> Mathematical Logic and Graph Theory 2022 Homework 5 Answers
>
> By Jingyi Chen with C and Songxiao Guo with G after each question number.

- 3.3.23 G
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3.3.23 G

> How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other?

用插空法，先组合再排列，将 8 个男士排成一排，算上首尾，共 9 个空。在其中选择 5 个放女士，再考虑人的不同，共 $A_8^8 A_5^5 C_9^5 = 609638400$ 种。

3.3.31 G

> How many $4-$permutations of the positive integers not exceeding 100 contain three consecutive integers $k, k+1, k+2$, in the correct order
>
> - a) where these consecutive integers can perhaps be separated by other integers in the permutation?
> - b) where they are in consecutive positions in the permutation?

- a)
  1. 包含恰好 3 个连续的整数：在前 100 个数中选择连续的 3 个数，有 $100-3+1=98$ 种。根据本题的需要，分成两类：$1,2,3$、$98,99,100$ 为第 1 类，其他 96 种为第 2 类。对于第 1 类，再选出一个与已选出的 3 个数不相连的种类有 96 种，而对于第 2 类，有 95 种。组合后再排列，每种选出的 4 个数，都有 4 种方式使其顺序相同。这种情况有 $(2\times 96+96\times 95)\times 4=37248$ 种。
  2. 包含 4 个连续的整数：在前 100 个数中选择连续的 4 个数，有 $100-4+1=97$ 种。每种选出的 4 个数，都含有 2 种连续的 3 个数，故有 $4\times 2-1=7$ 种方式使其中有连续的 3 个数顺序相同。（排除重复情况）。这种情况有 $97\times 7=679$ 种。

  共有 $37248+679=37927$ 种。
- b)
  1. 包含恰好 3 个连续的整数：在前 100 个数中选择连续的 3 个数，有 $100-3+1=98$ 种。根据本题的需要，分成两类：$1,2,3$、$98,99,100$ 为第一类，其他 96 种为第 2 类。对于第 1 类，再选出一个与已选出的 3 个数不相连的种类有 96 种，而对于第 2 类，有 95 种。组合

后再排列，每种选出的 4 个数，都有 2 种方式使其顺序相同且连续。这种情况有 $(2\times 96+96\times 95)\times 2=18624$ 种。

2. 包含 4 个连续的整数：在前 100 个数中选择连续的 4 个数，有 $100-4+1=97$ 种。每种选出的 4 个数，都含有 2 种连续的 3 个数，故有 $2\times 2-1=3$ 种方式使其中有连续的 3 个数顺序相同。（排除重复情况）。这种情况有 $97\times 3=291$ 种。

共有 $18624+291=18915$ 种。

4.1.17 G

> - a) Find a recurrence relation for the number of ternary strings of length n that do not contain consecutive
>   symbols that are the same.
> - b) What are the initial conditions?
> - c) How many ternary strings of length six do not contain consecutive symbols that are the same?

- a) 考虑长为 $n-1$ 的串，与其最后一位不同的符号有两个，从中选择一个，并加在长为 $n-1$ 的串后面，就得到了长为 n 的满足要求的串。另一方面，对每个长为 n 的满足要求的串，减去其末尾，都能得到一个对应的长为 $n-1$ 的满足要求的串。设长为 n 的满足要求的串的个数为 a_n，则 $a_n=2a_{n-1},n\geq 2$。
- b) $a_1=3$。
- c) $a_n=2^{n-1}\times 3\Rightarrow a_6=96$。

4.1.27 G

> - a) Find a recurrence relation for the number of ways to lay out a walkway with slate tiles if the tiles are red, green, or gray, so that no two red tiles are adjacent and tiles of the same color are considered indistinguishable.
> - b) What are the initial conditions for the recurrence relation in part (a)?
> - c) How many ways are there to lay out a path of seven tiles as described in p art (a)?

- a) 设 a_n 表示没有红色砖相邻的用 n 块砖铺路的方式，b_n 表示没有红色砖相邻，且最后一块为红色的用 n 块砖铺路的方式。易知 $a_n=2b_{n-1}+3(a_{n-1}-b_{n-1})$，$b_n=a_{n-1}-b_{n-1}$，得到 $a_n=2a_{n-1}+2a_{n-2},n\geq 3$。
- b) $a_1=3,a_2=8.$
- c) 利用特征方程法求得 $a_n=\frac{\sqrt{3}+2}{2\sqrt{3}}(1+\sqrt{3})^n+\frac{\sqrt{3}-2}{2\sqrt{3}}(1-\sqrt{3})^n$。故 $a_7=1224$。

4.1.29 G

> Let $S(m,n)$ denote the number of onto functions from a set with m elements to a set with n elements. Show that $S(m,n)$ satisfies the recurrence relation
>
> $$S(m,n)=n^m-\sum_{k=1}^{n-1}C_n^kS(m,k)$$
>
> whenever $m\geq n$ and $n>1$, with the initial condition $S(m,1)=1$.

