Scale spaces and Semigroup theory

Table of Contents

Motivational problems

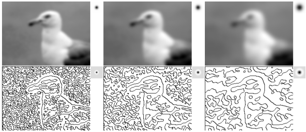
1. Enhancing (Deblurring)

(a) Blurry Data

(b) Enhanced Data

2. Smoothing (Denoising, Edge detection)

(c) Gaussian Filter



(d) Edge Detection

PDE Methods

Idea: The image we want is a solution of some PDE

Figure: Gaussian filter

PDE Methods

Idea: The image we want is a solution of some PDE

- Smoothing: (Gaussian filter)

$$\partial_t u + \Delta u = 0 \text{ on } \Omega \times (0, T)$$
$$u(\cdot, 0) = u^\delta \text{ on } \Omega$$

Figure: Gaussian filter

PDE Methods

Idea: The image we want is a solution of some PDE

- Smoothing: (Gaussian filter)

$$\partial_t u + \Delta u = 0 \text{ on } \Omega \times (0, T)$$
$$u(\cdot, 0) = u^\delta \text{ on } \Omega$$

- Enhancing: (Backward linear diffusion)

$$\partial_t u + \Delta u = 0 \text{ on } \Omega \times (0, T)$$
$$u(\cdot, T) = u^\delta \text{ on } \Omega$$

Figure: Gaussian filter

PDE Methods

Idea: The image we want is a solution of some PDE

- Smoothing: (Gaussian filter)

$$\partial_t u + \Delta u = 0 \text{ on } \Omega \times (0, T)$$
$$u(\cdot, 0) = u^\delta \text{ on } \Omega$$

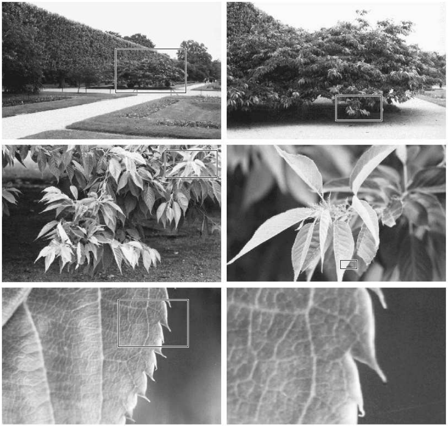
- Enhancing: (Backward linear diffusion)

$$\partial_t u + \Delta u = 0 \text{ on } \Omega \times (0, T)$$
$$u(\cdot, T) = u^\delta \text{ on } \Omega$$

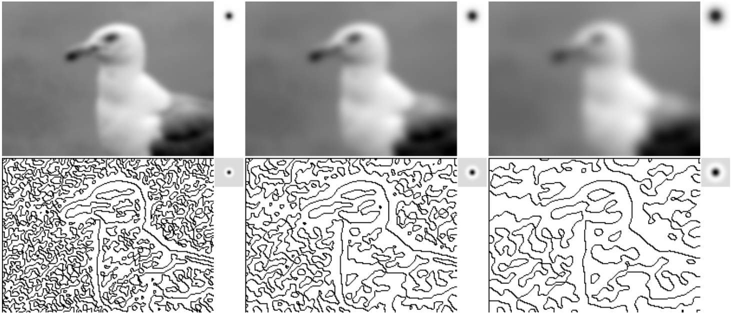
Figure: Gaussian filter

Why scale spaces

Example problem: Edge detection
Observation: We perceive things in different scales.
Solution: Smooth things to make the different scales pop out.



Example problem: Edge detection
Observation: We perceive things in different scales.
Solution: Smooth things to make the different scales pop out.



Scale spaces: Definition

Definition

Let U be a space of functions on $\Omega \subseteq \mathbb{R}^n$. A **scale space** on U is a family of mappings $\{T_t : U \to U\}_{t \geq 0}$. It is called **pyramidal** if there exists a family of operators $T_{t+h,t} : U \to U_{t,h \geq 0}$ such that:

$$T_{t+h,t} T_t = T_{t+h}, \quad T_0 = \mathrm{Id}$$

Scale spaces: Definition

Definition

Let U be a space of functions on $\Omega \subseteq \mathbb{R}^n$. A **scale space** on U is a family of mappings $\{T_t : U \to U\}_{t \geq 0}$. It is called **pyramidal** if there exists a family of operators $T_{t+h,t} : U \to U_{t,h \geq 0}$ such that:

$$T_{t+h,t} T_t = T_{t+h}, \quad T_0 = \mathrm{Id}$$

Definition

A pyramidal scale space satisfies the **local comparison principle** if for all $u, v \in U$ the following are true:

- $u \leq v$ around $x \in \Omega \Rightarrow T_{t+h,t} u(x) \leq T_{t+h,t} v(x) + o(h)$
- $u \leq v$ in Ω, then $T_{t+h,t} u \leq T_{t+h,t} v, \forall h \geq 0$

Scale spaces and PDEs

Definition

A pyramidal scale space is **regular** if there exists a function $F : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \to \mathbb{R}$ continuous with respect to its last component such that

$$\lim_{h \to 0^+} \frac{T_{t+h,t}u(x) - u(x)}{h} = F(t, x, u(x), \nabla u(x), \nabla^2 u(x)) \qquad \text{(R)}$$

for all quadratic functions u around x, i.e.

$$u(y) = c + p^T(y - x) + \frac{1}{2}(y - x)^T A(y - x)$$

for y around x. A regular pyramidal scale space that satisfies the local comparison principle is called **causal**.

