Lista 3 - Processos Estocásticos

1 Bernoulli $N=3$ $\quad P(x) = \binom{3}{x} p^x (1-p)^{3-x}$

a) $\{0,1,2,3\}$

b)

b) $p(0) = \frac{3!}{0!\,3!} p^0 (1-p)^3 = (1-p)^3 \qquad p(1) = \frac{3!}{1!\,2!} p(1-p)^2 = 3p(1-p)^2$

$p(2) = \frac{3!}{2!\,1!} p^2(1-p) = 3p^2(1-p) \qquad p(3) = \frac{3!}{3!\,1!} p^3(1-p)^0 = p^3$

2 a) $x \in \mathbb{R} \mid x \leq 1$

b) Para a CDF (cumulativa), pode-se interpretar como todos os perímetros até o raio designado, sobre a área total:

c) $f_x(x) = \frac{2\pi x}{\pi} = 2x = \frac{d}{dx}F_x(x) \qquad F_x(x) = \frac{\pi x^2}{\pi 1^2} = x^2$

d) $P(x<a) = F_x(a) = a^2$ $\quad$ e) $P(a<x<b) = b^2 - a^2$ $\quad$ f) $P(0{,}1<x<0{,}9) = 0{,}9^2 - 0{,}1^2 = 0{,}8$

g) $E(x) = \int_0^1 x f(x) = \int_0^1 x(2x) = 2\frac{x^3}{3}\Big|_0^1 = \frac{2}{3}$

h) $Var[x] = E(x^2) - E(x)^2 \Rightarrow \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$

$\int_0^1 x^2 f(x) = \frac{2x^4}{4}\Big|_0^1 = \frac{1}{2}$

3 $f_x(x) = \begin{cases} 2\ln(a)x^2, & |x| \leq 1 \\ 0, & \text{etc} \end{cases}$

a) $\int_{-1}^{1} 2\ln(a)x^2 dx = 1 \rightarrow 2\ln(a)\cdot\frac{x^3}{3}\Big|_{-1}^{1} = \frac{4}{3}\ln(a) \equiv 1 \rightarrow a = e^{3/4}$

b) $x \in \mathbb{R} \mid -1 \leq x \leq 1$

c) $E[x] = \int_{-1}^{1} x f_x(x) dx = \frac{3}{2}\cdot\frac{x^4}{4}\Big|_{-1}^{1} = \varnothing$

d) $Var(x) \Rightarrow 2\ln(a)\int \frac{x^4}{4} dx = \frac{4}{5}\ln(a)$

$Var(x) = \frac{4}{5}\ln(a) - 0^2 = 3/5$

e) $P(x>0) = \int_0^1 2\ln(a)x^2 dx = 2\cdot\frac{3}{4}\cdot\frac{x^3}{3}\Big|_0^1 = \frac{1}{2}[1-0] = \frac{1}{2}$

4 $f_x = \begin{cases} 5x^4, & 0 \leq x \leq 1 \\ \varnothing, & \text{etc} \end{cases}$

a) $P(x>1/2) = \int_{1/2}^{1} 5x^4 dx = x^5\Big|_{1/2}^{1} = (1 - \varnothing{,}5^5)$

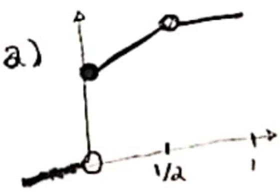
c) $E(x) = \int_0^1 x f_x(x) dx = \frac{5}{6}x^6\Big|_0^1 = \frac{5}{6}$

b) $F_x(x) = \int f_x(x) dx = x^5$

d) $Var\{X\} = \int_0^1 5x^6 dx - (5/6)^2 = \frac{5}{252}$

5 $F_x = \begin{cases} \varnothing, & x<0 \\ x+1/2, & 0 \leq x \leq 1/2 \\ 1, & 1/2 \leq x \end{cases}$

a)



b) $f_x = \frac{dF_x(x)}{dx} = \begin{cases} \varnothing, & x<\varnothing \\ 1, & 0 \leq x < 1/2 \\ \varnothing, & 1/2 \leq x \end{cases}$

