Random Process or Stochastic Process

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Signal and Data

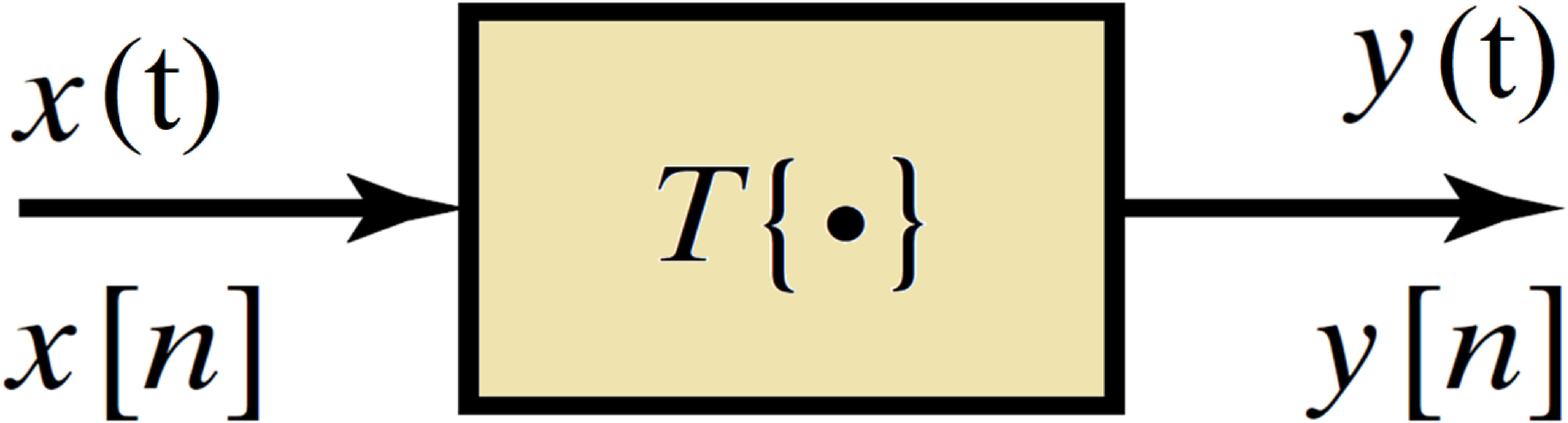
- Signal: The signal is related to the variable "time", "space", both or other related quantity.
- Pure data: The data usually consists of a sequence where its elements are not related to each other by the temporal variable, a spatial variable or both.

System definition

- The “**system**” is a piece (a portion, a part) of the “**universe**” that it individualizes from the rest through an “edge” (a border, a frontier).
- The “system” has an interface with the “universe” (it communicates with the universe).
- The system accepts “**inputs**” to the universe and, “**through its own laws**”, produces “**outputs**” to the universe.

System

continuous-time



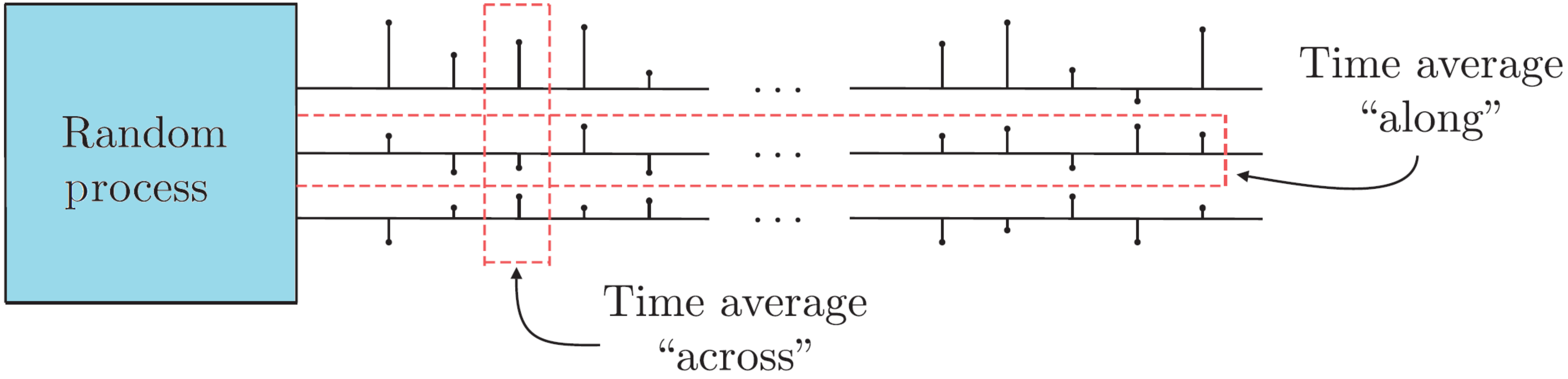
discrete-time

Random Processes

- A probability system, which is composed of a sample space where of multiple observations of random values is taken over **a period of time**, is called a **random process** or a **stochastic process.**
- In signal processing jargon, a stochastic process is also known as a random signal.

Random Processes

- Suppose to every outcome (sample point) ζ in the sample space Ω of a random experiment, according to some rule, a function of time t is assigned.



Random Processes

- The ensemble or collection of all such functions that result from a random experiment, denoted by $X(t, \zeta)$, is a ***random process*** or a ***stochastic process***.
- A conjunct of output observation $\boldsymbol{X}(t, \zeta)$ of the function versus $\boldsymbol{t}$, for $\boldsymbol{\zeta}$ fixed, is called a **realization**, **sample path**, ***ensemble member*** or **sample function** of the random process.

$$\{X(t,\zeta), t \in T, \zeta \in \Omega\}$$

Random Processes

- For a given $\zeta = \zeta_i$, a specific function of time t, i.e. $X(t, \zeta_i)$, is thus produced, and denoted by $x(t)$.
- For a specific time $t = t_k$, $X(t_k, \zeta)$ is a random variable, and is called a ***time sample***.
- For a specific ζ_i and a specific t_k, $X(t_k, \zeta_i)$ is simply a nonrandom constant.

Classification of Random Processes

- The set of possible values that the random process $X(t)$ may take on is called its ***state space*** or ***sample space*** **Ω**.
- The ***sample space*** **Ω** could be **finite**, **countably infinite** or **uncountable**.
- The variable **t** (time variable) can also be **continuous** or **discrete**: it means continuous-time/discrete-time.
- If the possible values are **finite** or **countably infinite**, the random process $X(t)$ is then called a ***discrete-state random process*** or a ***discrete-valued random process***.

Classification of Random Processes

- If the possible values of the random process $X(t)$ are part of a **finite** or **infinite continuous interval** (or set of such intervals), the random process $X(t)$ is then called a ***continuous-state random process*** or a ***continuous-valued random process***.
- It is also possible to have a ***mixed-valued random process***, where possible values of $X(t)$ at any time can be continuous or discrete.

Classification of Random Processes

- In a ***continuous-time random process***, the time parameter *t* is continuous that is it can take real values in an interval or set of intervals on the real line and thus comes from an uncountably infinite set.
- In a ***discrete-time random process***, the time parameter ***t*** is a countable set.
- A discrete-time random process is often called a ***random sequence*** and generally denoted by $X(n)$ (in discrete-time signal processing we use the notation: $x[n]$), where n is an integer.

Classification of Random Processes

- $\{X(t,\zeta), t \in T, \zeta \in \Omega\}$: *let* *T* and **Ω** represent the **time parameter** and the **sample space**, respectively, we can have the following four cases:
- 1 - If both *T* and **Ω** are continuous, the random process $X(t)$ is then called a ***continuous random process***.
- 2 - If *T* is discrete and **Ω** is continuous, the random process $X(n)$ (or $X[n]$)is then called a ***continuous random sequence***.

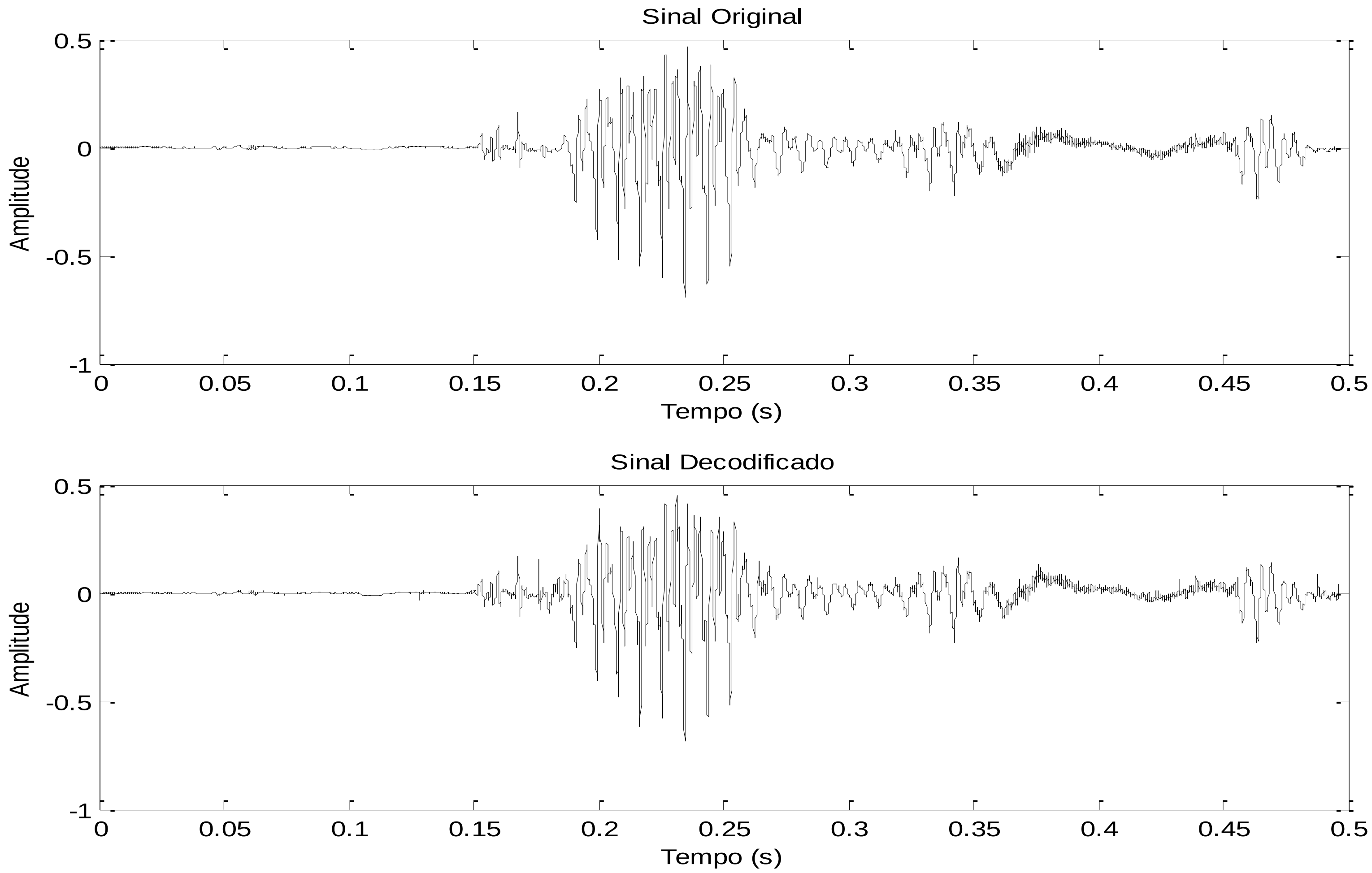
Classification of Random Processes

3 - If T is continuous and **Ω** is discrete, the random process $X(t)$ is then called a ***discrete random process***.

4 - If both T and **Ω** are discrete, the random process $X[n]$ is then called a ***discrete random sequence***.

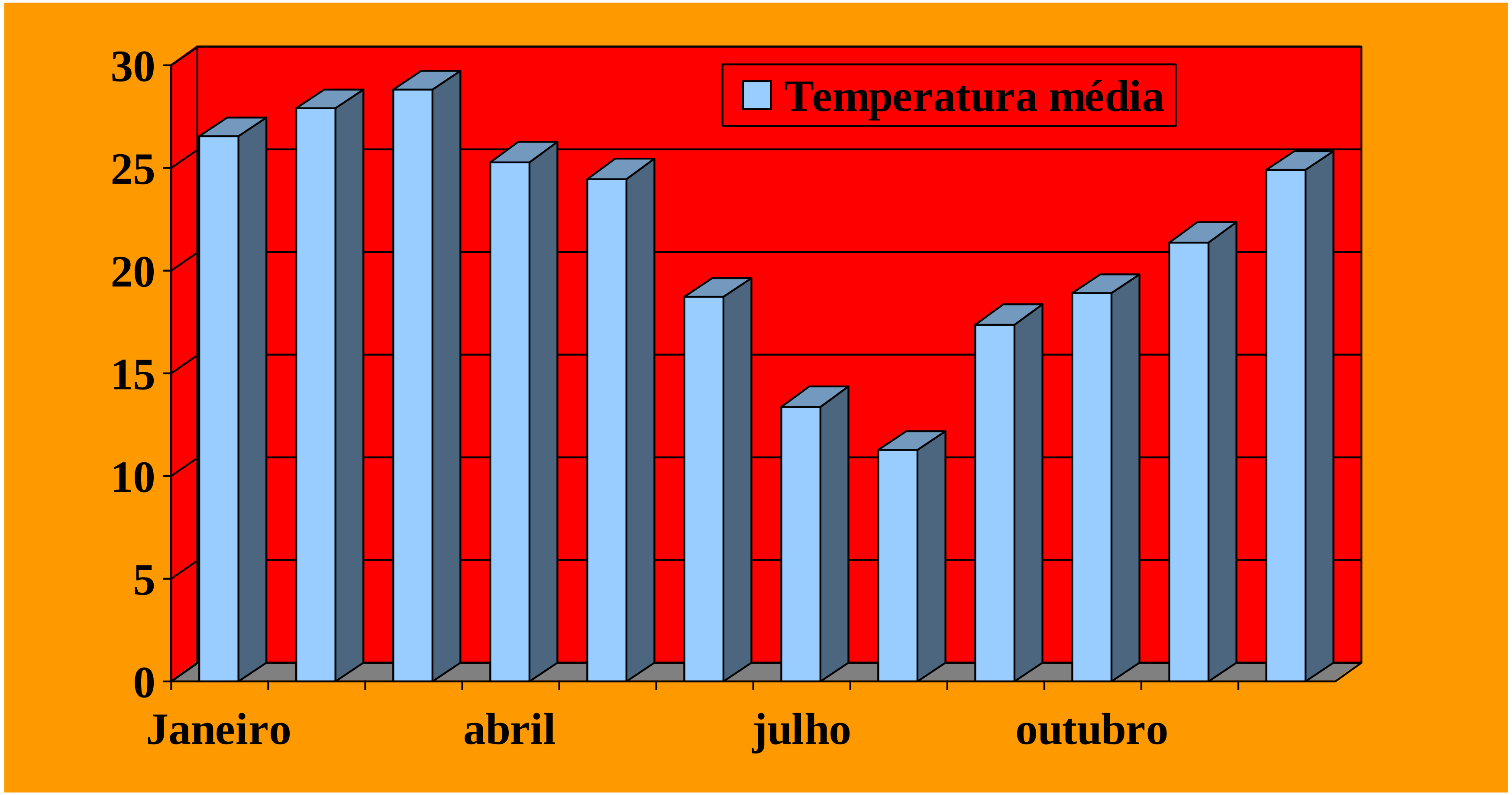
Classification of Random Processes

- **Continuous random process**: T and Ω continuous (voice signal original and decoded at a rate of 32kbps.)

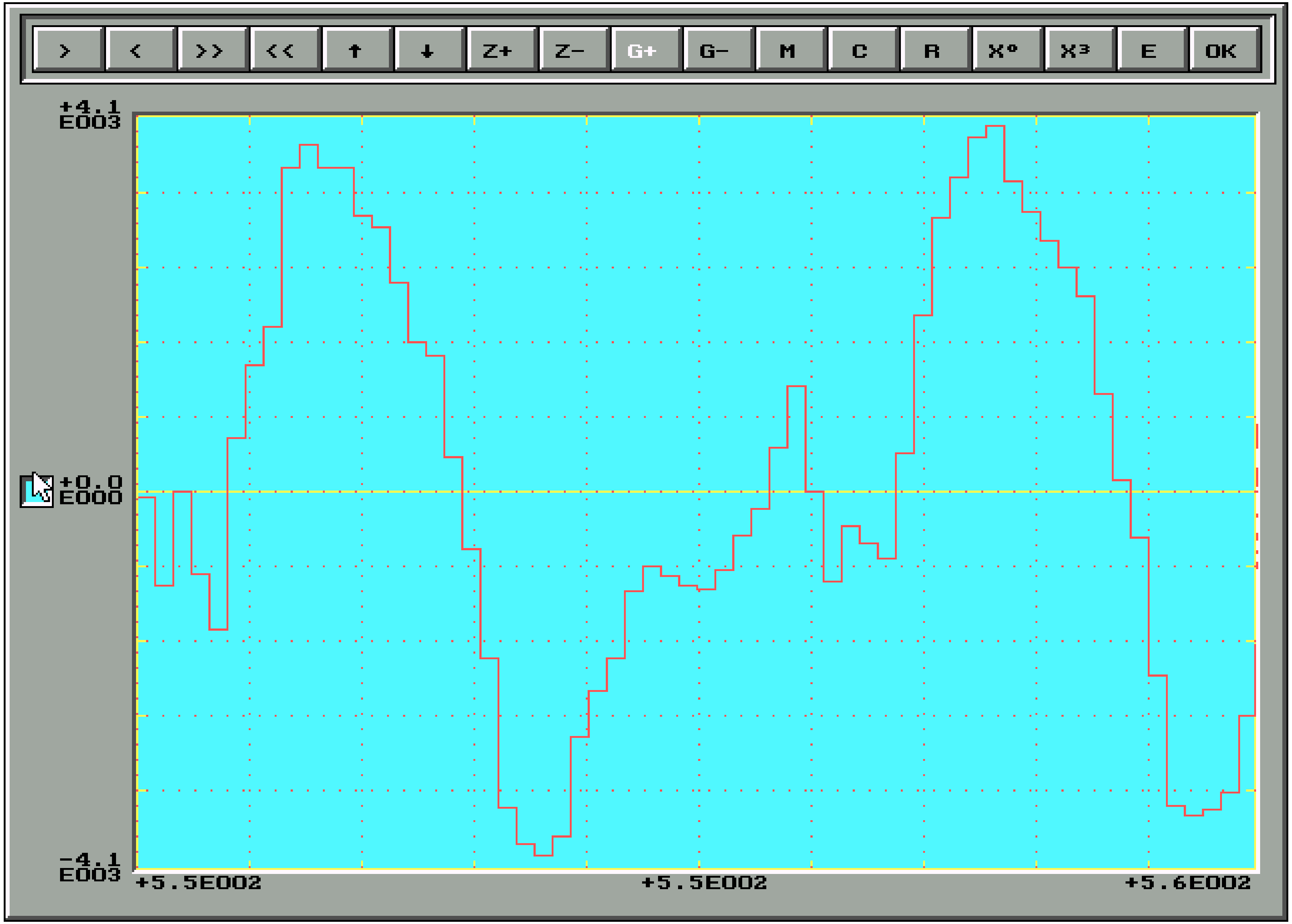


Examples of Random Processes

- Example of **continuous random sequence**: T discrete and Ω continuous (Discrete Domain Signal: continuous function codomain of a discrete variable domain)

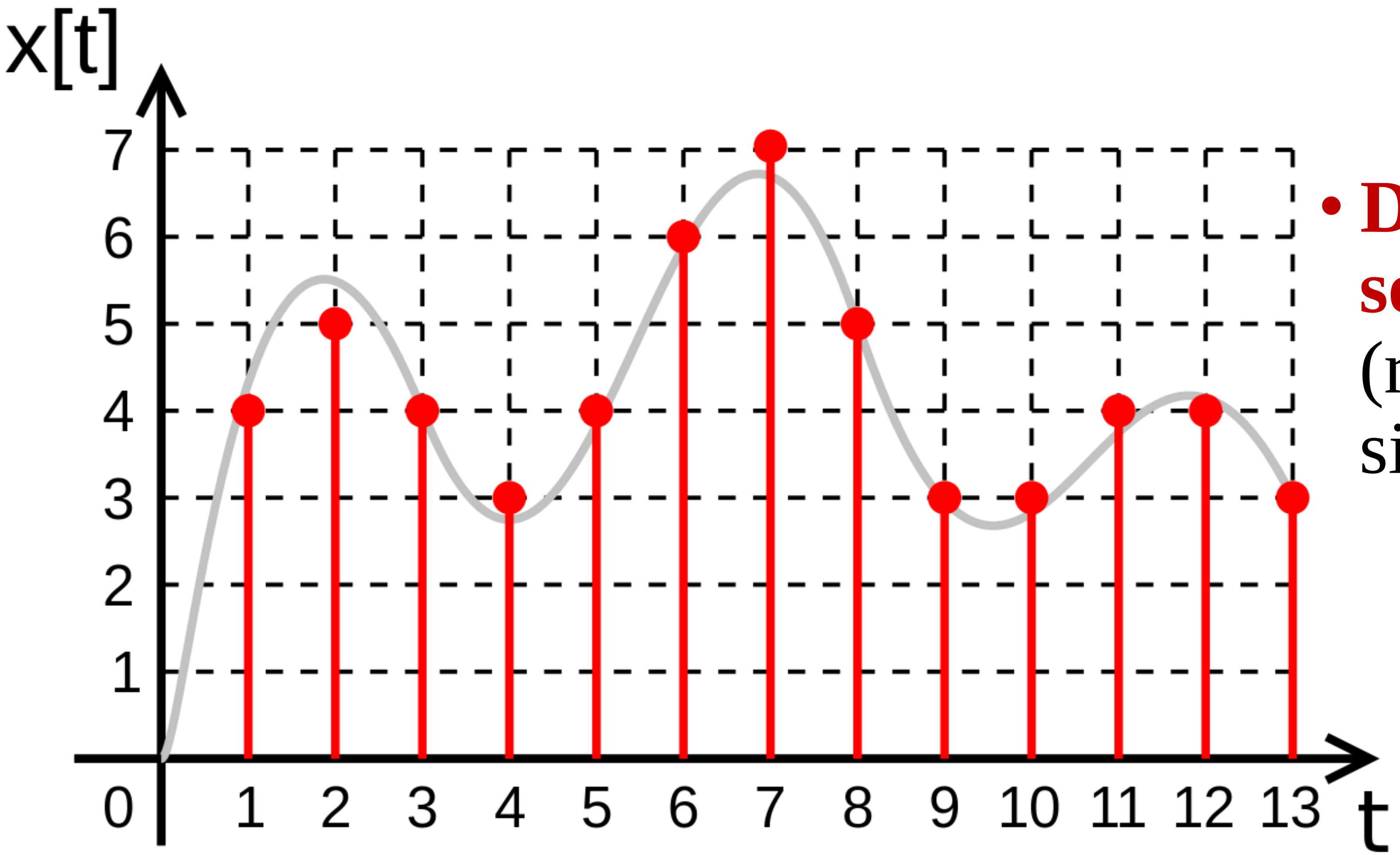


Examples of Random Processes



Discrete random process (discrete signal of continuous variable): ECG signal obtained in the output of a sample-and-hold quantized with 8 bits.

