Joint distributions

Stochastic Process

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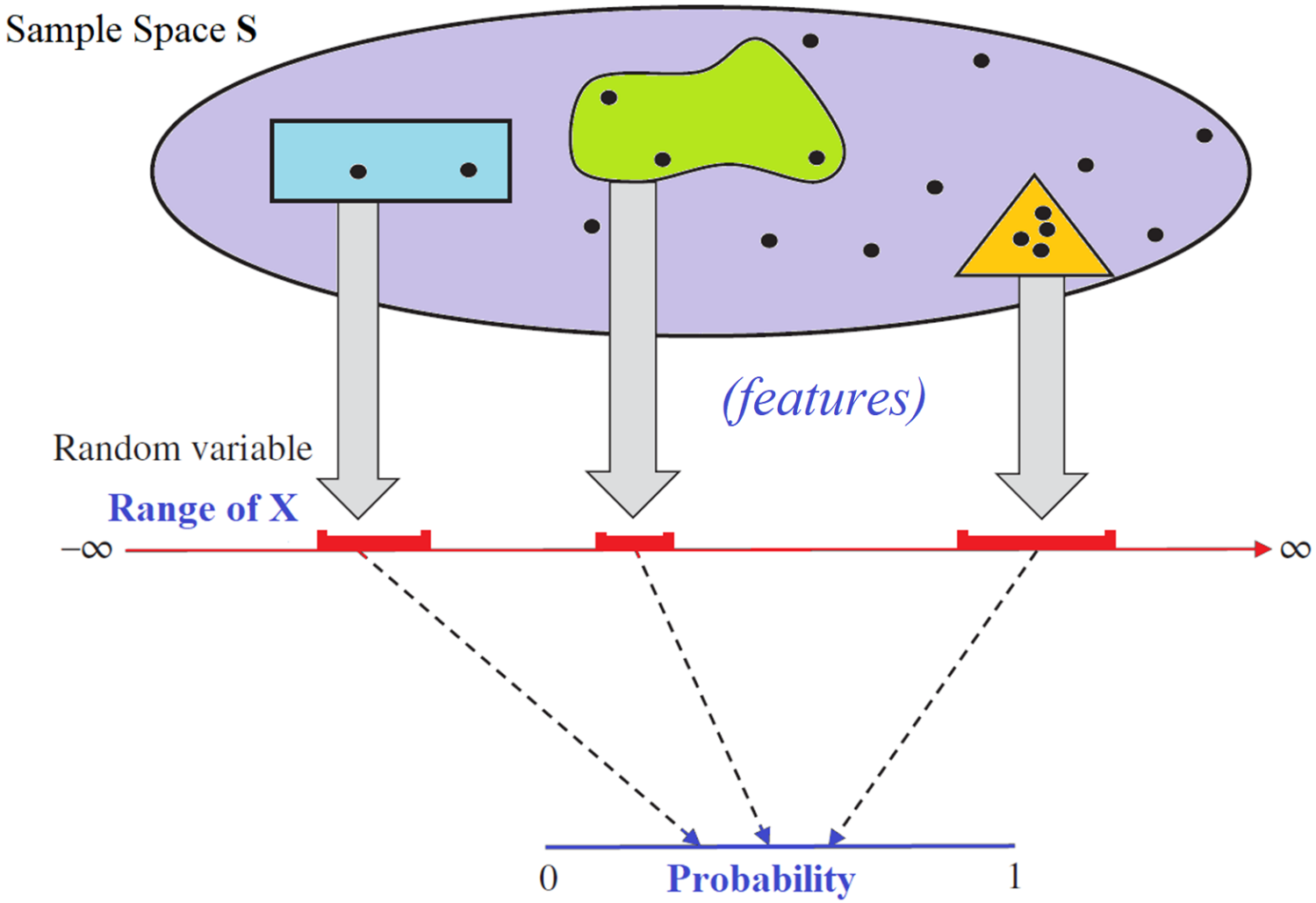
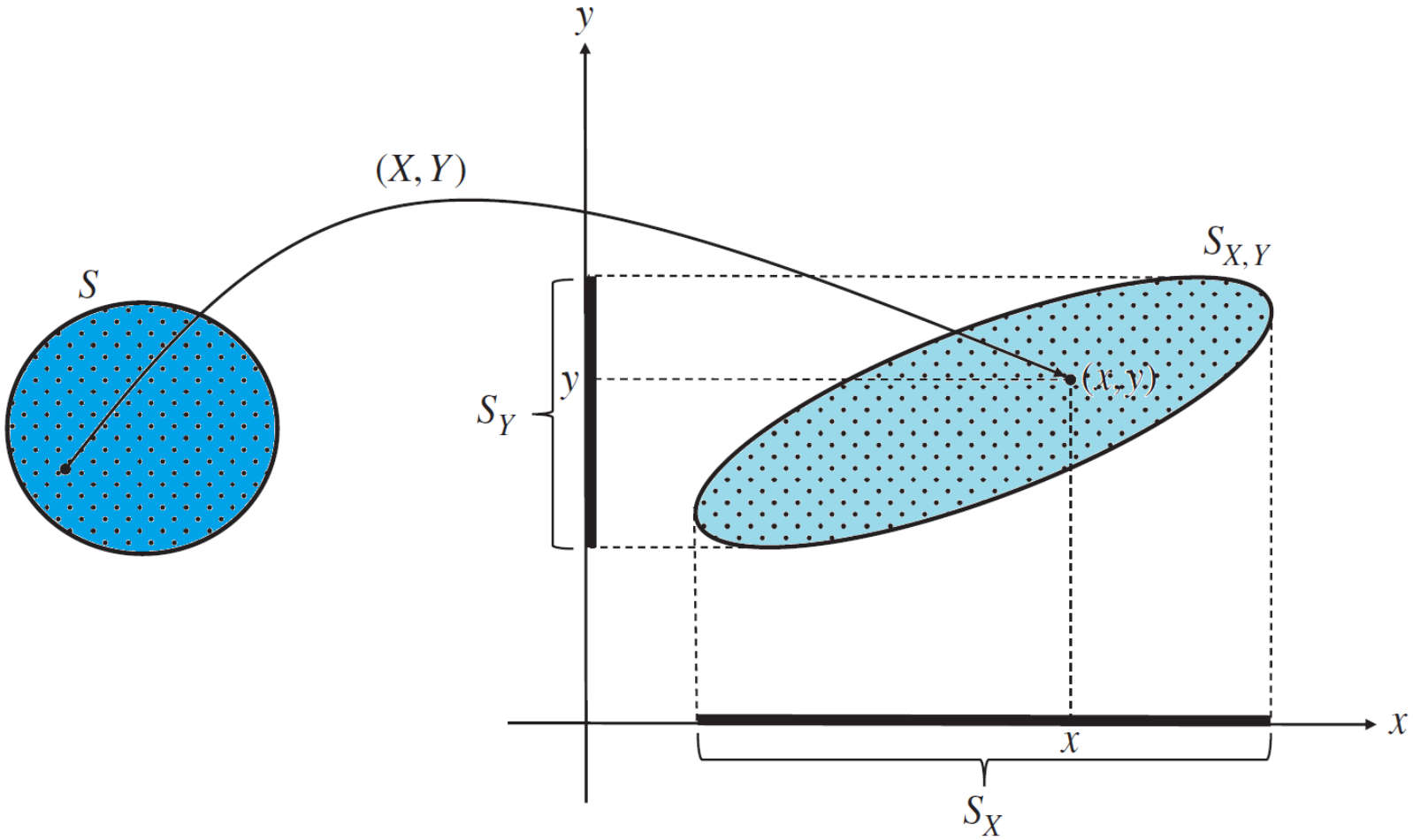
The relationship among sample space, random variable, and probability.



