Definition 2.1.1. Let V be a complex or real vector space. A symplectic form $\omega : V \times V \to \mathbb{R}$ is a non-degenerate, skew-symmetric (i.e. $\omega(x, y) = -\omega(y, x)$) bilinear form. Equivalently, the associated matrix $M_B(\omega)$ is skew-symmetric and nonsingular.

Proposition 2.1.1. *Every symplectic form on a finite-dimensional vector space V can be written as*

$$\omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

with respect to some basis of V.

Proof. We proceed by induction on $\dim V$. If $\dim V = 2$, then we let e_1 be some non-zero vector and using non-degeneracy, we let e_2 be such that $\omega(e_1, e_2) = 1$. Then $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Supposing the statement holds for $\dim V = 2n$, we consider $\dim V = 2n+2$, and using the same arguements we can find $e_1, e_2 \in V$ such that $\omega(e_1, e_2) = 1$. In particular, $W = \operatorname{span}\{e_1, e_2\}$ is a non-degenerate subspace, so the same will be true for $W^\perp$. By the inductive hypothesis, we can find a basis of $W^\perp$ such that ω is given by

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then we can extend this basis to V by adding e_1, e_2, and after rearranging the order of the basis elements, we obtain the desired form. $\square$

Definition 2.1.2. Let V be a symplectic vector space. A subspace $W \subseteq V$ is called *Lagrangian* if $W = W^\perp$, while a subspace $W \subseteq V$ is called *isotropic* if $W \subseteq W^\perp$.

Proposition 2.1.2. *Let V be a symplectic vector space. Then:*

(i) A subspace is lagrangian subspace if and only if it is maximally isotropic.

(ii) Every isotropic subspace is contained in a lagrangian subspace.

(iii) Symplectic vector spaces have even-dimension.

Proof. 1. Let W be a Lagrangian subspace and W' be an isotropic subspace containing W. If $W \neq W'$, then there exists $v \in W' \setminus W$. Then $v \in W^\perp = W$, a contradiction. Letting W be maximal isotropic, we have that $W \subseteq W^\perp$. But for every $v \in W^\perp$, we have that $\mathbb{C}v + W$ is isotropic (here we use skew-symmetry of ω to obtain that v is isotropic), so $v \in W$. That is $W = W^\perp$ and W is Lagrangian.

2. Let W' be isotropic and not Lagrangian. Then W is not maximal isotropic, that is, there exists $v \notin W$ such that $W + \mathbb{C}v$ is isotropic. Repeating this process, we obtain an increasing chain of isotropic subspaces containing W, which will terminate at a Lagrangian subspace.

3. Using the identity
$$\dim W + \dim W^\perp = \dim V$$
that holds for all subspaces W, we obtain that $2 \dim W = \dim V$ for any lagrangian subspace V.

□

Proposition 2.1.3. *Let ω, ω' be symplectic forms on V. Then there exists a subspace $W \subseteq V$ that is Lagrangian with respect to both forms.*

Proof. Let $A = M_B(\omega')M_B(\omega)^{-1}$ with respect to some basis B of V. This is exactly the matrix for which
$$\omega'(v, w) = \omega(Av, w)$$
Using skew-symmetry, it is easy to see that $A^* = A$ with respect to ω' (where the $*$ denotes the adjoint with respect to ω). Then we can similarly show that it is also symmetric with respect to ω', that is:
$$\omega(Av, w) = \omega(v, Aw), \omega'(Av, w) = \omega'(v, Aw).$$

Consider the generalized eigenvalue decomposition with respect to A:
$$V = \bigoplus_\lambda V_\lambda,$$
where $V_\lambda = \ker\left((A - \lambda I)^{k_\lambda}\right)$ and the sum is taken over all generalized eigenvalues of A. Moreover, the following lemma implies that the decomposition is orthogonal with respect to ω, ω'.

