Qnt Macro II - PS2: Krussel & Smith with Endogenous Labor Choice

Gabriela Barbosa

November 23, 2018

Introduction

Utility now includes endogenous labor choice:

$$U(c, n) = \log(c) - \Gamma \frac{n^{1+\gamma}}{1+\gamma}$$

Besides this, the framework is the same as in K&S.

Production is given by:

$$y = zk^{\alpha}l^{1-\alpha}$$

where z is the stochastic aggregate shock to productivity.

This TFP shock has two possible realizations: z_g, z_b and it is a Markov, described by the transition probability $\pi_{zz'}$.

At the individual level, there is an idiosyncratic shock to labor $\epsilon = \{0, 1\}$: that is, agent can be employed ($\epsilon = 1$) or unemployed.

Idiosyncratic and aggregate shocks are correlated. The unemployed are a (constant) fraction u_g in good times and $u_b$in bad times. Define $\pi_{zz',\epsilon\epsilon'}$ as the transition probability of the joint distribution.

Markets are incomplete and the only asset is capital k.

I. Model without aggregate risk

Let TFP shock be fixed at $z_t = \bar{z}$ for all t[1].

Then, this is just Aiyagari with endogeous labor - which is supplied elastically.

1 Household problem

With a given TFP shock, the individual state is defined by the idiosyncratic shock: $x = (e, a)$.

Given (w, r), the recursive formulation of consumer's problem is given by:

[1] To save notation I will supress the depence of variables on $\bar{z}$

$$V(e,a) = \max_{c,n,a'} \left\{ \log(c) - \Gamma\frac{n^{1+\gamma}}{1+\gamma} + \beta \underbrace{\sum_{e'=0,1} \pi_{ee'} V(e',k')}_{E_t[V(e',a')]} \right\}$$

$$\begin{aligned} &\text{s.t.} \quad c + a' = ra + e_i wn + (1-\delta)a \\ &n = \in [0,1] \\ &c \geq 0 \\ &a' \geq \underline{a} \end{aligned} \tag{1.1}$$

The optimal decision rules (policy functions) are: $g_a \equiv a'(e,a), g_c \equiv c(e,a), g_n \equiv n(e,a)$.

A) Optimality conditons

FOCs of the problem above gives:

$$\begin{aligned} &(c) \quad \frac{1}{c} - \lambda = 0 \\ &(a') \quad \beta E_t[V'(e',a')] - \lambda = 0 \\ &(n) \quad -\Gamma n^\gamma + \lambda(ew) = 0 \\ &(ET) \quad V'(e,a) = \frac{1}{c}(1+r-\delta) \end{aligned}$$

The *optimal intratemporal and inter-temporal conditions (Euler Equation)* are then, respectively:

$$\boxed{n = \left[\frac{1}{\Gamma}\frac{ew}{c}\right]^{1/\gamma}} \tag{1.2}$$

$$\frac{1}{c} = \beta\frac{1}{E(c')}(1+r-\delta) \rightarrow \boxed{\frac{1}{c} = \beta(1+r-\delta)\frac{1}{\left[\sum_{e'} \pi_{ee'}\Big((1+r-\delta)a' + n'e'w - a''\Big)\right]}} \tag{1.3}$$

where

$$\sum_{e'} \pi_{ee'}\Big((1+r-\delta)a' + n'e'w - a''\Big) = \pi_{e1}\Big((1+r-\delta)a' + n'w - a''\Big) + \pi_{e0}\Big((1+r-\delta)a' - a''\Big)$$

B) Problem in term of savings decision

From the above we can see that labor-leisure choice is an instantenous decision. In this sense, we can use its intratemporal condition together with the budget constraint to implicitly define the optimal labor n as a functions of states x and future savings a'. That is, we can construct a function:

$$n = \tilde{g_n}(e,a,a')$$

We can plug in this function into the optimization problem, which will then become an equivalent problem of just choosing the opitmal savings decision a' and get the policy function

$$a' = g_a(e,a)$$

Hence, we transform the problem into a 'non-endogenous-labor' one, just like on PS1 or Aiyagari.

