Analogous Dynamics of Boolean Network

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Analogous dynamics

1. Different Boolean networks $\rightarrow$ similar dynamics
2. From the observation standpoint they behave similarly.
3. Possibly, different predictions for unobserved behaviours due to the lack of facts discriminating them.

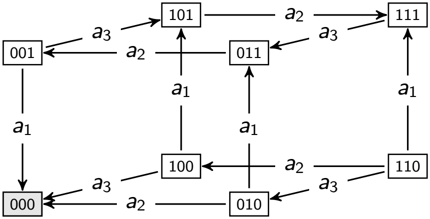
Analogy ?

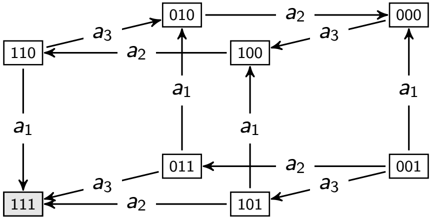
1. Equivalence on trajectories
2. Formalized by isomorphism on model dynamics:

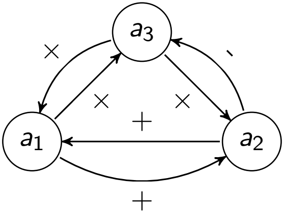
Two networks behave analogously if and only if
their model (of dynamics) is isomorphic.

Example on Boolean Networks

$$f_{\{a_3,a_2,a_1\}} = \begin{cases} f_{a_1} = a_3 + a_2 \\ f_{a_2} = a_1 . a_3 \\ f_{a_3} = a_1 . \overline{a}_2 \end{cases} \qquad f_{\{a_3,a_2,a_1\}} = \begin{cases} f_{a_1} = a_3 . a_2 \\ f_{a_2} = a_1 + a_3 \\ f_{a_3} = a_1 + \overline{a}_2 \end{cases}$$







Objectives

- Formalization of the class of equivalence of analogous Boolean networks.
- Sensitivity to the variation of the updating policy (mode) .
- Structural invariant properties of a class.

Boolean Network & Dynamics

Evolution function

An evolution function is a boolean function on states, $f : \mathbb{B}^{|A|} \to \mathbb{B}^{|A|}$ defined as a sequence of propositional formulas.

Example (Evolution function)

$$f_A = (a_4, a_4 + a_2, \bar{a}_3, a_2), A = \{a_4, a_3, a_2, a_1\}$$

Definition

$f_{A'}, A' \subseteq A$ is a sub-function whose co-domain is restricted to states of A' *i.e.*, $f_{A'} : \mathbb{B}^{|A|} \to \mathbb{B}^{|A'|}$.

Example

$$f_A = (a_4, a_4 + a_2, \bar{a}_3, a_2).$$

$$f_{\{a_3, a_1\}} = (a_4 + a_2, a_2).$$

Interaction

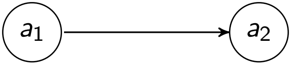
- interaction = causal relation on two agents.
- interaction relates to dynamics.
- An agent interacts with another if and only if the variation of the former leads to the variation of the latter for some states.

Definition

Let f_A be an evolution function, we define an interaction on agents as:

$$a_i \longrightarrow a_j \stackrel{\text{def}}{=} \exists s_1, s_2 \in \mathbb{B}^n : s_1(a_i) \neq s_2(a_i) \wedge s_1(A \setminus a_i) = s_2(A \setminus a_i) \wedge f_{a_j}(s_1) \neq f_{a_j}(s_2).$$

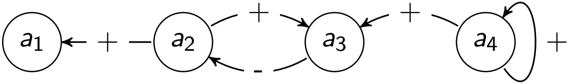
Interaction graph



Signed interaction graph

$$f_A(a_4, a_3, a_2, a_1) = (a_4, a_4 + a_2, \bar{a}_3, a_2).$$

Sign = monotonous property: **+** increasing, - decreasing



Remark

The signed interaction graph can be directly deduced from f_A where proposition are in sound disjunctive normal form.

