> Mathematical Logic and Graph Theory 2022 Homework 6 Answers
>
> By Jingyi Chen with C and Songxiao Guo with G after each question number.

5.3.15 C

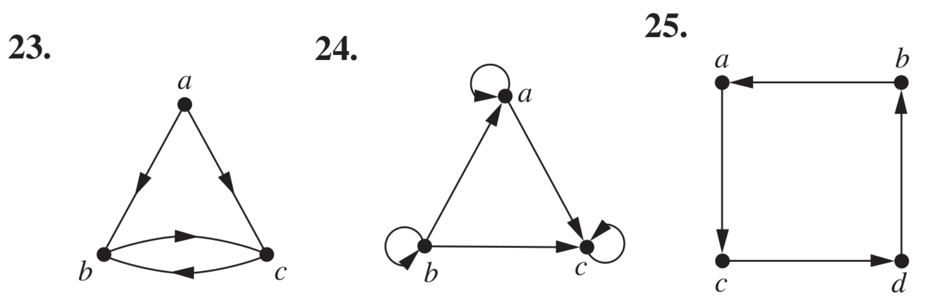
> Let R be the relation represented by the matrix
>
> $$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$
>
> Find the matrices that represent
> a) R^2 .
>
> b) R^3 .
> c) R^4 .

a) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

5.3.31 C

> Determine whether the relations represented by the directed graphs shown in Exercises 23–25 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.
>
> 
>

23：反自反的；

24：自反的，反对称的，传递的；

25：反自反的，反对称的。

5.4.13 C

> Suppose that the relation R on the finite set A is represented by the matrix $\mathbf{M}_R$. Show that the matrix that represents the symmetric closure of R is $\mathbf{M}_R \vee \mathbf{M}_R^t$.

定理：关系矩阵的转置矩阵就是关系矩阵的逆矩阵，而对称闭包就是$R \cup R^{-1}$，所以 $M_{R \cup R^{-1}} = M_R \vee M_R^T$。

5.4.23 C

> Suppose that the relation R is symmetric. Show that $R^*$is symmetric.

已知$R = R^{-1}$，则$(R^*)^{-1} = (\cup R^n)^{-1} = (\cup (R^n)^{-1}) = \cup R^n = R^*$。图论来说就是，边是双向的，连通性自然也是对称的。

5.4.29 C

> Find the smallest relation containing the relation $\{(1,2),(1,4),(3,3),(4,1)\}$ that is
>
> - a) reflexive and transitive.
> - b) symmetric and transitive.
> - c) reflexive, symmetric, and transitive.

a){(1,1),(1,2),(1,4),(2,2),(3,3),(4,1),(4,2),(4,4)}

b){(1,1),(1,2),(1,4),(2,1),(2,2),(2,4),(3,3),(4,1),(4,2),(4,4)}

c){(1,1),(1,2),(1,4),(2,1),(2,2),(2,4),(3,3),(4,1),(4,2),(4,4)}

用图示说清楚也行。

5.5.7 G

> Show that the relation of logical equivalence on the set of all compound propositions is an equivalence relation. What are the equivalence classes of F and of T?

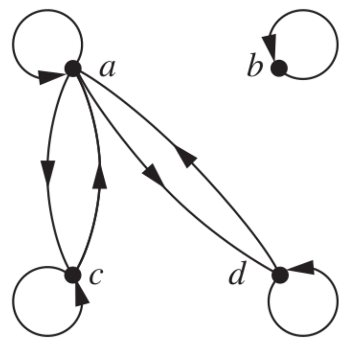
用 $R(p,q)$ 表示在复合命题集合里面有 $p \leftrightarrow q$。$p \leftrightarrow q$ 表示 p 与 q 有相同的真值。下面证明 R 是一个等价关系：

- 自反性：p 与 p 显然有相同的真值。
- 对称性：如果 p 与 q 有相同的真值，则显然有 q 与 p 有相同的真值。
- 传递性：如果 p 与 q 有相同的真值，且 q 与 r 有相同的真值，显然有 p 与 r 有相同的真值。

T 的等价类是所有永真式构成的集合；F 的等价类是所有永假式构成的集合。

5.5.21 G

> Determine whether the relation with the directed graph shown is an equivalence relation.