By counting dimensions, we see that if W_λ is a lagrangian subspace of V_λ, then $W = \bigoplus_\lambda W_\lambda$ is a lagrangian subspace of V. Hence it suffices to consider each V_λ separately. There, we may take $W_\lambda = \ker(A - \lambda I)^{k_\lambda - 1}$ and check that it is isotropic with respect to both forms. To do this, we proceed inductively on $\dim V_\lambda$. If $\dim V_\lambda = 2$, then every isotropic subspace is lagrangian, so we can take W_λ to be the span of any vector in V_λ. In particular, W_λ is lagrangian. If $\dim V_\lambda = 2n$, then the quotient space $W_\lambda^\perp / W_\lambda$ is symplectic and by the inductive hypothesis, there exist $v_1, \cdots, v_r \in W^\perp - W$ such that $W + \mathbb{C}v_1 \oplus \cdots \oplus W + \mathbb{C}v_r$ is lagrangian. Then $W \oplus \mathbb{C}v_1 + \cdots \oplus \mathbb{C}v_r$ is isotropic and by dimension counting, we see it is lagrangian as well. □

Lemma 2.1.1. *Let V be a vector space with bilinear form ω. Then every matrix A that is self-adjoint with respect to ω has orthogonal generalized eigenspaces.*

Proof. Let $u \in V_\lambda, v \in V_\mu$ and consider polynomials $P(x) = (x-\lambda)^{k_\lambda}, Q(x) = (x-\mu)^{k_\mu}$. Then P, Q are prime with each other, so there exist polynomials U, V such that $UP + VQ = 1$. Then $\omega(u,v) = \omega((UP+VQ)(A)u, v) = \omega(VQ(A)u, v) = \omega(u, VQ(A)v) = 0$, where we use that since $P(A), Q(A)$ are polynomials of A, they are self-adjoint as well. □

2.2 Proximal elements in $\mathrm{GL}(n, \mathbb{R})$

Here we will talk about basic definitions and dynamics of proximal elements in $\mathrm{GL}(n, \mathbb{R})$ and its subgroups.

Definition 2.2.1. We say that $g \in \mathrm{PGL}(d, \mathbb{R})$ is *proximal* (in $\mathbb{P}(\mathbb{R}^d)$) if it admits an attractive fixed point in $\mathbb{P}(\mathbb{R}^d)$, i.e. there exists a line $x_g^+ \in \mathbb{P}(\mathbb{R}^d)$ and a compact neighborhood $b^+ \subseteq \mathbb{P}(\mathbb{R}^d)$ of $x + g^+$ such that $g^n x \to x$ as $n \to \infty$ uniformly for $x \in b^+$. We say that g is biproximal if both g and g^{-1} are proximal.

To better understand the dynamics of proximal elements, we will recall the Jordan decomposition of a matrix. Let $A \in \mathrm{GL}(d, \mathbb{R})$. Then A admits a Jordan canonical form $A = BJB^{-1}$, where $B \in \mathrm{GL}(d, \mathbb{R})$ and J is a block diagonal matrix with Jordan blocks $J_1, \ldots, J_k$:

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix},$$

where each J_i is either a single real entry j_i, or a (real) Jordan block of the form

$$J_i = \begin{pmatrix} C_i & 1 & & \\ & C_i & \ddots & \\ & & \ddots & 1 \\ & & & C_i \end{pmatrix},$$

with C_i being a 2×2 matrix of the form

$$C_i = \begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix},$$

and $b_i \neq 0$. In the latter case, we let j_i be one of the complex eigenvalues $j_i = \sqrt{\det C_i} e^{i \arccos(a_i)}$ of C_i.

Remark 2.2.1. Each C_i is merely a similitude (in case C_i is not a single entry) by $|j_i| = \sqrt{\det C_i}$ of a rotation by $\arccos a_i$, so J_i rotates and multiplies the plane corresponding to its first two collumns.

We call each j_i a *generalised eigenvalue* of A and the subspace E_i preserved by each J_i the *generalised eigenspace* of j_i. By changing B we may assume that $|j_1| \geq \ldots \geq |j_m|$.

We are now ready to describe the dynamics of a Jordan block, which is the same as the dynamics of A in the respective generalised eigenspace.