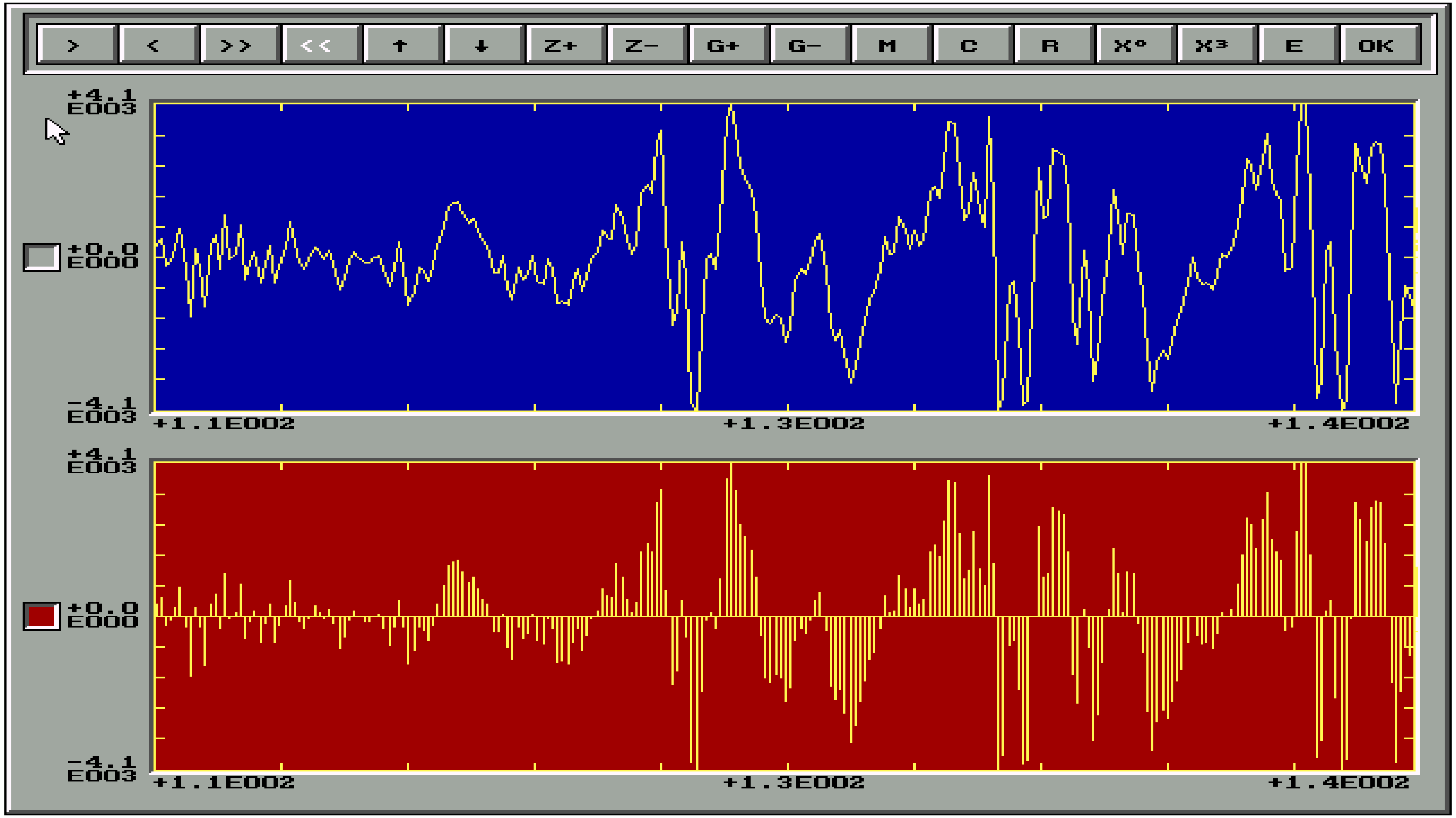
Examples of Random Processes



- **Discrete random sequence** (random digital signal)

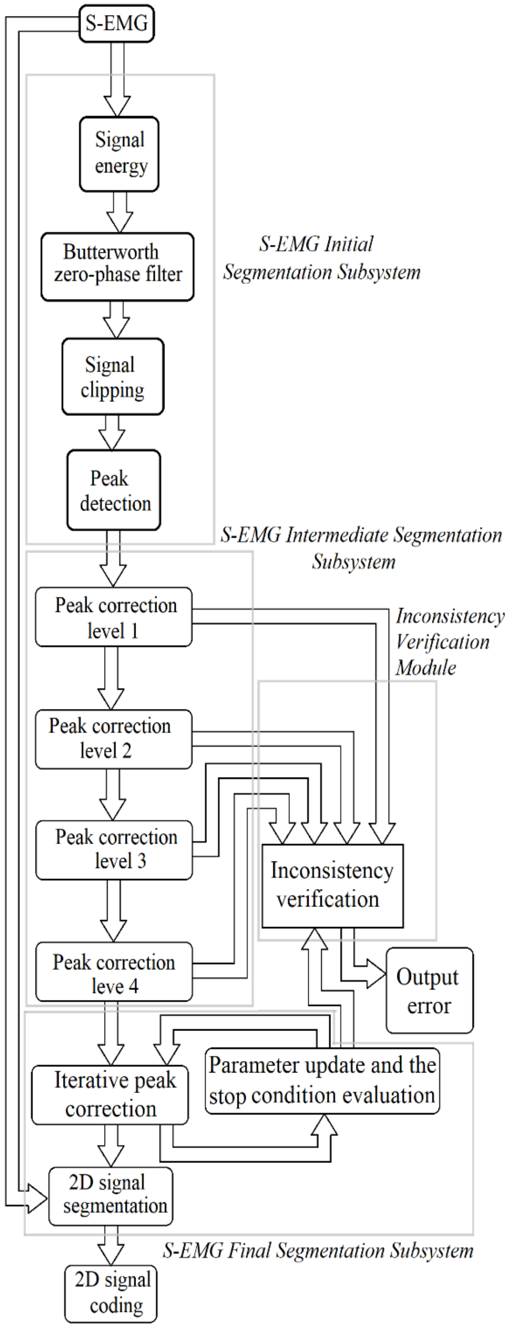
Examples of Random Processes

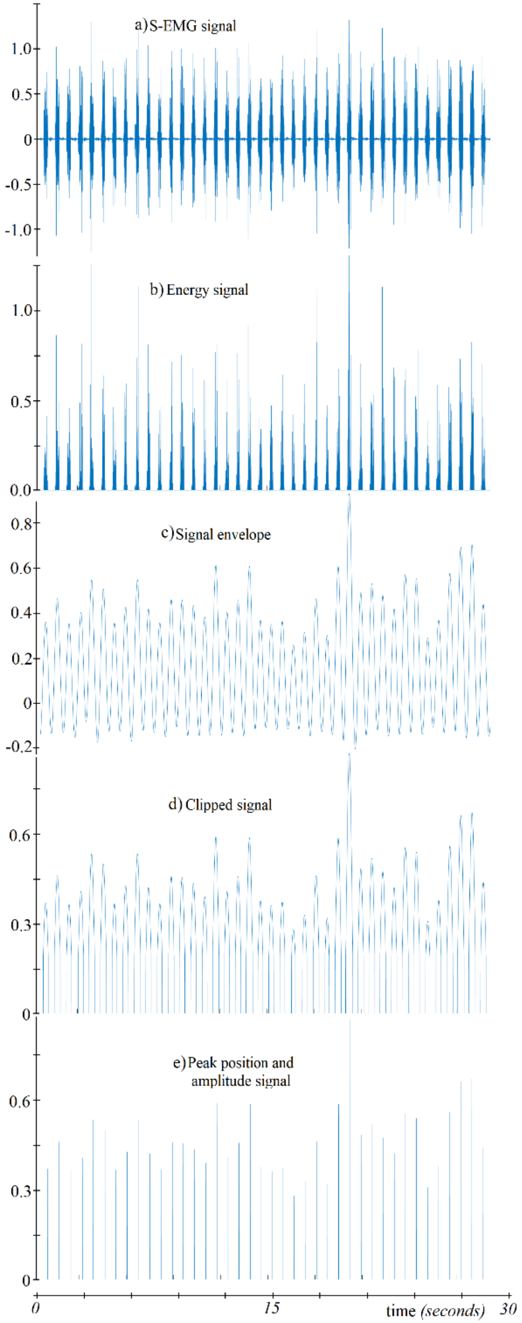
- Voice signal represented as a variable of continuous domain on top (***continuous random process***) and as a discrete domain variable on bottom (***discrete random sequence***).



Examples of Random Processes

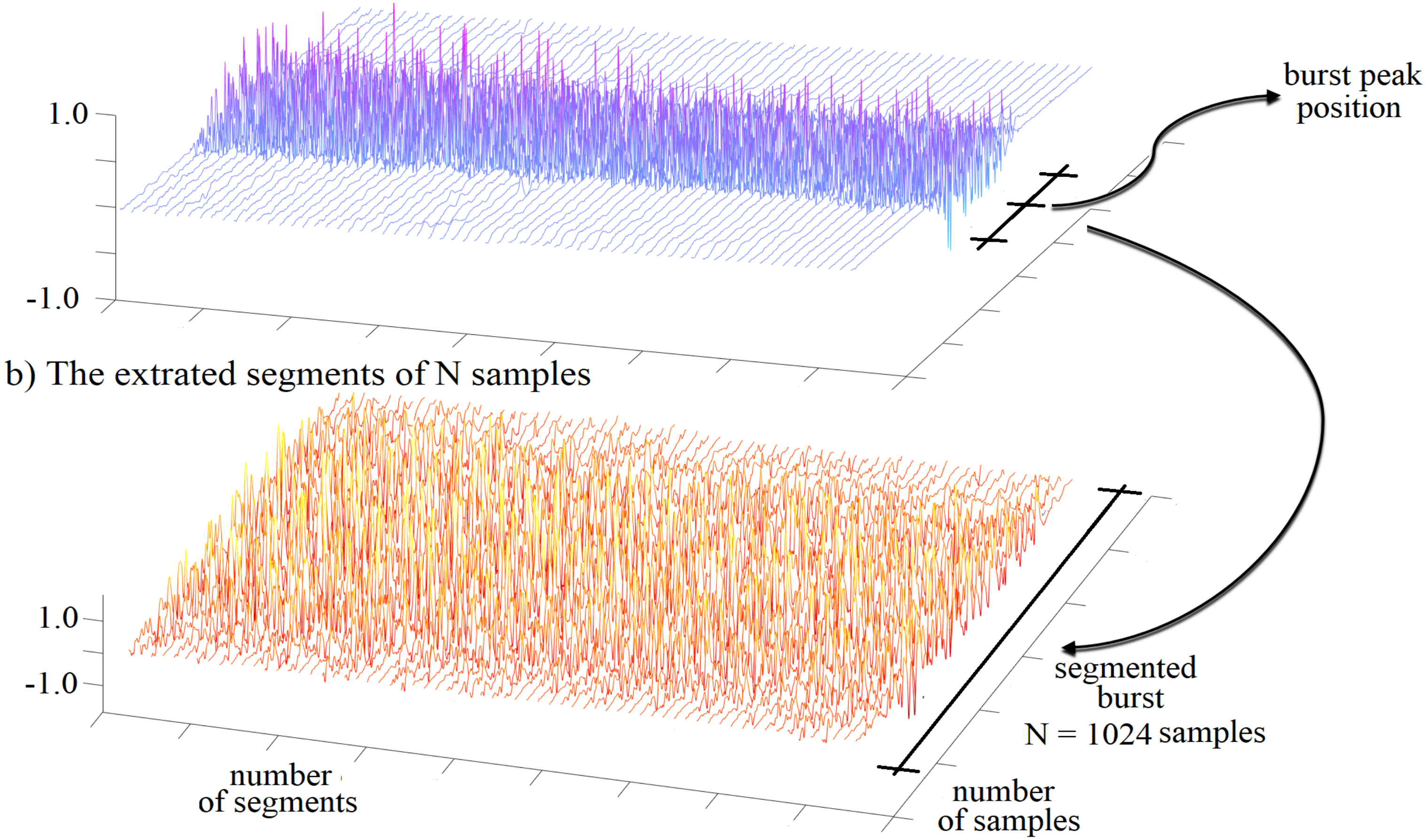
Example of mixed-valued random process: X((t,n), Ω) (time, window number), variable range)





a) 2D S-EMG signal representation - interpolated (2:1)

b) The extrated segments of N samples



Joint Distributions of Time Samples

- Let $X(t_1)$, …, $X(t_k)$ be the k random variables obtained by sampling the random process $X(t)$ at time instants t_1, …, t_k, where k is a positive integer.
- The ***kth-dimension joint* CDF *of a random process*** is then defined as follows:

$$F_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k) = P[X(t_1) \leq x_1,\ldots,X(t_k) \leq x_k]$$

Independent Random Process

- A discrete-time random process consisting of a sequence of independent random variables with common CDF $F_X(x)$ is called the ***independent random process***.

$$F_{X(t_1),\ldots,X(t_k)}(x_1, \ldots, x_k) = F_X(x_1) \times F_X(x_2) \times \cdots \times F_X(x_k)$$

Mean Function of a Random Process

- For a random process $\{X(t), t \in \mathrm{T}\}$, the **mean function** $\mu_X(t) : \mathrm{T} \rightarrow \mathbb{R}$, is defined as

$$\mu_X(t) = E[X(t)]$$

- Analogically, for discrete-time random processes, if $\{X_n, n \in \mathrm{T}\}$ is a discrete-time random process, then

$$\mu_X(n) = E[X_n], \text{ for all } n \in T.$$

Linear Convolution

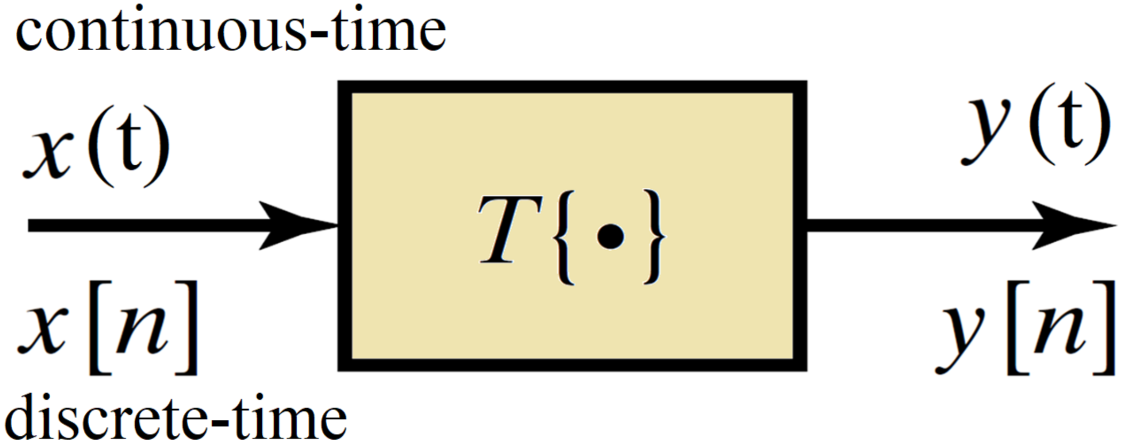
- Continuous-time linear convolution

$$x_1(t) * x_2\,(t) \equiv \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau$$

- Discrete-time linear convolution

$$x_1[n] * x_2[n] \equiv \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$$

Time-invariant linear systems (deterministic)



- Response os Continuous-Time Linear System:

$y(t) = \mathrm{T}\{x(\mathrm{t})\} = x(\mathrm{t}) * h(\mathrm{t})$

$$y(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = x(t) * h(t)$$

- Response os Discrete-Time Linear System:

$y[\mathrm{n}] = \mathrm{T}\{x[\mathrm{n}]\} = x[\mathrm{n}] * h[\mathrm{n}]$

$$y[n] \equiv \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] * h[n]$$

Autocorrelation function

- For a random process $\{X(t), t \in T\}$, the **autocorrelation function** or, simply, the **correlation function**, $R_X(t_1, t_2)$, is defined by

$$R_X(t_1, t_2) = E[X(t_1)\, X(t_2)], \text{ for } t_1, t_2 \in T.$$

Autocorrelation function

- The autocorrelation function of 1(one) dimension in continuous-time can be defined by

$$R_X(t) = \int_{-\infty}^{\infty} x(\tau)x(t+\tau)\,d\tau = x(t) * x(-t)$$

- For a N samples length sequence of 1(one) dimension in discrete-time the autocorrelation can be defined with

$$R_X[n] = \sum_{m=-\infty}^{\infty} x[m]x[n+k] = x[n] * x[-n]$$

Autocovariance function

- For a random process $\{X(t), t \in J\}$, the **autocovariance function** or, simply, the **covariance function**, $C_X(t_1, t_2)$, is defined by

$$C_X(t_1, t_2) = \mathrm{Cov}\,(X(t_1), X(t_2)) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2), \text{ for } t_1, t_2 \in \mathrm{T}.$$

Multiple Random Processes: Cross-correlation

- Given two random processes $\{X(t), t \in T\}$ e $\{Y(t), t \in T\}$, the cross-correlation, $R_{XY}(t_1, t_2)$, is defined by

$$R_{XY}(t_1, t_2) = E[X(t_1)\, Y(t_2)], \text{ for } t_1, t_2 \in T.$$

Multiple Random Processes: Cross-correlation

- For two continuous-time signals of 1(one) dimension in continuous-time the cross-correlation can be written as

$$R_{XY}(t,\tau) = \int_{-\infty}^{\infty} x(\tau)y(t+\tau)\,d\tau = x(t) * y(-t)$$

- For two sequences of 1(one) dimension in discrete-time the cross-correlation can be defined as

$$R_{XY}[n,m] \equiv \sum_{m=-\infty}^{\infty} x[m]y[n+m] = x[n] * y[-n]$$

Multiple Random Processes: cross-covariance

- Given two random processes $\{X(t), t \in T\}$ e $\{Y(t), t \in T\}$, the **cross-covariance**, $C_{XY}(t_1, t_2)$, is defined by

$$C_{XY}(t_1, t_2) = \mathrm{Cov}(X(t_1), Y(t_2)) = E[(X(t_1) - \mu_X(t_1))(Y(t_2) - \mu_Y(t_2))]$$
$$= R_{XY}(t_1, t_2) - \mu_X(t_1)\,\mu_Y(t_2), \text{ for } t_1, t_2 \in T.$$

Cross-correlation and Cross-covariance Functions

- The random processes $X(t)$ and $Y(t)$ are ***uncorrelated*** if their cross-covariance function is zero, and they are ***orthogonal*** if their cross-correlation function is zero. In other words, for all t_1 and t_2, we have the following:

$$C_{XY}(t_1, t_2) = 0 \Leftrightarrow \text{Uncorrelated random processes}$$

$$R_{XY}(t_1, t_2) = 0 \Leftrightarrow \text{Orthogonal random processes}$$

Independent Random Processes

- Two random processes $\{X(t), t \in T\}$ and $\{Y(t), t \in T'\}$ are said to be **independent** if, for all

$$t_1, t_2, \ldots, t_m \in T$$

- and

$$t'_1, t'_2, \ldots, t'_n \in T',$$

- the set of random variables

$$X(t_1), X(t_2), \cdots, X(t_m)$$

- are independent of the set of random variables

$$Y(t'_1), Y(t'_2), \cdots, Y(t'_n).$$

Independent Random Processes

- The two random processes are independent definition implies that for all real numbers $x_1, x_2, \cdots, x_m$ and $y_1, y_2, \cdots, y_n$, we have

$$F_{X(t_1),X(t_2),\cdots,X(t_m),Y(t_1'),Y(t_2'),\cdots,Y(t_n')}(x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n)$$
$$= F_{X(t_1),X(t_2),\cdots,X(t_m)}(x_1, x_2, \cdots, x_m) \cdot F_{Y(t_1'),Y(t_2'),\cdots,Y(t_n')}(y_1, y_2, \cdots, y_n).$$

- If two random processes $X(t)$ and $Y(t)$ are independent, then their covariance function, $C_{XY}(t_1, t_2)$, for all t_1 and t_2 is given by

$$\begin{aligned} C_{XY}(t_1, t_2) &= \mathrm{Cov}\big(X(t_1), Y(t_2)\big) \\ &= 0 \quad (\text{since } X(t_1) \text{ and } Y(t_2) \text{ are independent}). \end{aligned}$$

Fundamental concepts: *Strict-Sense Stationary*

- A continuous-time random process $X(t)$ is said to be a ***stationary process*** or a ***strict-sense stationary process***, if the **PDF** of any set of samples does not vary with time.
- In other words, the joint distribution of $X(t_1), \ldots, X(t_k)$ is the same as the joint distribution of $X(t_1 + \tau), \ldots, X(t_k + \tau)$, for any positive integer k, any time shift τ, and all choices of $t_1, \ldots, t_k$.
- A ***nonstationary process***, on the other hand, is characterized by a joint distribution that depends on time instants $t_1, \ldots, t_k$.

Fundamental concepts: *Strict-Sense Stationarity*

- ***Strict-sense Stationarity*****:** A stochastic process, X_n, is said to be *strict-sense stationary* (**SSS**) if its statistical properties are **invariant to a shift of the origin**, or if $\forall k \in Z$.

$$P(x_n, x_m, \ldots, x_r) = P(x_{n-k}, x_{m-k}, \ldots, x_{r-k}),$$

and for *any* possible combination of time instants, $n, m, \ldots, r \in Z$.

Stationarity: A continuous-time random process

- A continuous-time random process $\{X(t), t \in \mathbb{R}\}$ is **strict-sense stationary** or simply **stationary** if, for all $t_1, t_2, \cdots, t_r \in \mathbb{R}$ and all $\Delta t \in \mathbb{R}$, the joint CDF of

$$X(t_1), X(t_2), \cdots, X(t_r)$$

- is the same as the joint CDF of

$$X(t_1 + \Delta t), X(t_2 + \Delta t), \cdots, X(t_r + \Delta t)$$

Fundamental concepts: *Wide-Sense Stationary*

- A random process $X(t)$ is a ***wide-sense stationary process*** or ***weakly stationary process***, if it has a constant **mean**, i.e.:
  - it is **independent of time**,
  - and also, its **autocorrelation function** depends only on the time difference $\tau = t_2 - t_1$ and not on t_1 and t_2 individually.

Fundamental concepts: *Wide-Sense Stationary*

- In other words, in a wide-sense stationary process, the **mean** and **autocorrelation functions** do not depend on the choice of the time origin.

- All random processes that are stationary in the strict sense are wide-sense stationary, but the converse, in general, is not true.

Fundamental concepts: *Wide-Sense Stationary*

- ***Wide-sense Stationarity***: A stochastic process, is said to be *wide-sense stationary* (WSS) if the mean value is constant over all time instants and the autocorrelation (autocovariance) signals/sequences depend on the difference of the involved time indices, or (for continuous random process)

$$\mu_X(t) = E[X(t)] = \text{constant and}$$

$$R_X(t_1, t_2) = E[X(t_1)\, X(t_2)] = R_X(t_2 - t_1) = R_X(\tau)$$

- For discrete-time random process

$$\mu_X = \mu, \text{ and } R[n-k] = R[k].$$

Wide-Sense Stationary: autocorrelation function $R_X(\tau)$ properties.