> **21.**
>
> 
>

不是。该关系包括$(d,a),(a,c)$，但不包括 (d,c)，不满足传递性。

5.5.47 G

> List the ordered pairs in the equivalence relations produced by these partitions of $\{0,1,2,3,4,5\}$.
>
> - a) $\{0\},\{1,2\},\{3,4,5\}$
> - b) $\{0,1\},\{2,3\},\{4,5\}$
> - c) $\{0,1,2\},\{3,4,5\}$
> - d) $\{0\},\{1\},\{2\},\{3\},\{4\},\{5\}$

- a)

$$(0,0),(1,1),(2,2),(1,2),(2,1),(3,3),(4,4),(5,5),(3,4),(4,3),(3,5),(5,3),(4,5),(5,4)$$

- b)

$$(0,0),(1,1),(0,1),(1,0),(2,2),(3,3),(2,3),(3,2),(4,4),(5,5),(4,5),(5,4)$$

- c)

$$(0,0),(1,1),(2,2),(1,2),(2,1),(0,2),(2,0),(0,1),(1,0),\\(3,3),(4,4),(5,5),(3,4),(4,3),(3,5),(5,3),(4,5),(5,4)$$

- d)

$$(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)$$

5.5.59 G

> Let R be the relation on the set of all colorings of the 2×2 checkerboard where each of the four squares is colored either red or blue so that (C_1,C_2), where C_1 and C_2 are 2×2 checkerboards with each of their four squares colored blue or red, belongs to R if and only if C_2 can be obtained from C_1 either by rotating the checkerboard or by rotating it and then reflecting it.
>
> - a) Show that R is an equivalence relation.
> - b) What are the equivalence classes of R?

- a)
  - 自反性：C 可以通过 C 旋转 360° 得到。
  - 对称性：如果 C_1 由 C_2 旋转 θ 得到，则 C_2 由 C_1 旋转 $360^\circ-\theta$ 得到；如果 C_1 由 C_2 旋转 θ，再翻转得到，则 C_2 由 C_1 旋转 $360^\circ-\theta$，再翻转得到。
  - 传递性：如果 C_1 由 C_2 旋转 θ_1 得到， C_2 由 C_3 旋转 θ_2 得到，则 C_1 由 C_3 旋转 $\theta_1+\theta_2$ 得到；如果 C_1 由 C_2 旋转 θ_1 ，再翻转得到， C_2 由 C_3 旋转 θ_2 得到，则 C_1 由 C_3 旋转 $\theta_1+\theta_2$ ，再翻转得到；如果 C_1 由 C_2 旋转 θ_1 得到， C_2 由 C_3 旋转 θ_2 ，再翻转得到，则 C_1 由 C_3 旋转 $\theta_1+\theta_2$ ，再翻转得到；如果 C_1 由 C_2 旋转 θ_1 ，再翻转得到， C_2 由 C_3 旋转 θ_2 ，再翻转得到，则 C_1 由 C_3 旋转 $\theta_1+\theta_2$ 得到。

- b)

$$\{\begin{pmatrix} r & r \\ r & r \end{pmatrix}\}, \{\begin{pmatrix} b & b \\ b & b \end{pmatrix}\}, \{\begin{pmatrix} r & r \\ b & r \end{pmatrix}, \begin{pmatrix} b & r \\ r & r \end{pmatrix}, \begin{pmatrix} r & b \\ r & r \end{pmatrix}, \begin{pmatrix} r & r \\ r & b \end{pmatrix}\}, \{\begin{pmatrix} b & b \\ r & b \end{pmatrix}, \begin{pmatrix} r & b \\ b & b \end{pmatrix}, \begin{pmatrix} b & r \\ b & b \end{pmatrix}, \begin{pmatrix} b & b \\ b & r \end{pmatrix}\},$$
$$\{\begin{pmatrix} b & b \\ r & r \end{pmatrix}, \begin{pmatrix} r & b \\ r & b \end{pmatrix}, \begin{pmatrix} r & r \\ b & b \end{pmatrix}, \begin{pmatrix} b & r \\ b & r \end{pmatrix}\}, \{\begin{pmatrix} r & b \\ b & r \end{pmatrix}, \begin{pmatrix} b & r \\ r & b \end{pmatrix}\}.$$

5.5.63 G

> Do we necessarily get an equivalence relation when we form the transitive closure of the symmetric closure of the reflexive closure of a relation?

能。一个关系的自反闭包显然有自反性，在其形成对称闭包时，不会改变自反性。在其形成传递闭包时，不会改变其自反性和对称性，故其具有自反性、对称性和传递性，也就是一个等价关系。