Lemma 2.2.1 (Dynamics of a Jordan block)**.** *Let J_i be a Jordan block, and let j_i, E_i be its generalised eigenvalue and eigenspace respectively. We denote with $e_1, \cdots, e_k$ part of the standard basis that spans E_i.*

(i) There exists vectors $v \in E_i$ such that

$$|J_i^n v| = |j_i|^n |v|.$$

When C_i is a real entry, this holds for $v \in \mathbb{R}v_1$, while when C_i is a 2×2 matrix, this holds for $v \in \mathbb{R}e_1 \oplus \mathbb{R}e_2$.

(ii) If J_i is upper-triangular but not a single entry, then E_i contains an actual eigenvector (namely e_1) such that for $v \in E_i$ we have $J_i^n \mathbb{R}v \to \mathbb{R}e_1$.

Proof. While we will not give a concrete proof of this fact, we can consider examples.

(i) This follows by the fact that for v in the respective subspaces, J_i is a similitude by j_i and perhaps a rotation.

(ii) Suppose J is 2×2, so

$$J_i = \begin{pmatrix} a_i & 1 \\ 0 & a_i \end{pmatrix}.$$

Then for $w = w_1 e_1 + w_2 e_2 \in E_i$ we have

$$J_i \mathbb{R}w = \mathbb{R}(w + a_i w_2 e_1).$$

Then it is clear to see that $J_i^n \mathbb{R}w \to \mathbb{R}e_1$ and that J_i moves lines clockwise when $a_1 > 0$ and counterclockwise when $a_1 < 0$. (see Figure 2.1).

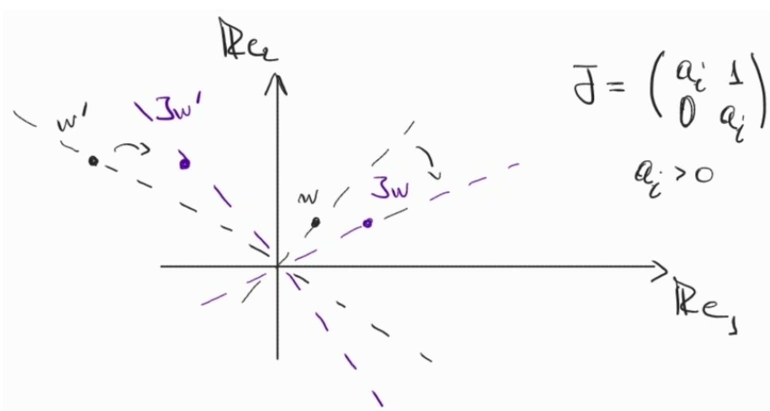


Figure 2.1: Dynamics of a Jordan block

$\square$

Remark 2.2.2. Even when a Jordan block upper triangular and is not a single entry, the eigenline may not be an attracting fixed point since the convergence is not uniform. Take for example

$$J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } v_n = (n, -1).$$

Then $J^n v_n = (0, -1)$, even though $v_n \to e_1, J^n \mathbb{R}v_{n_0} \to \mathbb{R}e_1$ as $n \to \infty$.

We now consider the Jordan matrix (see Figure 2.2):

Proposition 2.2.1 (Dynamics of a Jordan matrix)**.** *Let J be a Jordan matrix with Jordan blocks $J_1, \cdots J_m$. Then J is proximal if and only if $|j_1| > |j_2|$ and J_1 is a real entry. In that case, for any $x \in \operatorname{span}\{e_2, \cdots, e_d\}^c$ we have*

$$J^n x \to \mathbb{R}e_1,$$

and the convergence is uniform in compact neighborhoods of $\operatorname{span}\{e_2, \cdots, e_d\}^c$*. Similarly, J is biproximal if and only if $|j_1| > |j_2|, |j_{m-1}| > |j_m|$ and J_1, J_m are real entries. In that case, for any $x \in \operatorname{span}\{e_1, \cdots, e_{d-1}\}^c$ we have*

$$J^{-n} x \to \mathbb{R}e_d,$$

and the convergence is uniform in compact neighborhoods of $\operatorname{span}\{e_1, \cdots, e_{d-1}\}^c$.

