Scale spaces and Semigroup theory

https://tinyurl.com/scalegroups

Table of Contents

Motivational problems

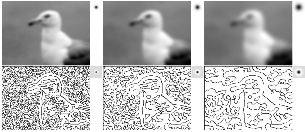
1. Enhancing (Deblurring)

(a) Blurry Data

(b) Enhanced Data

2. Smoothing (Denoising, Edge detection)

(c) Gaussian Filter



(d) Edge Detection

PDE Methods

Idea: The image we want is a solution of some PDE

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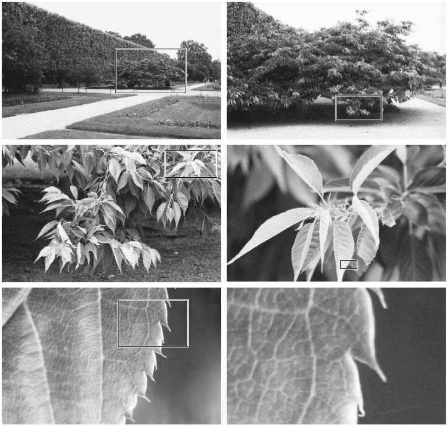
Why scale spaces

Example problem: Edge detection
Observation: We perceive things in different scales.
Solution: Smooth things to make the different scales pop out.



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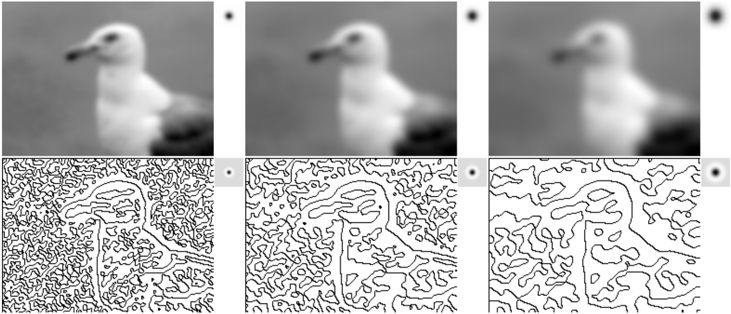

Scale spaces: Definition

Definition

Let U be a space of functions on $\Omega \subseteq \mathbb{R}^n$. A **scale space** on U is a family of mappings $\{T_t : U \to U\}_{t \geq 0}$. It is called **pyramidal** if there exists a family of operators $\{T_{t+h,t} : U \to U\}_{t,h \geq 0}$ such that:

$$T_{t+h,t} T_t = T_{t+h}, \quad T_0 = \mathrm{Id}$$

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Definition

A pyramidal scale space satisfies the **local comparison principle** if for all $u, v \in U$ the following are true:

- $u \leq v$ around $x \in \Omega \Rightarrow T_{t+h,t}u(x) \leq T_{t+h,t}v(x) + o(h)$
- $u \leq v$ in Ω, then $T_{t+h,t}u \leq T_{t+h,t}v, \forall h \geq 0$

Scale spaces and PDEs

Definition

A pyramidal scale space is **regular** if there exists a function $F : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ continuous with respect to its last component such that

$$\lim_{h\to 0^+} \frac{T_{t+h,t}u(x) - u(x)}{h} = F(t, x, u(x), \nabla u(x), \nabla^2 u(x)) \qquad \text{(R)}$$

for all quadratic functions u around x, i.e.

$$u(y) = c + p^T(y-x) + \frac{1}{2}(y-x)^T A(y-x)$$

for y around x. A regular pyramidal scale space that satisfies the local comparison principle is called **causal**.

Scale Spaces and PDEs

Theorem

If T_t is causal, then (R) holds for all $u \in C^2(\Omega), x \in \Omega, t \geq 0$. Moreover the function F is non-decreasing with respect to its last component in the sense that: $F(t, x, c, p, A) \leq F(t, x, c, p, B)$ when $B - A \geq 0$

Scale Spaces and PDEs

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Example

Let $U = C_b(\mathbb{R}^n)$ and define $T_t u^\delta = u(t, \cdot)$ is the convolution with the Gaussian with variance t. Then $u(t, x) = T_t u^\delta(x)$ solves

$$\partial_t u = \Delta u, t > 0$$
$$u(0, \cdot) = u^\delta$$

Then T_t it a causual scale space that satisfies the local comparison principle.

