On the affect of the Laplacian in equlibration dynamics of the Spherical Model

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1 Intorduction and Motivation

1.1 Motivation for the Model

Consider N nodes in a undirected network passing around a finite resource. Eventually the resource will equilibrate within the network solely due it its finiteness. We wish to study the equilibration timescales of such a problem, where finiteness of the resource and connectedness of the network may be expressed is a variety of ways, giving rise different flavours of the Sherrington-Kirkpatrick spherical model.

Exact dynamical solutions can be obtained if finiteness of the resource is the expressed as a constant L_2 norm across all the nodes. In addition, if the connectedness of the network is known, it may be used to obtain explicit expressions — at least in some limit — for the dynamics of the resource, and hence timescales can be extracted.

The Wigner ensemble is used to generate the connectivity in an attempt to minimise a priori assumptions on it. Then intorducing the M-dimensional periodic lattice laplacian allows the study of the model as it departs from randomness and gains spatial structure. Is possible to extract the dimension of the manifold that the nodes lie on simply from observed correlations and equilibration timescales in the resource? This work may have applications in dimensionality reduction methods.

1.2 Analytical Solution of the Langevin Equation

Let the resource across N nodes be represented by vector $s(t) \in \mathbb{R}^N$, and the weighted undirected connections between them be a Hermitian $N \times N$ matrix $\mathbf{J}$. This matrix represents the rate at which the resource is passed between nodes. Without finiteness, we can solve the linear equations, revealing that the resource at different nodes would diverge or go to zero depending on the sign of the eigenvalues of $\mathbf{J}$. This does not seem reasonable.

Therefore a lagrange multiplier $\mu(t) \in \mathbb{R}$ is introduced which holds the L_2 norm of vector $s(t)$ constant. This implicitly introduces a nonlinearity. The dynamics is cast as a Langevin Equation with white noise $\xi(t) \in \mathbb{R}^N$ which is characterised by its moments $\langle \dots \rangle$. Eventually the resource will equilibrate due it its finiteness, subject to the amplitude of the noise. The symbol $\mathbb{1}$ represents the diagonal identity.

$$\partial_t s = \big[\mathbf{J} - \mathbb{1}\mu(t)\big] s + \xi \tag{1.1}$$

$$\langle \xi(t) \rangle = \mathbb{0} \qquad \langle \xi(t)\xi(t') \rangle = 2T\mathbb{1}\delta(t-t') \tag{1.2}$$

$$\text{where} \quad \mu(t) = \frac{1}{N} s^\top \left(\mathbf{J}s + \xi\right) \quad \text{enforces constraint} \quad |s(t)|^2 = N \tag{1.3}$$

It is possible to solve the inhomogenous time-dependent ordinary linear system of differential equations for $s(t)$ using integrating factor method and rotate into the eigenbasis $\sigma_\lambda(t) \in \mathbb{R}$ of matrix $\mathbf{J}$. This does not yield a closed form for $\mu(t)$ however; the derivation by Cugliandolo, L. F. and Dean D. S. [] yields the moments below for the uniform initial condition $\sigma_\lambda(0) = 1$. Here the complexity lies in performing the inverse laplace transform $\mathcal{L}^{-1}$ which in turn depends on the eigenvalue distribution $p(\lambda \in \mathbf{J})$.

$$\langle \sigma_\lambda(t) \rangle = \frac{\mathbb{e}^{\lambda t}}{\sqrt{\Gamma(t)}} \qquad \langle \sigma_\lambda(t)\sigma_{\lambda'}(t') \rangle = \tag{1.4}$$

$$\Gamma(t) = \sum_{k=0}^{\infty} (2T)^k \mathcal{L}^{-1}\left[\Phi(s)^k\right] \quad \text{where} \quad \Phi(s) = \int \frac{p(\lambda \in \mathbf{J})}{s - 2\lambda} \mathrm{d}\lambda \tag{1.5}$$

