

On the affect of the Laplacian in equilibration dynamics of the Spherical Model

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Consider a system of N linearly interacting degrees of freedom. We collect them into a vector $\bar{\mathbf{s}}(t) = (s_1(t), \dots, s_N(t))$ and represent their interactions with random coupling matrix $\mathbf{J}(t)$. We subject them to a global constraint: lying on an N -dimensional sphere of radius N , enforced by lagrange multiplier μ . We write down the Langevin Equation of motion with external noise $\boldsymbol{\xi}(t)$, dropping the explicit time dependence for brevity.

$$\partial_t \bar{\mathbf{s}} = (\mathbf{J} - \mu) \bar{\mathbf{s}} + \boldsymbol{\xi} \quad (0.1)$$

$$\text{where } \mu = \frac{1}{N} \bar{\mathbf{s}}^\top (\mathbf{J} \bar{\mathbf{s}} + \boldsymbol{\xi}) \quad \text{enforces constraint } |\bar{\mathbf{s}}(t)|^2 = N \quad (0.2)$$

The following is an attempt to characterise the statistical differences between spatial and non-spatial interactions, and their effect on the equilibration dynamics by introducing the discretised Laplacian Δ as a diffusive term.

1 Analytical Solutions

Here we follow the methods used to obtain exact solutions[] in the field theoretic setting of the model.

$$\bar{\mathbf{u}}(t) = e^{\Lambda t} \mathbf{\Gamma}^{1/2} \bar{\mathbf{u}}(0)$$

1.1 Spectra of Discrete Laplacians

Using the Circular Diagonalization Theorem [1] one can derive the eigenvalues $\lambda_N(k)$ of an $N \times N$ matrix \mathbf{X} which represents the second-order central difference approximation to the second derivative along N sites of a one dimensional ring.

$$\mathbf{X} := \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix} \quad (1.1)$$

$$\begin{aligned} \lambda_N(k) &= 2 \left(\cos \left(\frac{2\pi k}{N} \right) - 1 \right) \\ k &\in \{0, 1, \dots, N-1\} \end{aligned} \quad (1.2)$$

As the number of sites $N \rightarrow \infty$ the argument $k/N \in [0, 1]$ and the eigenvalues remain bounded $-2 < \lambda < 0$. By shifting and scaling the index $k \rightarrow \frac{k-N\pi}{2\pi}$ the eigenvalues are expressed as the familiar dispersion relation [1].

$$\lambda(x) = -2(\cos x + 1) \quad x \in [-\pi, \pi] \quad (1.3)$$

The discrete M -dimensional laplacian is simply the kronecker sum of one dimensional cases $\Delta = \mathbf{X} \oplus \mathbf{X} \oplus \dots \oplus \mathbf{X}$ and thus its eigenvalues is simply the sum one dimensional dispersions [1].

$$\lambda(\bar{\mathbf{x}}) = -2 \sum_{x \in \bar{\mathbf{x}}} (\cos x + 1) \quad \bar{\mathbf{x}} \in [-\pi, \pi]^M \quad (1.4)$$

The probability density $\rho_M(\lambda)$ can be expressed as an integral over the M -dimensional hypercube region $\Omega = [-\pi, \pi]^M$ in complete analogue with the density of states.

$$\rho_M(\lambda') = \frac{1}{Z_M} \int_{\Omega} \delta(\lambda' - \lambda(\bar{\mathbf{x}})) d\bar{\mathbf{x}}$$

We proceed with an element-wise change of variables $\bar{\mathbf{u}} = 2 \cos \bar{\mathbf{x}}$ and recognise that the integration region is M -fold symmetric across each component axis, which allows restriction of the domain of integration to a hyperoctant. Given coordinates $\bar{\mathbf{u}}$ the region becomes $\Omega' = [-2, 2]^M$.

$$\begin{aligned}
\rho_M(\lambda) &= \frac{1}{Z_M} \int_{\Omega'} \frac{\delta(\Lambda_M + \sum_{u \in \bar{\mathbf{u}}} u)}{\sqrt{\prod_{u \in \bar{\mathbf{u}}} (1 - u^2/4)}} d\bar{\mathbf{u}} & \Lambda_M = \lambda + 2M \\
& & |\Lambda_M| \leq 2M \\
&= \frac{1}{2\pi Z_M} \int_{-\infty}^{\infty} \int_{\Omega'} \frac{e^{\Lambda_M \mathfrak{i}k} \exp[\sum_{u \in \bar{\mathbf{u}}} u \mathfrak{i}k]}{\sqrt{\prod_{u \in \bar{\mathbf{u}}} (1 - u^2/4)}} d\bar{\mathbf{u}} dk \\
&= \frac{1}{2\pi Z_M} \int_{-\infty}^{\infty} e^{\Lambda_M \mathfrak{i}k} \prod_{u \in \bar{\mathbf{u}}} \int_{-2}^2 \frac{e^{u \mathfrak{i}k}}{\sqrt{1 - u^2/4}} du dk
\end{aligned}$$