Proof. Note that if $|j_1| > |j_2|$ and J_1 is a real entry, then for $w \notin \operatorname{span} e_2, \cdots, e_d$, we can write $w = w_1 + \cdots + w_m$ with each $w_i \in E_i$. Then

$$J^n \mathbb{R}w = \mathbb{R}(J_1^n w_1 + \cdots + J_m^n w_m) = \mathbb{R}(w_1 + \frac{J_2^n}{j_1^n} w_2 + \cdots + \frac{J_m^n}{j_1^n} w_m) \to \mathbb{R}e_1,$$

where the last convergence holds since each of the eigenvalues j_i are less than j_1.

If on the other hand J is proximal, we have seen that in the remark above, that J_1 needs to be a single entry, otherwise the convergence is not uniform. On the other hand, if $|j_1| = |j_2|$, then J will rotate the plane spanned by e_2, e_3, like for instance when

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, v = (1, x, y), J^n v = \begin{pmatrix} 1 \\ \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix}$$

□

Noting that being proximal is invariant under conjugation, we can now describe the dynamics of a proximal element in $\mathrm{GL}(d, \mathbb{R})$:

Corollary 2.2.1. *Let $g \in \mathrm{PGL}(d, \mathbb{R})$ and denote with g any of its representatives. Then the following are equivalent:*

(i) g is proximal if and only if g has a Jordan decomposition $g = BJB^{-1}$ with J being proximal.

(ii) Denoting with $\lambda_1(g), \cdots \lambda_d(g)$ the (possibly complex) eigenvalues of g in decreasing order of their modulus, g is proximal if and only if $|\lambda_1(g)| > |\lambda_2(g)|$.

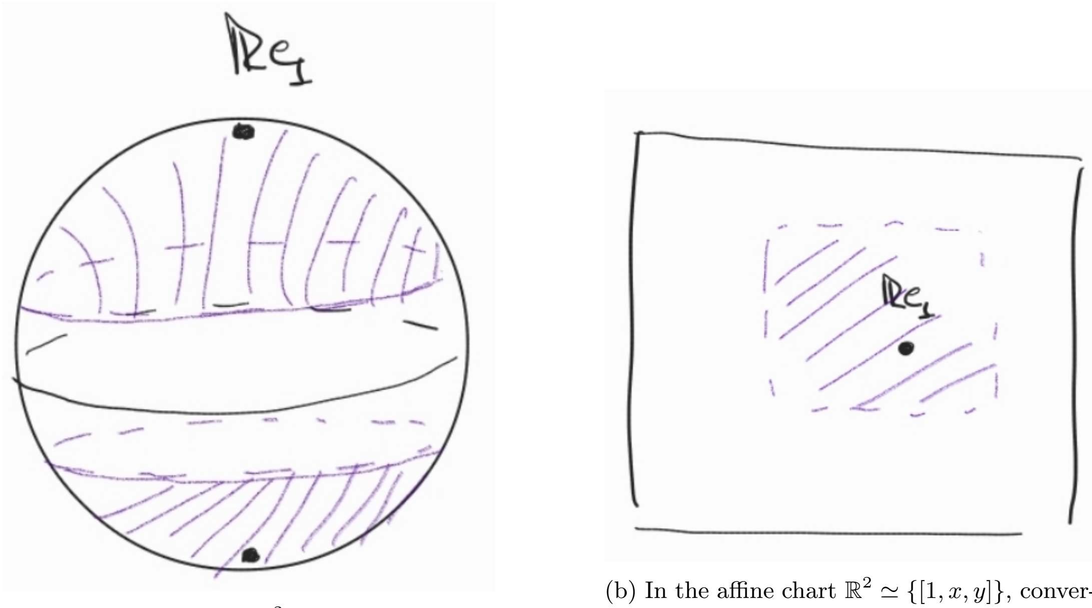
and similarly for biproximal elements:

(i) g is biproximal if and only if g has a Jordan decomposition $g = BJB^{-1}$ with J being biproximal.

(ii) Denoting with $\lambda_1(g), \cdots \lambda_d(g)$ the (possibly complex) eigenvalues of g in decreasing order of their modulus, g is proximal if and only if $|\lambda_1(g)| > |\lambda_2(g)|$ and $|\lambda_{d-1}(g)| > |\lambda_d(g)|$.

If this is the case, the attracting fixed point of g is the line spanned by the eigenvector Be_1 and the convergence is uniform in compact neighborhoods of $\operatorname{span}\{Be_2, \cdots, Be_d\}^c$.

