Integration of differential equations

Let's take a differential equation of the form:

$$\frac{dY}{dx} = f(x, y)$$

Simpler discretization:

$$\frac{Y_{n+1} - Y_n}{h} = f(x_n, y_n)$$

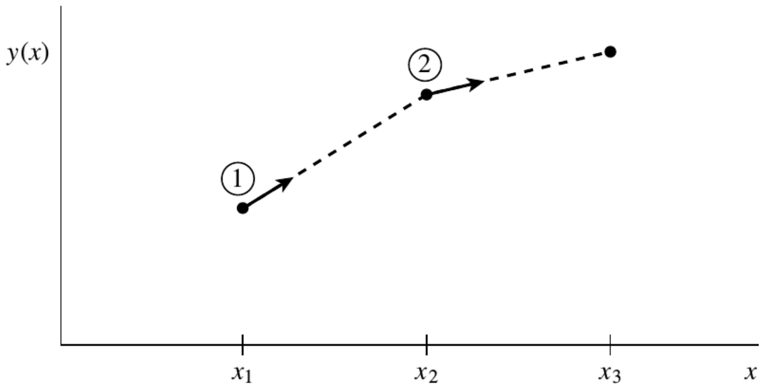
where h is the integration step, and we can advance the solution from n to n+1 using:

$$y_{n+1} = Y_n + hf(x_n, y_n)$$

This is the **Euler method.**

Problems:

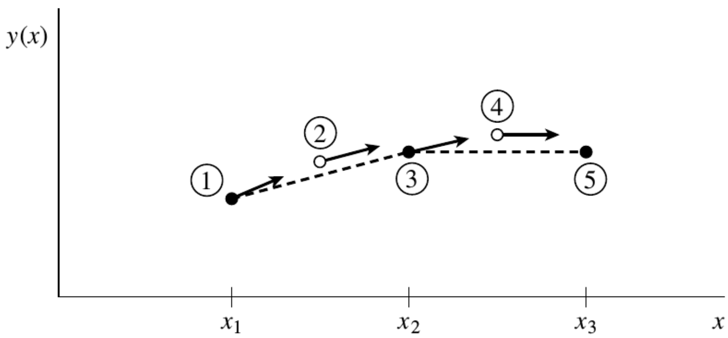
- uses the derivatives information only at the beginning of the integral
- non simmetric
- unstable
- $O(h^2)$



Midpoint method:

Let's use a "trial" step in the middle point of the integration step:

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}\right.$$
$$y_{n+1} = y_n + k_2 + O(h^3)$$

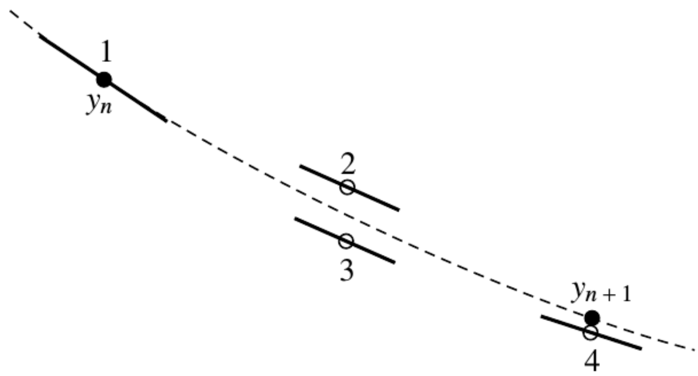


This is the **second-order Runge-Kutta** method.

More stable than Euler and uses more information on the derivatives.

Clearly, higher orders can be used. One of the most used is the **fourth-order RK**:

$$
\begin{aligned}
k_1 &= hf(x_n, y_n) \\
k_2 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}) \\
k_3 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}) \\
k_4 &= hf(x_n + h, y_n + k_3) \\
y_{n+1} &= y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5)
\end{aligned}
$$



Exercize: A simple star formation law can be written as:

$$\frac{d\rho_*}{dt} = -\frac{d\rho_g}{dt} = \frac{c_* \rho_g}{t_g}$$

Where t_g is a typical gas consumption time, say 2 Gyr, and ρ_g is the available gas mass, let's take 10^6 Msol (with an arbitrary volume). c_{star} is the star formation efficiency (let's use 0.01). See more on this on Katz, Weinberg & Hernquist, 1996, ApJ, 105, 19

Note that the above equation is easily integrable!

Write a code that integrates the star formation law, using RK4, and compare the result with the analytical solution. Repeat the integration, supposing a different (unphysical!) time evolution:

$$\frac{dM_{star}}{dt} = \frac{c_{star}\rho_{gas}}{t_{gas}} t^2$$

For more informations on the subject of integrating ODEs, see e.g. "Numerical Recipes in C"