# Stochastic Process

## Random Variables - Module 3

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# One Random Variable - topics

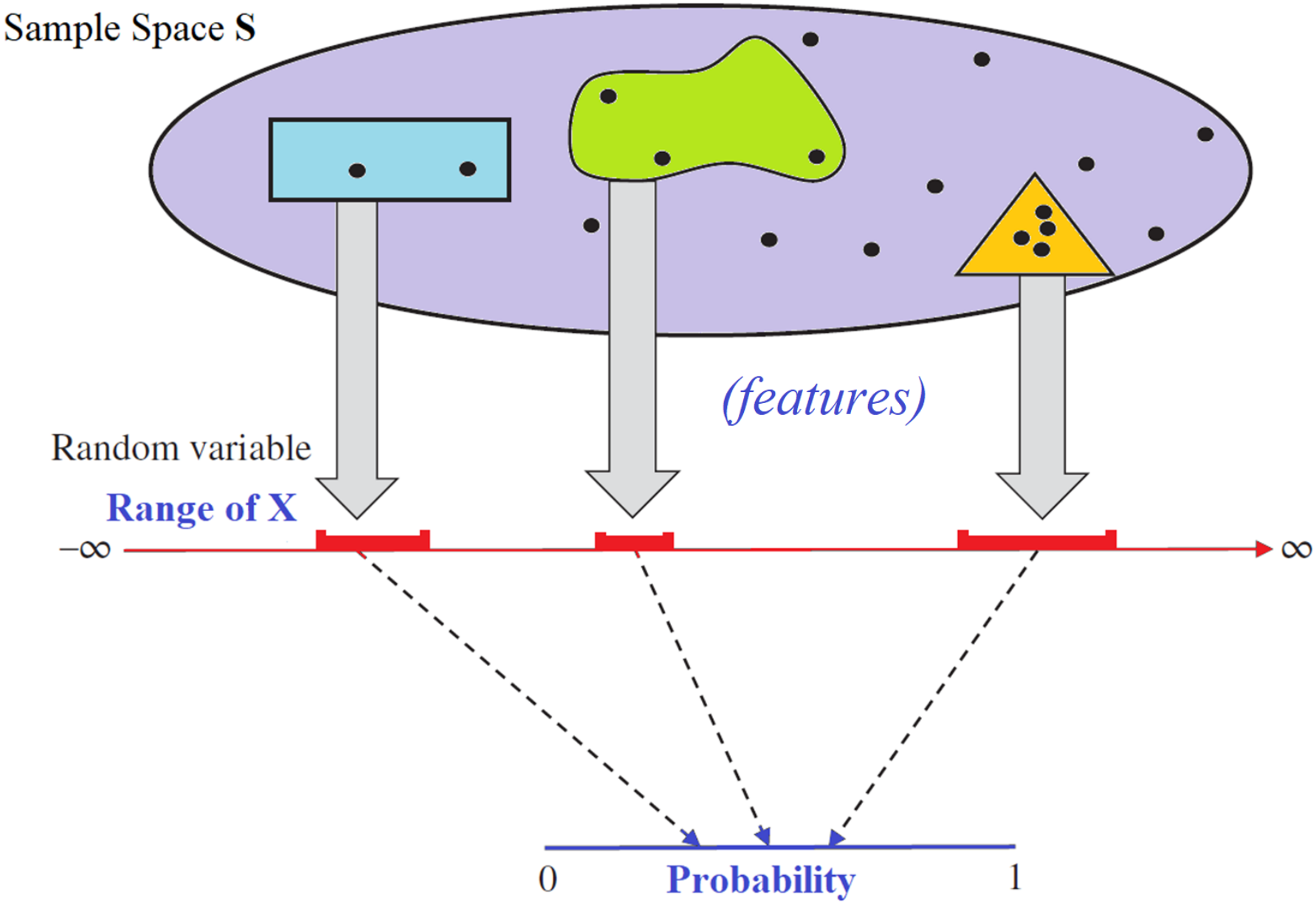
- *Continuous random variable*
- *Discrete random variable*
- Cumulative Distribution Function (CDF)
- Probability Density Function (PDF)
- Probability Mass Function (PMF)

# Defining a Random Variable:

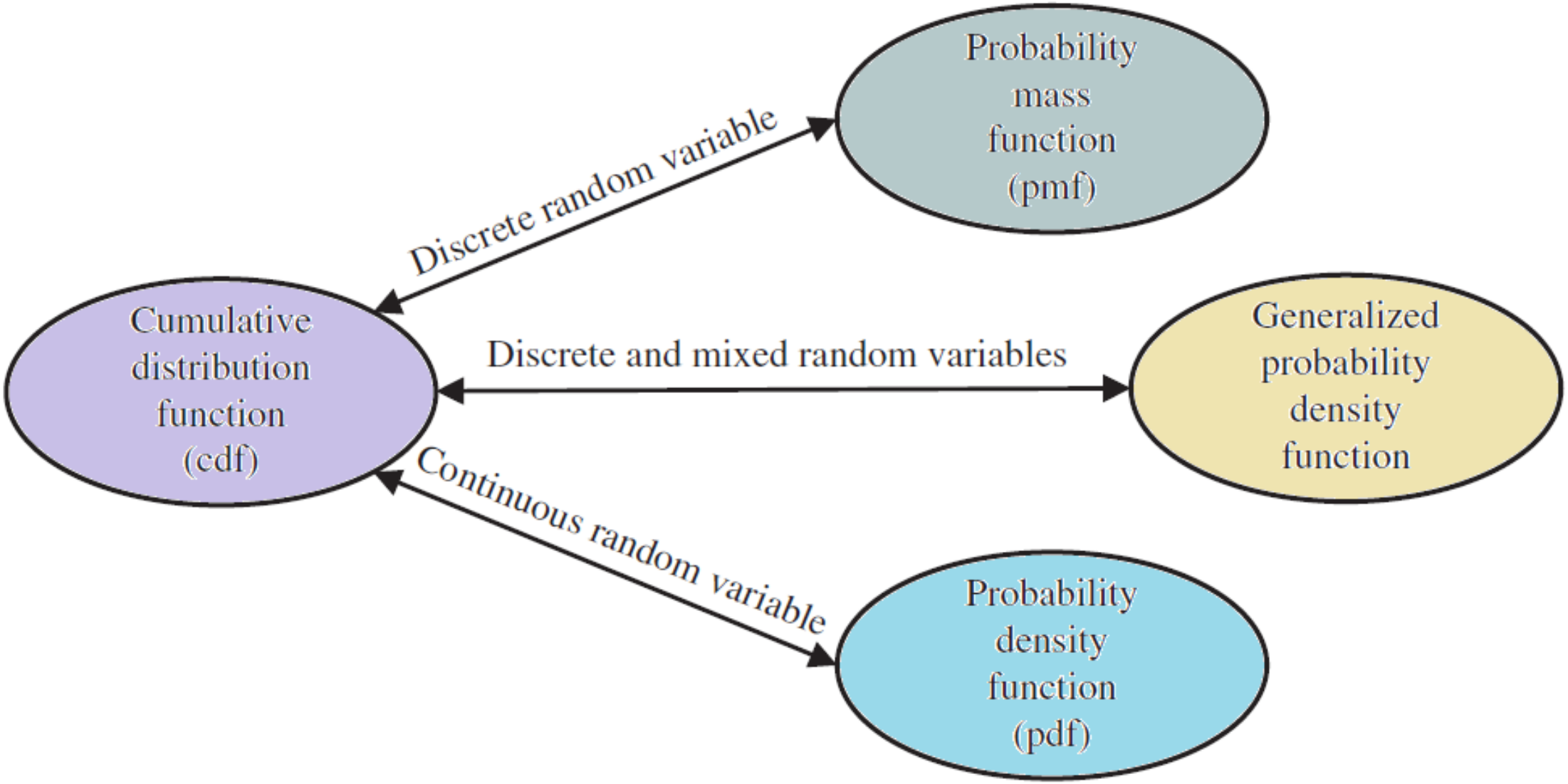
- We usually show random variables by capital letters such as **X**, **Y**, and **Z**.

- Since a random variable is a function, we can talk about its dynamic range.

# The relationship among sample space, random variable, and probability.



# The relationship among **CDF**, **PMF**, and **PDF**.



# Continuous random variable

# Definition: Functions of Continuous Random Variables

- If $\boldsymbol{X}$ is a continuous random variable and $\boldsymbol{Y} = g(\boldsymbol{X})$ is a function of $\boldsymbol{X}$, then $\boldsymbol{Y}$ itself is a random variable.

- It is possible find the **CDF** and **PDF** of $\boldsymbol{Y}$.

- The function of a continuous random variable might be **a non-continuous random variable**.

# Continuous Random Variables

- Cumulative Distribution Function (CDF)
- Probability Density Function (PDF)
- Expected Value
- Standart Deviation/Variance
- High-order statistics
- Continuous Distributions

# *Cumulative Distribution Function (CDF) of a* Continuous Random Variable

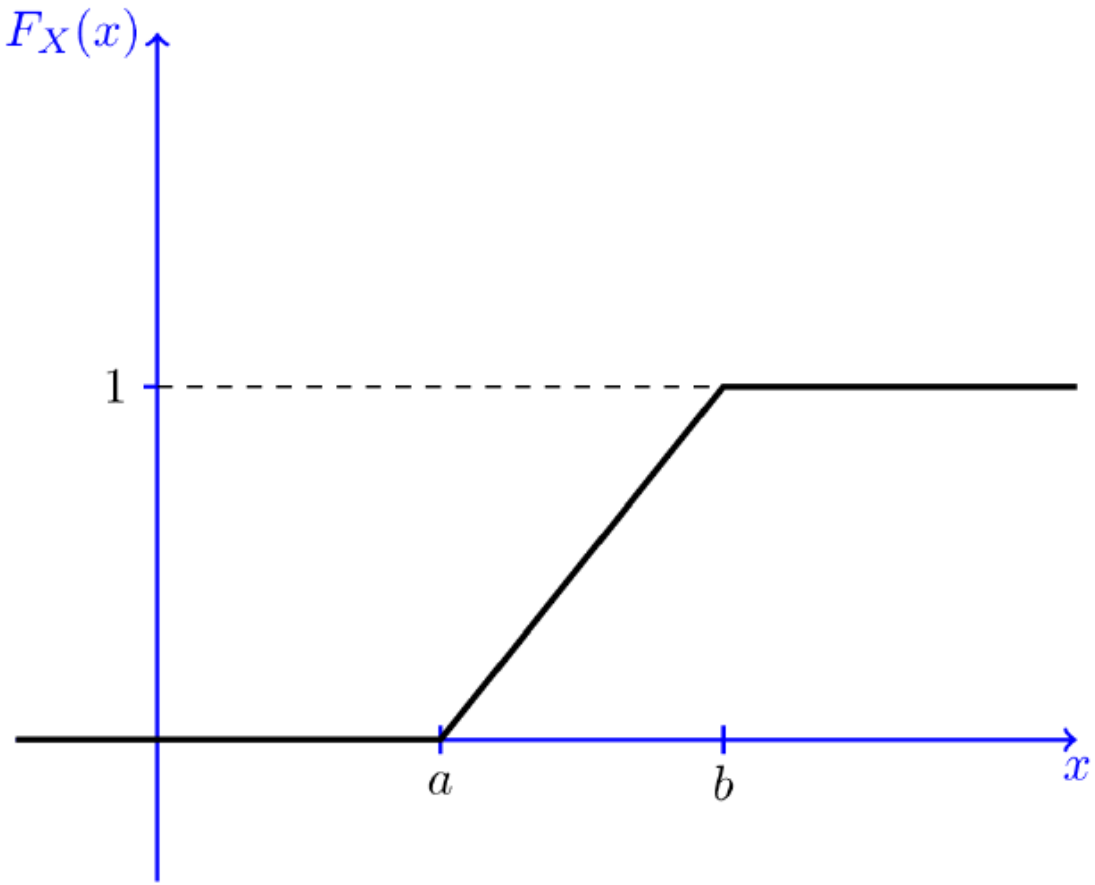
- The ***cumulative distribution function*** of a random variable ***X*** expresses the complete probability model of a random experiment as the following mathematical function:

$$F_X(x) = P(X \leq x); -\infty < x < \infty$$

- The random variable ***X*** takes on a value in the interval $(-\infty, x]$.

# Example of **Cumulative Distribution Function** (**CDF**)

- CDF for a continuous random variable uniformly distributed over [*a*, *b*].



# Properties of **CDF**

$$0 \leq F_X(x) \leq 1$$

$$a < b \rightarrow F_X(a) \leq F_X(b)$$

$$\lim_{x \to -\infty} F_X(x) = 0$$

$$\lim_{x \to +\infty} F_X(x) = 1$$

$$F_X(b-) = F_X(b+)$$

# Properties of **CDF**

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

$$P(X = b) = F_X(b+) - F_X(b-) = 0$$

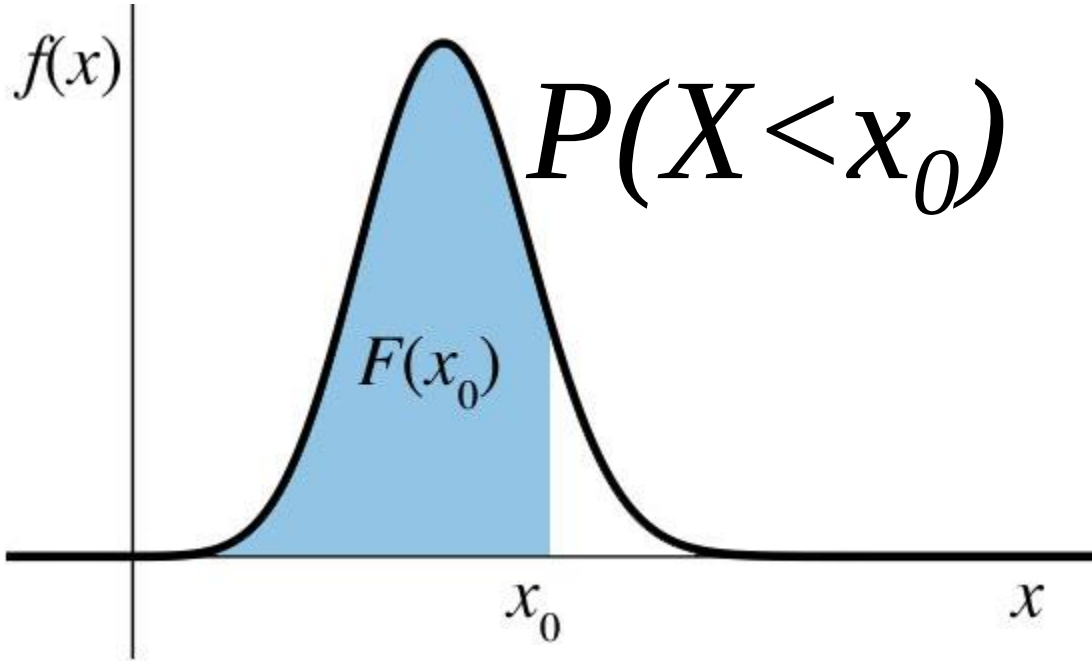
$$P(X > x) = 1 - F_X(x)$$

# Probability Density Function (PDF) of a Continuous Random Variable

- The ***probability density function*** of a continuous random variable ***X***, if it exists, expresses the complete probability model of a random experiment as:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$P(X \leq x_0) = F_X(x_0) = \int_{-\infty}^{x_0} f_X(x)dx$$



# Properties of PDF

$$f_X(x) \geq 0 \ \forall x \ and \int_{-\infty}^{\infty} f_X(x)dx = 1$$

$$P(a \leq X \leq b) = \int_a^b f_X(x)dx$$

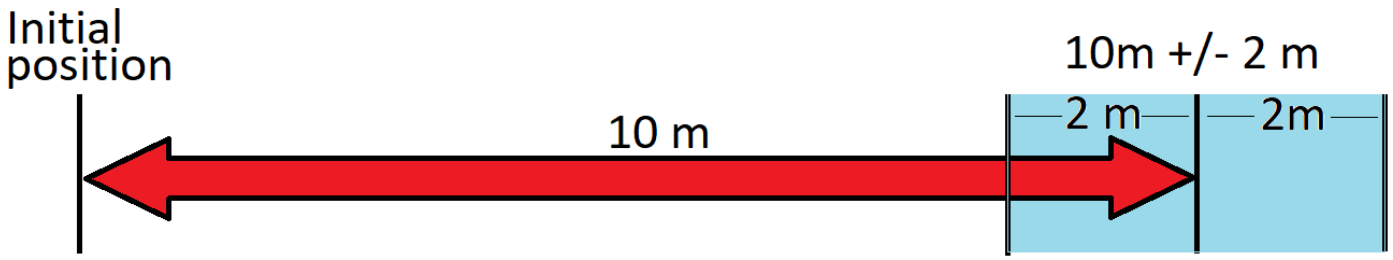
$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

# Example: building a PDF

- In a random experiment shot-putters (arremessadores de massa) **were asked to aim at a line 10m away**.

- **They threw the shot 200 times** and **thows were measured within 2m either side of the line**.

# Building a PDF



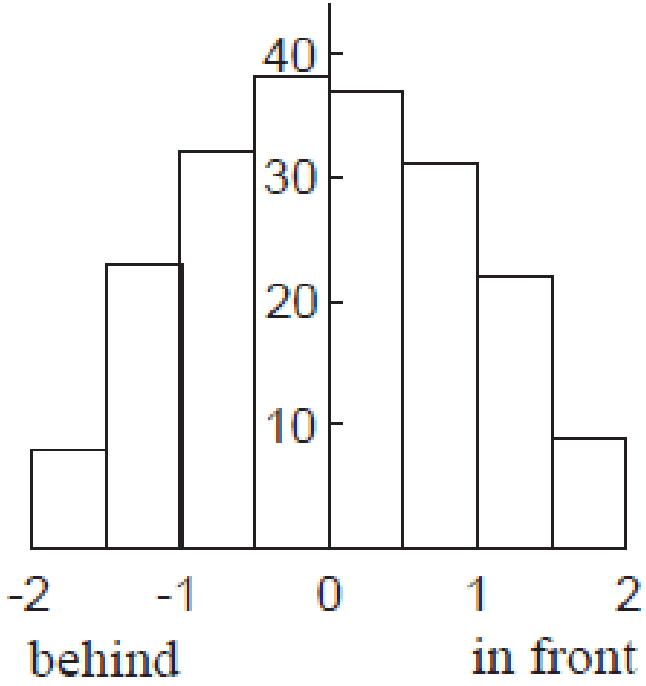

**in front**

| metres | 1.99-1.50 | 1.49-1.00 | 0.99-0.5 | 0.49-0 |
|---|---|---|---|---|
| frequency | 9 | 22 | 31 | 37 |

**behind**

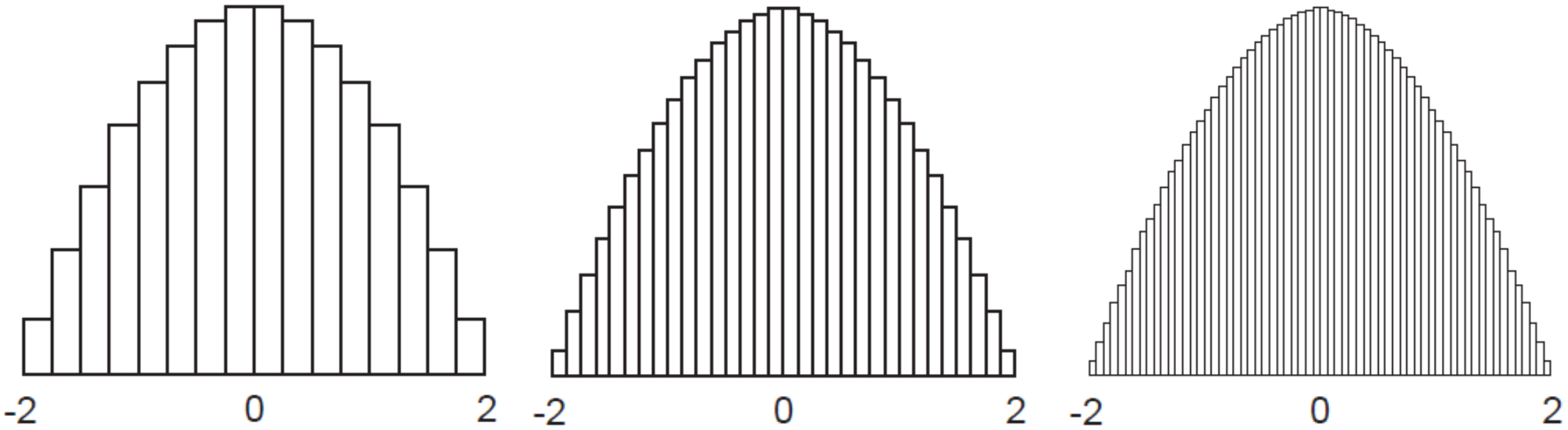
| metres | 0-0.49 | 0.50-0.99 | 1.00-1.49 | 1.50-1.99 |
|---|---|---|---|---|
| frequency | 38 | 32 | 23 | 8 |

- A histogram represent these data.



# Building a PDF: what could be done

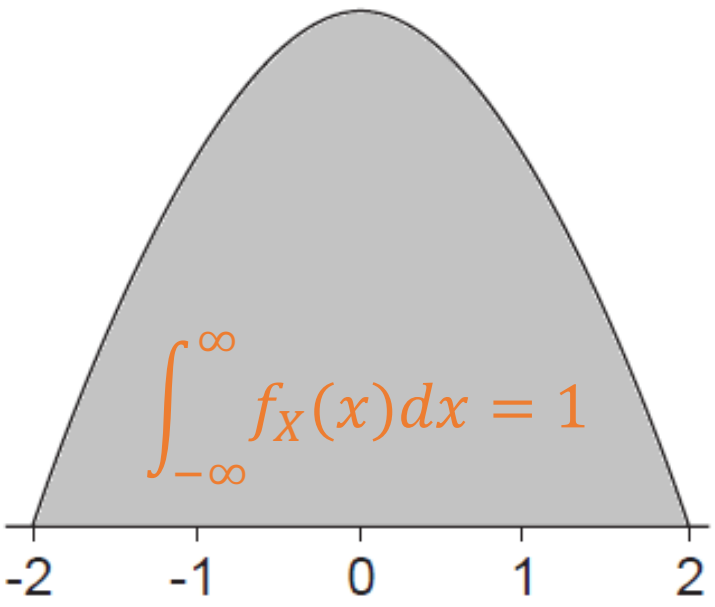
- To obtain a more accurate result it might be possible to collect more data.
- A possible pattern might emerge.



# Building a PDF

- When the number of outcomes tends to infinity, the curve converges to a continuous representation.

- If you normalize the curve so that $\int_{-\infty}^{\infty} f_X(x)dx = 1$ then as a result you will get a **PDF**



# Using a PDF from the experiment acquired data

- What is the probability of the mass falling by 0 m and 1m beyond the 10 m line?
- We can estimate the probability by computing the trapezium area.

$$P(0.0 \leq x \leq 1.0) = \frac{38 + 32}{200} = 0.350$$

**in front**

| metres | 1.99-1.50 | 1.49-1.00 | 0.99-0.5 | 0.49-0 |
|---|---|---|---|---|
| frequency | 9 | 22 | 31 | 37 |

**behind**

| metres | 0-0.49 | 0.50-0.99 | 1.00-1.49 | 1.50-1.99 |
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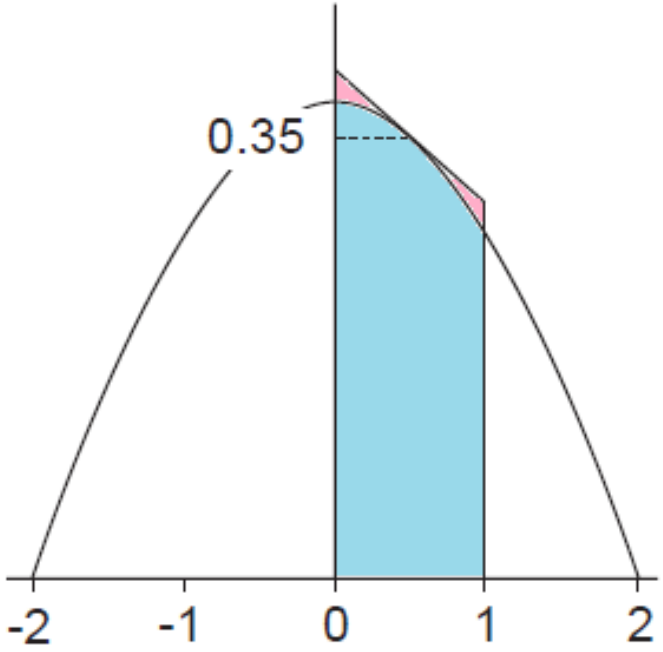
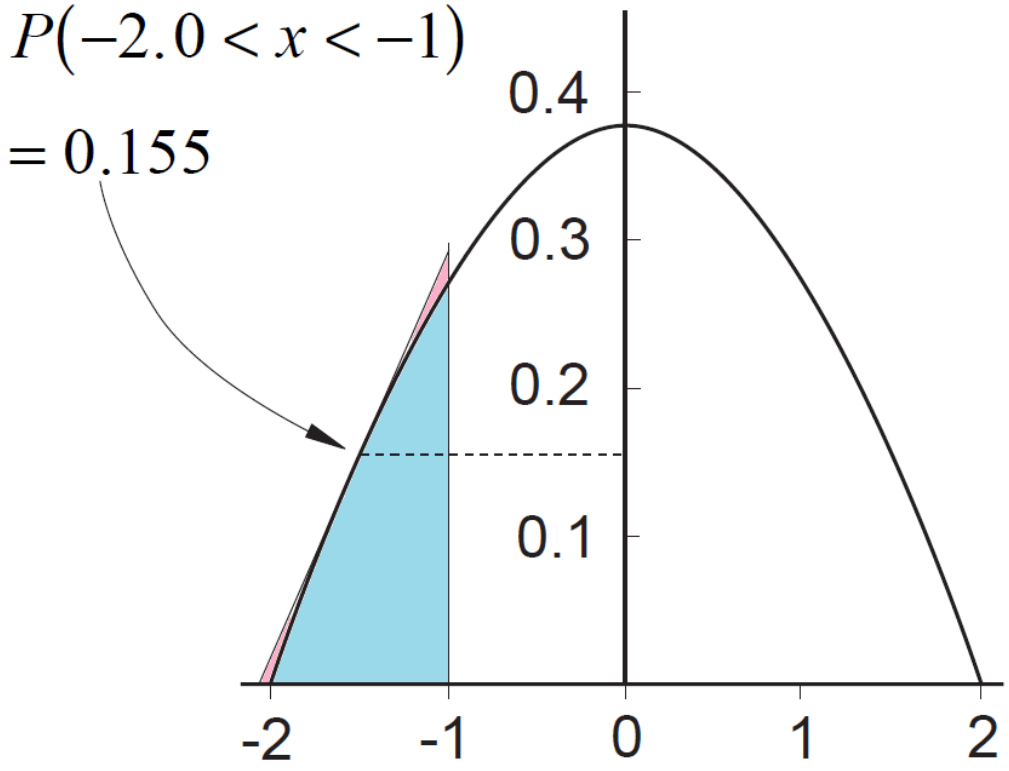


# Using PDF from acquired data

- What is the probability of the mass falling by -2.0 m and -1.0 m before the 10 m line?

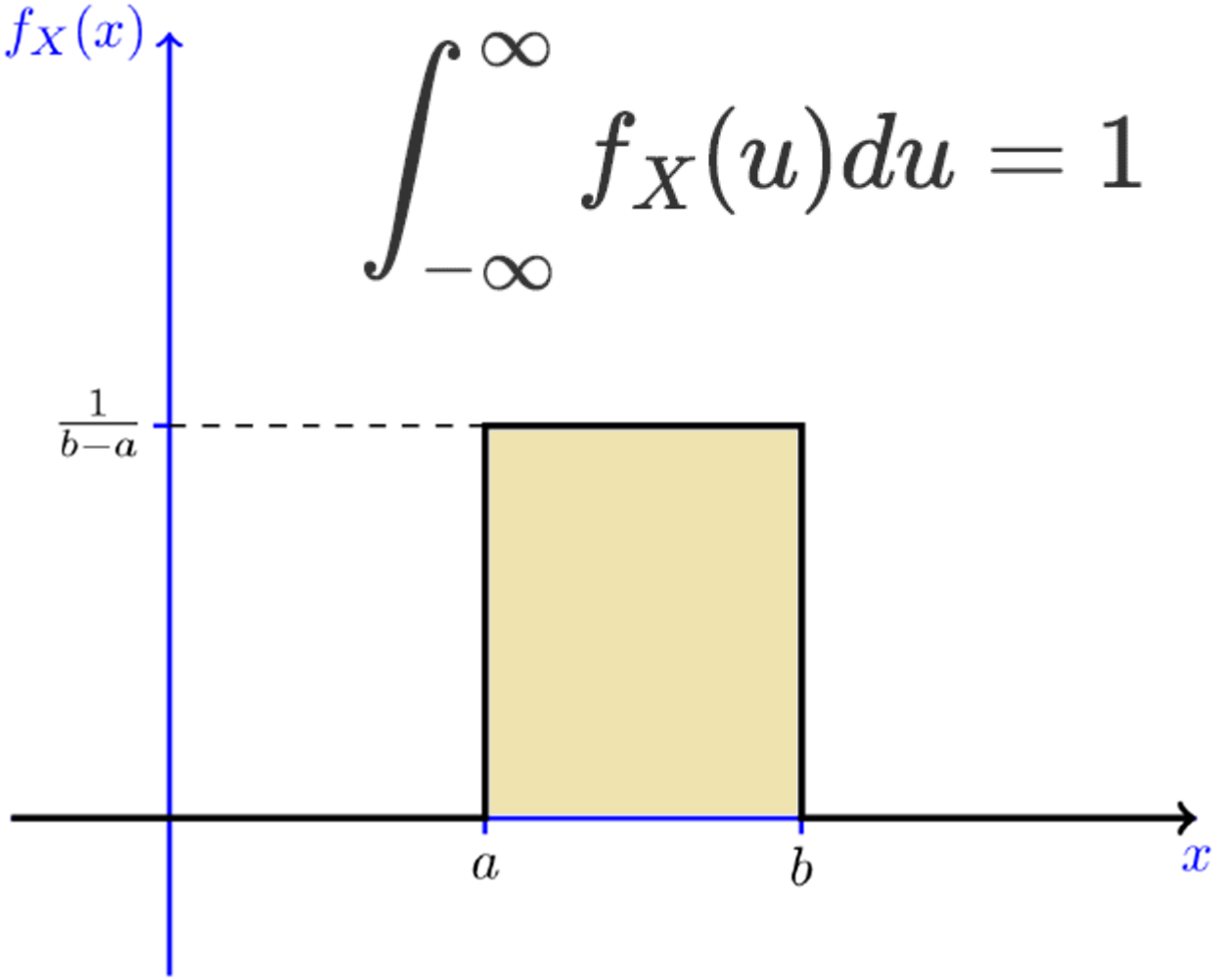



$$P(-2.0 \leq x \leq -1.0) = \frac{9 + 22}{200} = 0.155$$

# An Example of **Probability Density Function (PDF)**

- PDF for a continuous random variable **uniformly** distributed over [*a*, *b*].

