On the affect of the Laplacian in equlibration dynamics of the Spherical Model

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March 3, 2018

Consider a system of N linearly interacting degrees of freedom. We collect them into a vector $\bar{\mathbf{s}}(t) = (\, s_1(t), \cdots, s_N(t)\,)$ and represent their interactions with random coupling matrix $\mathbf{J}(t)$. We subject them to a global constraint: lying on an N-dimensional sphere of radius N, enforced by lagrange multiplier μ. We write down the Langevin Equation of motion with external noise $\boldsymbol{\xi}(t)$, dropping the explicit time dependence for brevity.

$$\partial_t \bar{\mathbf{s}} = (\mathbf{J} - \mu)\bar{\mathbf{s}} + \boldsymbol{\xi} \tag{0.1}$$

$$\text{where} \quad \mu = \frac{1}{N}\bar{\mathbf{s}}^\top (\mathbf{J}\bar{\mathbf{s}} + \boldsymbol{\xi}) \quad \text{enforces constraint} \quad |\bar{\mathbf{s}}(t)|^2 = N \tag{0.2}$$

The following is an attempt to characterise the statistical differences between spatial and non-spatial interactions, and their effect on the equilibration dynamics by introducting the discretised Laplacian Δ as a diffusive term.

1 Analytical Solutions

Here we follow the methods used to obtain exact solutions[] in the field theoretic setting of the model.

$$\bar{\mathbf{u}}(t) = \mathbb{e}^{\boldsymbol{\Lambda} t}\boldsymbol{\Gamma}^{1/2}\bar{\mathbf{u}}(0) + \int^t \mathbb{e}^{\boldsymbol{\Lambda}(t-t')\boldsymbol{\Gamma}^{1/2}(t')\boldsymbol{\Gamma}^{1/2}(t)}\boldsymbol{\xi}(t')\,\mathrm{d}t'$$

1.1 Spectra of Discrete Laplacians

Using the Circular Diagonalization Theorem [] one can derive the eigenvalues $\lambda_N(k)$ of an $N \times N$ matrix $\mathbf{X}$ which represents the second-order central difference approximation to the second derivative along N sites of a one dimensional ring.

$$\mathbf{X} := \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix} \tag{1.1}$$

$$\begin{gathered} \lambda_N(k) = 2\left(\cos\left(\tfrac{2\pi k}{N}\right) - 1\right) \\ k \in \{0, 1, \cdots, N-1\} \end{gathered} \tag{1.2}$$

As the number of sites $N \to \infty$ the argument $k/N \in [0,1]$ and the eigenvalues remain bounded $-2 < \lambda < 0$. By shifting and scaling the index $k \to \frac{k - N\pi}{2\pi}$ the eigenvalues are expressed as the familiar dispersion relation [].

$$\lambda(x) = -2\left(\cos x + 1\right) \quad x \in [-\pi, \pi] \tag{1.3}$$

The discrete M-dimensional laplacian is simply the kronecker sum of one dimensional cases $\mathbf{\Delta} = \mathbf{X} \oplus \mathbf{X} \oplus \cdots \oplus \mathbf{X}$ and thus its eigenvalues is simply the sum one dimensional dispersions [].

$$\lambda(\bar{\mathbf{x}}) = -2 \sum_{x \in \bar{\mathbf{x}}} \left(\cos x + 1\right) \quad \bar{\mathbf{x}} \in [-\pi, \pi]^M \tag{1.4}$$

The probability density $\rho_M(\lambda)$ can be expressed as an integral over the M-dimensional hypercube region $\Omega = [-\pi, \pi]^M$ in complete analogue with the density of states.

$$\rho_M(\lambda') = \frac{1}{Z_M} \int_\Omega \delta(\lambda' - \lambda(\bar{\mathbf{x}}))\, \mathrm{d}\bar{\mathbf{x}}$$

We proceed with an element-wise change of variables $\bar{\mathbf{u}} = 2\cos\bar{\mathbf{x}}$ and recognise that the integration region is M-fold symmetric across each component axis, which allows restriction of the domain of integration to a hyperoctant. Given coordinates $\bar{\mathbf{u}}$ the region becomes $\Omega' = [-2, 2]^M$.

$$\begin{aligned}\rho_M(\lambda) &= \frac{1}{Z_M}\int_{\Omega'}\frac{\delta(\Lambda_M+\sum_{u\in\bar{\mathbf{u}}}u)}{\sqrt{\prod_{u\in\bar{\mathbf{u}}}(1-u^2/4)}}\,\mathrm{d}\bar{\mathbf{u}} \qquad \begin{matrix}\Lambda_M=\lambda+2M\\ |\Lambda_M|\le 2M\end{matrix}\\ &= \frac{1}{2\pi Z_M}\int_{-\infty}^{\infty}\int_{\Omega'}\frac{\mathbb{e}^{\Lambda_M \mathbb{i}k}\exp[\sum_{u\in\bar{\mathbf{u}}}u\mathbb{i}k]}{\sqrt{\prod_{u\in\bar{\mathbf{u}}}(1-u^2/4)}}\,\mathrm{d}\bar{\mathbf{u}}\mathrm{d}k\\ &= \frac{1}{2\pi Z_M}\int_{-\infty}^{\infty}\mathbb{e}^{\Lambda_M\mathbb{i}k}\prod_{u\in\bar{\mathbf{u}}}\int_{-2}^{2}\frac{\mathbb{e}^{u\mathbb{i}k}}{\sqrt{1-u^2/4}}\,\mathrm{d}u\mathrm{d}k\end{aligned}$$