c) $P(x>1/4) = 1 - (1/4 + 1/2) = 1/4$

⑥ $F_x(x) = \begin{cases} 0 & , x < 0 \\ 1 - e^{-x} & , 0 \leq x \leq 1 \\ 1 & , 1 < x \end{cases}$

a) $P(x < \pi) = 1 \quad , \pi > 1$

b) $P(\frac{\pi}{4} < x \leq \frac{\pi}{3}) = (1 - e^{-\frac{\pi}{3}})(1 - e^{-\frac{\pi}{4}}) = e^{-\frac{\pi}{4}} - e^{-\frac{\pi}{3}}$

c) $f_x(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & 0 \leq x \leq 1 \\ 0 & 1 < x \end{cases}$

⑦ $F_x(x) = \begin{cases} a - a e^{-x/b} & , x > 0 \\ 0 & , x < 0 \end{cases}$

$\lim_{x \to -\infty} F_x(x) = 0$ ✓

$\lim_{x \to \infty} F_x(x) = 1 \rightarrow \lim_{x \to \infty} a(1 - e^{-x/b}) = a \rightarrow a = 1$

sendo monotônico não decrescente

$\frac{d}{dx} F_x(x) \geq 0 \rightarrow \frac{a}{b} e^{-x/b} = 0, \quad e^{-x/b} \geq 0 \ \forall \ b > 0$

⑧ $f_x(x) = \begin{cases} \alpha \pi x^2 & , 0 \leq x \leq 1 \\ 0 & , \text{etc} \end{cases}$

a) $\int_{-\infty}^{\infty} \alpha \pi x^2 dx = \alpha \pi \frac{x^3}{3} \Big|_0^1 = \frac{\alpha \pi}{3} \stackrel{!}{=} 1 \Leftrightarrow \alpha = \frac{3}{\pi}$

b) $F_x(x) = \begin{cases} x^3 & , 0 \leq x \leq 1 \\ 1 & , \text{etc} \end{cases}$

c) $P(1 < x < 2) = 1 - \frac{\alpha \pi}{3} = 0$

d) $P(0{,}6 < x < 0{,}9) = 1 \cdot (0{,}9^3 - 0{,}6^3) = 0{,}513$

e) $P((x \leq 0{,}5) \cup (x > 0{,}3)) = 0{,}613$

$P(x \leq 0{,}5) = 0{,}5^3$

$P(x > 0{,}3) = 1 - P(x < 0{,}3) = 1 - 0{,}3^3$

f) $f_x(x) = \begin{cases} \alpha \pi x^2 & , 0 \leq x \leq 1 \\ 0 & , \text{etc} \end{cases}$

$\int_0^1 x f(x) dx = \alpha \pi \frac{x^4}{4} \Big|_0^1 = \frac{3}{4}$

⑨ $f_x(x) = \begin{cases} |x| & , -1 \leq x \leq 1 \\ 0 & , \text{etc} \end{cases} = \begin{cases} x & , 0 \leq x \leq 1 \\ -x & , -1 \leq x < 0 \\ 0 & , \text{etc} \end{cases}$

a) $F_x(x) = \int_{-1}^{x} -x dx = -\frac{x^2}{2} \Big|_{-1}^{x} = -\frac{x^2}{2} + \frac{1}{2}$,

$\int_0^x x dx = \frac{x^2}{2} + \frac{1}{2}$

$F_x(x) = \begin{cases} 0 & x < -1 \\ 1/2 - x^2/2 & -1 < x \leq 0 \\ 1/2 + x^2/2 & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$

b) $\mu_x = \int_{-1}^{0} x(-x) dx + \int_0^1 x(x) dx = -\frac{1}{3} + \frac{1}{3} = 0$

c) $Y = X^3 - X^2 - X + 1$

$E\{Y\} = E\{X^3\} - E\{X^2\} - E\{X\} + 1 = 1/2$

$E\{X^3\} = \int_{-1}^{0} x^3(-x) dx + \int_0^1 x^3(x) dx = 0$

$E\{X^2\} = \int_{-1}^{0} x^2(-x) dx + \int_0^1 x^2(x) dx = \frac{1}{2}$

⑩ questão igual à questão ⑥

(11) $X = A\cos(2\pi f_c + \theta) + 1 \quad A > 0,5$

$f_\theta(x) = \begin{cases} \frac{1}{2\pi}, & 0 < \theta \leq 2\pi \\ 0 & \text{c.c.} \end{cases}$

a) $\mathcal{E}\{X\} = \mathcal{E}\{A\cos(\sim) + 1\} = \mathcal{E}\{A\} \cdot \underbrace{\mathcal{E}\{\cos(\sim)\}}_{0} + \mathcal{E}\{1\}$

