Extremal Kähler metrics

Gábor Székelyhidi

University of Notre Dame

Outline

Canonical metrics

Question: Is there a "canonical" or "best" metric on a manifold M?

or

Canonical metrics

Question: Is there a "canonical" or "best" metric on a manifold M?

Uniformization Theorem

Let (Σ, g_0) be a closed Riemann surface. There exists a metric $g = e^{\phi} g_0$ in the conformal class of g with constant curvature. This metric is unique up to isometry and scaling.

Higher dimensional generalization? E.g.

- Yamabe problem
- Thurston geometrization

Kähler metrics

On a complex manifold (M, J), a metric g is Kähler if:

- g is Hermitian, i.e. $g(Jv, Jw) = g(v, w)$,
- Associated form $\omega(v, w) = g(Jv, w)$ is closed, i.e. $d\omega = 0$.

In local holomorphic coordinates

$$g_{j\bar{k}} = \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k},$$

for a real valued function f.

Kähler classes

A Kähler metric ω determines a class $[\omega] \in H^2(M, \mathbf{R})$.

Theorem ($\partial\overline{\partial}$-lemma)

Let M be a compact complex manifold. If ω_1 and ω_2 are two Kähler forms with $[\omega_1] = [\omega_2]$, then there is function $\phi : M \to \mathbf{R}$ such that

$$\omega_2 = \omega_1 + i\partial\overline{\partial}\phi.$$

If $\dim_{\mathbf{C}}(M) = 1$, then

$$\omega + i\partial\overline{\partial}\phi = (1 + \Delta\phi)\omega,$$

so "fixed Kähler class" $=$ "fixed conformal class & area"

Extremal metrics

Definition (**Calabi, 1982**)

A Kähler metric ω on a compact complex manifold M^n is extremal if it is a critical point of the functional

$$\eta \mapsto \int_M S(\eta)^2 \eta^n,$$

for η in the Kähler class $[\omega]$. Here $S(\eta)$ is the scalar curvature.

Equivalently: gradient $\nabla S(\omega)$ is holomorphic.

Examples

- Constant scalar curvature Kähler (cscK) metrics, e.g. on Riemann surfaces.
- Kähler-Einstein metrics with non-positive Ricci curvature – **Yau and Aubin 1978**
- Non-cscK example on blowup $\mathrm{Bl}_p\mathbf{CP}^2$ – **Calabi 1982**
- Non-cscK examples on $\mathbf{CP}^1$-bundles over high genus curves – **Tønnesen-Friedman 1997**
- Non-cscK example on $\mathrm{Bl}_{p,q}\mathbf{CP}^2$ **Chen-LeBrun-Weber 2007**
- Both cscK and non-cscK examples on toric surfaces – **Donaldson 2008, Chen-Li-Sheng 2010**
- Kähler-Einstein metrics with positive Ricci curvature – **Chen-Donaldson-Sun and Tian 2012**

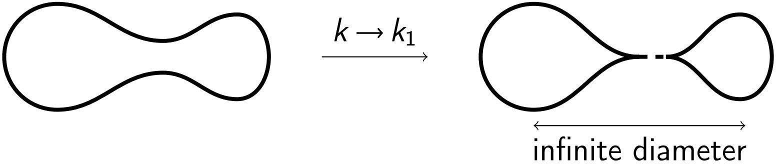
Ruled surface example

Let $\mathrm{genus}(\Sigma) = 2$, and $\mathscr{L} \to \Sigma$ a degree 1 line bundle.

$$M = \mathbf{P}(\mathscr{O} \oplus \mathscr{L})$$

Up to scaling, Kähler classes on M are parametrized by $k > 0$ (volume of fibre relative to base).

Tønnesen-Friedman: For $k < k_1 \approx 18.89...$, there is an extremal metric on M. As $k \to k_1$, the fiber metrics degenerate.



