> Mathematical Logic and Graph Theory 2022 Homework 1 Answers
>
> By Jingyi Chen with C and Songxiao Guo with G after each question number.

1.1.13 G

> Let p and q be the propositions
> p: It is below freezing.
> q: It is snowing.
> Write these propositions using p and q and logical connectives (including negations).
> a) It is below freezing and snowing.
> b) It is below freezing but not snowing.
> c) It is not below freezing and it is not snowing.
> d) It is either snowing or below freezing (or both).
> e) If it is below freezing, it is also snowing.
> f) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.
> g) That it is below freezing is necessary and sufficient for it to be snowing.

- a) $p \land q$
- b) $p \land \neg q$
- c) $\neg p \land \neg q$
- d) $p \lor q$
- e) $p \rightarrow q$
- f) $(p \lor q) \land (p \rightarrow \neg q)$
- g) $q \leftrightarrow p$

1.1.19 G

> Determine whether each of these conditional statements is true or false.
> a) If 1 + 1 = 2, then 2 + 2 = 5.
> b) If 1 + 1 = 3, then 2 + 2 = 4.
> c) If 1 + 1 = 3, then 2 + 2 = 5.
> d) If monkeys can fly, then 1 + 1 = 3.

- a) F
- b) T
- c) T
- d) T

1.1.35 G

> Construct a truth table for each of these compound propositions.
> a) $(p \vee q) \rightarrow (p \oplus q)$
> b) $(p \oplus q) \rightarrow (p \wedge q)$
> c) $(p \vee q) \oplus (p \wedge q)$
> d) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
> e) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$
> f) $(p \oplus q) \rightarrow (p \oplus \neg q)$

- a)

p	q	$p \vee q$	$p \oplus q$	$(p \vee q) \rightarrow (p \oplus q)$
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

- b)

p	q	$p \oplus q$	$p \wedge q$	$(p \oplus q) \rightarrow (p \wedge q)$
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

- c)

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \oplus (p \wedge q)$
T	T	T	T	F
T	F	T	F	T
F	T	T	F	T
F	F	F	F	F

- d)

p	q	$\neg p$	$p \leftrightarrow q$	$\neg p \leftrightarrow q$	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
T	T	F	F	F	T
T	F	F	T	T	T
F	T	T	T	T	T
F	F	T	F	F	T

- e)

p	q	r	$\neg p$	$\neg r$	$p \leftrightarrow q$	$\neg p \leftrightarrow \neg r$	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$
T	T	T	F	F	T	T	F
T	T	F	F	T	T	F	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	T	F	T	T
F	F	T	T	F	T	F	T
F	F	F	T	T	T	T	F

- f)

p	q	$\neg q$	$p \oplus q$	$p \oplus \neg q$	$(p \oplus q) \to (p \oplus \neg q)$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

1.1.43 G

> Explain, without using a truth table, why $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true when at least one of p, q, and r is true and at least one is false, but is false when all three variables have the same truth value.

- 至少有一个为真时，析取式前半的合取式为真；至少有一个为假时，析取式后半的合取式为真；故析取式为真。
- 三个变量具有相同的真值时，析取式的前半与后半的合取式的真值相反，故析取式为假。

1.2.11 G

> Are these system specifications consistent? “The router can send packets to the edge system only if it supports the new address space. For the router to support the new address space it is necessary that the latest software release be installed. The router can send packets to the edge system if the latest software release is installed. The router does not support the new address space.”

令 p 为“路由器能向边缘系统发送分组”， q 为“路由器支持新的地址空间”，令 r 为“路由器安装了最新版本的软件”。则上面几个规范说明可以写为：

$$p \to q, q \to r, r \to p, \neg q.$$

使规范说明都为真时， $\neg q$为真， q 必为假。由 $p \to q$ ， p 也为假。由 $r \to p$ 为真，得 r 为假。此时 $q \to r$ 也为真，故这些规范说明是一致的。

1.2.29 G

> Exercises 28–35 relate to inhabitants of an island on which there are three kinds of people: knights who always tell the truth, knaves who always lie, and spies (called normals by 39. Smullyan [Sm78]) who can either lie or tell the truth. You encounter three people, A, B, and C. You know one of these people is a knight, one is a knave, and one is a spy. Each of the three people knows the type of person each of other two is. For each of these situations, if possible, determine whether there is a unique solution and determine who the knave, knight, and spy are. When there is no unique solution, list all possible solutions or state that there are no solutions.
>
> A says "I am the knight," B says "I am the knave," and C says "B is the knight."