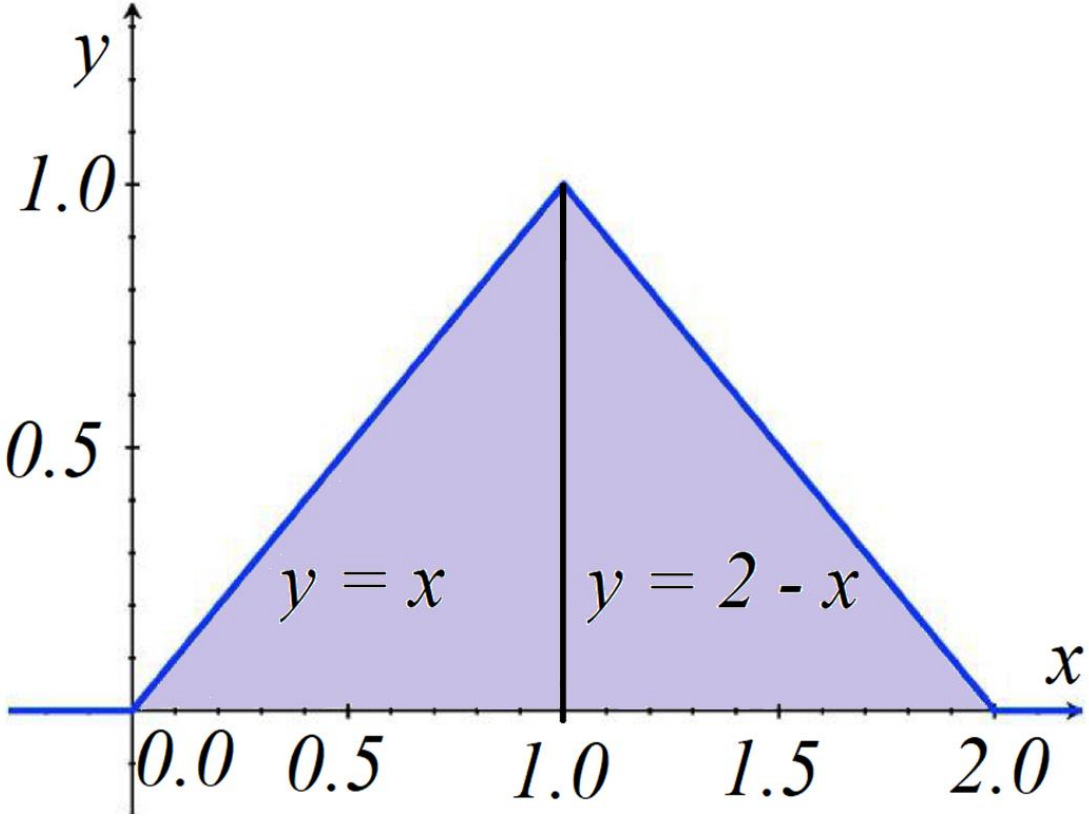
$$\int_{-\infty}^{\infty} f_X(u)du = 1$$



# Exercício 1

- Let the random variable denote **the time a person waits for an elevator to arrive**. Suppose the longest one would need to wait for the elevator is 2 minutes, so that the possible values of (in minutes) are given by the interval. A possible pdf for is given by

$$f_X(x) = \begin{cases} x, & for\ 0 \le x \le 1 \\ 2 - x, & for\ 1 \le x \le 2 \\ 0, & otherwise \end{cases}$$



# Exercise 1: solution

- a)Verify if $f_X(x)$ is a PDF.
- b) Calculate the probability that a person waits less than 30 seconds (or 0.5 minutes) for the elevator to arrive.
- c) Compute the probability value that a person will wait form $x = 0.5$ minutes to $x = 1.5$ minutes. Use the CDF to obtain the probability.
- d) Determine an analytical formulation for the CDF.

# Exercise 1:Solution

- a)Verify if $f_X(x)$ is a PDF.

From the graph, it is clear that $f_X(x) \geq 0$ , for all $x \in \boldsymbol{R}$.
Since there are no holes, jumps, asymptotes, we see that is (piecewise) continuous.
To $f_X(x)$ be a PDF it is necessary: $\int_{-\infty}^{\infty} f_X(x)dx = 1$

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_{0.0}^{1.0} xdx + \int_{1.0}^{2.0} (2-x)dx \quad = \left.\frac{x^2}{2}\right|_{0}^{1.0} + \left.\left(2x - \frac{x^2}{2}\right)\right|_{1.0}^{2.0}$$

$$= 0.5 + (2.0 - 1.5) = 1.0$$

# Exercise 1:Solution

- b) Calculate the probability that a person waits less than 30 seconds (or 0.5 minutes) for the elevator to arrive.

$$P(0.0 \leq X \leq 0.5) = F_X(0.5) = \int_{0.0}^{0.5} f_X(x)dx$$

$$= \int_{0.0}^{0.5} xdx$$

$$= \left.\frac{x^2}{2}\right|_0^{0.5} = 0.125$$

# Exercise 1:Solution

- c) Compute the probability value that a person will wait form $x = 0.5$ minutes to $x = 1.5$ minutes. Use a CDF to calculate

$$P(0.5 \leq X \leq 1.5) = F_X(x = 1.5) - F_X(x = 0.5)$$

$$F_X(x = 0.5) = \int_{0.0}^{0.5} x dx = \left.\frac{x^2}{2}\right|_0^{0.5} = 0.125$$

$$F_X(x = 1.5) = \int_{0.0}^{1.0} x dx + \int_{1.0}^{1.5} (2 - x) dx = \left.\frac{x^2}{2}\right|_0^{1.0} + \left.\left(2x - \frac{x^2}{2}\right)\right|_{1.0}^{1.5}$$

$$= 0.5 + (1.875 - 1.5) = 0.875$$

$$P(0.5 \leq \boldsymbol{X} \leq 1.5) = F_X(x = 1.5) - F_X(x = 0.5)$$

$$= 0.875 - 0.125 = 0.750$$

# Exercise 1:Solution

- Other solution of (c) using the PDF:

$$P(0.5 \leq X \leq 1.5) = \int_{0.5}^{1.5} f_X(x)dx = \int_{0.5}^{1.0} xdx + \int_{1.0}^{1.5} (2-x)dx$$

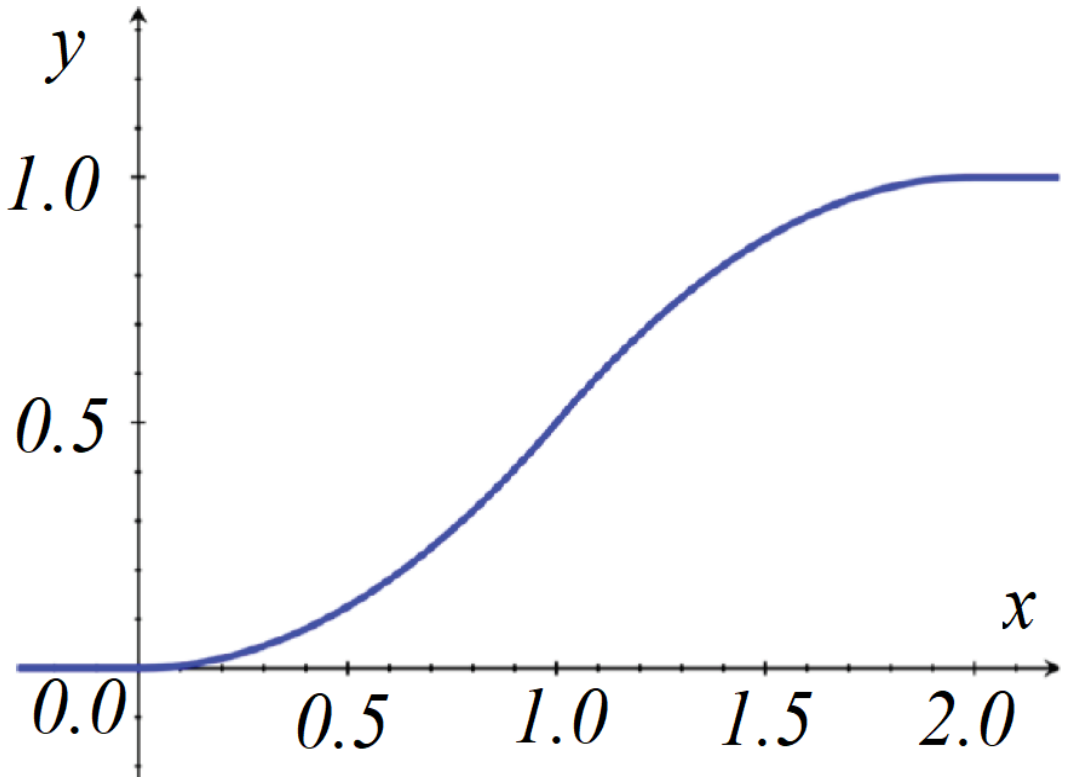
$$= \frac{x^2}{2}\Bigg|_{0.5}^{1.0} + \left(2x - \frac{x^2}{2}\right)\Bigg|_{1.0}^{1.5}$$

$$= 0.375 + 0.375 = 0.750$$

# Exercise 1:Solution

- d) Determine an analytical formulation for the CDF.

$$F_X(X) = \begin{cases} 0.0, & for \;\; x \;\leq 0.0 \\ \dfrac{x^2}{2}, & for\; 0.0 \leq x \leq 1.0 \\ 2x - \dfrac{x^2}{2} - 1, & for\; 1 \leq x \leq 2.0 \\ 1.0, & for\; x \geq \;\; 2.0 \end{cases}$$



# Expected value (or mean) of a continuous random variable X

- **Expected Value, Expectancy, Expectance, Expectation, Mean**

- The expected value of a continuous random variable is defined as

$$E[\boldsymbol{X}] = \mu_{\boldsymbol{X}} \equiv \int_{-\infty}^{\infty} x f_{\boldsymbol{X}}(x) dx$$

# Properties of the **expected value** of a continuous random variable

- $E[E[\boldsymbol{X}]] = E[\boldsymbol{X}] = \mu_{\boldsymbol{X}}$

- $E\,[\boldsymbol{a}\boldsymbol{X}+\boldsymbol{b}] = \boldsymbol{a}E[\boldsymbol{X}] + \boldsymbol{b}$, for all $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{R}$.

- $E[\boldsymbol{X}_1+\boldsymbol{X}_2+\ldots+\boldsymbol{X}_n] = E[\boldsymbol{X}_1]+E[\boldsymbol{X}_2]+\ldots+E[\boldsymbol{X}_n]$, for any set of random variables $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$.

# Properties of the **expected value** of a **continuous random variable**

- Linear combination of randomic variables

$$\boldsymbol{E}[\boldsymbol{\alpha}_1\boldsymbol{X}_1 + \boldsymbol{\alpha}_2\boldsymbol{X}_2 + \ldots + \boldsymbol{\alpha}_n\boldsymbol{X}_n] =$$

$$= \boldsymbol{\alpha}_1 E[\boldsymbol{X}_1] + \boldsymbol{\alpha}_2 E[\boldsymbol{X}_2] + \ldots + \boldsymbol{\alpha}_n E[\boldsymbol{X}_n]$$

# Law of the unconscious statistician (LOTUS) for continuous random variables

- The expected value of a function g(**X**) of a random variable **X** when one knows the probability distribution of **X** but one does not know the distribution of g(**X**) is defined as

$$E[g(\boldsymbol{X})] \equiv \int_{-\infty}^{\infty} g(x) f_{\boldsymbol{X}}(x) dx$$

# Variance of a Continuous Random Variable

$$\sigma_X^2 = Var(\boldsymbol{X}) = E[(\boldsymbol{X} - \mu_{\boldsymbol{X}})^2]$$

$$= \int_{-\infty}^{\infty} (x - \mu_{\boldsymbol{X}})^2 f_{\boldsymbol{X}}(x) dx$$

$$= E[g(\boldsymbol{Y})]; g(\boldsymbol{Y})\Big|_{\mathbf{Y}=(\boldsymbol{X}-\mu_{\boldsymbol{X}})^2}$$

# Properties of the **Variance**

. Variance is non-negative because the squares are positive or zero: $Var(X) = \sigma_X^2 \geq 0.$

.The variance of a constant is zero: $Var(a) = 0$ , $a \in \mathrm{R}.$

.Variance is invariant with respect to changes in a location parameter: $Var(X + a) = Var(X) = \sigma_X^2$

. $Var(X) = \sigma_X^2 = E[(X - \mu_{\boldsymbol{X}})^2] = E[\boldsymbol{X}^2] - (E[\boldsymbol{X}])^2$

# Properties of the **Variance**

.$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X),\ a, b \in \mathrm{R}$

.The variance of a sum of two independent random variables:

$$Var(X + Y) = Var(X) + Var(Y).$$

# The **standard deviation** of a Continuous random variable $X$

- The **standard deviation** of a continuous random variable $X$ is defined as:

$$\sigma_X = \sqrt{\sigma_X^2}$$

The *m*-th moment of a Continuous Random Variable is defined as

$$E[\boldsymbol{X}^{\boldsymbol{m}}] \equiv \int_{-\infty}^{\infty} x^m f_{\boldsymbol{X}}(x)dx$$

The *m*-th central moment of a continuous random variable is defined as

$$E[(X-\mu_{\boldsymbol{X}})^m] \equiv \int_{-\infty}^{\infty} (x-\mu_{\boldsymbol{X}})^m f_{\boldsymbol{X}}(x)dx$$

# Observe that

- The first moment is the expectation

$$E\,[X\,] = \mu_X.$$

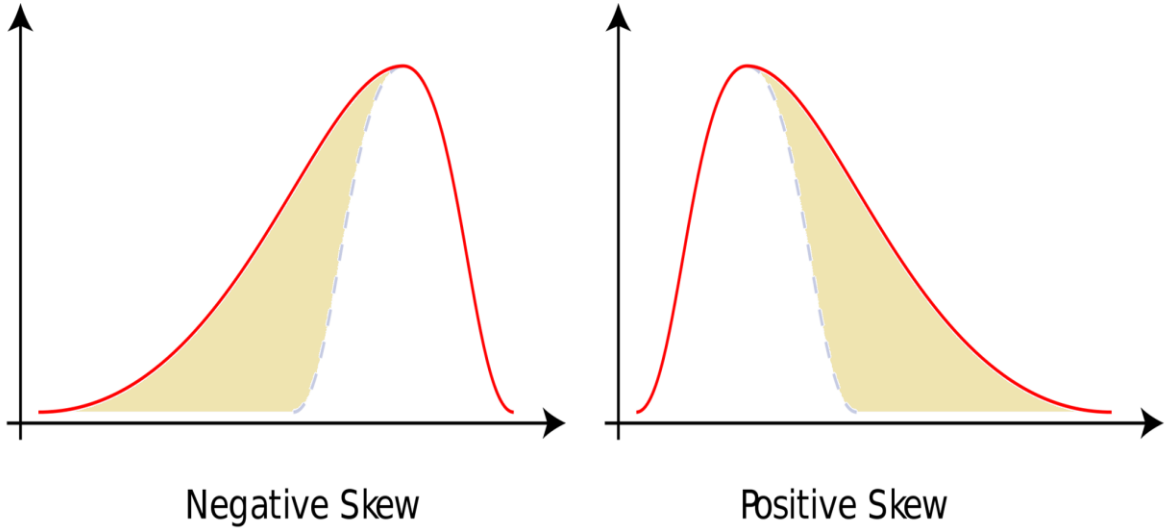
- The second central moment is the variance.

$$Var(X) = \sigma_X^2$$

# Third order central moment: Skewness

- Skewness ("inclinação") is a measure of the asymmetry of the probability distribution of a real-valued random variable about its mean.
- The skewness value can be positive, zero, negative, or undefined.

$$\tilde{\mu}_3 = \mathrm{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3}{\sigma^3} = \frac{\mathrm{E}\left[(X-\mu)^3\right]}{(\mathrm{E}[(X-\mu)^2])^{3/2}}$$

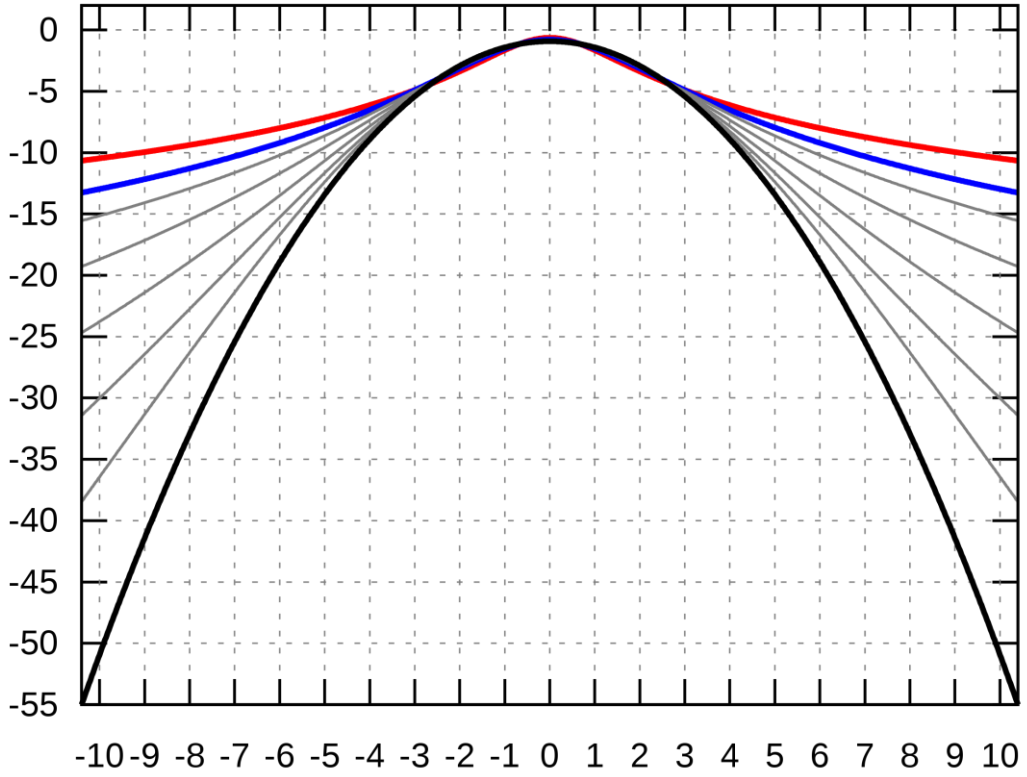


# Fourth order central moment: **Kurtosis**

- kurtosis is a measure of shape that characterizes the flattening of the probability distribution function (PDF) curve.

Higher kurtosis corresponds to greater extremity of deviations (or outliers), and not the configuration of data near the mean. It is usually (mostly) defined as:

$$\frac{\mu_4}{\mu_2^2} = \frac{E\left(X - E(X)\right)^4}{\left[E\left(X - E(X)\right)^2\right]^2}$$



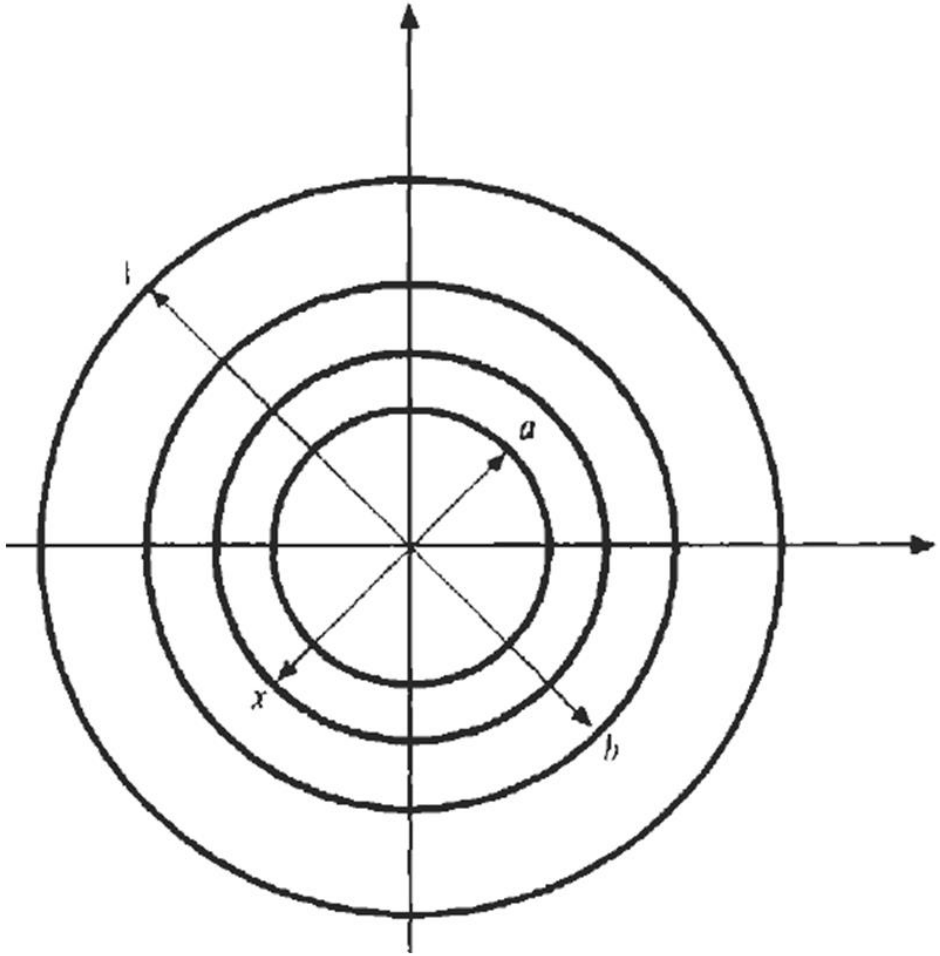

kurtosis of infinity (red); 2 (blue); 1, 1/2, 1/4, 1/8, and 1/16 (gray); and 0 (black)

# Exercício 2

- Considere o experimento de lançar um dardo em um alvo circular com raio unitário (1m).
- Seja **X** a variável randômica representando a distância do ponto em que o dardo acerta o alvo a partir do ponto central do alvo.

- Suponha que o ponto onde o dardo sempre acerte o alvo seja **igualmente provável sobre toda a área do alvo**.

# Exercício 2




a) Qual é a faixa dinâmica da variável randômica $\boldsymbol{X}$?

b) Determine analiticamente a CDF $F_X(x)$ da variável randômica X e esboce o seu respectivo gráfico.

c) Determine analiticamente a PDF $f_X(x)$ da variável randômica $\boldsymbol{X}$ e esboce o seu respectivo gráfico.

d) Determine $P(\boldsymbol{X} > 0.5)$

e) Determine $P(a \leq x \leq b)$; $a < b < 1$.

f) Determine $E[\boldsymbol{X}]$.

g) Determine $Var[\boldsymbol{X}]$.

h) Calcule o momento de terceira ordem: $E[\boldsymbol{X}^3]$.

# Exercício 2: Solução.

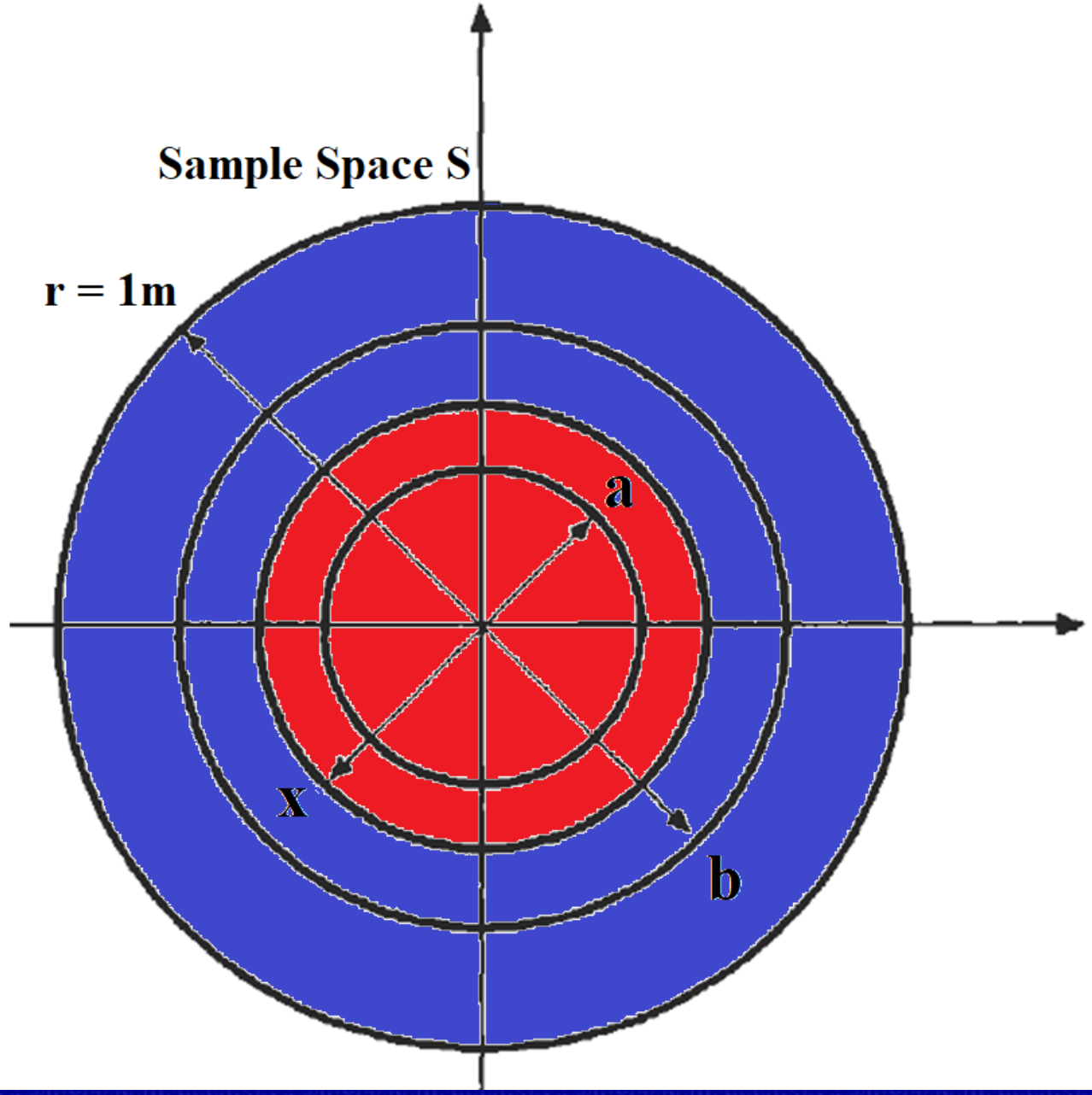
- a) Qual é a faixa dinâmica da variável randômica $\boldsymbol{X}$?

$$\boldsymbol{X} = \{x | x \in R; 0.0 \ \leq x \ \leq 1.0\} \ (metros)$$

# Exercício 2: Solução.

- b) Determine analiticamente a CDF $F_X(x)$ da variável randômica $\boldsymbol{X}$ e esboce o seu respectivo gráfico.

$$P(\boldsymbol{X} \leq x) = \frac{\pi x^2}{\pi r^2}$$

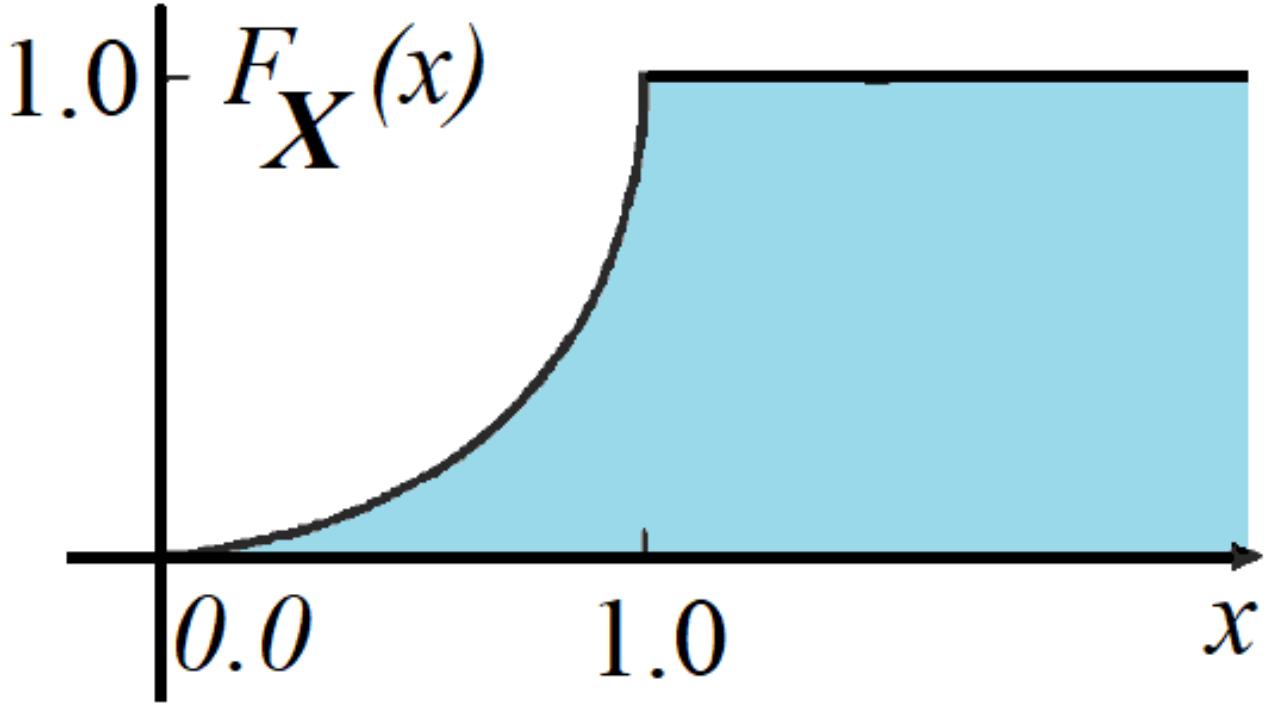


# Exercício 2: Solução.

- b) Determine analiticamente a CDF $F_{\boldsymbol{X}}(x)$ da variável randômica $\boldsymbol{X}$ e esboce o seu respectivo gráfico.

$$P(\boldsymbol{X} \leq x) = \frac{\pi x^2}{\pi r^2} = \frac{x^2}{r^2} = x^2; 0 \ \leq x \ \leq 1$$

$$F_{\boldsymbol{X}}(\boldsymbol{X}) = \begin{cases} 0\,, for\ x < 0 \\ x^2, for\ 0 \leq x \leq 1 \\ 1, for\ x > 1.0 \end{cases}$$

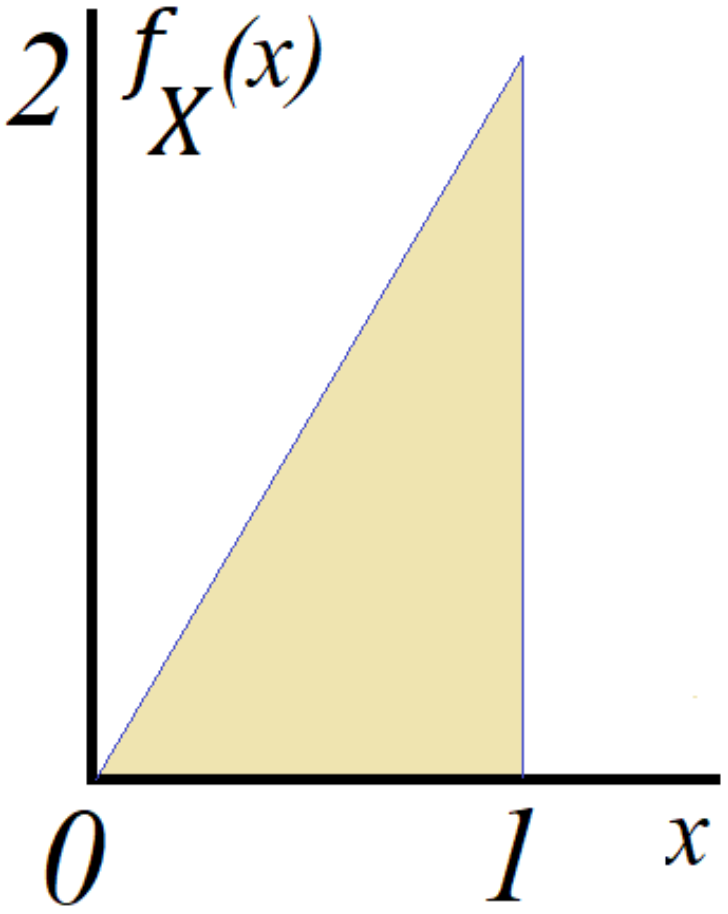


# Exercício 2: Solução.

- c) Determine analiticamente a PDF $f_X(x)$ da variável randômica $\boldsymbol{X}$ e esboce o seu respectivo gráfico.