K-stability

Tian '97, Donaldson '02

Let $L \to M$ be an ample line bundle. Using a basis of sections, embed $M \subset \mathbf{CP}^N$, and let $\lambda : \mathbf{C}^* \hookrightarrow GL(N+1, \mathbf{C})$. Can define "flat" limit

$$M_0 = \lim_{t \to 0} \lambda(t) \cdot M.$$

This is a test-configuration. Using M_0 and λ, define the generalized Futaki invariant $\mathrm{Fut}(M, \lambda)$.

Definition

(M, L) is K-stable, if $\mathrm{Fut}(M, \lambda) > 0$ for all "non-trivial" λ, and for all embeddings of M using powers L^k.

When $\mathrm{Aut}(M)$ is non-trivial, a modification of Fut leads to *relative K-stability* (**Sz. '07**).

Yau-Tian-Donaldson conjecture

Conjecture

M admits an extremal metric in $c_1(L)$ $\Leftrightarrow$ *(M, L) is relatively K-stable.*

Analogous to Donaldson-Uhlenbeck-Yau theorem for vector bundles.

- $\Rightarrow$ holds – **Tian, Donaldson, Mabuchi, Stoppa, Sz., Berman**
- $\Leftarrow$ holds for toric surfaces – **Donaldson, Chen-Li-Sheng**
- $\Leftarrow$ holds in the Kähler-Einstein case – **Yau, Aubin, Chen-Donaldson-Sun, Tian**, e.g. if $L = K_M^{-1}$ and $\mathrm{Aut}(M)$ is trivial.

Note: probably $\Leftarrow$ is false in general – example by **Apostolov - Calderbank - Gauduchon - Tønnesen-Friedman**
Need strengthening of relative K-stability.

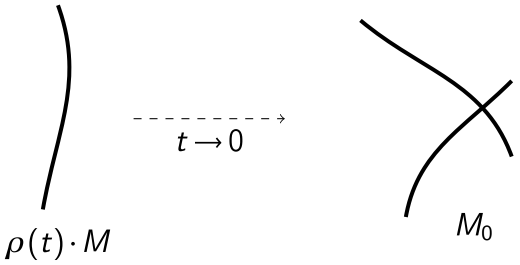
Example

Let $M = \mathbf{CP}^1$, and $M \hookrightarrow \mathbf{CP}^2$ as conic $xz - y^2 = 0$.
Let $\rho(t)(a,b,c) = (ta, b, c)$. Then $\rho(t)\cdot M$ has equation

$$xz - ty^2 = 0.$$

Limit M_0 as $t \to 0$ is union of lines $xz = 0$.

Can compute $\mathrm{Fut}(M,\rho) = \frac{1}{2} > 0$. However, similar degeneration of a family of $\mathbf{CP}^1$s can destabilize a ruled surface (**Sz. '07**).



Filtrations

When $M \subset \mathbf{P}H^0(L^k)^*$ and $\lambda : \mathbf{C}^* \hookrightarrow GL(H^0(L^k))$, then only the weight-filtration on $H^0(L^k)$ matters for $\mathrm{Fut}(M,\lambda)$.

This data can be encoded (**Witt Nyström '10**), independently of k, in a filtration of the co-ordinate ring

$$R = \bigoplus_{k\geqslant 0} H^0(L^k).$$

We can then allow general filtrations χ, and define $\mathrm{Fut}(M,\chi)$ and a norm $\|\chi\|$ – **Sz. '11**.

Theorem (Boucksom-Sz. '12)

If M admits a cscK metric in $c_1(L)$ and $\mathrm{Aut}(M) = \{1\}$, then $\mathrm{Fut}(M,\chi) > 0$ for all filtrations with $\|\chi\| > 0$.