Mode and modalities

- Modality: set of agents updated jointly.
- Mode: set of modalities.

Some frequent modes

- Asynchronous mode: $W = \{\{a_n\}, \ldots, \{a_1\}\}$
  the state of one agent only is updated by a transition.
- Parallel/Synchronous mode: $W = \{\{a_n, \ldots, a_1\}\}$
  The states of all the agents is updated.
- Generalized mode: $W = \mathbf{2}^A \setminus \{\emptyset\}$
  All the possible combinations of update.

Definition (Boolean Network = Evolution function + Mode)

$$\langle f_A, W \rangle, \bigcup_{w_i \in W} w_i \subseteq A$$

Spectrum

The spectrum of a mode, $\textbf{spx}\, W$ = multi-set representing the number of modalities with the same cardinalities.

Example

$$\textbf{spx}\, \{\{a_1\}, \{a_2\}, \{a_3, a_4\}, \{a_5, a_6, a_7\}\} = \{2 \bullet 1, 1 \bullet 2, 1 \bullet 3\}$$

Definition (Regular mode)

Partition of the agents A into modalities with same cardinality m.

$$\textbf{spx}\, W = \{k \bullet m\}, km = |A|.$$

Example (Regular modes of $\{a_6, a_5, a_4, a_3, a_2, a_1\}$)

- $\textbf{spx}\, \{\{a_6\}, \{a_5\}, \{a_4\}, \{a_3\}, \{a_2\}, \{a_1\}\} = 6 \bullet 1$
- $\textbf{spx}\, \{\{a_6, a_5\}, \{a_4, a_3\}, \{a_2, a_1\}\} = 3 \bullet 2$
- $\textbf{spx}\, \{\{a_6, a_5, a_4\}, \{a_3, a_2, a_1\}\} = 2 \bullet 3$
- $\textbf{spx}\, \{\{a_6, a_5, a_4, a_3, a_2, a_1\}\} = 1 \bullet 6$

Model

The dynamics associated to network is modelled by a labelled transition system (LTS) where:

- Labels are modalities (set of agents updated jointly).
- Transitions describes the evolution w.r.t. the modality (label).

Definition (Model)

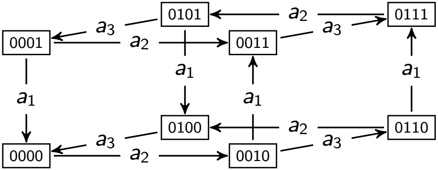
Let f_A be an evolution function for a set A of agents, a LTS $\mathcal{M} = \langle S, W, \longrightarrow \rangle$ models $N = \langle f_A, W \rangle$, $\mathcal{M} \models N$, if and only if:

- $S = \mathbf{dom}\ f_A = \mathbb{B}^{|A|}$;
- $\forall w \in W, \forall s_1, s_2 \in S : s_1 \xrightarrow{w} s_2 \iff$
  $$s_2(w) = f_w(s_1) \wedge s_2(A \setminus w) = s_1(A \setminus w) \wedge s_1 \neq s_2.$$

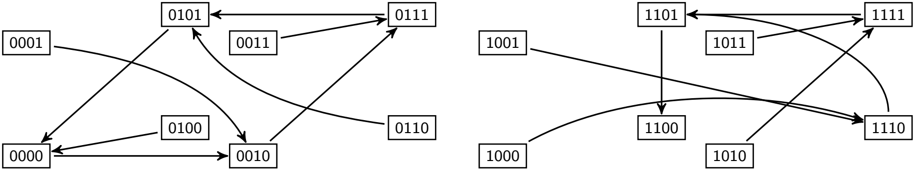
4 agents example

$$f_A = (a_4, a_4 + a_2, \bar{a}_3, a_2).$$

Asynchronous mode: $\{\{a_4\}, \{a_3\}, \{a_2\}, \{a_1\}\}$



Parallel mode : $\{\{a_4, a_3, a_2, a_1\}\}$



Equilibrium

Equilibria = asymptotic states.

