

# Scale spaces and Semigroup theory

<https://tinyurl.com/scalegroups>

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1 PDEs in Image processing

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# Motivational problems

## ① Enhancing (Deblurring)



(a) Blurry Data

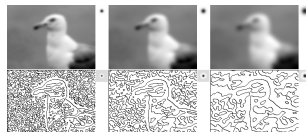


(b) Enhanced Data

## ② Smoothing (Denoising, Edge detection)



(c) Gaussian Filter



(d) Edge Detection

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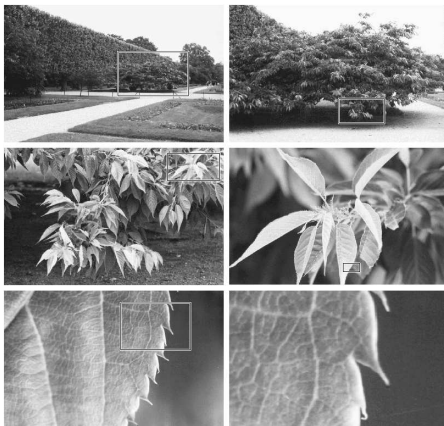


# Why scale spaces

Example problem: Edge detection

Observation: We perceive things in different scales.

Solution: Smooth things to make the different scales pop out.



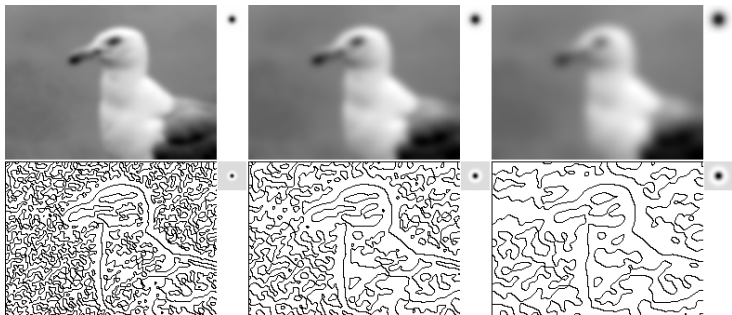


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# Scale spaces: Definition

## Definition

Let  $U$  be a space of functions on  $\Omega \subseteq \mathbb{R}^n$ . A **scale space** on  $U$  is a family of mappings  $\{T_t : U \rightarrow U\}_{t \geq 0}$ . It is called **pyramidal** if there exists a family of operators  $T_{t+h,t} : U \rightarrow U_{t,h \geq 0}$  such that:

$$T_{t+h,t} T_t = T_{t+h}, \quad T_0 = \text{Id}$$

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## Definition

A pyramidal scale space satisfies the **local comparison principle** if for all  $u, v \in U$  the following are true:

- $u \leq v$  around  $x \in \Omega \Rightarrow T_{t+h,t}u(x) \leq T_{t+h,t}v(x) + o(h)$
- $u \leq v$  in  $\Omega$ , then  $T_{t+h,t}u \leq T_{t+h,t}v, \forall h \geq 0$

## Definition

A pyramidal scale space is **regular** if there exists a function  $F : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$  continuous with respect to its last component such that

$$\lim_{h \rightarrow 0^+} \frac{T_{t+h,t} u(x) - u(x)}{h} = F(t, x, u(x), \nabla u(x), \nabla^2 u(x)) \quad (\text{R})$$

for all quadratic functions  $u$  around  $x$ , i.e.

$$u(y) = c + p^T(y - x) + \frac{1}{2}(y - x)^T A(y - x)$$

for  $y$  around  $x$ . A regular pyramidal scale space that satisfies the local comparison principle is called **causal**.

## Theorem

If  $T_t$  is causal, then (R) holds for all  $u \in C^2(\Omega)$ ,  $x \in \Omega$ ,  $t \geq 0$ . Moreover the function  $F$  is non-decreasing with respect to its last component in the sense that:  $F(t, x, c, p, A) \leq F(t, x, c, p, B)$  when  $B - A \geq 0$

## Example

Let  $U = C_b(\mathbb{R}^n)$  and define  $T_t u^\delta = u(t, \cdot)$  is the convolution with the Gaussian with variance  $t$ . Then  $u(t, x) = T_t u^\delta(x)$  solves

$$\begin{aligned}\partial_t u &= \Delta u, t > 0 \\ u(0, \cdot) &= u^\delta\end{aligned}$$

Then  $T_t$  is a causal scale space that satisfies the local comparison principle.

