Lista 4 — Processos Estocásticos

1 - a) Para construir a PMF conjunta, desenvolve-se uma tabela com as frequências de dados e suas classes

X \ Y	0	1	2	3
0	1	1	1	1
1	1	4	4	1
2	1	4	8	1
3	1	1	1	1

$N = 32$

Agora, é necessário normalizar para se obter as frequências:

PMF:

X \ Y	0	1	2	3
0	$1/32$	$1/32$	$1/32$	$1/32$
1	$1/32$	$4/32$	$4/32$	$1/32$
2	$1/32$	$4/32$	$8/32$	$1/32$
3	$1/32$	$1/32$	$1/32$	$1/32$

b) Para a CDF, basta somar iterativamente conforme X e Y aumentam.

CDF

X \ Y	0	1	2	3
0	$1/32$	$2/32$	$3/32$	$4/32$
1	$2/32$	$7/32$	$12/32$	$14/32$
2	$3/32$	$12/32$	$25/32$	$31/32$
3	$4/32$	$14/32$	$23/32$	$32/32$

② PMF marginal:

$P_x(x) = P_{xy}(x, y_i) = \sum_i P_{xy}(x, y_i)$, de modo que

$P_x(x) =$

X	$P_x(x)$
0	4/32
1	10/32
2	14/32
3	4/32

$P_y(y) =$

Y	$P_y(y)$
0	4/32
1	10/32
2	14/32
3	4/32

③ CDF marginal:

$F_x(x) =$

X	$F_x(x)$
0	4/32
1	14/32
2	31/32
3	32/32

$F_y(y) =$

Y	$F_y(y)$
0	4/32
1	14/32
2	28/32
3	32/32

$F_x = \lim_{y \to \infty} F_{xy}(x, y)$ $\quad F_y = \lim_{x \to \infty} F_{xy}(x, y)$

④ Condição para independência: $P(A \cap B) = P(A, B) = P(A) \cdot P(B)$

ou, $P(A|B) = P(A)$

contra-exemplo: $P(X=1, Y=2) = 4/32 \neq P(X=1) = 10/32$

logo, são variáveis dependentes

⑤ PMF condicional $P_{X|Y}(x_i | y_i)$ de X dado Y

sabendo que $P_{X|Y}(x) = \dfrac{P(X=x, Y)}{P(Y)} = \dfrac{P_{xy}(x, y)}{P_y}$

PMF condicional de X dado Y

X \ Y	0	1	2	3
0	1/4	1/10	1/14	1/4
1	1/4	4/10	4/14	1/4
2	1/4	4/10	3/14	1/4
3	1/4	1/10	1/14	1/4

exemplo: $P_{X|Y}(0,0) = \dfrac{P_{XY}(0,0)}{P_Y(0)} = \dfrac{1/32}{4/32} = 1/4$

⑥ CDF condicional

segue o mesmo raciocínio, contudo, utiliza-se de $X < x$,

X \ Y	0	1	2	3
<0	0	0	0	0
<1	1/4	1/10	1/14	1/4
<2	2/4	5/10	5/14	2/4
<3	3/4	9/10	13/14	3/4
<4	4/4	10/10	1/14	4/4

⑦ PMF condicional de Y dado X

$P_{Y|X} = \dfrac{P_{XY}(0,0)}{P_X(0)} = \dfrac{1/32}{4/32} = 1/4$

X \ Y	0	1	2	3
0	1/4	1/4	1/4	1/4
1	1/10	4/10	4/10	1/10
2	1/14	4/14	3/14	1/14
3	1/4	1/4	1/4	1/4

(8) CDF condicional $Y < y$:

$X \backslash Y$	<0	<1	<2	<3	<4
0	0	1/4	2/4	3/4	4/4
1	0	1/10	5/10	9/10	10/10
2	0	1/14	9/14	13/14	14/14
3	0	1/4	2/4	3/4	4/4

(9) Calculando as probabilidades condicionais

a) $P_{X|Y}(x=0 \mid 1 \leq y \leq 3)$: $\sum_j \frac{P_{XY}(0, y_j)}{P_Y(1 \leq y \leq 3)} = \frac{P(0,1) + P(0,2) + P(0,3)}{P_Y(1) + P_Y(2) + P_Y(3)}$

$$= \frac{1/32 + 1/32 + 1/32}{10/32 + 14/32 + 4/32} = \frac{3}{28}$$

b) $P_{Y|X}(y=0, 1 \leq x \leq 3) = \frac{\sum P_{XY}(x_i, 0)}{P_X(1 \leq x \leq 3)} = \frac{1/32 + 1/32 + 1/32}{10/32 + 14/32 + 4/32} = \frac{3}{28}$

c) $P_{Y|X}(y=2, x \leq 2) = \frac{P_{XY}(0,2) + P_{XY}(1,2) + P_{XY}(2,2)}{P_X(x \leq 2)} = \frac{1/32 + 4/32 + 8/32}{28/32} = \frac{13}{28}$

d) $P_{X|Y}(x \leq 1, y \geq 2) = \frac{P_{XY}(0,2) + P_{XY}(0,3) + P_{XY}(1,2) + P_{XY}(1,3)}{P_Y(y \geq 2)} =$

$$\frac{\frac{1}{32} + \frac{1}{32} + \frac{4}{32} + \frac{1}{32}}{18/32} = \frac{7}{18}$$

e) $P_{Y|X}(y=3 \mid x \leq 1) = \frac{1/32 + 1/32}{14/32} = \frac{2}{14} = \frac{1}{7}$

f) $P_{X|Y}(1 \le x \le 3 \mid 1 \le y \le 3) = \dfrac{25/32}{28/32} = \dfrac{25}{28}$

