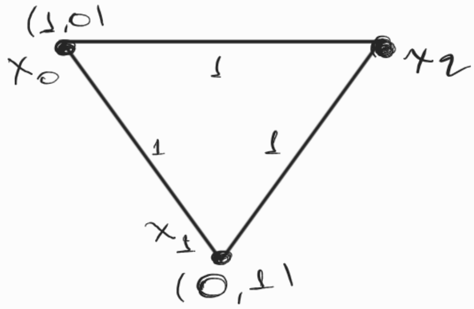
Simple Poisson Example



$$u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \nabla u = \begin{pmatrix} 0 & (-1,+1) & (-1/2,+1/2) \\ (+1,-1) & 0 & (+1/2,-1/2) \\ (+1/2,-1/2) & \underset{\substack{\| \\ (\nabla u)_{21} = u_1 - u_2}}{(-1/2,+1/2)} & 0 \end{pmatrix}$$

$$\|\nabla u\|^2_{\ell^2} = 2\cdot 2 + 4\cdot 1/2 = 6$$

$$\bar{y} = \left(\frac{1}{2}, \frac{1}{2}\right), \quad y - \bar{y} := \begin{pmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

$$I_2[u] = \frac{6}{4} - \left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}$$

$$I_3[u] = \frac{1}{6}\cdot\left(\sqrt{2}^3\cdot 2 + \frac{1}{\sqrt{2}^3}\cdot 4\right) - \left(\frac{1}{2} + \frac{1}{2}\right) \approx 0{,}1785$$

$$I_2[0] = 0$$

Linear constraints: $\sum_{i=1}^{n} d_i u(x_i) = 0 \iff \forall j = 1,\dots,k : \sum_{i=1}^{n} d_i u(x_i)_j = 0$

$$\sum_{i=1}^{n} d_i u(x_i)_1 = \langle \underbrace{[\underbrace{d_1 \cdots 0}_{k}\ \underbrace{d_2\, 0 \cdots 0}_{k} \cdots]}_{n}, [u(x_1)_1 \cdots u(x_1)_k \cdots u(x_n)_1 \cdots u(x_n)_k] \rangle \stackrel{!}{=} 0$$

$$\iff k\left\{ \underbrace{\underbrace{\begin{bmatrix} \underbrace{d_1 0 \cdots 0}_{k}\ d_2 0 \cdots 0 & \cdots & d_n 0 \cdots 0 \\ 0 d_1 0 \cdots 0\ 0 d_2 0 \cdots 0 & \cdots & 0 d_n 0 \cdots 0 \\ \vdots & & \\ 0 \cdots & 1 & \cdots \end{bmatrix}}_{nk}}_{\substack{\| \\ [d_1 I_k\ d_2 I_k \cdots d_n I_k]}} \right. \begin{bmatrix} u(x_1)_1 \\ \vdots \\ u(x_1)_k \\ \vdots \\ u(x_n)_1 \\ \vdots \\ u(x_n)_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{cases} -\Delta u(x_i) = \sum_{j=1}^{m} (y_j - \overline{y}_u)\,\delta_{ij} \quad \forall i = 1, \dots, n \\ \sum_{i=1}^{n} d_i u(x_i) = 0 \end{cases}$$

Graph Calculus

$u, v : X \to \mathbb{R}^k$, $\nabla u(x_i, x_j) = u(x_j) - u(x_i)$

$V, W : X^2 \to \mathbb{R}^k$, $\operatorname{div} V = \sum_{i,j=1}^{n} w_{ij} V(x_i, x_j)$

↳ vector field

i.e. $V(x_i, x_j) = -V(x_j, x_i)$

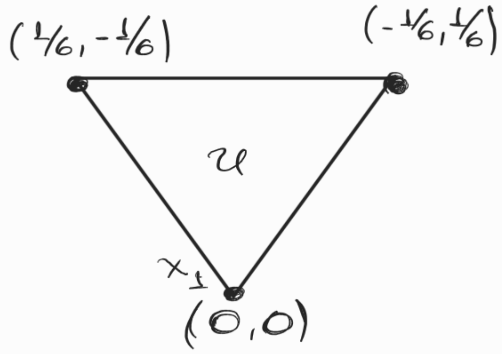
$$(u, v)_{\ell^2(X)} = \sum_{i=1}^{n} u(x_i) v(x_i)$$

