> Mathematical Logic and Graph Theory 2022 Homework 2 Answers
>
> By Jingyi Chen with C and Songxiao Guo with G after each question number.

1.4.7 C

> Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.
> a) $\forall x(C(x) \rightarrow F(x))$
> b) $\forall x(C(x) \wedge F(x))$
> c) $\exists x(C(x) \rightarrow F(x))$
> d) $\exists x(C(x) \wedge F(x))$

- a)每个喜剧演员都很有趣；
- b)每个人都是很有趣的喜剧演员；
- c)存在某个人，如果他是喜剧演员，那么他是很有趣的；
- d)某些喜剧演员是很有趣的（某些很有趣的人是喜剧演员）。

1.4.27 C

> Translate each of these statements into logical expressions in three different ways by varying the domain and by using predicates with one and with two variables.
> a) A student in your school has lived in Vietnam.
> b) There is a student in your school who cannot speak Hindi.
> c) A student in your school knows Java, Prolog, and C++.
> d) Everyone in your class enjoys Thai food.
> e) Someone in your class does not play hockey.

- a)令谓词 $Y(x)$ 表示语句"x 是在你学校上的学生"。
  - 如果我们令 $V(x)$ 为"x 曾在越南居住"，则
    - 论域是你学校同学时：$\exists x V(x)$.
    - 论域是所有人时：$\exists x(Y(x) \wedge V(x))$.
  - 如果令 $D(x, y)$ 表示 x 曾在国家 y 居住，论域是所有人时：$\exists x(Y(x) \wedge D(x, \text{越南}))$.
- b)令谓词 $Y(x)$ 表示语句"x 是在你学校上的学生"。
  - 如果我们令 $V(x)$ 为"x 会说印地语"，则
    - 论域是你学校同学时：$\exists x \neg V(x)$.

- 论域是所有人时：$\exists x(Y(x) \wedge \neg V(x))$.
- 如果令 $D(x,y)$ 表示 x 会说语言 y，论域是所有人时：$\exists x(Y(x) \wedge \neg D(x, \text{印地语}))$.

- c)令谓词 $Y(x)$ 表示语句“x 是在你学校上的学生”。
  - 如果我们令 $V(x)$ 为“x 会用 Java、Prolog、C++”，则
    - 论域是你学校同学时：$\exists x V(x)$.
    - 论域是所有人时：$\exists x(Y(x) \wedge V(x))$.
  - 如果令 $D(x,y)$ 表示 x 会用语言 y，论域是所有人时：$\exists x(Y(x) \wedge D(x, Java) \wedge D(x, Prolog) \wedge D(x, C++))$.
- d)令谓词 $Y(x)$ 表示语句“x 是在你班上的学生”。
  - 如果我们令 $V(x)$ 为“x 喜欢泰国食物”，则
    - 论域是你班同学时：$\forall x V(x)$.
    - 论域是所有人时：$\forall x(Y(x) \rightarrow V(x))$.
  - 如果令 $D(x,y)$ 表示 x 喜欢食物 y，论域是所有人时：$\forall x(Y(x) \rightarrow D(x, \text{泰国}))$.
- e)令谓词 $Y(x)$ 表示语句“x 是在你班上的学生”。
  - 如果我们令 $V(x)$ 为“x 打曲棍球”，则
    - 论域是你班同学时：$\exists x \neg V(x)$.
    - 论域是所有人时：$\exists x(Y(x) \wedge \neg V(x))$.
  - 如果令 $D(x,y)$ 表示 x 玩游戏 y，论域是所有人时：$\exists x(Y(x) \wedge \neg D(x, \text{曲棍球}))$.

1.4.63 C

> Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements “x is a baby,” “x is logical,” “x is able to manage a crocodile,” and “x is despised,” respectively. Suppose that the domain consists of all people. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.
> a) Babies are illogical.
> b) Nobody is despised who can manage a crocodile.
> c) Illogical persons are despised.
> d) Babies cannot manage crocodiles.
> e) Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

- a)$\forall x(P(x) \rightarrow \neg Q(x))$
- b)$\forall x(R(x) \rightarrow \neg S(x))$
- c)$\forall x(\neg Q(x) \rightarrow S(x))$
- d)$\forall x(P(x) \rightarrow \neg R(x))$
- e)可以得出结论。把b)变为$\forall x(S(x) \rightarrow \neg R(x))$，反复利用$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$即可由a、b、c得到d。