The Sample Space ***S*** is the ***Domain of the Pair of Random Variables***

- $S_{X,Y}$ is the ***range of the pair of random variables***.
- The range $\boldsymbol{S_{X,Y}}$ is a subset of all possible points in the real plane with nonzero probability.



Discrete random variables

- Joint probability mass function (PMF)
- Marginal PMF
- Joint cumulative distribution function (CDF)
- Marginal CDF
- Conditional PMF
- Conditional CDF
- Discrete random independent variables

Discrete random variables

- Conditional expectation
- Law of total expectation
- Law of total variance
- Law of the unconscious statistician (LOTUS) for joint variables
- Conditional expectation as a function of a random variable
- Expectation for independent random variables
- Conditional variance
- Conditional variance as a function of a random variable

Joint Probability Mass Function (PMF) of a discrete random variables

- The **joint probability mass function** of two discrete random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ is defined as

$$P_{\boldsymbol{XY}}(x,y) = P(\boldsymbol{X}=x, \boldsymbol{Y}=y)$$

$$= P((\boldsymbol{X}=x) \text{ and } (\boldsymbol{Y}=y))$$

$$= P((\boldsymbol{X}=x) \cap (\boldsymbol{Y}=y))$$

Joint Probability Mass Function (PMF) of a discrete random variables

- We can define the joint range for ***X*** and ***Y*** as

$$R_{XY} = \{(x, y) | P_{XY}(x, y) > 0\}$$

- In particular, if $R_X = \{x_1, x_2, \ldots\}$ and $R_Y = \{y_1, y_2, \ldots\}$, then we can always write (ordered sampling (permutations) with replacement):

$$R_{XY} \subset R_X \times R_Y = \{(x_i, y_j) | x_i \in R_X, y_j \in R_Y\}.$$

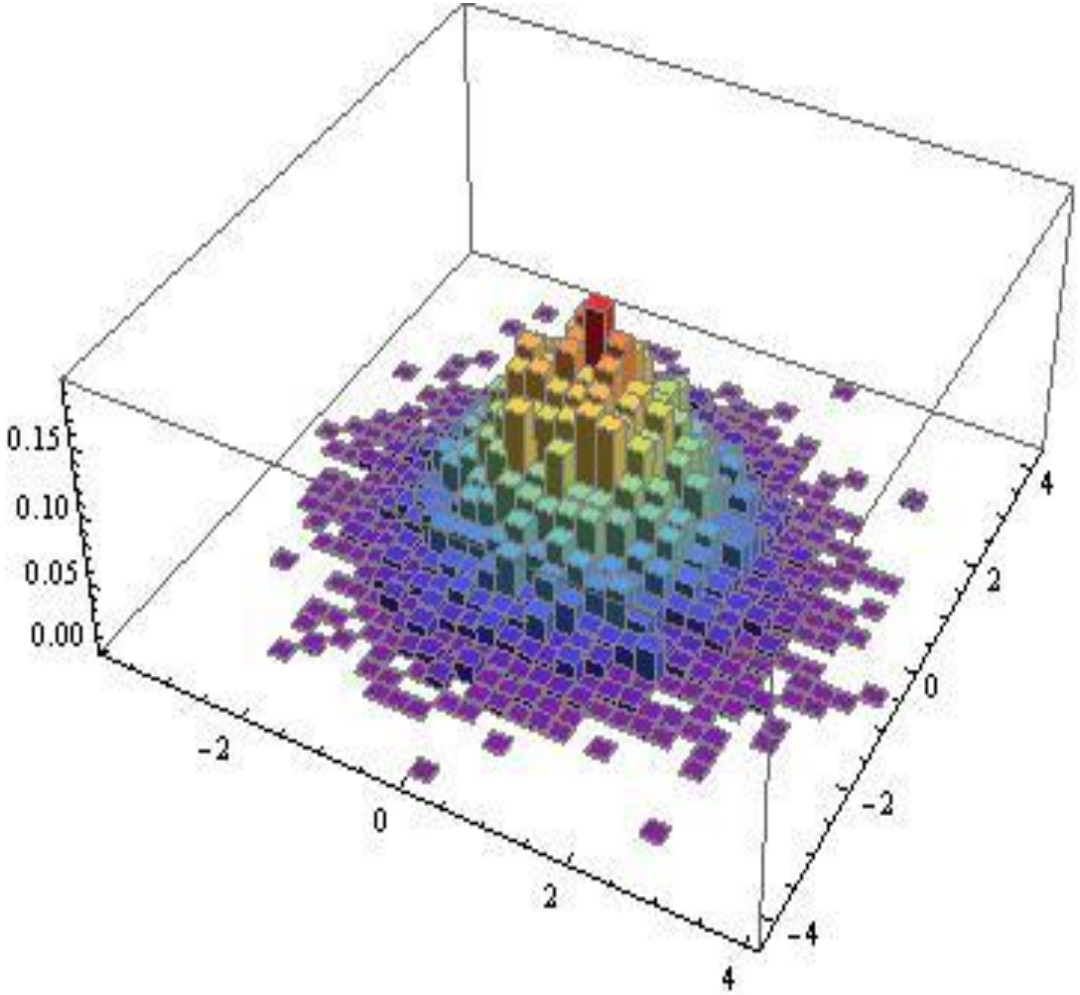
$R_X \times R_Y \rightarrow$ Cartesian product = Multiplication principle

Joint Probability Mass Function (PMF) of a discrete random variables

- Example countable two-dimensional data

Index	X	Y	Class
1	1	1	0
2	2	1	0
3	3	2	1
4	1	3	1
5	2	1	1
6	1	1	0
...	...	...	...