Futaki invariant

Given a filtration $F_i R$, define d_k and "total weight" w_k on $H^0(L^k)$:

$$d_k = \dim H^0(L^k),$$
$$w_k = \sum_i i \cdot (\dim F_i H^0(L^k) - \dim F_{i-1} H^0(L^k)).$$

Define

$$d_\infty = \lim_{k\to\infty} \frac{\dim H^0(L^k)}{k^n}, \quad w_\infty = \lim_{k\to\infty} \frac{w_k}{k^{n+1}}.$$

Then

$$\mathrm{Fut} = \liminf_{k\to\infty} \left(\frac{w_k}{d_k} - \frac{k w_\infty}{d_\infty} \right).$$

Test-configurations vs. filtrations

- Every rational, piecewise linear convex function $f : [0,1] \to \mathbf{R}$ defines a test-configuration for $(\mathbf{CP}^1, \mathscr{O}(1))$:

  Embed $\mathbf{CP}^1 \hookrightarrow \mathbf{CP}^N$ for large enough N, and use the values of f for the weights of a diagonal $\mathbf{C}^*$-action.

- Every convex function $f : [0,1] \to \mathbf{R}$ defines a filtration for $(\mathbf{CP}^1, \mathscr{O}(1))$:

  Co-ordinate ring is $\mathbf{C}[x,y]$. The i^{th} filtered piece is

$$F_i\mathbf{C}[x,y] = \mathrm{span}\left\{x^p y^{k-p} : f(p/k) < i/k\right\}.$$

In both cases: $\quad \mathrm{Fut}(f) = f(0) + f(1) - 2\int_0^1 f(x)\,dx.$

Existence results on blowups

Let (M,ω) be extremal, and $p\in M$. For small $\epsilon>0$, the blowup $\mathrm{Bl}_p M$ has Kähler classes

$$\Omega_\epsilon = \pi^*[\omega] - \epsilon^2[E].$$

Question: Does $\mathrm{Bl}_p M$ admit an extremal metric in Ω_ϵ? Goes back to work of **LeBrun-Singer, Rollin-Singer**

Theorem (Arezzo-Pacard, '06, '09, Arezzo-Pacard-Singer '11)

- *If $\mathrm{Aut}(M)=\{1\}$, then $\mathrm{Bl}_p M$ admits a cscK metric in Ω_ϵ for $\epsilon \ll 1$.*
- *In general $\mathrm{Bl}_p M$ admits an extremal metric in Ω_ϵ only under certain conditions. These conditions are easier to satisfy if we blow up many points.*

When $\mathrm{Aut}(M)$ is non-trivial

Let $G = \mathrm{Isom}_H(M, \omega)$ – Hamiltonian isometries

$$\text{Moment map: } \mu : M \to g.$$

$g = \mathrm{Lie}(G) \subset \{\text{Hamiltonian functions}\}$, $g \cong g^*$ using L^2-product.

Theorem (Arezzo-Pacard-Singer '11, Sz. '12)

If (M, ω) is extremal and

- *$\mu(p)$ vanishes at p,*

then $\mathrm{Bl}_p M$ admits an extremal metric in Ω_ϵ for $\epsilon \ll 1$.

Corollary (Sz. '12)

If (M, ω) is Kähler-Einstein, then for $\epsilon \ll 1$, if $(\mathrm{Bl}_p M, \Omega_\epsilon)$ is K-stable, then $\mathrm{Bl}_p M$ admits a cscK metric in Ω_ϵ.

Sharper results – assume $\dim M > 2$

Theorem (Sz. '13)

If (M,ω) is extremal and

- $\mu(p)+\delta\Delta\mu(p)$ *vanishes at p for some $\delta \ll 1$,*

then $\mathrm{Bl}_p M$ admits an extremal metric in Ω_ϵ for $\epsilon \ll 1$.

