> Mathematical Logic and Graph Theory 2022 Homework 1 Answers
>
> By Jingyi Chen with C and Songxiao Guo with G after each question number.

3.1.41 C

> A palindrome is a string whose reversal is identical to the string. How many bit strings of length n are palindromes?

n为偶数时，$2^{n/2}$；n为奇数时，$2^{(n+1)/2}$。

3.1.53 C

> How many bit strings of length eight contain either three consecutive 0s or four consecutive 1s?

147个。

包含3个连续的0的个数是104，4个连续1的个数是48，都包含的个数是8，容斥原理得147。

3.1.77 C

> How many diagonals does a convex polygon with n sides have? (Recall that a polygon is convex if every line segment connecting two points in the interior or boundary of the polygon lies entirely within this set and that a diagonal of a polygon is a line segment connecting two vertices that are not adjacent.)

$n(n-3)/2$，从每个顶点往另外$n-3$个顶点连线，每条线都重复了一次。

3.2.13 C

> Let $(x_i, y_i, z_i), i = 1, 2, 3, 4, 5, 6, 7, 8, 9$, be a set of nined istinct points with integer coordinates in xyz space.
> Show that the midpoint of at least one pair of these points has integer coordinates.

一对点的3个维度分别有相同的奇偶性，他们的中点坐标就是整数的；每个点的3个维度的奇偶性一共有8中情况，鸽巢原理必有2个点的坐标奇偶性一致，中点坐标为整数。

3.2.15 G

> - a) Show that if five integers are selected from the first eight positive integers, there must be a pair of these
>   integers with a sum equal to 9.
> - b) Is the conclusion in part (a) true if four integers are selected rather than five?

- a) 将集合 $\{1, 2 \cdots 8\}$ 划分成 $\{1, 8\}$、$\{2, 7\}$、$\{3, 6\}$ 和 $\{4, 5\}$ 4个不相交的集合。由抽屉原理，在 $\{1, 2 \cdots 8\}$ 中任取 5 个元素，必有至少 2 个元素在被划分成的某个集合中。另一方面，被

划分成的任意集合，里的两个元素的和为 9 ，故结论成立。

- b) 不成立。如 $\{1,2,3,4\}$，可以枚举验证其中没有两个数的和为 9 。

3.2.25 G

> Show that whenever 25 girls and 25 boys are seated around a circular table there is always a person both of
> whose neighbors are boys.

将圆桌的座位从 $1 \sim 50$ 编号， 50 号座位与 1 号座位相邻。有 25 个奇数号座位， 25 个偶数号座位。如果最多有 12 个男生获得了奇数号座位，那么至少有 13 个男生获得了偶数号座位。反之亦然。故不失一般性，假设至少有 13 个男生获得了 25 个奇数座位中的一些位置，那么这些男生中至少有 2 个会获得连续的奇数号，这样坐在他们中间的人就会是左右都与一个男生相邻。

3.2.49 G

> An alternative proof of Theorem 3 based on the generalized pigeonhole principle is outlined in this exercise. The
> notation used is the same as that used in the proof in the text.
>
> - a) Assume that $i_k \leq n$ for $k = 1, 2, \ldots, n^2+1$. Use the generalized pigeonhole principle to show that there are $n+1$ terms $a_{k_1}, a_{k_2}, \cdots, a_{k_{n+1}}$ with $i_{k_1} = i_{k_2} = \cdots = i_{k_n} + 1$, where $1 \leq k_1 < k_2 < \cdots < k_{n+1}$.
> - b) Show that $a_{k_j} > a_{k_{j+1}}$ for $j = 1, 2, \cdots, n$.
> - c) Use parts (a) and (b) to show that if there is no increasing subsequence of length $n+1$, then there must be a decreasing subsequence of this length.

- a) 根据广义鸽巢原理，至少有 $\lceil (n^2+1)/n \rceil = n+1$ 个数字 $i_{k_1}, i_{k_2}, \cdots, i_{k_{n+1}}$ 相等。
- b) 如果 $a_{k_j} < a_{k_{j+1}}$ ，那么长度为 i_{k_j} 以 a_{k_j} 开头的子序列包含长度为 i_{k_j+1} ，以 $a_{k_{j+1}}$ 开头的递增子序列。这与 $i_{k_j} = i_{k_{j+1}}$ 矛盾。
- c) 如果没有长度超过 n 的递增子序列，那么就有 $i_k \leq n$ ，便满足 a) 的条件。根据 b) ，有 $a_{k_1} > a_{k_2} > \cdots > a_{k_{n+1}}$，构造出了长度为 $n+1$ 的递减子序列。