Scale Spaces and PDEs

Theorem

If T_t is causal, then (R) holds for all $u \in C^2(\Omega), x \in \Omega, t \geq 0$. Moreover the function F is non-decreasing with respect to its last component in the sense that: $F(t, x, c, p, A) \leq F(t, x, c, p, B)$ when $B - A \geq 0$

Example

Let $U = C_b(\mathbb{R}^n)$ and define $T_t u^\delta = u(t, \cdot)$ is the convolution with the Gaussian with variance t. Then $u(t, x) = T_t u^\delta(x)$ solves

$$\partial_t u = \Delta u, t > 0$$
$$u(0, \cdot) = u^\delta$$

Then T_t it a causual scale space that satisfies the local comparison principle.

Scale spaces: Invariance

Definition

Let T_t be a causal scale space on U. Then:

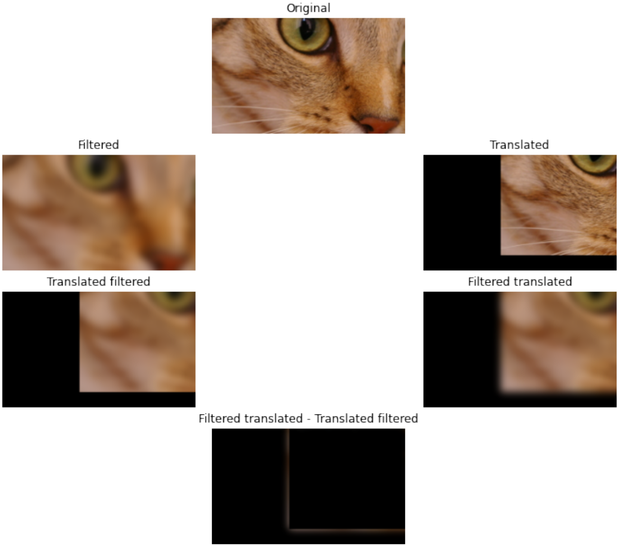
- T_t is **translation invariant** if

$$T_{t+h,t} \circ \tau_z = \tau_z \circ T_{t+h,t}$$

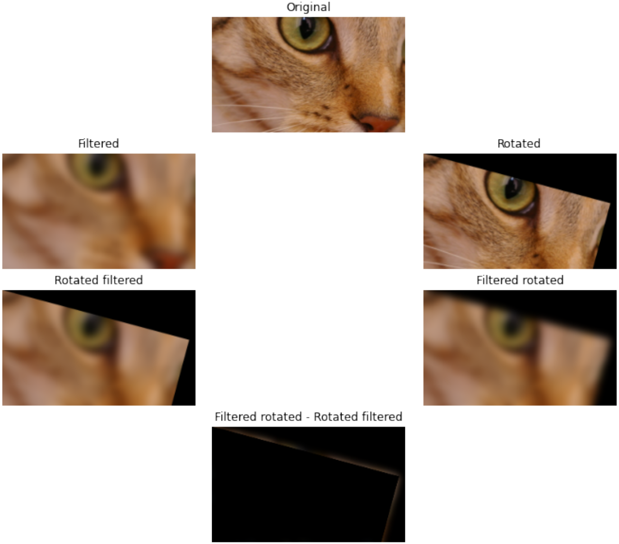
- T_t is **Euclidean invariant** if for every orthogonal matrix O:

$$T_{t+h,t} \circ \rho_O = \rho_O \circ T_{t+h,t}$$

Translation invariance for heat scale space



Euclidean invariance for heat scale space



Scale spaces: Invariance

Definition

- T_t is **scale invariant** if there exists a rescaling function $\theta : (0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying the following conditions:
  - θ is differentiable and $\partial_c \theta(t, 1) > 0$ and is continuous for all $t > 0$.
  - $T_{t+h,t} \circ \sigma_c = \sigma_c \circ T_{\theta(c,t+h),\theta(c,t)}, c > 0$
- A scale invariant T_t is **affine invariant** if there exists $\hat{\theta} : \mathrm{GL}^n \times [0, \infty) \to [0, \infty)$ such that $\theta(c, \cdot) := \hat{\theta}(c\mathrm{Id}, \cdot)$ satsfies the conditions above and

$$T_{t+h,t} \circ \rho_A = \rho_A \circ T_{\hat{\theta}(A,t+h),\hat{\theta}(A,t)}, A \in \mathrm{GL}^n$$

where $(\sigma_c u)(x) := u(cx), (\rho_A u)(x) = u(Ax)$.

Scale spaces: Invariance

Definition

- T_t is **invariant by gray level translations** if

$$T_{t+h,t}(0) = 0, \quad T_{t+h,t}(u + C) = T_{t+h,t}(u) + C, C \in \mathbb{R}$$

- T_t is **contrast invariant** if for every non-decreasing continuous function $g : \mathbb{R} \to \mathbb{R}$ and $u \in U$:

$$g(T_{t+h,t}u(x)) = T_{t+h,t}(g \circ u)(x), x \in \mathbb{R}^n$$

Heat scale space is not contrast invariant