Joint Probability Mass Function (PMF) of a discrete random variables



$$P_{XY}(x,y) = P(X = x, Y = y)$$
$$= P\big((X = x) \cap (Y = y)\big).$$

$$\sum_{(x_i,y_j)\in R_{XY}} P_{XY}(x_i, y_j) = 1$$

- If $P((X,Y) \in A$ for any event (a set) $A \subset \boldsymbol{R}^2$

$$P\big((X,Y) \in A\big) = \sum_{(x_i,y_j)\in(A\cap R_{XY})} P_{XY}(x_i, y_j)$$

Marginal PMF

- The **marginal PMF** of $\boldsymbol{X}$ *is* called $P_X(x)$ and it can be expressed as:

$$P_X(x) = P_X(X = x) = \sum_{y_j \in R_Y} P(X = x, Y = y_j)$$

$$= \sum_{y_j \in R_Y} P_{XY}(x, y_j) \quad \text{(law of total probability)}$$

Marginal PMF

- Similarly, the **marginal PMF** of ***Y*** *is* called $P_Y(y)$ and it can be expressed as:

$$P_Y(y) = P_y(Y = y) = \sum_{x_k \in R_X} P(X = x_k, Y = y)$$

$$= \sum_{x_k \in R_X} P_{XY}(x_k, y) \qquad \text{(law of total probability)}$$

Marginal PMF

- Marginal PMFs of ***X*** and ***Y*** :

$$P_X(x) = \sum_{y_j \in R_Y} P_{XY}(x, y_j), \quad \text{for any } x \in R_X$$

$$P_Y(y) = \sum_{x_i \in R_X} P_{XY}(x_i, y), \quad \text{for any } y \in R_Y$$

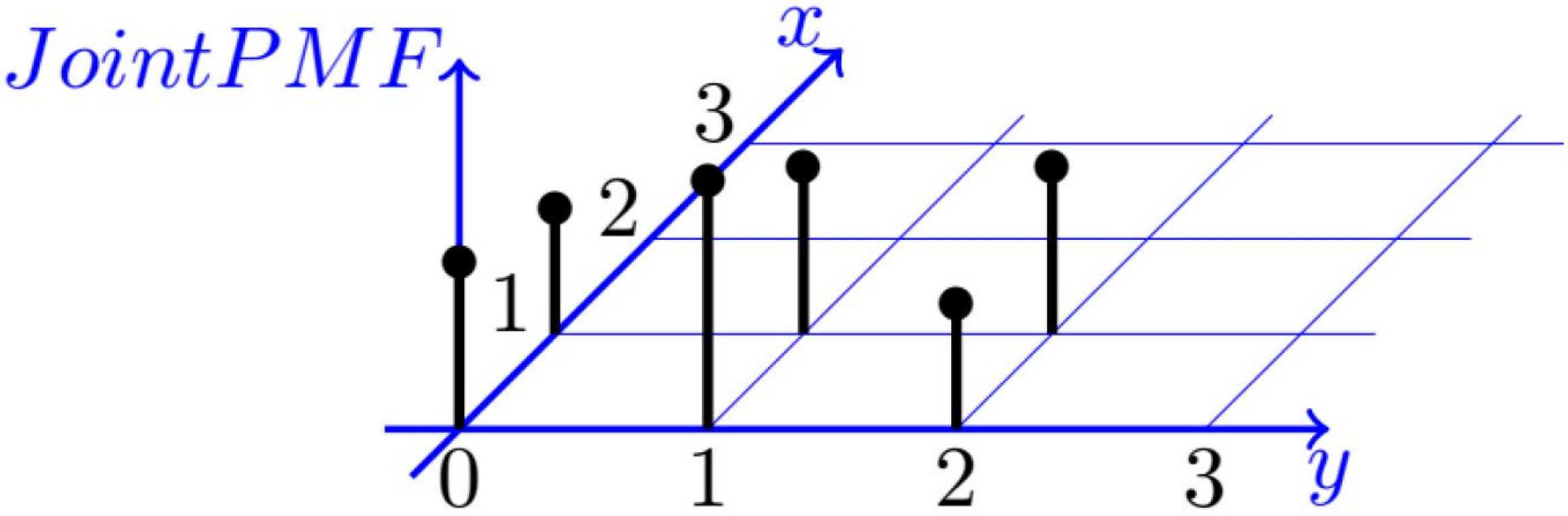
Exercício 4.1

- Considere duas variáveis randômicas $\boldsymbol{X}$ e $\boldsymbol{Y}$ com PMF conjunta $P_{XY}(x, y)$ dada na tabela abaixo:

	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

Exercício 4.1

- PMF conjunta de ***X*** e ***Y***: $P_{XY}(x, y)$.



Exercício 4.1

- a) Encontre $P(X = 0, Y \leq 1)$.
- b) Encontre os PMFs marginais de X e Y.
- c) Encontre a probabilidade condicional $P(Y = 1|X = 0)$.
- d) Verifique se ***X*** e ***Y*** são independentes?

Exercício 4.1: Solução

- a) Encontre $P(X = 0, Y \leq 1)$.

Pode ser calculado direto da tabela

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

$$P(X=0, Y \leq 1) = P_{XY}(0,0) + P_{XY}(0,1) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$$

Exercício 4.1: Solução

- b) Encontre os PMFs marginais de ***X*** e ***Y***.

Para a variável ***X***:

$$P_X(x) = \sum_{y_j \in R_Y} P_{XY}(x, y_j)$$

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

- PMF Marginal da variável ***X***:

X	0	1
$P_x(x)$	$\frac{13}{24}$	$\frac{11}{24}$

$$P_X(0) = P_{XY}(0,0) + P_{XY}(0,1) + P_{XY}(0,2) = \frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$$

$$P_X(1) = P_{XY}(1,0) + P_{XY}(1,1) + P_{XY}(1,2) = \frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$$

Exercício 4.1: Solução

- Para a variável ***Y***:

$$P_Y(y=0) = P_{XY}(0,0) + P_{XY}(1,0) = \frac{1}{6} + \frac{1}{8} = \frac{7}{24}$$

$$P_Y(y=1) = P_{XY}(1,0) + P_{XY}(1,1) = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

$$P_Y(y=2) = P_{XY}(2,0) + P_{XY}(2,1) = \frac{1}{8} + \frac{1}{6} = \frac{7}{24}$$

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

Observe que as PMFs marginais estão associadas a faixa dinâmica das variáveis.

$$P_Y(y) = \sum_{x_k \in R_X} P_{XY}(x_k, y)$$

Exercício 4.1: Solução

- c) Encontre a probabilidade condicional P(Y = 1|X = 0).

