> Mathematical Logic and Graph Theory 2022 Homework 3 Answers
>
> By Jingyi Chen with C and Songxiao Guo with G after each question number.

1.8.5 C

> Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$.

假设$x \geq y$，则$\max(x, y) = x, \min(x, y) = y$；假设$x \leq y$，则$\max(x, y) = y, \min(x, y) = x$，
$\therefore x + y = \max(x, y) + \min(x, y)$。

1.8.13 C

> Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.

8和9。

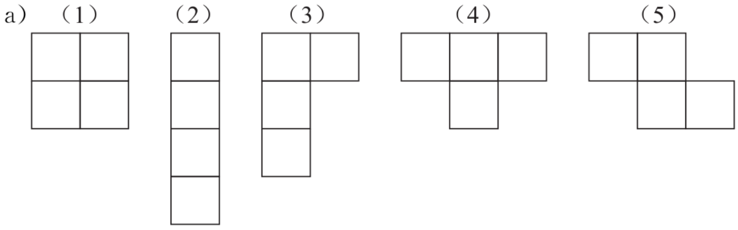
1.8.27 G

> Write the numbers $1, 2, \cdots, 2n$ on a blackboard, where n is an odd integer. Pick any two of the numbers, j and
> k, write $|j - k|$ on the board and erase j and k. Continue this process until only one integer is written on the board. Prove that this integer must be odd.

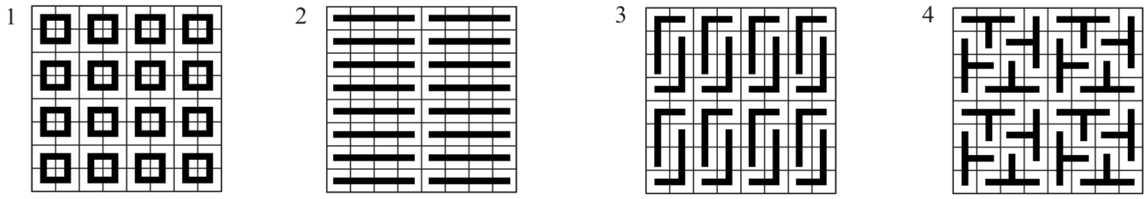
数字和的奇偶性不变。最初情况为$1 + 2 + \cdots + 2n = n(2n + 1) \equiv n \equiv 1 (\mod 2)$。每次减少一个数，最后达到只剩一个数的情况，此时该数为奇数。

1.8.51 G

> a) Draw each of the five different tetrominoes, where a tetromino is a polyomino consisting of four squares.
> b) For each of the five different tetrominoes, prove or disprove that you can tile a standard checkerboard us-
> ing these tetrominoes.



b)下图展示了用前四种四联骨牌的拼接。



假设第(5)种可以覆盖，则左上角必定有一块。根据其形状，则第一行或第一列的每个位置都有唯一的放置方式，将导致第一行或第一列最后一个格子无法被覆盖。

2.1.11 G

> Determine whether each of these statements is true or false.
> a) $0 \in \emptyset$
> b) $\emptyset \in \{0\}$
> c) $\{0\} \subset \emptyset$
> d) $\emptyset \subset \{0\}$
> e) $\{0\} \in \{0\}$
> f) $\{0\} \subset \{0\}$
> g) $\{\emptyset\} \subseteq \{\emptyset\}$

- a) F
- b) F
- c) F
- d) T
- e) F
- f) F
- g) T

2.1.27 G

> Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

- 充分性：已知 $A \subseteq B$，对 $\forall x \in \mathcal{P}(A)$，有 $x \subseteq A \subseteq B$，故有 $x \in \mathcal{P}(B)$。由 x 的任意性得 $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
- 必要性：已知 $\mathcal{P}(A) \subseteq \mathcal{P}(B)$，对 $\forall x \in A$，有 $\{x\} \in \mathcal{P}(A) \subseteq \mathcal{P}(B)$，故有 $x \in B$。由 x 的任意性得 $A \subseteq B$.