Definition (Equilibrium)

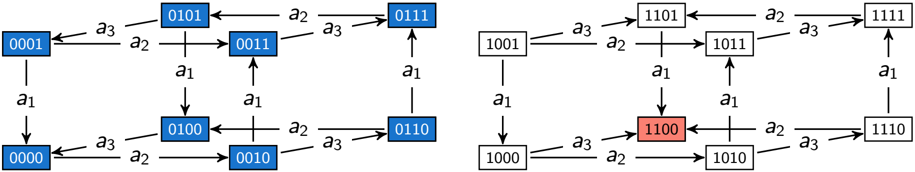
Let $\langle S, W, \longrightarrow \rangle$ be a LTS, a state s_1 is an equilibrium if and only if it is infinitely often reached once met, that is:

$$Eq(s_1) \stackrel{\text{def}}{=} \forall s_2 \in S : s_1 \longrightarrow^* s_2 \implies s_2 \longrightarrow^* s_1.$$

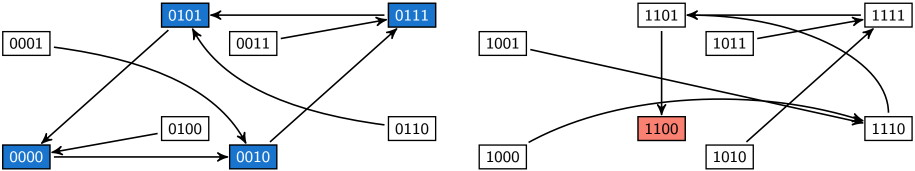
4 agents example - equilibrium

$$f_A = (a_4, a_4 + a_2, \bar{a}_3, a_2).$$

Asynchronous mode



Parallel mode



Attractors in Blue, Stable states in Red

Model isomorphisms

Model isomorphism

Two models $\mathcal{M} = \langle \mathbb{B}^n, W, \longrightarrow \rangle$ and $\mathcal{M}' = \langle \mathbb{B}^n, W', \longrightarrow' \rangle$ are isomorphic, $\mathcal{M} \simeq \mathcal{M}'$, if and only if there exists a bijection $\varphi : \mathbb{B}^n \to \mathbb{B}^n$ preserving the transitions:

$$\forall s_1, s_2 \in S : s_1 \longrightarrow s_2 \iff \varphi(s_1) \longrightarrow' \varphi(s_2).$$

Remark

- Preserve the trajectories but differ on states.
- The equilibria of isomorphic models are structurally identical.

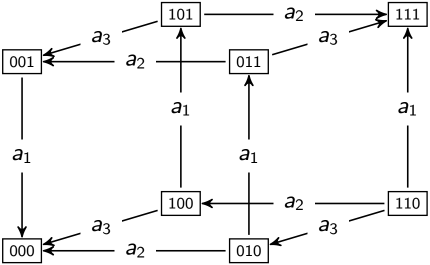
Model isomorphism

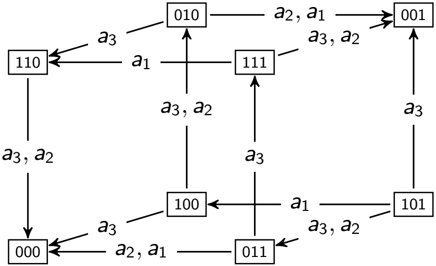
Bad News 1

The models are not closed under Boolean isomorphism.

No mode can be found for the LTS in r.h.s!

$f_{\{a_3,a_2,a_1\}} = (a_1.a_2, a_1.a_3, a_2 + a_3)$ $\qquad$ $\varphi = (001\,110\,0101\,010\,011\,111)$

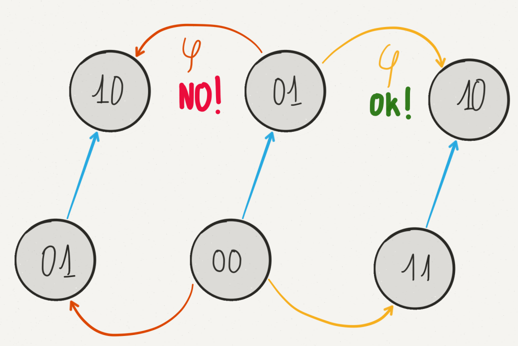




Check whether a mode exists for a LTS isomorphic to a model.