A probabilidade condicional de dois eventos A e B em um espaço amostral S pode ser calculada por meio de: $P(A|B) = \dfrac{P(A,B)}{P(B)}$

- Aplicando para P(Y = 1|X = 0), obtém-se:

$$P(Y=1|X=0) = \frac{P(X=0,Y=1)}{P(X=0)} = \frac{P_{XY}(0,1)}{P_X(0)} = \frac{\frac{1}{4}}{\frac{13}{24}} = \frac{6}{13}$$

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

- Joint PMF

X	0	1
$P_x(x)$	$\frac{13}{24}$	$\frac{11}{24}$

- Marginal PMF

Exercício 4.1: Solução

- d) Verifique se ***X*** e ***Y*** são independentes?
- Dois eventos *A* and *B* que fazem parte do espaço amostral *S* são independents se:

$$P(A \cap B) = \mathrm{P(A,B)} = P(A)P(B).$$

- O que leva a:

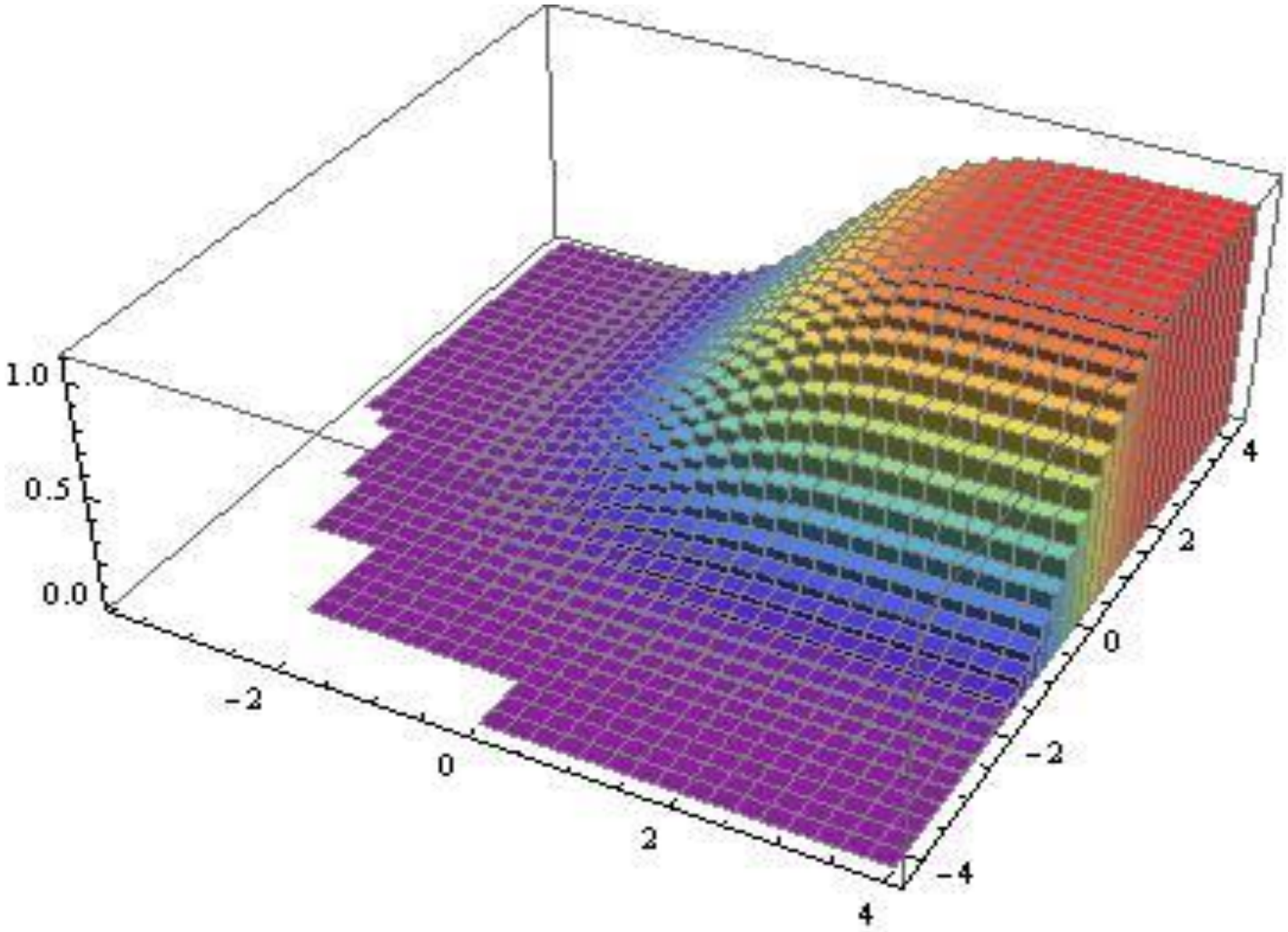
$$P(A|B) = \frac{P(A,B)}{P(B)} = \frac{P(A)P(\,B)}{P(B)} = \mathrm{P(A)}$$

- Podemos concluir que X e Y não são independentes, por exemplo:

$$P(Y=1|X=0) = \frac{6}{13} \neq P(Y=1) = \frac{5}{12}$$

Joint Cumulative Distributive Function (CDF)

- The shape of the discrete **joint cumulative distribution function** of two random variables ***X*** and ***Y*.**

