> Mathematical Logic and Graph Theory 2022 Homework 7 Answers
>
> By Jingyi Chen with C and Songxiao Guo with G after each question number.

5.6.3 G

> Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if
>
> - a) a is taller than b?
> - b) a is not taller than b?
> - c) $a = b$ or a is an ancestor of b?
> - d) a and b have a common friend?

- a) 不是，没有自反性。
- b) 不是，没有反对称性。
- c) 是。
- d) 不是，没有传递性。

5.6.33 G

> Answer these questions for the poset ($\{3, 5, 9, 15, 24, 45\}$, |).
>
> - a) Find the maximal elements.
> - b) Find the minimal elements.
> - c) Is there a greatest element?
> - d) Is there a least element?
> - e) Find all upper bounds of $\{3, 5\}$.
> - f) Find the least upper bound of $\{3, 5\}$, if it exists.
> - g) Find all lower bounds of $\{15, 45\}$.
> - h) Find the greatest lower bound of $\{15, 45\}$, if it exists.

- a) 24、45。
- b) 3、5。
- c) 没有。
- d) 没有。
- e) 15、45。
- f) 15。
- g) 3、5、15。

- h) 15。

5.6.43 G

> Determine whether the posets with these Hasse diagrams are lattices.
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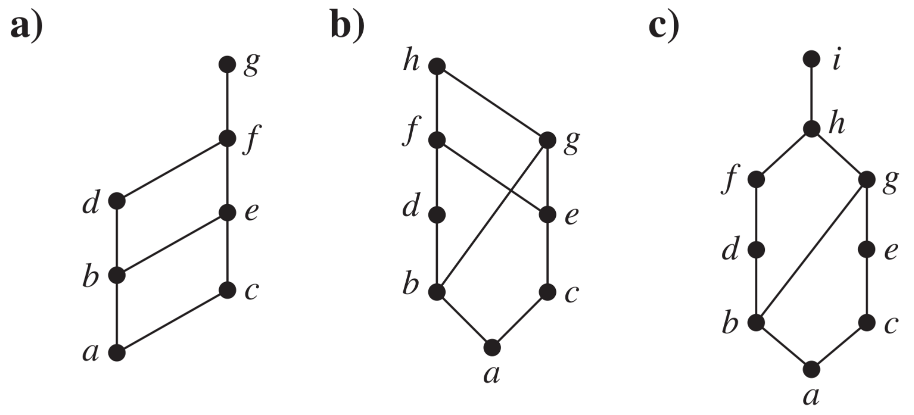
> 
>

- a) 是。
- b) 不是，$\{b, e\}$ 没有最小上界。
- c) 是。

5.6.49 G

> Show that the set of all partitions of a set S with the relation $P_1 \preccurlyeq P_2$ if the partition P_1 is a refinement of the partition P_2 is a lattice.

设 Π 是集合 S 的所有划分组成的集合。先证明 $(\Pi, \preccurlyeq)$ 是偏序集：

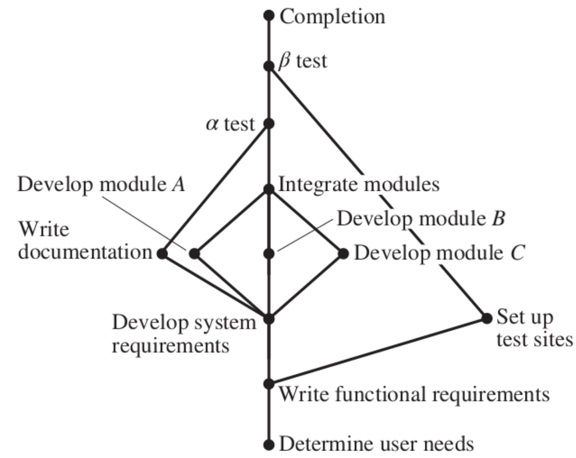
- 自反性：显然。
- 反对称性：对 $P_1, P_2 \in \Pi$ ，假设 $P_1 \preccurlyeq P_2$ 且 $P_2 \preccurlyeq P_1$，取 $T \in P_1$，则 $\exists T' \in P_2$，有 $T \subseteq T'$。进一步 $\exists T'' \in P_1$，有 $T' \subseteq T''$。$T \neq \emptyset$ 且 $T'' \neq \emptyset$ 时，由于 P_1 是划分，若 $T \neq T''$，则 $T \cap T'' = \emptyset$，故有 $T = T''$。进一步，$T = T'$。由 T 的任意性，以及 P_1, P_2 是划分，有$P_1 = P_2$。
- 传递性：假设 $P_1 \preccurlyeq P_2$ 且 $P_2 \preccurlyeq P_3$，取 $T \in P_1$，则 $\exists T' \in P_2$，有 $T \subseteq T'$。进一步 $\exists T'' \in P_3$，有 $T' \subseteq T''$。故 $T \in T''$。由 T 的任意性，以及 P_1, P_3 是划分，有 $P_1 \preccurlyeq P_3$。