1.5.7 C

> Let $T(x,y)$ mean that student x likes cuisine y, where the domain for x consists of all students at your school andt he domain for y consists of all cuisines. Express each of these statements by a simple English sentence.
> a) $\neg T(AbdallahHussein, Japanese)$
> b) $\exists x T(x, Korean) \wedge \forall x T(x, Mexican)$
> c) $\exists y(T(MoniqueArsenault, y) \vee T(JayJohnson, y))$
> d) $\forall x \forall z \exists y((x \neq z) \rightarrow \neg(T(x,y) \wedge T(z,y)))$
> e) $\exists x \exists z \forall y(T(x,y) \leftrightarrow T(z,y))$
> f) $\forall x \forall z \exists y(T(x,y) \leftrightarrow T(z,y))$

- a)Abdallah Hussein 不喜欢日本菜。

- b)学校的某些学生喜欢韩国菜，并且学校的每个人都喜欢墨西哥菜。
- c)存在一些菜，不是 Monique Arsenault 喜欢，就是 Jay Johnson 喜欢。
- d)学校中每一对不同的学生，总有一种菜，他们不都喜欢。
- e)*
- f)学校中任意两个学生（可能相同），总有一种菜，他们有相同的看法。

1.5.11 C

> Let $S(x)$ be the predicate “x is a student,” $F(x)$ the predicate “x is a faculty member,” and $A(x, y)$ the predicate“x has asked y a question,” where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.
> a) Lois has asked Professor Michaels a question.
> b) Every student has asked Professor Gross a question.
> c) Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.
> d) Some student has not asked any faculty member a question.
> e) There is a faculty member who has never been asked a question by a student.
> f) Some student has asked every faculty member a question.
> g) There is a faculty member who has asked every other faculty member a question.
> h) Some student has never been asked a question by a faculty member.

- a)$A(Lois, Michaels\text{教授})$.
- b)$\forall x(S(x) \rightarrow A(x, Gross\text{教授}))$.
- c)$\forall x(F(x) \rightarrow (A(x, Miller\text{教授}) \vee A(Miller\text{教授}, x)))$.
- d)$\exists x(S(x) \wedge \forall y(F(y) \rightarrow \neg A(x, y)))$.
- e)$\exists x(F(x) \wedge \forall y(S(y) \rightarrow \neg A(y, x)))$.
- f)$\forall y(F(u) \rightarrow \exists x(S(x) \wedge A(x, y)))$.
- g)$\exists x(F(x) \wedge \forall y((F(y) \wedge (y \neq x)) \rightarrow A(x, y)))$.
- h)$\exists x(S(x) \wedge \forall y(F(y) \rightarrow \neg A(y, x)))$.

1.5.21 C

> Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that every positive integer is the sum of the squares of four integers.

$$\forall x \exists a \exists b \exists c \exists d((x > 0) \rightarrow x = a^2 + b^2 + c^2 + d^2),$$

论域为全体整数。

1.5.35 C

> Find a common domain for the variables x, y, z, and w for which the statement $\forall x \forall y \forall z \exists w((w \neq x) \wedge (w \neq y) \wedge (w \neq z))$ is true and another common domain for these variables for which it is false.

任意具有4个或更多不同成员的论域会使命题为真，任意具有3个或更少成员的论域会使命题为假。

1.6.11 G

> Show that the argument form with premises $p_1, p_2, \cdots, p_n$ and conclusion $q \rightarrow r$ is valid if the argument form with premises $p_1, p_2, \ldots, p_n$,q, and conclusion r is valid.

当 q 为真时，由给定论证形式的有效性（$p_1 \wedge p_2 \cdots \wedge p_n \wedge q \rightarrow r$）知当 $p_1, p_2, \cdots, p_n$ 为真时，r 为真，故 $q \rightarrow r$ 为真。

当 q 为假时，$q \rightarrow r$ 为真自然成立。

综上可知结论成立。

1.6.23 G

> Identify the error or errors in this argument that supposedly shows that if $\exists x P(x) \wedge \exists x Q(x)$ is true then $\exists x(P(x) \wedge Q(x))$ is true.
>
> 1.$\exists x P(x) \wedge \exists x Q(x)$ Premise
> 2.$\exists x P(x)$ Simplification from (1)
> 3.$P(c)$ Existential instantiation from (2)
> 4.$\exists x Q(x)$ Simplification from (1)
> 5.$Q(c)$ Existential instantiation from (4)
> 6.$P(c) \wedge Q(c)$ Conjunction from (3) and (5)
> 7.$\exists x(P(x) \wedge Q(x))$ Existential generalization

第 5 步存在实例错误。c 已经被使用过，在此处的 c 没有理由是第 3 步所使用的 c 。

1.6.29 G

> Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$,$\forall x(\neg Q(x) \vee S(x))$, $\forall x(R(x) \rightarrow \neg S(x))$, and $\exists x \neg P(x)$ are true, then $\exists x \neg R(x)$ is true.