$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{d(x^2)}{dx} = 2x,$$

$$0 \leq x \leq 1$$



# Exercício 2: Solução.

- d) Determine P($\boldsymbol{X}$ > 0.5).

$$\boldsymbol{Solução\ 1}: \boldsymbol{Usando\ a\ CDF}:\ \boldsymbol{F_X(x)}$$

$$P(\boldsymbol{X} > 0.5) = 1 - P(\boldsymbol{X} < 0.5) = 1 - F_{\boldsymbol{X}}(x = 0.5) = 1 - 0.5^2 = 0.75$$

$$\boldsymbol{Solução\ 2}: \boldsymbol{Usando\ a\ PDF}:\ \boldsymbol{f_X(x)}$$

$$P(\boldsymbol{X} > 0.5) = \int_{0.5}^{1.0} f_X(x)dx = \int_{0.5}^{1.0} (2x)dx = x^2\Big|_{0.5}^{1.0} = \{1 - 0.25\} = 0.75$$

# Exercício 2: Solução.

- e) Determine P(a ≤ x ≤ b); a < b < 1.

$$\boldsymbol{Solução\ 1}: \boldsymbol{Usando\ a\ CDF}:\ \boldsymbol{F_X(x)}$$

$$P(\mathrm{a} \leq \mathrm{x} \leq \mathrm{b}) = F_{\boldsymbol{X}}(x = b) - F_{\boldsymbol{X}}(x = a) = b^2 - a^2$$

$$\boldsymbol{Solução\ 2}: \boldsymbol{Usando\ a\ PDF}:\ \boldsymbol{f_X(x)}$$

$$P(\mathrm{a} \leq \mathrm{x} \leq b) = \int_a^b f_X(x)dx = \int_a^b (2x)dx = x^2\Big|_a^b = b^2 - a^2$$

# Exercício 2: Solução.

- f) Determine E[$\boldsymbol{X}$].

$$E[\boldsymbol{X}] = \mu_{\boldsymbol{X}} = \int_{-\infty}^{\infty} x f_{\boldsymbol{X}}(x) dx = \int_{0}^{1} (2x^2) dx = \frac{2}{3} x^3 \Big|_{0}^{1} = \frac{2}{3}$$

# Exercício 2: Solução.

- g) Determine Var[$\boldsymbol{X}$].

$$\sigma_X^2 = Var(\boldsymbol{X}) = E[(\boldsymbol{X} - \mu_{\boldsymbol{X}})^2] = \int_{-\infty}^{\infty} (x - \mu_{\boldsymbol{X}})^2 f_{\boldsymbol{X}}(x) dx$$

$$= \int_0^1 \left(x - \frac{2}{3}\right)^2 (2x) dx = \int_0^1 2\left(x^3 - \frac{4}{3}x^2 + \frac{4}{9}x\right) dx$$

$$= 2\left(\frac{1}{4}x^4 - \frac{4}{9}x^3 + \frac{2}{9}x^2\right)\Bigg|_0^1$$

$$= 2(0.250 - 0.444 + 0.222) = 0.056$$

# Exercício: Solução.

- h) Calcule o momento de terceira ordem.

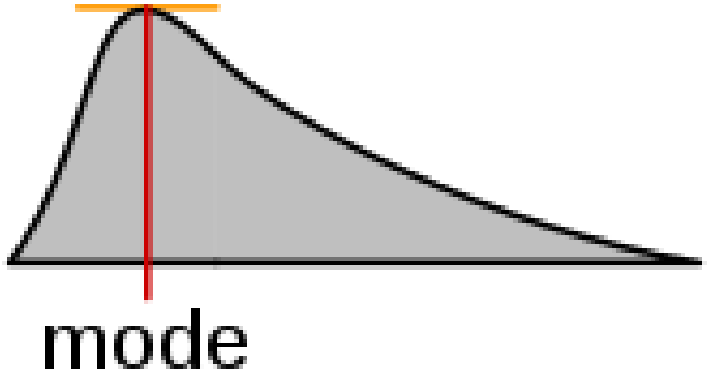
Using the **Law of the unconscious statistician** (LOTUS) we can express

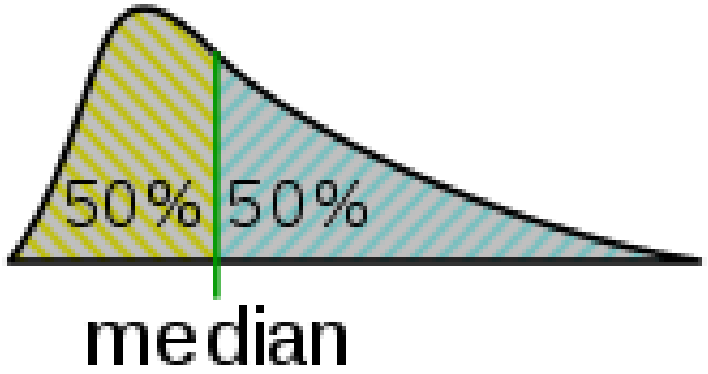
$$E[g(\boldsymbol{X})] = \int_{-\infty}^{\infty} g(x) f_{\boldsymbol{X}}(x) dx = E\left[\boldsymbol{X}^{\mathbf{3}}\right] = \int_{0}^{1} x^3 (2x) dx =$$

$$= 2\int_{0}^{1} x^4 dx \quad = \frac{2}{5} x^5 \Big|_{0}^{1} = \frac{2}{5}$$

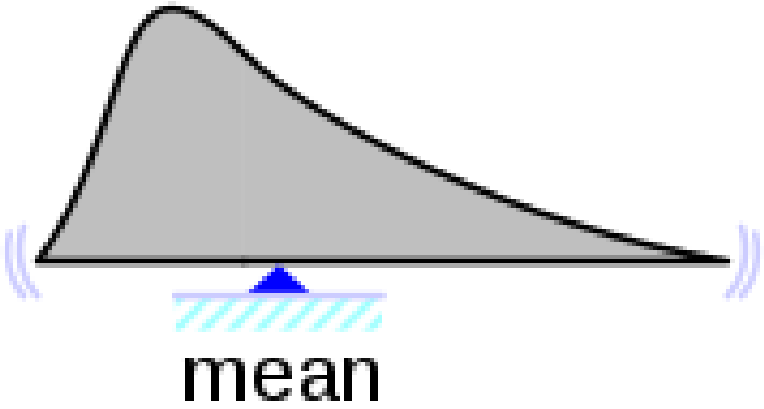
# Other concepts

The value that appears most often is called **mode**. The mode can be verified in PDF as its maximum.




The **median** is the value separating the higher half from the lower half of data set.

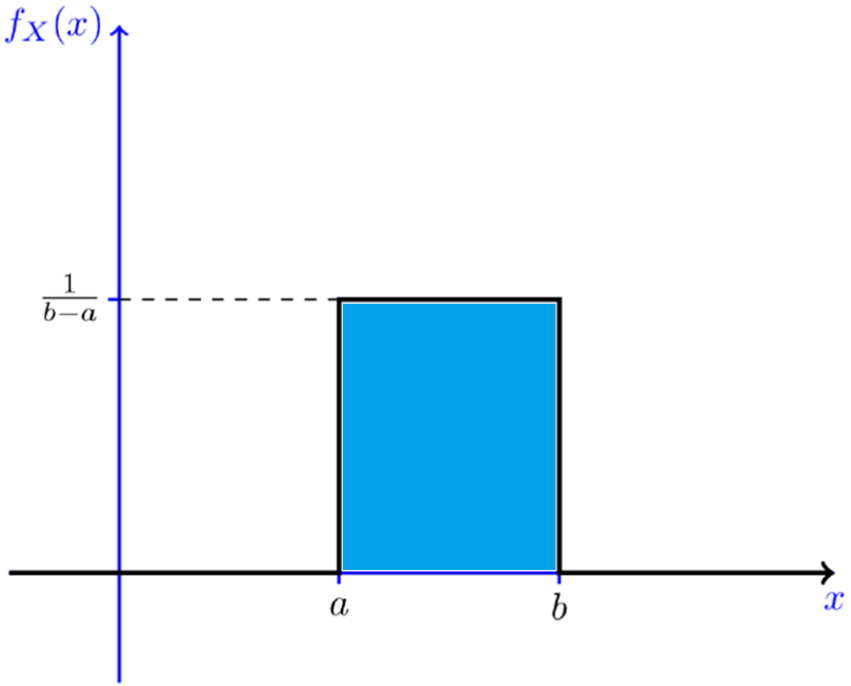


# Classical *Continuous* Distributions

# PDF for a continuous random variable **uniformly** distributed over $S_X = [a, b]$.

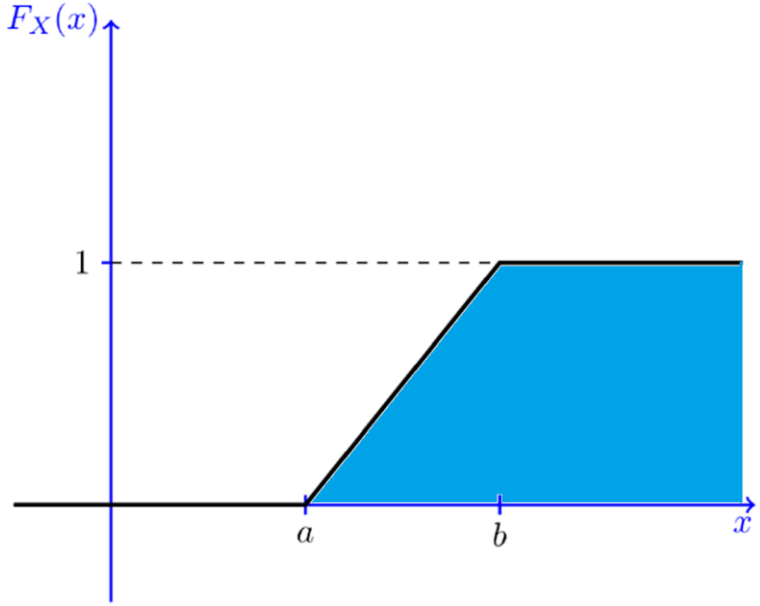
$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, a \leq x \leq b \\ 0, x < \ a \ e \ x > \ b \end{cases}$$

$$F_X(x) = \begin{cases} 0, x < a \\ \dfrac{x-a}{b-a}, a \leq x \leq b \\ 1, x > b \end{cases}$$




$$E[X] = \frac{a+b}{2}$$

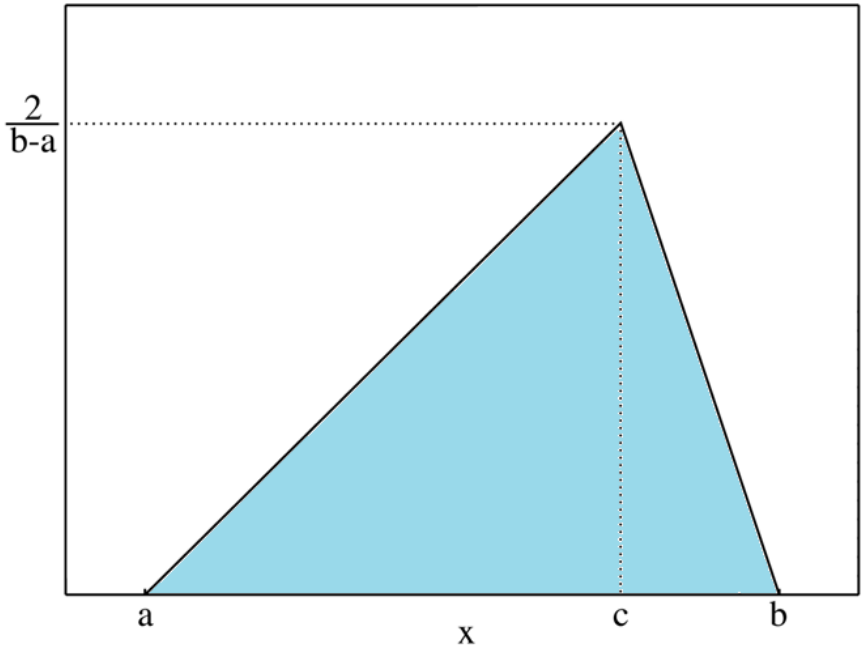
$$\sigma_X^2 = \frac{(b-a)^2}{12}$$



# Triangular distribution of a continuous random variable $S_X = [a, b]$.

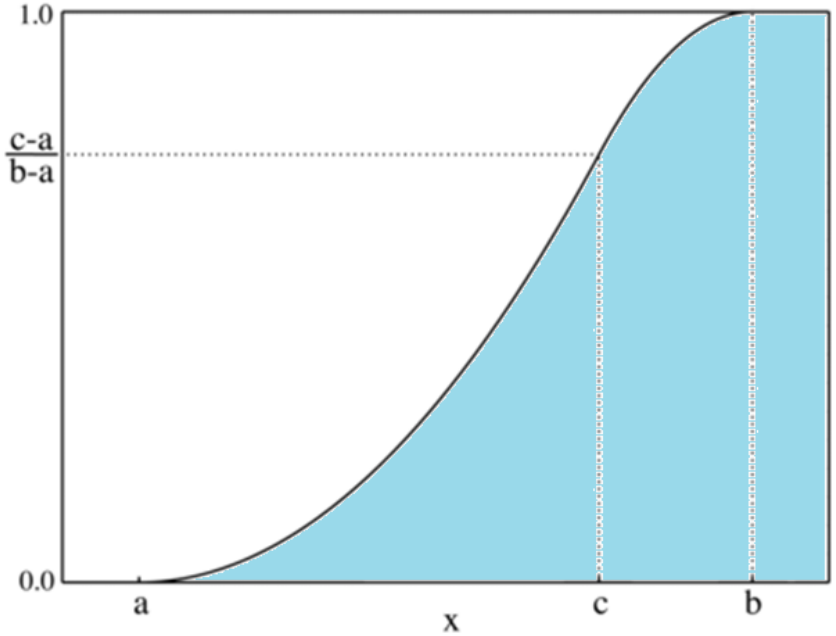
$$f_X(x) = \begin{cases} 0, x < a \\ \dfrac{2(x-a)}{(b-a)(c-a)}; a \le x < c \\ \dfrac{2(b-x)}{(b-a)(b-c)}; c \le x \le b \\ 0, x < b \end{cases}$$

$$F_X(x) = \begin{cases} 0, x < a \\ \dfrac{(x-a)^2}{(b-a)(c-a)}; a \le x < c \\ 1 - \dfrac{(b-x)^2}{(b-a)(b-c)}; c \le x \le b \\ 1, x > b \end{cases}$$




$$E[X] = \frac{a+b+c}{3}$$

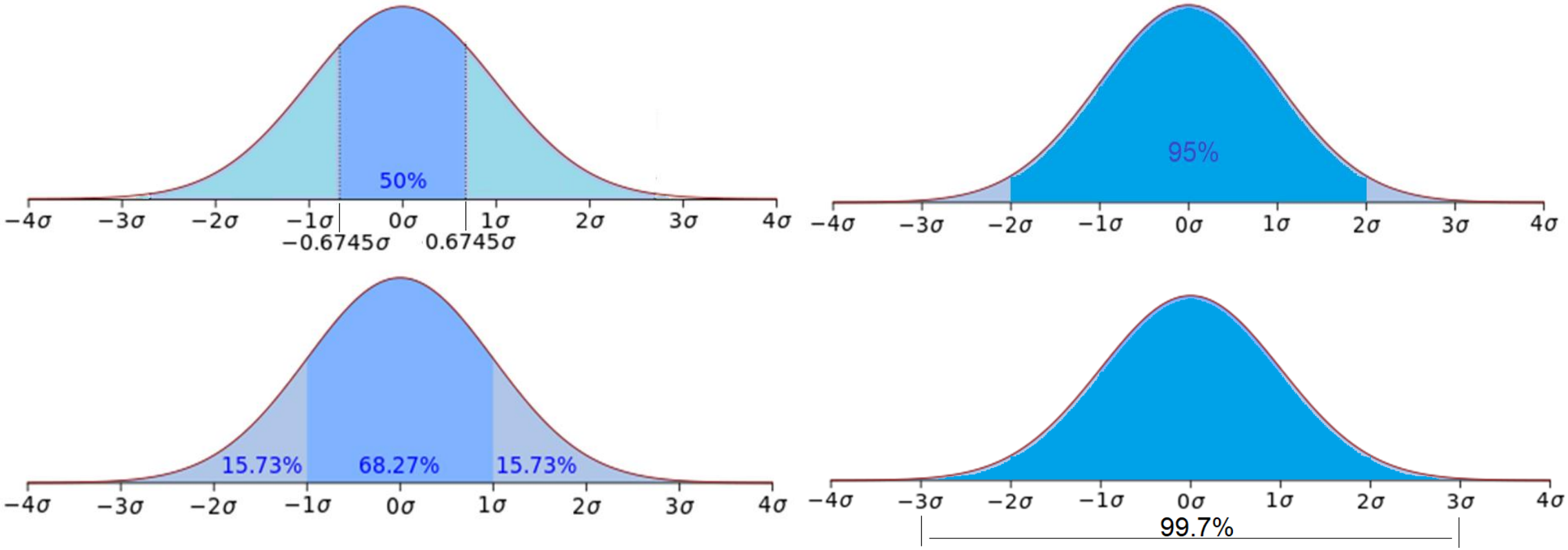
$$\sigma_X^2 = \frac{a^2+b^2+c^2-ab-ac-bc}{18}$$



# **Gaussian** (or normal) **distribution** of a continuous random variable $S_X = (-\infty, \infty)$.

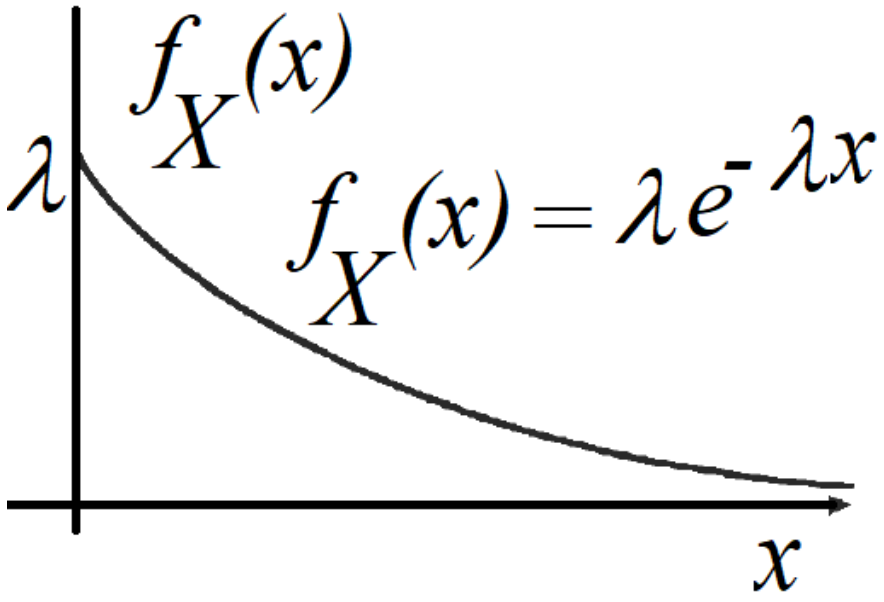


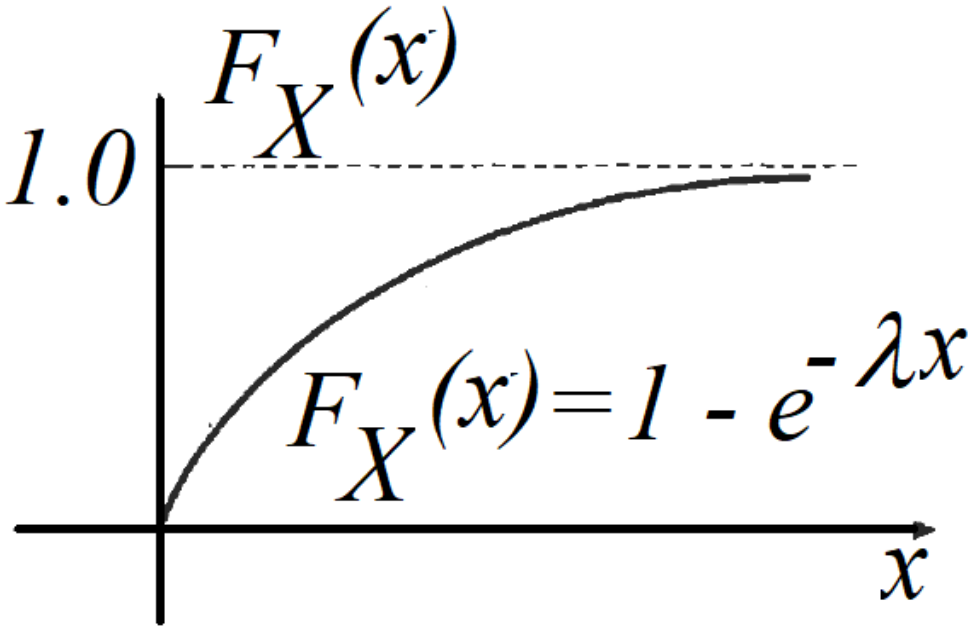
$$\boldsymbol{f_X(x)} = \mathcal{N}(X \mid \mu, \sigma_X^2) = \frac{\mathbf{1}}{\sqrt{\mathbf{2\pi\sigma_X^2}}} \boldsymbol{e}^{-\frac{\mathbf{1}}{\mathbf{2\sigma_X^2}}(\boldsymbol{x}-\boldsymbol{\mu_X})^2}, -\infty < \boldsymbol{x} < +\infty$$



# Exponential Distribution of a continuous random variable $S_X = [0, \infty)$.

$$f_X(x) = \begin{cases} 0, x \leq 0 \\ \lambda e^{-\lambda x}, x > 0 \end{cases} ; \quad \lambda > 0 \quad F_X(x) = \begin{cases} 0, x \leq 0 \\ 1 - e^{-\lambda x}, x > 0 \end{cases}$$



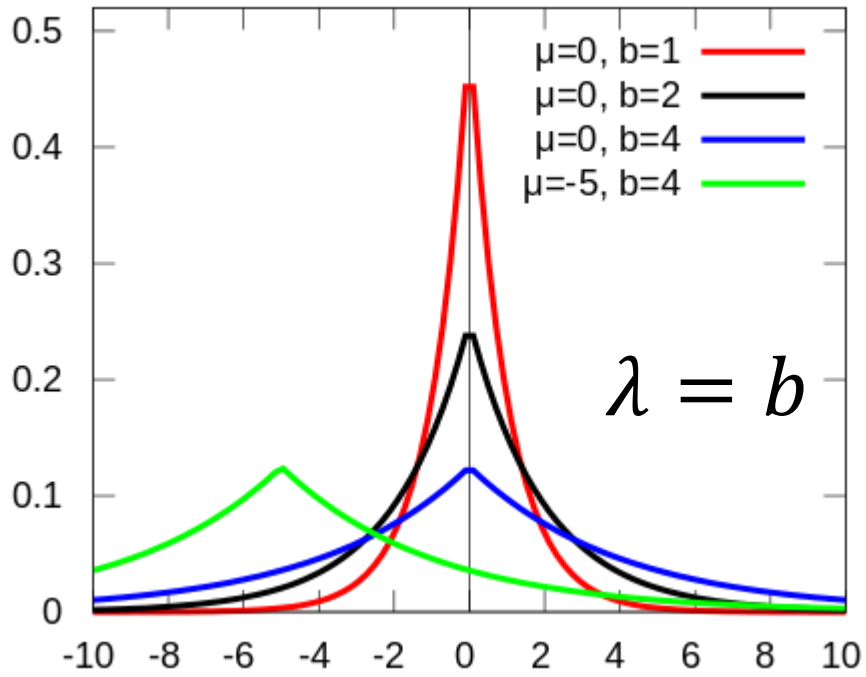
# Laplace distribution $S_X = (-\infty, +\infty)$.



$$f_X(x) = \frac{1}{2\lambda} e^{-\frac{1}{\lambda}|x-\mu_X|}, -\infty < x < +\infty,$$

$$-\infty < \mu_X < +\infty, \ \lambda > 0$$

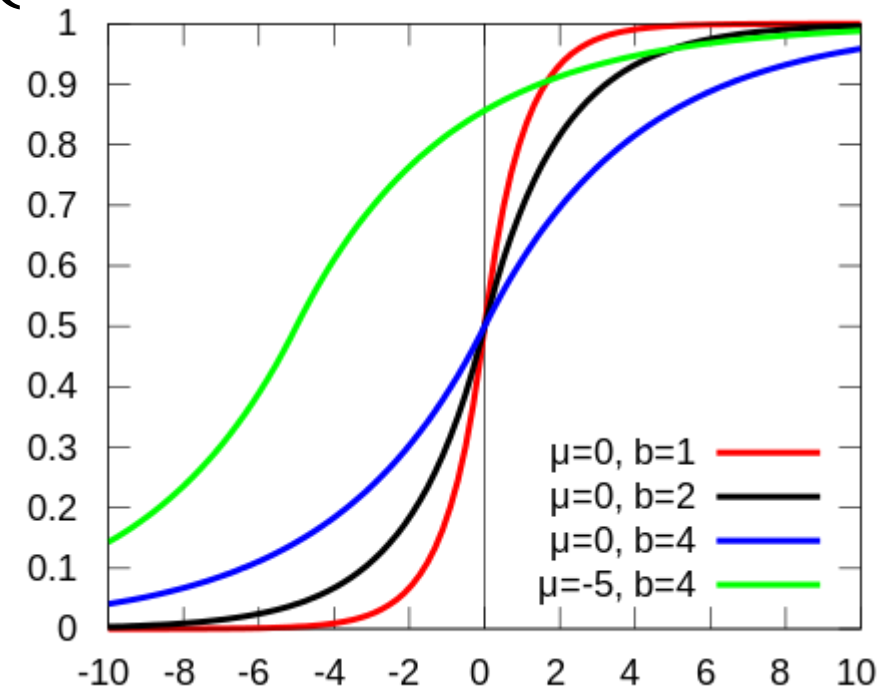
$$F_X(x) = \begin{cases} \frac{1}{2} e^{-\frac{1}{\lambda}(\mu_X - x)}, x < \mu_X \\ 1 - \frac{1}{2} e^{-\frac{1}{\lambda}(x-\mu_X)}, x \geq \mu_X \end{cases}$$




$$\lambda = b$$

$$E(\boldsymbol{X}) = \mu_X$$

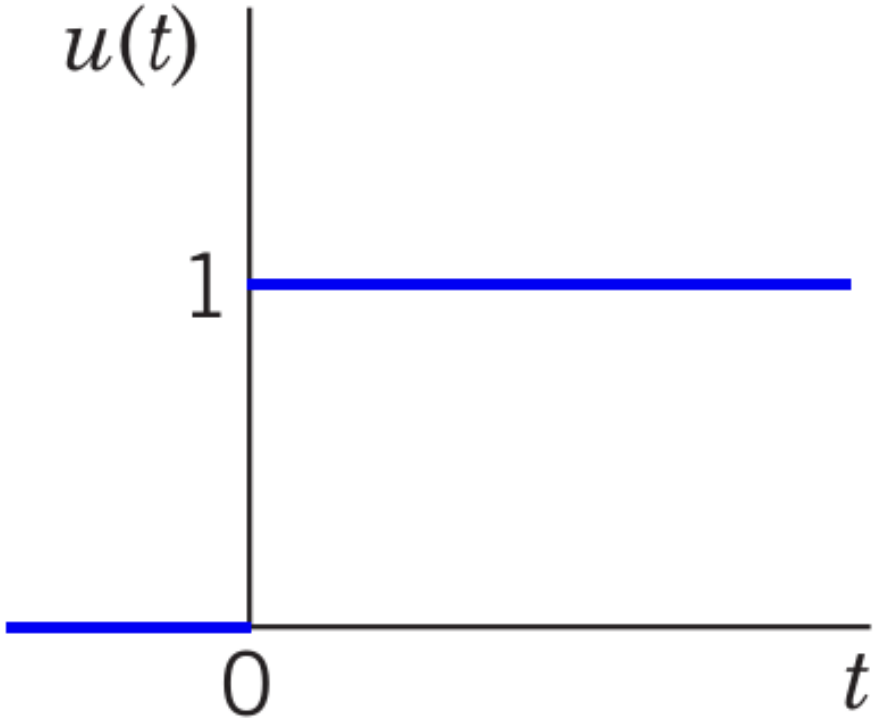
$$Var(\boldsymbol{X}) = \frac{2}{\lambda^2}$$