设 $x_1, x_2, x_3, x_4, x_5, x_6$ 分别表示 (A, B, C) 是 (骑士, 无赖, 间谍)，(骑士, 间谍, 无赖)，(间谍, 骑士, 无赖)，(间谍, 无赖, 骑士)，(无赖, 骑士, 间谍)，(无赖, 间谍, 骑士)。

则 $x_1 \vee x_2$ 表示"A 是骑士"，$x_1 \vee x_4$ 表示"B 是无赖"，$x_3 \vee x_5$ 表示"B 是骑士"。

情况可以形式化地表述为：

$$\begin{cases} (x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6) \vee (\neg x_1 \wedge x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6) \vee \\ (\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6) \vee (\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6) \vee \\ (\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge x_5 \wedge \neg x_6) \vee (\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge x_6) = 1, \\ (x_1 \wedge (x_1 \vee x_2) \wedge (\neg(x_1 \vee x_4))) \vee (x_2 \wedge (x_1 \vee x_2) \wedge (\neg(x_3 \vee x_5))) \vee \\ (x_3 \wedge (x_1 \vee x_4) \wedge (\neg(x_3 \vee x_5))) \vee (x_4 \wedge (x_3 \vee x_5) \wedge (\neg(x_1 \vee x_4))) \vee \\ (x_5 \wedge (x_1 \vee x_4) \wedge (\neg(x_1 \vee x_2))) \vee (x_6 \wedge (x_3 \vee x_5) \wedge (\neg(x_1 \vee x_2))) = 1. \end{cases}$$

由第一式可知 $(x_1, \cdots, x_6)$ 的取值仅有 $(0,0,0,0,0,1)$、$(0,0,0,0,1,0)$、$(0,0,0,1,0,0)$、$(0,0,1,0,0,0)$、$(0,1,0,0,0,0)$、$(1,0,0,0,0,0)$ 六种。

分别带入第二式，得到仅有 $(0,1,0,0,0,0)$ 时成立，即该问题有唯一解：A 是骑士，B 是间谍，C 是无赖。

1.2.39 C

> A detective has interviewed four witnesses to a crime. From the stories of the witnesses the detective has concluded that if the butler is telling the truth then so is the cook; the cook and the gardener cannot both be telling the truth; the gardener and the handyman are not both lying; and if the handyman is telling the truth then the cook is lying. For each of the four witnesses, can the detective determine whether that person is telling the truth or lying? Explain your reasoning.

记男管家、厨师、园丁、杂役说真话为p, q, r, s，已知的真命题有$p \to q$,$\neg(q \wedge r)$,$\neg(\neg r \wedge \neg s)$, $s \to \neg q$.这里发现q, r, s互相在一个命题中产生联系，先假设q为真，易于推出矛盾，而q为假不产生矛盾，因此q为假。由$p \to q$可知p也为假。这时分别假设r, s为真或假都不会产生矛盾，因此不能确定其真值。

所以男管家、厨师在说假话，园丁、杂役不能确定。

1.3.19 C

> Determine whether $(\neg q \wedge (p \to q)) \to \neg p$ is a tautology.

使用真值表当然可行。使用逻辑恒等式，例如：

$$
\begin{aligned}
(\neg q \land (p \to q)) \to \neg p &\equiv \neg(\neg q \land (p \to q)) \lor \neg p \\
&\equiv q \lor (\neg(p \to q)) \lor \neg p \\
&\equiv (p \land \neg q) \lor \neg p \lor q \\
&\equiv (p \land \neg q) \lor (\neg(p \land \neg q)) \\
&\equiv T
\end{aligned}
$$

所以它是永真式。

1.3.29 C

> Show that $(p \to r) \lor (q \to r)$ and $(p \land q) \to r$ are logically equivalent.

使用真值表当然可行。使用逻辑恒等式，例如：

$$
\begin{aligned}
(p \to r) \lor (q \to r) &\equiv \neg p \lor r \lor \neg q \lor r \\
&\equiv (\neg p \lor \neg q) \lor r \\
&\equiv \neg(p \land q) \lor r \\
&\equiv (p \land q) \to r
\end{aligned}
$$

1.3.33 C

> Show that $(p \to q) \land (q \to r) \to (p \to r)$ is a tautology.

使用真值表当然可行。使用逻辑恒等式，例如：

$$
\begin{aligned}
(p \to q) \land (q \to r) \to (p \to r) &\equiv \neg(\neg(\neg p \lor q) \lor \neg(\neg q \lor r)) \to (\neg p \lor r) \\
&\equiv (\neg(\neg p \lor q)) \lor (\neg(\neg q \lor r)) \lor (\neg p \lor r) \\
&\equiv \neg p \lor (p \land \neg q) \lor r \lor (q \land \neg r) \\
&\equiv ((\neg p \lor p) \land (\neg p \lor \neg q)) \lor ((r \lor \neg r) \land (q \lor \neg r)) \\
&\equiv \neg p \lor \neg q \lor q \lor \neg r \\
&\equiv T
\end{aligned}
$$

1.3.45 C

> Find a compound proposition involving the propositional variables p, q, and r that is true when exactly two of p, q, and r are true and is false otherwise.

依据字面意思即可写出：

propositional$(p \land q \land \neg r) \lor (p \land \neg q \land r) \lor (\neg p \land q \land r)$

1.3.55 C

> Find a compound proposition logically equivalent to $p \to q$ using only the logical operator $\downarrow$.

$$
\begin{aligned}
p \to q &\equiv \neg p \lor q \\
&\equiv \neg(p \land \neg q) \\
&\equiv \neg(\neg\neg p \land \neg q) \\
&\equiv \neg(\neg p \downarrow q) \\
&\equiv \neg((p \downarrow p) \downarrow q) \\
&\equiv ((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q)
\end{aligned}
$$