- $R_X(\tau)$ is an even function, it means $R_X(\tau) = R_X(-\tau)$.

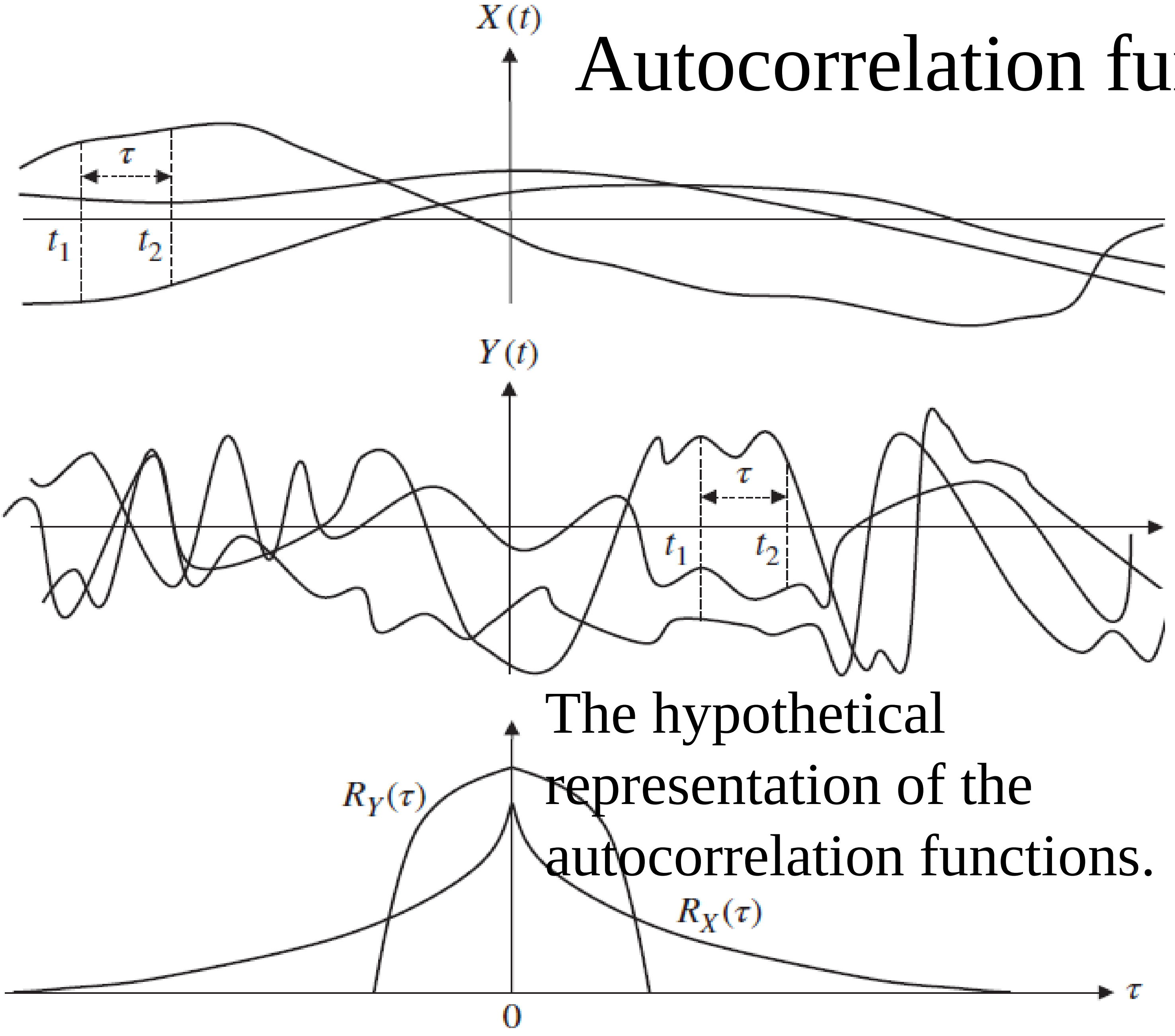
- $R_X(0) = E[X^2(t)] \geq 0$ gives the mean-square value of the random process (the energy of a signal/sequence).

- The maximum value of $R_X(\tau)$ occurs at $\tau = 0$, i.e. $|R_X(\tau)| \leq R_X(0)$.

Wide-Sense Stationary: autocorrelation function $R_X(\tau)$ properties.

- The more rapidly $X(t)$ changes with time t, the more rapidly $R_X(\tau)$ decreases from its maximum $R_X(0)$ as τ increases.
- It happens simply because the autocorrelation function is a measure of the rate of change of a random process and measures the predictability of a random process.

Autocorrelation function $R_X(\tau)$ properties



The hypothetical representation of the autocorrelation functions.

- The figure on the left shows two sets of sample functions: X(t) and Y(t).
- X(t) and Y(t) present different temporal behaviors. X (t) presents smoother transitions (low frequencies), while Y(t) presents abrupt transitions (high frequencies).

Wide-Sense Stationary: autocorrelation function $R_X(\tau)$ properties.

- The autocorrelation function $R_X(0)$ approaches the square of the mean, $(E[X(t)])^2$, as $\tau \to \infty$.

- For a zero-mean process, this means that samples become uncorrelated for large lags.

Wide-Sense Stationary: autocorrelation function $R_X(\tau)$ properties.

- The autocorrelation function does not have to be positive or negative for all values of τ.

- The autocorrelation function cannot have an arbitrary shape, as it must have a Fourier transform (CTFT or DTFT).

Wide-Sense Stationary: autocorrelation function $R_X(\tau)$ properties.

- The autocorrelation function may have the following components:

  - A component that approaches zero as $\tau \to \infty$ (e.g. an exponential function).
  - A periodic component (e.g. a sinusoidal function).
  - A constant component representing the square of the mean (e.g. any nonnegative real number).

Wide-Sense Stationary: autocorrelation function $R_X(\tau)$ properties.

- It is important to note that a wide-sense stationary random process yields a unique autocorrelation function.

- But the opposite is not true, i.e. two different wide-sense stationary random processes may have the same autocorrelation function.

Example: Non-Stationary measurement applied for muscle fatigue estimation

- The description of fatigue phenomenon through non-invasive tools is an issue of interest to many areas of science such as biomechanics, clinical and orthopedic medicine, physiotherapy and rehabilitation, control of interfaces, intelligent prosthetics and exoskeletons, and as expert systems to support medical diagnosis
- Noninvasive investigation of muscle fatigue through the use of surface electromyography (S-EMG) signals processing has been the subject of many studies in the last three decades.
- The human body can perceive in different ways increasing fatigue during physical or cognitive activities, such as decreasing performance, increasing heartbeat, increasing respiratory frequency, pain, and failure.

Example: Non-Stationary measurement applied for muscle fatigue estimation

- Modifications measured in myoelectric signals during physical exertion may be associated with a spectral signature of the muscle fatigue phenomenon and can be quantified.

- The **magnitude spectrum displacement to low frequencies** (non-stationary) and **the increase in the dynamic range** (non-stationary) of the S-EMG signal during physical activities with fatigue production have been reported by numerous studies and accepted as indicators of the muscular fatigue intensity

Example: Non-Stationary measurement applied for muscle fatigue estimation *y*

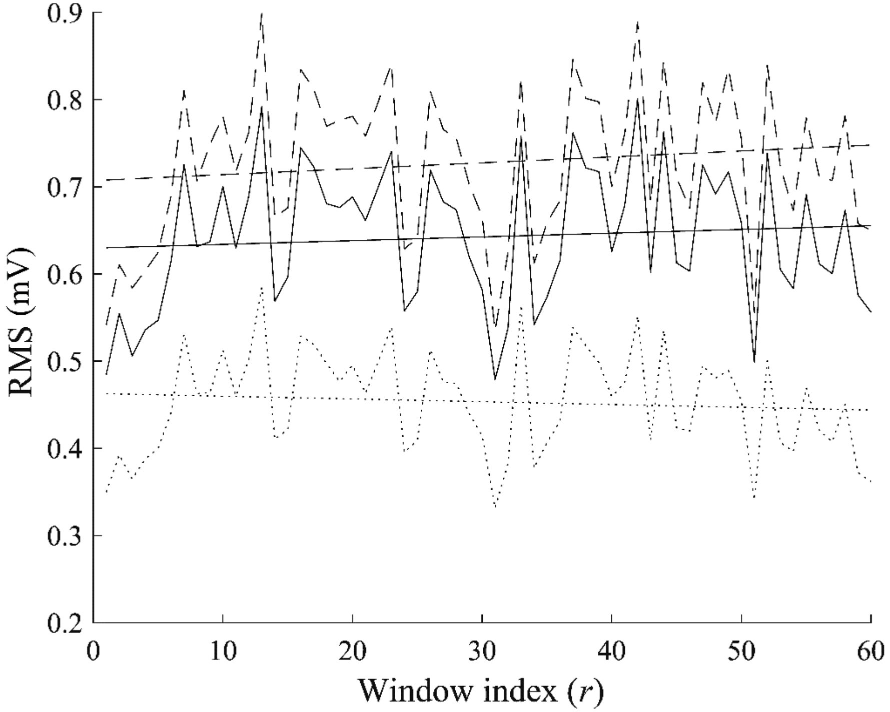
- In time domain analysis, the amplitude of the Root Mean Square (RMS) values of the EMG signal, the integrated EMG (iEMG), or zero crossing rate are commonly used as objective estimators of fatigue.
- In frequency domain, the mean power frequency (MPF) or median frequency (MDF) of the EMG signal and muscle fiber conduction velocity (MCV) are used as classical fatigue indicators.

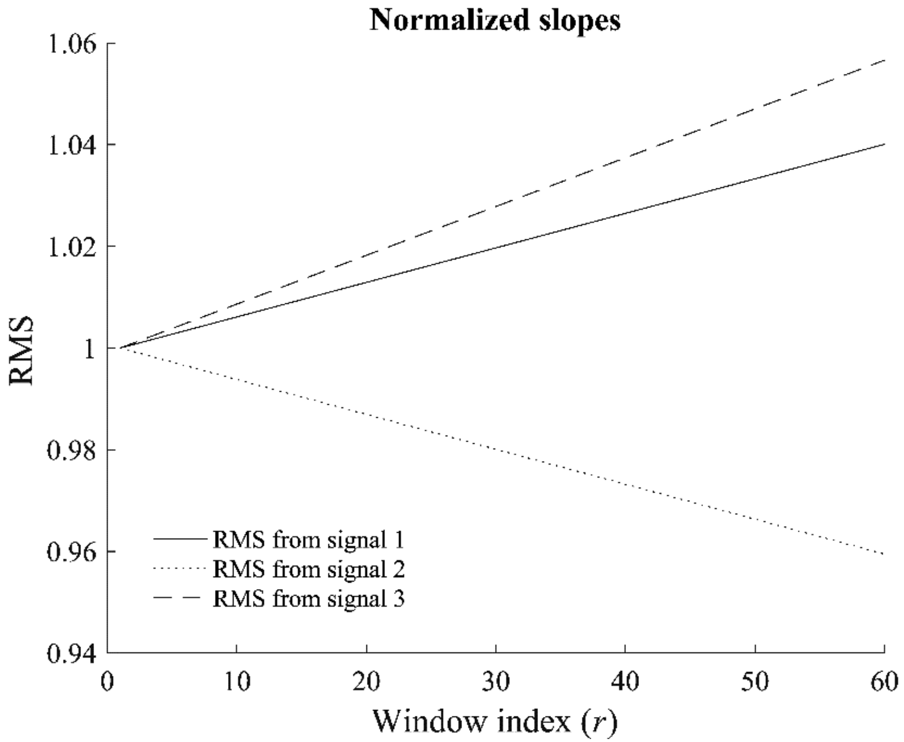
Example: Non-Stationary measurement applied for muscle fatigue estimation

- Other estimators have also been proposed to investigate muscle fatigue, for example objective parameters based on Hilbert transform, Wavelet transform, principal component methods, complex covariance, and multicomponent AM–FM Decomposition.

Example: Non-Stationary measurement applied for muscle fatigue estimation

- Fig on the left shows the RMS value calculated in successive windows on three distinct S-EMG signal channels (channels 1, 2, and 3, respectively). Fig on the right presents the classical least-square regression line used to estimate fatigue.





Although the signals were obtained from the same subject during the same experiment, the slope of the regression line computed with a classical methodology (RMS value) present discrepant results.

Example: Non-Stationary measurement applied for muscle fatigue estimation

- Some points are listed in scientific literature include:
  - 1) the variation of the temporal indices when the same experiment is performed under the same conditions on different dates.
  - 2) the variation of spatial indices because the results depend on the electrode fixation position and indices from signals received at different points in muscle belly generate low repeatable information.
  - 3) low reproducibility of fatigue indexes reported for different muscle groups and in physical experimental protocols varying contraction intensities.

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- Consider a stochastic electromyographic signal segment $x_m[n], n = 0,1,\ldots,N-1$, which is a set of N samples *m* corresponding to segment index *m* = 1, 2,..., L (consecutive or overlapping time window) and

$$\hat{x}_m[n] = x_m[n]w[n], n = 0,1,\ldots,N-1,$$

where $w[n],\ n = 0,1,\ldots,N-1$ is a specific window type.

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- Considering the stationarity of the process (homogeneous), the expectancy is independent of position $\vec{v}$, i.e., $\mu_\lambda(\vec{v}) = \mu_\lambda$.

$$\mu_\lambda = \sum_{m=1}^{r} p(\lambda)\lambda[m]$$

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- In a **wide stationary process ,** the autocorrelation ($R_\lambda[k]$) and the autocovariance ($C_\lambda[k]$) functions could be expressed as

$$R_\lambda[k] = (R_\lambda[0] - {\mu_\lambda}^2)e^{-\alpha|k|} + {\mu_\lambda}^2$$

$$C_\lambda[k] = R_\lambda[0]e^{-\alpha|k|}$$

where α is a decay factor of the autocorrelation function (and of the autocovariance function) that provides the statistic dependence of explicit temporal parameters.

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- Consider that a random variable (a certain time, frequency, or time-frequency parameter) $\lambda[m]$ computed over $\hat{x}_m[n]$. In a **wide stationary process**, the sum of Expectancy values computed over all the r windows (accumulated), for $r \gg 1$, will approach

$$\sum_{m=1}^{r} E\{\lambda[m]\} \cong \sum_{r=1}^{r} \mu_\lambda \cong \mu_\lambda \sum_{r=1}^{r} 1 \cong r\mu_\lambda = \lambda_c[r]$$

$\lambda_c[r]$ is the r-th cumulated parameter.

estimation.

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- It can be normalized by $\lambda_c[r]|_{r=1}$ or by an estimative based on few initial values of r
- The weighted-cumulated parameter, in a stationary process for $r \gg 1$ could be expressed as an approaching values of r.

$$\lambda_a[r] = \frac{1}{\lambda_c[1]} \sum_{m=1}^{r} E\{\lambda[m]\} = \frac{\lambda_c[r]}{\lambda_c[1]} \cong r$$

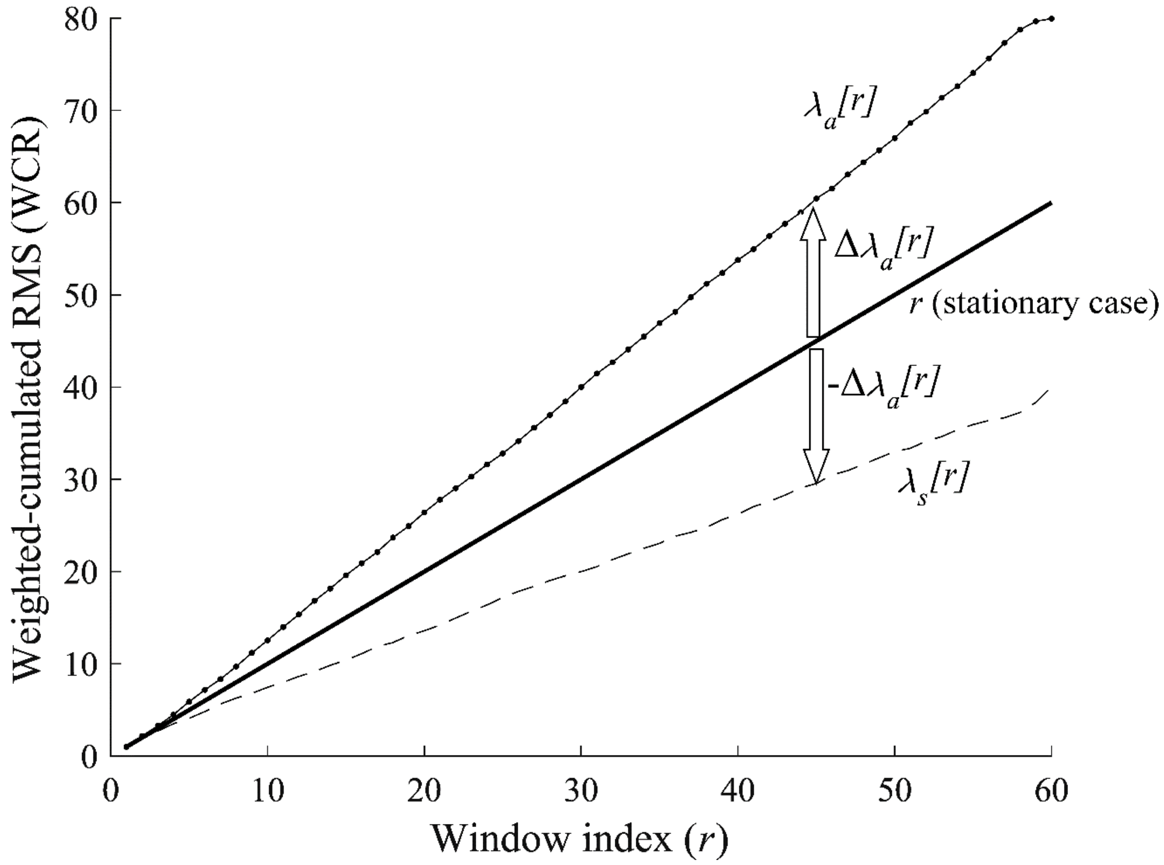
where $\lambda_a[r]$ represents the weighted-cumulated parameter.

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- Thus, $\lambda_a[r]$ corresponds to the cumulated quantity over the m analysis windows and is normalized by the first outcome value.

- If $\lambda_a[r]$ representing a strictly/wide stationary random process, the curve of $\lambda_a[r]$ could be represented by a first-degree function crossing the axes through (0,1) and a slope equal to 1 (angular coefficient equal to $\pi/4$ radians).

Example: Weighted-Cumulated Methodology for Fatigue Estimation



- This illustrative example shows the weighted-cumulative stationary case behavior by solid line plot.

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- However, in a real physical isometric or dynamic flexion test with fatigue production, the behavior of the S-EMG signal is not stationary.
- This non-stationarity phenomenon has a certain “inertia”, that is associated to the increasing physiological muscle fatigue.
- These changes are due to **the increase in the dynamic range** of the S-EMG signal and the **spectral shift to low frequencies**.

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- These behavioral characteristics, which correspond to a non-stationarity phenomenon, are perceived by $y_a[m]$ computed over the fatiguing test.
- Consequentially, the $y_a[m]$ **curve diverges slowly and progressively** downwards proportionally to the physiological fatigue increasing process.
- The curve distortion observed a first-degree function crossing the axes through (0,1) and a slope equal to 1 (angular coefficient equal to π/4 radians) is due to non-stationarity caused by the physiological phenomenon of fatigue.

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- As the object of the study is muscle fatigue, and this process is relatively slow with respect to sampling rate, the expected non-stationarity has a certain "inertia", which is manifesting as time elapses from the beginning of the experiment.
- The objective estimator of fatigue must discriminate the non-stationarity in S-EMG signal to associate it with this physiological phenomenon.

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- Desirable boundary conditions for objective computational fatigue estimators established for the present work are:
- 1) reduce the sensitivity of the estimator to spatial location of surface electrodes;
- 2) reduce variation in dynamic range of signal observed in surface electromyography signals when experiments are performed on different dates using the same experimental protocol;

The fatigue estimator was normalized by its first estimation, which creates a dimensionless magnitude parameter in a way that minimizes the effect of the temporal variance of the physical procedure and the spatial variance of the location of the electrodes generally observed in the S-EMG signals.

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- The normalization should adjust the dynamic range of the fatigue descriptor parameters so that regardless of electrodes placement or the date of the experiment, it should assume the same initial value.
- The fact that muscle fatigue descriptors are dimensionless makes comparisons between them easy, allowing identification of the sensitivity of the objective estimator for muscular fatigue process.

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- 3) reduce spurious broadband noise (white noise) and other local spurious short-term phenomena that occasionally occur during the experiment;
- 4) preserve the spectral signature of individuals at different experimental times.

- The objective fatigue indicator was chosen as a cumulative estimator. This approach aims to reduce the effects of white Gaussian noise, which by intrinsic characteristic is orthogonal to itself and self-cancels in a process that involves averaging.
- It also minimizes the effects of local spurious noises (such as electrode displacement), as the cumulative process dilutes the effects of local phenomena.