2.1.41 G

> Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.

$$A \times B \times C = \{(a, b, c) | a \in A, b \in B, c \in C\},$$
$$(A \times B) \times C = \{((a, b), c) | a \in A, b \in B, c \in C\}.$$

2.1.43 G

> Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

结论错误。设 $A = B = \emptyset$，则 $\mathcal{P}(A \times B) = \{\emptyset\}$，$\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset, \emptyset)\}$，二者明显不相等。

2.1.49 G

> The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are
> equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the

> ordered pair (a, b) to be $\{\{a\}, \{a, b\}\}$, then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

- 充分性：已知 $a = c, b = d$ 时，$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ 显然成立。
- 必要性：已知 $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ 时，
    - $a = b$，则 $\{\{a\}, \{a, b\}\} = \{a\}$，$\{\{c\}, \{c, d\}\} = \{c\}$，即已知 $\{a\} = \{c\}$，便有 $a = c$，推知 $b = d$。
        - $a \neq b$，则 $\{\{a\}, \{a, b\}\}$ 是二元集，含有一个一元集和一个二元集。两集合相等，故 $\{\{c\}, \{c, d\}\}$ 也含有一个一元集和一个二元集。这样便有 $c \neq d$。故 $\{a\} = \{c\} \Rightarrow a = c$。又$\{a, b\} = \{c, d\}$，易得 $b = d$。

2.2.19 G

> Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$
> a) by showing each side is a subset of the other side.
> b) using a membership table.

- a)

$$\begin{aligned} x \in \overline{A \cap B \cap C} &\equiv x \in \overline{A} \cup \overline{B} \cup \overline{C} \\ &\equiv x \notin A \vee x \notin B \vee x \notin C \\ &\equiv x \in \overline{A} \vee x \in \overline{B} \vee x \in \overline{C} \\ &\equiv x \in \overline{A} \cup \overline{B} \cup \overline{C}. \end{aligned}$$

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A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	0	1	0	1	0	1	0	1
1	0	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	0	1	1
0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1

2.2.35 G

> Let A, B, and C be sets. Use the identities in Table 1 to show that $\overline{(A \cup B)} \cap \overline{(B \cup C)} \cap \overline{(A \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}$.

$$\begin{aligned} \overline{(A \cup B)} \cap \overline{(B \cup C)} \cap \overline{(A \cup C)} &= \overline{A} \cap \overline{B} \cap \overline{B} \cap \overline{C} \cap \overline{A} \cap \overline{C} && \text{德·摩根律} \\ &= \overline{A} \cap (\overline{B} \cap \overline{B}) \cap \overline{C} \cap \overline{A} \cap \overline{C} && \text{结合律} \\ &= \overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{A} \cap \overline{C} && \text{幂等律} \\ &= \overline{A} \cap \overline{B} \cap (\overline{C} \cap \overline{A}) \cap \overline{C} && \text{结合律} \\ &= \overline{A} \cap \overline{B} \cap (\overline{A} \cap \overline{C}) \cap \overline{C} && \text{交换律} \\ &= \overline{A} \cap \overline{B} \cap \overline{A} \cap (\overline{C} \cap \overline{C}) && \text{结合律} \\ &= \overline{A} \cap \overline{B} \cap \overline{A} \cap \overline{C} && \text{幂等律} \\ &= \overline{B} \cap \overline{A} \cap \overline{A} \cap \overline{C} && \text{交换律} \\ &= \overline{B} \cap (\overline{A} \cap \overline{A}) \cap \overline{C} && \text{结合律} \\ &= \overline{B} \cap \overline{A} \cap \overline{C} && \text{幂等律} \\ &= \overline{A} \cap \overline{B} \cap \overline{C} && \text{交换律.} \end{aligned}$$

2.2.47 C

> Suppose that A, B, and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

是的。

对$\forall x \in A$，如果$x \in C$则有$x \notin A \oplus C$，因此$x \notin B \oplus C$，从而$x \in B$;

如果$x \notin C$则有$x \in A \oplus C$，因此$x \in B \oplus C$，从而$x \in B$;

对$\forall x \in B$，同理有$x \in A$，所以$A = B$。

2.3.3 C

> Determine whether f is a function from the set of all bit strings to the set of integers if
> a) $f(S)$ is the position of a 0 bit in S.
> b) $f(S)$ is the number of 1 bits in S.
> c) $f(S)$ is the smallest integer i such that the ith bit of S is 1 and $f(S) = 0$ when S is the empty string, the string with no bits.