Isomorphism preserving the mode

$$\mathbf{md}\ \mathcal{M} = \mathbf{md}\ \mathcal{M}^{\varphi}$$



The models are closed by isomorphisms because a mode exists !

Proposition

The set of isomorphisms preserving a mode forms a group (or a spectrum).

Groups of permutation

- Boolean permutations on $\mathbb{B}^l, l \in \mathbb{N} = \mathsf{S}_{\mathbb{B}^l}$, (notation β)
- Integer permutations on $[\![l]\!] = \{1, \ldots, l\}, l \in \mathbb{N} = S_l$ (notation π)
- Signed permutations of rank $l, l \in \mathbb{N}$ denoted BC_l. (notation σ)
- $\varphi =$ isomorphism preserving the mode.
- The action is suffixed: $\varphi(x)$ is denoted x^φ.

A permutation: factorization in disjoint cycles
where each cycle, $(c_1 \ \ldots \ c_k) = \{c_1 \mapsto c_2, \ldots, c_k \mapsto c_1\}$

Example

$\{1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 4, 4 \mapsto 3, 5 \mapsto 5\} \equiv (1\,2)(3\,4)$.

Group of signed permutations

Definition (Group of signed permutations)

BC_I is the group of signed permutations of rank n such that:

$$\sigma(-i) = -\sigma(i), i \in \{-n, \dots, -1, 1, \dots, n\}$$

Definition (Action of a signed permutation on $\mathbb{B}^n$)

$\sigma = (p, \pi_\sigma)$, with $\pi_\sigma(i) = |\sigma(i)|$ and $p \in \mathbb{B}^n, p(i) = \sigma(i) < 0$.
The action is defined as:

$$c = (b \oplus p)^{\pi_\sigma} \qquad \text{where } c_i = b[\pi_\sigma^{-1}(i)] \oplus p[\pi_\sigma^{-1}(i)], i \in [\![n]\!].$$

Example

- $\sigma = \{1 \mapsto -2, 2 \mapsto 1, -1 \mapsto 2, -2 \mapsto -1\} = (p = (0, 1), \pi = (1\,2))$.
- $(1, 0)^\sigma = (1, 1)$.
- bit inversion in $\mathbb{B}^3$: $\sigma = ((1, 1, 1), e_{S_3})$.

Characterization of Groups preserving the mode

Group of isomorphisms preserving the mode whatever the model.

$$\forall \mathcal{M}, \forall \varphi \in S_? : \textbf{md } \mathcal{M} = \textbf{md } \mathcal{M}^{\varphi}.$$

1. Method: illustrated for the group preserving the Asynchronous mode.
2. Basic bricks: Generalization to group preserving a regular mode.
3. Result: Extension to group preserving mode forming a partition of the agents.

Remark

- The groups are defined up to an isomorphism on groups.
- Essentially the action matters.

Method for regular modes

Two steps

1. Collect all the models of a particular mode with the same number of agents $\rightarrow$ Graph union of models.
2. Find the group of automorphisms of this graph $\rightarrow$ group preserving the regular mode.

Remark

- The automorphisms preserve the structure of the graph by definition.
- Hence the action on a model corresponding to a sub-graph necessary maps in another model (sub-graph).
- if an isomorphism does not preserve the mode then it is not an automorphism of the graph union.

Application to the asynchronous mode (n agents)

1. Graph union of models
    - By definition of the asynchronous mode and a model: $s \longrightarrow s' \implies s$ differs to s' in 1 position only.
    - Hence collecting all the models with the asynchronous mode corresponds the hypercube of dimension n, Q_n
2. Group preserving the asynchronous mode.
    - The group of automorphisms of the hypercube of dimension n is known to be isomorphic to the signed permutations of rank n [1].
    - the result generalizes those presented in [3].