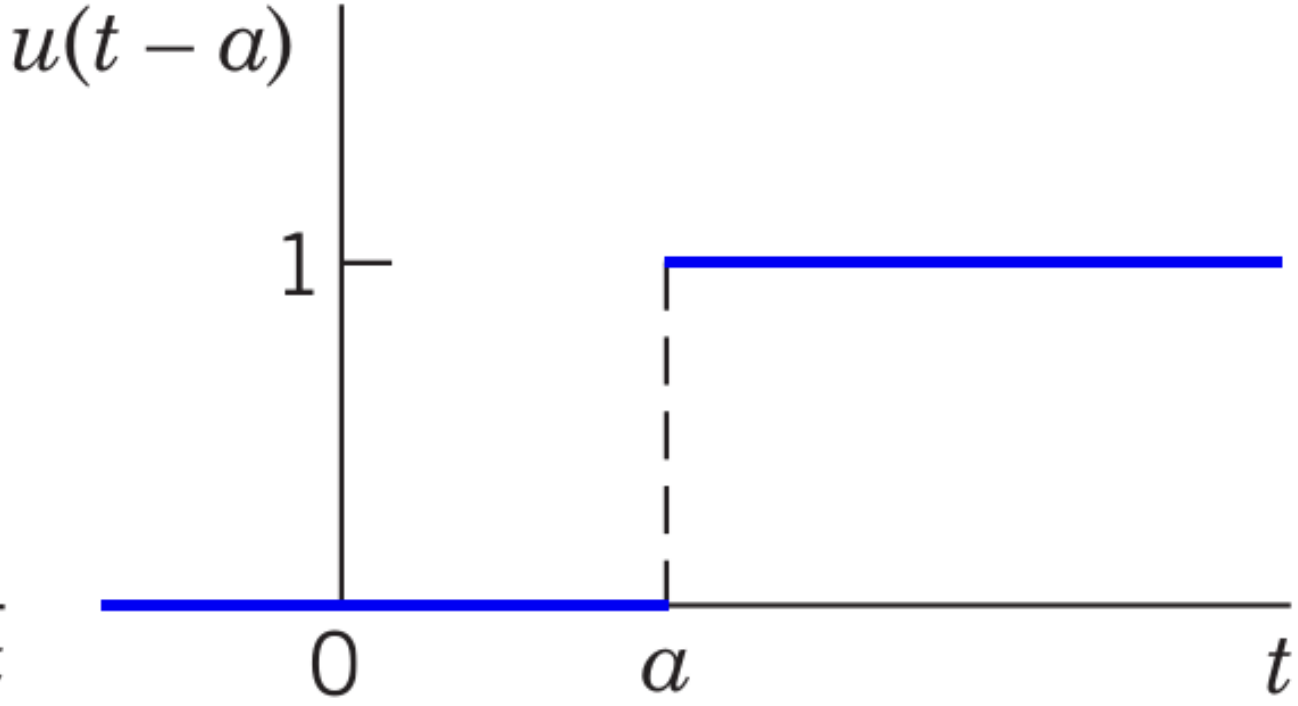


# Função degrau unitário

- Vamos nos lembrar da função degrau unitário $u(t)$:

$$u(t-\mathrm{a}) = \begin{cases} 0, se\ t < a \\ 1, se\ t \geq a \end{cases}$$



# Rayleigh Distribution $S_X = [0, +\infty)$.

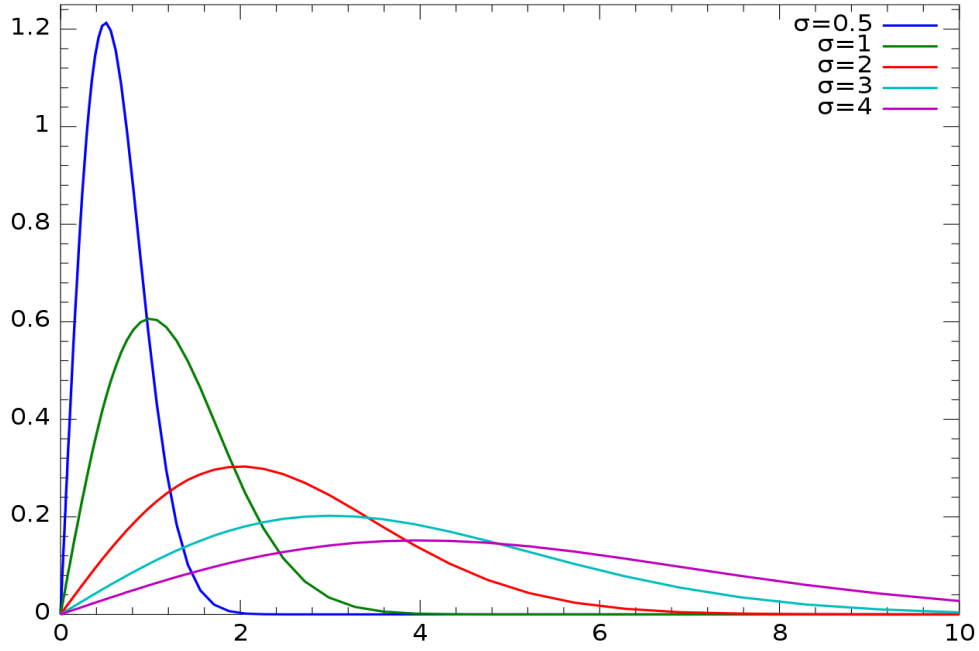
- The probability density function of the Rayleigh distribution is

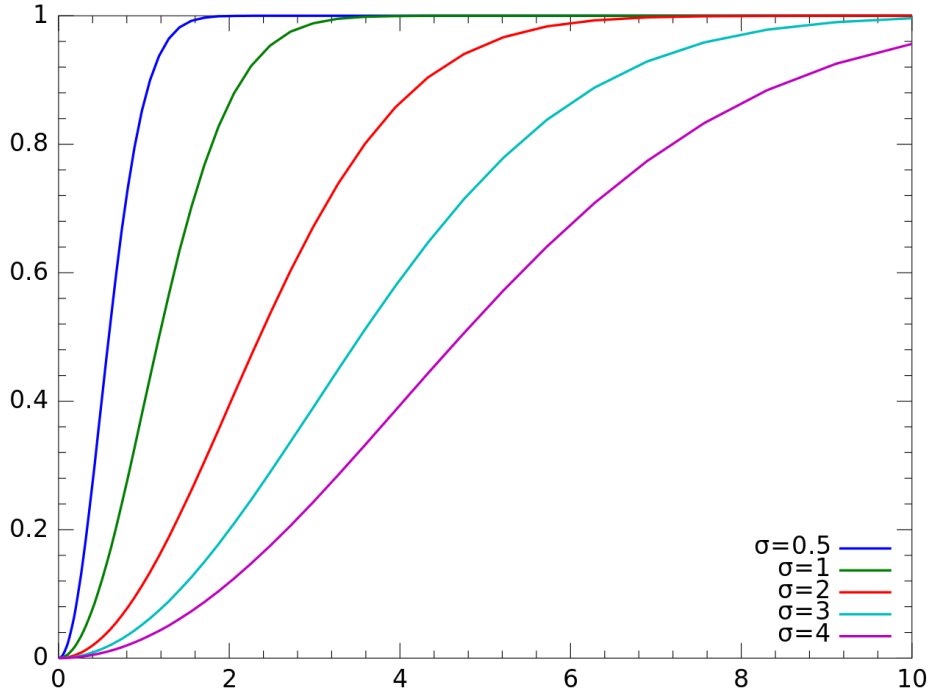
$$f_X(x) = \frac{x}{\sigma_X^2} e^{-\frac{x^2}{2\sigma_X^2}} u(x)$$

$$F_X(x) = \left(1 - e^{-\frac{x^2}{2\sigma_X^2}}\right) u(x)$$

$$E[\boldsymbol{X}] = \mu_X = \alpha\sqrt{\frac{\pi}{2}}$$

$$\mathrm{Var}[\boldsymbol{X}] = \sigma_X^2 = \frac{2 - \frac{\pi}{2}}{\alpha^2}$$



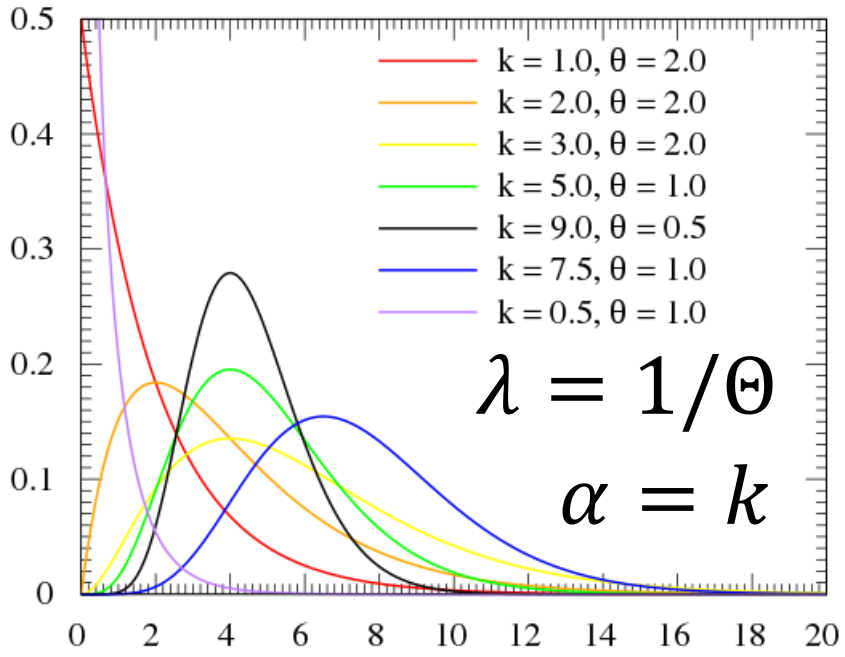
# Gamma Distribution - γ $\quad S_X = [0, +\infty)$.

$$f_X(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1}, x > 0 \\ 0, \text{otherwise} \end{cases}$$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \alpha > 0$$

$x \geq 0, \lambda > 0$

$$F_X(x) = 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda x)^k}{k!} e^{-\lambda x}, if\ \alpha \in Z\ and\ x > 0$$

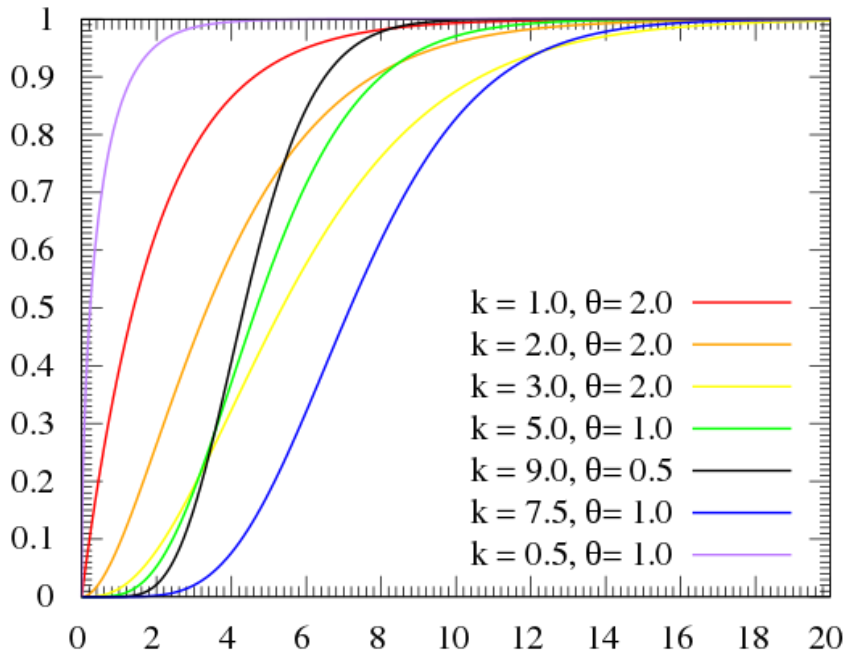



$$\lambda = 1/\Theta$$

$$\alpha = k$$

$$E(\boldsymbol{X}) = \frac{\alpha}{\lambda}$$

$$Var(\boldsymbol{X}) = \frac{\alpha}{\lambda^2}$$



# Properties of the gamma function

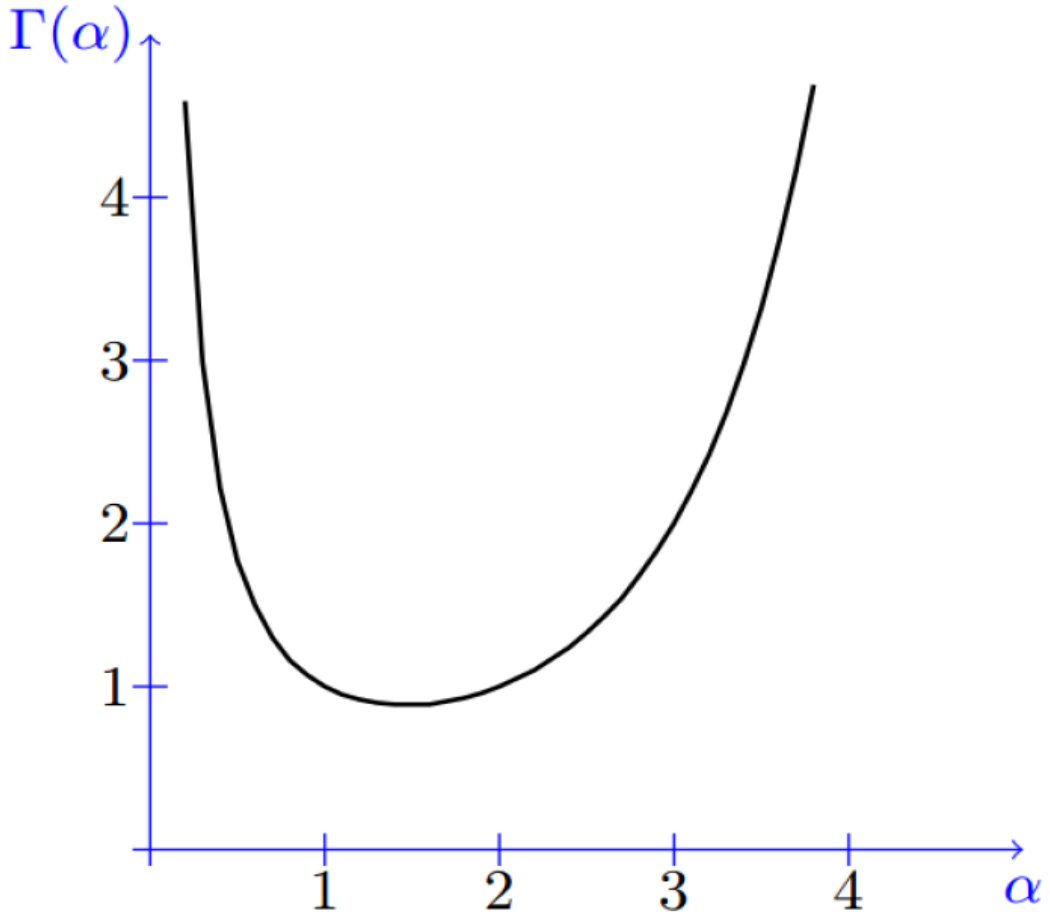
1. $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx;$

2. $\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}, \qquad \text{for } \lambda > 0;$

3. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha);$

4. $\Gamma(n) = (n-1)!, \text{ for } n = 1, 2, 3, \cdots;$

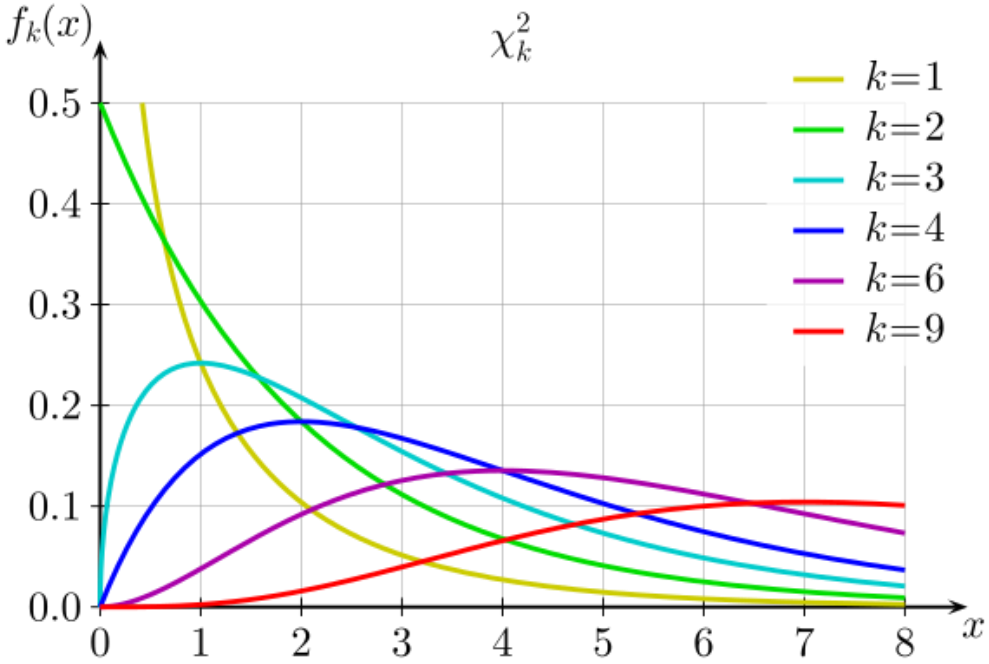
5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}.$

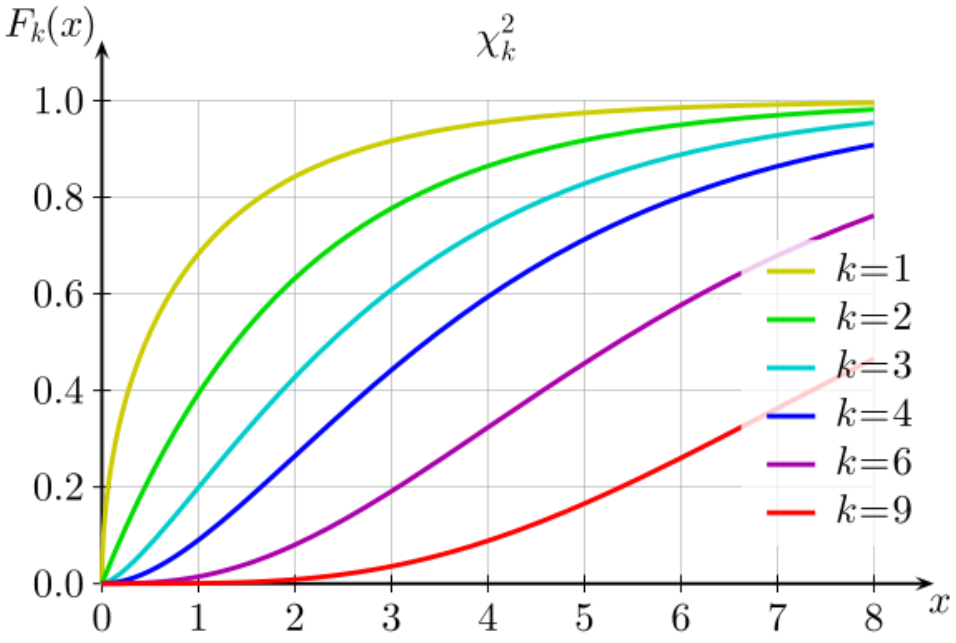


# Chi-Square Distribution $S_X = [0, +\infty)$.

$$f_X(x) = \begin{cases} \dfrac{x^{\frac{k}{2}-1}}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} e^{-\frac{x}{2}}, x > 0, \mathrm{k} > 0 \\ 0, otherwise \end{cases}$$

$$F(x) = \frac{\gamma(\frac{\nu}{2}, \frac{x}{2})}{\Gamma(\frac{\nu}{2})} \quad \text{for } x \geq 0$$



# Exercício 3

- Considere a tabela à direita. Nesta tabela forma feitas 16 medidas associadas a um exame voltado para identificar determinada enfermidade.
- Na coluna do muito se tem a grandeza mensurada no exame e na coluna da direita o diagnóstico do especialista em medicina.

| Medida | X | Diagnóstico |
|---|---|---|
| 1 | 1.3 | 0 |
| 2 | 5.0 | 1 |
| 3 | 5.3 | 1 |
| 4 | 2.8 | 0 |
| 5 | 7.5 | 2 |
| 6 | 7.6 | 2 |
| 7 | 3.8 | 1 |
| 8 | 9.6 | 2 |
| 9 | 2.0 | 0 |
| 10 | 10.6 | 2 |
| 11 | 8.8 | 2 |
| 12 | 7.7 | 1 |
| 13 | 3.8 | 0 |
| 14 | 7.8 | 2 |
| 15 | 6.2 | 1 |
| 16 | 9.1 | 2 |

# Exercício 3

- Os sistema de aprendizado de máquina são baseados em *aprendizado por exemplo* (amostragem randômica) e, neste caso específico, queremos utilizar a **Lei de Bayes** para classificar se um paciente não tem a enfermidade (Classe 0), se o resultado do exame é indeterminado (Classe 1) ou se ele possui a enfermidade (Classe 2).
- Use o banco de dados para o aprendizado e depois estime um diagnóstico para os resultados de exames dos pacientes 1, 2 e 3:

$$x_1 = 2.9;\ x_2 = 7.4 \text{ e } x_3 = 9.8.$$

- Considere que os dados produzidos a partir dos exames médicos possuem comportamento Gaussiano.

# Exercício 3: Solução (aprendizado supervisionado)

| Medida | Classe 0 | Medida | Classe 1 | Medida | Classe 2 |
|---|---|---|---|---|---|
| 1 | 1.30 | 1 | 5.00 | 1 | 7.50 |
| 2 | 2.80 | 2 | 5.30 | 2 | 7.60 |
| 3 | 2.00 | 3 | 3.80 | 3 | 9.60 |
| 4 | 3.80 | 4 | 7.70 | 4 | 10.60 |
| | | 5 | 6.20 | 5 | 8.80 |
| | | | | 6 | 7.80 |
| | | | | 7 | 9.10 |
| | Classe 0 | | Classe 1 | | Classe 2 |
| Média | 2.48 | | 5.60 | | 8.71 |
| Var | 1.16 | | 2.12 | | 1.34 |

# Exercício 3: solução.

- Usando a **Lei de Bayes**:

$$P(C_k|x) = \frac{P(x|C_k)P(C_k)}{P(x)} = \frac{P(x|C_k)P(C_k)}{\sum_k P(x|C_k)P(C_k)}$$

$$P(C_0) = \frac{4}{16} = 0.250; P(C_1) = \frac{5}{16} = 0.313; P(C_2) = \frac{7}{16} = 0.437$$

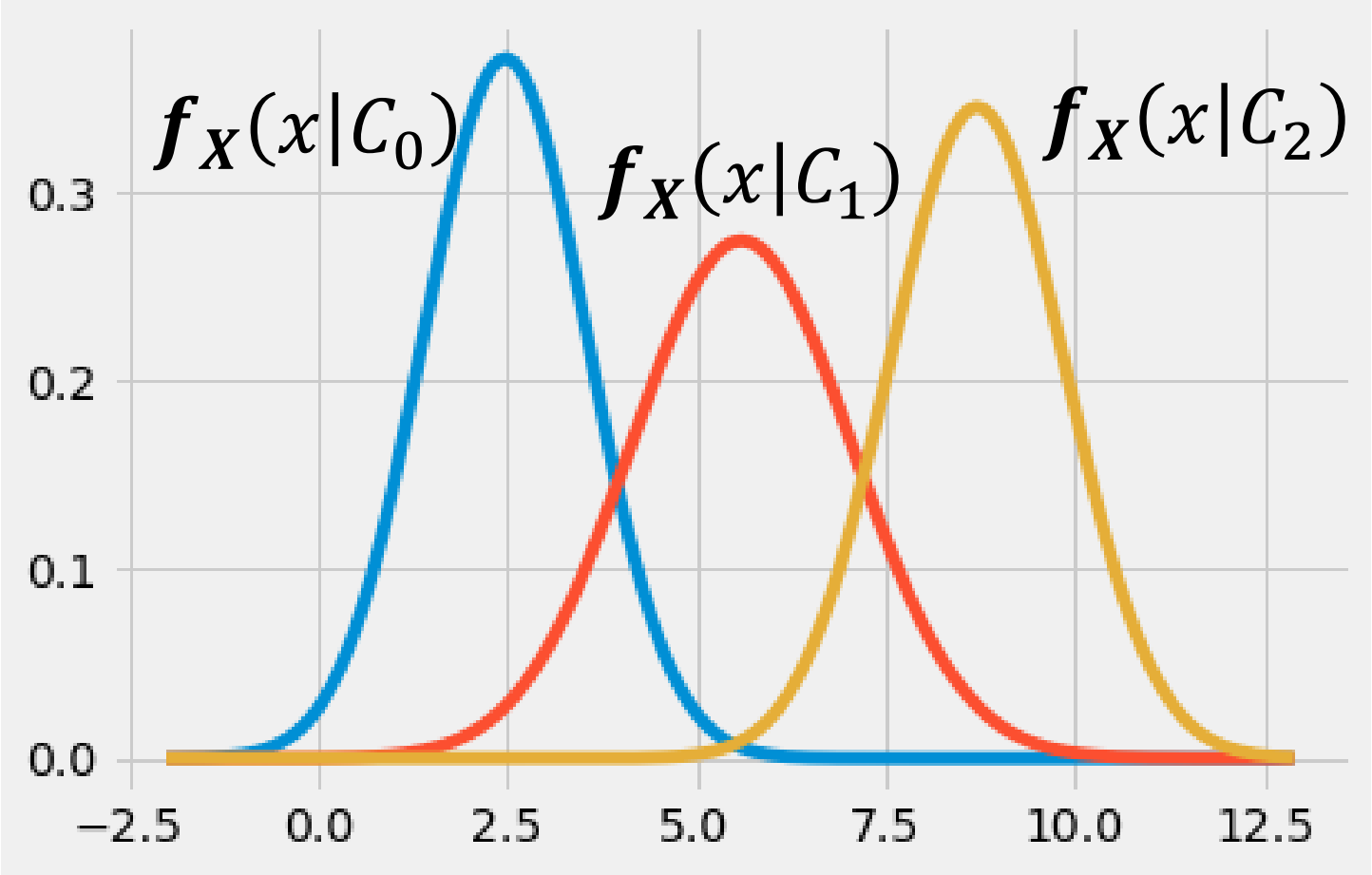
$$P(C_0) + P(C_1) + P(C_2) = 1$$

# Exercício 3: solução.

$$\boldsymbol{f_X(x|C_k)} = \mathcal{N}(X \mid \mu, \sigma_X^2) = \frac{\mathbf{1}}{\sqrt{\mathbf{2}\boldsymbol{\pi}\boldsymbol{\sigma}_X^2}} \boldsymbol{e}^{-\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\sigma}_X^2}(\boldsymbol{x}-\boldsymbol{\mu}_X)^2}, -\infty < \boldsymbol{x} < +\infty$$

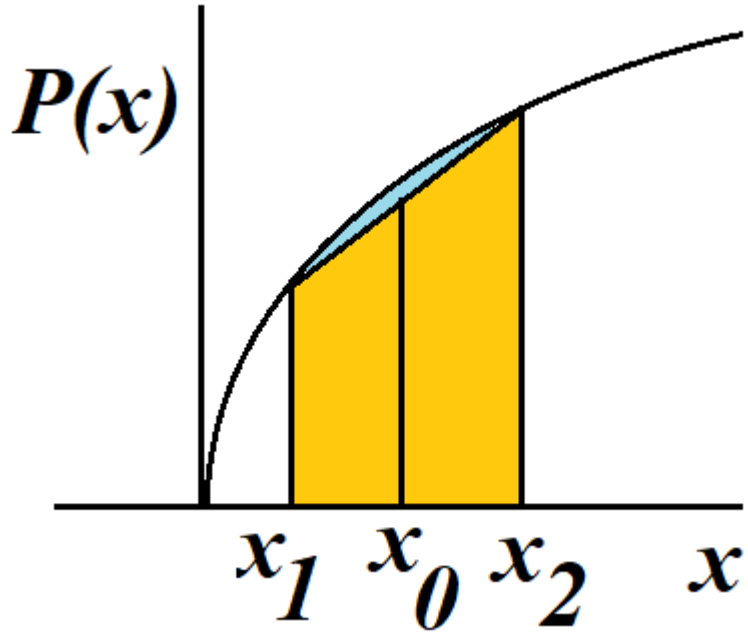
- Com a informação da média e do desvio padrão para cada classe, podemos computar a curva Gaussiana para as três classes:

$$P(x|C_k) = \boldsymbol{f_X}(\boldsymbol{x_i} - \Delta \leq \boldsymbol{x} \leq \boldsymbol{x_i} + \Delta | C_k)$$



# Exercício 3: solução.

- Comentário: Aproximação no cálculo das probabilidades para variáveis contínuas: considere a área do trapézio abaixo




$$P(x|C_k) = f_X(x_i - \Delta \leq x \leq x_i + \Delta|C_k)$$

Uma aproximação: área do retângulo

$$P(x < X \leq x + \delta) \approx f_X(x)\delta$$

Outra aproximação: área do trapezio

$$P(x_1 \leq \mathrm{x} \leq x_2) = \int_{x_1}^{x_2} f_X(x)dx \cong \frac{\{P(x_2)+P(x_1)\}(x_2-x_1)}{2} = \mathrm{P}(x_0)(x_2 - x_1)$$

--- método de integração de Euler

# Exercício 3: solução

- $C_k$ (classe) – partição do espaço amostral.
- Observe que o termo $(x_2 - x_1)$ aparecerá no numerador e no denominador da equação de Bayes.

$$P(C_k|x_1 \leq \mathrm{x} \leq x_2) = \frac{P(x_1 \leq \mathrm{x} \leq x_2|C_k)P(C_k)}{P(x_1 \leq \mathrm{x} \leq x_2)}$$

$$= \frac{P(x_0\,|C_k)(x_2 - x_1)P(C_k)}{\sum_k P(x_0|C_k)P(C_k)(x_2 - x_1)}$$

- Assim, ele se cancelará, restando então o $\mathrm{P}(x_0)$.

$$= \frac{P(x_0\,|C_k)(x_2 - x_1)P(C_k)}{(x_2 - x_1)\sum_k P(x_0|C_k)P(C_k)}$$

$$= \frac{P(x_0\,|C_k)P(C_k)}{\sum_k P(x_0|C_k)P(C_k)}$$

# Exercício 3: solução.

- Deseja-se maximizar a probabilidade de se estimar corretamente o diagnóstico, ou seja:

$$\text{Diagnóstico} = \max_k \{P(C_k|x_i)\}$$

- Onde:

$$P(C_0|x_i) = \frac{P(x_i|C_0)P(C_0)}{P(x_i|C_0)P(C_0) + P(x_i|C_1)P(C_1) + P(x_i|C_2)P(C_2)}$$

$$P(C_1|x_i) = \frac{P(x_i|C_1)P(C_1)}{P(x_i|C_0)P(C_0) + P(x_i|C_1)P(C_1) + P(x_i|C_2)P(C_2)}$$

$$P(C_2|x_i) = \frac{P(x_i|C_2)P(C_2)}{P(x_i|C_0)P(C_0) + P(x_i|C_1)P(C_1) + P(x_i|C_2)P(C_2)}$$

# Exercício 3: solução.

- Calculando as probabilidades $P(C_k|x_1 = 2.9)$:

$$P(C_0|x_1 = 2.9) = \frac{(0.343)(0.250)}{(0.343)(0.250) + (0.0490)(0.313) + (1.167\ 10^{-6})(0.437)} = 0.848$$

$$P(C_1|x_1 = 2.9) = \frac{(0.0490)(0.313)}{(0.343)(0.250) + (0.0490)(0.313) + (1.167\ 10^{-6})(0.437)} = 0.151$$

$$P(C_2|x_1 = 2.9) = \frac{(1.167\ 10^{-6)}(0.437)}{(0.343)(0.250) + (0.0490)(0.313) + (1.167\ 10^{-6)}(0.437)} = 0.48010^{-6}$$

▶ O diagnóstico é que o paciente provavelmente não tem a enfermidade:

$$\max\{0.848; 0.151; 0.480\ 10^{-6}\} = \mathrm{k} = \mathbf{0}$$

# Exercício 3: solução.

- Calculando as probabilidades $P(C_k|x_2 = 7.4)$:

$$P(C_0|x_2 = 7.4) = \frac{(7.100\ 10^{-6})(0.250)}{(7.100\ 10^{-6})(0.250)+(0.116)(0.313)+(0.199)(0.437)} = 1.433\ 10^{-5}$$

$$P(C_1|x_2 = 2.9) = \frac{(0.116)(0.313)}{(7.100\ 10^{-6})(0.250)+(0.116)(0.313)+(0.199)(0.437)} = 0.295$$

$$P(C_2|x_2 = 2.9) = \frac{(0.199)(0.437)}{(7.100\ 10^{-6})(0.250)+(0.116)(0.313)+(0.199)(0.437)} = 0.704$$

- O diagnóstico é que o paciente (possivelmente) tem a enfermidade:

$$\text{Diagnóstico} = \max\{1.433\ 10^{-5}; 0.295; 0.704\} = \text{k} = \mathbf{2}$$

# Exercício 3: solução.

- Calculando as probabilidades $P(C_k|x_3 = 9.8)$:

$$P(C_0|x_3 = 9.8) = \frac{(3.453\ 10^{-11})(0.250)}{(3.453\ 10^{-11})(0.250)+(0.0042)(0.313)+(0.221)(0.437)} = 8.799\ \ 10^{-11}$$

$$P(C_1|x_3 = 9.8) = \frac{(0.0042)(0.313)}{(3.453\ 10^{-11})(0.250)+(0.0042)(0.313)+(0.221)(0.437)} = 0.013$$

$$P(C_2|x_3 = 9.8) = \frac{(0.221)(0.437)}{(3.453\ 10^{-11})(0.250)+(0.0042)(0.313)+(0.221(0.437)} = 0.986$$

▸ O diagnóstico é que o paciente tem a enfermidade:

$$\text{Diagnóstico} = \max\{8.799\ \ 10^{-11}; 0.013; 0.986\} = \text{k} = \mathbf{2}$$

# Discrete Random Variables

- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expected Value
- Variance
- Discrete Distributions
- Mixed random variables

# Probability Mass Function (PMF) of a Discrete Random Variable

- Consider $\boldsymbol{X}$ *as* a discrete random variable with range $R_X = \{x_1, x_2, x_3, \ldots\}$ (finite or countably infinite).

