

# On the affect of the Laplacian in equilibration dynamics of the Spherical Model

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Consider a system of  $N$  linearly interacting degrees of freedom. We collect them into a vector  $\bar{\mathbf{s}}(t) = (s_1(t), \dots, s_N(t))$  and represent their interactions with random coupling matrix  $\mathbf{J}(t)$ . We subject them to a global constraint: lying on an  $N$ -dimensional sphere of radius  $N$ , enforced by lagrange multiplier  $\mu$ . We write down the Langevin Equation of motion with external noise  $\boldsymbol{\xi}(t)$ , dropping the explicit time dependence for brevity.

$$\partial_t \bar{\mathbf{s}} = (\mathbf{J} - \mu) \bar{\mathbf{s}} + \boldsymbol{\xi} \quad (0.1)$$

$$\text{where } \mu = \frac{1}{N} \bar{\mathbf{s}}^\top (\mathbf{J} \bar{\mathbf{s}} + \boldsymbol{\xi}) \quad \text{enforces constraint } |\bar{\mathbf{s}}(t)|^2 = N \quad (0.2)$$

The following is an attempt to characterise the statistical differences between spatial and non-spatial interactions, and their effect on the equilibration dynamics by introducing the discretised Laplacian  $\Delta$  as a diffusive term.

## 1 Analytical Solutions

Here we follow the methods used to obtain exact solutions[] in the field theoretic setting of the model.

$$\bar{\mathbf{u}}(t) = e^{\Lambda t} \mathbf{\Gamma}^{1/2} \bar{\mathbf{u}}(0)$$

## 1.1 Spectra of Discrete Laplacians

Using the Circular Diagonalization Theorem [1] one can derive the eigenvalues  $\lambda_N(k)$  of an  $N \times N$  matrix  $\mathbf{X}$  which represents the second-order central difference approximation to the second derivative along  $N$  sites of a one dimensional ring of unit circumference.

$$\mathbf{X} := \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix} \quad (1.1)$$

$$\begin{aligned} \lambda_N(k) &= 2 \left( \cos \left( \frac{2\pi k}{N} \right) - 1 \right) \\ k &\in \{0, 1, \dots, N-1\} \end{aligned} \quad (1.2)$$

As the number of sites  $N \rightarrow \infty$  the argument  $k/N \in [0, 1]$  and the eigenvalues remain bounded  $-2 < \lambda < 0$ . By shifting and scaling the index  $k \rightarrow \frac{k-N\pi}{2\pi}$  the eigenvalues are expressed as the familiar dispersion relation [1].

$$\lambda(x) = -2(\cos x + 1) \quad x \in [-\pi, \pi] \quad (1.3)$$

The discrete  $M$ -dimensional laplacian is simply the kronecker sum of one dimensional cases  $\Delta = \mathbf{X} \oplus \mathbf{X} \oplus \dots \oplus \mathbf{X}$  and thus its eigenvalues is simply the sum one dimensional dispersions [1].

$$\lambda(\bar{\mathbf{x}}) = -2 \sum_{x \in \bar{\mathbf{x}}} (\cos x + 1) \quad \bar{\mathbf{x}} \in [-\pi, \pi]^M \quad (1.4)$$

The probability density  $\rho(\lambda)$  can be expressed as an integral over the region  $\Omega = [-\pi, \pi]^M$  in complete analogue with the density of states.

$$\rho(\lambda') = \frac{1}{Z_M} \int_{\Omega} \delta(\lambda' - \lambda(\bar{\mathbf{x}})) d\bar{\mathbf{x}}$$

We proceed with an element-wise change of variables  $\bar{\mathbf{u}} = 2 \cos \bar{\mathbf{x}}$  and recognise that the integration region is  $M$ -fold symmetric across each component axis, which allows restrict the domain of integration to a hyperoctant. Thus in the new coordinates  $\bar{\mathbf{u}}$  the  $M$ -dimensional hypercube

region becomes  $\Omega' = [-2, 2]^M$ .

$$\rho(\lambda) = \frac{1}{Z_M} \int_{\Omega'} \frac{\delta(\Lambda_M + \sum_{u \in \bar{\mathbf{u}}} u)}{\sqrt{\prod_{u \in \bar{\mathbf{u}}} (1 - u^2/4)}} d\bar{\mathbf{u}} \quad \begin{array}{l} \Lambda_M = \lambda + 2M \\ |\Lambda_M| \leq 2M \end{array}$$

The tricky aspect of this integral writing down how the domain of integration  $\Omega' \rightarrow \partial\Omega'(\Lambda_M)$  becomes of a function of  $\Lambda_M$  after using the delta function filtering property.

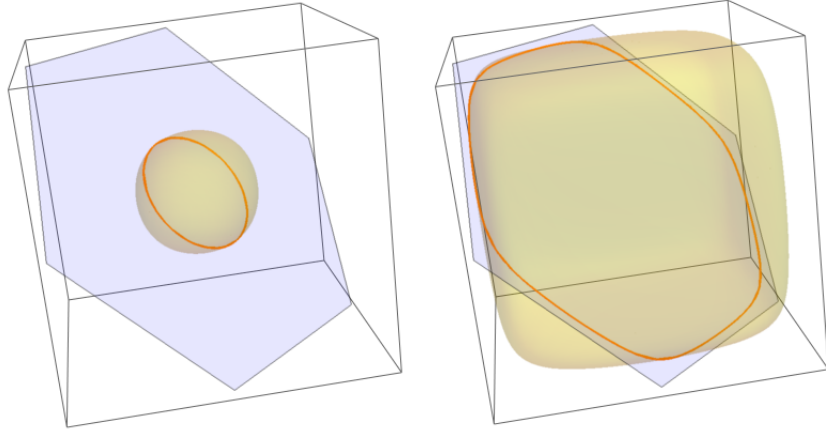


Figure 1: Isosurfaces of the integrand  $M = 3$  within the cubic integration region  $\Omega'$  with delta constraint represented by a plane shifted from the origin by parameter  $\Lambda_3$ . This illustration