On the affect of the Laplacian in equlibration dynamics of the Spherical Model

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Consider a system of N linearly interacting degrees of freedom. We collect them into a vector $\bar{\mathbf{s}}(t) = (\, s_1(t), \cdots, s_N(t)\,)$ and represent their interactions with random coupling matrix $\mathbf{J}(t)$. We subject them to a global constraint: lying on an N-dimensional sphere of radius N, enforced by lagrange multiplier μ. We write down the Langevin Equation of motion with external noise $\boldsymbol{\xi}(t)$, dropping the explicit time dependence for brevity.

$$\partial_t \bar{\mathbf{s}} = (\mathbf{J} - \mu)\bar{\mathbf{s}} + \boldsymbol{\xi} \tag{0.1}$$

$$\text{where} \quad \mu = \frac{1}{N}\bar{\mathbf{s}}^\top (\mathbf{J}\bar{\mathbf{s}} + \boldsymbol{\xi}) \quad \text{enforces constraint} \quad |\bar{\mathbf{s}}(t)|^2 = N \tag{0.2}$$

The following is an attempt to characterise the statistical differences between spatial and non-spatial interactions, and their effect on the equilibration dynamics by introducting the discretised Laplacian Δ as a diffusive term.

1 Analytical Solutions

Here we follow the methods used to obtain exact solutions[] in the field theoretic setting of the model.

$$\bar{\mathbf{u}}(t) = \mathbb{e}^{\boldsymbol{\Lambda} t}\boldsymbol{\Gamma}^{1/2}\bar{\mathbf{u}}(0)$$

1.1 Spectra of Discrete Laplacians

Using the Circular Diagonalization Theorem [] one can derive the eigenvalues $\lambda_N(k)$ of an $N \times N$ matrix $\mathbf{X}$ which represents the second-order central difference approximation to the second derivative along N sites of a one dimensional ring of unit circumference.

$$\mathbf{X} := \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix} \tag{1.1}$$

$$\begin{gathered} \lambda_N(k) = 2\left(\cos\left(\tfrac{2\pi k}{N}\right) - 1\right) \\ k \in \{0, 1, \cdots, N-1\} \end{gathered} \tag{1.2}$$

As the number of sites $N \to \infty$ the argument $k/N \in [0, 1]$ and the eigenvalues remain bounded $-2 < \lambda < 0$. By shifting and scaling the index $k \to \frac{k-N\pi}{2\pi}$ the eigenvalues are expressed as the familiar dispersion relation [].

$$\lambda(x) = -2\left(\cos x + 1\right) \quad x \in [-\pi, \pi] \tag{1.3}$$

The discrete M-dimensional laplacian is simply the kronecker sum of one dimensional cases $\mathbf{\Delta} = \mathbf{X} \oplus \mathbf{X} \oplus \cdots \oplus \mathbf{X}$ and thus its eigenvalues is simply the sum one dimensional dispersions [].

$$\lambda(\bar{\mathbf{x}}) = -2 \sum_{x \in \bar{\mathbf{x}}} \left(\cos x + 1\right) \quad \bar{\mathbf{x}} \in [-\pi, \pi]^M \tag{1.4}$$

The probability density $\rho(\lambda)$ can be expressed as an integral over the region $\Omega = [-\pi, \pi]^M$ in complete analogue with the density of states.

$$\rho(\lambda') = \frac{1}{Z_M} \int_\Omega \delta(\lambda' - \lambda(\bar{\mathbf{x}}))\, \mathrm{d}\bar{\mathbf{x}}$$

We proceed with an element-wise change of variables $\bar{\mathbf{u}} = 2\cos\bar{\mathbf{x}}$ and recognise that the integration region is M-fold symmetric across each component axis, which allows restrict the domain of integration to a hyperoctant. Thus in the new coordinates $\bar{\mathbf{u}}$ the M-dimensional hypercube region becomes $\Omega' = [-2, 2]^M$.

$$\rho(\lambda) = \frac{1}{Z_M} \int_{\Omega'} \frac{\delta(\Lambda_M + \sum_{u \in \bar{\mathbf{u}}} u)}{\sqrt{\prod_{u \in \bar{\mathbf{u}}} (1 - u^2/4)}} \, \mathrm{d}\bar{\mathbf{u}} \qquad \begin{array}{c} \Lambda_M = \lambda + 2M \\ |\Lambda_M| \leq 2M \end{array}$$

The tricky aspect of this integral writing down how the domain of integration $\Omega' \to \partial\Omega'(\Lambda_M)$ becomes of a function of Λ_M after using the delta function filtering property.

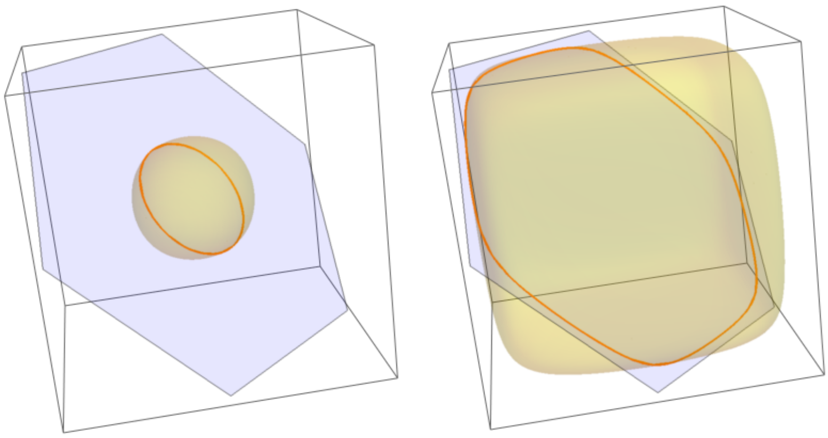

Figure 1: Isosurfaces of the integrand $M = 3$ within the cubic integration region Ω' with delta constraint represented by a plane shifted from the origin by parameter Λ_3. This illustration