1.3 Hermitian Wigner Ensemble

Suppose the connections between nodes are given by a random matrix $\mathbf{H}$ taken from the Hermitian Wigner ensemble, whos upper triangle elements $[\mathbf{H}]_{ij}$ are distributed according to a density $\mathbb{P}$ with subexponential tails [] with expectation $\mathbb{E}$ and variance $\mathbb{V}$, such that in the limit $N \to \infty$ the eigenvalue spectrum converges to the semicircle law $p(\lambda \in \mathbf{H}) \to \cap(\lambda|J)$. Below the symbols $\mathbf{0}$ and $\mathbf{1}$ denote constant matrices of zeros and ones respectively.

$$\cap(\lambda|J) = \frac{1}{2\pi J^2}\sqrt{4J^2 - \lambda^2} \tag{1.6}$$

$$\mathbb{E}[\mathbf{H}] = \mathbf{0} \quad \mathbb{V}[\mathbf{H}] = \frac{J^2}{N}(\mathbf{1} + \mathbb{1}) \tag{1.7}$$

$$\mathbb{P}\left(t^\alpha \leq |[\mathbf{H}]_{ij}|\right) \leq e^{-t} \quad \forall t \geq \alpha, \forall i, j \tag{1.8}$$

To match the terms of the expansion in (1.5) to known inverse laplace transforms some algebraic trickery is required: after substituting the limiting density (1.6) into the second equation in (1.5) the infinite sum is evaluated first. Then the result is expanded in terms the denominator part that is a function of s. Only then do the terms match the laplace transforms of modified Bessel functions of the first kind $\frac{I_l(4Jt)}{t}$. These have well-known asymtotics for $z \to \infty$

$$\Phi(s) = \frac{s - \sqrt{s^2 - 16J^2}}{8J^2} \qquad \Gamma(t) = \frac{1}{2T}\sum_{l=1}^{\infty} l \left(\frac{T}{J}\right)^l \frac{I_l(4Jt)}{t} \tag{1.9}$$

1.4 Discrete Laplacians

Using the Circular Diagonalization Theorem [] one can derive the eigenvalues $\lambda_N(k)$ of an $N \times N$ matrix $\mathbf{X}$ which represents the second-order central difference approximation to the second derivative along N sites of a one dimensional ring.

$$\mathbf{X} := \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix} \tag{1.10}$$

$$\begin{gathered} \lambda_N(k) = 2\left(\cos\left(\tfrac{2\pi k}{N}\right) - 1\right) \\ k \in \{0, 1, \cdots, N-1\} \end{gathered} \tag{1.11}$$

As the number of sites $N \to \infty$ the argument $k/N \in [0, 1]$ and the eigenvalues remain bounded $-2 < \lambda < 0$. By shifting and scaling the index $k \to \frac{k-N\pi}{2\pi}$ the eigenvalues are expressed as the familiar dispersion relation [].

$$\lambda(x) = -2\left(\cos x + 1\right) \quad x \in [-\pi, \pi] \tag{1.12}$$

The discrete M-dimensional laplacian is simply the kronecker sum of one dimensional cases $\mathbf{\Delta} = \mathbf{X} \oplus \mathbf{X} \oplus \cdots \oplus \mathbf{X}$ and thus its eigenvalues is simply the sum one dimensional dispersions [].

$$\lambda(\bar{\mathbf{x}}) = -2 \sum_{x \in \bar{\mathbf{x}}} \left(\cos x + 1\right) \quad \bar{\mathbf{x}} \in [-\pi, \pi]^M \tag{1.13}$$

The probability density $\Delta_M(\lambda)$ can be expressed as an integral over the M-dimensional hypercube region $\Omega = [-\pi, \pi]^M$ in complete analogue with the density of states.

$$\Delta_M(\lambda') = \frac{1}{Z_M} \int_\Omega \delta(\lambda' - \lambda(\bar{\mathbf{x}}))\, \mathrm{d}\bar{\mathbf{x}} \tag{1.14}$$

