On the affect of the Laplacian in equlibration dynamics of the Spherical Model

Gregory Szep

King's College London

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Consider a system of N linearly interacting degrees of freedom. We collect them into a vector $\bar{\mathbf{s}}(t) = (\, s_1(t), \cdots, s_N(t)\,)$ and represent their interactions with random coupling matrix $\mathbf{J}(t)$. We subject them to a global constraint: lying on an N-dimensional sphere of radius N, enforced by lagrange multiplier μ. We write down the Langevin Equation of motion with external noise $\boldsymbol{\xi}(t)$, dropping the explicit time dependence for brevity.

$$\partial_t \bar{\mathbf{s}} = (\mathbf{J} - \mu)\bar{\mathbf{s}} + \boldsymbol{\xi} \tag{0.1}$$

$$\text{where} \quad \mu = \frac{1}{N}\bar{\mathbf{s}}^\top (\mathbf{J}\bar{\mathbf{s}} + \boldsymbol{\xi}) \quad \text{enforces constraint} \quad |\bar{\mathbf{s}}(t)|^2 = N \tag{0.2}$$

The following is an attempt to characterise the statistical differences between spatial and non-spatial interactions, and their effect on the equilibration dynamics by introducting the discretised Laplacian Δ as a diffusive term.

1 Analytical Solutions

Here we follow the methods used to obtain exact solutions[] in the field theoretic setting of the model.

$$\bar{\mathbf{u}}(t) = \mathbb{e}^{\boldsymbol{\Lambda} t}\boldsymbol{\Gamma}^{1/2}\bar{\mathbf{u}}(0) +$$

1.1 Spectra of Discrete Laplacians

Using the Circular Diagonalization Theorem [] one can derive the eigenvalues $\lambda_N(k)$ of an $N \times N$ matrix $\mathbf{X}_N$ which represents the second-order central difference approximation to the second derivative along N sites of a one dimensional ring of unit circumference.

$$\mathbf{X}_N := \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix} \tag{1.1}$$

$$\begin{gathered} \lambda_N(k) = 2\left(\cos\left(\tfrac{2\pi k}{N}\right) - 1\right) \\ k \in \{0, 1, \cdots, N-1\} \end{gathered} \tag{1.2}$$

As the number of sites $N \to \infty$ the argument $k/N \in [0,1]$ and the eigenvalues remain bounded. The probability density $\rho(\lambda)$ can be expressed as a continuous integral:

$$\begin{aligned} \rho(\lambda') &= \int \delta(\lambda' - \lambda(x)) p(x)\,\mathrm{d}x \\ &= \sum_{x, \lambda' = \lambda(x)} \frac{p(x)}{|\partial_x \lambda(x)|} \\ &= \sum_{\alpha = 0,1} \frac{\int_0^1 \delta\left(x - \alpha\right)\,\mathrm{d}x}{Z|\partial_x \lambda|_{x=\alpha}} \end{aligned}$$