Considering the case of $\mathrm{PO}(p, q)$ for $p, q \geq 0$, we have that every proximal element is biproximal:



(a) In the double cover S^2 convergence is uniform when uniformly away from the equator.

(b) In the affine chart $\mathbb{R}^2 \simeq \{[1, x, y]\}$, convergence is uniform when uniformly away from the circle at infinity.

Figure 2.2: Dynamics of Jordan matrix on $\mathbb{P}(\mathbb{R}^3)$

Proposition 2.2.2. *In* $\mathrm{PO}(p, q)$*, proximality and biproximality are equivalent.*

Proof. Let $g \in \mathrm{PO}(p, q)$ be proximal. Then the eigenvalues of g are stable under taking inverse: $\lambda(g) = \lambda(g^{-1})$. Indeed, $g \in \mathrm{O}(p, q)$ implies that $g^t I_{p,q} g = I_{p,q}$, so for an eigenvector v of g with eigenvalue λ, we multiply $gv = \lambda v$ by $I_{p,q}$ to obtain that

$$\lambda I_{p,q} v = I_{p,q} g v = (g^t)^{-1} I_{p,q} v,$$

so $\sigma(g) \subseteq \sigma((g^t)^{-1}) = \sigma(g^{-1})$. $\square$

2.3 Complexification of vector spaces

Definition 2.3.1. Let V be a real vector space and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of a real Lie algebra $\mathfrak{g}$.

(i) The complexification $V^{\mathbb{C}}$ of V is the complex vector space defined setwise by $V \times V$, with the addition structure of $V \oplus V$ and the scalar multiplication rule $(x + iy)(v_1, v_2) = (xv_1 - yv_2, xv_2 + yv_1)$. We write (v_1, v_2) as $v_1 + iv_2$.

(ii) If W is a complex vector space isomorphic to the complexification of V, then we say that V is a real form of W.

Recall that a map $f : V \to V$ on a complex vector space V is called *antilinear* if it it satisfies $f(u + v) = f(u) + f(v)$ and $f(\lambda v) = \overline{\lambda} f(v)$.

Definition 2.3.2. Let V be a complex vector space. A conjugation on V is an antilinear map $C : V \to V$ such that $C^2 = \mathrm{id}$.

Example 2.3.1. If E is a real vector space and $V = E^{\mathbb{C}}$, then we can define the conjugation $C : V \to V$ by $C(v) = \overline{v}$, where $\overline{u + iv} = u - iv$.

Proposition 2.3.1. *Let V be a complex vector space. Then real forms of V are in bijective correspondence with conjugations on V.*

Proof. To each real form E of V, we can associate the conjugation $C(u + iv) = u - iv$, for $u, v \in E$. Conversely, for each conjugation C on V, we consider the real vector space $E = \{v \in V \mid C(v) = v\}$, which turns out to be a real form of V. $\square$

Proposition 2.3.2. *If $\mathfrak{E}$ is a representation of $\mathfrak{g}$ that is not of real type, then there exists an irreducible complex representation $V \subseteq E^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$, such that*

$$E^{\mathbb{C}} = V \oplus \overline{V},$$

whre $\overline{V}$ is the conjugate representation of V, i.e. $\overline{V} = \{\overline{v} \mid v \in V\}$ and the action of $\mathfrak{g}$ on $\overline{V}$ is given by $\rho(X)(\overline{v}) = \overline{\rho(X)v}$.

Proof. Let V be a complex subspace of $E^{\mathbb{C}}$ that is invariant under $\mathfrak{g}^{\mathbb{C}}$. Then $\overline{V}$ is also invariant under $\mathfrak{g}^{\mathbb{C}}$, so the same is true for $V \cap \overline{V}$ and $V + \overline{V}$. Since these are invariant under the conjugation of $E^{\mathbb{C}}$, we have that $V + \overline{V}, V \cap \overline{V}$ admit real forms. Moreover, because the action of $\mathfrak{g}^{\mathbb{C}}$ commutes with conjugation,the real forms are invariant under the action of $\mathfrak{g}$. By irreducibility of E, we obtain that $V \oplus \overline{V} = E^{\mathbb{C}}$. $\square$