We proceed with an element-wise change of variables $\bar{\mathbf{u}} = 2\cos\bar{\mathbf{x}}$ and recognise that the integration region is M-fold symmetric across each component axis, which allows restriction of the domain of integration to a hyperoctant. Given coordinates $\bar{\mathbf{u}}$ the region becomes $\Omega' = [-2, 2]^M$.

$$\Delta_M(\lambda) = \frac{1}{Z_M}\int_{\Omega'} \frac{\delta(\Lambda_M + \sum_{u\in\bar{\mathbf{u}}} u)}{\sqrt{\prod_{u\in\bar{\mathbf{u}}}(1-u^2/4)}}\,\mathrm{d}\bar{\mathbf{u}} \qquad \begin{matrix}\Lambda_M = \lambda + 2M \\ |\Lambda_M| \le 2M\end{matrix}$$

$$= \frac{1}{2\pi Z_M}\int_{-\infty}^{\infty}\int_{\Omega'} \frac{\mathbb{e}^{\Lambda_M \mathbb{i} k}\exp[\sum_{u\in\bar{\mathbf{u}}} u\mathbb{i}k]}{\sqrt{\prod_{u\in\bar{\mathbf{u}}}(1-u^2/4)}}\,\mathrm{d}\bar{\mathbf{u}}\mathrm{d}k$$

$$= \frac{1}{2\pi Z_M}\int_{-\infty}^{\infty} \mathbb{e}^{\Lambda_M \mathbb{i} k}\prod_{u\in\bar{\mathbf{u}}}\int_{-2}^{2}\frac{\mathbb{e}^{u\mathbb{i}k}}{\sqrt{1-u^2/4}}\,\mathrm{d}u\mathrm{d}k$$

The fourier representation of the delta function allowed the factorisation of the integral. We recognise a repeated Bessel integral and replace it with the Bessel function of the first kind $J_n(k)$, leaving only a fourier transform which we define $\mathcal{F} : f \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(k)e^{\mathbb{i}\Lambda k}\mathrm{d}k$. To clean the formula up even further we may use the convolution theorem to deal with the powers of M, leaving only the fourier transform of the Bessel function $J_0(k)$, which is the arcsine distribution $\alpha(\lambda)$. It becomes clear that the eigenvalue density of a kronecker sum of matrices is the convolution of the densities of those matrices.

$$\Delta_M(\lambda) = \underbrace{\alpha(\lambda) * \alpha(\lambda) * \cdots * \alpha(\lambda)}_{M} \tag{1.15}$$

$$\alpha(\lambda) = \frac{1}{2\pi\sqrt{1-\left(\frac{\lambda+2}{2}\right)^2}}\Pi\left(\frac{\lambda+2}{2}\right) \qquad \Pi(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \ge 1 \end{cases} \tag{1.16}$$

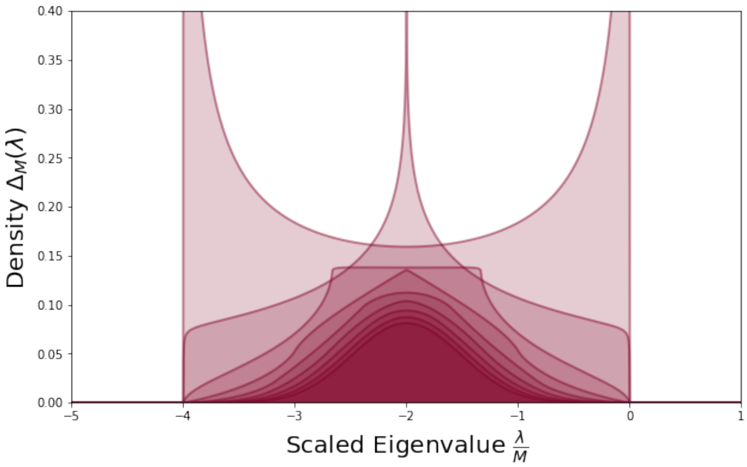

Figure 1: Eigenvalue distributions $\Delta_M(\lambda)$ of the M-dimensional Laplacian

The one dimensional density has two Van Hove singularities at $\lambda = -4, 0$ given by the arcsine law $\alpha(\lambda)$, whereas the two dimensional case has one at $\lambda = -4$ given by the complete elliptic integral of the first kind $K(m)$. Figure 1 reveals that in higher dimensions singularities do not occur; instead there appear to be discontinuities in the higher order derivatives. The density smooths out as repeated convolutions bring it to a normal distrubtion; this is another way to state the Central Limit Theorem [].