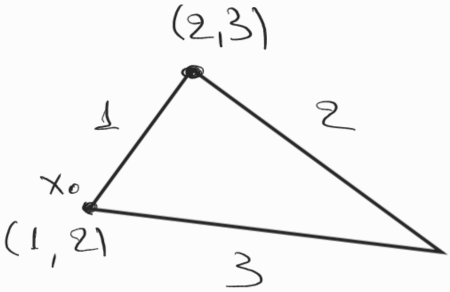
$$(V, W)_{\ell^2(X^2)} = \frac{1}{2} \sum_{i,j=1}^{n} V(x_i, x_j) W(x_i, x_j)$$

becomes:

$$\left\{ \begin{array}{l} -\Delta u(x_0) := 2u(x_0) - u(x_1) - u(x_2) = \left(\frac{1}{2}, -\frac{1}{2}\right) \\ -\Delta u(x_1) := 2u(x_1) - u(x_0) - u(x_2) = \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\Delta u(x_2) = 2u(x_2) - u(x_0) - u(x_1) = 0 \\ u(x_0) + u(x_1) + u(x_2) = 0 \end{array} \right\} \iff$$

$$\iff \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \end{bmatrix} = \begin{bmatrix} 1/6 & -1/6 \\ -1/6 & 1/6 \\ 0 & 0 \end{bmatrix}$$





$$\bar{y} = \left(\frac{3}{2}, \frac{5}{2} \right), \; y - \bar{y} = \begin{pmatrix} -1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$u = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & 2 \end{pmatrix}$$

$$\nabla u = \begin{pmatrix} 0 & (1,2) & (-2,1) \\ (-1,-2) & 0 & (-3,-1) \\ (2,-1) & (3,1) & 0 \end{pmatrix}$$

$$\|\nabla u\|_{\ell^2}^2 = 5 + 5 + 4 + 10 + 5 + 10 = 39$$

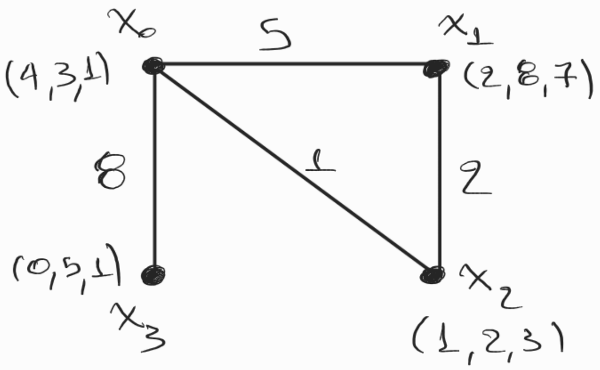
$$I_2[u] = \frac{39}{2} - \left(-1 + \left(1 + 3/2 \right) \right) = \frac{36}{2} = 18$$

Gradient calculation

- $A, B \in \mathbb{R}^{n\times d}$, $C = \text{substract.outter}(A,B) \in \mathbb{R}^{n\times d\times n\times d}$
  $C[i_0, i_1, j_0, j_1] = A[i_0, i_1] - B[j_0, j_1]$
  Let $A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, $F = \begin{bmatrix} a_1-b_1 & \cdots & a_1-b_n \\ \vdots & & \\ a_n-b_1 & \cdots & a_n-b_n \end{bmatrix} \in \mathbb{R}^{n\times d\times n}$
  $a_i, b_j \in \mathbb{R}^d$, $F[i,j,k] = (a_i - b_j)[k] = A[i,k] - B[j,k] =$
  $= C[i,k,j,k]$

- Thus given $u \in \mathbb{R}^{n\times d}$, we want $\text{grad}\,u \in \mathbb{R}^{n\times n\times d}$ where
  $\text{grad}\,u[i,j,k] = u[i,k] - u[j,k] = C[i,k,j,k]$
  $C = \text{np.substraction.outter}(u,u)$

- For $A \in \mathbb{R}^{n_0\times n_1\times n_2\times n_3}$, $B = A.\text{flatten}() \in \mathbb{R}^{n_0 n_1 n_2 n_3}$ and
  $A[i_0, i_1, i_2, i_3] = A[n_0 n_1 n_2 i_0 + n_1 n_2 i_1 + n_2 i_2 + i_3]$



$$W = \begin{pmatrix} 0 & 5 & 1 & 8 \\ 5 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 8 & 0 & 0 & 0 \end{pmatrix}$$

$$u = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 8 & 7 \\ 1 & 2 & 3 \\ 0 & 5 & 1 \end{pmatrix}$$