- The function $P_X(x_k) = \mathrm{P}(\boldsymbol{X} = x_k)$, for $k = 1, 2, 3, \ldots,$ is called the ***probability mass function (PMF)*** of the discrete random variable $\boldsymbol{X}$.

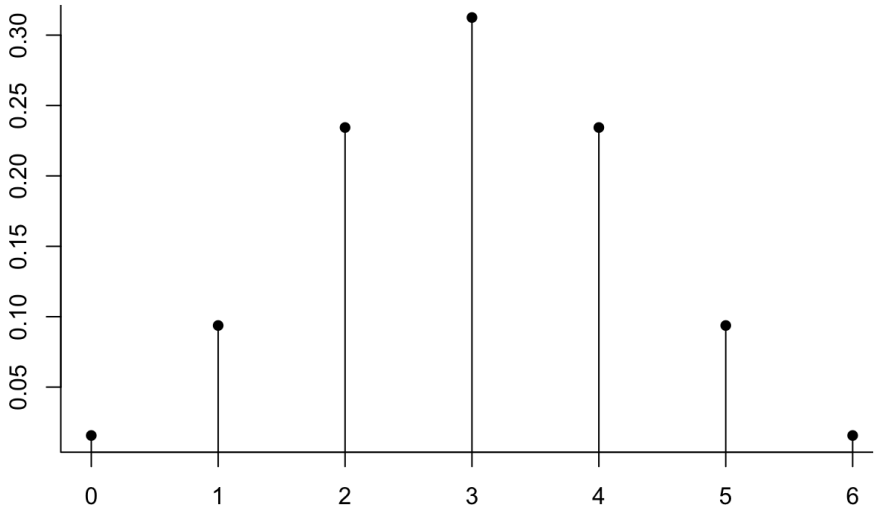
# Properties of PMF

$$0 \leq P_X(x) \leq 1 \text{ for all } \boldsymbol{x}.$$

A given outcome of the random experiment may have a non-zero probability, it means $P(x = x_i) \neq 0$.

$$\sum_{x \in R_x} P_X(x) = 1$$

Example of PMF.

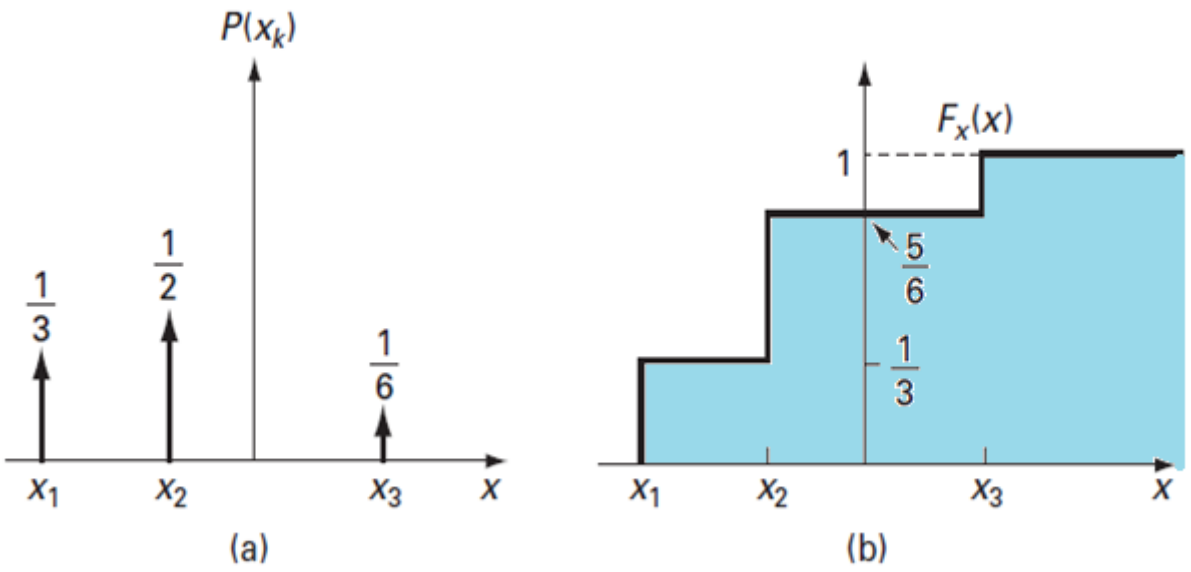


# Cumulative Distribution Function (**CDF**)

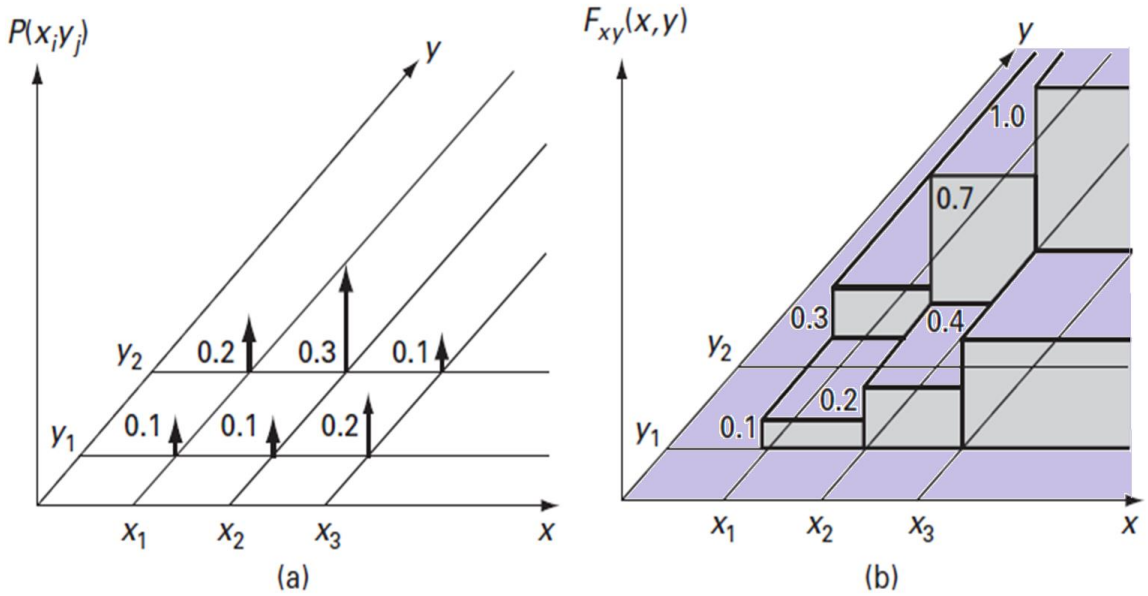
- The cumulative distribution function (CDF) of random variable ***X*** is defined as:

$$F_X(x) = P(X \leq x), \text{ for all } x \in \mathbb{R}.$$

# Examples of **PMF** and **CDF**.




(a) The probability distribution and (b) the distribution function of a discrete random variable.




(a) The joint probability distribution and (b) the joint distribution function.

# Exercício 4

- Suponha que temos uma moeda que não é viciada (P(H) = P(T) = 0.5). Vamos criar um experimento randômico, denotando a variável $\boldsymbol{X}$ como o número de **o número de caras (H) em três lançamentos de moeda** (distribuição binomial).
- a) Esboce o PMF $P_{\boldsymbol{X}}(x)$ para a variável aleatória discreta $\boldsymbol{X}$.
- b) Encontre a CDF $F_{\boldsymbol{X}}(x)$ para variável aleatória discreta $\boldsymbol{X}$.

# Exercício 4: Solução

- Primeira coisa: O tamanho do espaço amostral pode ser dados pelo produto cartesiano (princípio da multiplicação): $|S| = N^k = 2^3 = 8$.
- **S** = {***TTT***,***TTH***,***THT***,***THH***,***HTT***,***HTH***,***HHT***,***HHH***}.

- Observe que a condição colocada pelo problema nos leva aos conceitos estudados no módulo 2: análise combinatória: |S| corresponde a uma "**amostragem ordenada com reposição**";
- Além disso, cada evento (o número de caras em três jogadas sucessivas) pode ser visto como uma "**amostragem não ordenada com reposição**".

# Exercício 4: Solução

- Para uma "**amostragem não ordenada com reposição**", você poderia usar as expressões abaixo, obtidas no módulo 2, para calcular o tamanho dos eventos (o evento corresponde a se obter no experimento: 0, 1, 2 ou 3 caras).

$$P(\boldsymbol{k}\ times \text{ is } true\ and\ (\boldsymbol{n}-\boldsymbol{k})\ times \text{ is } false)$$

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for } 0 \leq k \leq n.$$

# Exercício 4: Solução

- Observe que o experimento randômico é estatisticamente independente. Pode-se então calcular a probabilidade para cada valor possível de nossa variável aleatória.
- **S = {*TTT*,*TTH*,*THT*,*THH*,*HTT*,*HTH*,*HHT*,*HHH*}.**

$$P(X = 0) = P(TTT) = \frac{1}{8}$$

$$P(X = 1) = P(TTH) + P(THT) + P(HTT) = \frac{3}{8}$$

$$P(X = 2) = P(THH) + P(HHT) + P(HTH) = \frac{3}{8}$$

$$P(X = 3) = P(HHH) = \frac{1}{8}$$

{1/8,3/8,3/8,1/8} the probability

X = {0, 1, 2, 3}, the random variable and its dynamic range

# Using Bernoulli (Binomial distribution) to compute probability

- Toss a coin three times n = 3, **two heads** k = 2

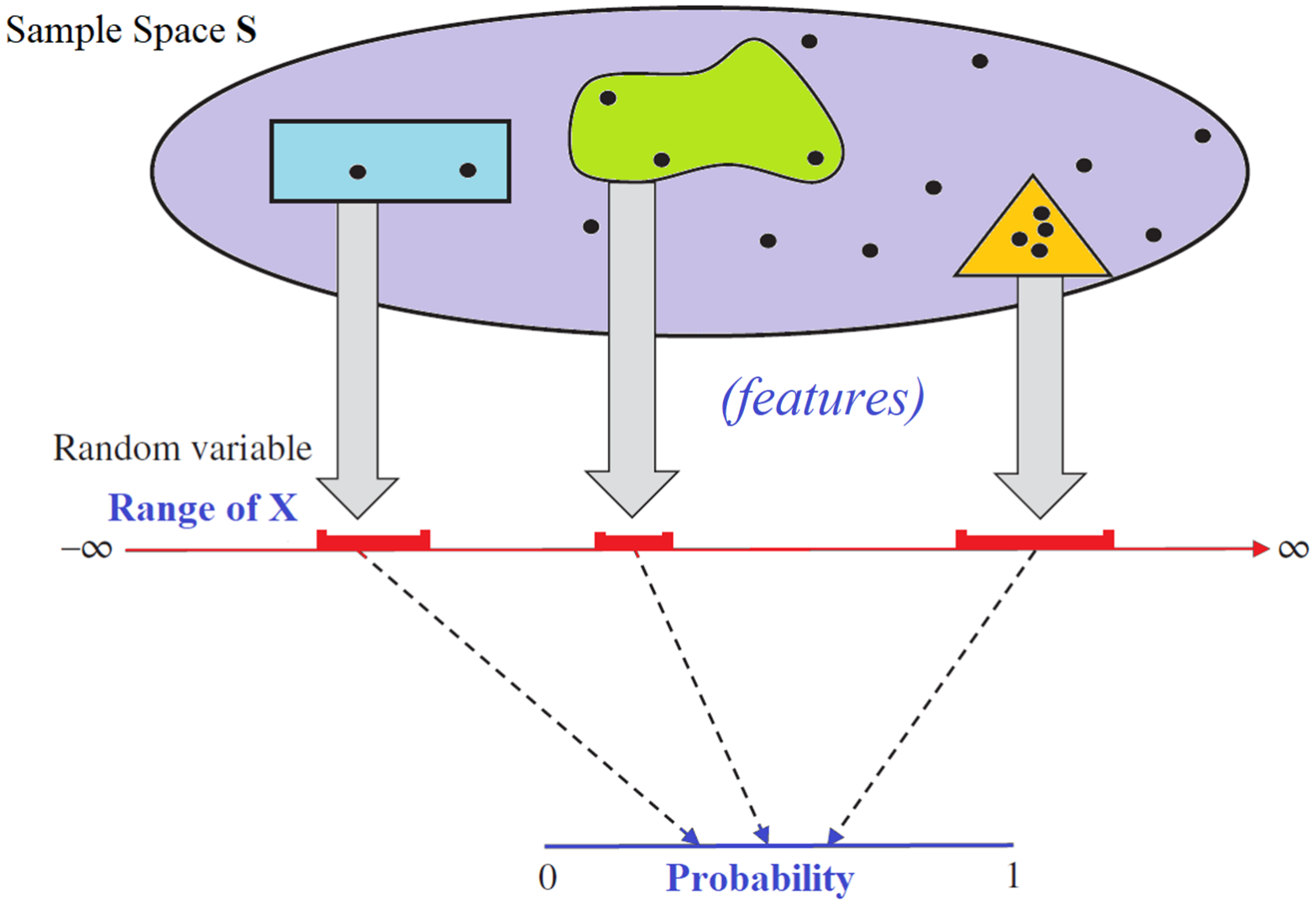
$$P(\boldsymbol{k} = 2\ times \text{ is } true\ and\ (\boldsymbol{n} - \boldsymbol{k}) = 1\ times \text{ is } false)$$

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for } 0 \leq k \leq n.$$

$$P(k=2) = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{3-2} = \frac{3!}{2!\,1!}\frac{1}{2^3} = \frac{3}{8}$$

# The relationship among sample space, random variable, and probability.



# Exercício 4: Solução

- A PMF de nossa variável randômica ***X*** pode ser representada como:

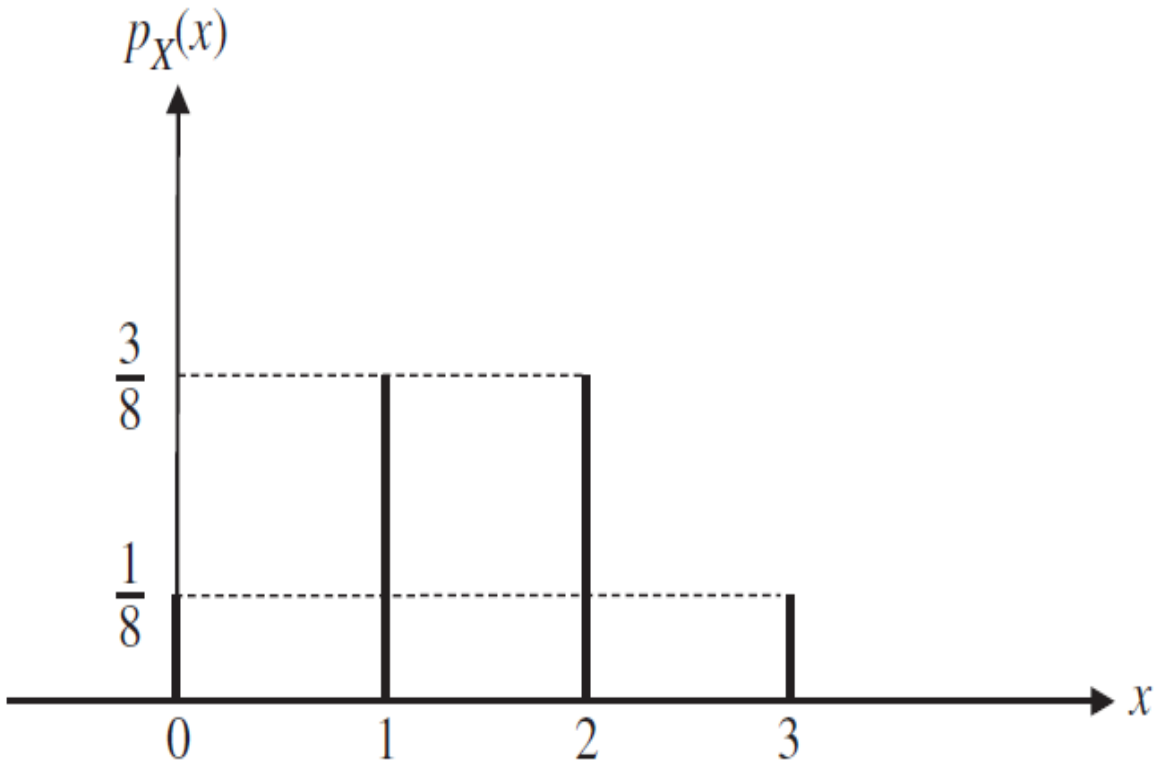
$$p_X(x) = P(X = x) = \begin{cases} 0.125 & x = 0 \\ 0.375 & x = 1 \\ 0.375 & x = 2 \\ 0.125 & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

# Exercício 4: Solução

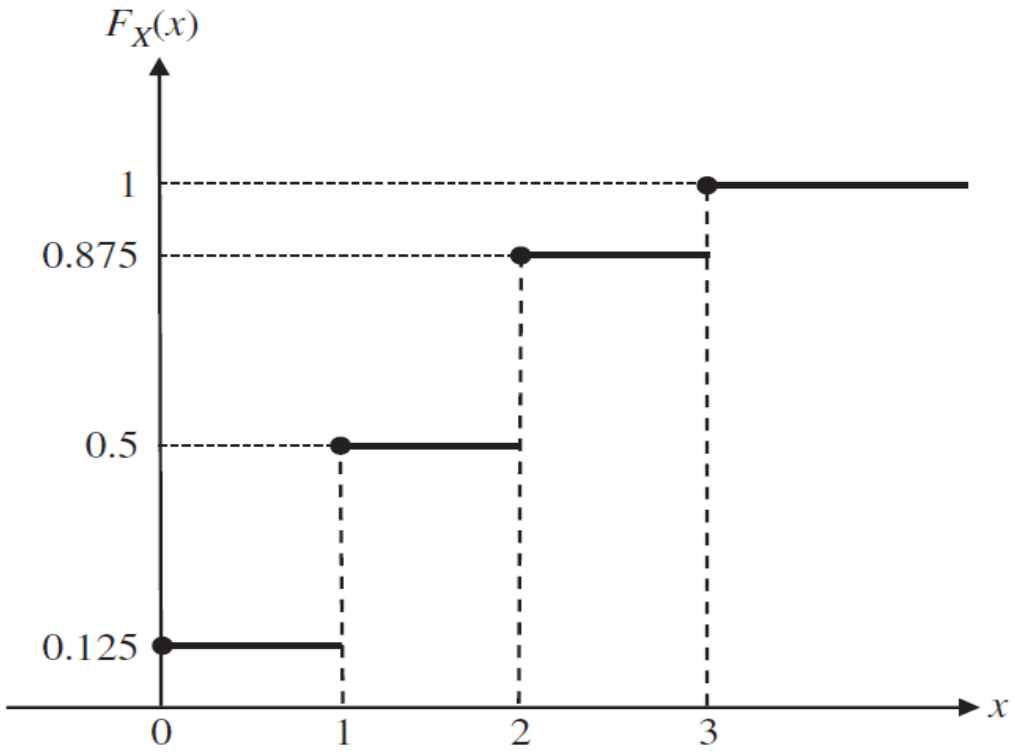
- A CDF $F_X(x)$ pode ser escrita como:

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & x < 0 \\ 0.125 & 0 \leq x < 1 \\ 0.5 & 1 \leq x < 2 \\ 0.875 & 2 \leq x < 3 \\ 1 & 3 \leq x \end{cases}$$

# Exercício 4: Solução




PMF $p_X(x)$

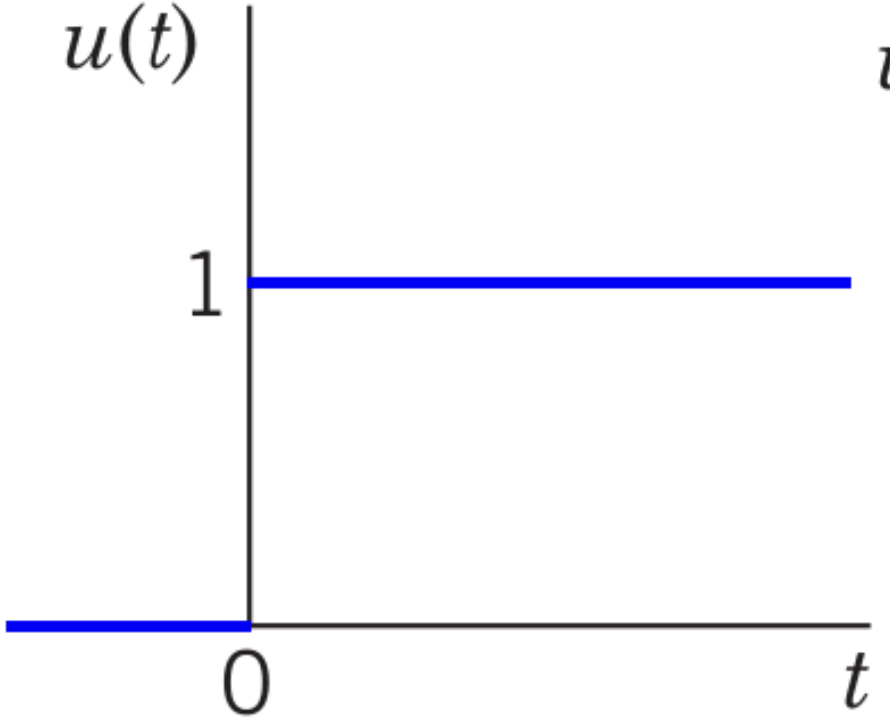


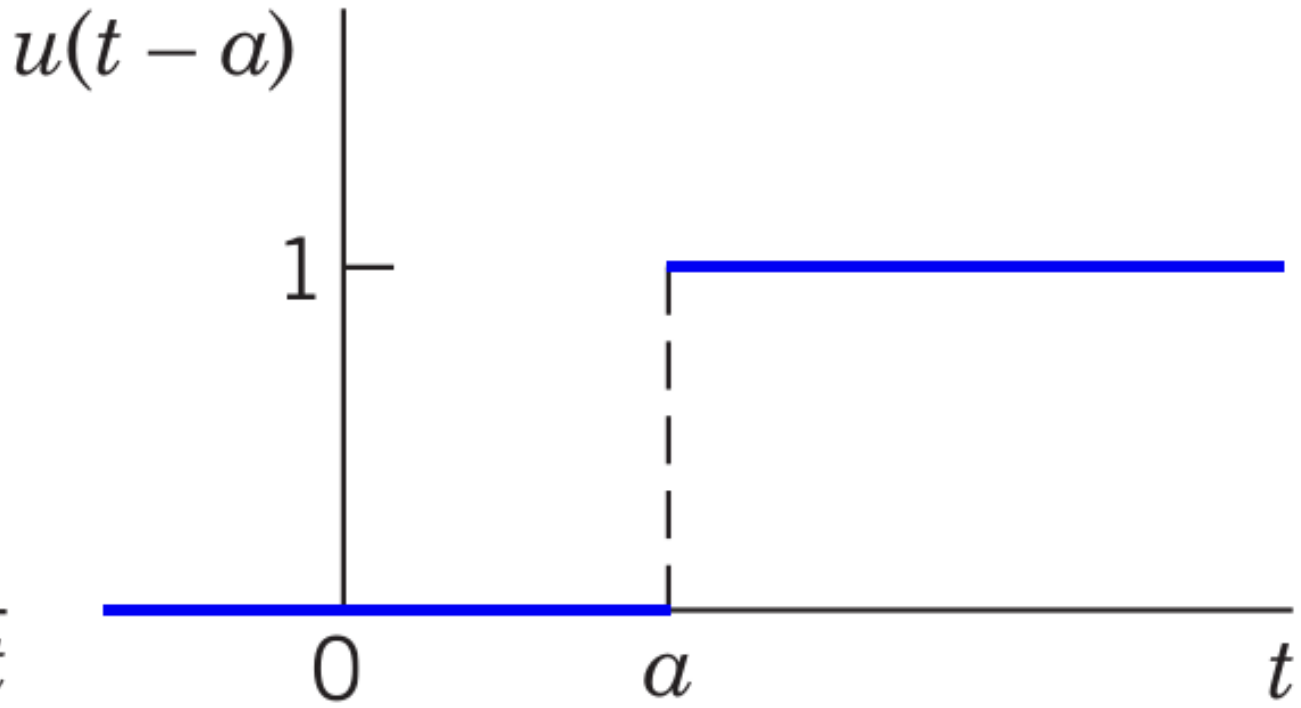

CDF $F_X(x)$

# Exercício: Outra representação de $F_X(x)$

- Vamos nos lembrar da função degrau unitário $u(t)$:

$$u(t-a)=\begin{cases}0, se\ t<a\\ 1, se\ t\geq a\end{cases}$$

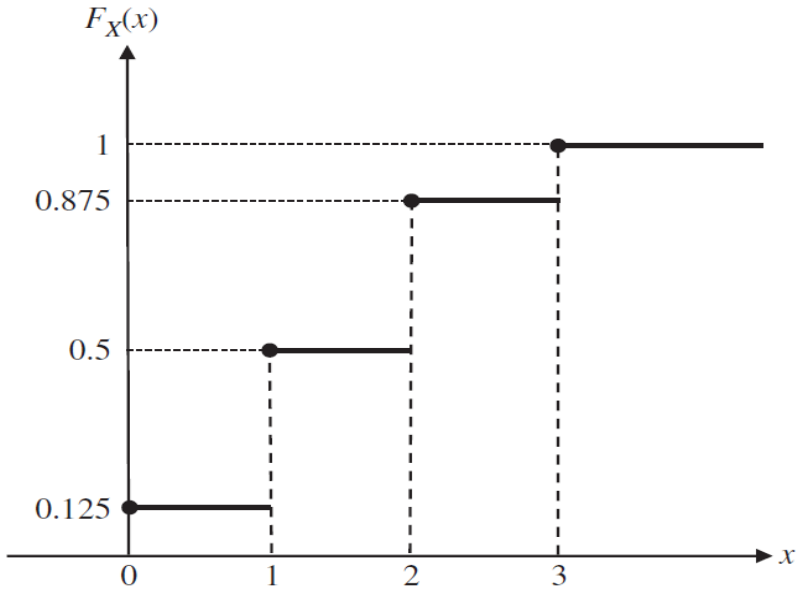


# Exercício 4: outra forma de representar $F_X(x)$.

- Observe que $F_X(x)$ pode ser representada analiticamente como uma **combinação linear** de degraus unitários, na forma:

$$F_X(x) = 0.125\,u(x) + 0.375\,u(x-1) + 0.375\,u(x-2) + 0.125\,u(x-3)$$



# Expectation of a discrete random variable

- Consider the random variable ***X***. The expectancy of **X** can be compared by:

$$E[X] = \sum_{x_k \in R_X} x_k \mathrm{P}(X = x_k) = \sum_{x_k \in R_X} x_k P_X(x_k).$$

# Properties of the expected value of a *discrete random variable*

- $E[E[\boldsymbol{X}]] = E[\boldsymbol{X}] = \mu_{\boldsymbol{X}}$

- $E[\boldsymbol{a}\boldsymbol{X} + \boldsymbol{b}] = \boldsymbol{a}E[\boldsymbol{X}] + \boldsymbol{b}$, for all $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{R}$.

- $E[\boldsymbol{X}_1 + \boldsymbol{X}_2 + \ldots + \boldsymbol{X}_n] = E[\boldsymbol{X}_1] + E[\boldsymbol{X}_2] + \ldots + E[\boldsymbol{X}_n]$, for any set of random variables $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$.

# Expected Value of a Function of a Random Variable (LOTUS)

- **The law of the unconscious statistician** (**LOTUS**) for discrete random variables can be expressed as:

$$E[g(X)] = \sum_{x_k \in R_X} g(x_k) P_X(x_k)$$

# Variance of a discrete random variable

- The variance of a discrete random variable is computed using:

$$\mathrm{Var}(X) = E\left[(X-\mu_X)^2\right] = \sum_{x_k \in R_X} (x_k - \mu_X)^2 P_X(x_k).$$

# Variance Properties of a discrete random variable

. Variance is non-negative because the squares are positive or zero: $Var(X) = \sigma_X^2 \geq 0$.

.The variance of a constant is zero: $Var(a) = 0$ , $a \in$ R.