Substituting the limiting density (**??**) into the second equation in (1.5) the inverse laplace transform of the powers can be performed immediately.

$$\Phi(s) = \frac{1}{\sqrt{8s + s^2}} \qquad \Gamma(t) = 2Te^{-4t} \sum_{k=0}^{\infty} \frac{(2Tt)^{k-1}}{(k-1)!} {}_0F_1(\frac{1+k}{2}, 4t^2) \tag{1.17}$$

1.5 Sum of the Wigner and Laplacian Matrices

In the formalism of Free Probability it is possible to express the $N \to$ limiting eigenvalue density ρ of a sum of matrices in terms of the free convolutions the individual limiting densities, provided that these densities have compact support [**?**, **?**]. The following expressions for the free convolution of a semicircle distrubtion $\cap(\lambda|J)$ with an arbitrary compact distrubtion $\mu(\lambda)$ can be used when one of these matrices comes from a Wigner Ensemble [].

$$\mathcal{H}[\rho \circ \psi(\lambda|J)] = \int_{\mathbb{R}} \frac{(\lambda - \lambda')\mu(\lambda')\mathrm{d}\lambda'}{(\lambda - \lambda')^2 + E(\lambda|J)^2} \qquad \psi(\lambda|J) = \lambda + J\mathcal{H}[\rho \circ \psi(\lambda|J)] \tag{1.18}$$

$$E(\lambda|J) = \inf \left\{ E \geq 0 \;\middle|\; \int_{\mathbb{R}} \frac{\mu(\lambda')\mathrm{d}\lambda'}{(\lambda - \lambda')^2 + E^2} \leq \frac{1}{J} \right\} \tag{1.19}$$

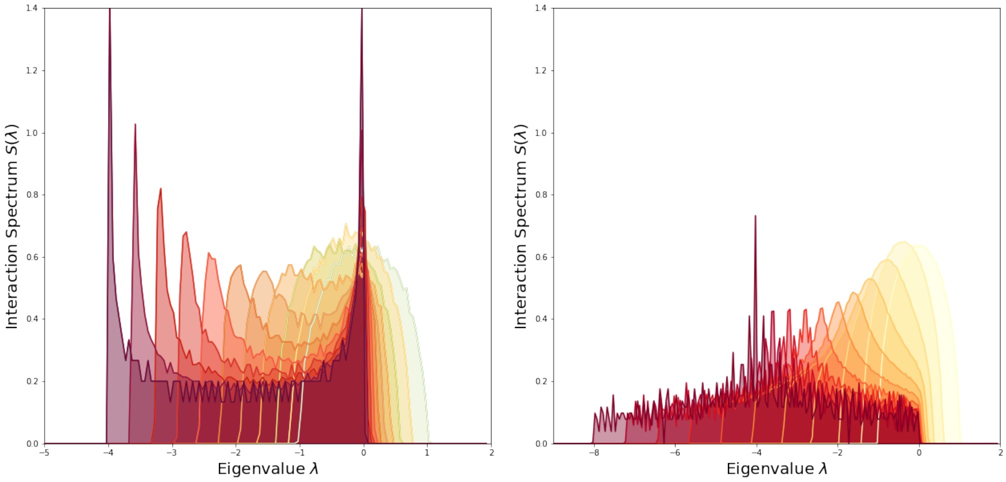

Figure 2: Left/Right: Random interaction spectra $S(\lambda|\mu)$ for variable values of the order parameter μ which introduces one/two dimensional laplacian