NEW

1. Define potimal labor supply as a function of optimal consumption and employment state

From intertemporal condition:

$$\mathcal{F}_1(c,e) \equiv \left[\Gamma^{-1}\frac{ew}{c}\right]^{1/\gamma} \tag{1.4}$$

2. Guess a decision rule for the optimal consumption as a function of futere states

$$g_c^0(a'e') \tag{1.5}$$

3. Define the that measures the current consumption level as a function of the future asset holdings and current employment status state, and the optimal consumption guess

From the Euler Equation - letting $(1+r-\delta)=R$:

$$c = (\beta R)^{-1}E(c')$$

$$\mathcal{F}_2^0(a',e) = (\beta R)^{-1}E\Big(g_c^0(a',e')\Big) = (\beta R)^{-1}\Big[\sum_{e'}\pi_{ee'}g_c^0(a',e')\Big] \tag{1.6}$$

4. Find current asset holdings consistent with the guess of consumption policy function

From the budget constraint:

$$a = \frac{1}{R}\Big[c + a' - ewn\Big]$$

$$\begin{aligned}\mathcal{F}_3^0 &= \frac{1}{R}\Big\{\mathcal{F}_2^0(a',e) + a' - ew\Big[\mathcal{F}_1\Big(\mathcal{F}_2^0(a',e),e\Big)\Big]\Big\} \\ &= \frac{1}{R}\Big\{(\beta R)^{-1}\Big[\sum_{e'}\pi_{ee'}g_c^0(a',e')\Big] + a' - ew\Big[\Gamma^{-1}ew\Big((\beta R)^{-1}\Big[\sum_{e'}\pi_{ee'}g_c^0(a',e')\Big]\Big)\Big]^{1/\gamma}\Big\}\end{aligned} \tag{1.7}$$

Thus, we can obtain the optimal savings rule $a' = g_a(e,a)$ by solving the above. And with this we can get the policy function for labor:

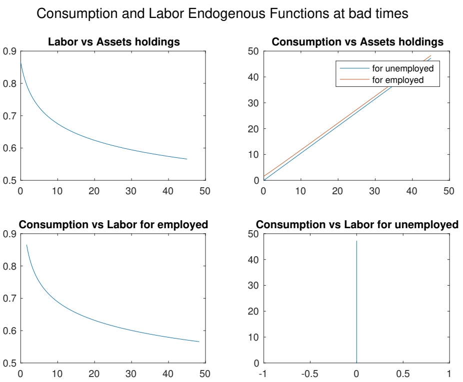
$$\boxed{n = \tilde{g}_n(e,a,g_a(e,a')) = g_l(e,a)} \tag{1.8}$$

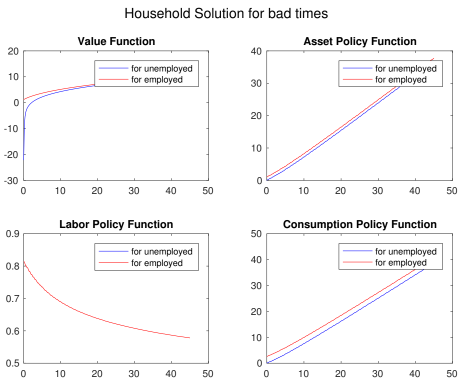
C) Numerical solution: Matlab code

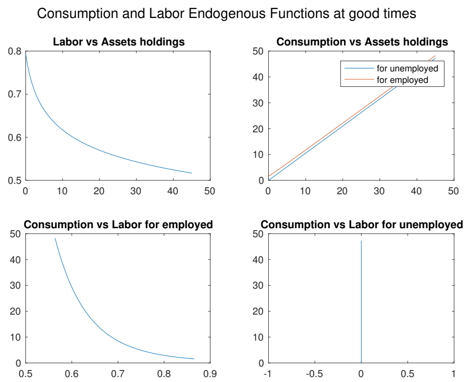
The code delivers the policy functions $a' = g_a(a,e), n = g_n(a,e)$, for a given price system (w,r) and a given TFP shock $z = z_g, z_b$.