Lemma

The group of the signed permutations of rank n is isomorphic to the group preserving the asynchronous mode: $\mathrm{Aut}(Q_n) \simeq BC_n \simeq SRM_1^n$.

Characterization of SRM_k^m $(k = \frac{n}{m})$

Union of models = complete modal graph of W, KM_W

$V(KM_W) = \mathbb{B}^n$ and,
$E(KM_W) = \{(b_1, b_2) \mid b_1[w] \neq b_2[w] \wedge b_1[A \backslash \{w\}] = b_2[A \backslash \{w\}], w \in W\}$.

Group preserving the regular mode

Proposition

Let W be a regular mode of length m, the complete modal graph KM_W is isomorphic to $\frac{n}{m}$ products of complete graphs with 2^m vertices: $K_{2^m}^{\square \frac{n}{m}}$.

Lemma

The wreath product of $\mathrm{S}_{\mathbb{B}^m} \wr S_k, k = \frac{n}{m}$ is isomorphic to the group preserving the regular mode of length m:

$$SRM_m^k \simeq \mathrm{Aut}((K_{2^m})^{\square \frac{n}{m}}) \simeq \mathrm{S}_{\mathbb{B}^m} \wr S_k$$

The order is $(2^m!)^k\, k!$ (order of $S_l \wr S_k = (l!)^k k!$.)

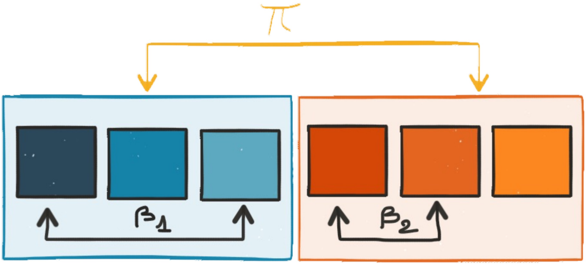
Action of $\varphi \in SRM_m^k$ in $\mathbb{B}^n$

$\varphi \in SRM_m^k : \varphi = (\beta, \pi)$ where:

- $\beta : [\![k]\!] \to \mathrm{S}_{\mathbb{B}^m}$, such that $\beta_i \in \mathrm{S}_{\mathbb{B}^m}$ acts on $b[w_i]$;
- $\pi : [\![k]\!] \to [\![k]\!]$, exchange the position of the modalities,

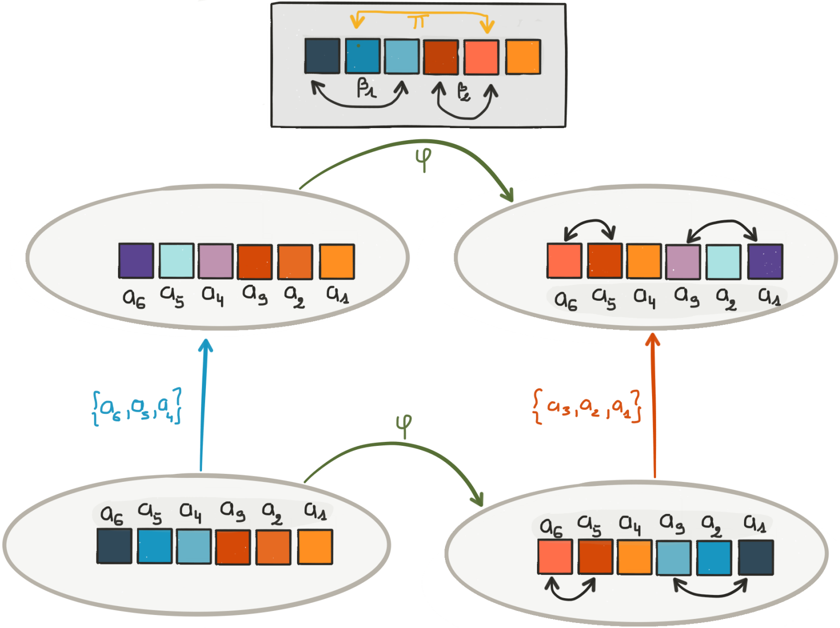
Permutations β_i local to modalities + permutation π on blocks of modalities

