On the affect of the Laplacian in equlibration dynamics of the Spherical Model

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Consider a system of N linearly interacting degrees of freedom. We collect them into a vector $\bar{\mathbf{s}}(t) = (\, s_1(t), \cdots, s_N(t)\,)$ and represent their interactions with random coupling matrix $\mathbf{J}(t)$. We subject them to a global constraint: lying on an N-dimensional sphere of radius N, enforced by lagrange multiplier μ. We write down the Langevin Equation of motion with external noise $\boldsymbol{\xi}(t)$, dropping the explicit time dependence for brevity.

$$\partial_t \bar{\mathbf{s}} = (\mathbf{J} - \mu)\bar{\mathbf{s}} + \boldsymbol{\xi} \tag{0.1}$$

$$\text{where} \quad \mu = \frac{1}{N}\bar{\mathbf{s}}^\top (\mathbf{J}\bar{\mathbf{s}} + \boldsymbol{\xi}) \quad \text{enforces constraint} \quad |\bar{\mathbf{s}}(t)|^2 = N \tag{0.2}$$

The following is an attempt to characterise the statistical differences between spatial and non-spatial interactions, and their effect on the equilibration dynamics by introducting the discretised Laplacian Δ as a diffusive term.

1 Analytical Solutions

Here we follow the methods used to obtain exact solutions[] in the field theoretic setting of the model.

$$\bar{\mathbf{u}}(t) = \mathbb{e}^{\boldsymbol{\Lambda} t}\boldsymbol{\Gamma}^{1/2}\bar{\mathbf{u}}(0)$$

1.1 Spectra of Discrete Laplacians

Using the Circular Diagonalization Theorem [] one can derive the eigenvalues $\lambda_N(k)$ of an $N \times N$ matrix $\mathbf{X}$ which represents the second-order central difference approximation to the second derivative along N sites of a one dimensional ring.

$$\mathbf{X} := \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix} \tag{1.1}$$

$$\begin{gathered} \lambda_N(k) = 2\left(\cos\left(\tfrac{2\pi k}{N}\right) - 1\right) \\ k \in \{0, 1, \cdots, N-1\} \end{gathered} \tag{1.2}$$

As the number of sites $N \to \infty$ the argument $k/N \in [0,1]$ and the eigenvalues remain bounded $-2 < \lambda < 0$. By shifting and scaling the index $k \to \frac{k-N\pi}{2\pi}$ the eigenvalues are expressed as the familiar dispersion relation [].

$$\lambda(x) = -2\left(\cos x + 1\right) \quad x \in [-\pi, \pi] \tag{1.3}$$

The discrete M-dimensional laplacian is simply the kronecker sum of one dimensional cases $\mathbf{\Delta} = \mathbf{X} \oplus \mathbf{X} \oplus \cdots \oplus \mathbf{X}$ and thus its eigenvalues is simply the sum one dimensional dispersions [].

$$\lambda(\bar{\mathbf{x}}) = -2 \sum_{x \in \bar{\mathbf{x}}} \left(\cos x + 1\right) \quad \bar{\mathbf{x}} \in [-\pi, \pi]^M \tag{1.4}$$

The probability density $\rho_M(\lambda)$ can be expressed as an integral over the M-dimensional hypercube region $\Omega = [-\pi, \pi]^M$ in complete analogue with the density of states.

$$\rho_M(\lambda') = \frac{1}{Z_M} \int_\Omega \delta(\lambda' - \lambda(\bar{\mathbf{x}}))\, \mathrm{d}\bar{\mathbf{x}}$$

We proceed with an element-wise change of variables $\bar{\mathbf{u}} = 2\cos\bar{\mathbf{x}}$ and recognise that the integration region is M-fold symmetric across each component axis, which allows restriction of the domain of integration to a hyperoctant. Given coordinates $\bar{\mathbf{u}}$ the region becomes $\Omega' = [-2, 2]^M$.

$$
\begin{aligned}
\rho_M(\lambda) &= \frac{1}{Z_M}\int_{\Omega'} \frac{\delta(\Lambda_M + \sum_{u\in\bar{\mathbf{u}}} u)}{\sqrt{\prod_{u\in\bar{\mathbf{u}}}(1-u^2/4)}}\mathrm{d}\bar{\mathbf{u}} \qquad \begin{array}{c}\Lambda_M = \lambda + 2M \\ |\Lambda_M| \leq 2M\end{array} \\
&= \frac{1}{2\pi Z_M}\int_{-\infty}^{\infty}\int_{\Omega'} \frac{\mathbb{e}^{\Lambda_M \mathbb{i} k}\exp[\sum_{u\in\bar{\mathbf{u}}} u\mathbb{i}k]}{\sqrt{\prod_{u\in\bar{\mathbf{u}}}(1-u^2/4)}}\mathrm{d}\bar{\mathbf{u}}\mathrm{d}k \\
&= \frac{1}{2\pi Z_M}\int_{-\infty}^{\infty} \mathbb{e}^{\Lambda_M \mathbb{i} k}\prod_{u\in\bar{\mathbf{u}}}\int_{-2}^{2}\frac{\mathbb{e}^{u\mathbb{i}k}}{\sqrt{1-u^2/4}}\mathrm{d}u\mathrm{d}k
\end{aligned}
$$