$I = (1, 2, 3, 0, 2, 0, 1, 0) \rightsquigarrow$ nodes w/ at least 1 edge starting there

$J = (0, 0, 0, 1, 1, 2, 2, 3) \rightsquigarrow$ nodes w/ at least 1 edge ending there

$u[I]$ = values of u at nodes w/ at least 1 neighb.

$$u[\underline{I}] - u[J] = \begin{bmatrix} u(x_1) - u(x_0) \\ u(x_2) - u(x_0) \\ u(x_3) - u(x_0) \\ \vdots \end{bmatrix} = \begin{bmatrix} -2 & 5 & 6 \\ & & \\ & & \end{bmatrix}$$

$$I : \mathbb{R}^{nk} \to \mathbb{R}, \quad I[u] = \frac{1}{2p}\sum_{i,j=1}^{n} w_{ij}|u(x_i)-u(x_j)|^p - \sum_{j=1}^{m}(y_j-\bar{y})u(x_j)$$

$$= I_A[u] - I_B[u]$$

where $u \simeq \begin{bmatrix} u(x_1) \\ \vdots \\ u(x_n) \end{bmatrix} \simeq \begin{bmatrix} u(x_1)_1 \cdots u(x_n)_k \\ \vdots \end{bmatrix} \simeq [u_{11} \cdots u_{1k} \cdots u_{n1} \cdots u_{nk}]$

$$\frac{\partial I_A}{\partial u_{rs}} = \frac{1}{2p}\sum_{i,j} w_{ij}\, p\, |u(x_i)-u(x_j)|^{p-2}\sum_{l=1}^{k}(u_{il}-u_{jl})\underbrace{(\delta_{il,rs}-\delta_{jl,rs})}_{\text{0 unless } l=s} =$$

$$= \frac{1}{2}\sum_{i,j=1}^{n} w_{ij}|u(x_i)-u(x_j)|^{p-2}(u_{is}-u_{js}) \quad (\delta_{ir}-\delta_{jr})$$

$$= \frac{1}{2}\sum_{j=1}^{n} w_{rj}|u(x_r)-u(x_j)|^{p-2}(u_{rs}-u_{js})$$

$$- \frac{1}{2}\sum_{i=1}^{n} w_{ir}|u(x_i)-u(x_r)|^{p-2}(u_{is}-u_{rs})$$

$$= \sum_{i=1}^{n} w_{ir}|u(x_i)-u(x_r)|^{p-2}(u_{rs}-u_{is})$$

$$\frac{\partial I_B}{\partial u_{rs}} = \frac{\partial}{\partial u_{rs}}\sum_{j=1}^{m}\sum_{l=1}^{k}(y_{jl}-\bar{y}_l)u_{jl} - \begin{cases} y_{rs}-\bar{y}_s, & r \le m \\ 0, & r > m \end{cases}$$

Hence

$$\frac{\partial I}{\partial u_{rs}} = \sum_{i=1}^{n} w_{ir}|\nabla u(x_i,x_r)|^{p-2}(u_{rs}-u_{is}) - \begin{cases} y_{rs}-\bar{y}_s, & r \le m \\ 0, & r > m \end{cases}$$

$$\left(w[:,r]\cdot|\nabla u|^{p-2}[:,r]\ (u[r,s]-u[:,s])\right).\text{sum}$$

$$\begin{pmatrix} w_{1r} \\ \vdots \\ w_{nr} \end{pmatrix}\begin{pmatrix} |\nabla u(x_1,x_r)|^{p-2} \\ \vdots \\ |\nabla u(x_n,x_r)|^{p-2} \end{pmatrix}\left[u_{rs} - \begin{pmatrix} u_{1s} \\ \vdots \\ u_{ns} \end{pmatrix}\right].\text{sum}$$

$$w[:,r]\ |\nabla u|^{p-2}[:,r]\,(u_{rs}-u[:,s]).\text{sum}$$

$$B_{rs} = \begin{cases} y_{rs} - \bar{y}_s, & r \le m \\ 0, & \text{else} \end{cases}, \quad B \in \mathbb{R}^{n \times m}$$

