

Integration of differential equations

Let's take a differential equation of the form:

$$\frac{dY}{dx} = f(x, y)$$

Simpler discretization:

$$\frac{Y_{n+1} - Y_n}{h} = f(x_n, y_n)$$

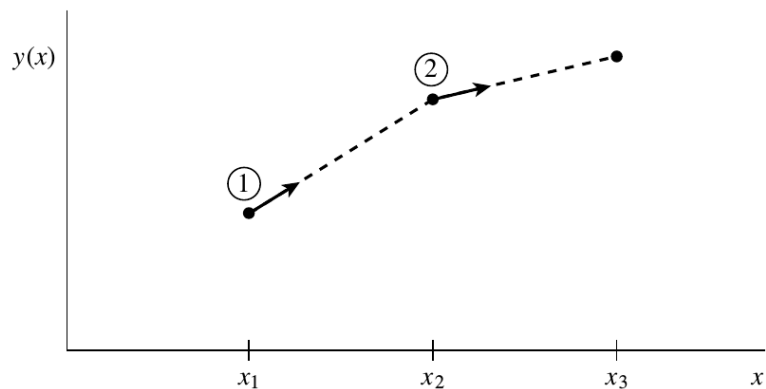
where h is the integration step, and we can advance the solution from n to $n+1$ using:

$$y_{n+1} = Y_n + hf(x_n, y_n)$$

This is the **Euler method**.

Problems:

- uses the derivatives information only at the beginning of the integral
- non symmetric
- unstable
- $O(h^2)$



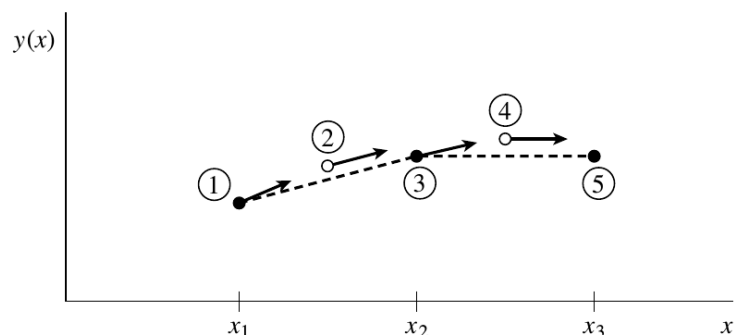
Midpoint method:

Let's use a "trial" step in the middle point of the integration step:

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

$$y_{n+1} = y_n + k_2 + O(h^3)$$

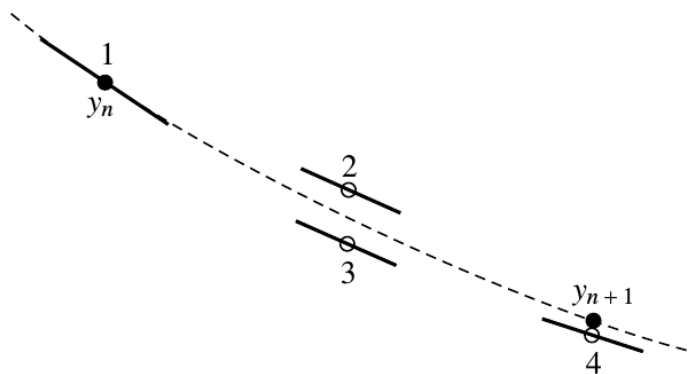


This is the **second-order Runge-Kutta** method.

More stable than Euler and uses more information on the derivatives.

Clearly, higher orders can be used. One of the most used is the **fourth-order RK**:

$$\begin{aligned}k_1 &= hf(x_n, y_n) \\k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\k_4 &= hf(x_n + h, y_n + k_3) \\y_{n+1} &= y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5)\end{aligned}$$



Exercise: A simple star formation law can be written as:

$$\dot{M}_{star} = M_g \frac{t}{t_{star}}$$

Where t_{star} is a typical gas consumption time, say 2 Gyr, and M_g is the available gas mass, let's take 10^6 Msol.

Note that the above equation is integrable.

Write a code that integrates the star formation law, using RK4, and compare the result with the analytical solution.

For more informations on this subject see e.g. "Numerical Recipes in C"