## Definition

Let  $T_t$  be a causal scale space on  $U$ . Then:

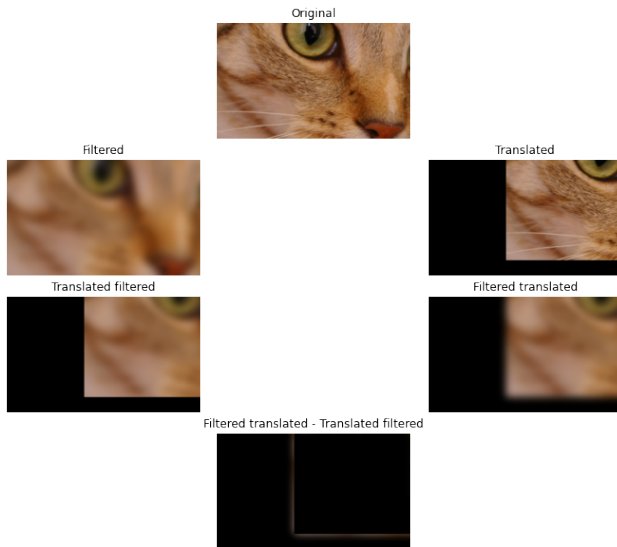
- $T_t$  is **translation invariant** if

$$T_{t+h,t} \circ \tau_z = \tau_z \circ T_{t+h,t}$$

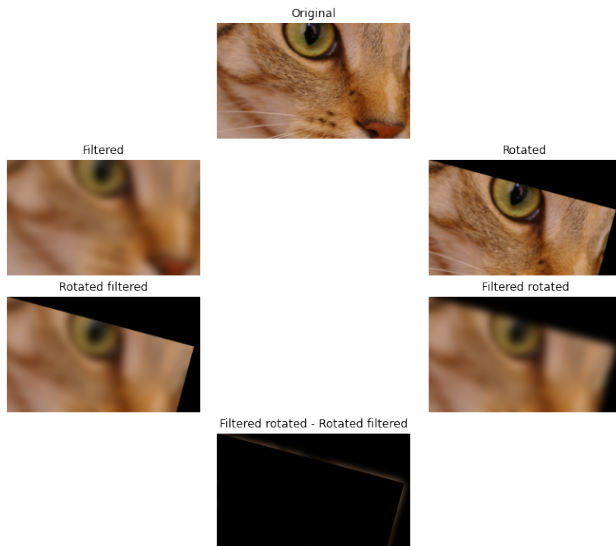
- $T_t$  is **Euclidean invariant** if for every orthogonal matrix  $O$ :

$$T_{t+h,t} \circ \rho_O = \rho_O \circ T_{t+h,t}$$

# Translation invariance for heat scale space



# Euclidean invariance for heat scale space





## Definition

- $T_t$  is **scale invariant** if there exists a rescaling function  $\theta : (0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  satisfying the following conditions:
  - $\theta$  is differentiable and  $\partial_c \theta(t, 1) > 0$  and is continuous for all  $t > 0$ .
  - $T_{t+h,t} \circ \sigma_c = \sigma_c \circ T_{\theta(c,t+h), \theta(c,t)}, c > 0$
- A scale invariant  $T_t$  is **affine invariant** if there exists  $\hat{\theta} : \text{GL}^n \times [0, \infty) \rightarrow [0, \infty)$  such that  $\theta(c, \cdot) := \hat{\theta}(\text{cld}, \cdot)$  satisfies the conditions above and

$$T_{t+h,t} \circ \rho_A = \rho_A \circ T_{\hat{\theta}(A,t+h), \hat{\theta}(A,t)}, A \in \text{GL}^n$$

where  $(\sigma_c u)(x) := u(cx)$ ,  $(\rho_A u)(x) = u(Ax)$ .

## Definition

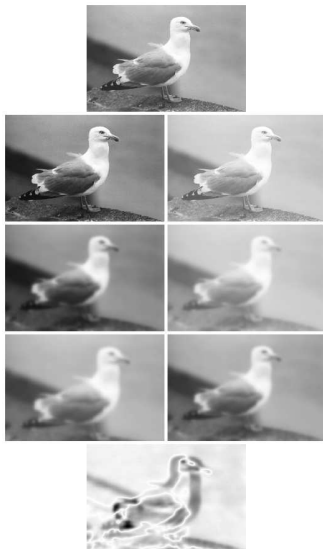
- $T_t$  is **invariant by gray level translations** if

$$T_{t+h,t}(0) = 0, \quad T_{t+h,t}(u + C) = T_{t+h,t}(u) + C, C \in \mathbb{R}$$

- $T_t$  is **contrast invariant** if for every non-decreasing continuous function  $g : \mathbb{R} \rightarrow \mathbb{R}$  and  $u \in U$ :

$$g(T_{t+h,t}u(x)) = T_{t+h,t}(g \circ u)(x), x \in \mathbb{R}^n$$

# Heat scale space is not contrast invariant



- [GMR12] Frédéric Guichard, Jean-Michel Morel, and Robert Ryan. “Contrast invariant image analysis and PDE’s (Preprint)”. In: (2012).
- [Sch+09] Otmar Scherzer et al. “Variational methods in imaging”. In: (2009).