$\mathcal{E}\{x\} = A \cdot 0 + 1 = 1$

b) $\mathcal{E}\{X^2\} = \int_0^{2\pi} [A\cos(\sim) + 1]^2 d\theta = \int_0^{2\pi} [A^2\cos^2(\sim) + 2A\cos(\sim) + 1] d\theta = 2A0 + 1 + A^2 \int_0^{2\pi} \cos^2(\sim) d\theta$

$= (A^2/2) + 1$

$\cos^2(x) = \frac{1 + \cos(2x)}{2}$

c) $Var\{X\} = \mathcal{E}\{x^2\} - \mathcal{E}\{x\}^2 = \frac{A^2}{2} + 1 - 1^2 = \frac{A^2}{2}$

(12) $\mu_{c_0} = \frac{1,5 + 2,3 + 2,2 + 2,2 + 2,3}{5} \approx 2,32$ $\quad \mu_{c_1} = \frac{1,2 + 3,3 + 5,4 + 5,1 + 4,4 + 4,5 + 3,2}{7} \approx 4,028$

$\sigma_{c_0}^2 = \frac{(1,5 - 2,32)^2 + (2,3 - 2,32)^2 + (2,2 - 2,32)^2 + (2,8 - 2,32)^2 + (2,3 - 2,32)^2}{7} \approx 0,2296$

$\sigma_{c_1}^2 = 1,617$

deste modo, tendo μ e σ, pode-se modelar as Gaussianas.

probabilidade a priori: $P(C_i) \rightarrow P(C_0) = 5/12 \quad P(C_1) = 7/12$

verossimilhança: $p(x|C_i) \rightarrow \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

$\rightarrow p(5,7|C_0) \approx 3,87 \cdot 10^{-12}$

$p(5,7|C_1) \approx 0,128$

$p(C_0|5,7) \approx 5,2 \cdot 10^{-11}$

$p(C_1|5,7) \approx 1$

$\boxed{\text{logo, } 5,7 \rightarrow \text{enfermo}}$

$p(2,6|C_0) = 0,683 \quad p(C_0|2,6) = 0,835$

$p(2,6|C_1) = 0,143 \quad p(C_1|2,6) \approx 0,165$

$\boxed{2,6 \rightarrow \text{sadio}}$

$p(3, C_0) \approx 0,16 \quad p(C_0|3) \approx 0,253 \rightarrow \boxed{3 \text{ é enfermo}}$

$p(3, C_1) \approx 0,196 \quad p(C_1|3) \approx 0,747$

(13) $P(a) = 1/2 \quad P(b) = 1/4 \quad P(c) = 1/8 = P(d)$

$a = 0$
$b = 10$
$c = 110$
$d = 111$

a) $\{1, 2, 3\}$

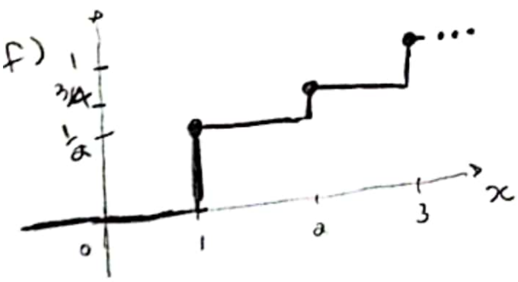
b) $P(x=1) = P(a) = 1/2$

c) $P(x=2) = P(b) = 1/4$

d) $P(x=3) = P(c) \cup P(d) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

e) $P(x>3) = 0$

f)