10) por X e Y serem equiprováveis, há igual probabilidade para suas classes $\{0,1,2,3\}$

a) PMF

4 opções de cada

$N = 4 \times 4 = 16$

X \ Y	0	1	2	3
0	1/16	1/16	1/16	1/16
1	1/16	1/16	1/16	1/16
2	1/16	1/16	1/16	1/16
3	1/16	1/16	1/16	1/16

b) CDF

X \ Y	0	1	2	3
0	1/16	2/16	3/16	4/16
1	2/16	4/16	6/16	8/16
2	3/16	6/16	9/16	12/16
3	4/16	8/16	12/16	16/16

c) PMF marginal

$P_X(x) =$

X	$P_X(x)$
0	4/16
1	4/16
2	4/16
3	4/16

$P_Y(y) =$

Y	$P_Y(y)$
0	4/16
1	4/16
2	4/16
3	4/16

d) CDF marginal

X	$F_X(x)$
0	4/16
1	8/16
2	12/16
3	16/16

Y	$F_Y(y)$
0	4/16
1	8/16
2	12/16
3	16/16

e) Verificando independência

$$P(A|B) \stackrel{?}{=} P(A) \rightarrow P_{xy}(x=1 \mid y=1) = P_x(x=1)$$

$$P_{xy}(x=1 \mid y=1) = \frac{P_{xy}(x=1, y=1)}{P_y(y=1)} = \frac{1/16}{4/16} = \frac{1}{4}$$

$$P_x(x=1) = \frac{4}{16} = \frac{1}{4}$$

Como a distribuição é toda equiprovável, p/ $\forall (x,y)$ escolhido, vale que $P_{xy}(x,y) = P_x(x) = P_y(y)$

(11) a) $E[X|Y=y] = \sum x P_{X|Y}(x, y=y) =$

$y=0$ $\quad E[X|Y=0] = 0 \cdot 1/4 + 1 \cdot 1/4 + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = 6/4 = 3/2$

$y=1$ $\quad E[X|Y=1] = 0 \cdot 1/10 + 1 \cdot 4/10 + 2 \cdot \frac{4}{10} + 3 \cdot \frac{1}{10} = 15/10 = 3/2$

$y=2$ $\quad E[X|Y=2] = 0 \cdot 1/14 + 1 \cdot 4/14 + 2 \cdot 8/14 + 3 \cdot \frac{1}{14} = 23/14$

$y=3$ $\quad E[X|Y=3] = E[X|Y=0] = 3/2$

b) $x=0$ $\quad E[Y|X=0] = 0 \cdot 1/4 + 1 \cdot 1/4 + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = 3/2$

$x=1$ $\quad E[Y|X=1] = 0 \cdot 1/10 + 1 \cdot \frac{4}{10} + 2 \cdot \frac{4}{10} + 3 \cdot \frac{1}{10} = 3/2$

$x=2$ $\quad E[Y|X=2] = 0 \cdot \frac{1}{14} + 1 \cdot \frac{4}{14} + 2 \cdot \frac{8}{14} + 3 \cdot \frac{1}{14} = 23/14$

$x=3$ $\quad E[Y|X=3] = 0 \cdot 1/4 + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = 3/2$

(12) $E[X|Y] = g(y) = \begin{cases} 3/2, & Y=0 \\ 3/2, & Y=1 \\ 23/14, & Y=2 \\ 3/2, & Y=3 \end{cases}$ $\qquad E[Y|X] = g(x) = \begin{cases} 3/2, & x=0 \\ 3/2, & x=1 \\ 23/14, & x=2 \\ 3/2, & x=3 \end{cases}$

(13) Sabendo que: $Var(X|Y=y) = E[X^2|Y=y] - (E[X|Y=y])^2$

calculando $E[X^2|Y=y]$:

$y=0: \to 0^2 \cdot 1/4 + 1^2 \cdot 1/4 + 2^2 \cdot 1/4 + 3^2 \cdot 1/4 = 14/4 = 7/2$

$y=1: \quad 0^2 \cdot 1/10 + 1^2 \cdot 4/10 + 2^2 \cdot 4/10 + 3^2 \cdot 1/10 = 29/10$

$y=2: \to 0^2 \cdot 1/14 + 1^2 \, 4/14 + 2^2 \cdot 8/14 + 3^2 \cdot 1/14 = 45/14$