再证明对 $\forall (P_1, P_2) \in \Pi$ ，P_1, P_2 都有最小上界和最大下界：

- 最小上界：$\forall (P_1, P_2) \in \Pi$，我们这样构造它们的上界 P_3：取 $P' = P_1 \cup P_2$，对 $\forall T \in P'$ ，若 $\exists T' \in P_1 \cup P_2 \wedge T \cap T' \neq \emptyset$，则从 P 中去掉 T, T'，然后用 $T \cup T'$代替它。（闭包）最终得到的 P' 便是 P_3。任取 P_1 和 P_2 的上界 P_4，$\forall T_1 \in P_1, \exists T_1' \in P_4; \forall T_2 \in P_2, \exists T_2' \in P_4$。由于 P_4 是划分，故只有 $T_1' = T_2'$ 或 $T_1' \cap T_2' = \emptyset$。由构造规则，若$T_1$ 的闭包与 T_2' 相交，则在某一步中，$\exists T' \in P_1 \cup P_2$，$T' \neq \emptyset$ ，有 $T' - T_1' - T_2' \neq \emptyset$，只能有 $T_1' = T_2'$，这样对 T_1, T_1' 有 T_1 的闭包属于 T_1'。另一方面，T_1 的闭包是划分 P_3 的元素，由 T_1 的任意性，证明了 P_3 是 P_1 和 P_2 的最小上界。
- 最大下界：$\forall (P_1, P_2) \in \Pi$，易知 $P_3 = \{T | T = T_1 \cap T_2, T_1 \in P_1, T_2 \in P_2\}$ 是P_1 和 P_2 的下界。任取 P_1 和 P_2 的下界 P_4 ，$\forall T \in P_4, \exists T_1 \in P_1, \exists T_2 \in P_2, T \in T_1 \wedge T \in T_2 \Rightarrow T \in T_1 \cap T_2$，故 $T \in P_3$。由 T 的任意性，以及 P_4, P_3 是划分，有 $P_4 \preccurlyeq P_3$。这就证明了 P_3 是 P_1 和 P_2 的最大下界。

5.6.67 G

> Find an ordering of the tasks of a software project if the Hasse diagram for the tasks of the project is as shown.



确定用户需求 $\prec$ 写出功能需求 $\prec$ 设置测试点 $\prec$ 开发系统需求 $\prec$ 写文档 $\prec$ 开发模块 A $\prec$ 开发模块 B $\prec$ 开发模块 C $\prec$ 模块集成 $\prec$ α 测试 $\prec$ β 测试 $\prec$ 完成。