- a) 不是，因为"某个0"的位置是不确定的。
- b)是的，1的个数由S确定。
- c)是的，i的只由S确定。

2.3.21 C

> Give an explicit formula for a function from the set of integers to the set of positive integers that is
> a) one-to-one, but not onto.
> b) onto, but not one-to-one.
> c) one-to-one and onto.
> d) neither one-to-one nor onto.

- a)例如$f(x) = 3x + 1 (x \geq 0), f(x) = -3x + 2 (x < 0)$。
- b)例如$f(x) = |x| + 1$。
- c)例如$f(x) = 2x + 1 (x \geq 0), f(x) = -2x (x < 0)$。
- d)例如$f(x) = x^2 + 1$。

2.3.37 C

> If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.

不一定，比如$A = a, B = b, c, C = d, g(a) = b, f(b) = d, f(c) = d$。类似这样构造一个中间的较大的集合和两侧较小的集合，使得g不能映上即可。

2.4.27 C

> Show that if a_n denotes the nth positive integer that is not a perfect square, then $a_n = n + \{\sqrt{n}\}$, where $\{x\}$ denotes the integer closest to the real number x.

把正整数分成两部分，一部分是完全平方数$1^2, 2^2, \ldots, k^2$，另一部分是去掉完全平方数后的序列$a_1, a_2, \ldots, a_n$，这时$k^2 < n + k < (k+1)^2$，即$(k - 1/2)^2 + 3/4 = k^2 - k + 1 \leq n \leq (k + 1/2)^2 - 1/4$，因此$k = \left\{\sqrt{n}\right\}$，$a_n = n + \left\{\sqrt{n}\right\}$。

2.5.11 C

> Give an example of two uncountable sets A and B such that $A \cap B$ is
> a) finite.
> b) countably infinite.
> c) uncountable.

- a)例如$A = [0, 1], B = [2, 3]$, 直接没交集。
- b)例如$A = [0, 1] \cup Z, B = [2, 3] \cup Z$，交集把不可数的去掉了。
- c)例如$A = B = [0, 1]$，交集有一段不可数的即可。

2.5.17 C

> If A is an uncountable set and B is a countable set, must $A-B$ be uncountable?

是的。如果$A-B$可数，则$(A-B)+(A\cap B)=A$可数；A不可数，所以$A-B$不可数。

2.5.31 G

> Show that $Z^+ \times Z^+$ is countable by showing that the polynomial function $f: Z^+ \times Z^+ \to Z^+$ with $f(m,n)=(m+n-2)(m+n-1)\ /\ 2+m$ is one-to-one and onto.

- 映上：对于任意$z=f(m,n)$，若$(x-2)(x-1)/2<z<(x-1)x/2$，则取 $m=z-(x-2)(x-1)/2, n=x-m$。
- 一对一：假设$f(m_1,n_1)=f(m_2,n_2)$，则

$$0=f(m_1,n_1)-f(m_2,n_2)=$$

$$m_1-m_2+\frac{1}{2}((m_1+m_2)(m_1-m_2)+(n_1+n_2)(n_1-n_2)+2n_1m_1-2n_2m_2-3(m_1-m_2)-3(n_1-n_2))$$

$$=(m_1-m_2)(\frac{1}{2}(m_1+m_2)+n_1-1)+(n_1-n_2)(\frac{1}{2}(n_1+n_2)+m_1-\frac{3}{2}).$$

由 $(\frac{1}{2}(m_1+m_2)+n_1-1)>0$，$(\frac{1}{2}(n_1+n_2)+m_1-\frac{3}{2})>0$，故有$m_1=m_2,n_1=n_2$。