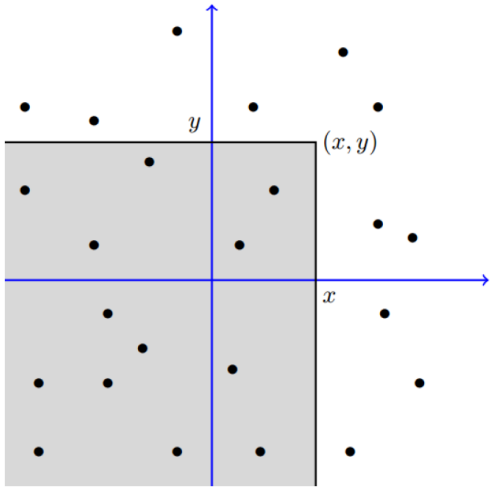


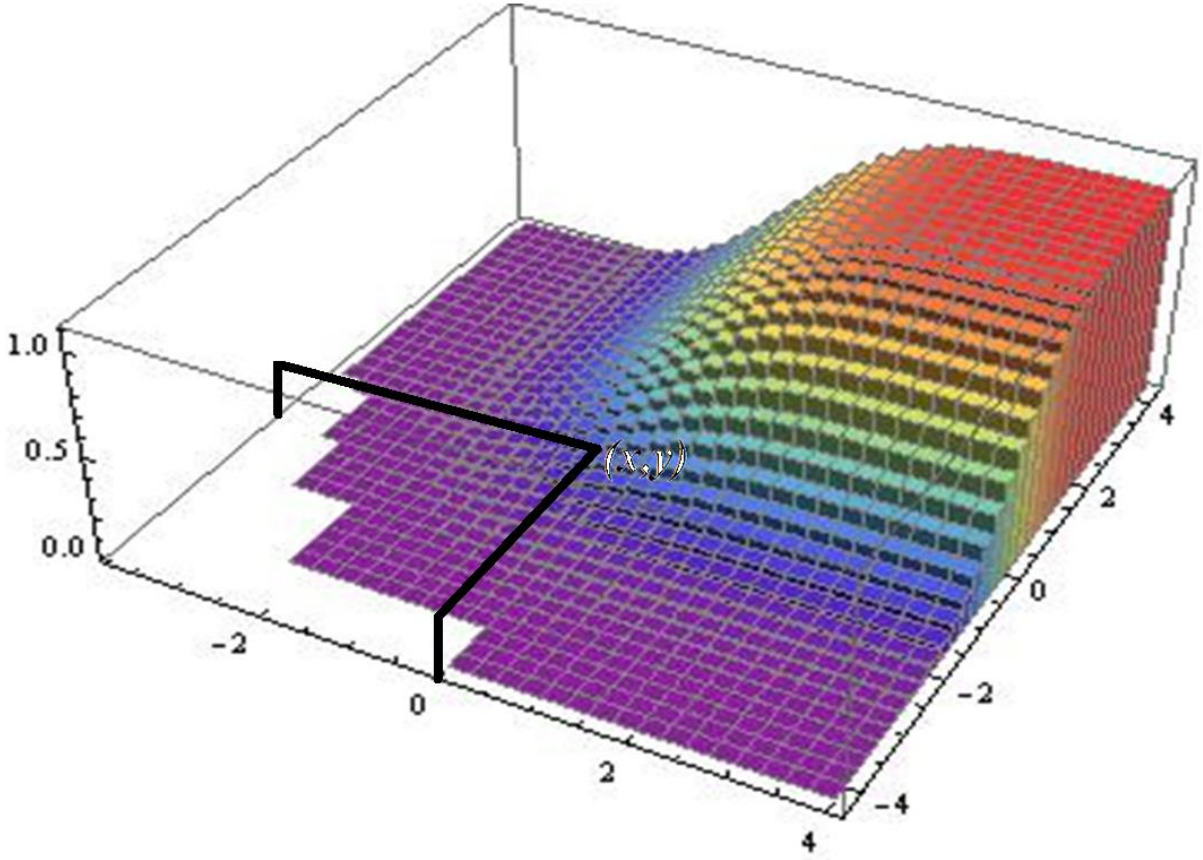
Joint Cumulative Distributive Function (CDF)

08/05/2024

- The **joint cumulative distribution function** of two random variables X and Y is defined as

$$F_{XY}(x,y) = P(X \leq x, Y \leq y) = P((X \leq x) \text{ and } (Y \leq y)) = P((X \leq x) \cap (Y \leq y))$$





The properties of joint distribution functions

$$0 \leq F_{X,Y}(x,y) \leq 1$$

$$x_1 \leq x_2 \text{ and } y_1 \leq y_2 \rightarrow F_{X,Y}(x_1,y_1) \leq F_{X,Y}(x_2,y_2)$$

$$\lim_{x \to a^+} F_{X,Y}(x,y) = F_{X,Y}(a,y)$$

$$\lim_{y \to b^+} F_{X,Y}(x,y) = F_{X,Y}(x,b)$$

$$F_{X,Y}(x,-\infty) = 0$$

$$F_{X,Y}(-\infty,y) = 0$$

$$F_{X,Y}(-\infty,-\infty) = 0$$

$$F_{X,Y}(\infty,\infty) = 1$$

Marginal CDFs of X and Y

- We can obtain the *marginal* CDFs, $F_X(x)$ and $F_Y(y)$

$$F_X(x) = F_{XY}(x, \infty) = \lim_{y \to \infty} F_{XY}(x, y), \quad \text{for any } x,$$

$$F_Y(y) = F_{XY}(\infty, y) = \lim_{x \to \infty} F_{XY}(x, y), \quad \text{for any } y$$

Exercício 4.2

- Para a PMF dada no exercício anterior, mostrada logo abaixo

	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

Exercício 4.2

- a) Determine a respectiva CDF.
- b) Determine the *marginal* CDFs, $F_X(x)$ and $F_Y(y)$.

Exercício 4.2: Solução

- a) Determine a respectiva CDF

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

$P_{XY}(x,y)$

$F_{XY}(x,y)$

	Y = 0	Y = 1	Y = 2
X = 0	$\frac{4}{24}$	$\frac{10}{24}$	$\frac{13}{24}$
X = 1	$\frac{7}{24}$	$\frac{17}{24}$	$\frac{24}{24}$

$$F_{XY}(x=1,y=1) = P(X \leq 1, Y \leq 1)$$
$$= \mathrm{P}(0,0)+\mathrm{P}(1,0)+\mathrm{P}(0,1)+\mathrm{P}(1,1)$$
$$= \frac{1}{6}+\frac{1}{8}+\frac{1}{4}+\frac{1}{6} = \frac{17}{24}$$

Exercício 4.2: Solução

- b) Determine the *marginal* CDFs, $F_X(x)$ e $F_Y(y)$

$$F_X(x) = F_{XY}(x, \infty) = \lim_{y \to \infty} F_{XY}(x, y), \quad \text{for any } x,$$

$$F_Y(y) = F_{XY}(\infty, y) = \lim_{x \to \infty} F_{XY}(x, y), \quad \text{for any } y$$

	Y = 2
X = 0	$\frac{13}{24}$
X = 1	$\frac{24}{24}$

Marginal CDF $F_X(x)$

	Y = 0	Y = 1	Y = 2
X = 1	$\frac{7}{24}$	$\frac{17}{24}$	$\frac{24}{24}$

Marginal CDF $F_Y(y)$

Exercício 4.3

- Considere um experimento realizado em laboratório, onde se tem duas variáveis randômicas sorteadas de forma conjunta. A variável ***X*** apresenta a faixa dinâmica ***X***={0,1} e a variável ***Y*** apresenta a respectiva faixa dinâmica ***Y***={1, 2, 3, 4, 5, 6}.
- Proceda conforme solicitado.

índice	X	Y
1	1	1
2	1	4
3	0	1
4	1	5
5	0	3
6	1	1
7	0	6
8	0	2
9	1	2
10	0	5
11	0	2
12	1	5
13	1	3
14	0	5
15	1	5
16	1	5
17	0	6
18	0	3
19	0	4
20	1	5
21	1	5
22	0	5
23	1	4
24	1	6

Exercício 4.3

- a) Construa a PMF conjunta que represente o experimento realizado.
- b) Determine a CDF conjunta.
- c) Encontre a PMF marginal $P_X(x)$ e $P_Y(y)$
- d) Encontre a CDF marginal $F_X(x)$ e $F_Y(y)$
- e) Verifique se $\boldsymbol{X}$ e $\boldsymbol{Y}$ são independentes.