$n=1$ 时，易知$S(m,1)=1$。$n\geq 2$ 时，考虑从 m 元集到 n 元集的所有映射，共有 n^m 个，如果一个映射不是映上的，等价于它的值域可以取 $1,2,\cdots,n-1$。对每个 $i\in\{1,2,\cdots,n-1\}$，从 n 元集中取 i 个的种类有 C_n^k 种。这样便有 $S(m,n)=n^m-\sum_{k=1}^{n-1}C_n^kS(m,k),m\geq n,n>1$。

4.1.35 G

> This problem is based on an account by the historian Flavius Josephus, who was part of a band of 41 Jewish rebels trapped in a cave by the Romans during the Jewish-Roman war of the first century. The rebels preferred suicide to capture; they decided to form a circle and to repeatedly count off around the circle, killing every third rebel left alive. However, Josephus and another rebel did not want to be killed this way; they determined the positions where they should stand to be the last two rebels remaining alive. The variation we consider begins with n people, numbered 1 to n, standing around a circle. In each stage, every second person still left alive is eliminated until only one survives. We denote the number of the survivor by $J(n)$.
>
> Show that $J(n)$ satisfies the recurrence relation $J(2n)=2J(n)-1$ and $J(2n+1)=2J(n)+1$, for $n\geq 1$, and $J(1)=1$.

- 当人数是偶数时，首先偶数位置的人被排除，则原来站在 $2i-1$ 位置的人，现在站在 i 位置（$i=1,2,\cdots,n$）。此时剩下 n 人。那么最后剩下的人在 $J(n)$ 位置的人，原来站在 $2J(n)-1$ 位置。
- 当人数是奇数时，首先偶数位置的人被排除，再排除 1 号位置的人，则原来站在 $2i+1$ 位置的人，现在站在 i 位置$i=1,2,\cdots,n$）。此时剩下 n 人。那么最后剩下的人在 $J(n)$ 位置的人，原来站在 $2J(n)+1$ 位置。

4.5.17 G

> How many permutations of the 10 digits either begin with the 3 digits 987, contain the digits 45 in the fifth and sixth positions, or end with the 3 digits 123?

根据容斥原理，满足条件的排列数为

$$A_7^7+A_8^8+A_7^7-A_5^5-A_5^5-A_4^4+A_2^2=50138.$$

4.6.15 C

> A machine that inserts letters into envelopes goes haywire and inserts letters randomly into envelopes. What is
> the probability that in a group of 100 letters
>
> - a) no letter is put into the correct envelope?
> - b) exactly one letter is put into the correct envelope?
> - c) exactly 98 letters are put into the correct envelopes?
> - d) exactly 99 letters are put into the correct envelopes?
> - e) all letters are put into the correct envelopes?

- a) $D_{100}/100!$.
- b) $100D_{99}/100!$.
- c) $C(100,2)/100!$.
- d) 0.
- e) $1/100!$.

4.6.17 C

> How many ways can the digits $0,1,2,3,4,5,6,7,8,9$ be arranged so that no even digit is in its original position?

由逐步淘汰原理：

$$A_{10}^{10}-C_5^1A_9^9+C_5^2A_8^8-C_5^3A_7^7+C_5^4A_6^6-C_5^5A_5^5=2170680.$$

5.1.5 C

> Determine whether the relation R on the set of all Web pages is reflexive, symmetric, antisymmetric, and/or
> transitive, where $(a,b) \in R$ if and only if
>
> - a) everyone who has visited Web page a has also visited Web page b.
> - b) there are no common links found on both Web page a and Web page b.
> - c) there is at least one common link on Web page a and Web page b.
> - d) there is a Web page that includes links to both Web page a and Web page b.

- a) 自反的，传递的。
- b) 对称的。
- c) 对称的。
- d) 对称的。

5.1.9 C

> Show that the relation $R = \emptyset$ on the empty set $S = \emptyset$ is reflexive, symmetric, and transitive.

自反性$\forall a((a,a) \in R)$对，因为没这个a；对称性和传递性也对，也是因为没有a,b。（否定前件律）

5.1.49 C

> How many relations are there on a set with n elements that are
>
> - a) symmetric?
> - b) antisymmetric?
> - c) asymmetric?
> - d) irreflexive?
> - e) reflexive and symmetric?
> - f) neither reflexive nor irreflexive?

- a)$2^{n(n+1)/2}$，因为每两者（包括自己和自己）之间可能有这个对称关系。
- b)$2^n 3^{n(n-1)/2}$，因为每个元素都可以和自己有该关系也可以没有，同时，每两不同元素之间可能有两种方向的关系或没有关系。
- c)$3^{n(n-1)/2}$，因为每不同两者之间可能有两种方向的该关系或者没有关系。
- d)$2^{n(n-1)}$，因为自己和自己没有，每有序两者之间可能有该关系或没有。
- e)$2^{n(n-1)/2}$，因为自己和自己只能有，每无序两者之间可能有关系或没有。
- f)$2^{n^2} - 2 \times 2^{n(n-1)}$，所有关系减去自反的、反自反的，自反的和反自反的一样多。

5.1.57 C

> Let R be a relation that is reflexive and transitive. Prove that $R^n = R$ for all positive integers n.

归纳法，如果R^n是自反的、传递的，则因为是传递的，$R^{n+1} \subseteq R$；$(a,b) \in R$时，$(b,b) \in R^n$，所以$(a,b) \in R^{n+1}$。

5.2.23 C

> Show that if C is a condition that elements of the $n-$ary relations R and S may satisfy, then $sC(R \cap S) = sC(R) \cap sC(S)$.

如果一个n元组在$R \cap S$中且满足条件C，那么它或者在R中或者在S中且满足条件C，因此它也属于右侧；类似的如果一个n元组属于右侧，则它既在R中且满足条件C又在S中且满足条件C，因此它在$R \cap S$中且满足条件C，因此它也属于左侧。