variável aleatória discreta

$F_x(x) = \begin{cases} 0 & x < 1 \\ 1/2 & 1 \leq x < 2 \\ 3/4 & 2 \leq x < 3 \\ 1 & x > 3 \end{cases}$

g) $P(x \leq 1) = 1/2$

h) $P(1 < x \leq 2) = 3/4 - 1/2 = 1/4$

i) $P(1 \leq x \leq 2) = 3/4 - 0 = 3/4$

Digitalizado com CamScanner

(14) $p = 0{,}5$ $\quad X = \binom{6}{x} p^{x}(1-p)^{6-x}$

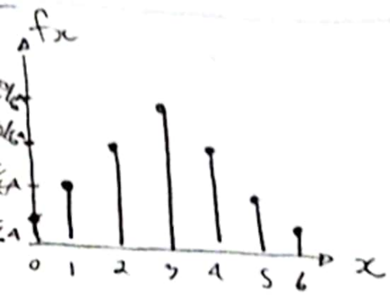
Faixa dinâmica: $\{0,1,2,3,4,5,6\}$

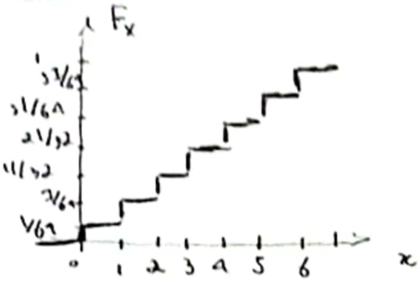
a) $f_x(x) = \binom{6}{x} 0{,}5^6 = \binom{6}{x} \cdot \frac{1}{64}$

$P(0) = 1/64$ $\quad P(5) = 3/32$
$P(1) = 3/32$ $\quad P(6) = 1/64$
$P(2) = 15/64$
$P(3) = 5/16$
$P(4) = 15/64$

b) $F_x(x) = \begin{cases} 0 & x < 0 \\ 1/64 & 0 \le x < 1 \\ 7/64 & 1 \le x < 2 \\ 11/32 & 2 \le x < 3 \\ 21/32 & 3 \le x < 4 \\ 51/64 & 4 \le x < 5 \\ 57/64 & 5 \le x < 6 \\ 1 & 6 \le x \end{cases}$

$F_x(6) = P(0) + P(1) + \dots + P(6)$





c) $P(x \ge 6) = 1/64$

d) $P(x \le 2) = 11/32$

e) $P(1 \le x \le 5) = \frac{57}{64} - \frac{1}{64} = 7/8$

f) $E(x) = n \cdot p = 6 \cdot 0{,}5 = 3$

g) $E(x^2) = \sum_0^6 x^2 P_x(x) = 10{,}5$

h) $\sigma_x^2 = 10{,}5 - 3^2 = 1{,}5$

(15) $S_x = \{1,2,3,4,5,6\}$ $\quad y = \cos(x \cdot \pi/6)$

$S_y = \{-1, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{1}{2}\}$

a) $y(1) = \cos(\pi/6) = \sqrt{3}/2$
$y(2) = 1/2$ $\quad y(3) = 0$
$y(4) = -1/2$ $\quad y(5) = -\sqrt{3}/2$ $\quad y(6) = -1$

b) Assumindo a distribuição uniforme, $P(y) = 1/6$, com: $f(y) = \begin{cases} 1/6 & \forall y \in S_y \\ 0 & , \text{c.c.} \end{cases}$

c) $F_y(y) = \sum_0^N f(i) = \begin{cases} 0 & y < -1 \\ 1/6 & -1 \le y < -\sqrt{3}/2 \\ 1/3 & -\frac{\sqrt{3}}{2} \le y < -1/2 \\ 1/2 & -1/2 \le y < 0 \\ 2/3 & 0 \le y < 1/2 \\ 5/6 & 1/2 \le y < \sqrt{3}/2 \\ 1 & \frac{\sqrt{3}}{2} \le y \end{cases}$

d) $E\{y\} = \sum_0^N y\, P(y) = -1/6$

e) $\sigma^2 = E\{y^2\} - E\{y\}^2 = \underline{17/36}$

$E\{y^2\} = \sum y^2 P_y = 1/2$