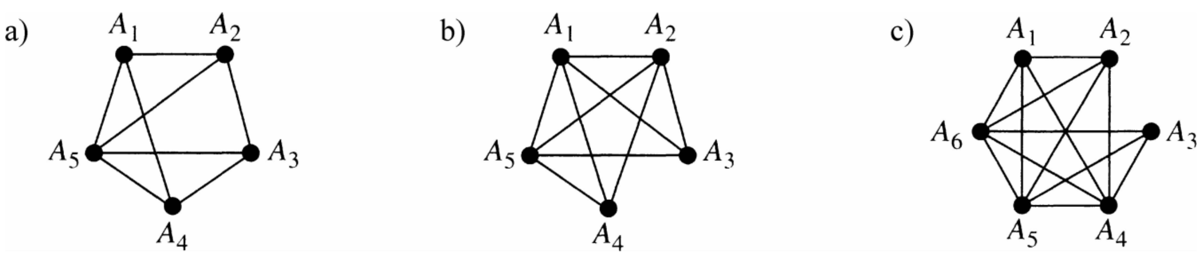
6.1.11 G

> Let G be a simple graph. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G.

- 对称的：由简单图边的无向性，知 uRv 则 vRu。
- 反自反的：由简单图无重边知 $u \not R u$。

6.1.13 G

> The intersection graph of a collection of sets $A_1, A_2, \cdots, A_n$ is the graph that has a vertex for each of these
> sets and has an edge connecting the vertices representing two sets if these sets have a nonempty intersection.
> Construct the intersection graph of these collections of sets.
>
> - a)
> $A_1 = \{0, 2, 4, 6, 8\}, A_2 = \{0, 1, 2, 3, 4\}, A_3 = \{1, 3, 5, 7, 9\}, A_4 = \{5, 6, 7, 8, 9\}, A_5 = \{0, 1, 8, 9\}$
> - b)
> $$A_1 = \{\cdots, -4, -3, -2, -1, 0\}, A_2 = \{\cdots, -2, -1, 0, 1, 2, \cdots\},$$
> $$A_3 = \{\cdots, -6, -4, -2, 0, 2, 4, 6, \cdots\}, A_4 = \{\cdots, -5, -3, -1, 1, 3, 5, \cdots\}, A_5 = \{\cdots, -6, -3, 0, 3, 6, \cdots\}$$
> - c)
> $$A_1 = \{x \mid x < 0\}, A_2 = \{x \mid -1 < x < 0\}, A_3 = \{x \mid 0 < x < 1\},$$
> $$A_4 = \{x \mid -1 < x < 1\}, A_5 = \{x \mid x > -1\}, A_6 = R$$



6.1.29 G

> Describe a graph model that represents whether each person at a party knows the name of each other person at the party. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?

令 V 是参加聚会的人的集合，E 是 $V \times V$ 中满足 u 知道 v 的名字的有序对 (u, v) 的集合。不允许多重边、有向图、允许环。

6.2.5 G

> Can a simple graph exist with 15 vertices each of degree five?

不存在。总度数等于边数的二倍，为偶数，与 $15\times = 75$ 为奇数矛盾。

6.2.27 C

> Suppose that there are four employees in the computer support group of the School of Engineering of a large university. Each employee will be assigned to support one of four different areas: hardware, software, networking, and wireless. Suppose that Ping is qualified to support hardware, networking, and wireless; Quiggley is qualified to support software and networking; Ruiz is qualified to support networking and wireless, and Sitea is qualified to support hardware and software.
>
> - a) Use a bipartite graph to model the four employees and their qualifications.
> - b) Use Hall's theorem to determine whether there is an assignment of employees to support areas so that each employee is assigned one area to support.
> - c) If an assignment of employees to support areas so that each employee is assigned to one support area exists, find one.

a)人是$PQRS$，工作是$hsnw$，则二分图是$\{\{P,n\},\{P,w\},\{Q,s\},\{Q,n\},\{R,n\},\{R,w\},\{S,h\},\{S,s\}\}$；

b)存在，因为任抽几个人，相应能完成的工作都比人数多，所以定理条件成立。

c)$\{Pw,Qs,Rn,Sh\}$或者$\{Pn,Qs,Rw,Sh\}$两种答案。

6.2.33 C

> Suppose that m people are selected as prize winners in a lottery, where each winner can select two prizes from a collection of different prizes. Show if there are 2m prizes that every winner wants, then every winner is able to select two prizes that they want.