Exercício 4.3: solução

- a) Construa a PMF conjunta que represente o experimento realizado.

X/Y - PMF	1	2	3	4	5	6
0	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{2}{24}$
1	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{6}{24}$	$\frac{1}{24}$

Exercício 4.3: solução

- b) Determine a CDF conjunta.

- A CDF conjunta poderia até ser construída a partir da tabela que relata os resultados dos experimentos.

X/Y	1	2	3	4	5	6
0	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{5}{24}$	$\frac{6}{24}$	$\frac{9}{24}$	$\frac{11}{24}$
1	$\frac{3}{24}$	$\frac{6}{24}$	$\frac{9}{24}$	$\frac{12}{24}$	$\frac{21}{24}$	$\frac{24}{24}$

$$F_{XY}(x=1, y=3) = P(X \leq 1, Y \leq 3) =$$

$$\mathrm{P}(0,1)+(1,1)+\mathrm{P}(0,2)+\mathrm{P}(1,2)+\mathrm{P}(0,3)+\mathrm{P}(1,3)=\frac{1}{24}+\frac{2}{24}+\frac{2}{24}+\frac{1}{24}+\frac{2}{24}+\frac{1}{24}=\frac{9}{24}$$

Exercício 4.3: solução

- c) Encontre a PMF marginal $P_X(x)$ e $P_Y(y)$.

  Para $P_X(x)$ temos:

$$P_X(x) = \sum_{y_j \in R_Y} P_{XY}(x, y_j), \text{for any } x \in R_X$$

$$P_X(0) = PXY(0, 1) + P_{XY}(0, 2) + P_{XY}(0, 3) + P_{XY}(0, 4) + P_{XY}(0, 5) + P_{XY}(0, 6) = 0.45833$$

$$P_X(1) = PXY(1, 1) + P_{XY}(1, 2) + P_{XY}(1, 3) + P_{XY}(1, 4) + P_{XY}(1, 5) + P_{XY}(1, 6) = 0.54167$$

X	$P_X(x)$
0	0.45833
1	0.54167

Exercício 4.3: solução

- (Solução 2) Outra forma de encontrar a **PMF marginal** $P_X(x)$: **Elimine a variável Y na tabela original** e calcule $P_X(x)$.

$$P_X(x=0) = \frac{11}{24} = 0.45833$$

$$P_X(x=1) = \frac{13}{24} = 0.54167$$

índice	X
1	1
2	1
3	0
4	1
5	0
6	1
7	0
8	0
9	1
10	0
11	0
12	1
13	1
14	0
15	1
16	1
17	0
18	0
19	0
20	1
21	1
22	0
23	1
24	1

Exercício 4.3: solução

- A PMF marginal $P_{\boldsymbol{Y}}(y)$ pode ser computada na forma:

$$P_Y(y) = \sum_{x_k \in R_X} P_{XY}(x_k, y), \text{for any } y \in R_Y$$

$P_Y(y=1) = P_{XY}(0, 1) + P_{XY}(1, 1) = 0.12500$

$P_Y(y=2) = P_{XY}(0, 2) + P_{XY}(1, 2) = 0.12500$

$P_Y(y=3) = P_{XY}(0, 3) + P_{XY}(1, 3) = 0.12500$

$P_Y(y=4) = P_{XY}(0, 4) + P_{XY}(1, 4) = 0.12500$

$P_Y(y=5) = P_{XY}(0, 5) + P_{XY}(1, 5) = 0.37500$

$P_Y(y=6) = P_{XY}(0, 6) + P_{XY}(1, 6) = 0.12500$

Y	$P_{\mathbf{Y}}(y)$
1	0.125
2	0.125
3	0.125
4	0.125
5	0.375
6	0.125

Exercício 4.3: solução

- (Solução 2) Outra forma de encontrar a PMF marginal $P_Y(y)$: **Elimine a variável X na tabela original** e calcule $P_Y(y)$

índice	Y
1	1
2	4
3	1
4	5
5	3
6	1
7	6
8	2
9	2
10	5
11	2
12	5
13	3
14	5
15	5
16	5
17	6
18	3
19	4
20	5
21	5
22	5
23	4
24	6

Y	$P_Y(y)$
1	$\frac{3}{24}$
2	$\frac{3}{24}$
3	$\frac{3}{24}$
4	$\frac{3}{24}$
5	$\frac{9}{24}$
6	$\frac{3}{24}$

Exercício 4.3: solução

- d) Encontre a CDF marginal $F_X(x)$ e $F_Y(y)$

$$F_X(x) = F_{XY}(x, \infty) = \lim_{y \to \infty} F_{XY}(x, y), \quad \text{for any } x,$$

$$F_Y(y) = F_{XY}(\infty, y) = \lim_{x \to \infty} F_{XY}(x, y), \quad \text{for any } y$$

$$F_X(x) = F_{XY}(x, 6)$$

$$F_Y(y) = F_{XY}(1, y)$$

X	$F_X(x)$
0	$\frac{11}{24}$
1	$\frac{24}{24}$

Y	$F_Y(y)$
1	$\frac{3}{24}$
2	$\frac{6}{24}$
3	$\frac{9}{24}$
4	$\frac{12}{24}$
5	$\frac{21}{24}$
6	$\frac{24}{24}$

Exercício 4.3: Solução

- e) Verifique se $\boldsymbol{X}$ e $\boldsymbol{Y}$ são independentes?
- Dois eventos A and B que fazem parte do espaço amostral S são independents se:

$$P(X \cap Y) = \mathrm{P(X,Y)} = P(X)P(Y).$$

- O que leva a: $$P(Y|X) = \frac{P(X,Y)}{P(X)} = \frac{P(Y)P(X)}{P(X)} = \mathrm{P(Y)}$$

- Podemos concluir que X e Y não são independentes, por exemplo:

$$P(Y=1|X=0) = \frac{P(X=0,Y=1)}{P(X=0)} = \frac{\frac{1}{24}}{\frac{11}{24}} = \frac{1}{11} \neq P(Y=1) = \frac{3}{24} = \frac{1}{8}$$

Exercício 4.3: Solução

- Novamente se verifica que X e Y não são independentes:

$$P(Y=5|X=0)=\frac{P(X=0,Y=5)}{P(X=0)}=\frac{\frac{9}{24}}{\frac{11}{24}}=\frac{9}{11}\neq P(Y=5)=\frac{9}{24}=\frac{3}{8}$$