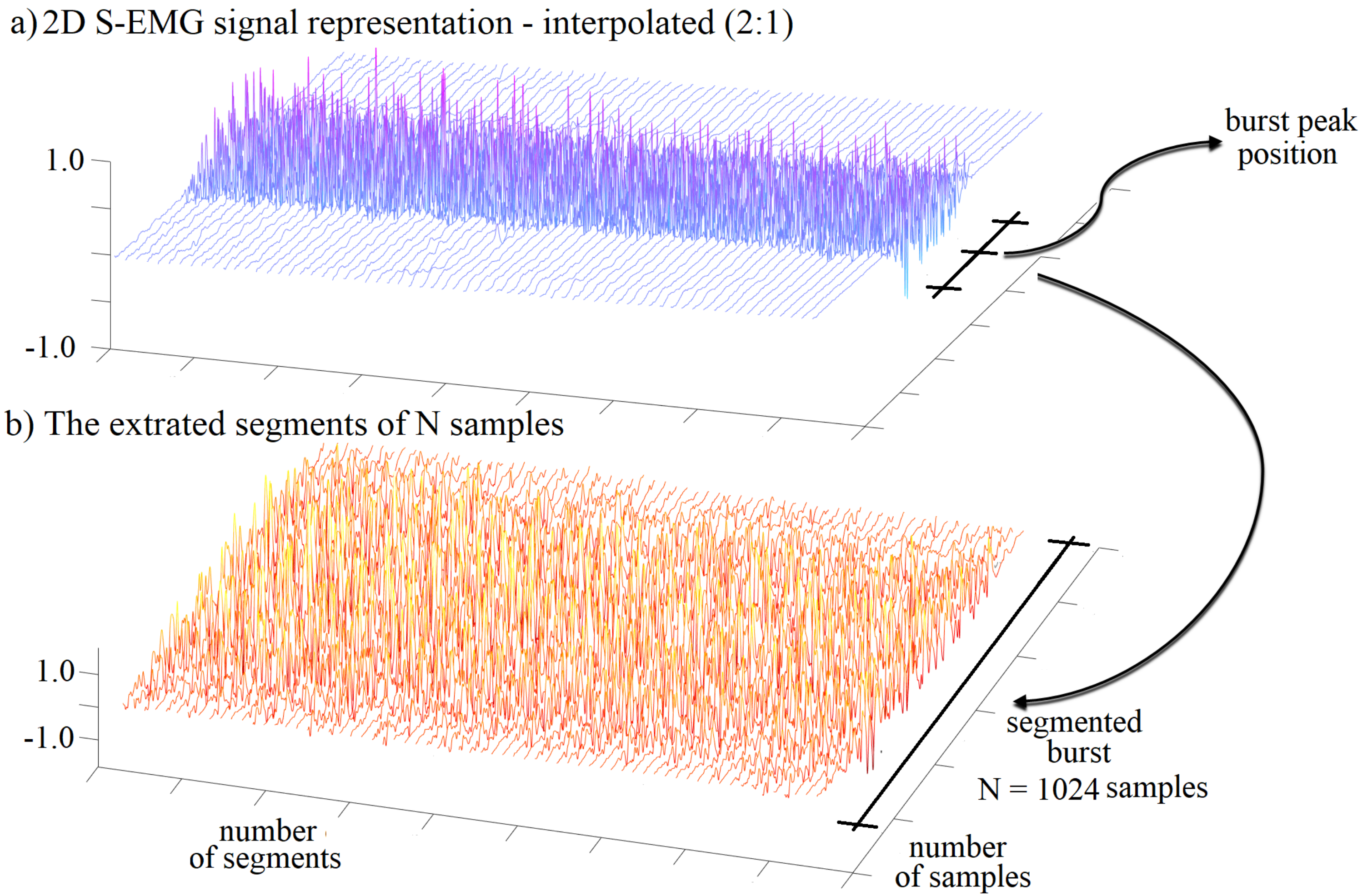
Example: Weighted-Cumulated Methodology for Fatigue Estimation

The example shown in the above plotted 3D figure has the sampling frequency $f_s = 2$kHz, and a spline interpolation algorithm of ratio 2:1 is applied.

In the upper part (a) of the figure, the segmented S-EMG signal is centered with respect to the burst peak position.

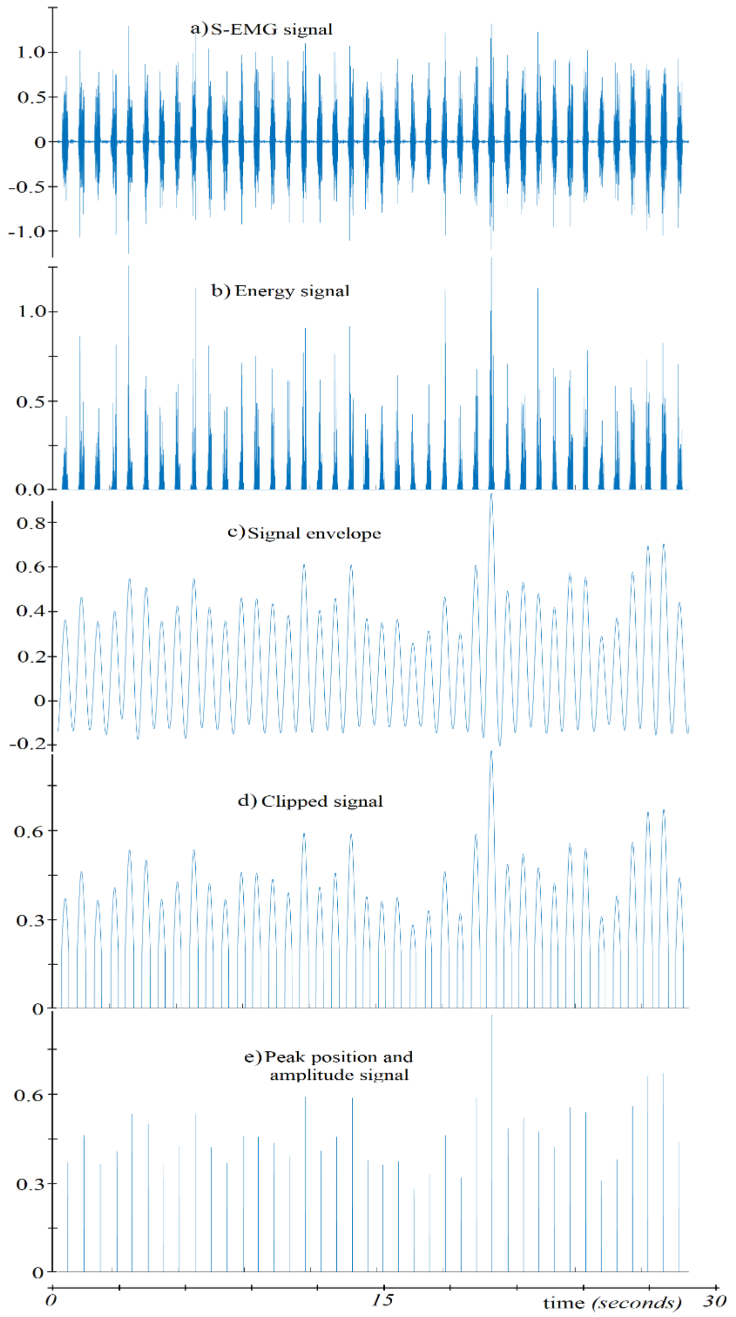
At the bottom of this figure, the extracted segments of length N = 1024 samples are shown.

These extracted S-EMG signal segments of N samples are used in the Weighted-Cumulated Methodology.



Example: Weighted-Cumulated Methodology for Fatigue Estimation

- The normalized waveforms obtained in several steps of the dynamic S-EMG signal segmentation algorithm in cycling.

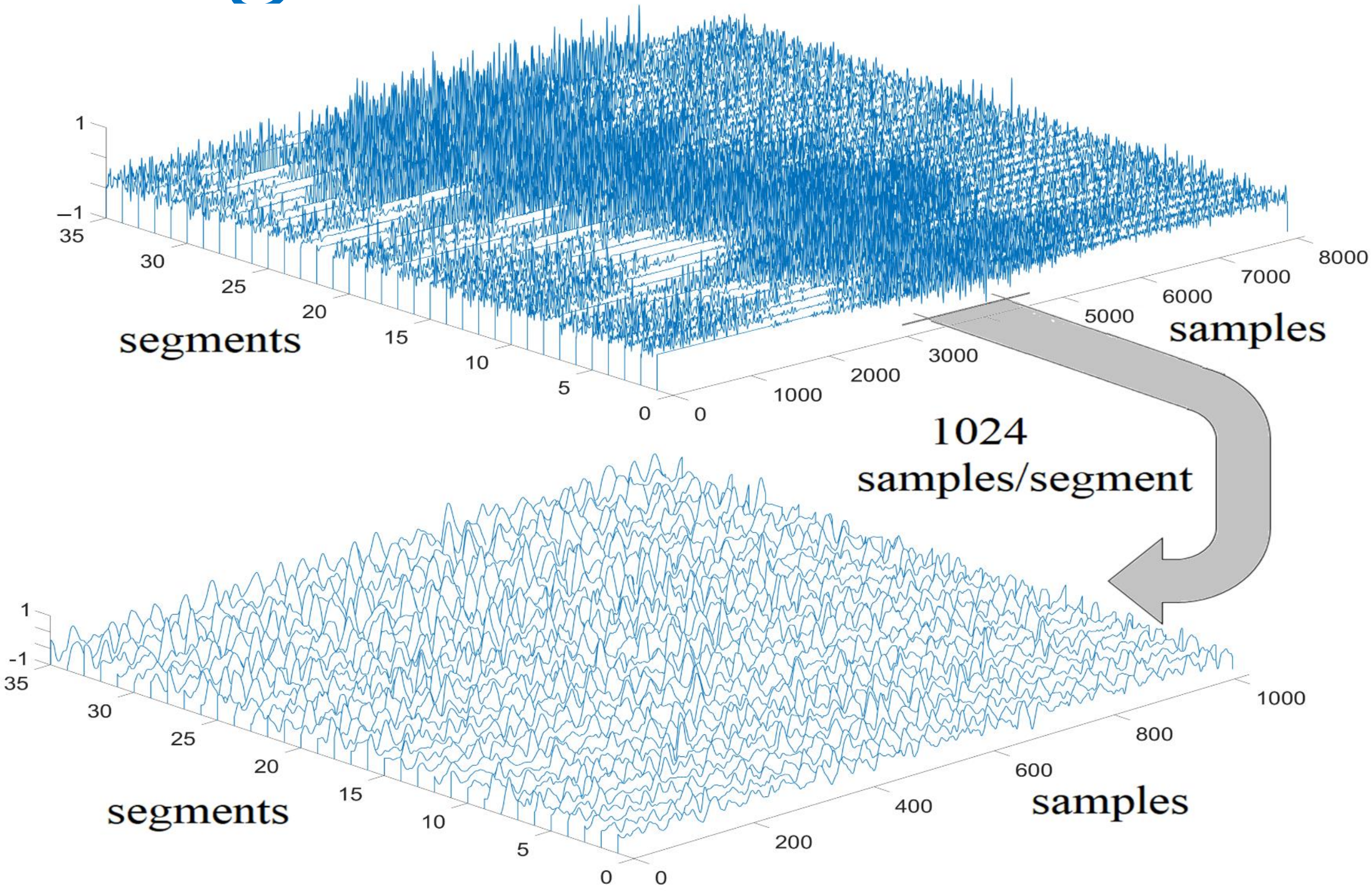


Example: Weighted-Cumulated Methodology for Fatigue Estimation

The S-EMG signal example collected from *biceps brachii*.

The exercise protocol cadence was two seconds for the concentric phase and two seconds for the eccentric phase.

The signal segmented based on burst peak location is presented on top, and below, is the second segmentation, which is the input of the weighted-cumulated methodology.



Example: Weighted-Cumulated Methodology for Fatigue Estimation

- Weighted-Cumulated Zero-Crossing estimator (WCZ)

$$\lambda_Z[r] = \frac{1}{\gamma}\sum_{m=1}^{r}\frac{N_Z[m]}{N} \qquad\qquad \gamma = \frac{N_Z[1]}{N}$$

where $N_Z[m]$ is the zero-crossing number of the *m-th* segment of S-EMG signal, N in the number of samples in each segment, $\lambda_Z[r]$ is the weighted-cumulated zero-crossing rate estimator computed until the *r-th* S-EMG segment, and γ is the zero-crossing rate value of the first segment.

- Weighted-Cumulated Root Mean Square estimator (WCR)

$$\lambda_R[r] = 2r - \frac{1}{\gamma}\sum_{m=1}^{r}\sqrt{\frac{1}{N}\sum_{n=0}^{N-1}\hat{x}_m[n]^2} \qquad\qquad \gamma = \sqrt{\frac{1}{N}\sum_{n=0}^{N-1}\hat{x}_1[n]^2}$$

where $\lambda_R[r]$ is the Weighted-Cumulated RMS (WCR) value estimator computed until the *r-th* S-EMG segment and γ is the RMS value of the first segment.

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- Weighted-Cumulated MDF estimator (WCM)

$$\lambda_M[r] = \frac{1}{\gamma}\sum_{m=1}^{r} \lambda_{MDF}[m] \qquad\qquad \gamma = \lambda_{MDF}[1]$$

where λ_MDF [m] is the MDF value computed over the m-th segment of S-EMG, λ_M [r] is the weighted-cumulated MDF estimator computed until the r-th S-EMG segment, and γ is the MDF value computed over the first segment

- Weighted-Cumulated Fourier estimator (WCF)

$$\lambda_F[r] = 2r - \frac{1}{\gamma}\sum_{m=1}^{r}\sqrt{\frac{1}{N-1}\sum_{k=1}^{N-1}(N-k)|X_m[k]|^2} \qquad\qquad \gamma = \sqrt{\frac{1}{N-1}\sum_{k=1}^{N-1}(N-k)|X_1[k]|^2}$$

where is the $\lambda_F[r]$ is the weighted-cumulated Fourier estimator computed until the *r-th* S-EMG segment, and γ is the normalization factor which is computed over the first time-signal window.

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- Weighted-Cumulated Wavelet estimator (WCW)

$$\lambda_W[r] = 2r - \frac{1}{\gamma}\sum_{m=1}^{r}\sqrt{\sum_{l=0}^{log_2(N)-1}\sum_{k=0}^{2^l-1}\frac{|W_m[2^l+k-1]|^2}{2^l}} \qquad \gamma = \sqrt{\sum_{l=0}^{log_2(N)-1}\sum_{k=0}^{2^l-1}\frac{|W_1[2^l+k-1]|^2}{2^l}}$$

where $\lambda_W[r]$ is the weighted-cumulated wavelet estimator computed until the r-th S-EMG segment and γ is the normalization factor, analogous to the weighted-cumulated Fourier estimator definition. The decimated Wavelet transform can be a real or a complex base. Symlet 5 base was used in the simulation results shown in this work.

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- Scalable Weighted-Cumulated Fourier Estimator

$$y_F[m] \equiv 2m - \frac{1}{\gamma}\sum_{r=1}^{m}\sqrt[p]{\sum_{k=1}^{N-1}(N-k)^p|X_m[k]|^p} \qquad \gamma = \sqrt[p]{\sum_{k=1}^{N-1}(N-k)^p|X_1[k]|^p}$$

- Scalable Weighted-Cumulated Wavelet Estimator

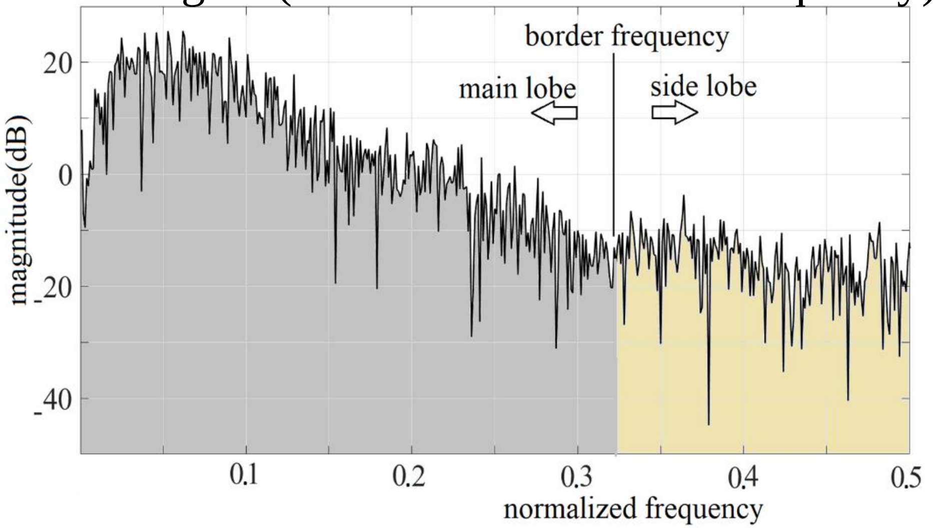
$$y_W[m] \equiv 2m - \frac{1}{\gamma}\sum_{r=1}^{m}\sqrt[p]{\sum_{l=0}^{log_2(N)-1}\sum_{k=0}^{2^l-1}(N-2^l)^p|W_m[2^l+k-1]|^p} \qquad \gamma = \sqrt[p]{\sum_{l=0}^{log_2(N)-1}\sum_{k=0}^{2^l-1}(N-2^l)^p|W_1[2^l+k-1]|^p}$$

Example: Weighted-Cumulated Methodology for Fatigue Estimation

- P-Side Side Attenuation Algorithm

$$y_{SL}[m] = 2m - \frac{1}{\gamma}\sum_{r=1}^{m}(y_{BW}[\mathrm{r}])^{q}\left(\sqrt[p]{\sum_{m=0}^{\beta[\mathrm{r}]-1}\ \sum_{k=m\frac{N}{L}}^{(m+1)\frac{N}{L}-1}|\psi[r,k]|^{p}}\right) \qquad \gamma = (y_{BW}[1])^{q}\left(\sqrt[p]{\sum_{m=0}^{\beta[1]-1}\ \sum_{k=m\frac{N}{L}}^{(m+1)\frac{N}{L}-1}|\psi[1,k]|^{p}}\right)$$

The DFT magnitude spectrum for N = 1024 samples, typical of an S-EMG signal (decibels × normalized frequency).



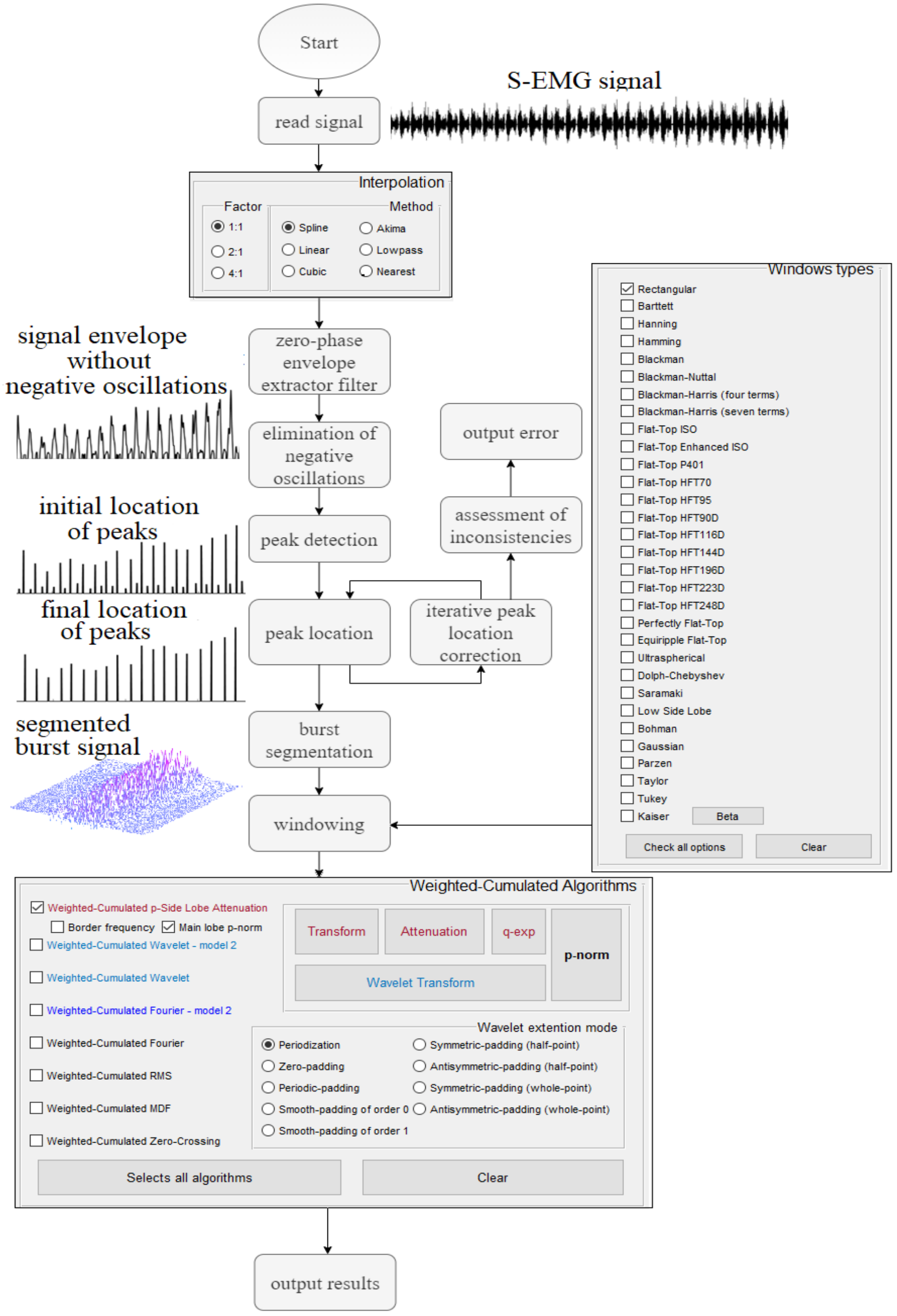
$$S_a^{p}[r,R] = \sum_{q=0}^{R-1}\ \sum_{k=q\frac{N}{L}}^{(q+1)\frac{N}{L}-1}|\psi[r,k]|^{p}\ for\ 0 \le R \le L-1.$$

$$S^{p}[r] = \sum_{k=0}^{N-1}|\psi[r,k]|^{p} \qquad 20log_{10}\left(\sqrt[p]{1-\frac{S_a^{p}[r,v]}{S^{p}[r]}}\right) = A\ dB$$

$$y_{BW}[\mathrm{r}] = N - \beta[\mathrm{r}] \qquad \beta[\mathrm{r}] = \frac{N}{L}(v+1) \qquad for\ \frac{N}{L} \le \beta[r] \le N,$$

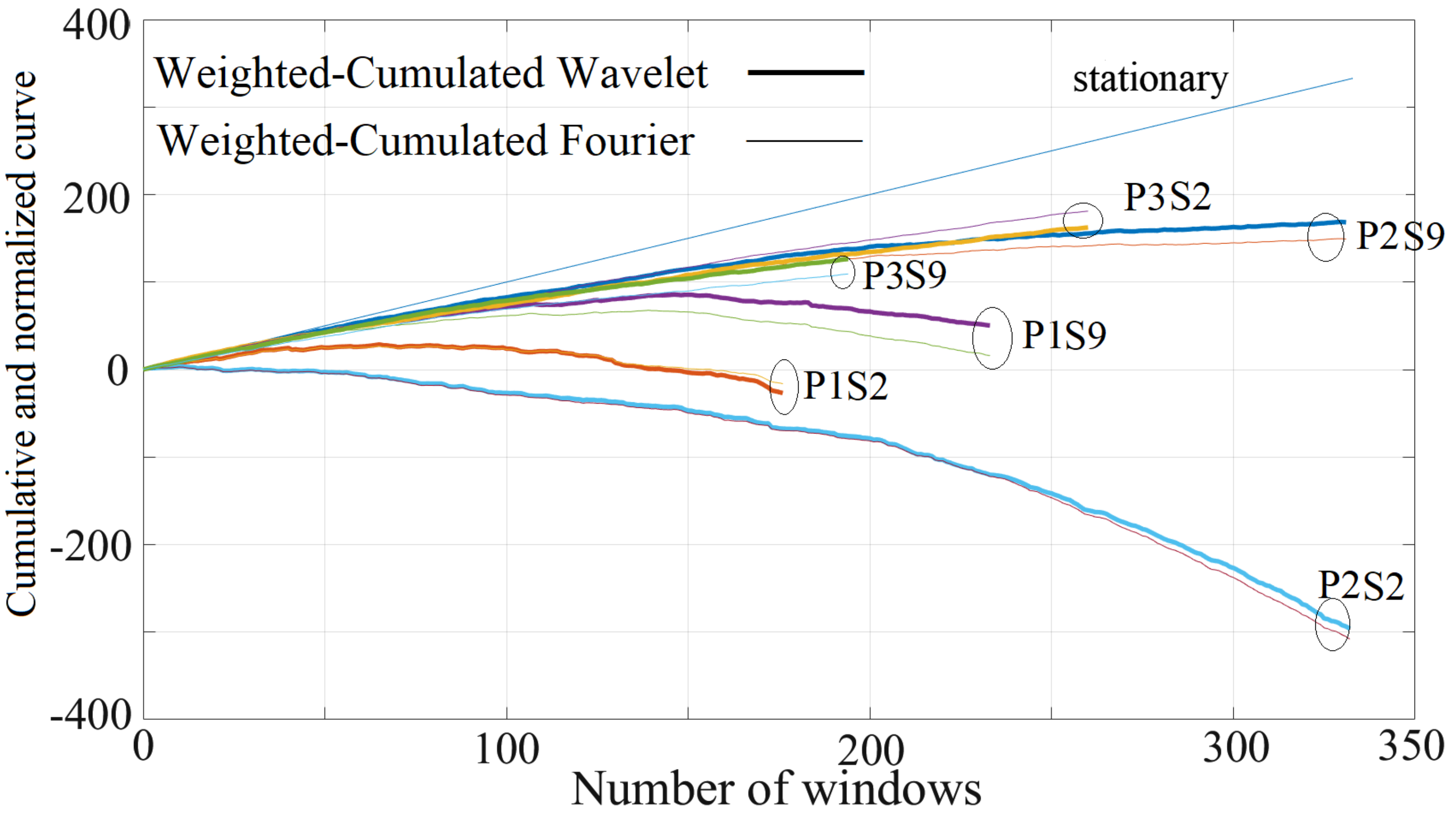
Example: Weighted-Cumulated Methodology for Fatigue Estimation