$$c = b^{\varphi} \text{ where } c = \left(b[w_{\pi^{-1}(1)}]^{\beta_{\pi^{-1}(1)}}, \dots, b[w_{\pi^{-1}(k)}]^{\beta_{\pi^{-1}(k)}} \right), b, c \in \mathbb{B}^n.$$



$$\{\{a_6, a_5, a_4\}, \{a_3, a_2, a_1\}\}, \varphi = ((\beta_1, \beta_2), \pi) \in SRM_3^2.$$

Application



Remark on regular mode

- Asynchronous mode: the group $\mathsf{S}_{\mathbb{B}} \wr S_n$ is isomorphic to the group of the signed permutations BC_n,

$$\mathsf{S}_{\mathbb{B}} \wr S_n \simeq S_2 \wr S_n \simeq BC_n \text{ [2]}.$$

- Parallel mode: The group of isomorphisms preserving the parallel mode SRM_n^1 is obviously isomorphic to the group of the Boolean permutations on $\mathbb{B}^n$, $\mathsf{S}_{\mathbb{B}^n}$,

$$\mathsf{S}_{\mathbb{B}^n} \wr S_1 = \mathsf{S}_{\mathbb{B}^n} .$$

Group of isomorphisms preserving partition, $SP_{\mathbf{spx}\, W}$

Remark

mode = partition of A = union of regular sub-modes
$W = \bigcup_{i=1}^{l} W_i$ such that $\mathbf{spx}\, W = \{k_i \bullet m_i\}_{1 \leq i \leq l}, \mathbf{spx}\, W_i = k_i \bullet m_i$.

Theorem

The set of isomorphisms on models preserving a mode partitioning the agents with a spectrum $\{k_i \bullet m_i\}_{1 \leq i \leq l}$, is the Cartesian product of their regular modes:

$$SP_\gamma = \bigtimes_{i=1}^{l} SRM_{m_i}^{k_i}, SRM_{m_i}^{k_i} \simeq \mathsf{S}_{\mathbb{B}^{m_i}} \wr S_{k_i}.$$

the order of this group is $\prod_{i=1}^{l} (2^{m_i}!)^{k_i}\, k_i!$.

Action of $SP_{\mathbf{spx}\,W}$ on transitions

$$\varphi \in SP_{\mathbf{spx}\,W}, \varphi = (\beta, \pi).$$

- β: a family $\beta = \{\beta_i\}$ of Boolean permutations on modalities s.t. β_i acts on $b(w_i)$.
- π: a permutation on modalities restricted to modalities with the same cardinalities.

$$\forall w_i \in W, \forall s_1, s_2 \in \mathbb{B}^n : (s_1 \xrightarrow{w_i} s_2)^\varphi = s_1^\varphi \xrightarrow{w_{\pi(i)}} s_2^\varphi.$$

Remark

The action can be also defined on evolution functions for the asynchronous mode.

Equivalence on networks

Equivalence on Networks

1. Analogy/Similarity on the behaviours of networks:
   - same mode;
   - different evolution functions.
2. Two networks N, N' are dynamically equivalent, $N \sim N'$, if and only if their model is isomorphic (w.r.t $\varphi \in SP$).

Definition (Dynamical Network Equivalence)

$$N \sim N' \stackrel{\text{def}}{=} \textbf{md } N = \textbf{md } N' \wedge (\exists \varphi \in SP_{\textbf{spx md } N} : \mathcal{M} \models N \wedge \mathcal{M}^{\varphi} \models N').$$

Objectives

- Structural invariance property of equivalent networks:
  characterization of a structure abstracting a network such that these structures are isomorphic for equivalent networks.
- Conditions for preserving the equivalence across different modes:
  properties insuring that an equivalence found for a mode is also preserved for another.