$$
\begin{array}{lr}
1.\exists x \neg P(x) & \text{前提引入} \\
2.\neg P(c) & \text{存在实例，由}(1) \\
3.\forall x(P(x) \vee Q(x)) & \text{前提引入} \\
4.P(c) \vee Q(c) & \text{全称实例，由}(3) \\
5.Q(c) & \text{析取三段论，由}(2)\text{和}(4) \\
6.\forall x(\neg Q(x) \vee S(x)) & \text{前提引入} \\
7.\neg Q(c) \vee S(c) & \text{全称实例，由}(6) \\
8.\neg(\neg Q(c)) & \text{双重否定律，由}(5) \\
9.S(c) & \text{析取三段论，由}(7)\text{和}(8) \\
10.\forall x(R(x) \rightarrow \neg S(x)) & \text{前提引入} \\
11.R(c) \rightarrow \neg S(c) & \text{存在实例，由}(10) \\
12.\neg(\neg S(c)) & \text{双重否定律，由}(9) \\
13.\neg R(c) & \text{取拒式，由}(11)\text{和}(12) \\
14.\exists x \neg R(x) & \text{存在引入，由}(13)
\end{array}
$$

1.6.35 G

> Determine whether this argument, taken from Kalish and Montague [KaMo64], is valid. If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil,he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

用 x_1 表示超人能够防止邪恶，用 x_2 表示超人愿意防止邪恶，用 x_3 表示超人防止邪恶，用 x_4 表示超人是无能的，用 x_5 表示超人是恶意的，用 x_6 表示超人存在。则该论证的前提是

$$
x_1 \wedge x_2 \rightarrow x_3, \neg x_1 \rightarrow x_4, \neg x_2 \rightarrow x_5, \neg x_3, x_6 \rightarrow (\neg x_4 \wedge \neg x_5),
$$

期望的结论是

$$
\neg x_6.
$$

这样的论证形式证明这些前提导出期望的结论：

$$
\begin{array}{lr}
1.x_1 \land x_2 \to x_3 & \text{前提引入} \\
2.\neg x_3 & \text{前提引入} \\
3.\neg(x_1 \land x_2) & \text{取拒式，由}(1)\text{和}(2) \\
4.\neg(\neg(x_1 \to \neg x_2)) & \text{条件命题的逻辑等价式，由}(3) \\
5.x_1 \to \neg x_2 & \text{双重否定律，由}(4) \\
6.\neg x_2 \to x_5 & \text{前提引入} \\
7.x_1 \to x_5 & \text{假言三段论，由}(5)\text{和}(6) \\
8.\neg x_5 \to \neg x_1 & \text{条件命题的逻辑等价式，由}(7) \\
9.\neg x_1 \to x_4 & \text{前提引入} \\
10.\neg x_5 \to x_4 & \text{假言三段论，由}(8)\text{和}(9) \\
11.x_5 \lor x_4 & \text{条件命题的逻辑等价式，由}(10) \\
12.\neg(\neg x_5 \land \neg x_4) & \text{德·摩根律，由}(11) \\
13.x_6 \to (\neg x_4 \land \neg x_5) & \text{前提引入} \\
14.\neg x_6 & \text{取拒式，由}(13)
\end{array}
$$

1.7.15 G

> Prove that if x is an irrational number and $x > 0$, then $\sqrt{x}$ is also irrational.

假设结论不成立，即 $\sqrt{x}$ 是有理数。由 $x > 0$，可设 $\sqrt{x} = \frac{p}{q}, p, q \in \mathbb{Z}^*$，故有 $x = \sqrt{x}\sqrt{x} = \frac{p^2}{q^2}, p^2, q^2 \in \mathbb{Z}^*$，即 x 是有理数，矛盾！故结论成立。

1.7.19 G

> Show that if n is an integer and $n^3 + 5$ is odd, then n is even using
> a) a proof by contraposition.
> b) a proof by contradiction.

- a)假设 n 是奇数。则 $n^3 + 5 \equiv 1 + 5 \equiv 0 \pmod 2$，即 $n^3 + 5$ 不是奇数，原条件也不成立。故结论成立。
- b)假设 n 是奇数，并且 $n^3 + 5$ 也是奇数，则 $1 \equiv n^3 + 5 \equiv 1 + 5 \equiv 0 \pmod 2$，矛盾！故结论成立。

1.7.41 G

> Prove that at least one of the real numbers $a_1, a_2, \cdots, a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

设这些数的平均值为 A，假设结论不成立，即 $a_i < A, i = 1, \cdots, n$。由 $A = \frac{1}{n}\sum_{i=1}^{n} a_i < \frac{nA}{n} = A$，矛盾！故结论成立。

我使用的是归谬法证明。