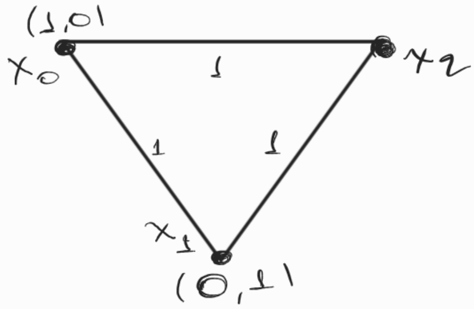
Simple Poisson Example



$$u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \nabla u = \begin{pmatrix} 0 & (1,-1) & (1/2,-1/2) \\ (-1,1) & 0 & (-1/2,1/2) \\ (-1/2,1/2) & (1/2,-1/2) & 0 \end{pmatrix}$$

$$\|\nabla u\|^2_{\ell^2} = 2 \cdot 2 + 4 \cdot 1/2 = 6$$

$$\bar{y} = \left(\frac{1}{2}, \frac{1}{2}\right), \quad y - \bar{y} := \begin{pmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

$$I_2[u] = \frac{6}{4} - \left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}$$

$$I_3[u] = \frac{1}{6} \cdot \left(\sqrt{2}^3 \cdot 2 + \frac{1}{\sqrt{2}^3} \cdot 4\right) - \left(\frac{1}{2} + \frac{1}{2}\right) \simeq 0{,}1785$$

$$I_2[0] = 0$$

Linear constraints: $\sum_{i=1}^{n} d_i u(x_i) = 0 \iff \forall j = 1, \dots, k: \sum_{j=1}^{m} d_i u(x_i)_j = 0$

$$\sum_{i=1}^{n} d_i u(x_i)_1 = \Big\langle \underbrace{[\underbrace{d_1 \cdots 0}_{k}\ \underbrace{d_2\, 0 \cdots 0}_{k} \cdots]}_{n}, [u(x_1)_1 \cdots u(x_1)_k \cdots u(x_n)_1 \cdots u(x_n)_k] \Big\rangle = 0$$

$$\iff k\left\{ \underbrace{\begin{bmatrix} \underbrace{d_1 0 \cdots 0}_{k} & d_2 0 \cdots 0 & \cdots & d_n 0 \cdots 0 \\ 0 d_1 0 \cdots 0 & 0 d_2 0 \cdots 0 & \cdots & 0 d_n 0 \cdots 0 \\ \vdots & & & \\ 0 \cdots & 1 & \cdots & \end{bmatrix}}_{nk} \right. \begin{bmatrix} u(x_1)_1 \\ \vdots \\ u(x_1)_k \\ \vdots \\ u(x_n)_1 \\ \vdots \\ u(x_n)_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underbrace{\qquad\qquad}_{\parallel}$$

$$[d_1 I_k\ d_2 I_k \cdots d_n I_k]$$

$$\begin{cases} -\Delta u(x_i) = \sum_{j=1}^{m} (y_j - \bar{y}_u)\,\delta_{ij} \quad \forall i = 1, \dots, n \\ \sum_{i=1}^{n} d_i u(x_i) = 0 \end{cases}$$

Graph Calculus

$u, v : X \to \mathbb{R}^k$, $\nabla u(x_i, x_j) = u(x_j) - u(x_i)$

$V, W : X^2 \to \mathbb{R}^k$, $\operatorname{div} V = \sum_{i,j=1}^{n} w_{ij} V(x_i, x_j)$

↳ vector field

i.e. $V(x_i, x_j) = -V(x_j, x_i)$

$(u, v)_{\ell^2(X)} = \sum_{i=1}^{n} u(x_i) v(x_i)$

$(V, W)_{\ell^2(X^2)} = \frac{1}{2} \sum_{i,j=1}^{n} V(x_i, x_j) W(x_i, x_j)$

becomes:

$$\left\{\begin{aligned} -\Delta u(x_0) &:= 2u(x_0) - u(x_1) - u(x_2) = \left(\tfrac{1}{2}, -\tfrac{1}{2}\right) \\ -\Delta u(x_1) &:= 2u(x_1) - u(x_0) - u(x_2) = \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \\ -\Delta u(x_2) &= 2u(x_2) - u(x_0) - u(x_1) = 0 \\ u(x_0) &+ u(x_1) + u(x_2) = 0 \end{aligned}\right\} \iff$$

$$\iff \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \end{bmatrix} = \begin{bmatrix} 1/6 & -1/6 \\ -1/6 & 1/6 \\ 0 & 0 \end{bmatrix}$$