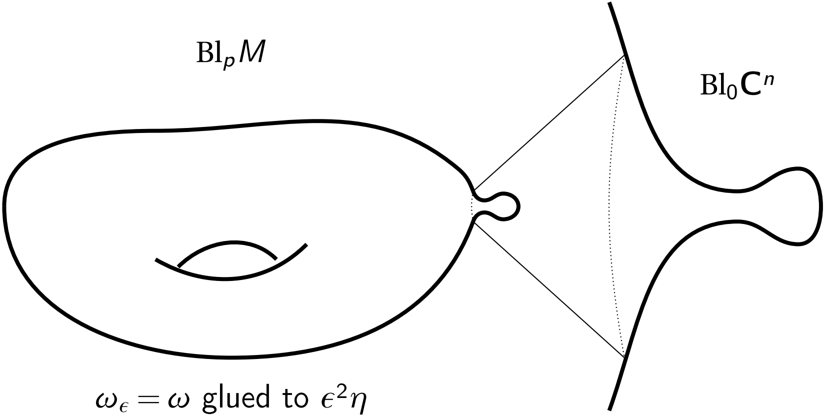
Corollary (Sz. '13)

If (M,ω) is cscK, then for $\epsilon \ll 1$, if $(\mathrm{Bl}_p M, \Omega_\epsilon)$ is K-stable, then $\mathrm{Bl}_p M$ admits a cscK metric in Ω_ϵ.

So the Yau-Tian-Donaldson conjecture holds for such "perturbed" manifolds. Similar existence results hold for $\mathbf{CP}^k$-bundles and fibered manifolds (**Hong '02, Fine '04**), but no relation to K-stability has been shown.

Method of proof – gluing

Step 1. Construct approximate solution ω_ϵ



$\omega_\epsilon = \omega$ glued to $\epsilon^2\eta$

scalar flat metric η
(**Burns-Simanca**)

Glue metrics near p using Kähler potentials:

$$\omega = i\partial\overline{\partial}\left[|z|^2 + O(|z|^3)\right]$$

$$\epsilon^2\eta = i\partial\overline{\partial}\left[|z|^2 + O(\epsilon^{2n-2}|z|^{4-2n})\right]$$

Cut off these terms.

For sharper results use better expansions and cut off fewer terms.

Step 2. Perturb ω_ϵ

Use Implicit Function Theorem to find ϕ, such that $\omega_\epsilon + i\partial\overline{\partial}\phi$ is extremal. This is obstructed if $\mathrm{Aut}(M)$ is non-trivial.

Obstruction is encoded in an element $f_\epsilon(p) \in \mathfrak{g}$:

- If $f_\epsilon(p)$ vanishes at p, then we get an extremal metric on $\mathrm{Bl}_p M$.

Conclusions obtained by computing expansion of $f_\epsilon(p)$ in ϵ.

$$f_\epsilon(p) = \mu(p) + \epsilon^2 \Delta\mu(p) + o(\epsilon^2).$$

A-P-S '11, Sz. '12 **Sz. '13**

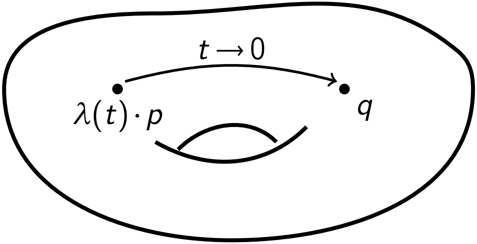
Sharper results (e.g. $\dim M = 2$) need more terms.

Step 3. Relate to K-stability

If there is no $q \in G^c \cdot p$ with $\mu(q) + \delta\Delta\mu(q)$ vanishing at q, then the Hilbert-Mumford criterion in Geometric Invariant Theory gives a $\mathbf{C}^*$-action $\lambda \subset \mathrm{Aut}(M)$, s.t.

$$q = \lim_{t\to 0} \lambda(t)\cdot p, \quad \text{and} \quad \mathrm{Fut}(\mathrm{Bl}_q M, \lambda) \leqslant 0.$$

This implies that $\mathrm{Bl}_p M$ is not K-stable.



Test-configuration:

$$\mathrm{Bl}_{\lambda(t)\cdot p} M \xrightarrow{t \to 0} \mathrm{Bl}_q M$$

What if no extremal metric exists?