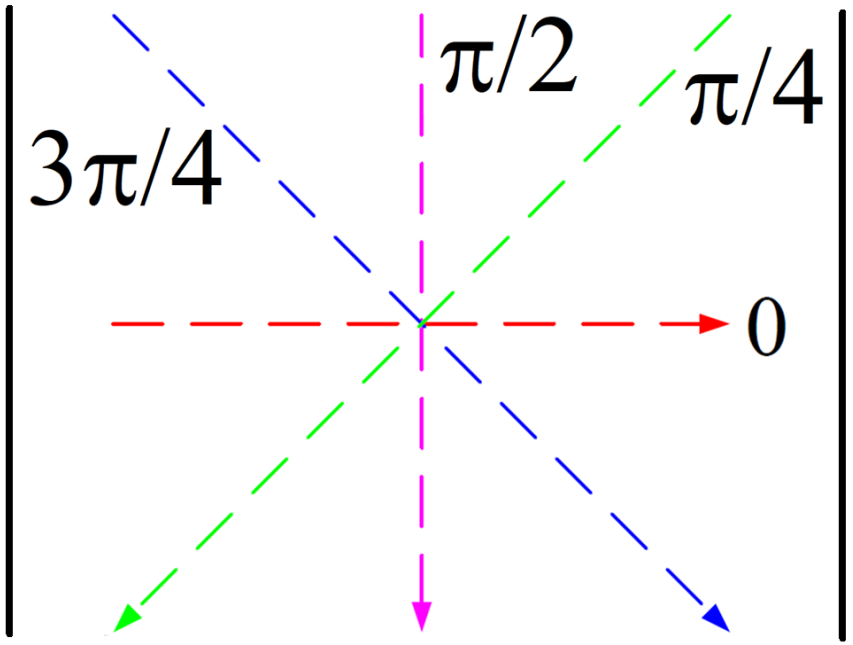
Example: Gray Level Co-occurrence Matrices (GLCM)

- Haralick, RM.; Shanmugam, K., “Textural features for image classification” IEEE Transactions on systems, man, and cybernetics 6 (1973): 610-621. DOI:10.1109/TSMC.1973.4309314.
- It is used to extract textures features in images.
- To build GLCM it is considered two pixels at a time, called the **reference** and the **neighbour** pixel.
- It is defined a particular spatial relationship between the reference and neighbour (pixel distance) pixel before calculating the GLCM.

Gray Level Co-occurrence Matrices (GLCM)

- GLCM measures a histogram of the repetition of pixel values in a given direction and considering a given neighborhood.
- Four-way Gray Level Co-occurrence Matrices are calculated.

1. Direction: 0 radians - scan over GLCM lines from left to right.
2. Direction: $\pi/4$ radians - scan over GLCM secondary diagonal from top to bottom and from right to left.
3. Direction: $\pi/2$ radians - scan over GLCM columns from top to bottom.
4. Direction: $3\pi/4$ radians - scan over GLCM primary diagonal from top to bottom and from left to right.



Example 1: Gray Level Co-occurrence Matrix (GLCM)

- Direction: $3\pi/4$ rad

neighbour pixel value

reference pixel value	0	1	2	3
0	0	1	0	1
1	0	1	1	0
2	0	1	0	0
3	0	0	1	3

- Direction: $\pi/2$ rad

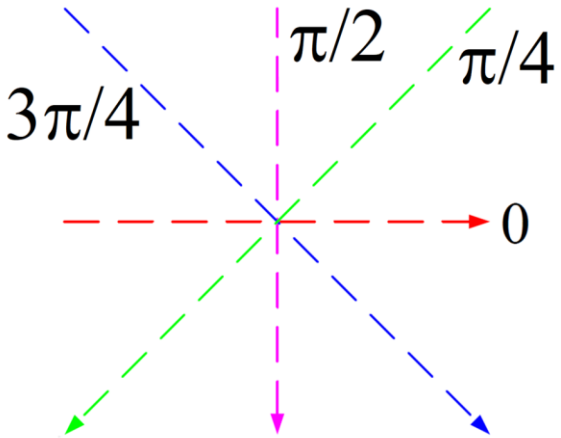
neighbour pixel value

reference pixel value	0	1	2	3
0	0	0	0	2
1	0	0	0	2
2	0	1	0	2
3	0	3	2	0

- Direction: $\pi/4$ rad

neighbour pixel value

reference pixel value	0	1	2	3
0	0	0	0	1
1	0	0	1	0
2	0	2	0	1
3	0	2	0	2



- Image pixels

0	**0**	**3**	**2**
3	**3**	**1**	**3**
1	**2**	**3**	**2**
3	**1**	**1**	**3**

- Direction: 0 rad

neighbour pixel value

reference pixel value	0	1	2	3
0	1	0	0	1
1	0	1	1	2
2	0	0	0	1
3	0	2	2	1

Example 1: Gray Level Co-occurrence Matrices (GLCM)

- PMF Conjunta P[n, m]: $3\pi/4$ rad

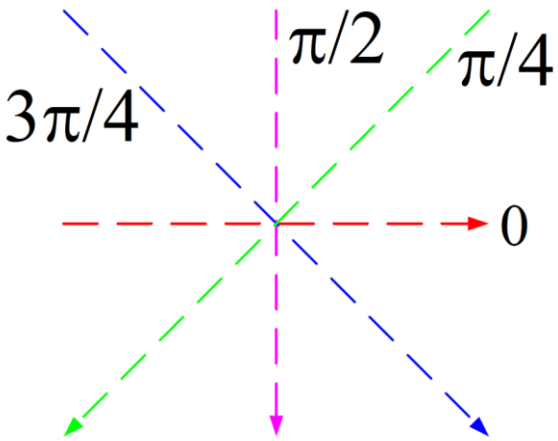
0.000	0.111	0.000	0.111
0.000	0.111	0.111	0.000
0.000	0.111	0.000	0.000
0.000	0.000	0.111	0.333

- PMF Marginal P[n]

0.222 0.222 0.111 0.444

- PMF Marginal P[m]

0.000 0.333 0.222 0.444



- PMF Conjunta P[n, m]: $\pi/2$ rad

0.000	0.000	0.000	0.167
0.000	0.000	0.000	0.167
0.000	0.083	0.000	0.167
0.000	0.250	0.167	0.000

- PMF Marginal P[n]

0.167 0.167 0.250 0.417

- PMF Marginal P[m]

0.000 0.333 0.167 0.500

- Image pixels

0	**0**	**3**	**2**
3	**3**	**1**	**3**
1	**2**	**3**	**2**
3	**1**	**1**	**3**

- PMF Conjunta P[n, m]: $\pi/4$ rad.

0.000	0.000	0.000	0.111
0.000	0.000	0.111	0.000
0.000	0.222	0.000	0.111
0.000	0.222	0.000	0.222

- PMF Marginal P[n]

0.111 0.111 0.333 0.444

- PMF Marginal P[m]

0.000 0.444 0.111 0.444

- PMF Conjunta P[n, m]: 0 rad.