- Framework block diagram

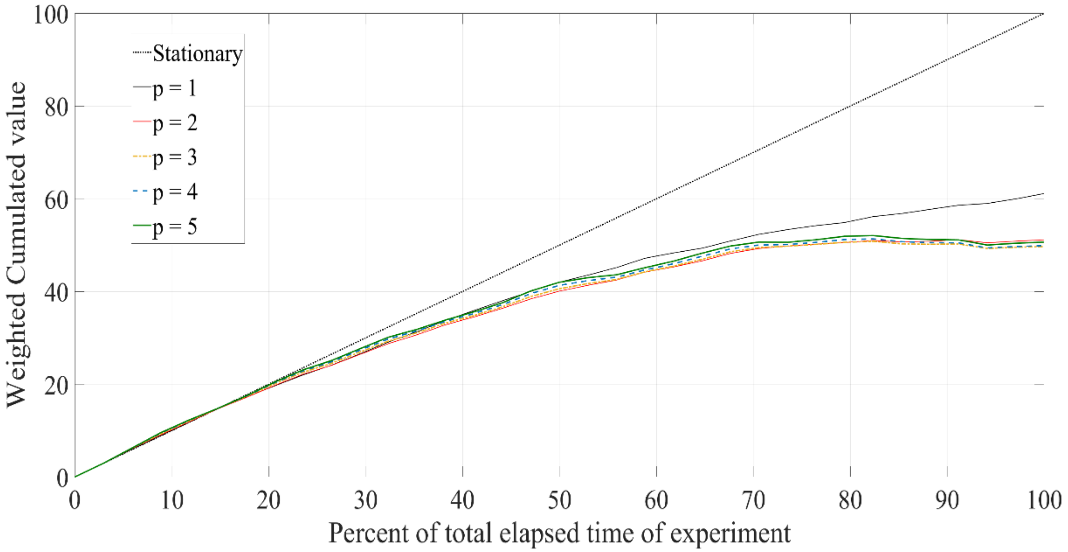


Example: Weighted-Cumulated Methodology for Fatigue Estimation

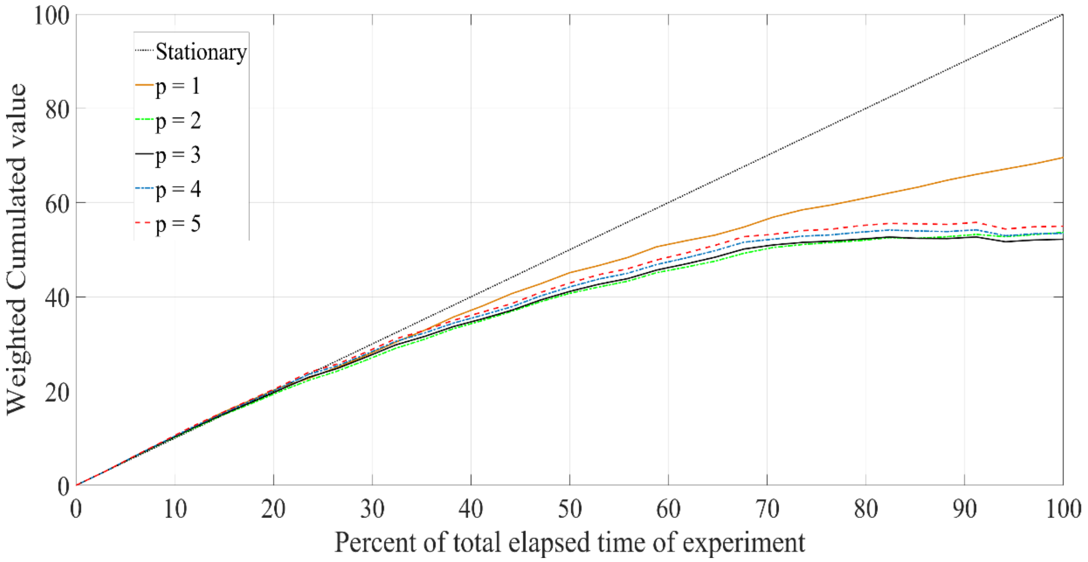
- The cumulative and normalized curves for participants 2 and 9 to compare behavior for the three proposed protocols for the WCF and WCW algorithms.
- The window length is 1024 samples. As an example of figure notation: P2S9 means Protocol 2 performed by subject 9.



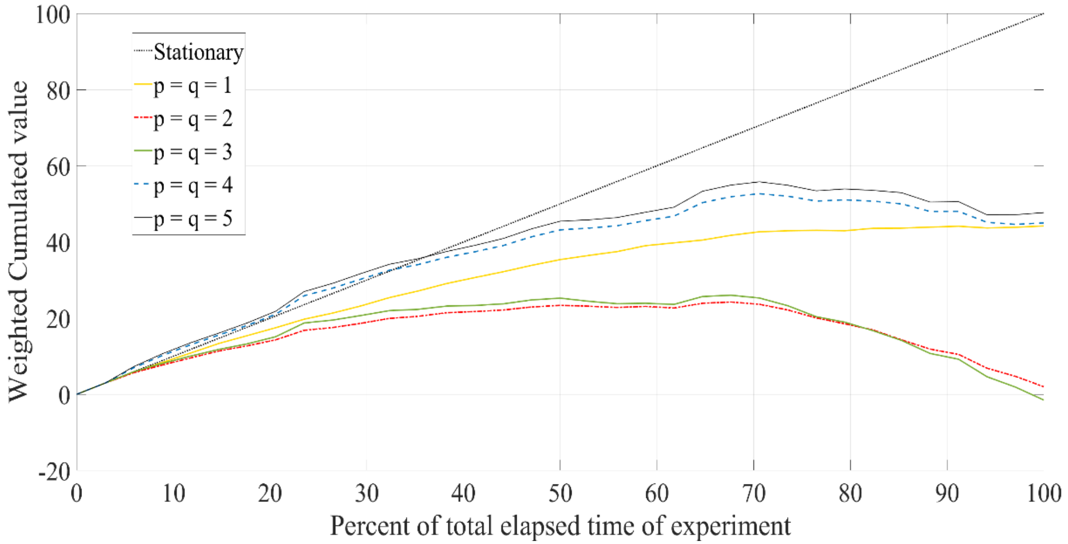
Example: Weighted-Cumulated Methodology for Fatigue Estimation



Results for the SWCW fatigue estimator.



Results for the SWCF fatigue estimator.



Results for the p-SL fatigue estimator for $p = q$.

Jointly Wide-Sense Stationary Processes

- The random processes $X(t)$ and $Y(t)$ are ***jointly wide-sense stationary***, if $X(t)$ and $Y(t)$ are each wide-sense stationary and if their **cross-correlation** depends only on $\tau = t_2 - t_1$, i.e. we have

$$R_{XY}(t_1, t_2) = R_{XY}(t_2 - t_1) = R_{XY}(\tau)$$

- We also have $R_{XY}(\tau) = R_{YX}(-\tau)$, which in turn means a shift of $Y(t)$ in one direction (in time) is equivalent to a shift of $X(t)$ in the other direction.

JointlyWide-Sense Stationary Processes

- $R_{XY}(0) = R_{YX}(0)$, but unlike autocorrelation function, they have no particular physical significance, nor do they represent mean-square values.

- The cross-correlation does not necessarily have its maximum at $\tau = 0$, but is upper-bounded by $\sqrt{R_X(0)R_Y(0)}$, and may not achieve this value anywhere.

JointlyWide-Sense Stationary Processes

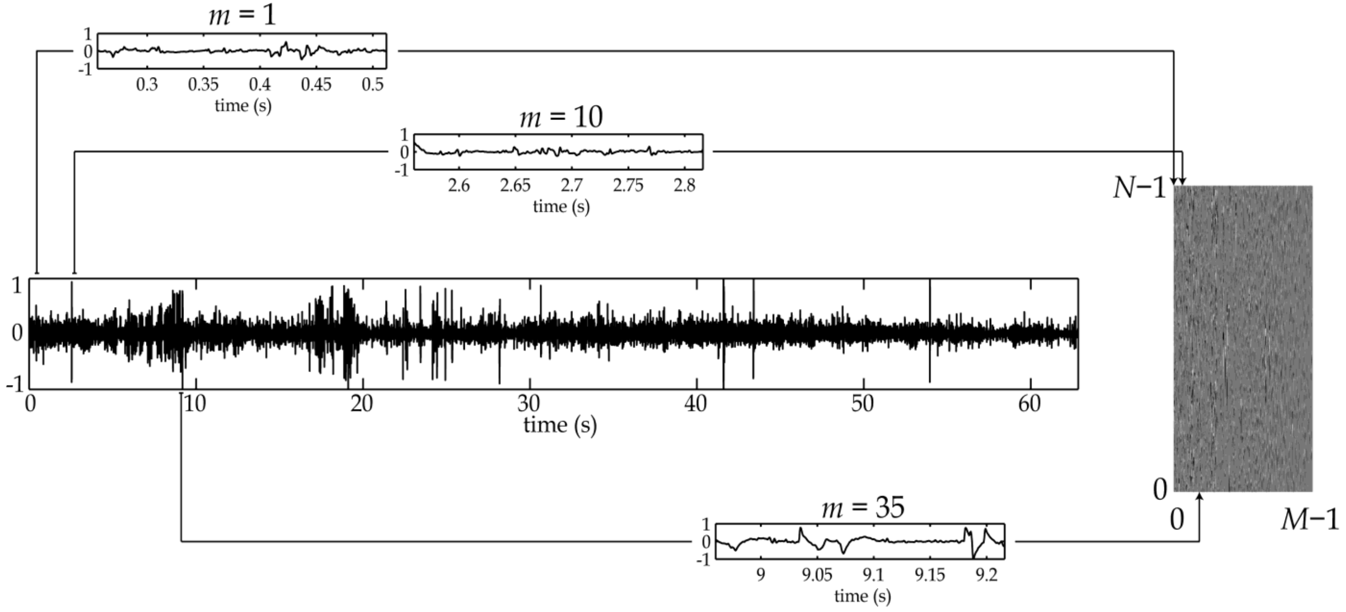
- If $X(t)$ and $Y(t)$ are statistically independent, then

$$\mathrm{R}_{XY}(\tau) = \mathrm{R}_{YX}(\tau) = E[X(t)]\, E[Y(t)].$$

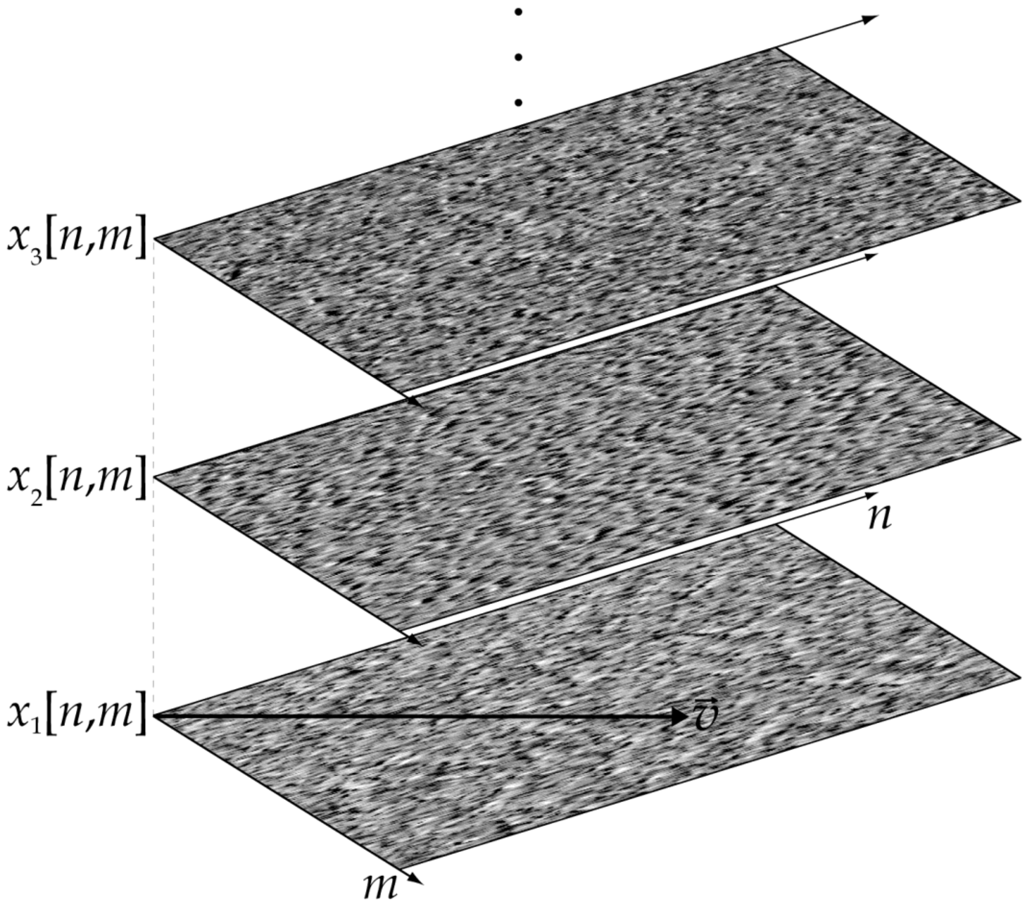
- The absolute value of the cross-correlation function is upper-bounded, as we have $|\mathrm{R}_{XY}(\tau)| \leq 0.5(\mathrm{R}_X(0) + \mathrm{R}_Y(0))$.

Example: JointlyWide-Sense Stationary Processes

- One-dimensional Surface electromyographic (S-EMG) signal rearranged into a two-dimensional matrix.



- Two-dimensional real **random field**, $x_s[n,m]$, showing multiple realizations of two-dimensionally arranged S-EMG signals.



Example: Jointly Wide-Sense Stationary Processes

- The autocorrelation function can be expressed as

$$R_x(n,m,i,j) = R_x[n-i, m-j] = R_x[i-n, j-m] \, .$$

- If we use the discrete variables k and r to denote the coordinate differences $n-i$ and $m-j$, respectively, the above equation can be rewritten as

$$R_x(n,m,i,j) = R_x[k,r] = R_x[-k,-r] \, .$$

Example: JointlyWide-Sense Stationary Processes

- The autocorrelation function of a homogeneous random field (wide sense stationary stochastic process) (e.g., S-EMG data) is a function of only two variables, *k* and *r*:

$$R_x[k,r] = E\{x[n,m]x[n+k,m+r]\}$$
$$= \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} x[n,m]x[n+k,m+r]$$

Example: Jointly Wide-Sense Stationary Processes

- Also, the autocovariance function, $C_x[k,r]$, is can be written as

$$C_x[k,r] = E\{x[n,m]x[n+k,m+r] - \mu^2\}$$
$$= \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} (x[n,m]x[n+k,m+r] - \mu^2)$$

Example: Jointly Wide-Sense Stationary Processes

- The two-dimensional S-EMG data can be considered as a homogeneous random field (shift-invariant property), the autocorrelation function, $R_x[k,r]$, may be assumed to be of the form

$$R_x[k,r] = (R_x[0,0] - \mu^2)e^{-\alpha|k|-\beta|r|} + \mu^2$$

- where α and β are positive constants. For the Surface electromyographic (S-EMG) signal we have

$$R_x[0,0] = E\{(x[n,m])^2\} = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} x[n,m]^2$$

$$\mu = E\{x[n,m]\} = \frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} x[n,m] = 0$$

Example: JointlyWide-Sense Stationary Processes

- For rearranged S-EMG data, μ = 0 , and the autocorrelation function is reduced to:

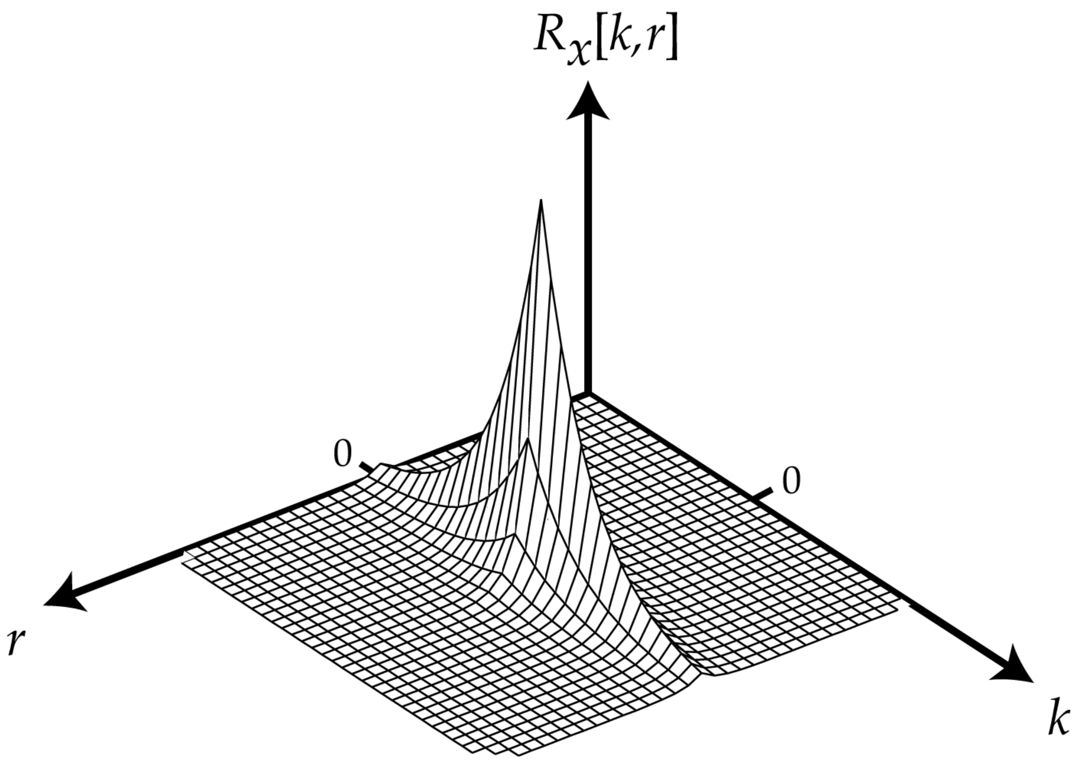
$$R_x[k,r] = R_x[0,0]e^{-\alpha|k|-\beta|r|} = C_x[k,r]$$

- Constants α and β can be distinct, due to the nature of rearranged S-EMG data.
- This means that the autocorrelation function can be used to model two-dimensional data with different degrees of correlation in the horizontal and vertical directions, by specifying the values of α and β.
- In this example, one direction corresponds to linear time data sampling, with strong correlation, and the other corresponds to window step, and leads to weak correlation.

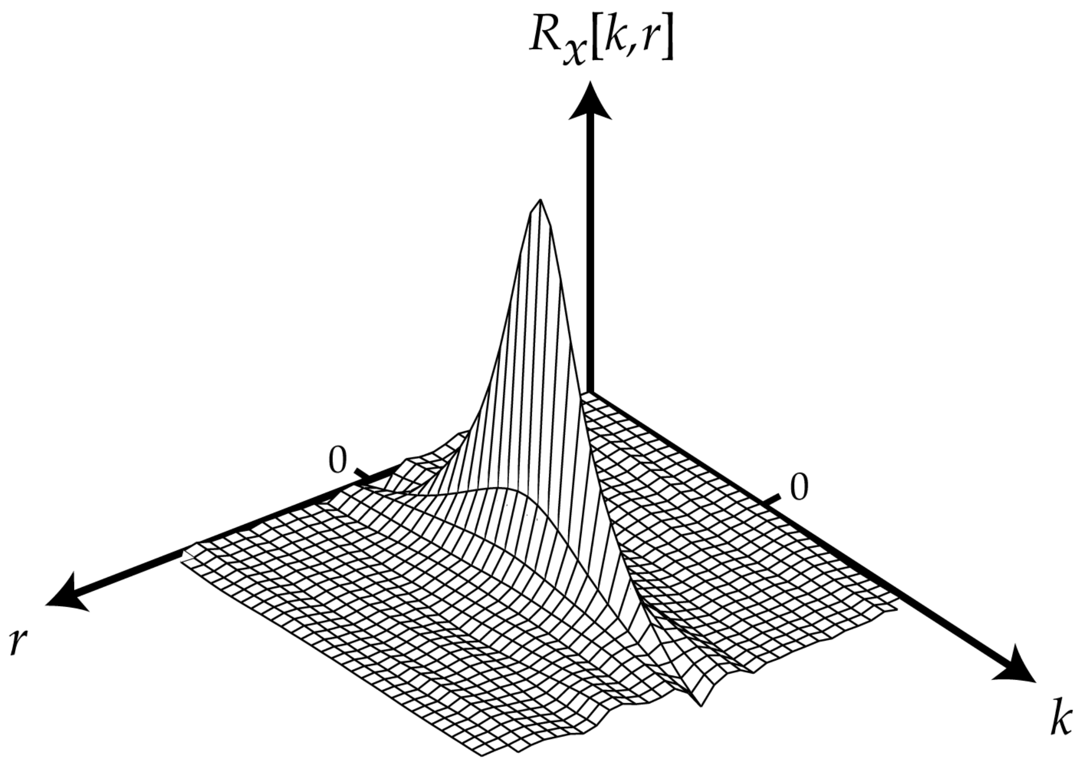
Example: JointlyWide-Sense Stationary Processes

- Autocorrelation functions: (a) computed from the theoretical model, using $\alpha = 0.215$ and $\beta = 0.95$; (b) computed from the real random signal.