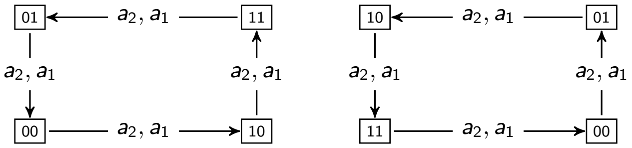
Invariance

Bad News 2

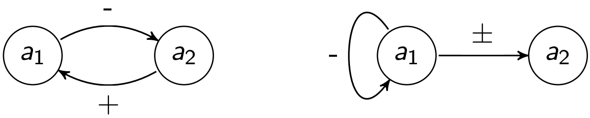
The interaction graph of equivalent networks is not invariant.

$$f_{\{a_2,a_1\}} = \begin{cases} f_{a_1} = a_2 \\ f_{a_2} = \overline{a_1} \end{cases} \qquad f'_{\{a_2,a_1\}} = \begin{cases} f'_{a_1} = \overline{a_1} \\ f'_{a_2} = a_1 \oplus a_2 \end{cases}$$

Parallel mode $\{\{a_2, a_1\}\}$ $\qquad \varphi = (00\ 11\ 01\ 10)$



Interaction graph



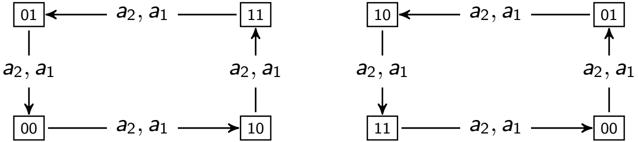
Preserving the equivalence over mode variation

Bad News 3

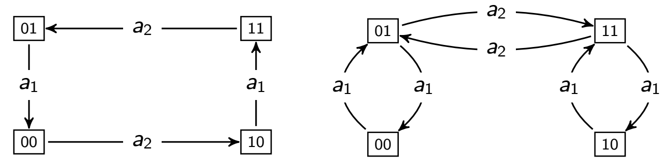
The equivalence on networks is not preserved when the mode varies.

$$f_{\{a_2,a_1\}} = \begin{cases} f_{a_1} = a_2 \\ f_{a_2} = \overline{a_1} \end{cases} \qquad f'_{\{a_2,a_1\}} = \begin{cases} f'_{a_1} = \overline{a_1} \\ f'_{a_2} = a_1 \oplus a_2 \end{cases}$$

Parallel mode: $\{\{a_2, a_1\}\}$



Asynchronous mode: $\{\{a_2\}, \{a_1\}\}$



Structural Invariance

Lemma

The interaction modal graph of equivalent networks are isomorphic:

$$\langle f, W\rangle \sim \langle f', W\rangle \implies G_f/W \simeq G_{f'}/W.$$

interaction modal graph, G_f/W

Quotient graph of the unsigned interaction graph w.r.t. mode

- vertices = modalities
- edges = quotient relation of the interactions w.r.t. a mode.

Definition (Interaction modal graph)

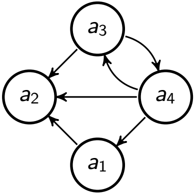
Let $G_f = \langle A, \longrightarrow\rangle$ be an interaction graph of f and W a mode partitioning A, G_f/W, is defined as:

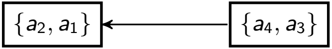
- $V(G_f/W) = W$;
- $E(G_f/W) = \{(w_i, w_j) \mid w_i \neq w_j \wedge (\exists a_i \in w_i, \exists a_j \in w_j : a_i \longrightarrow a_j)\}$.

Exhaustive analysis of all the equivalent networks

Germ/Representative: $f = (a_3, a_4, a_1.a_3 + a_4, a_4)$.
$W = \{\{a_4, a_3\}, \{a_2, a_1\}\}$, Regular group SRM_2^2 (order $= 1152$).

Interaction modal graphs





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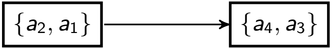

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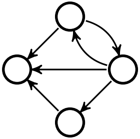
Patterns of interaction graphs for all equivalent functions.