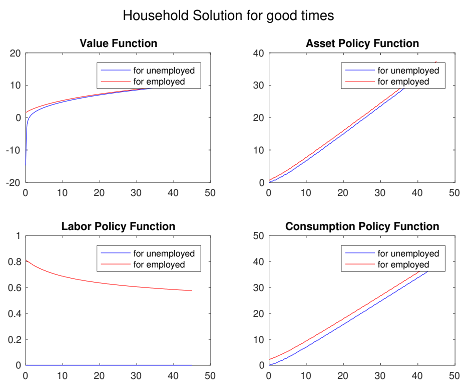
This means that we have transtion matrices for the employment state $e : \pi_{ee'}$, and the joint distrubition of shocks $\pi_{zz',ee'}$. So we can compare solutions between good and bad times.

Graphical solutions


Consumption and Labor Endogenous Functions at bad times
Labor vs Assets holdings
Consumption vs Assets holdings
for unemployed
for employed
Consumption vs Labor for employed
Consumption vs Labor for unemployed


Household Solution for bad times
Value Function
for unemployed
for employed
Asset Policy Function
for unemployed
for employed
Labor Policy Function
for unemployed
for employed
Consumption Policy Function
for unemployed
for employed


Consumption and Labor Endogenous Functions at good times
Labor vs Assets holdings
Consumption vs Assets holdings
for unemployed
for employed
Consumption vs Labor for employed
Consumption vs Labor for unemployed


Household Solution for good times
Value Function
for unemployed
for employed
Asset Policy Function
Labor Policy Function
Consumption Policy Function

2 Equilibrium

A) Markets Equilibrium condtions

There is a single representative firm with CRS technology. By profit maximization, prices satisfy the following optimal conditions:

$$r = f_k - \delta \rightarrow r = z\alpha k^{\alpha-1} l^{1-\alpha} - \delta \tag{2.1}$$

$$w = f_l \rightarrow w = z(1-\alpha)k^{\alpha} l^{-\alpha} \tag{2.2}$$

To the model, we need to specify the type measure/distribution of agents over e, a space $\mu = \mu(e, a)$and the its law of motion (updating function) $\mu' = H(\mu)$. Also, define the individual state space $X = A \times E$ where A is the possible asset holdings set and $E = \{0, 1\}$ is the employment status set; and let $\mathcal{X}$ be the set of all probability measures on the the measurable space of $(X, \mathcal{B}(X))$.

There are 3 markets in this economy: for asset, for labor, and for consumption goods.

Then, *market clearing* is given by:

$$K(H(\mu)) = \sum_e \int_{\mathcal{X}} g_a(a, e; \mu) \quad d\mu \tag{2.3}$$

$$L(\mu) = \sum_e \int_{\mathcal{X}} g_n(a, e; \mu)e \quad d\mu \tag{2.4}$$

$$\int_{\mathcal{X}} g_c(a, e; \mu) \quad d\mu + \int_{\mathcal{X}} g_a(a, e; \mu) \quad d\mu = \bar{z}K(\mu)L(\mu) + (1-\delta)K(\mu) \tag{2.5}$$

B) Solving for the equilbrium (GE)

Equilbrium concept here is the *Recursive Competitive Equilibrium*: it is defined as

i) given prices, $g^a(x), g_c(x), g_n(x)$ are the optimal decision rules for the consumer's UMP

ii) firm optimizes and prices are given by the marginal conditions of k, l

iii) markets clear

iv) consistency of the type-distribution μ with the optimal decidision rule $g_a(e, a)$ and the Markov process $\pi_{ee'}$[2].

$$\mu' = H(\mu)$$

<u>Remark</u>:

Unfourtunately, my code started crashing.

On my first attemps with it, I was able to run and get the GE. However, I started making some modification as I improved my knowledge of the code and then it started not running anymore. Even then, I believe my method is correct.

[2]The law of motion gives the probability that an individual with state x at t will have an state that lies in the measurable subset B of $\mathcal{X}$ at $t+1$: $H(x, t, B) = \pi(\{e' \in E : (g_a(x, t), e') \in B | e\})$

I am still working on it but what I am submitting is what I have so far.

Since GE is not running, neither is my economy with aggregate shocks. I will update its version soon.

Please refer to the code for all the details of my approach to this problem.

I found that the code is very sensitive to parameter values and to the endogenous function of consumption and labor. This was my main challenge.