$y=3 \quad = \quad y=0 \to 7/2$

$Var(X|Y=0) = 14/4 - (3/2)^2 = 5/4$

$Var(X|Y=1) = 29/10 - (3/2)^2 = 13/20$

$Var(X|Y=2) = 45/14 - (23/14)^2 = 101/196$

$Var(X|Y=3) = 14/4 - (3/2)^2 = 5/4$

$$Var(X|Y=y) = g(y) = \begin{cases} 5/4, & y=0 \\ 32/20, & y=1 \\ 101/196, & y=2 \\ 5/4, & y=3 \end{cases}$$

(14) Analogamente para $V(Y|X=x)$:

$x=0 \quad 0^2 \cdot 1/4 + 1^2 \cdot 1/4 + 2^2 \cdot 1/4 + 3^2 \cdot 1/4 = 14/4$

$x=1 \quad 0^2 \cdot 1/10 + 1^2 \, 4/10 + 2^2 \, 4/10 + 3^2 \, 1/10 = 29/10$

$x=2 \quad 0^2 \, 1/14 + 1^2 \cdot 4/14 + 2^2 \cdot 3/14 + 3^2 \cdot 1/14 = 45/14$

$x=3 \quad 0^2 \, 1/4 + 1^2 \, 1/4 + 2^2 \, 1/4 + 3^2 \, 1/4 = 7/2$

$$Var(Y|X=x) = g(x) = \begin{cases} 5/4, & x=0 \\ 32/20, & x=1 \\ 101/196, & x=2 \\ 5/4, & x=3 \end{cases}$$

(15) Referente ao problema equiprovável (com independência!)

a) $E[X|Y=y] = E[X] = \frac{0+1+2+3}{4} = \frac{6}{4} = \frac{3}{2}$

b) $E[Y|X=x] = E[Y] = \frac{0+1+2+3}{4} = \frac{3}{2}$

c) $g(x) = g(y) = 3/2 \sim N(3/2, 0)$

d) $Var(X|Y=y) = E[(X - \overset{3/2}{\cancel{E[X|Y=y]}})^2 | Y=y] = E[(X-3/2)|Y=y]$

$= Var[X]$ (independente!) $= E[X^2] - (E(X))^2$

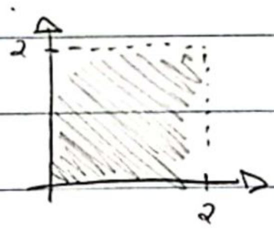
$E[X^2] = \frac{0^2+1^2+2^2+3^2}{4} = \frac{14}{4} = \frac{7}{2}$

$Var[X] = \frac{7}{2} - \left(\frac{3}{2}\right)^2 = \frac{5}{4}$

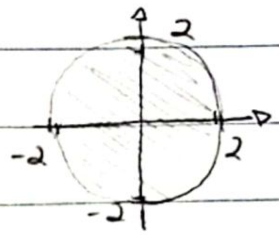
e) Análogo para $Var[Y|X=x] = 5/4$

f) $Var(X|Y=y) = Var(Y|X=x) : h(\cancel{t}) \sim N(5/4, \emptyset)$

(16) a) $A = \{(X,Y) | x+y \leq 2$

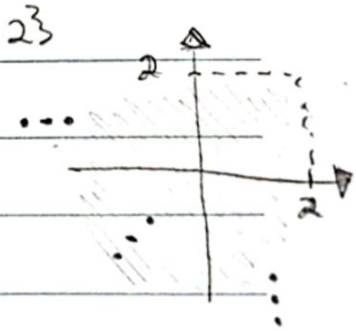


b) $B = \{(X,Y) | x^2+y^2 < 4\}$



c) $C = \{(X,Y) | \min(X,Y) \leq 2\}$

d) $D = \{(X,Y) | \max(X,Y) \leq 2\}$



(17) $F_{xy}(x,y) = \begin{cases} (1-e^{-\alpha x})(1-e^{-\beta y}) & , x \geq 0, y \geq 0, \alpha \geq 0, \beta \geq 0 \\ 0 & , \text{etc} \end{cases}$

a) $F_x(x) = \lim_{y \to \infty} F_{xy}(x,y) = 1 - e^{-\alpha x}$

$F_y(y) = \lim_{x \to \infty} F_{xy}(x,y) = 1 - e^{-\beta y}$

b) Se $F_{xy} = F_x \cdot F_y$, como é o caso, é independente.

c) $P(X \leq 1, Y \leq 1) = F_{xy}\Big|_{\substack{x=1 \\ y=1}} = (1-e^{-\alpha})(1-e^{-\beta})$

d) $P(X \leq 1) = F_x\Big|_{x=1} = 1 - e^{-\alpha}$

e) $P(Y > 1) = 1 - P(Y \leq 1) = 1 - F_y\Big|_{y=1} = 1-(1-e^{-\beta}) = e^{-\beta}$