(a)



(b)

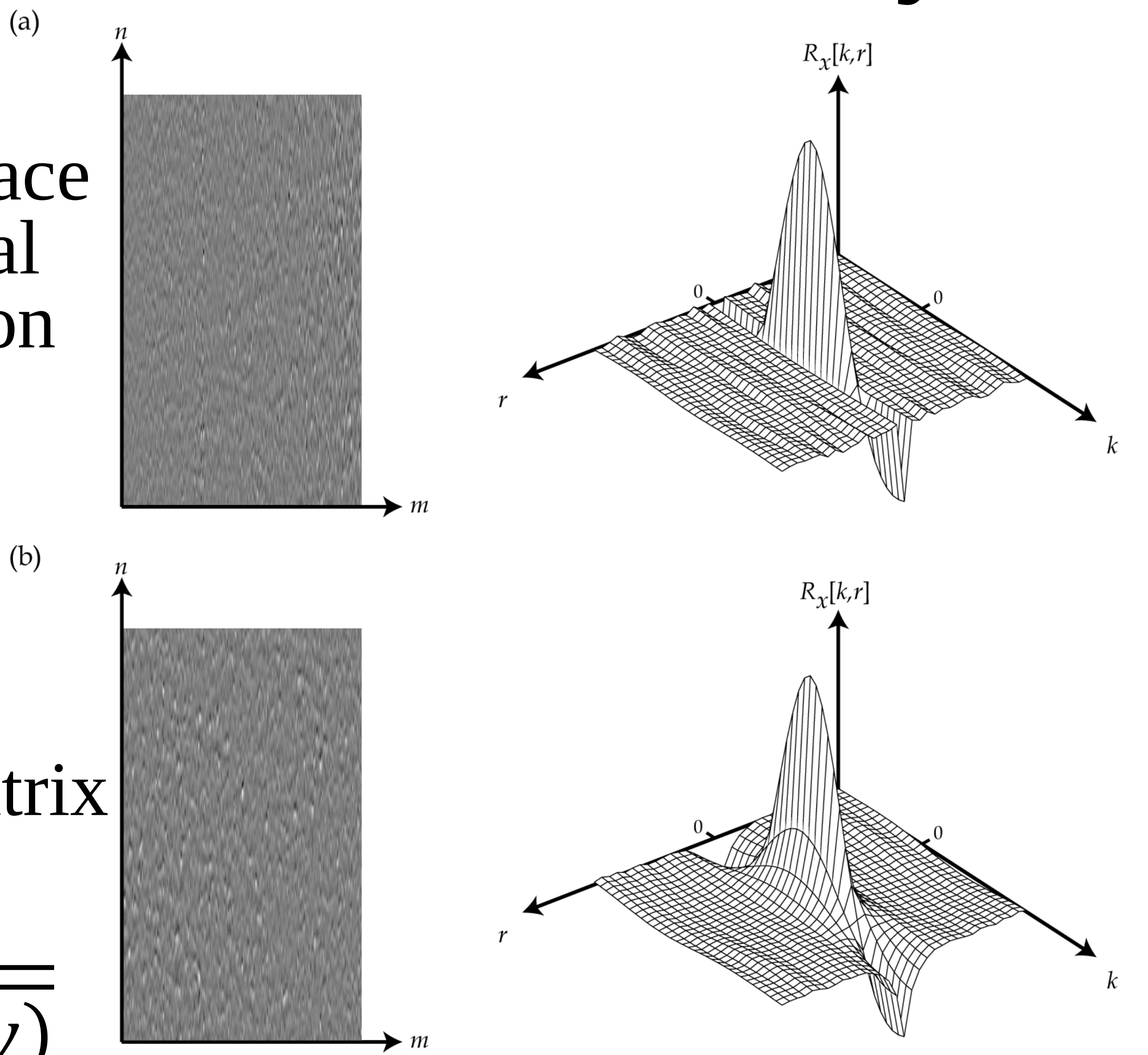


Example: JointlyWide-Sense Stationary Processes

- Two-dimensionally arranged Surface electromyographic (S-EMG) signal (left) and associated autocorrelation function (right): (a) without correlation sorting; (b) with correlation sorting.

- The matrix of column cross-correlation coefficients (R) is computed from the covariance matrix C, as follows:

$$R(u,w) = \frac{C(u,w)}{\sqrt{C(u,u)\cdot C(w,w)}}$$



Fundamental concepts: Cyclostationary Processes

- Some random processes have inherently a periodic nature, in that their **statistical properties are repeated every *T* units of time**, where *T* is the smallest possible positive real number.

Fundamental concepts: Cyclostationary Processes

- A random process $X(t)$ is called a ***cyclostationary process*** if the joint CDF of $X(t_1)$, …, $X(t_k)$ is **the same as the joint CDF** of

$$X(t_1 + mT), \ldots, X(t_k + mT)$$

for all integers m and k, and all choices of sampling instants, $t_1, \ldots, t_k$.

- Note that the sample functions of a cyclostationary random process need not be periodic.

Fundamental concepts: Cyclostationary Processes

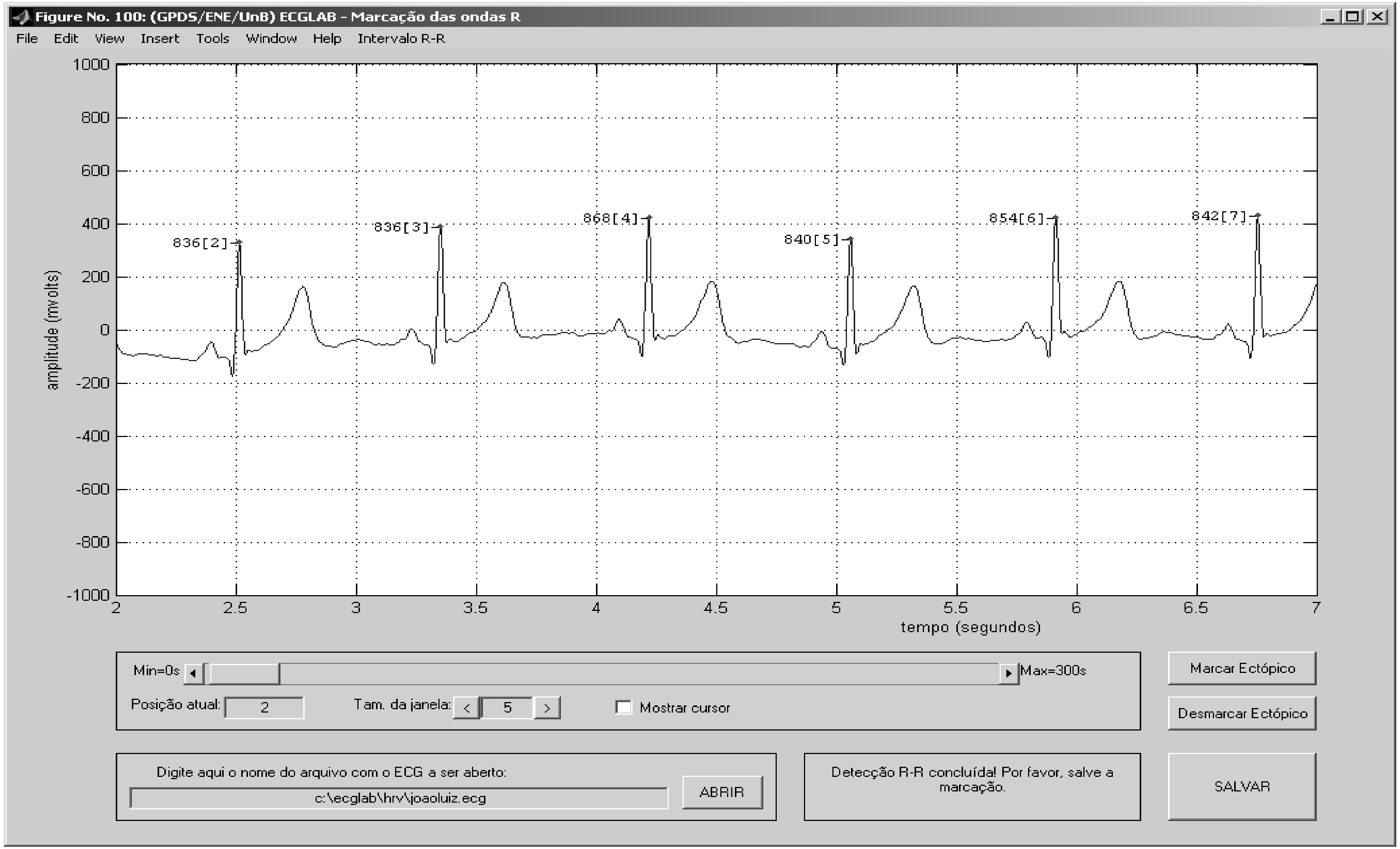
- A random process $X(t)$ is called a ***wide-sense cyclostationary process*** if the mean and autocorrelation functions of $X(t)$ remain the same with respect to time shifts that are multiples of T.
- In other words, for every integer m, $X(t)$ is a wide-sense cyclostationary process, if we have

$$\mu_X(t + mT) = \mu_X(t)$$

$$R_X(t_1 + mT, t_2 + mT) = R_X(t_1, t_2)$$

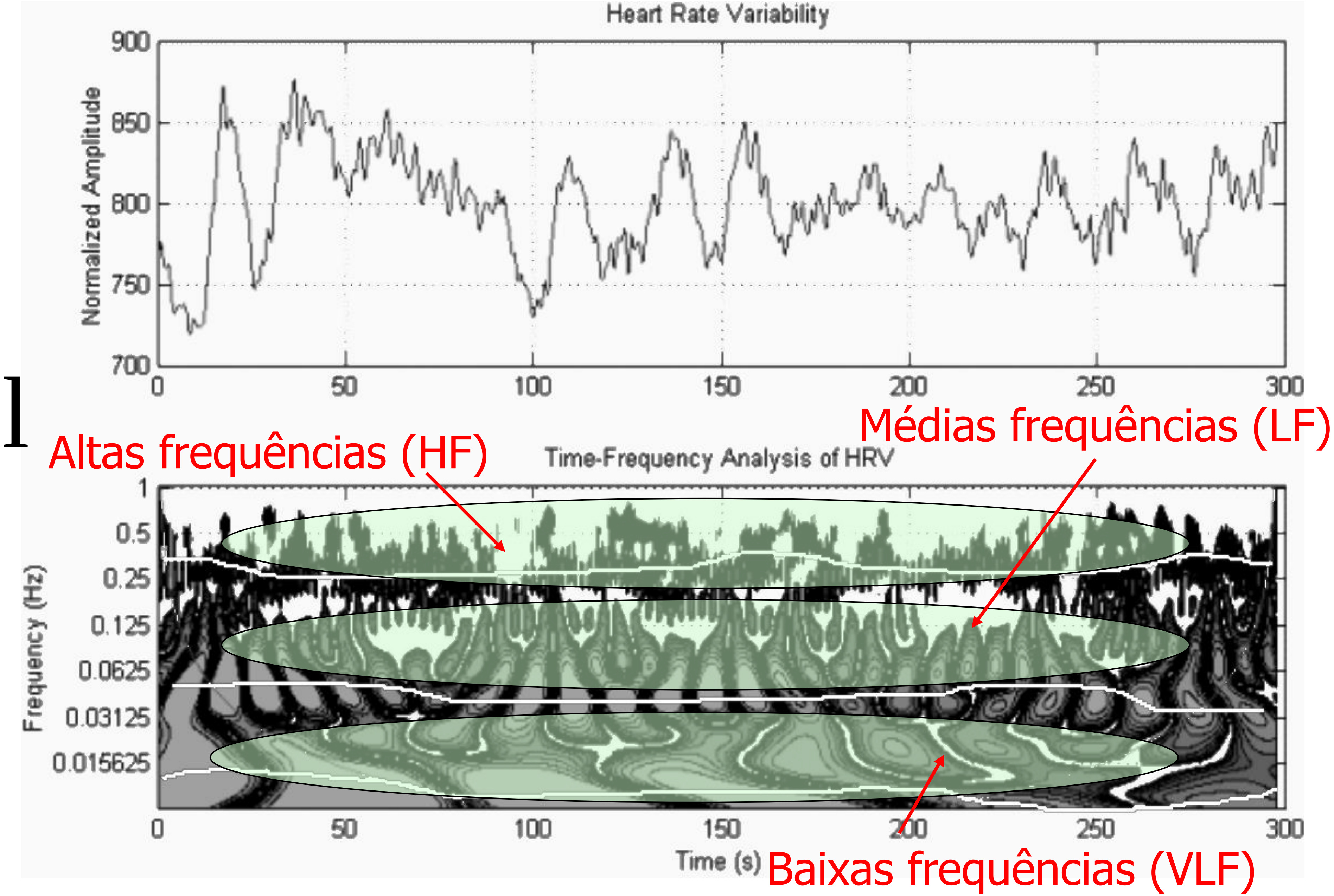
- Note that if a random process is cyclostationary, then it is also wide-sense cyclostationary.

Example: ECGLab - Detection module

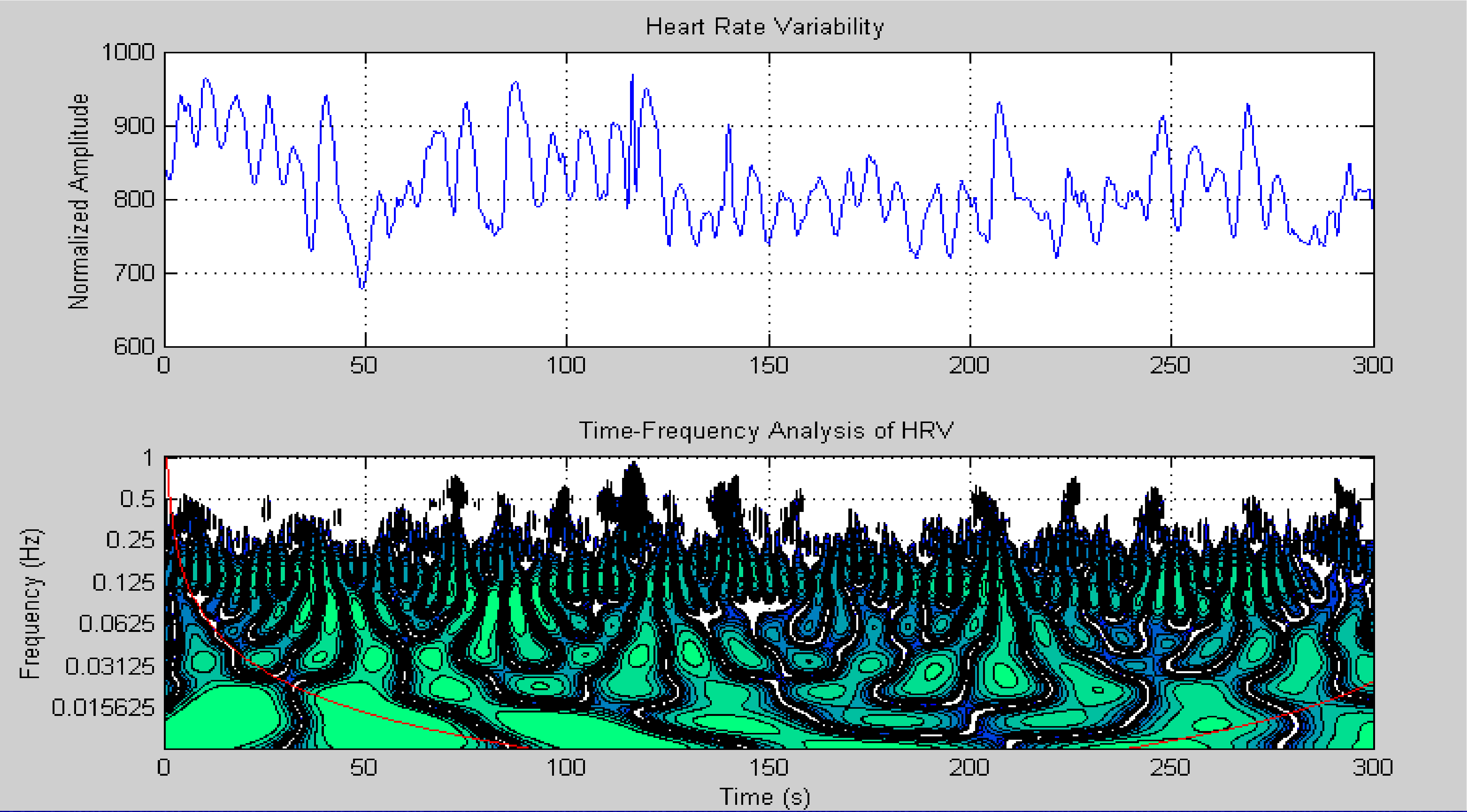


Example:

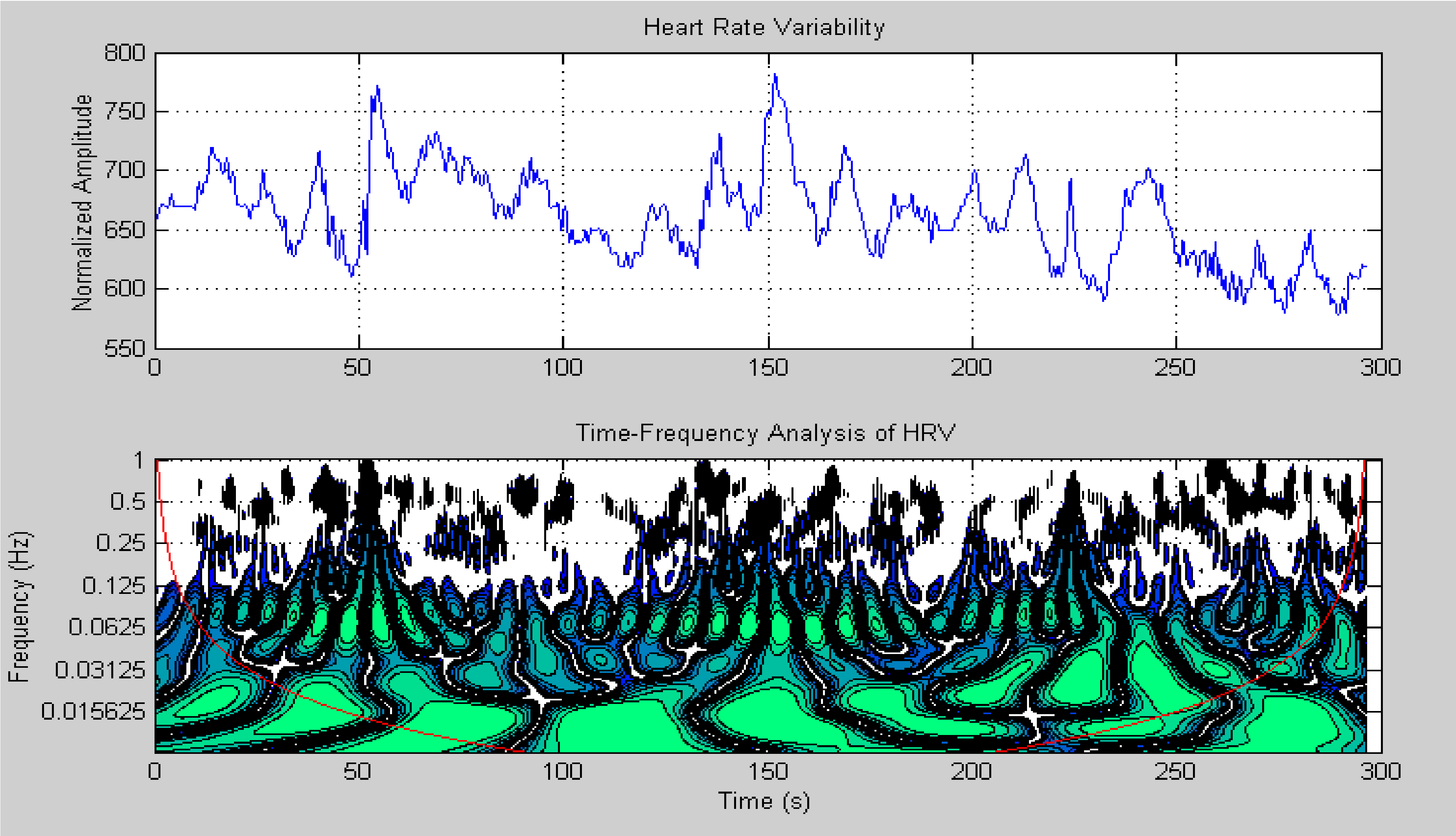
Heart Rate Variability HRV signal scalogram



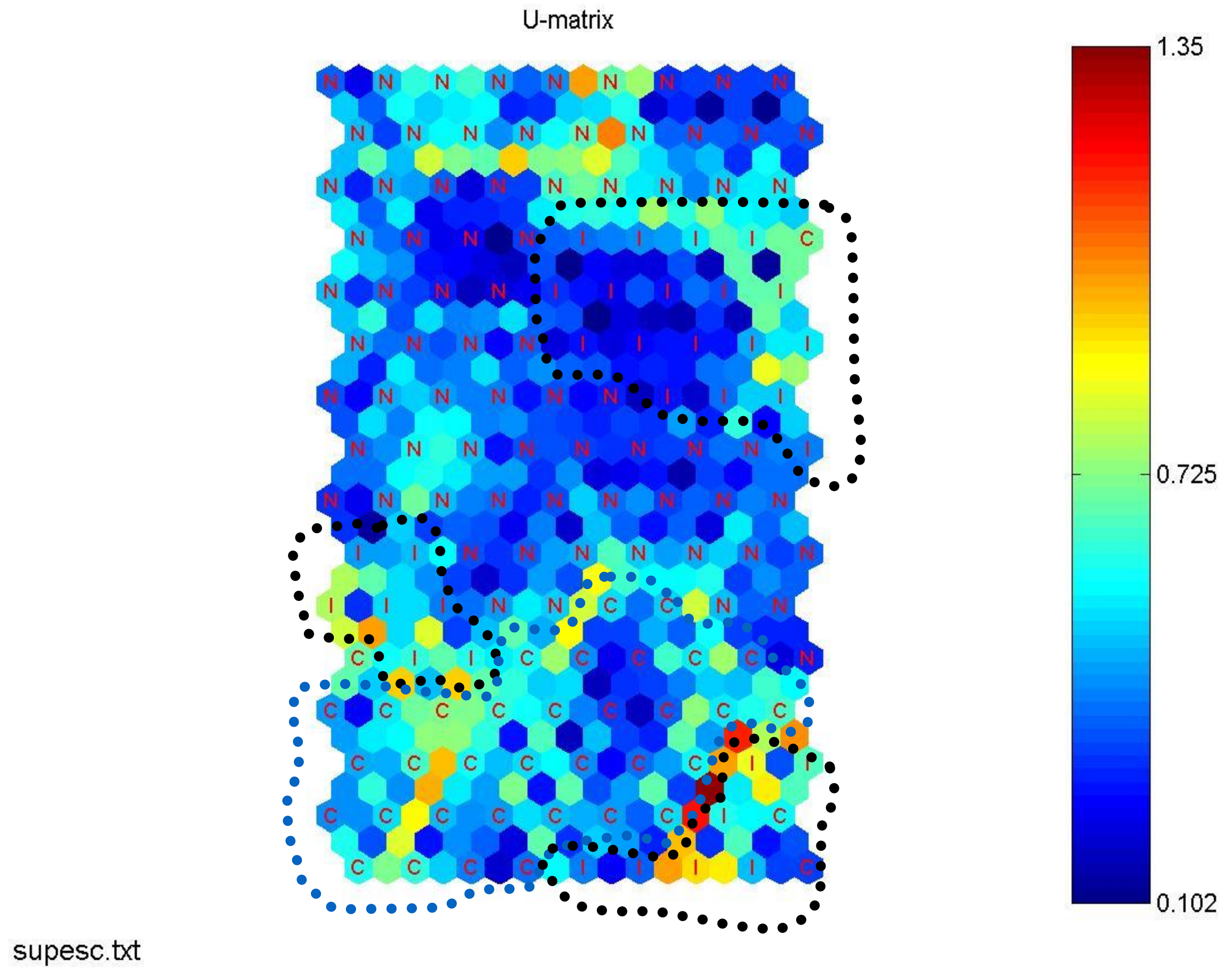
Example: DOG Wavelet Transform scalogram of a normal individual in the standing position



Example: DOG Wavelet Transform scalogram of a Cardiac Chagas patient in the standing position



Example: Kohonen's Map for Scalographic Indices



Fundamental concepts: Ergodicity

- Ergodicity is a stronger condition than stationarity.