建议构造二分图，一边是每个获奖者被表示两次的集合v_1，每次表示意味着他拿了一件奖品，另一边是2m件奖品的集合V_2。对于V_1的任一子集A，$N(A)=V_2$，所以$|N(A)|\geq |A|$，所以Hall定理（霍尔婚配定理）成立了。

6.2.47 C

> Show that a sequence $d_1,d_2,\cdots,d_n$ of nonnegative integers in nonincreasing order is a graphic sequence if and only if the sequence obtained by reordering the terms of the sequence $d_2-1,\cdots,d_{d_1+1}-1,d_{d_1}+2,\cdots,d_n$ so that the terms are in nonincreasing order is a graphic sequence.

前推后：如果原本的序列是成图序列，那么去掉第一个点，并把与第一个点相连的点的度数减一形成的当然是成图序列；这时对于前d_1个点中没有被度数减一的点（说明它没和第一个点连着），它度数必比“不在前d_1个点但被度数减一的点”度数要大，说明至少有一个点与前者相连而与后者不相连，那就把前者的这个连边给后者即可，如此调整即得到第二个成图序列。

后推前：后一个序列是成图序列，则直接把前d_1个点连一个新加入的点，就得到前者的成图序列了。

6.2.51 C

> How many subgraphs with at least one vertex does K_3 have?

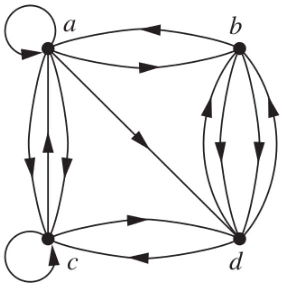
17个。只有一个顶点：3个；有两个顶点：6个；有三个顶点：8个。

6.2.63 C

> If the simple graph G has v vertices and e edges, how many edges does $\overline{G}$ have?

$v(v-1)/2-e$。总的可能边数减去G的边数就是$\overline{G}$的边数。

6.3.21 C

> Fnd the adjacency matrix of the given directed multigraph with respect to the vertices listed in alphabetic
> **21.**
> 
>
> order.

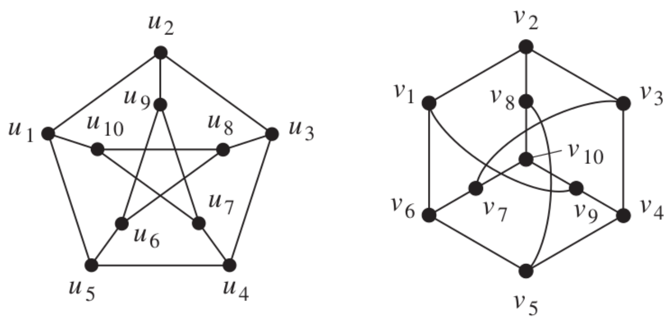
$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

6.3.33 C

> What is the sum of the entries in a column of the adjacency matrix for an undirected graph? For a directed graph?

前者是$deg(v)$减去v处的环数（自己和自己有环不算相邻），后者是$deg^-(v)$即入度。

6.3.47 C

> Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous
> **47.**
> 
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> argument that none exists.

是同构的。证明图的同构可以构造出相同的邻接矩阵，例如此题将左侧的点1~10映射到右侧的$1, 9, 4, 5, 6, 7, 8, 3, 10, 2$（答案不唯一）。

6.3.49 C

> Show that isomorphism of simple graphs is an equivalence relation.

自己肯定和自己同构，所以是自反的。G和H同构则存在点的一一映射关系保持了相邻和不相邻性，这个一一映射反过来也对，所以是对称的。G和H同构，则有一一映射f，H和G同构，则有一一映射g，那么$g \circ f$也是一一映射并保证了相邻和非相邻性的，所以是传递的。

6.3.59 C

> How many nonisomorphic simple graphs are there with five vertices and three edges?

4个。为了确保不同构，可以把度数按降序排列，4种情况分别是31110,21111,22110,22200。