The fourier representation of the delta function allowed the factorisation of the integral. We recognise a repeated Bessel integral and replace it with the Bessel function of the first kind $J_n(k)$, leaving only a fourier transform which we define $\mathcal{F} : f \rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(k)e^{\mathbb{i}\Lambda k}\mathrm{d}k$. To clean the formula up even further we may use the convolution theorem to deal with the powers of M, leaving only the fourier transform of the Bessel function $J_0(k)$, which is the arcsine distribution $\alpha(\lambda)$. It becomes clear that the eigenvalue density of a kronecker sum of matrices is the convolution of the densities of those matrices.

$$\rho_M(\lambda) = \underbrace{\alpha(\lambda) * \alpha(\lambda) * \cdots * \alpha(\lambda)}_{M} \tag{1.5}$$

$$\alpha(\lambda) = \frac{1}{2\pi\sqrt{1-\left(\frac{\lambda+2}{2}\right)^2}}\Pi\left(\frac{\lambda+2}{2}\right) \tag{1.6}$$

$$\Pi(x) = \begin{cases} s \end{cases} \tag{1.7}$$

The delta function defines a hyperplane region $\partial\Omega$ with normal vector $\bar{\mathbf{n}} = (1, \cdots, 1)$ and distance $\Lambda_M/\sqrt{M}$ from the origin. The final region of integration is the intersection between the constraint and hypercube $\partial\Omega' = \Omega' \cap \partial\Omega$ is thus becomes of a function of Λ_M. Figure 1 illustrates this dependence in the $M = 3$ case.

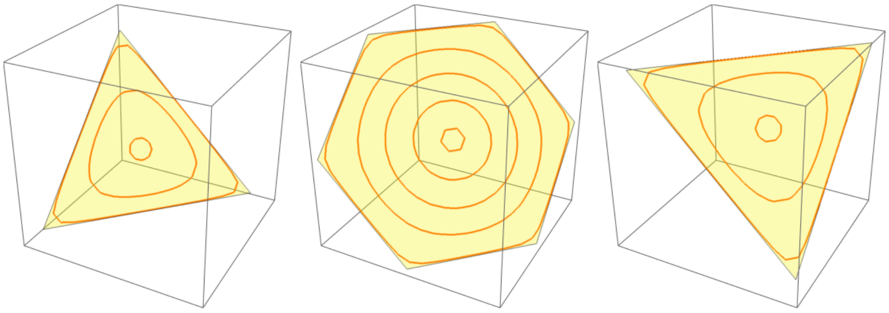

Figure 1: Integration region $\partial\Omega'$ with contour mesh representing isosurfaces of the integrand

In the one dimensional case the delta function filters the integrand and there are no more integrals left to do so one obtains the density directly. In the two dimensional case rotating into the plane of the constraint $\bar{\mathbf{v}} = \mathbf{R}(\frac{\pi}{4})\bar{\mathbf{u}}$ — where $\mathbf{R}(\theta)$ is the rotation matrix — simplifies the limits and reveals yet another symmetry $\Lambda \to -\Lambda$. The final integral yields a complete elliptic integral of the first kind $K(m)$.

$$
\begin{aligned}
\iint_{-2}^{2} \frac{\delta(\Lambda_2 + u + u')}{\sqrt{(1-\frac{1}{4}u^2)(1-\frac{1}{4}u'^2)}}\,\mathrm{d}u\mathrm{d}u' &= \iint_{-2}^{2} \frac{\delta(\Lambda_2 + v\sqrt{2})}{\sqrt{(1-\frac{1}{8}(v+v')^2)(1-\frac{1}{8}(v-v')^2)}}\,\mathrm{d}v\mathrm{d}v' \\
&= \int_{-2\sqrt{2}+\frac{\sqrt{2}}{2}|\Lambda_2|}^{2\sqrt{2}-\frac{\sqrt{2}}{2}|\Lambda_2|} \frac{1}{\sqrt{(1-\frac{1}{8}(v'+\frac{\Lambda_2}{\sqrt{2}})^2)(1-\frac{1}{8}(v'-\frac{\Lambda_2}{\sqrt{2}})^2)}}\,\mathrm{d}v' \\
&\sim \frac{1}{|\Lambda_2|+4} K\left(\left(\frac{|\Lambda_2|-4}{|\Lambda_2|+4}\right)^2\right)
\end{aligned}
$$

$$
\therefore \rho_1(\lambda) = \frac{1}{2\pi\sqrt{1-\frac{1}{4}(\lambda+2)^2}} \qquad \rho_2(\lambda) = \frac{4\pi^{-2}}{|\lambda+4|+4} K\left(\left(\frac{|\lambda+4|-4}{|\lambda+4|+4}\right)^2\right) \tag{1.8}
$$