The fourier representation of the delta function allowed the factorisation of the integral. We recognise a repeated Bessel integral and replace it with the Bessel function of the first kind $J_n(k)$, leaving only a fourier transform which we define $\mathcal{F} : f \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(k)e^{\mathbb{i}\Lambda k}\mathrm{d}k$. To clean the formula up even further we may use the convolution theorem to deal with the powers of M, leaving only the fourier transform of the Bessel function $J_0(k)$, which is the arcsine distribution $\alpha(\lambda)$. It becomes clear that the eigenvalue density of a kronecker sum of matrices is the convolution of the densities of those matrices.

$$\rho_M(\lambda) = \underbrace{\alpha(\lambda)*\alpha(\lambda)*\cdots*\alpha(\lambda)}_{M} \tag{1.5}$$

$$\alpha(\lambda) = \frac{1}{2\pi\sqrt{1-\left(\frac{\lambda+2}{2}\right)^2}}\Pi\left(\frac{\lambda+2}{2}\right) \qquad \Pi(x) = \begin{cases}1 & |x|<1\\ 0 & |x|\ge 1\end{cases} \tag{1.6}$$

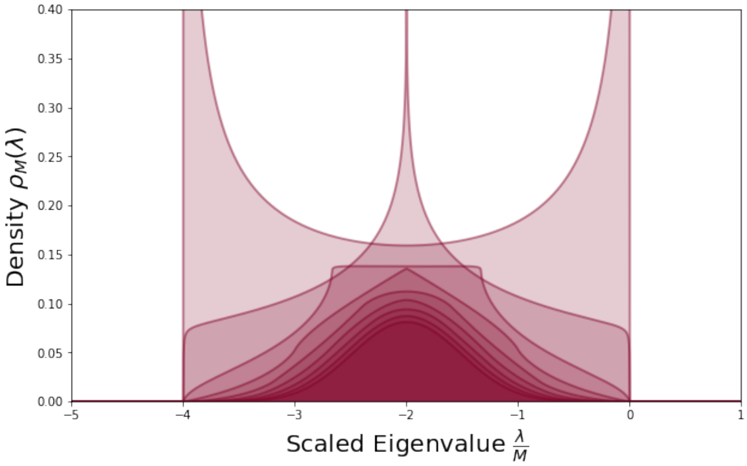

Figure 1: Eigenvalue distributions $\rho_M(\lambda)$ of the M-dimensional Laplacian

The one dimensional density has two Van Hove singularities at $\lambda = -4, 0$ given by the arcsine law $\alpha(\lambda)$, whereas the two dimensional case has one at $\lambda = -4$ given by the complete elliptic integral of the first kind $K(m)$. Figure 1 reveals that in higher dimensions singularities do not occur; instead there appear to be discontinuities in the higher order derivatives. The density smooths out as repeated convolutions bring it to a normal distrubtion; this is another way to state the Central Limit Theorem [].

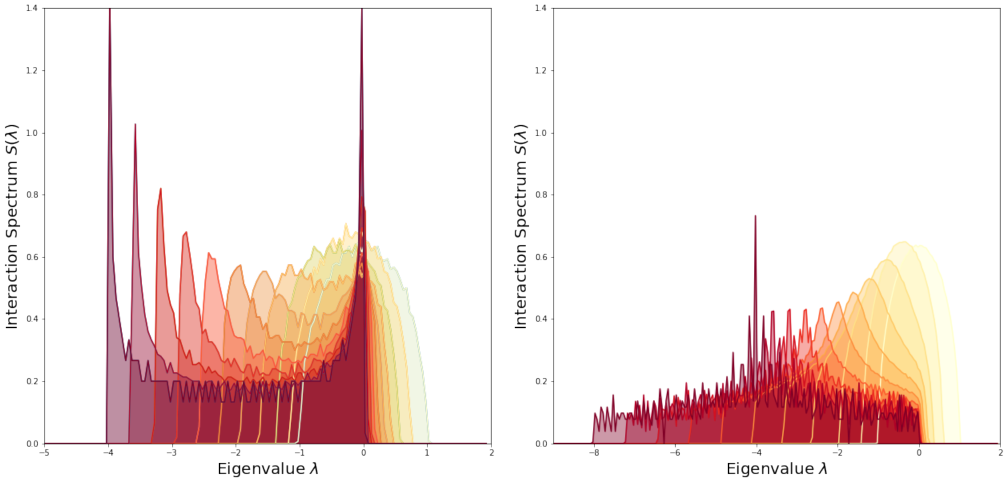

Figure 2: Left/Right: Random interaction spectra $S(\lambda|\mu)$ for variable values of the order parameter μ which introduces one/two dimensional laplacian