The fourier representation of the delta function allowed the factorisation of the integral. We recognise a repeated Bessel integral and replace it with the Bessel function of the first kind $J_n(k)$, leaving only a fourier transform which we define $\mathcal{F} : f \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k) e^{\mathfrak{i}\Lambda k} dk$. To clean the formula up even further we may use the convolution theorem to deal with the powers of M , leaving only the fourier transform of the Bessel function $J_0(k)$, which is the arcsine distribution $\alpha(\lambda)$. It becomes clear that the eigenvalue density of a kronecker sum of matrices is the convolution of the densities of those matrices.

$$\rho_M(\lambda) = \underbrace{\alpha(\lambda) * \alpha(\lambda) * \dots * \alpha(\lambda)}_M \quad (1.5)$$

$$\alpha(\lambda) = \frac{1}{2\pi \sqrt{1 - (\frac{\lambda+2}{2})^2}} \Pi\left(\frac{\lambda+2}{2}\right) \quad \Pi(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} \quad (1.6)$$

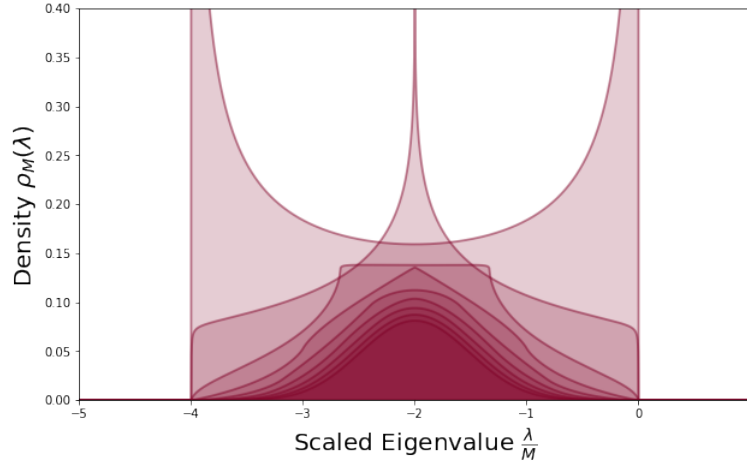


Figure 1: Eigenvalue distributions $\rho_M(\lambda)$ of the M -dimensional Laplacian

The one dimensional density has two Van Hove singularities at $\lambda = -4, 0$ given by the arcsine law $\alpha(\lambda)$, whereas the two dimensional case has one at $\lambda = -4$ given by the complete elliptic integral of the first kind $K(m)$. Figure 1 reveals that in higher dimensions singularities do not occur; instead there appear to be discontinuities in the higher order derivatives. The density smooths out as repeated convolutions bring it to a normal distribution; this is another way to state the Central Limit Theorem [1].

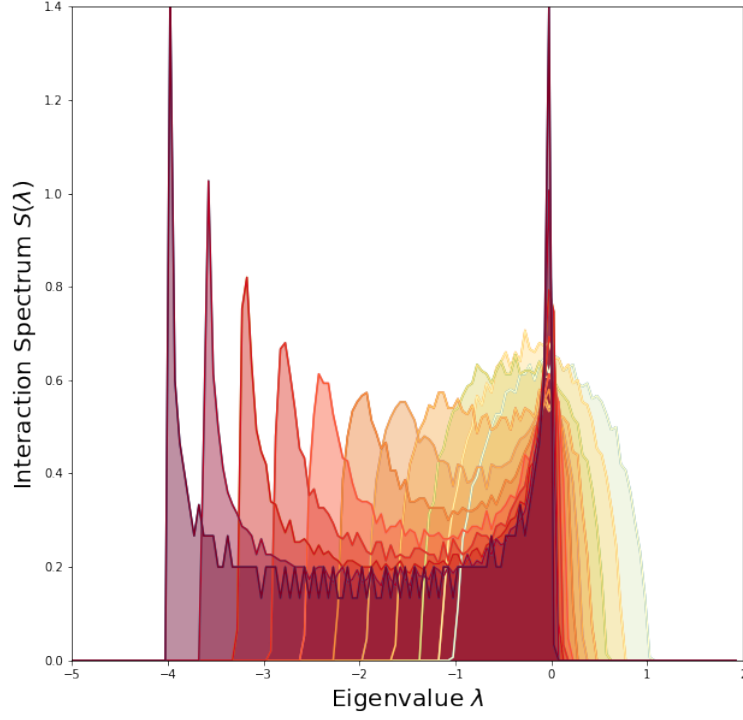


Figure 2: Eigenvalue distributions $\rho_M(\lambda)$ of the M -dimensional Laplacian