- ***Ergodicity*****:** A stochastic process is said to be ***ergodic*** if the complete statistics can be determined by any one of the realizations.

- If all the statistical properties of the random process $X(t)$ can be determined from a single sample function, then the random process is said to be a ***strict-sense ergodic process***.

Fundamental concepts: Ergodicity

- For ergodic processes, the moments can be determined by time averages as well as ensemble averages, simply put, all time and ensemble averages are interchangeable, not just the mean, variance, and autocorrelation functions.

- A random process that is ergodic in the mean, variance, and autocorrelation function is called a ***wide-sense ergodic process***.

Gaussian Random Processes

- A random process $\{X(t), t \in J\}$ is said to be a **Gaussian (normal) random process** if, for all

$$t_1, t_2, \ldots, t_n \in J,$$

- the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly normal.

$$f_X(\boldsymbol{X}) = \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} e^{-\frac{1}{2}\{(\boldsymbol{x}-\boldsymbol{\mu}_X)^T \boldsymbol{\Sigma}_X^{-1} (\boldsymbol{x}-\boldsymbol{\mu}_X)\}}$$

$$\boldsymbol{X} \sim \mathcal{N}(x | \boldsymbol{\mu}_X, \boldsymbol{\Sigma}_X)$$

Jointly Gaussian random processes

- Two random processes $\{X(t), t \in J\}$ and $\{Y(t), t \in J'\}$ are said to be **jointly Gaussian (normal)**, if for all

$$t_1, t_2, \ldots, t_m \in J$$

- and

$$t'_1, t'_2, \ldots, t'_n \in J',$$

- the random variables

$$X(t_1), X(t_2), \cdots, X(t_m), Y(t'_1), Y(t'_2), \cdots, Y(t'_n)$$

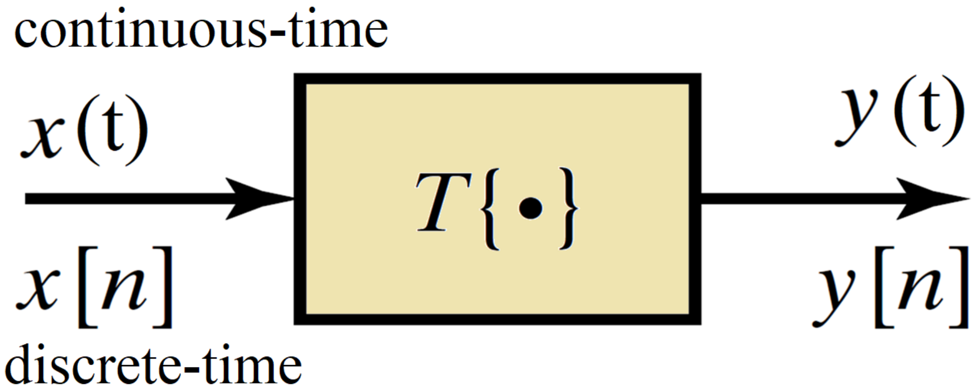
- are jointly normal.

Properties of a Gaussian process

- The Gaussian process makes mathematical analysis simple and analytic results possible, and in system analysis, the Gaussian process is often the only one for which a complete statistical analysis can be carried out.
- The Gaussian process, due to central limit theorem, provides a good mathematical model for numerous physically observed random time-varying phenomena.
- The Gaussian process from which Gaussian random variables are derived can be completely specified, in a statistical sense, from only first and second moments.

Properties of a Gaussian process

- If a Gaussian process is the input to a linear system, then its output is also a Gaussian process, as **the linear filter only modifies the first two moments**.
- The input and output of a linear system are jointly Gaussian processes.
- The linear operation, such as a sum, derivative, and integral (also linear shifting), on a Gaussian process results in another Gaussian process.



Properties of a Gaussian process

- The Gaussian process is one of the few random processes for which the moments at the output of a nonlinear transformation can be found.
- If a Gaussian process satisfies the conditions for wide-sense stationary, then the random process is also strictly stationary.
- If the set of random variables obtained by sampling a Gaussian process are all uncorrelated, the set of random variables are then statistically independent.

Properties of a Gaussian process

- If the joint **PDF** of the random variables resulting from sampling a Gaussian process is Gaussian, then the resulting marginal pdf's and conditional pdf's are all individually Gaussian.

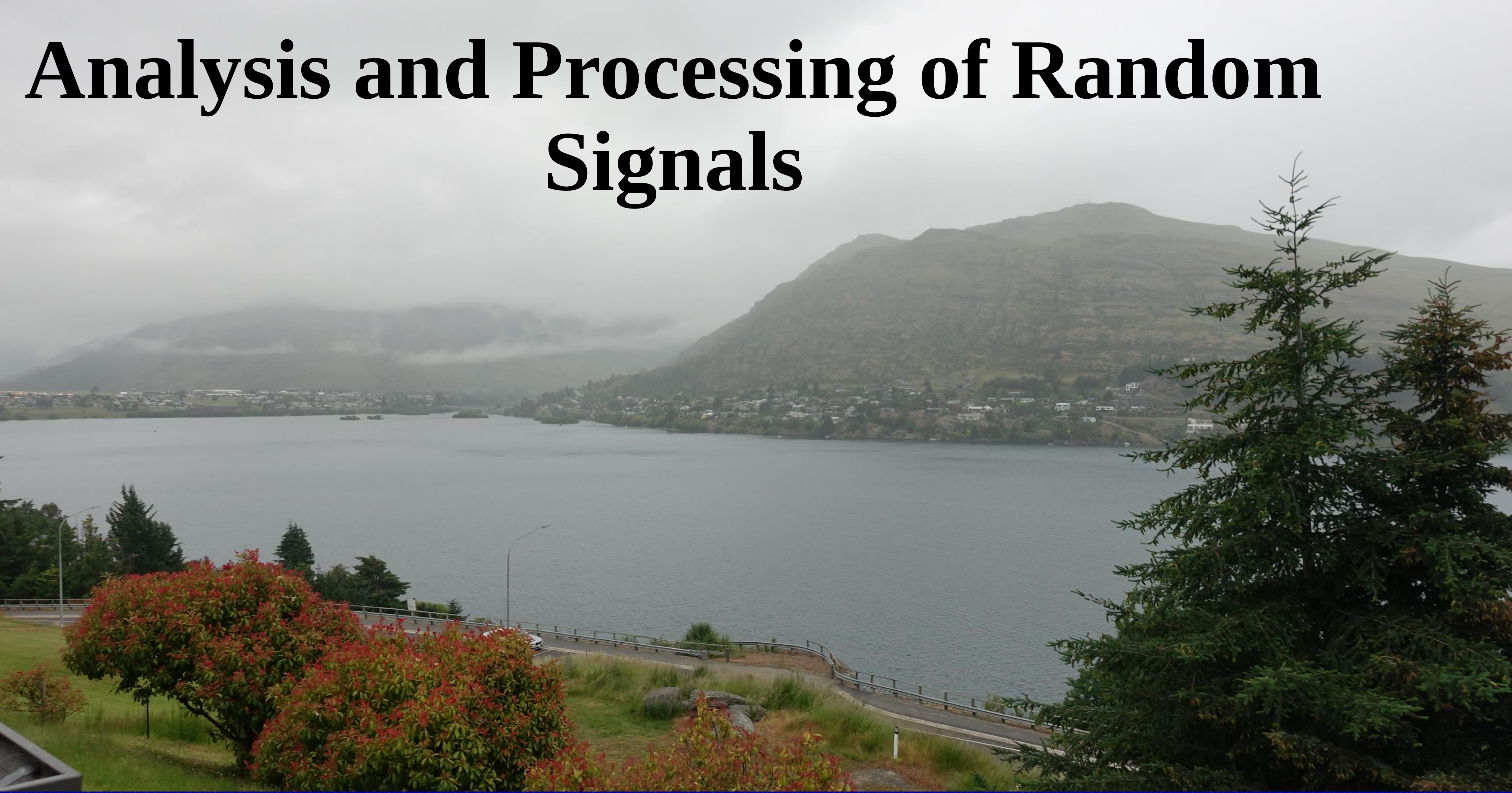
- Linear transformation of a set of Gaussian random variables, obtained by sampling a Gaussian process, produces another set of Gaussian random variables.

Properties of a Gaussian process

- The linear combinations of jointly Gaussian variables, resulting from sampling a Gaussian process, is also jointly Gaussian.

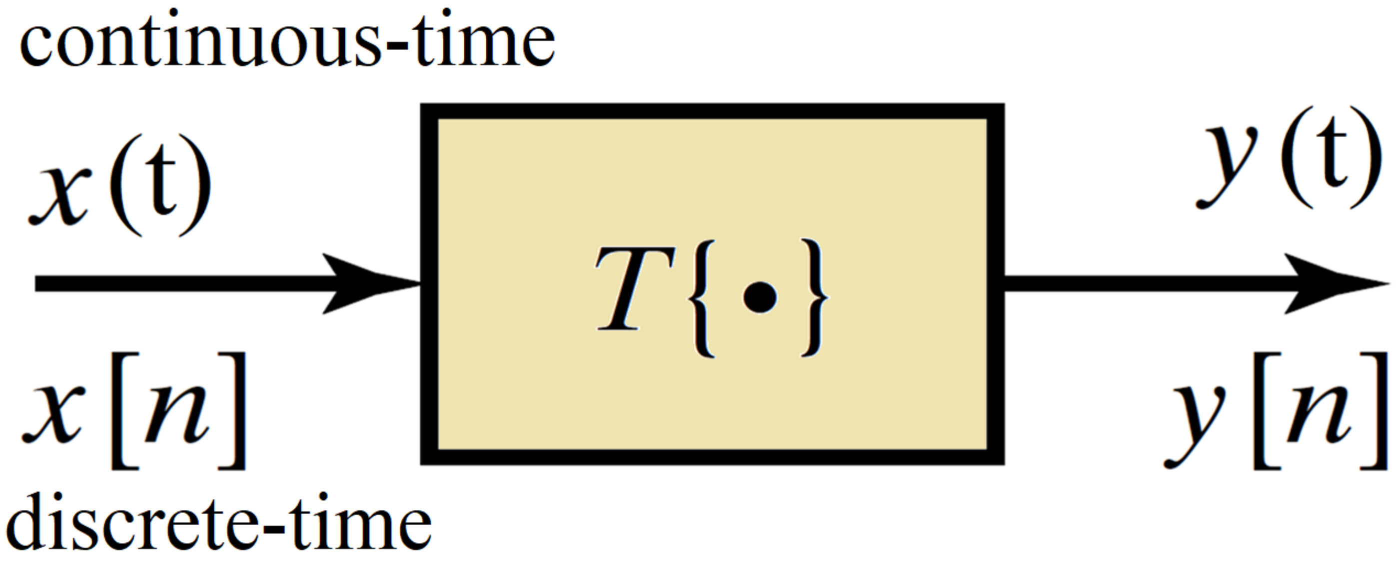
- The Gaussian independent, identically distributed random process (i.i.d) has the property that the value at every time instant is independent of the value at all other time instants.

Analysis and Processing of Random Signals



Random Signals Processing

- Many applications involving random processes are signals obtained from the output of deterministic systems:
  - Linear Time-Invariant (LTI) systems in continuous-time.
  - Linear Shift-Invariant (LSI) systems in discrete-time.



Relationship between analog and digital frequency

- $\Omega \rightarrow$ analog frequency $\left(\frac{radians}{second}\right)$
- $\Omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$;
- $f_0 = \frac{1}{T_0} \rightarrow$ analog frequency (Hz).
- $f_s = \frac{1}{T} \rightarrow$ sample frequency (Hz).
- $T = \frac{1}{f_s} \rightarrow$ sample $rate$ $(second)$.
- $\omega \longrightarrow$ digital frequency $\left(\frac{radians}{sample}\right)$.

$$\omega_0 = \Omega_0 T = \frac{\Omega_0}{f_s} = \frac{2\pi f_0}{f_s}$$

$$\Omega_0 = 2\pi f_0 = \frac{\omega_0}{T} = \omega_0\, f_s$$

$$f_0 = \frac{\omega_0}{2\pi T} = \frac{\omega_0}{2\pi} f_s$$

Relationship between analog and digital frequency

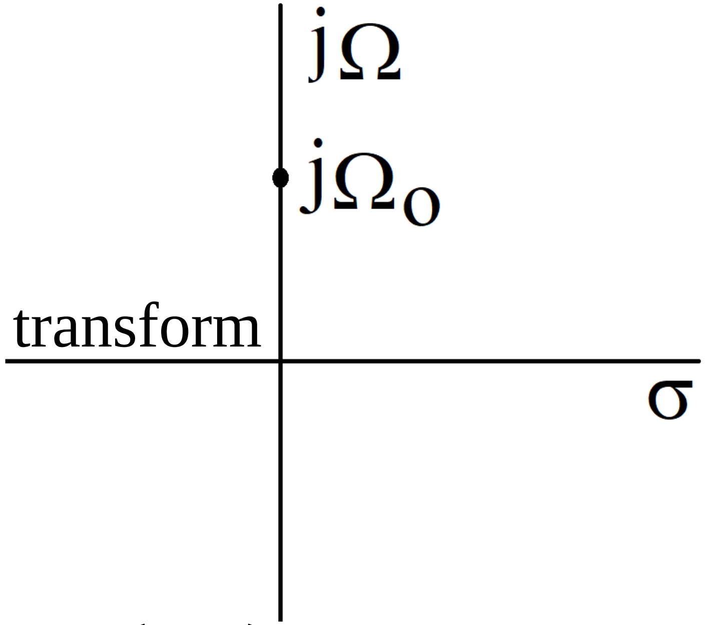
- $\Omega \rightarrow$ analogfrequency $\left(\frac{radians}{second}\right)$; $\omega \longrightarrow$ digital frequency $\left(\frac{radians}{sample}\right)$.

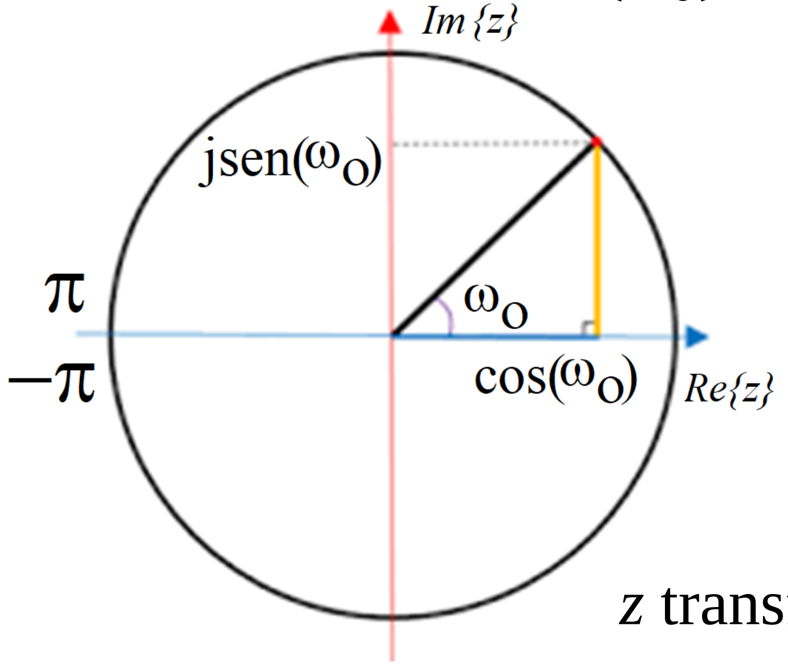
Continuous-Time

Discrete-Time

$$x[n]sen[\omega_0]u[n] \leftrightarrow X(z) = \frac{sen(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$$

Laplace transform





z transform

$$x(t) = sen(\Omega_0 t)u(t) \leftrightarrow X(s) = \frac{\Omega_0}{s + \Omega_0^2}$$

Power Spectral Density

- To analyze deterministic time-domain signals in the frequency domain, the Fourier transform is commonly used.
- Continuous-Time Fourier Transform (CTFT)

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t} d\Omega \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t} d\Omega$$

- Discrete-Time Fourier Transform (DTDF): $\omega = \Omega \mathrm{T}$

$$X(e^{j\omega}) \equiv \sum_{\mathrm{n}=-\infty}^{\infty} x[n] e^{-j\omega n} \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

Power Spectral Density

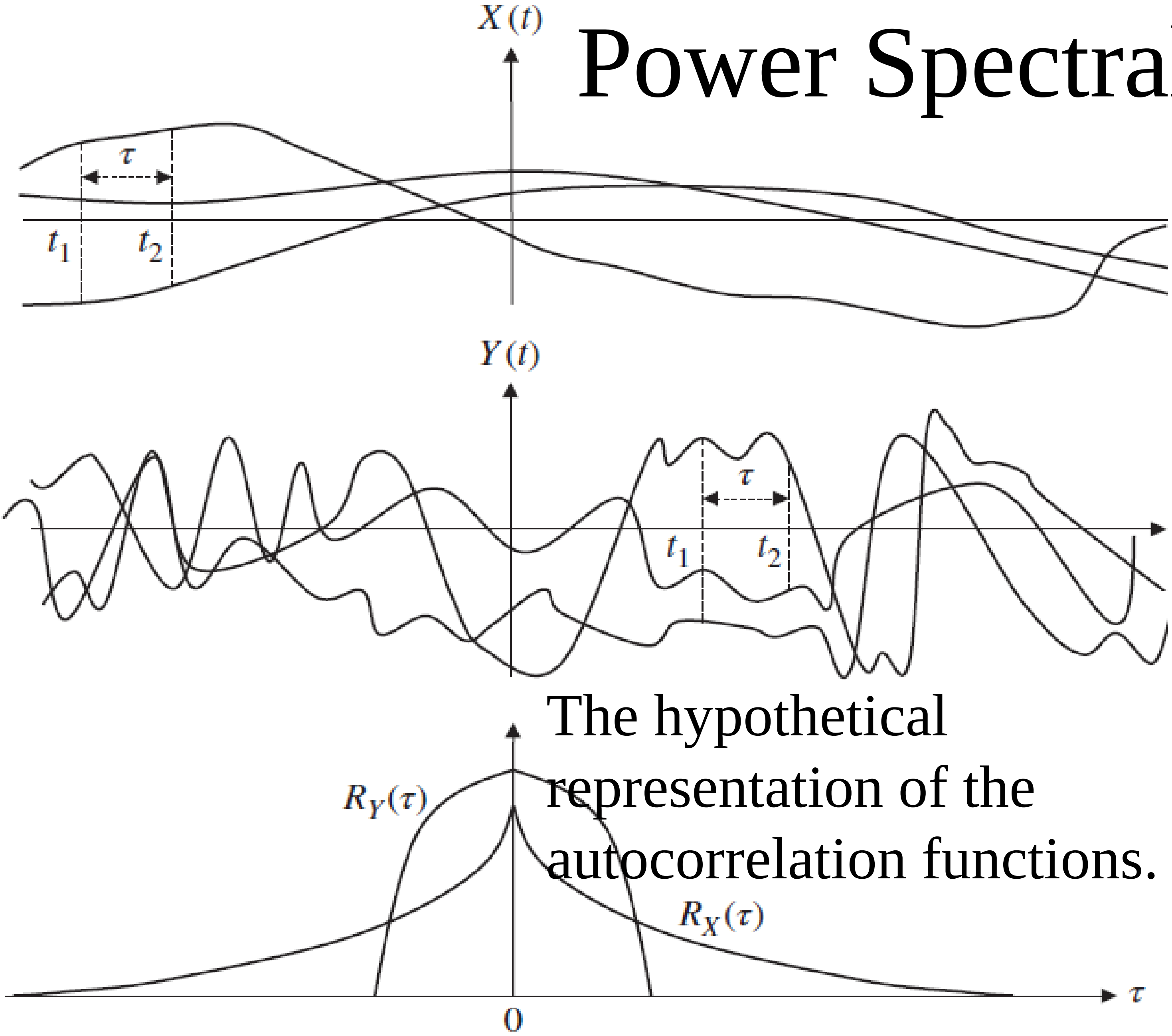
- A random process is an ensemble of sample functions, and **the spectral characteristics** of these signals determine **the spectral characteristics of the random process**.

- A **statistical average** of the **sample functions** is thus a more meaningful measure to reflect **the spectral components of the random process**.

Power Spectral Density

- For a **wide-sense stationary process** **(WSS)**, the autocorrelation function is an appropriate measure for the average rate of change of a random process.

Power Spectral Density



The hypothetical representation of the autocorrelation functions.

- The figure on the left shows two sets of sample functions: X(*t*) and Y(*t*).
- X(*t*) and Y(*t*) present different temporal behaviors. X (*t*) presents smoother transitions (low frequencies), while Y(*t*) presents abrupt transitions (high frequencies).

Power Spectral Density

- For random signals that have spectral concentration at high frequencies (or higher bandwidth signals), the autocorrelation function is short in duration and decreases rapidly as the time difference between two samples is increased.

- For limited “low-pass” signals (small band signals), the autocorrelation function is long in duration, and decreases slowly as the time difference between two samples is increased. It means high statistical dependence.