Theorem (Donaldson, '05)

For M with an ample line bundle L, we have

$$\inf_{\omega \in c_1(L)} \|S(\omega) - \widehat{S}\|_{L^2} \geqslant \sup_{\lambda \text{ test-config}} \frac{-\mathrm{Fut}(M,\lambda)}{\|\lambda\|}.$$

Here $\widehat{S}$ is the average scalar curvature.

Conjecture: Equality holds above.

Analogous conjecture for vector bundles is known.
(**Atiyah-Bott '83, Jacob '12**)

Toric manifolds

Theorem (Sz. '08)

Assume that the Calabi flow exists for all time on a toric manifold M, with a toric initial metric in $c_1(L)$. Then

$$\inf_{\omega \in c_1(L)} \|S(\omega) - \widehat{S}\|_{L^2} = \sup_{\lambda \text{ test-config}} \frac{-\mathrm{Fut}(M,\lambda)}{\|\lambda\|}.$$

Calabi flow:

$$\frac{\partial}{\partial t}\omega = i\partial\overline{\partial}S(\omega).$$

Unfortunately existence is rarely known.

Partial results by **Chen-He '06, Chen-Huang-Sheng '13**

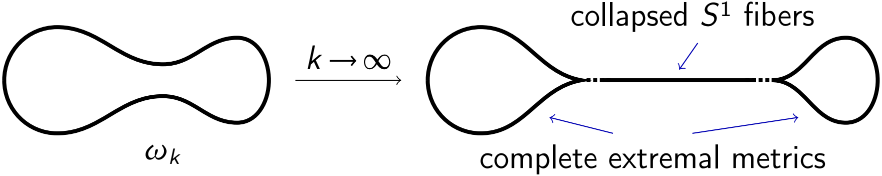
Ruled surface revisited

Theorem (Sz. '07)

Let $M = \mathbf{P}(\mathscr{O} \oplus \mathscr{L})$ over a genus 2 curve. For any ample $L \to M$,

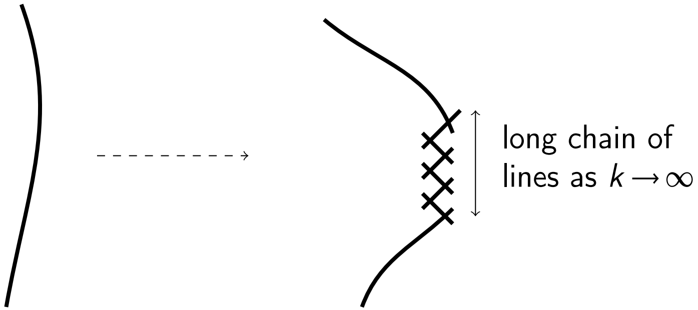
$$\inf_{\omega \in c_1(L)} \|S(\omega) - \widehat{S}\|_{L^2} = \sup_{\lambda \text{ test-config}} \frac{-\mathrm{Fut}(M, \lambda)}{\|\lambda\|}.$$

What does a minimizing sequence ω_k for the Calabi functional look like? In some Kähler classes, fiber metrics behave as:



The algebraic side – maximize $\frac{-\mathrm{Fut}(M,\lambda)}{\|\lambda\|}$

We can write down (**Sz. '07**) a sequence of test-configurations λ_k, realizing the supremum. On the fibers, λ_k degenerates a line into a chain of normal crossing lines:



The sequence λ_k can be encoded as a filtration.

Questions

- Does there always exist an "optimal destabilizing filtration"?

  Almost known in toric case (**Sz. '08**).

  Analogous to the Harder-Narasimhan filtration of unstable vector bundles (**Bruasse-Teleman, '05**).

- What relationship is there between metric degenerations and filtrations?

  Relevant for other problems in Kähler geometry, e.g. J-flow.

Thank you!