.Variance is invariant with respect to changes in a location parameter: $Var(X + a) = Var(X) = \sigma_X^2$

. $Var(X) = \sigma_X^2 = E[(X - \mu_{\boldsymbol{X}})^2] = E[\boldsymbol{X}^2] - (E[\boldsymbol{X}])^2$

# The **standard deviation** of a discrete random variable $\boldsymbol{X}$

- The **standard deviation** of a discrete random variable $X$ is defined as:

$$\sigma_X = \sqrt{\sigma_X^2}$$

# Special Discrete Random Distributions

- The Discrete Uniform Distribution
- The Bernoulli Distribution
- The Binomial Distribution
- The Geometric Distribution
- The Pascal Distribution
- The Hypergeometric Distribution
- The Poisson Distribution
- The Zipf (Zeta) Distribution

# The Discrete Uniform Distribution

- The discrete uniform random variable occurs when outcomes are equally likely.
- The **PMF** of a ***discrete uniform random variable*** is defined as follows:

$$P(X = k) = p_X(k) = \frac{1}{L} \quad k = m+1, \ldots, m+L$$
$$-\infty < m < \infty \quad L \in \{1, 2, \ldots\}$$

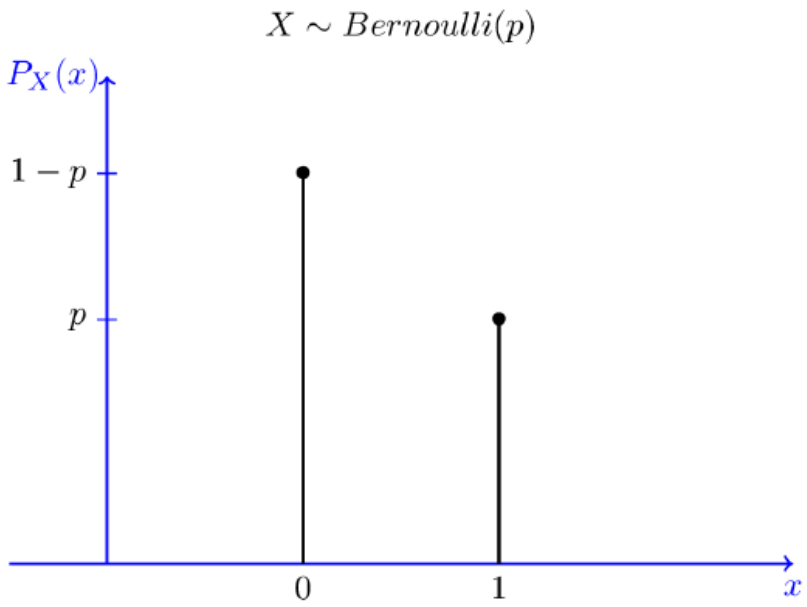
- The mean $\mu_X$ and variance $\sigma^2_X$ of the discrete uniform distribution $X$ are, respectively, as follows:

$$\mu_X = m + \frac{L+1}{2} \qquad \sigma_X^2 = \frac{L^2 - 1}{12}$$

# The Bernoulli Distribution: $X \sim Bernoulli\ (p)$

- The Bernoulli random variable $\boldsymbol{X}$ takes the value of 1 with probability $\boldsymbol{p}$ (also known as the probability of success) and the value of 0 with probability $\mathbf{1-}\boldsymbol{p}$ (also known as the probability of failure), where $\mathbf{0} < \boldsymbol{p} < \mathbf{1}$.
- The PMF of a ***Bernoulli random variable*** is defined as follows:

$$p_X(x) = \begin{cases} 1-p & x = 0 \\ p & x = 1 \\ 0 & x \neq 0, 1 \end{cases} \qquad 0 < p < 1$$

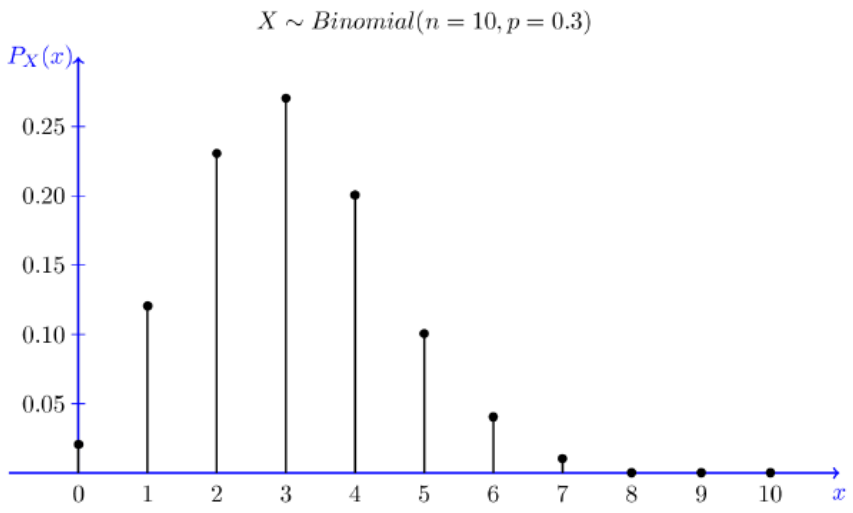


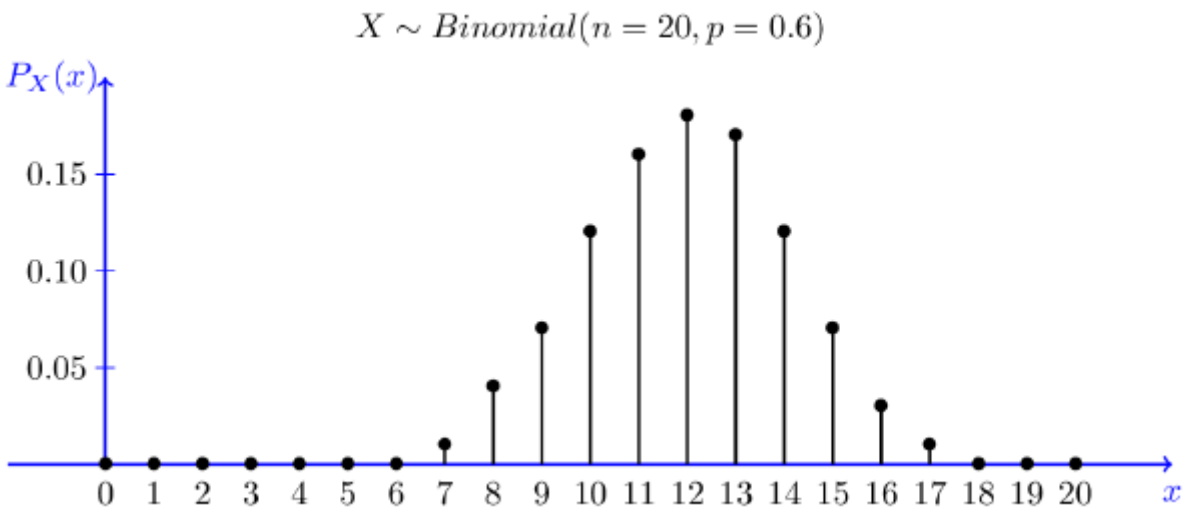
# The Binomial Distribution

- The PMF of a ***binomial random variable*** is defined as follows:

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \qquad 0 < p < 1$$

- The ***binomial coeficient*** is defined as: $\binom{n}{x} \triangleq \frac{n!}{x!(n-x)!}$



# Exercise 5

- The chance that a bit transmitted through a digital transmission channel is received in error is 0.1.
- Also, assume that the transmission trials are independent (Unordered Sampling (Combinations) without Replacement).

- Let $\boldsymbol{X}$ = the number of bits in error in the next four bits transmitted.

- Determine $P(\boldsymbol{X} = 2)$.

# Exercise 5: Solution

- Let the letter ***E*** denote a bit in error and let the letter ***O*** denote that the bit is okay, that is, received without error.
- We can represent the outcomes of this experiment as a list of four letters that indicate the bits that are in error and those that are okay. For example, the outcome *OEOE* indicates that the second and fourth bits are in error and the other two bits are okay.
- The corresponding values for $x$ correspond to the sample space of this problem:

| Outcome | $x$ | Outcome | $x$ |
|---|---|---|---|
| *OOOO* | 0 | *EOOO* | 1 |
| *OOOE* | 1 | *EOOE* | 2 |
| *OOEO* | 1 | *EOEO* | 2 |
| *OOEE* | 2 | *EOEE* | 3 |
| *OEOO* | 1 | *EEOO* | 2 |
| *OEOE* | 2 | *EEOE* | 3 |
| *OEEO* | 2 | *EEEO* | 3 |
| *OEEE* | 3 | *EEEE* | 4 |

# Exercise 5: Solution 1

- An outcome that contains $x$ errors can be constructed by partitioning the four trials (letters) in the outcome into two groups. One group is of size $\boldsymbol{x}$ and contains the errors, and the other group is of size $(\boldsymbol{n} - \boldsymbol{x})$ and consists of the trials that are okay.
- The event that $\boldsymbol{X} = 2$ consists of the six outcomes:

$$A = \{EEOO, EOEO, EOOE, OEEO, OEOE, OOEE\}$$

- Using the assumption that the trials are independent, the probability of $\{EEOO\}$ is

$$P(EEOO) = P(E)P(E)P(O)P(O) = (0.1)^2(0.9)^2 = 0.0081$$

- Also, any one of the six mutually exclusive outcomes for which $\boldsymbol{X} = 2$ has the same probability of occurring (law of total probability). Therefore,

$$\boldsymbol{P(X = 2) = 6(0.0081) = 0.0486}$$

# Exercise 5: Solution 2 – Using The Binomial Distribution

- $P(X = x)$ = (number of outcomes that result in $x$ errors): $(0.1)^x\,(0.9)^{4-x}$

  $P(\text{x } times \text{ is } true \text{ (bit error) } and\ (\boldsymbol{n} - x)\ times \text{ is } false)$

$$P(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}, \text{ for } 0 \leq \text{x} \leq n.$$

- The number of ways of partitioning four objects into two groups, one of which is of size $x$, is $\binom{n=4}{x=2} = \frac{n!}{x!(n-x)!} = \frac{4!}{2!2!} = 6$

$$P(x=2) = \binom{4}{2}(0.1)^2(1-0.1)^{4-2} = 0.0486$$

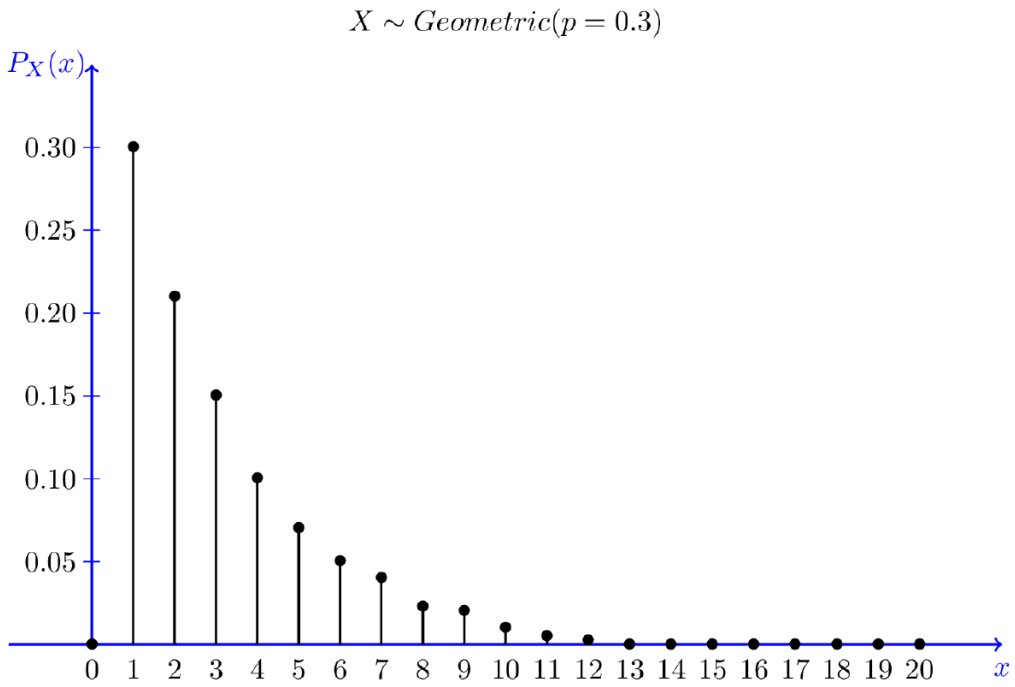
# Geometric Distribution

- The PMF of a ***geometric random variable*** is defined for $0 < p < 1$, as follows:

$$P_X(k) = \begin{cases} p(1-p)^{k-1} & \text{for } k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

The mean $\mu_X$ and variance $\sigma^2_X$ of the geometric random variable $\boldsymbol{X}$, are respectively as follows:

$$\mu_X = \frac{1}{p} \qquad \sigma_X^2 = \frac{1-p}{p^2}$$



# Exercício 6: distribuição geométrica

- O campeonato brasileiro começou, mas suponha que gostaríamos de prever o desempenho do seu time.
- Suponha que a chance (a probabilidade) de um time vencer uma partida de futebol (sucesso) seja de 0.4.

- a) Determine a probabilidade deste time vencer a quarta partida.

# Exercício 6: Solução

- a) Determine a probabilidade deste time vencer na quarta partida.
- Para determinar a probabilidade de que o time vença na quarta partida (tentativa), tem-se que contabilizar é simplesmente a probabilidade de que nas três primeira partidas ele perdeu (tentativas malsucedidas – observe que a ordem importa – se deseja a probabilidade da interseção), portanto, temos:

$$P_X(k) = \begin{cases} p(1-p)^{k-1} & \text{for } k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$P(x = 4) = (0.4\ )(1 - 04)^{4-1} = (0.4)(0.216) = 0.0864$$

# The Pascal Distribution

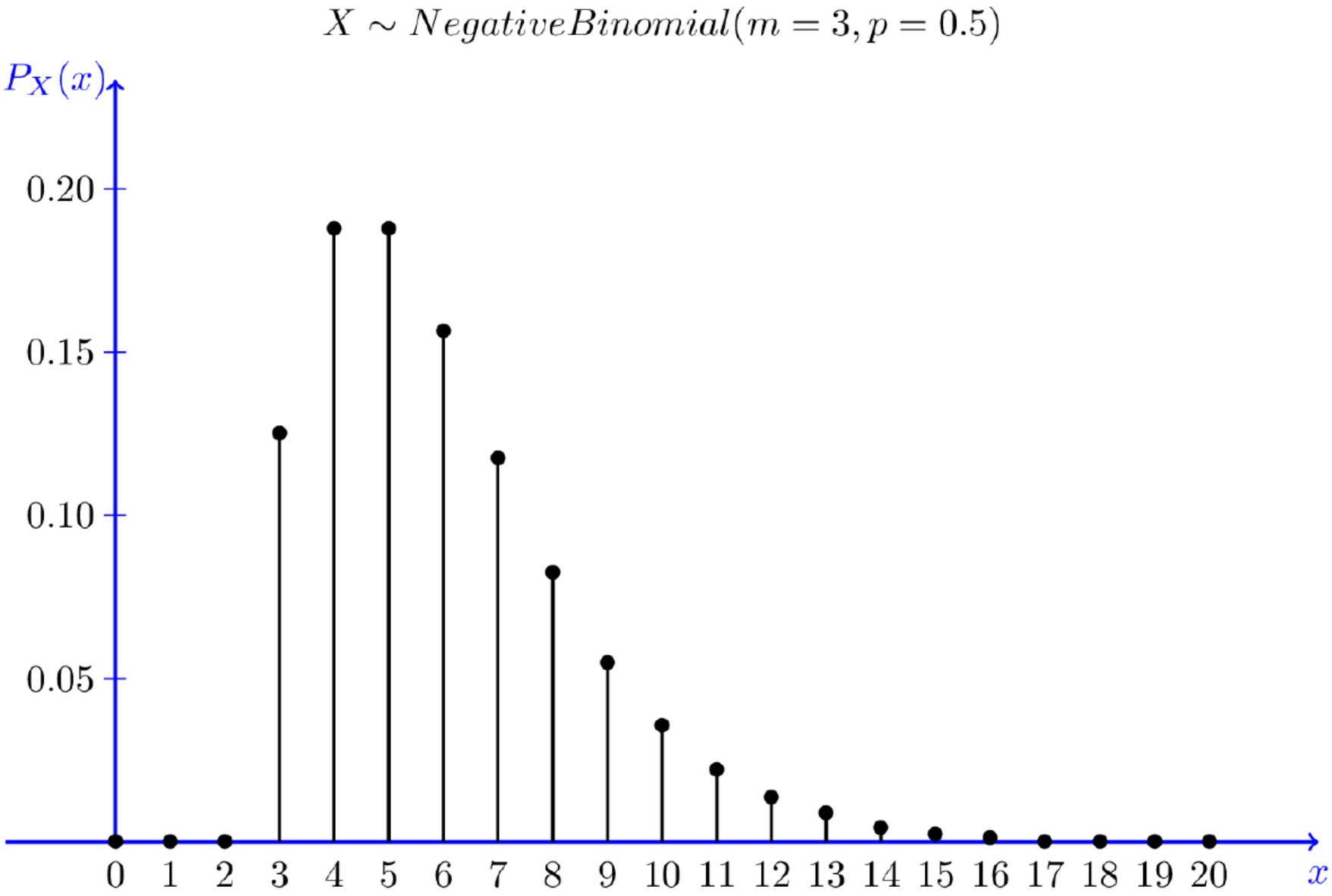
- The PMF of a ***Pascal random variable***, also known as a ***negative binomial random variable***, is defined for 0 < p < 1 as follows:

$$P_X(k) = \begin{cases} \binom{k-1}{m-1} p^m (1-p)^{k-m} & \text{for } k = m, m+1, m+2, m+3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

- The mean $\mu_X$ and variance $\sigma_X^2$ of the Pascal random variable ***X***, which can be independently set by the appropriate selection of ***p*** and ***k***, are respectively as follows:

$$\mu_X = \frac{k}{p} \qquad \sigma_X^2 = \frac{k(1-p)}{p^2}$$

# Example of The Pascal Distribution



# The Hypergeometric Distribution

- Suppose there is a finite population of **N** items of which **K** possess a certain attribute, the hypergeometric distribution then describes the probability that **a sample of *n* items, without replacement**, is selected of which ***x*** possess the attribute.
- The PMF of a ***hypergeometric random variable*** is defined as follows:

$$p_X(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \qquad \max(0, n+K-N) \le x \le \min(n, K)$$

$$0 \le n \le N \quad 0 \le x \le K$$

# The Hypergeometric Distribution

- The mean $\mu_X$ and variance $\sigma_X^2$ of the hypergeometric random variable ***X***, where each is a function of **n**, **K**, and **N**, are respectively as follows:

$$\mu_X = \frac{nK}{N} \qquad \sigma_X^2 = \frac{nK}{N}\left(\frac{N-K}{N}\right)\left(\frac{N-n}{N-1}\right)$$

# Exercício 7

- Suppose there are 100 exotic fishes in a small aquarium, 46 of these fish are red, 30 are blue, and the remaining 24 are black.
- A photo of twenty fishes is taken.

- Determine the probability that sixteen fishes in the photo are blue.

# Exercício 7: Solução

- **N = 100**, is a finite population;
- **k = 30**, the total that possess the atribute "***blue***".
- **n = 20**, the sample of 20 items in the total of N = 100 items, without replacement ($0 \leq n \leq N = 100$).
- in the foto ***x = 16*** possess same atribute "***blue***" (What is the value of that probability?)

$$(0 \leq x \leq K = 30).$$

$$P(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \qquad P(x = 16) = \frac{\binom{30}{16}\binom{100-30}{20-16}}{\binom{100}{20}}$$

# The Poisson Distribution

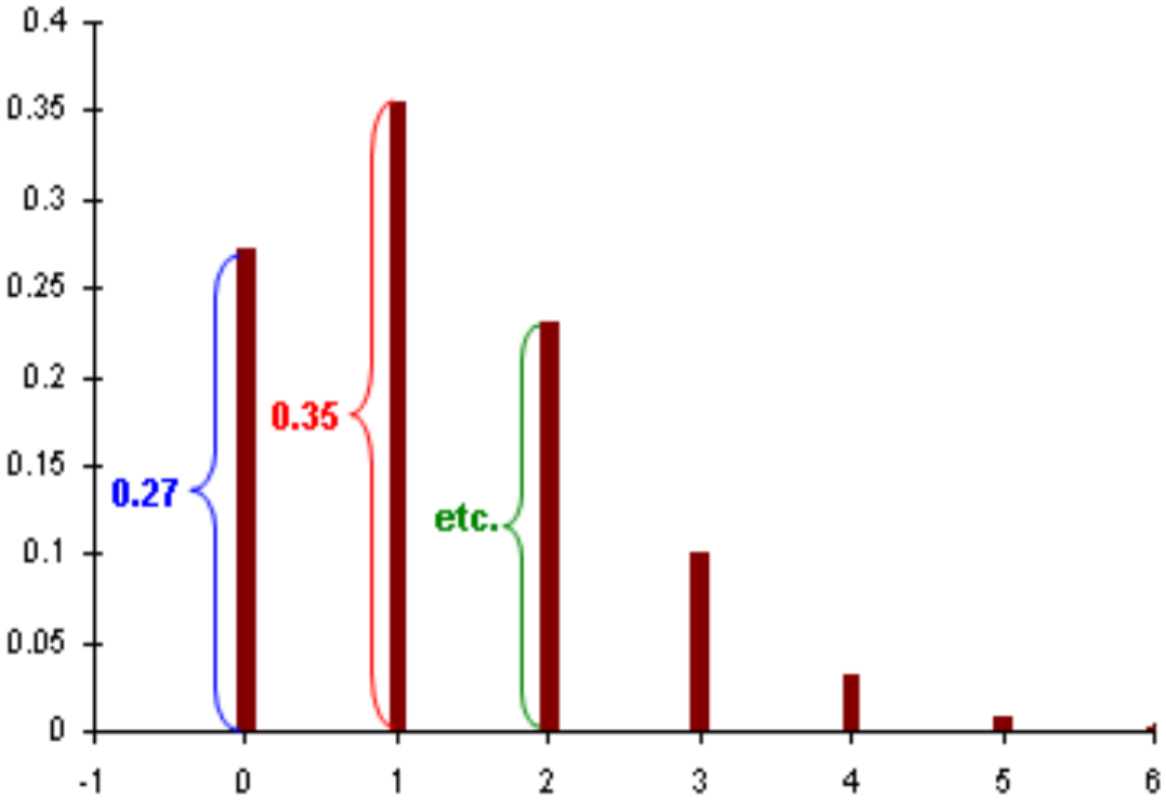
- The PMF of a ***Poisson random variable*** is defined as:

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0, 1, 2, \ldots \quad \lambda > 0$$

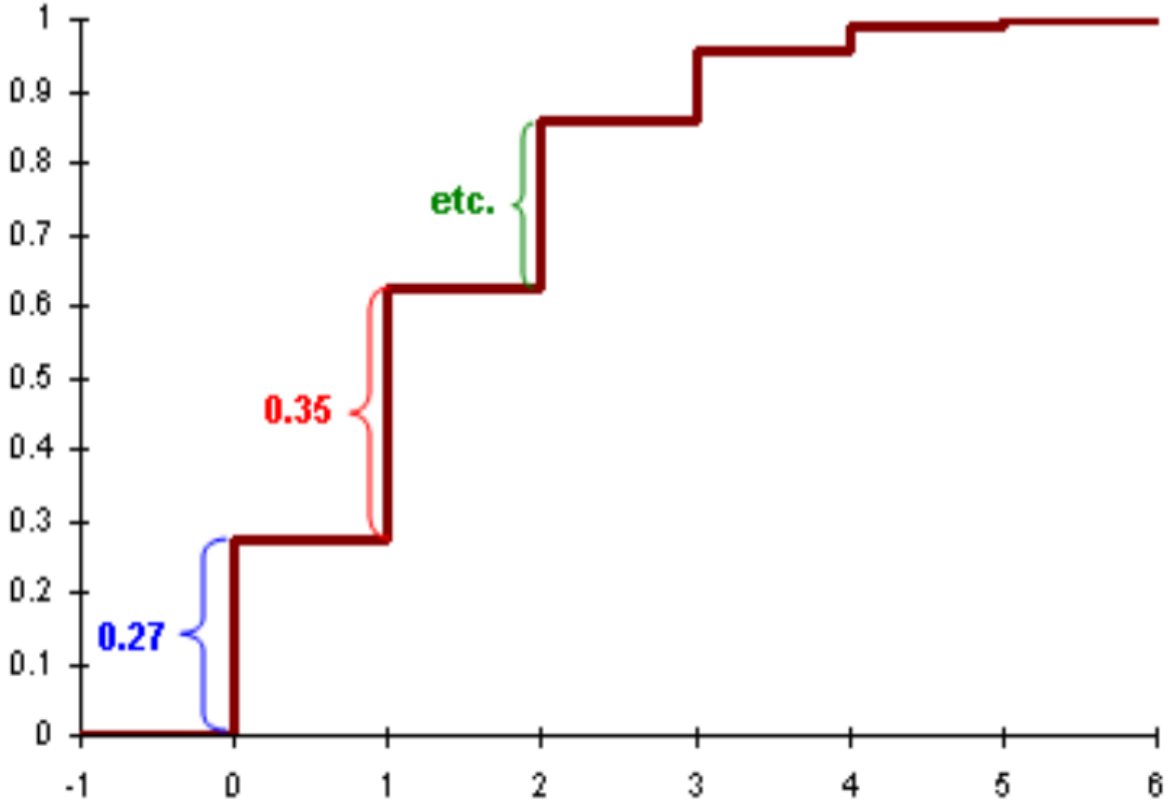
- where $\lambda$ is the parameter of the distribution reflecting the average rate of occurrence.
- The mean $\mu_X$ and variance $\sigma_X^2$ of the Poisson random variable $\boldsymbol{X}$ are respectively as

$$\mu_X = \lambda \quad \sigma_X^2 = \lambda$$

# Another Example of **PMF** and **CDF**: Poisson




PMF $p_X(x)$




CDF $F_X(x)$

# The Zipf (Zeta) Distribution

- The PMF of a ***Zipf random variable***, also known as a ***Zeta random variable***, is defined as:

$$p_X(x) = x^{-\alpha}(\zeta(\alpha))^{-1} \quad \alpha > 1 \quad x = 1, 2, \ldots$$

- where the ***zeta function*** $\zeta(\alpha)$ is defined as follows

$$\zeta(\alpha) = \sum_{k=1}^{\infty} \frac{1}{k^{\alpha}} \quad \alpha > 1$$

# The Zipf (Zeta) Distribution

- The mean $\boldsymbol{\mu_X}$ and variance $\boldsymbol{\sigma_X^2}$ of the Zipf random variable $X$ are, respectively, as follows:

$$\mu_X = \frac{\zeta(\alpha - 1)}{\zeta(\alpha)} \quad \alpha > 2 \qquad \sigma_X^2 = \frac{\zeta(\alpha)\zeta(\alpha - 2) - (\zeta(\alpha - 1))^2}{(\zeta(\alpha))^2} \quad \alpha > 3$$

# Exercise 8: An urn contains 3 red, 2 green and 5 white balls. Find:

- a) The probability that the 7th ball taken with replacement is the 1st red one.
- b) The probability that out of 10 balls drawn with replacement, 3 are red;
- c) The probability that of 5 balls drawn without replacement, 2 are red;
- d) The probability that the 10th ball drawn with replacement is the 3rd red one.

# Exercise 8: Solution - An urn contains 3 red, 2 green and 5 white balls. Find:

- a) The probability that the 7th ball taken with replacement is the 1st red one.
- $X = \{ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ \boldsymbol{v}\}$: The draw is independent.
- Ordered Sampling (Combinations) with replacement.
- What is the probability of p($x_1$) (the probability that the red ball is not drawn)?

$$p(x_1) = \frac{7}{10}$$

- What is the probability of p($v$) (the probability that the red ball is drawn)?

$$p(v) = \frac{3}{10}$$

- $p(z) = p(x_1)p(x_2)p(x_3)p(x_4)p(x_5)p(x_6)p(v)$, $p(v) = 0.7^6 0.3 = 0.0353$.

- The desired pmf fits with a ***geometric random variable which*** is defined for 0 < p < 1, as follows: $p = 0.3, 1 - p = 0.7$.

$$P_X(k) = \begin{cases} p(1-p)^{k-1} & \text{for } k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$p_X(x) = (0.3)(1 - 0.3)^6$$

# Exercise 8: Solution - An urn contains 3 red, 2 green and 5 white balls. Find:

- b) The probability that out of 10 balls drawn with replacement, 3 are red.
- $X = \{ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10} \}$ : Unordered Sampling (Combinations) with Replacement. Three of the 10 balls drawn are red.
- The draw can be of the form: $X = \{ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10} \}$
- *or* $X = \{ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10} \}$
- *or* $X = \{ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10} \}$
- *or* $X = \{ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10} \}$
- *or* $X = \{ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10} \}$
- *or* $X = \{ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10} \}$ …

# Exercise 8: Solution - An urn contains 3 red, 2 green and 5 white balls. Find:

- b) The probability that out of 10 balls drawn with replacement, 3 are red.
- $X = \{ x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10} \}$:(Unordered Sampling (Combinations) with Replacement). Three of the 10 balls drawn are red.

- The PMF that fits the problem is the ***binomial random variable***, and is defined as follows:

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \qquad 0 < p < 1 \qquad \binom{n}{x} \triangleq \frac{n!}{x!(n-x)!}$$

# Exercise 8: Solution - An urn contains 3 red, 2 green and 5 white balls. Find:

- The ***binomial random variable*** is defined as follows: p = 0.3

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \qquad 0 < p < 1 \qquad \binom{n}{x} \triangleq \frac{n!}{x!(n-x)!}$$

$$p_X(x) = \binom{10}{3} 0.3^3 (1-0.3)^7 = 120\,(0.3)^3 (0.7)^7 = 0.267$$

$$\binom{10}{3} = \frac{10!}{3!\,(10-3)!} = \frac{3628800}{(6)(5.040)} = 120$$

# Exercise 8: Solution - An urn contains 3 red, 2 green and 5 white balls. Find:

- c) The probability that of 5 balls drawn without replacement, 2 are red.
- The PMF that fits the problem is the ***hypergeometric random variabl*** , and it is defined as follows:

$$p_X(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \qquad \max(0, n+K-N) \le x \le \min(n, K)$$

$$0 \le n \le N \quad 0 \le x \le K$$

# Exercise 8: Solution - An urn contains 3 red, 2 green and 5 white balls. Find:



- c) The probability that of 5 balls drawn without replacement, 2 are red.
  - **N = 10**, is a finite population of 10 balls;
  - **k = 3**, the total balls that possess the atribute "***red***".
  - **n = 5**, the sample of 5 balls items in the total of N = 10 items, without replacement ($0 \leq n \leq N = 10$).
  - in the drawn ***x = 2 balls*** possess same atribute "***red***".

$$\binom{K}{x}$$

$$\binom{N-K}{n-x}$$

$$p_X(x) = \frac{\binom{3}{2}\binom{10-3}{5-2}}{\binom{10}{5}}$$

$$\binom{N}{n}$$

$$\binom{n}{x} \triangleq \frac{n!}{x!(n-x)!}$$

# Exercise 8: Solution - An urn contains 3 red, 2 green and 5 white balls. Find:

- c) The probability that of 5 balls drawn without replacement, 2 are red. $N = 10; K = 3; n = 5; x = 2$

$$p_X(x) = \frac{\binom{3}{2}\binom{10-3}{5-2}}{\binom{10}{5}} = \frac{\frac{3!}{2!\,(3-2)!}\frac{7!}{3!\,(7-3)!}}{\frac{10!}{5!\,(10-5)!}} = \frac{3\,\frac{5.040}{6(24)}}{\frac{3628800}{(120)(120)}}$$

$$= \frac{105}{252} = \frac{35}{84} = 0.4167$$

# Exercise 8: Solution - An urn contains 3 red, 2 green and 5 white balls. Find:

- d) The probability that the 10th ball drawn with replacement is the 3rd red one. The PMF that fits the problem is the Pascal random variable, which is defined as follows: $p = 0.3; 1 - p = 0.7$.