Scale spaces: Invariance

Definition

Let T_t be a causal scale space on U. Then:

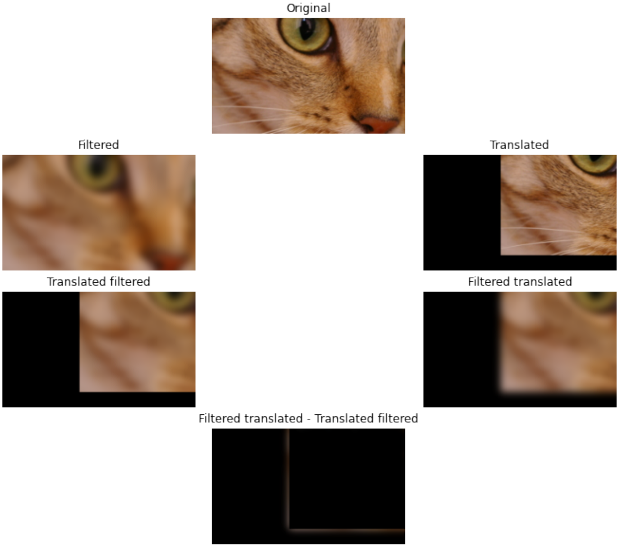
- T_t is **translation invariant** if

$$T_{t+h,t} \circ \tau_z = \tau_z \circ T_{t+h,t}$$

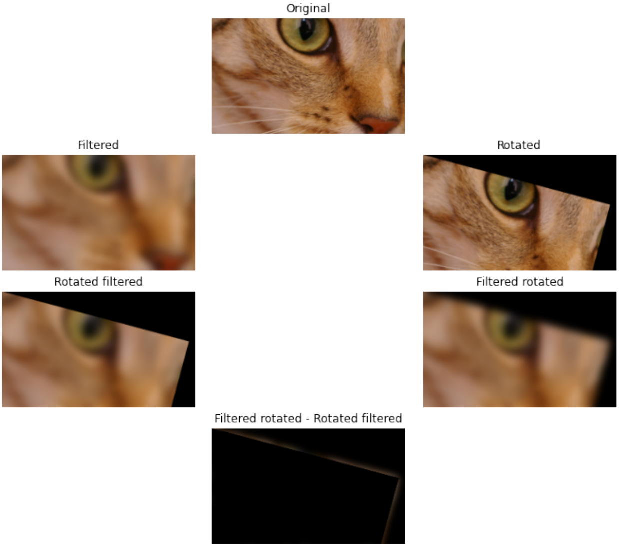
- T_t is **Euclidean invariant** if for every orthogonal matrix O:

$$T_{t+h,t} \circ \rho_O = \rho_O \circ T_{t+h,t}$$

Translation invariance for heat scale space



Euclidean invariance for heat scale space



Scale spaces: Invariance

Definition

- T_t is **scale invariant** if there exists a rescaling function $\theta : (0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying the following conditions:
  - θ is differentiable and $\partial_c \theta(t, 1) > 0$ and is continuous for all $t > 0$.
  - $T_{t+h,t} \circ \sigma_c = \sigma_c \circ T_{\theta(c,t+h),\theta(c,t)}, c > 0$
- A scale invariant T_t is **affine invariant** if there exists $\hat{\theta} : \mathrm{GL}^n \times [0, \infty) \to [0, \infty)$ such that $\theta(c, \cdot) := \hat{\theta}(c\mathrm{Id}, \cdot)$ satsfies the conditions above and

$$T_{t+h,t} \circ \rho_A = \rho_A \circ T_{\hat{\theta}(A,t+h),\hat{\theta}(A,t)}, A \in \mathrm{GL}^n$$

where $(\sigma_c u)(x) := u(cx), (\rho_A u)(x) = u(Ax)$.

Scale spaces: Invariance

Definition

- T_t is **invariant by gray level translations** if

$$T_{t+h,t}(0) = 0, \quad T_{t+h,t}(u + C) = T_{t+h,t}(u) + C, C \in \mathbb{R}$$

- T_t is **contrast invariant** if for every non-decreasing continuous function $g : \mathbb{R} \to \mathbb{R}$ and $u \in U$:

$$g(T_{t+h,t}u(x)) = T_{t+h,t}(g \circ u)(x), x \in \mathbb{R}^n$$

Heat scale space is not contrast invariant

Semigroups

Bibliography

[GMR12] Frédéric Guichard, Jean-Michel Morel, and Robert Ryan. "Contrast invariant image analysis and PDE's (Preprint)". In: (2012).

[Sch+09] Otmar Scherzer et al. "Variational methods in imaging". In: (2009).