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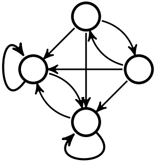

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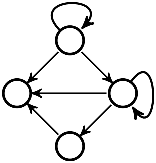

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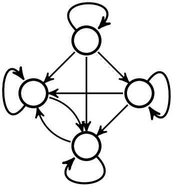

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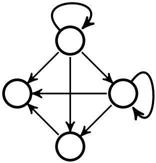

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Consequence on Asynchronous Mode

Interaction Modal Graph = Unsigned Interaction Graph.

Remark

Unsigned interaction graphs of equivalent networks are isomorphic and the sign of the interactions can be computed.

Lemma (Sign computation)

Let $\langle f, \{\{a_i\}\}_{a_i \in A}\rangle, \langle f', \{\{a_i\}\}_{a_i \in A}\rangle$ be two networks equivalent for the asynchronous mode with respect to the signed permutation $(p, \pi) \in SRM_1^n$, we have: $G_{f'} = \mu(G_f, (p, \pi))$ with:

$$\mu(a_i \xrightarrow{x} a_j, (p, \pi)) = a_{\pi(i)} \xrightarrow{x(-1)^{p_{\pi(i)}+p_{\pi(j)}}} a_{\pi(j)}, \quad p \in \mathbb{B}^n, \pi \in S_n .$$

Remark

The isomorphic cycles are of the same sign but not necessary the interactions.

Equivalence on mode variation

The equivalence on network is preserved if a mode is embedded in another.

- A mode W is embedded in another W' if and only if each modality of W is entirely included in a modality of W'.

$$\forall w \in W, \exists w' \in W' : w \subseteq w',\ W, W' \text{ are partitions of A.}$$

- The property can be generalized for an embedding involving a permutation on modalities ($\pi-$embedding).

Remark

- Asynchronous mode An equivalence found with the asynchronous mode also holds for any mode since the asynchronous mode is embedded in all modes.
- Parallel mode An equivalence found for any mode also holds for the parallel mode since the parallel mode embeds all modes.

Conclusion & Perspective

Conclusion

- Definition of group of isomorphism preserving the partitioned mode.
- Structural invariance of Interaction Modal Graph.
- condition for mode variation preserving the equivalence.

Properties on Boolean network vs. Properties on classes.

- Fact: Equilibria of equivalent networks are the same in structure and numbers.
- Thus some properties for a canonical representative are naturally extended to all equivalent networks.
- Canonicity of a network: an equivalent network with the smallest number of negative arcs.
- Proofs/properties on the canonical form $\rightarrow$ obvious extension to a class.
- Measure of complexity of architecture: number of equivalence classes of networks with the same unsigned interaction graph (architecture).

Canonicity

Bad News ... but the last

For Asynchronous mode finding a canonical form of a network seems to be NP-Complete.

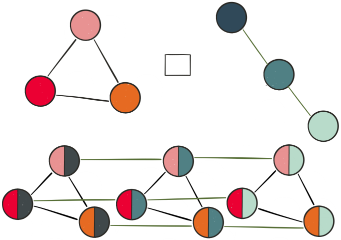
NP-Completeness by reduction to MAX-CUT.

However, it still remains an interesting notion for properties discovery !

Graph Product

Definition (Cartesian product $G_1 \square G_2$)

$$V(G_1 \square G_2) = V(G_1) \times V(G_2) \text{ and,}$$
$$E(G_1 \square G_2) = \begin{array}{l} \{(v_1 v_2, v_1' v_2) \mid (v_1, v_1') \in E(G_1)\} \cup \\ \{(v_1 v_2, v_1 v_2') \mid (v_2, v_2') \in E(G_2)\}. \end{array}$$



Remark

A graph is prime if it is decomposed as a product of trivial graphs only, *i.e.*, G is prime if and only if: $G = G_1 \square G_2$ implies that $G_1 = G$ or $G_2 = G$

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