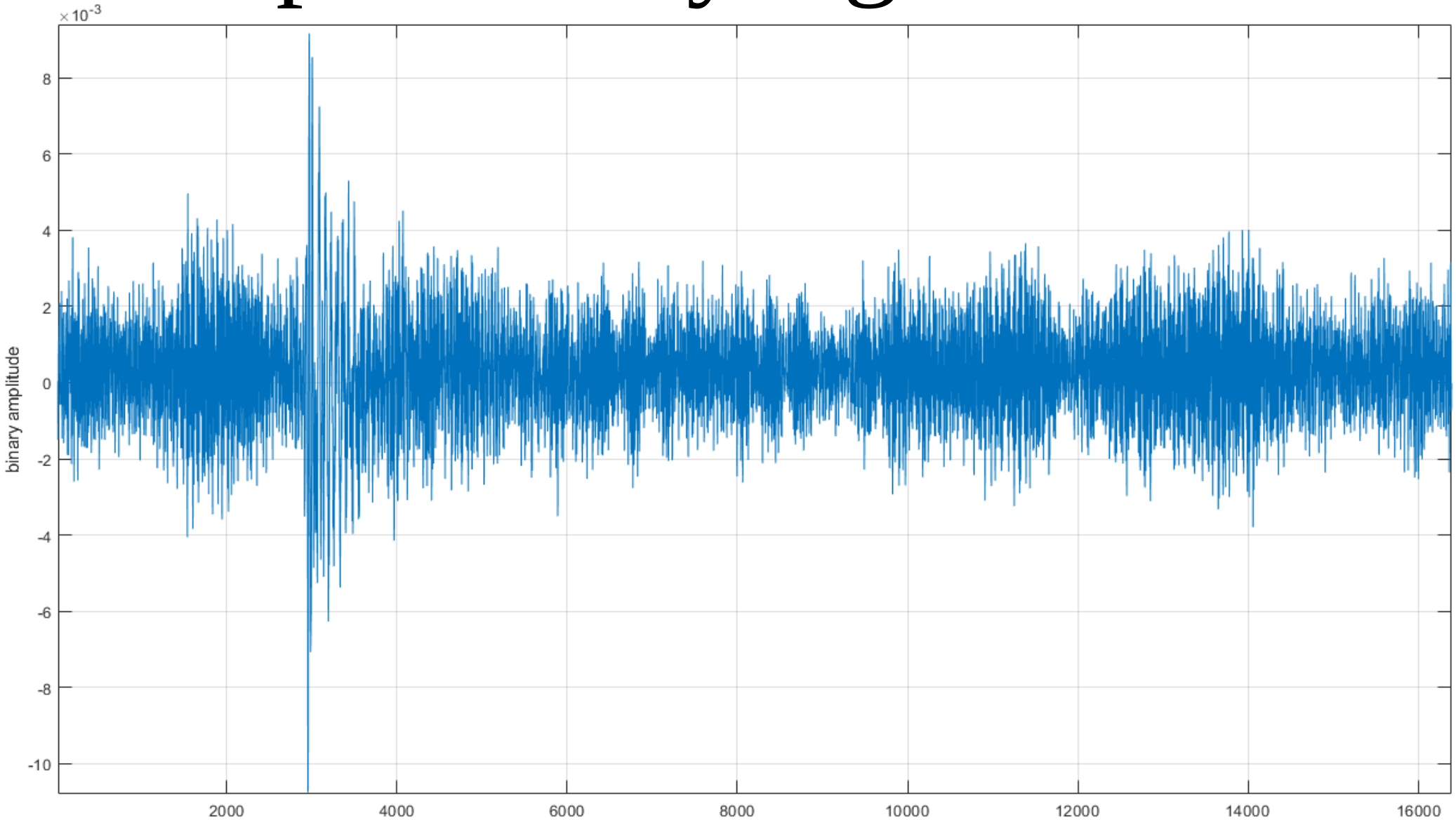
$$P_X(k) = \begin{cases} \binom{k-1}{m-1} p^m (1-p)^{k-m} & \text{for } k = m, m+1, m+2, m+3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{k} = 10; \mathrm{m} = 3$$

$$p_X(k) = \binom{9}{2}(0.3)^3(0.7)^7 = \frac{9!}{2!\,(9-2)!}(0.3)^3(0.7)^7$$

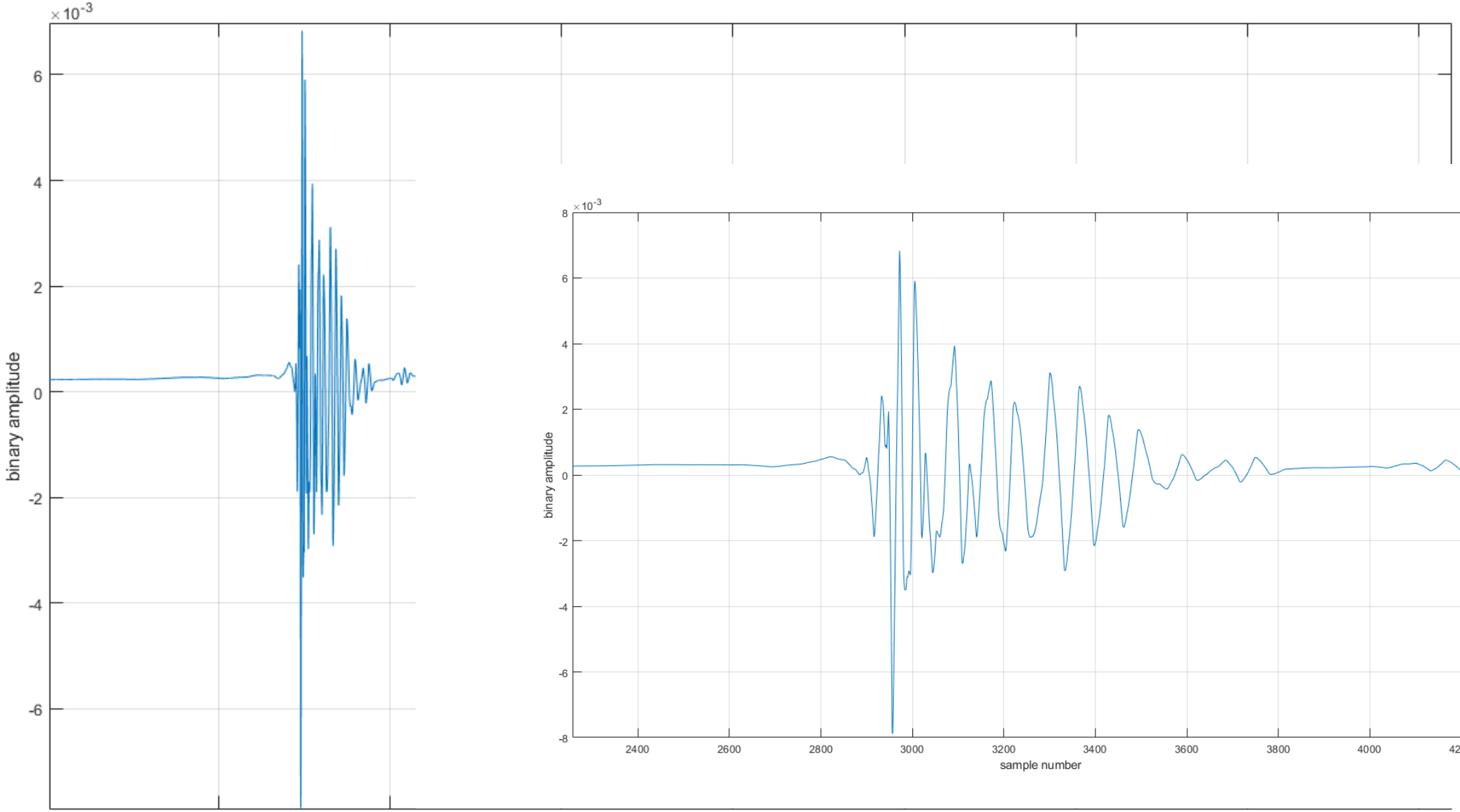
$$= \frac{362880}{2(5040)}(0{,}027)(0{,}0823543) = 0.08004$$

# Example - noisy signal



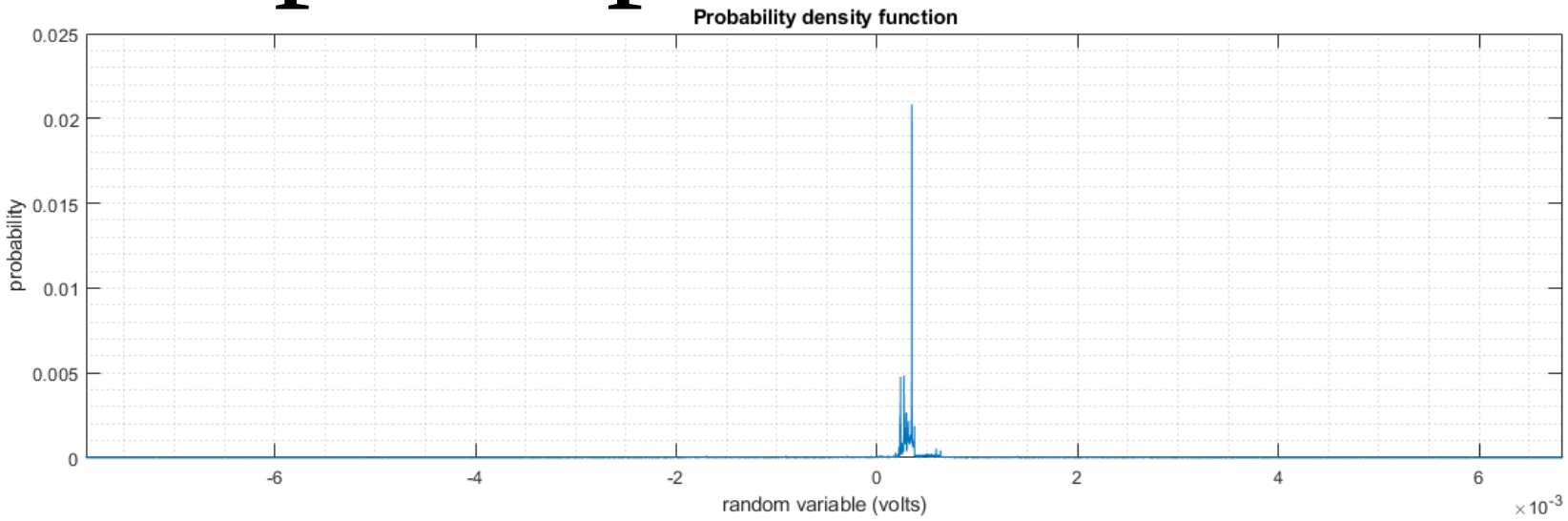
# Example - denoised signal

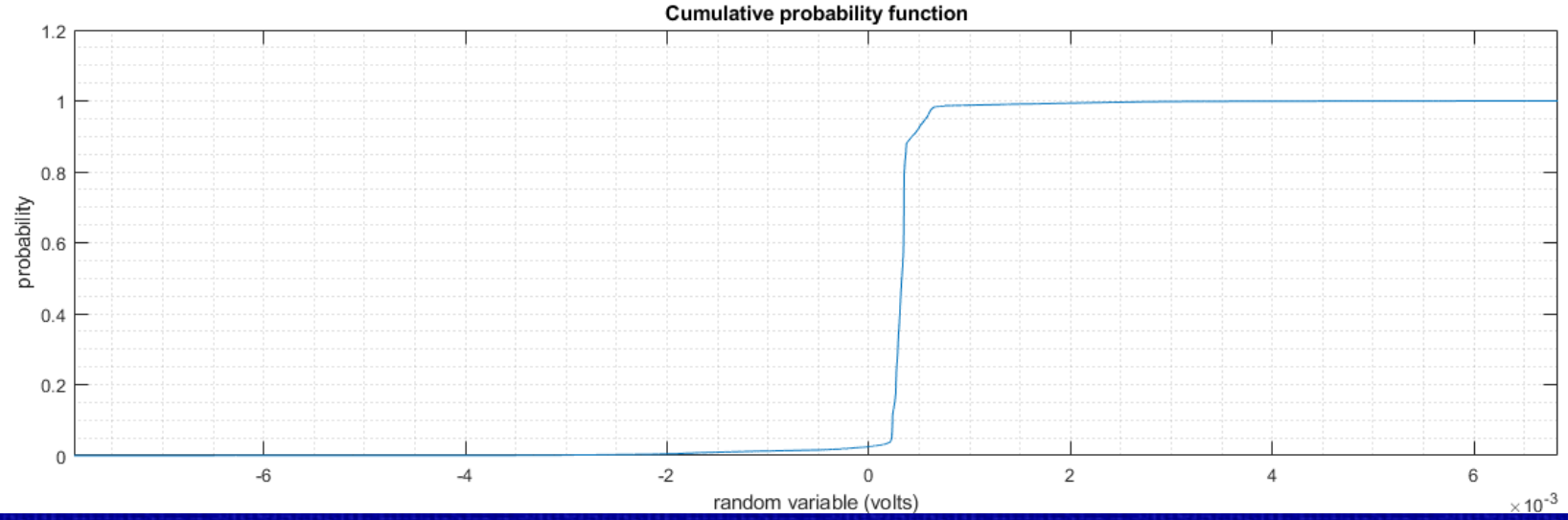




# Real example – pdf/cdf

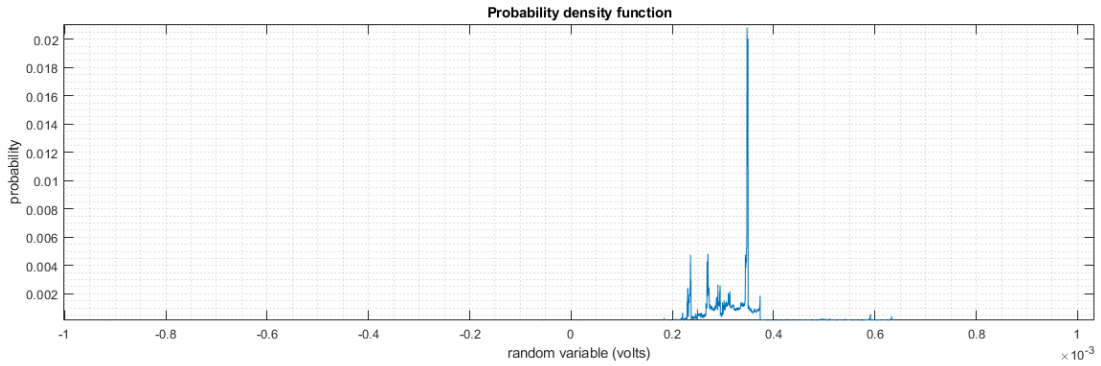


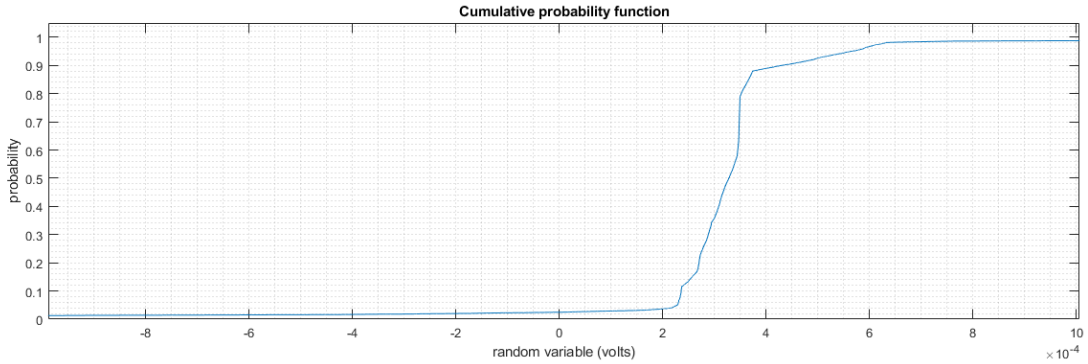


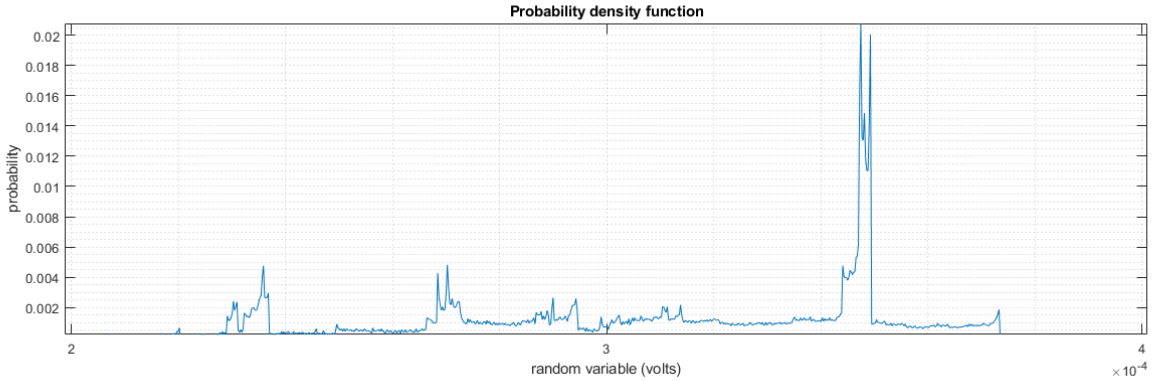



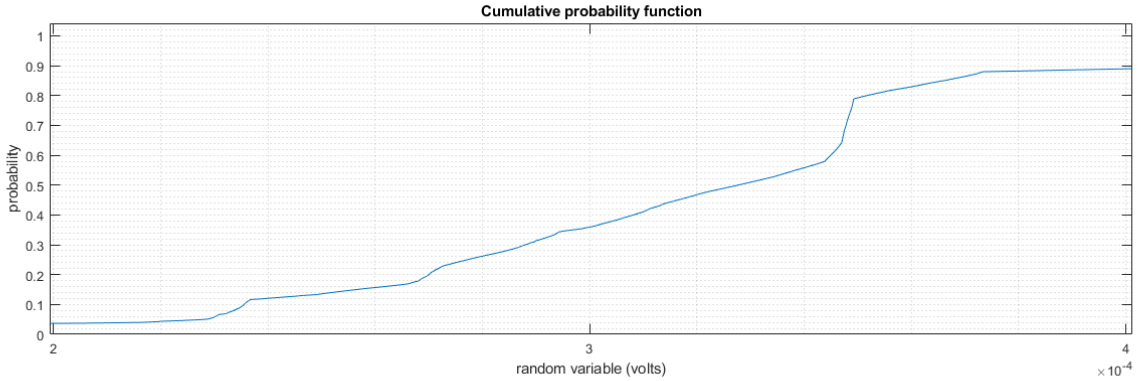
# Real example – pdf/cdf





# pmf/pdf

```
%-----------------------------------------------------------------------
%                           Compute_signal_pdf
%   Compute a pmf(pdf) of a one dimensional signal.
%   FAON - Apr 29th, 2023.
%-----------------------------------------------------------------------
function [signal_pdf, x, signal_average, signal_min_value, signal_max_value, p_moda, x_moda] = Compute_signal_pdf(signal, Bins)
    N = length(signal);
    signal_pdf = zeros(1, Bins, 'double');
    x = zeros(1, Bins, 'double');
    signal_bins = zeros(N, 'uint32');
    [signal_min_value, signal_max_value] = bounds(signal);
    %-----------------------------------------------------------------
    % Computes the signal dynamic range and the variable step length for
    % building the pmf(pdf).
    %-----------------------------------------------------------------
    signal_range = signal_max_value - signal_min_value;
    d_Bins = cast(Bins, 'double');
    x = linspace(signal_min_value, signal_max_value, Bins);
    %-----------------------------------------------------------------
    % Scales the signal to match the specified amount of bins.
    %-----------------------------------------------------------------
    signal_average = 0.0;
    for n = 1:N
        quantized_signal = (d_Bins-1) * ((signal(n) - signal_min_value)/signal_range) + 1;
        quantized_signal = cast(quantized_signal, 'uint32');
        signal_bins(n) = quantized_signal;
        signal_average = signal_average + signal(n);
    end
    signal_average = signal_average/N;
    %-----------------------------------------------------------------
    %   Compute the histogram using bins.
    %-----------------------------------------------------------------
    for n = 1:N
        signal_pdf(signal_bins(n)) = signal_pdf(signal_bins(n)) + 1;
    end
    signal_pdf = signal_pdf./N;
    [p_moda, p_index] =  max(signal_pdf);
    x_moda = x( p_index);
end
```

# cdf

```
%-----------------------------------------------------------------------
%                              Compute_cdf
%   Compute the Cumulative Probability Distributuin Function (CDF).
%   FAON - Apr 29th, 2023.
%-----------------------------------------------------------------------
function signal_cdf = Compute_cdf(signal_pdf)
    Bins = length(signal_pdf);
    signal_cdf = zeros(1,Bins, 'double');
    signal_cdf(1) = signal_pdf(1);
    for k = 2: Bins
        signal_cdf(k) = signal_cdf(k-1) + signal_pdf(k);
    end
end
```

# Real example – features

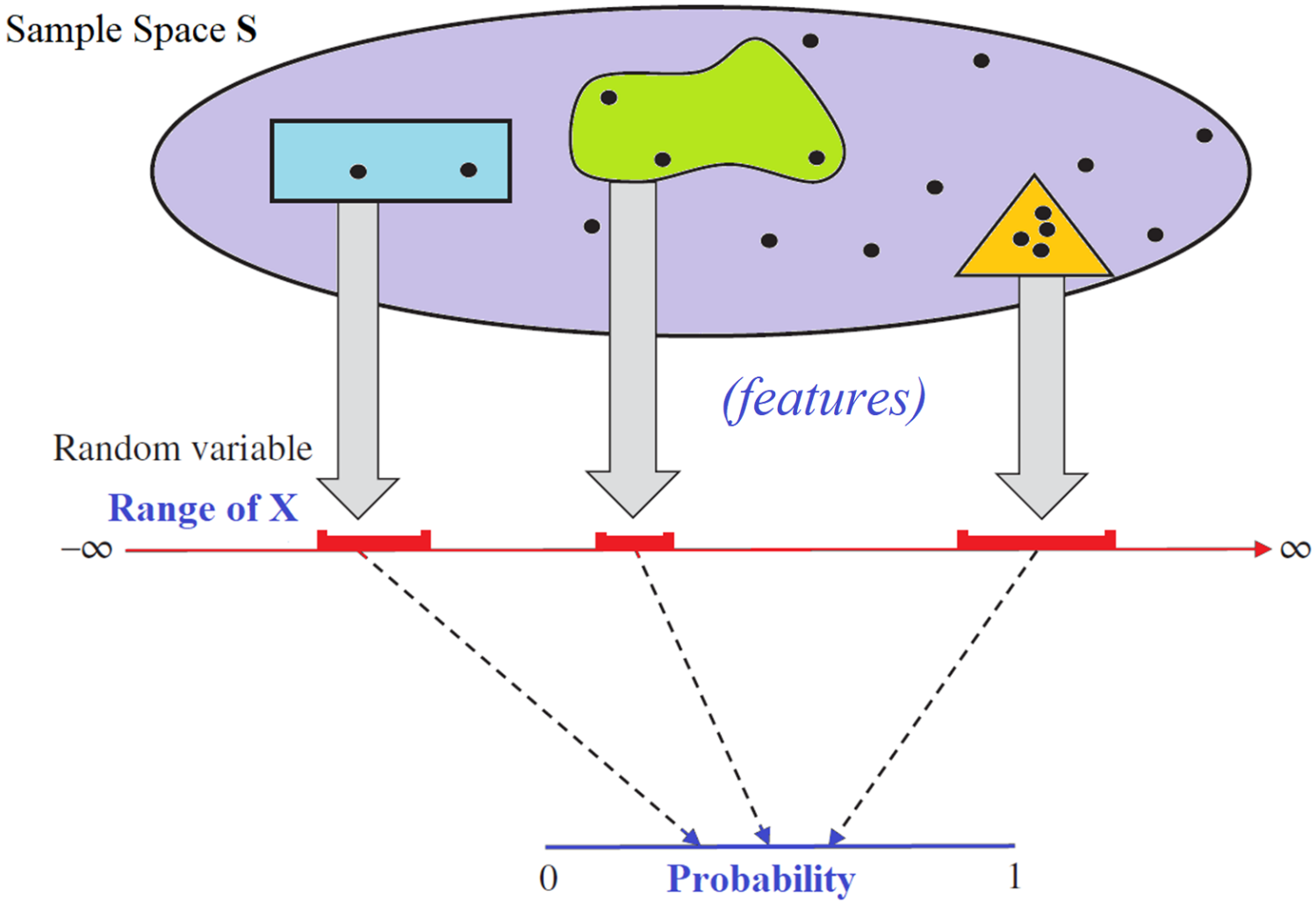


Features =

signal_histogram: [3 4 34 217 16159 127 14 5]
signal_min_value: -0.0079
signal_max_value: 0.0068
squared_range: 2.1648e-04
signal_average: 3.1701e-04
probability_of_mode: 0.0208
x_value_of_mode: 3.4744e-04
x1_for_probability_0333: 2.9333e-04
x2_for_probability_0666: 3.4722e-04
p_20_80: 0.9987
entropy: 9.7143
mean: 4.8372e-09
moment_2: 4.0155e-12
moment_3: 1.1174e-15
moment_4: 3.6718e-17
moment_5: -2.1967e-20
moment_c1: 4.8371e-09
variance: 4.0155e-12
moment_c3: 1.1173e-15
moment_c4: 3.6718e-17
moment_c5: -2.1968e-20

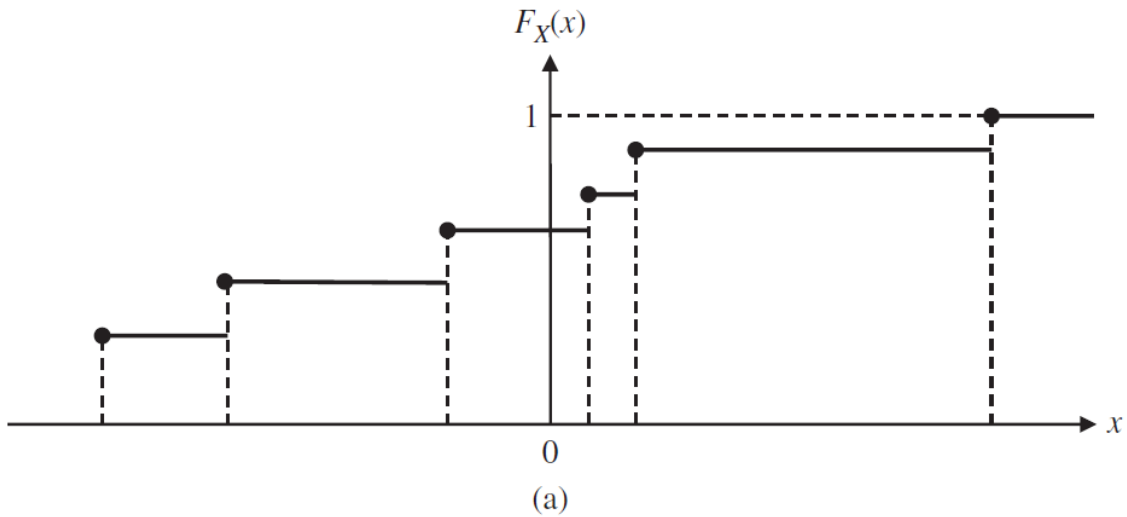
moment_nc1: 0.0024
moment_nc2: 1.0000
Skewness: 138.8585
Kurtosis: 2.2772e+06
moment_nc5: -6.7991e+08
normalized_energy: 2.6316e-07
rms_value: 5.1299e-04
absolute_deviation: 1.1709e-04
signal_uniformity: 0.0034
x_value_percentile_075: 2.3294e-04
x_value_percentile_150: 2.5652e-04
median: 3.2768e-04
x_value_percentile_850: 3.6608e-04
x_value_percentile_925: 4.9809e-04
interpercentile_mean_15_85: 0.0076
interpercentile_range_15_85: 1.0956e-04
interquartile_mean_25_75: 0.0084
interquartile_range_25_75: 7.1619e-05
interpercentile_mean_10_90: 3.1886e-04
interpercentile_range_10_90: 2.3564e-04
robust_mean_absolute_deviation: 8.4195e-09

# The relationship among sample space, random variable, and probability.

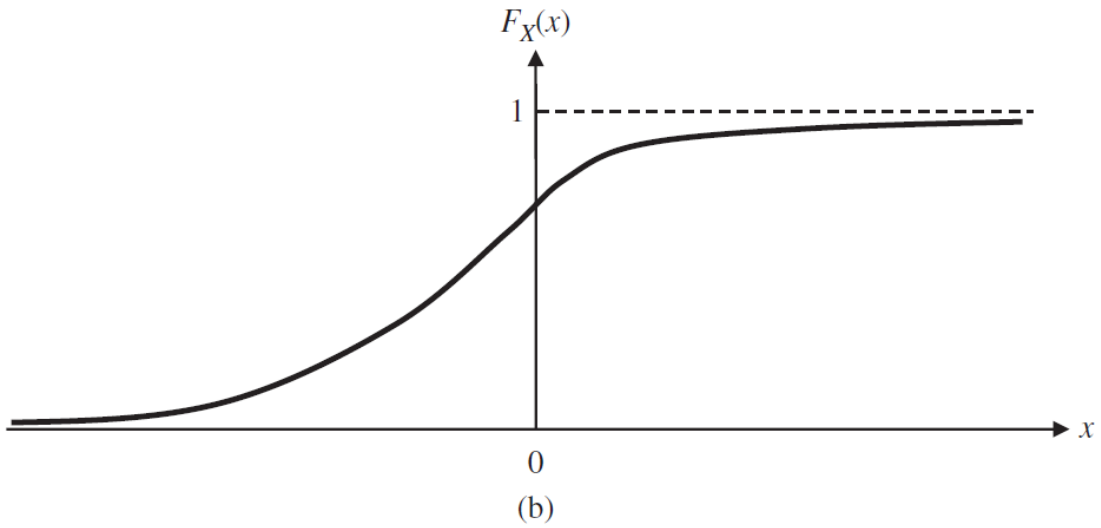


# Mixed random variable

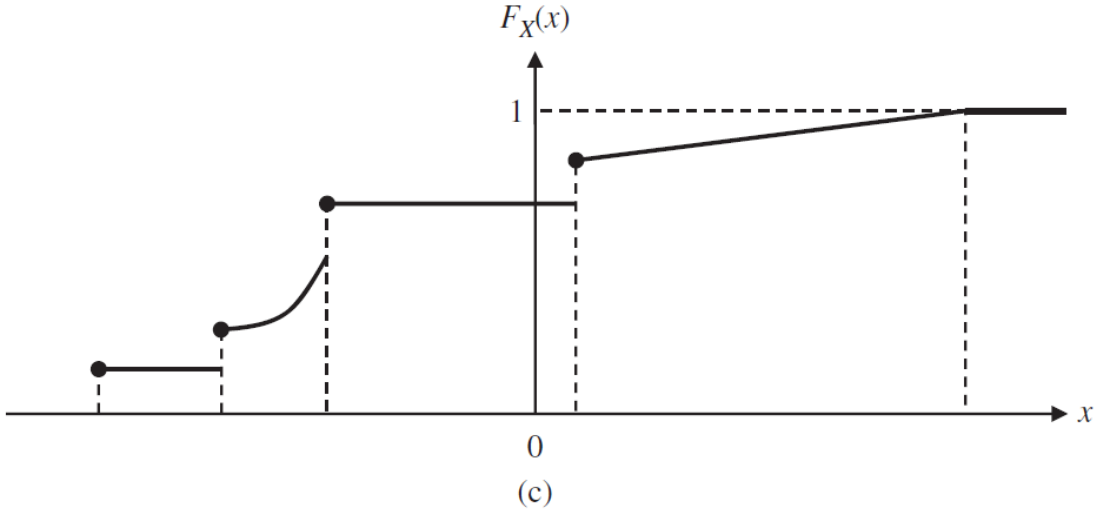
# Mixed random variable: cdf examples




(a) Discrete random variable.




(b) Continuous random variable.