In particular $\dfrac{\partial |\nabla u(x_i, x_j)|^p}{\partial u_{rs}} = p |\nabla u(x_i, x_j)|^{p-2} (u_{js} - u_{is})(\delta_{jr} - \delta_{ir})$

and $\dfrac{\partial^2 I}{\partial u_{rs} \partial u_{r's'}} =$

$$= (p-2) \sum_{i=1}^{n} w_{ir} |\nabla u(x_i, x_r)|^{p-4} (u_{rs'} - u_{is'})(\delta_{rr'} - \delta_{ir'}) +$$

$$+ \delta_{ss'} \sum_{i=1}^{n} w_{ir} |\nabla u(x_i, x_r)|^{p-2} (\delta_{rr'} - \delta_{ir'})$$

$$= (p-2) \delta_{rr'} \sum_{i=1}^{n} w_{ir} |\nabla u(x_i, x_r)|^{p-4} (u_{rs'} - u_{is'}) -$$

$$- (p-2) w_{r'r} |\nabla u(x_{r'}, x_r)|^{p-4} (u_{rs'} - u_{r's'})$$

$$+ \delta_{rr'} \delta_{ss'} \sum_{i=1}^{n} w_{ir} |\nabla u(x_i, x_r)|^{p-2} \quad + \quad \delta_{ss'} w_{r'r} |\nabla u(x_{r'}, x_r)|^{p-2}$$

$$= (p-2)\, Id[r, r'] \sum_{i=1}^{n} w |\nabla u|^{p-4} [i, r] (u_{rs'} - u_{is'})$$

Writing $A = w |\nabla u|^{p-4}$. sum(axis =

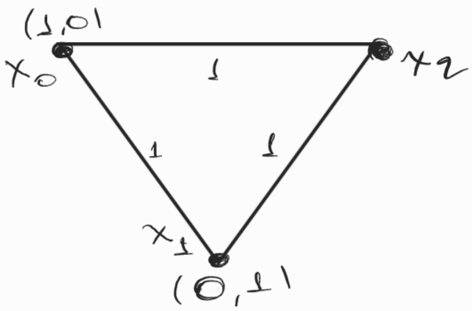
$B =$ diff.outter(u, u) : (n, k, n, k), $\quad B_{i_1 i_2 i_3 i_4} = u_{i_1 i_2} - u_{i_3 i_4}$

$C =$

$$\frac{\partial^2 I}{\partial u_{rs} \partial u_{r's'}} = \delta_{rr'} (p-2) \left(w |\nabla u|^{p-4} [:, r] \cdot (u_{rs'} - u[:, s']) \right).\text{sum}()$$
$$- (p-2) (w |\nabla u|^{p-4}) [r', r]\, B[r, s', r', s']$$
$$+ \delta_{rr'} \delta_{ss'} (w |\nabla u|^{p-2}) [:, r].\text{sum}()$$
$$+ \delta_{ss'} w |\nabla u|^{p-2} [r', r]$$

$$A_{rs'} = (p-2) \left(w |\nabla u|^{p-4} [:, r] \, (u_{rs'} - u[:, s']) \right).\text{sum}()$$

e.g.



$$p=3 \qquad y-\bar{y} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \nabla u = \begin{pmatrix} 0 & (-1,+1) & (-1/2,+1/2) \\ (+1,-1) & 0 & (+1/2,-1/2) \\ (+1/2,-1/2) & (-1/2,+1/2) & 0 \end{pmatrix}$$

$$\frac{\partial I}{\partial u_{11}} = w_{21}\,|u(x_2)-u(x_1)|\,(u_{21}-u_{11}) + w_{31}\,|u(x_3)-u(x_1)|\,(u_{31}-u_{11}) - y_{11} + \bar{y}_1$$

$$= 1\,\sqrt{2}\,(-1) + 1\cdot\frac{1}{\sqrt{2}}\left(-\frac{1}{2}\right) - 1 + \frac{1}{2} =$$

$$= -\sqrt{2} - \frac{1}{2\sqrt{2}} - \frac{1}{2} \approx -2{,}2677$$

$$\frac{\partial I}{\partial u_{31}} = w_{13}\,|\nabla u(x_3,x_1)|\,(u_{11}-u_{31}) + w_{23}\,|\nabla u(x_3,x_2)|\,(u_{21}-u_{31}) + 0 - 0$$

$$= 1\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{2} + 1\cdot\frac{1}{\sqrt{2}}\left(-\frac{1}{2}\right) = 0$$