Power Spectral Density

- The ***Wiener–Khintchine Theorem*** states that the **power spectral density** of a **wide-sense stationary** process and its autocorrelation function form a Fourier transform pair, as given by

Continuous-Time

$$R_X(t) \xleftrightarrow{CTFT} S_X(\Omega)$$

Discrete-Time

$$R_X[n] \xleftrightarrow{DTFT} S_X\left(e^{j\omega}\right)$$

Power Spectral Density

- For continuous-time the power spectral density for **wide-sense stationary process** can be written as

$$R_X(t) = \int_{-\infty}^{\infty} x(\tau)x(t+\tau)\,d\tau = x(t) * x(-t)$$

$$\updownarrow CTFT \qquad\qquad x(-t) \xleftrightarrow{CTFT} X^*(j\Omega)$$

$$S_X(\Omega) = X(j\Omega)X^*(j\Omega) = |X(j\Omega)|^2 = \mathcal{F}\{R_X(t)\}$$

- The autocorrelation function can be obtained through

$$R_X(t) = \mathcal{F}^{-1}\{S_X(\Omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\Omega)\,e^{j\Omega t}d\Omega$$

Power Spectral Density

- For discrete-time the power spectral density for **wide-sense stationary process** can be expressed as

$$R_X[n] \equiv \sum_{m=-\infty}^{\infty} x[m]x[n+\mathrm{m}] = x[n] * x[-n]$$

$$x[-n] \overset{DTFT}{\longleftrightarrow} X\left(e^{-j\omega}\right) = X^*\left(e^{j\omega}\right)$$

$$\updownarrow DTFT$$

$$S_X\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)X^*\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right|^2 = \mathcal{F}\{R_X[n]\}$$

- The autocorrelation function can be obtained through

$$R_X[n] = \mathcal{F}^{-1}\left\{S_X\left(e^{j\omega}\right)\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

Power Spectral Density: properties

- **Parseval theorem**

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1\,(e^{j\omega})X_2^*(e^{j\omega})d\omega$$

- The **expected power** (the energy of a sequence) be obtained by

$$R_X[0] = E[X^2(t)] = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})X^*(e^{j\omega})d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$$

Power Spectral Density: properties

- $S_X(\Omega) = S_X(-\Omega)$
- $S_X(\Omega) \geq 0$
- Linearity

- continous time

$$R_Z(t) = \alpha_1\ R_X(t) + \alpha_2 R_Y(t) \overset{CTFT}{\longleftrightarrow} S_Z(\Omega) = \alpha_1 S_X(\Omega) + \alpha_1 S_Y(\Omega)$$

- discrete time

$$R_Z[\mathrm{n}] = \alpha_1\ R_X[\mathrm{n}] + \alpha_2 R_Y[\mathrm{n}] \overset{DTFT}{\longleftrightarrow} S_Z\left(e^{j\omega}\right) = \alpha_1 S_X\left(e^{j\omega}\right) + \alpha_1 S_Y\left(e^{j\omega}\right)$$

Power Spectral Density: properties

- The autocorrelation and autocovariance functions are related by

$$R_X(t) = C_X(t) + \mu_X^2 \xleftrightarrow{CTFT} S_X(\Omega) = \mathcal{F}\{C_X(t) + \mu_X^2\}$$

$$R_X[n] = C_X[n] + \mu_X^2 \xleftrightarrow{DTFT} S_X\left(e^{j\omega}\right) = \mathcal{F}\{C_X[n] + \mu_X^2\}$$

Power Spectral Density

- The normalized autocovariance function in continuous-time

$$\rho_X(\tau) \equiv \frac{Cov[X(t)X(t-\tau)]}{\sqrt{Var[X(t)]Var[X(t-\tau)]}} = \frac{R_X(\tau) - \mu_X^2}{\sigma_X^2}$$

- which is related to the autocorrelation function according to

$$R_X(\tau) = \sigma_X^2 \rho_X(\tau) + \mu_X^2 \overset{CTFT}{\longleftrightarrow} S_X(\Omega)$$

Power Spectral Density

- The **power spectrum method applies only** when a **wide-sense stationary process** (WSS) process is observed over $(-\infty,\infty)$.

- The correlation function $\rho(\tau)$ decays to practically zero beyond $|\tau| > T$ for some finite T.

- For WSS, the autocorrelation function decays according to a power law (Pareto distribution - the Zeta distribution): $\boldsymbol{n^{-\alpha}}$, where $\alpha > 1$.

$$\rho_X(\tau) \propto |\tau|^{-\beta}, as\ |\tau| \longrightarrow \infty; 0 < \ \beta < 1$$

Power spectrum and periodogram of time series (signals)

- This situation occurs when either (i) the random process itself is a discrete-time process by its own nature or (ii) the random process is a continuous-time process but is observed at only discrete points in time (a sequence obtained from the digitization of a continuous time signal).
- Thus, instead of the process $\{x(t); -\infty < t < \infty\}$, we must deal with a **discrete-time random process** $\{x[n]; -\infty < n < \infty\}$.
- Such analysis is generally called "time-series analysis."!? (I think it's better to call it "signal processing" - at least for engineers).

Power spectrum and periodogram of time series

- "A **time series** is a sequence of data points, measured typically at successive times, spaced at (often uniform) time intervals" $\longrightarrow$ It is just a signal.
- We define the **serial correlation coefficient** (based on its autocovariance function) of *lag* k (or *order* k) $\{\rho_X[k]\}$ by

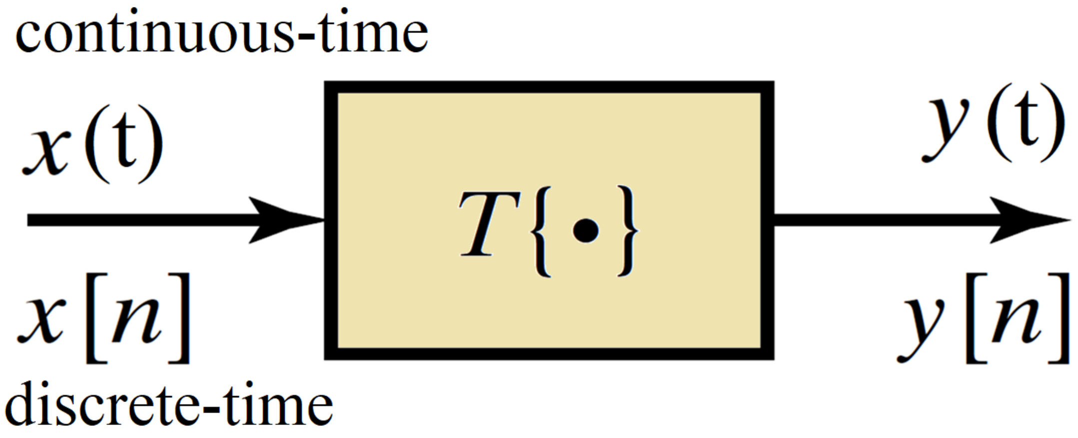
$$\rho_X[k] \equiv \frac{C_X[k]}{C_X[0]} = \frac{R_X[k] - \mu_X^2}{\sigma_X^2}, -\infty < k < \infty$$

$$0 \leq \rho_X[k] \leq 1, \rho_X[0] = 1$$

The cross-power spectral density

- The **cross-power spectral density** $S_{XY}(\Omega)/S_{XY}(e^{j\omega})$ is defined by

$$R_{XY}(\tau) = x(\tau) * y(-\tau) \overset{CTFT}{\longleftrightarrow} S_{XY}(\Omega)$$

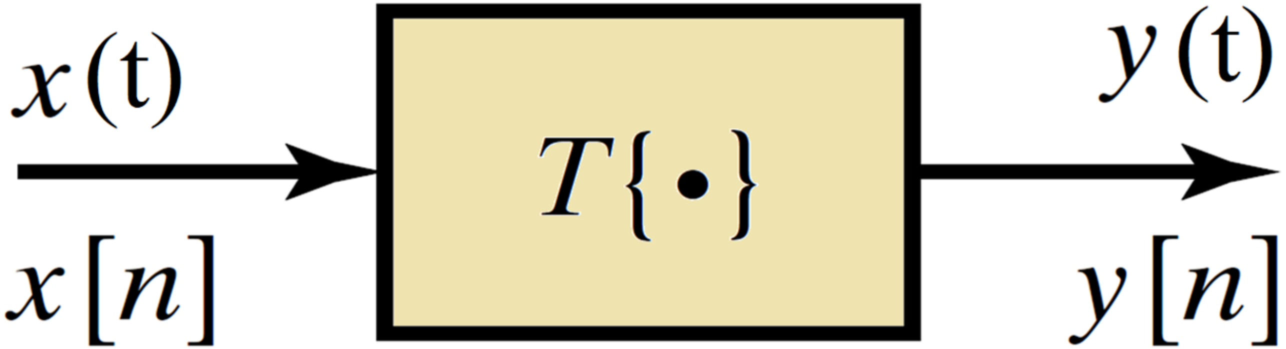


$$R_{XY}[n] = x[n] * y[-n] \overset{DTFT}{\longleftrightarrow} S_{XY}(e^{j\omega})$$

Response of a Linear Time-Invariant (LTI) Systems to Random Inputs

- The output of a LTI system can be calculated by linear convolution:
  - ODE/convolution $$y(t) = x(t) * h(t) = \int_{-\infty}^{-\infty} h(\tau)x(t-\tau)d\tau$$

continuous-time



discrete-time

h(*t*)/*h*[*n*] is the impulse response of the system.

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

  - LDE/convolution

Response of a Linear Time-Invariant (LTI)/ Linear Shift Invariant (LSI) Systems to Random Inputs

- **Time domain analysis**: Let $X(t)$ (the input of stable LTI/LSI) be a ***Wide-Sense Stationary process*** (WSS) random process and $Y(t)$ computed by the linear convolution

$$Y(t) = X(t) * h(t) \; or \; Y[n] = X[n] * h[n]$$

then

$$\mu_Y(t) = E[Y(t)] = \mu_Y = \mu_X \int_{-\infty}^{-\infty} h(\tau)d\tau \quad \text{(Continuous-Time)}$$

$$\mu_Y[n] = E[Y[n]] = \mu_Y = \mu_X \sum_{m=-\infty}^{\infty} h[m] \quad \text{(Discrete-Time)}$$

Response of LTI/LSI Systems to Random Inputs

- **Time domain analysis**: for a ***Wide-Sense Stationary process*** **(**WSS) **the cross-correlation/cross-power spectral density** $S_{XY}(\Omega)/S_{XY}\left(e^{j\omega}\right)$ is defined by

(Continuous-Time – Obs.: $R_{XY}(t) = h(-\tau) * x(\tau) * x(-\tau)$)

$$R_{XY}(t) = h(-t) * R_X(t) = \int_{-\infty}^{\infty} h(-\tau) R_X(t-\tau)\, d\tau \overset{CTFT}{\longleftrightarrow} S_{XY}(\Omega)$$

(Discrete-Time)

$$R_{XY}[n] = h[-n] * R_X[n] = \sum_{m=-\infty}^{\infty} h[-m] R_X[n-m] \overset{DTFT}{\longleftrightarrow} S_{XY}\left(e^{j\omega}\right)$$

Response of LTI/LSI Systems to Random Inputs

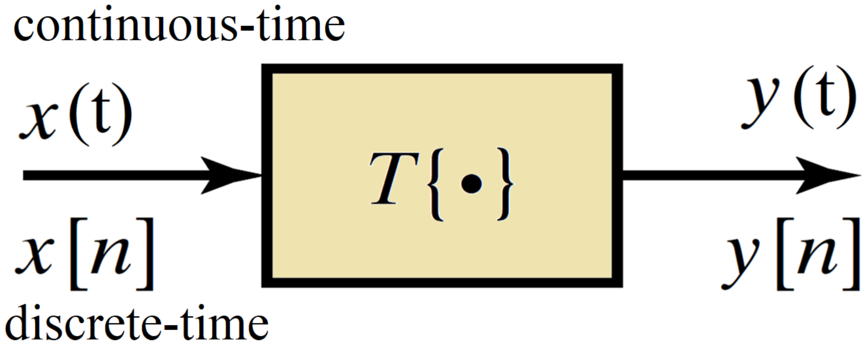
- **Time domain analysis**: for a ***Wide-Sense Stationary process*** (WSS) the **output power spectral density** $S_Y(\Omega)/S_Y\left(e^{j\omega}\right)$ computed using the impulsional renponse of a LTI/LSI system.

(Continuous-Time)

$$R_Y(t) = h(t) * h(-t) * R_X\,(t) \overset{CTFT}{\longleftrightarrow} S_Y(\Omega)$$

(Discrete-Time)

$$R_Y[n] = h[n] * h[-n] * R_X\,[n] \overset{DTFT}{\longleftrightarrow} S_Y\left(e^{j\omega}\right)$$



Observe: $R_Y(t) = h(t) * h(-t) * x(t) * x(-t) = y(t) * y(-t)$

Response of LTI/LSI Systems to Random Inputs

- **Frequency domain analysis**: Let $X(j\Omega)/X(e^{j\omega})$ (the LTI/LSI input) be a ***Wide-Sense Stationary process*** (WSS) random process and $Y(j\Omega)/Y(e^{j\omega})$ computed by algebraic product

$$Y(j\Omega) = X(j\Omega)H(j\Omega) \; or \; Y(e^{j\omega}) = X(e^{j\omega})H(e^{-j\omega})$$

then

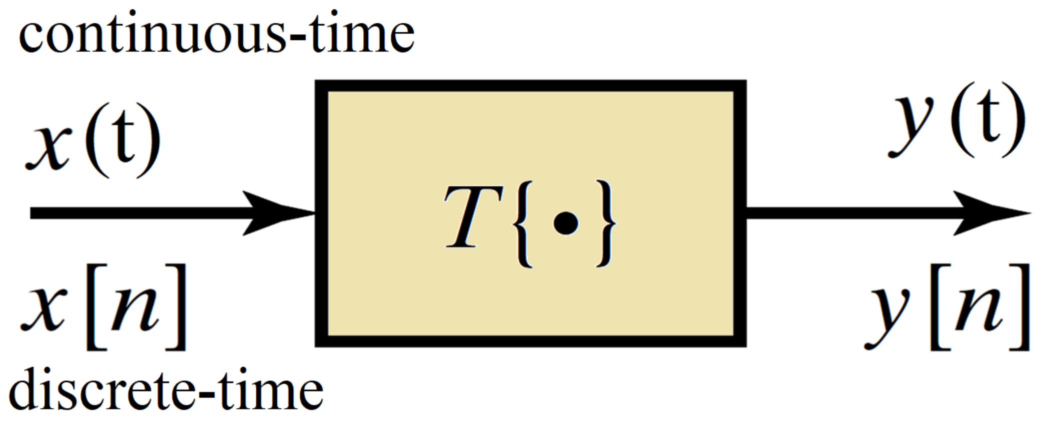
$$\mu_Y(t) = \mu_Y = \mu_X \, H(j\Omega)\Big|_{\Omega=0} \qquad \text{(Continuous-Time)}$$

$$\mu_Y[n] = \mu_Y = \mu_X \, H(e^{j\omega})\Big|_{\omega=0} \qquad \text{(Discrete-Time)}$$

Response of LTI/LSI Systems to Random Inputs

- **Frequency domain analysis**: The cross-correlation **cross-power spectral density** $S_{XY}(\Omega)/S_{XY}(e^{j\omega})$ can be computed in frequency domain.

$$S_{XY}(\Omega) = S_X(\Omega)H^*(j\Omega)$$

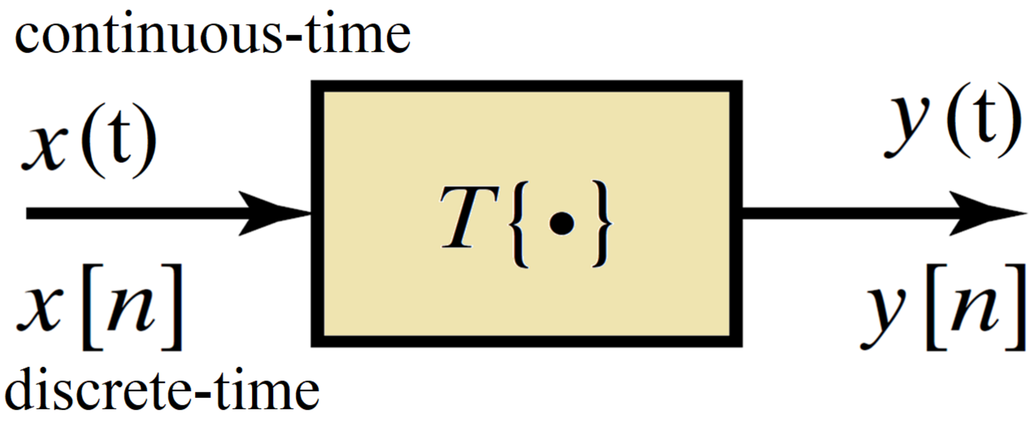


$$S_{XY}(e^{j\omega}) = S_X(e^{j\omega})H^*(e^{j\omega})$$

Response of LTI/LSI Systems to Random Inputs

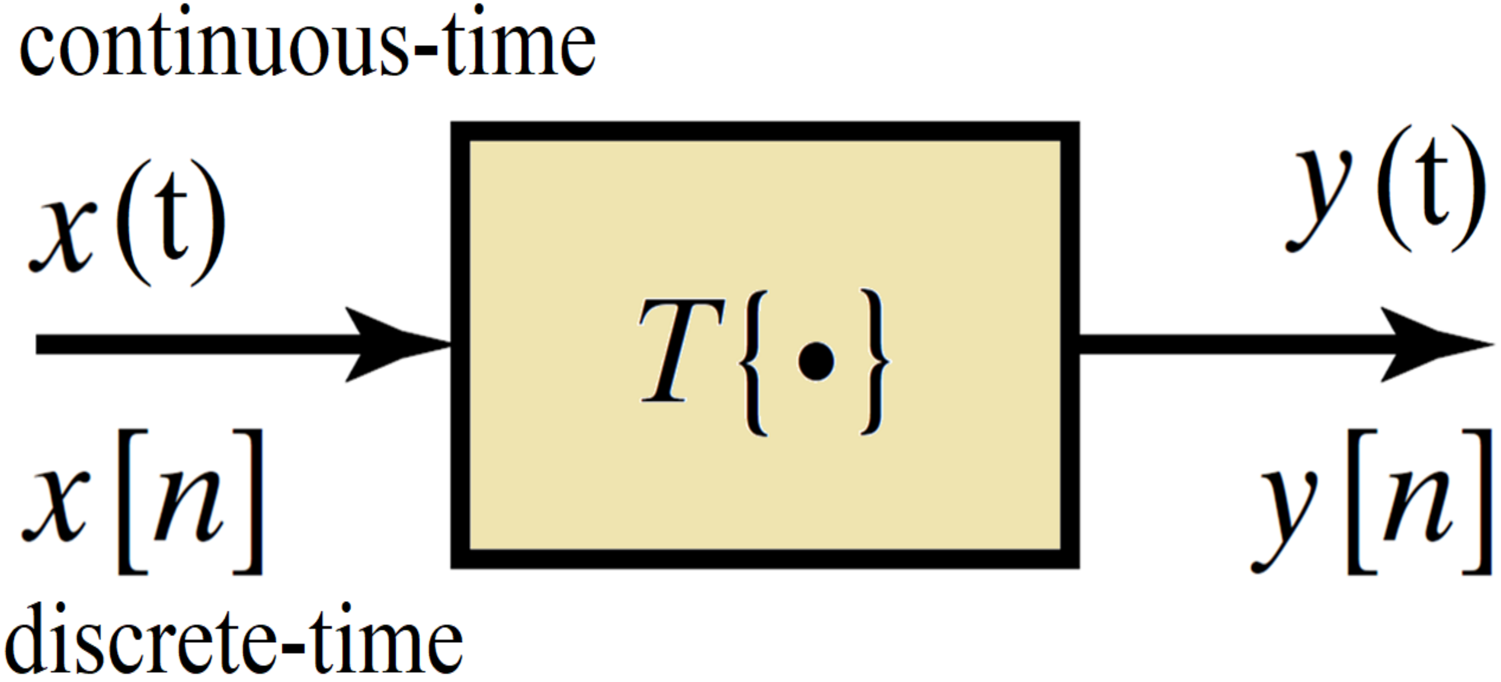
- **Frequency domain analysis**: The **power spectral density** $S_Y(\Omega)/S_Y(e^{j\omega})$ computed using the frequency response of a LTI/LSI system.

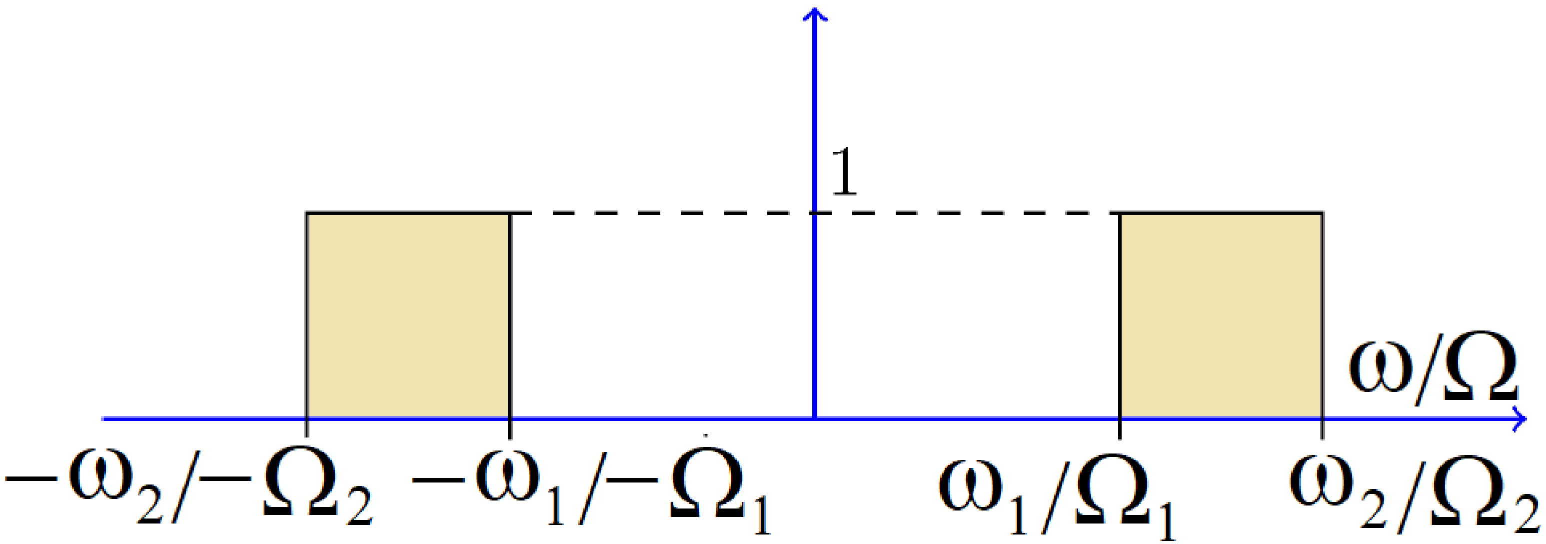
$$S_Y(\Omega) = S_X(\Omega)|H(j\Omega)|^2$$



$$S_Y(e^{j\omega}) = S_X(e^{j\omega})\left|H(e^{j\omega})\right|^2$$

Power in a Frequency Band





- Consider a WSS random process $X(t)$ that goes through an LTI system with the following transfer function.
- An ideal bandpass filter.

Power in a Frequency Band

- The expected power in $Y(t)/Y[n]$ is said to be the expected power in $X(t)/X[n]$ in the frequency range $\Omega_1 < |\Omega| < \Omega_2$/ $\omega_1 < |\omega| < \omega_2$.

(Contínuous-Time)

$$S_Y(\Omega) = S_X(\Omega)|H(j\Omega)|^2 \begin{cases} S_X(\Omega), \Omega_1 < |\Omega| < \Omega_2 \\ 0, \quad \text{otherwise} \end{cases}$$

(Discrete-Time)

$$S_Y(e^{j\omega}) = S_X(e^{j\omega})\left|H(e^{j\omega})\right|^2 \begin{cases} S_X(e^{j\omega}), \pi \leq \omega_1 < |\omega| < \omega_2 \leq \pi \\ 0, \quad \text{otherwise} \end{cases}$$


The end of module 7