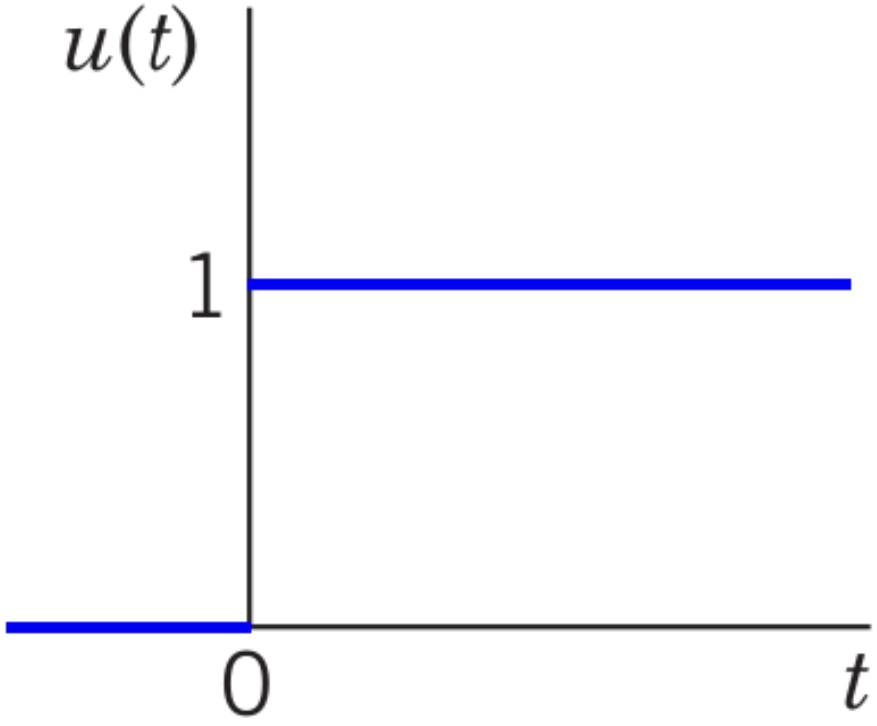


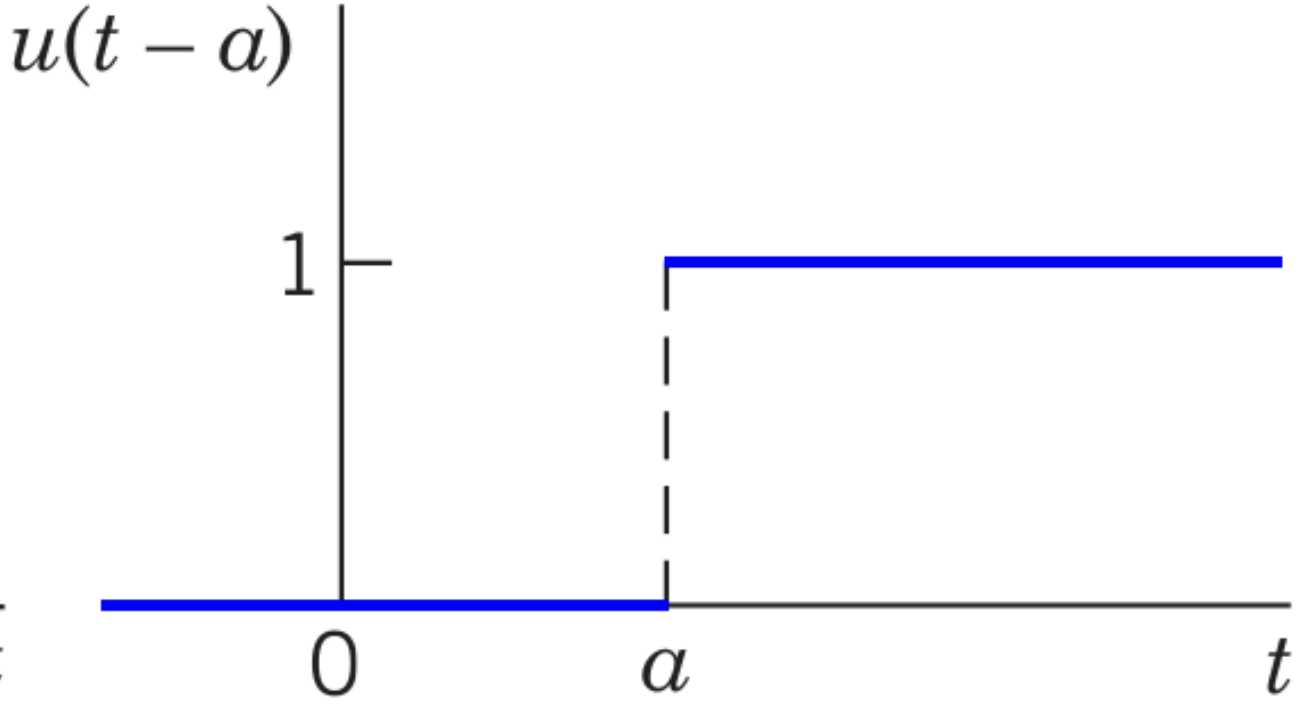

(c) Mixed random variable.

# Unit step function

- Vamos nos lembrar da função degrau u(t):

$$u(t-a)=\begin{cases}0, se\ t<a\\ 1, se\ t\geq a\end{cases}$$

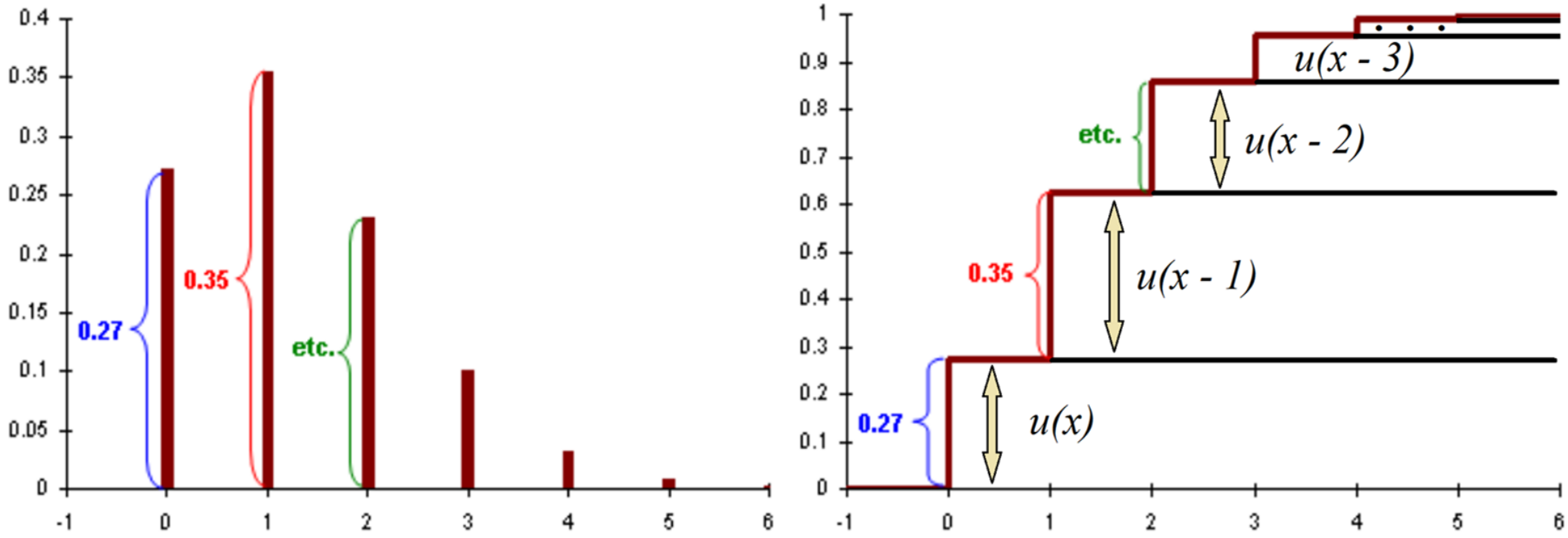


# Using unit step function to describe the CDF of a discrete random Variable

- Consider a discrete random variable $\boldsymbol{X}$ with range $R_X = \{x_1, x_2, x_3, \ldots\}$ and PMF $P_{\boldsymbol{X}}(x_k)$. Note that the CDF for $\boldsymbol{X}$ can be written as a linear combination

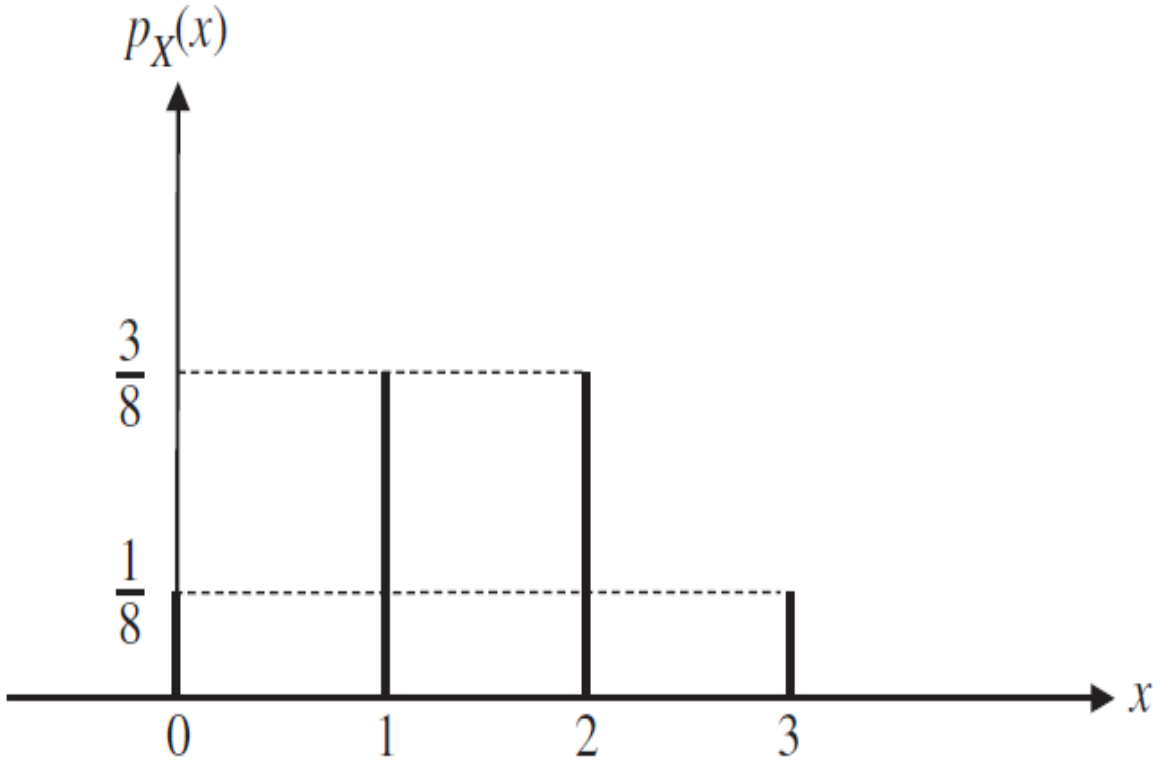
$$F_X(x) = \sum_{x_k \in R_X} P_X(x_k) u(x - x_k)$$



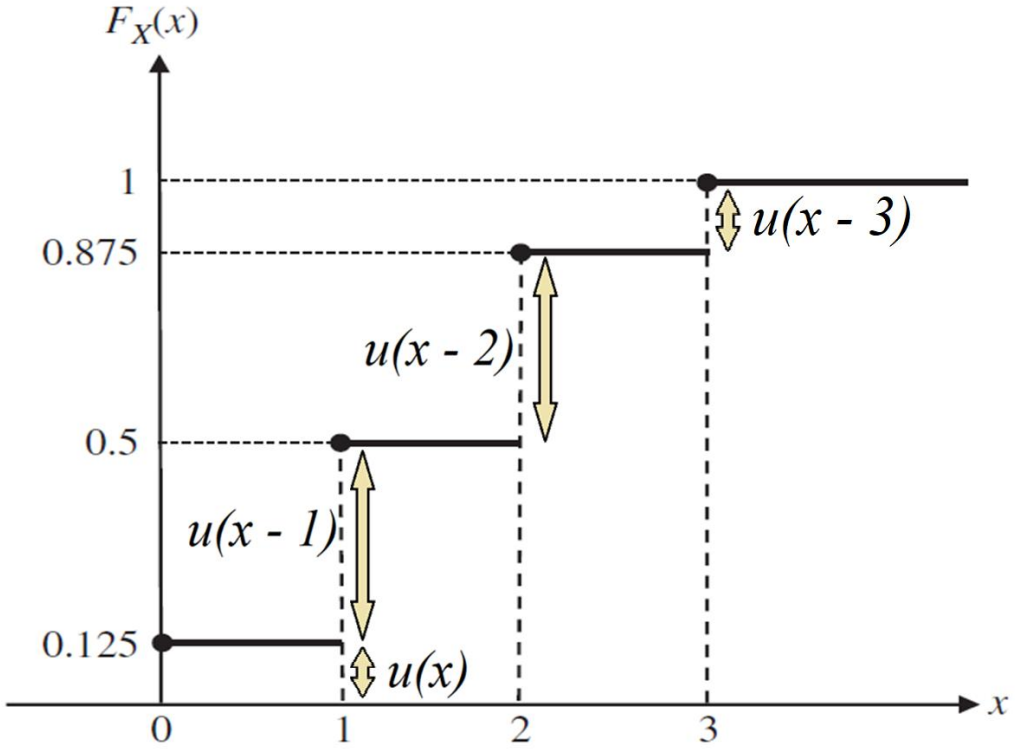

PMF $p_X(x)$ CDF $F_{\boldsymbol{X}}(x)$

# Using unit step function to describe the CDF of a discrete random Variable (Exercise 4)

$$F_X(x) = 0.125\,u(x) + 0.375\,u(x-1) + 0.375\,u(x-2) + 0.125\,u(x-3)$$



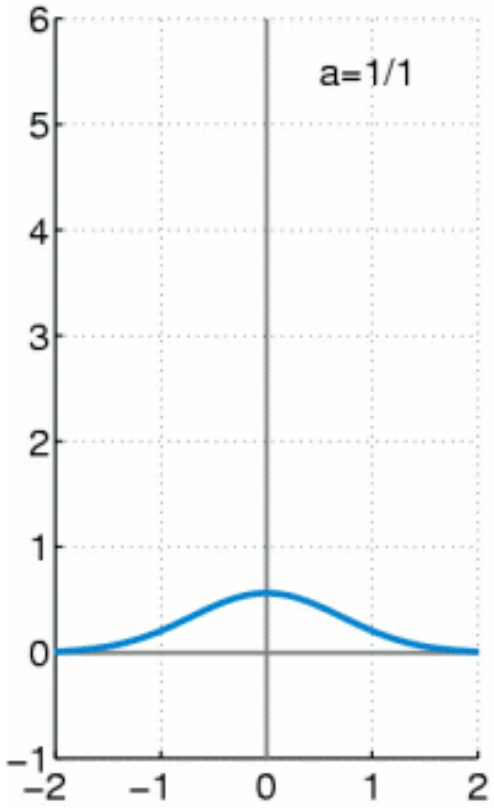

PMF $p_X(x)$

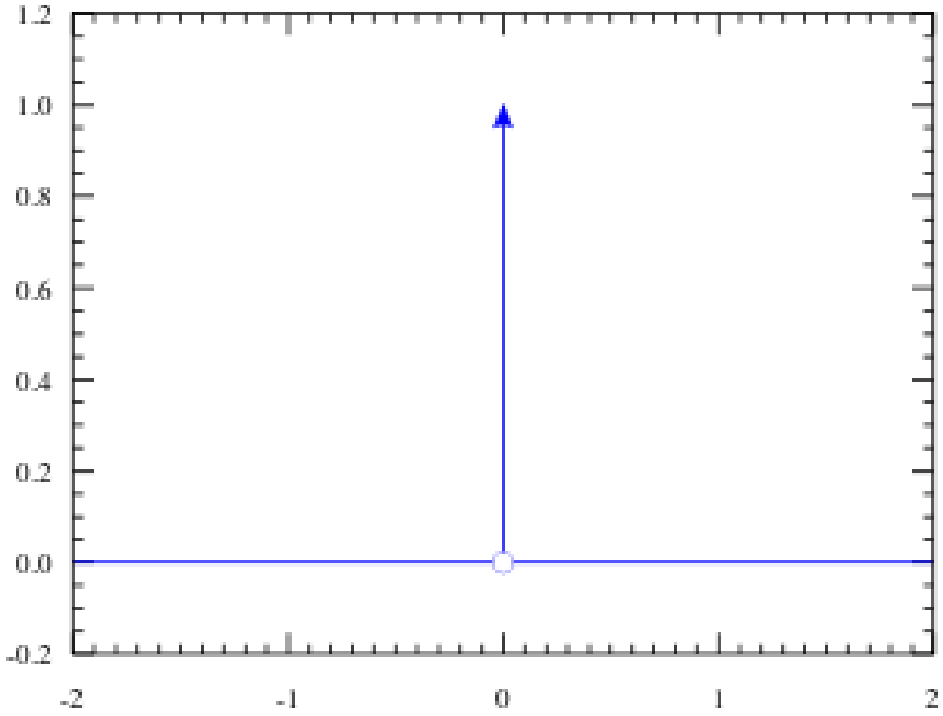



CDF $F_X(x)$

# Dirac delta function

- The $\delta$ is also known as the **unit impulse function.**




$$\delta_a(x) = \lim_{a \to 0} \left\{ \frac{1}{\sqrt{\pi}|a|} e^{-\left(\frac{x}{a}\right)^2} \right\}$$

# Properties of the delta function

1. $\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{otherwise} \end{cases}$
2. $\delta(x) = \frac{d}{dx}u(x)$, where $u(x)$ is the unit step function
3. $\int_{-\epsilon}^{\epsilon} \delta(x)dx = 1$, for any $\epsilon > 0$;
4. For any $\epsilon > 0$ and any function $g(x)$ that is continuous over $(x_0 - \epsilon, x_0 + \epsilon)$, we have

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)dx = \int_{x_0-\epsilon}^{x_0+\epsilon} g(x)\delta(x - x_0)dx = g(x_0).$$

# Generalized PDF of a discrete random variable

For a discrete random variable $X$ with range $R_X = \{x_1, x_2, x_3, \dots\}$ and PMF $P_X(x_k)$, we define the (generalized) probability density function (PDF) as

$$f_X(x) = \sum_{x_k \in R_X} P_X(x_k)\delta(x - x_k).$$

# Mixed random variable

- When you have mixed random variables, the generalized **CDF** and **PDF** are constructed through the additivity of a discrete (CDF/PMF) and a continuous (CDF/PDF) a random process.

- The generalized **CDF** can be expressed as (**C** for continuous and **D** for discrete):

$$\boldsymbol{F_X(x) = F_X^D + F_X^C}$$

- The generalized **PDF** can be given by:

$$\boldsymbol{f_X(x) = f_X^D + f_X^C}$$

# Exercício 9



Considere uma variável randômica $\boldsymbol{X}$ cuja CDF generalizada pode ser expressa analiticamente pela expressão:

$$\boldsymbol{F_X(x)} = \begin{cases} 0 & se\ x < 0 \\ \frac{1}{4} + \frac{1}{2}(1 - e^{-x}), & \text{se } 0 \leq x < 1 \\ \frac{1}{2} + \frac{1}{2}(1 - e^{-x}), & se\ x \geq 1 \end{cases}$$

a) Esboce a CDF generalizada de $\boldsymbol{X}$.

b) Determine analiticamente a PDF generalizada de $\boldsymbol{X}$.

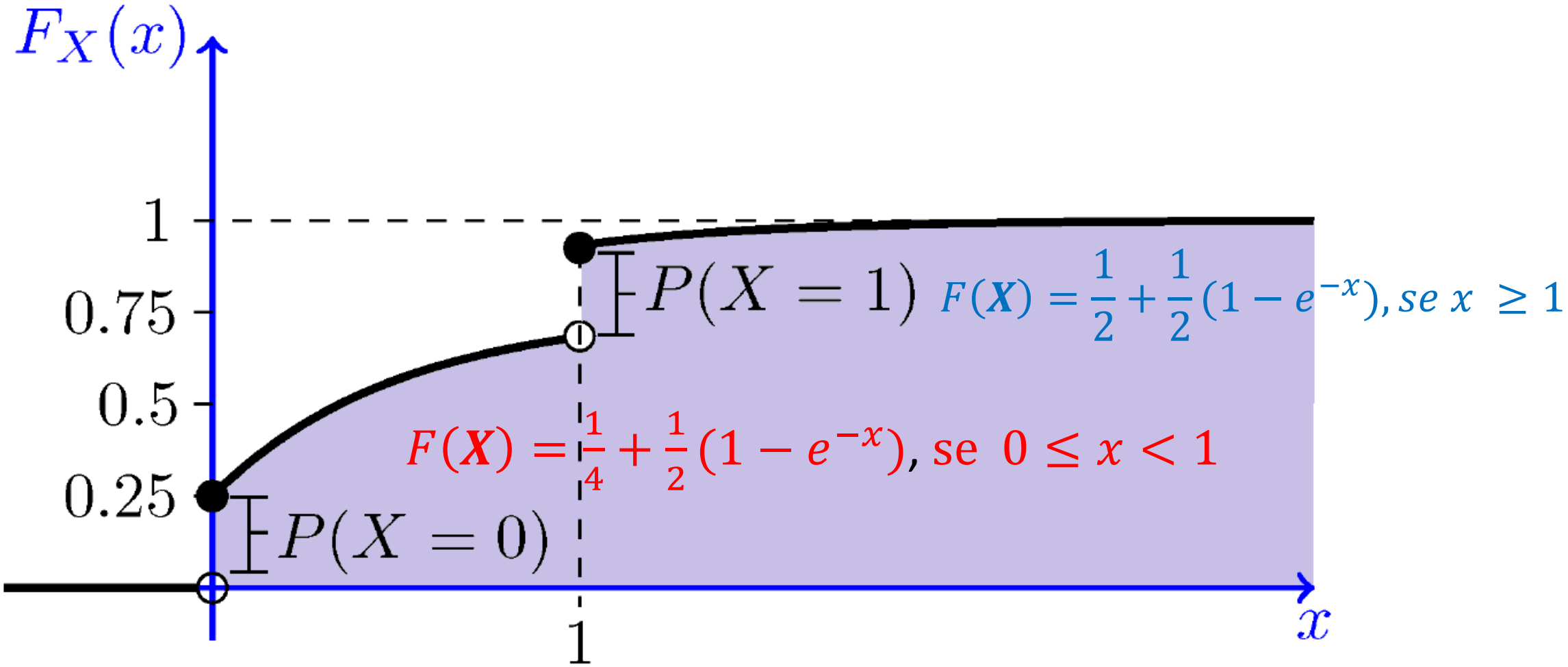
c) Encontre **P(X > 0.5)**, tanto usando a CDF e usando a PDF.

# Exercício 9

- d) Calcule a expectância da variável randômica X.
- e) Calcule a variância da variável randômica X.

# Exercício 9: Solução

- a) Esboce a CDF generalizada de $\boldsymbol{X}$.



# Exercício 9: Solução



b) Determine analiticamente a PDF generalizada de $\boldsymbol{X}$.

Observe que a CDF contínua pode ser representada por:

$$\boldsymbol{F_X^C}(\boldsymbol{x}) = \frac{1}{2}(1 - e^{-x}), \text{se } x \geq 0 \text{ ou } \boldsymbol{F_X^c} = \frac{1}{2}(1 - e^{-x})u(x)$$

Observe que a CDF discreta pode ser escrita como:

$$\boldsymbol{F_X^D}(\boldsymbol{x}) = \frac{1}{4}u(x) + \frac{1}{4}\,u(x-1)$$

A CDF generalizada pode ser dada por:

$$\boldsymbol{F_X}(\boldsymbol{x}) = \boldsymbol{F_X^D}(\boldsymbol{x}) + \boldsymbol{F_X^C}(\boldsymbol{x}) = \frac{1}{4}u(x) + \frac{1}{4}\,u(x-1) + \frac{1}{2}(1 - e^{-x})u(x)$$

# Exercício 9: Solução

- A PDF contínua pode ser computada na forma:

$$\boldsymbol{f_X^C(x)} = \frac{d\boldsymbol{F_X^C}(x)}{dx} = \frac{d\left\{\frac{1}{2}(1-e^{-x})u(x)\right\}}{dx} = \frac{1}{2}e^{-x}, \text{se } x \geq 0 \text{ ou}$$

$$\boldsymbol{f_X^c(x)} = \frac{1}{2}e^{-x}u(x)$$

- A PMF discreta também pode ser calculada como:

$$\boldsymbol{f_X^D} = \frac{d\boldsymbol{F_X^D}(x)}{dx} = \frac{1}{4}\delta\,(x) + \frac{1}{4}\delta(x-1)$$

# Exercício 9: Solução

- A PDF generalizada pode representada por:

$$\boldsymbol{f_X(x)} = \boldsymbol{f_X^D} + \boldsymbol{f_X^C} = \frac{1}{4}\,\delta\,(x) + \frac{1}{4}\delta(x-1) + \frac{1}{2}e^{-x}u(x)$$

- Observe que a PDF generalizada também pode ser obtida derivando-se a CDF generalizada:

$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{1}{4}\,\delta\,(x) + \frac{1}{4}\delta(x-1) + \frac{1}{2}e^{-x}u(x)$$

# Exercício 9: Solução

- c) Encontre **P(X> 0.5)**, tanto usando o CDF quanto usando o PDF.

Usando a CDF generalizada:

$$
\begin{aligned}
P(X > 0.5) &= 1 - F_X(0.5) \\
&= 1 - \left[\frac{1}{4} + \frac{1}{2}(1 - e^{-x})\right]\Bigg|_{x=0.5} \\
&= \frac{1}{4} + \frac{1}{2}e^{-0.5} \\
&= 0.5533
\end{aligned}
$$

# Exercício 9: Solução

- Calculando **P(X> 0.5),** por meio da **PDF** generalizada obtém-se:

$$
\begin{aligned}
P(X > 0.5) &= \int_{0.5}^{\infty} f_X(x)dx \\
&= \int_{0.5}^{\infty} \left( \frac{1}{4}\delta(x) + \frac{1}{4}\delta(x-1) + \frac{1}{2}e^{-x}u(x) \right) dx \\
&= 0 + \frac{1}{4} + \frac{1}{2}\int_{0.5}^{\infty} e^{-x}dx \\
&= \frac{1}{4} + \frac{1}{2}e^{-0.5} = 0.5533
\end{aligned}
$$

# Exercício 9: Solução

- d) Calcule a expectância $\mu_X$ da variável randômica $X$.

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx \\
&= \int_{-\infty}^{\infty} \left( \frac{1}{4} x\delta(x) + \frac{1}{4} x\delta(x-1) + \frac{1}{2} x e^{-x} u(x) \right) dx \\
&= \frac{1}{4} \times 0 + \frac{1}{4} \times 1 + \frac{1}{2} \int_0^{\infty} x e^{-x} dx \\
&= \frac{1}{4} + \frac{1}{2} \times 1 = \frac{3}{4}.
\end{aligned}
$$

$$\int u dv = uv - \int v du$$

# Exercício 9: Solução

- d) Calcule a variância da variável randômica $\boldsymbol{X}$.

$$\sigma_X^2 = Var(X) = E[(X - \mu_{\boldsymbol{X}})^2] =$$

$$\int_{-\infty}^{\infty} (x - \mu_{\boldsymbol{X}})^2 f_{\boldsymbol{X}}(x) dx = E[\boldsymbol{X}^2] - (E[X])^2 = E[\boldsymbol{X}^2] - \mu_{\boldsymbol{X}}{}^2$$

# Exercício 9: Solução

- Calculando E[$\boldsymbol{X}^2$]:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx$$

$$= \int_{-\infty}^{\infty} \left( \frac{1}{4}x^2\delta(x) + \frac{1}{4}x^2\delta(x-1) + \frac{1}{2}x^2 e^{-x}u(x) \right) dx$$

$$= \frac{1}{4} \times 0 + \frac{1}{4} \times 1 + \frac{1}{2}\int_0^{\infty} x^2 e^{-x} dx$$

$$= \frac{1}{4} + \frac{1}{2} \times 2 = \frac{5}{4}.$$

$$\int udv = uv - \int vdu$$

# Exercício 9: Solução

- Finalmente:

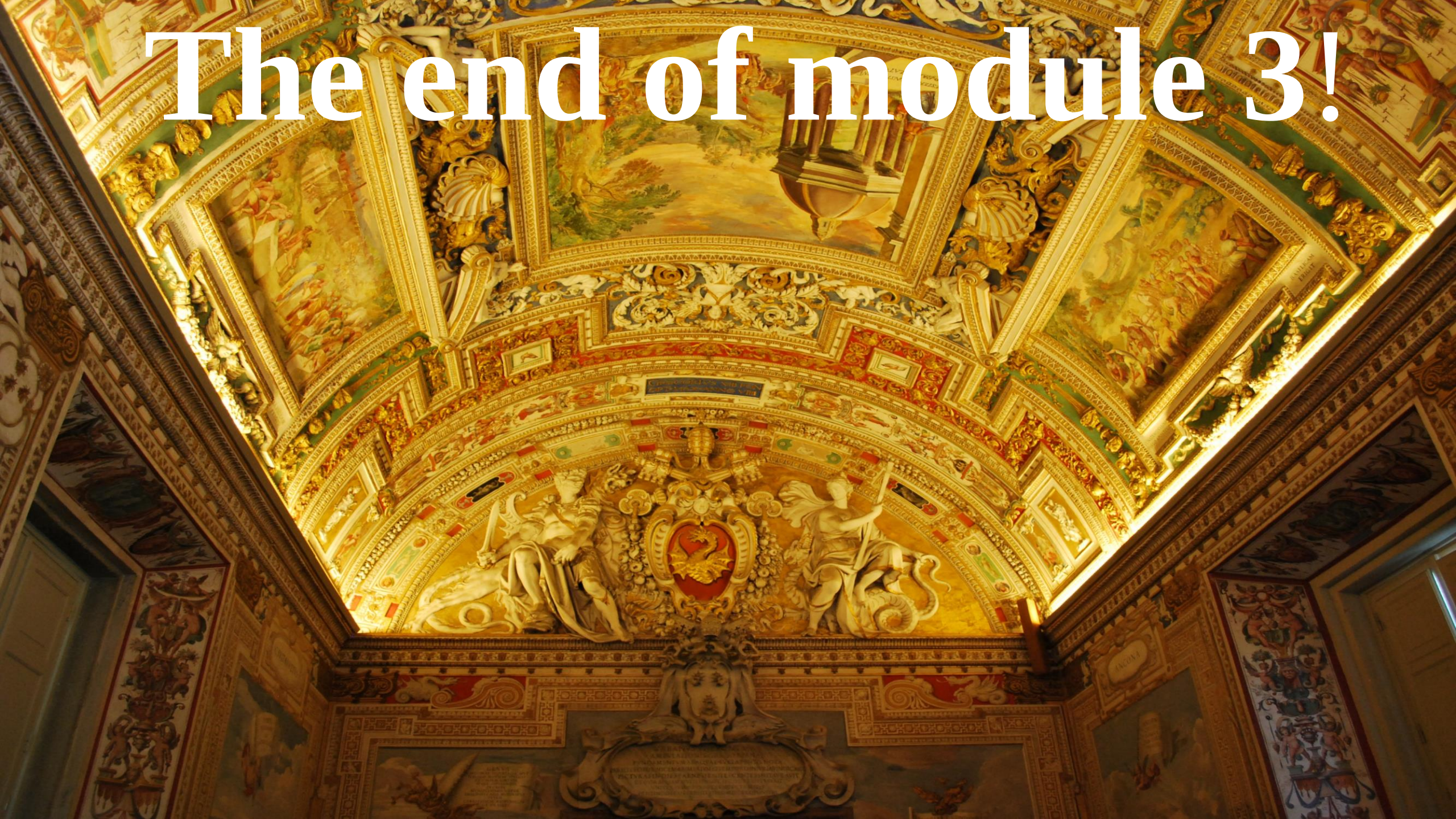
$$\sigma_X^2 = E[\boldsymbol{X}^2] - \mu_{\boldsymbol{X}}{}^2 = \frac{5}{4} - \left(\frac{3}{4}\right)^2 = \frac{11}{16}$$

# The generalized PDF of a mixed random variable

- The generalized **PDF** of a mixed random variable can be written in the form:

$$f_X(x) = \sum_k a_k \delta(x - x_k) + g(x),$$

where $a_k = P(X = x_k)$, and $g(x) \geq 0$ does not contain any delta functions. Furthermore, we have

$$\int_{-\infty}^{\infty} f_X(x)dx = \sum_k a_k + \int_{-\infty}^{\infty} g(x)dx = 1.$$

# The end of module 3!