0.083	0.000	0.000	0.083
0.000	0.083	0.083	0.000
0.000	0.000	0.000	0.083
0.000	0.167	0.167	0.083

- PMF Marginal P[n]

0.167 0.333 0.083 0.417

- PMF Marginal P[m]

0.083 0.250 0.250 0.417

Example 2: GLCM

neighbour pixel value

reference pixel value

	0	1	2	3
0	6	3	2	1
1	3	1	6	1
2	3	1	10	3
3	1	1	3	4

neighbour pixel value

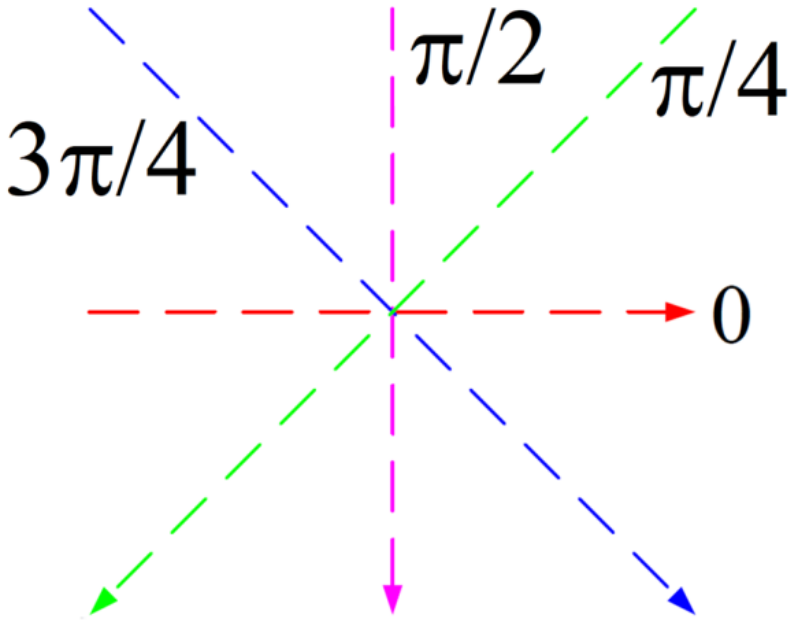
reference pixel value

	0	1	2	3
0	6	3	4	2
1	2	8	2	0
2	2	1	12	4
3	4	0	3	3

neighbour pixel value

reference pixel value

	0	1	2	3
0	3	3	5	3
1	3	1	1	1
2	3	6	8	2
3	3	1	4	2



- Image pixels

1 0 0 1 2 3 3 0
0 0 1 1 3 3 2 2
1 0 0 1 2 3 3 2
1 2 2 2 2 0 0 3
1 2 2 2 2 3 0 0
1 2 2 2 2 3 3 0
1 0 0 1 2 0 2 1
1 0 0 1 2 2 3 2

neighbour pixel value

reference pixel value

	0	1	2	3
0	7	5	1	1
1	4	1	7	1
2	2	1	11	5
3	3	0	3	4

PMF $P_{xy}(x,y)$ – $3\pi/4$ rad.

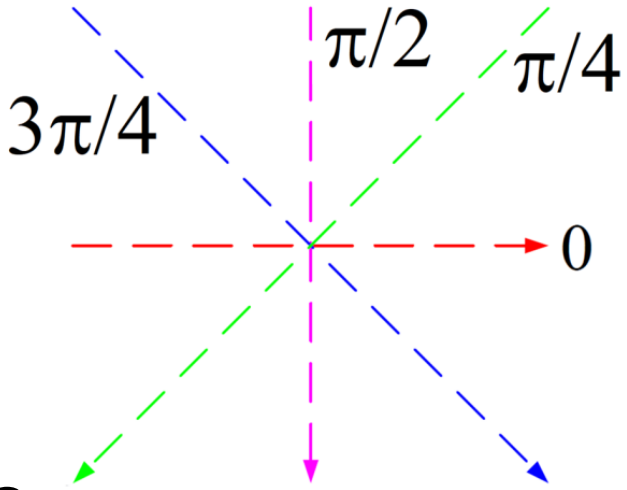
0.061	0.061	0.102	0.061
0.061	0.020	0.020	0.020
0.061	0.122	0.163	0.041
0.061	0.020	0.082	0.041

PMF $P_{xy}(x,y)$ – $\pi/2$ rad.

0.107	0.054	0.071	0.036
0.036	0.143	0.036	0.000
0.036	0.018	0.214	0.071
0.071	0.000	0.054	0.054

PMF $P_{xy}(x,y)$ – $\pi/4$ rad.

0.061	0.061	0.102	0.061
0.061	0.020	0.020	0.020
0.061	0.122	0.163	0.041
0.061	0.020	0.082	0.041



Example 2: GLCM

1	0	0	1	2	3	3	0
0	0	1	1	3	3	2	2
1	0	0	1	2	3	3	2
1	2	2	2	2	0	0	3
1	2	2	2	2	3	0	0
1	2	2	2	2	3	3	0
1	0	0	1	2	0	2	1
1	0	0	1	2	2	3	2

PMF $P_{xy}(x,y)$ – 0 rad.

0.125	0.089	0.018	0.018
0.071	0.018	0.125	0.018
0.036	0.018	0.196	0.089
0.054	0.000	0.054	0.071

Example: Gray Level Co-occurrence Matrix (GLCM) - Features computed

- 1. Autocorrelation.
- 2. Joint Mean.
- 3 Cluster Prominence.
- 4. Cluster Shade.
- 5. Cluster tendency.
- 6. Contrast.
- 7. Correlation.
- 8. Difference Average.
- 9. Difference Entropy.
- 10. Difference Variance.
- 11. Joint Energy.
- 12. Joint Entropy.
- 13. Informational Measure of Correlation (IMC) 1.
- 14. Informational Measure of Correlation (IMC) 2.
- 15. Inverse Difference Moment.
- 16. Maximal Correlation Coefficient.
- 17. Eigenvalues Energy Ratio.
- 18. Inverse Difference Moment Normalized.
- 19. Inverse Difference.
- 20. Inverse Difference Normalized.
- 21. Inverse Variance.
- 22. Maximum Probability.
- 23. Sum Average.
- 24. Sum Entropy.
- 25. Cluster Prominence.
- 26. Skewness.
- 27. Kurtosis.

Example: Gray Level Co-occurrence Matrix (GLCM) – Values of computed features

- [8.25000e+00 9.11100e+00 8.50000e+00 9.11100e+00 2.75000e+00 2.77800e+00 2.91700e+00 3.11100e+00 8.59790e+01 2.38120e+01 7.41700e+00 1.47167e+02 3.48000e+01 8.65400e+00 5.13210e+01 1.74420e+01 6.21700e+00 8.13960e+01 2.20230e+01 6.49500e+00 2.41700e+00 1.44400e+00 3.58300e+00 2.33300e+00 -2.59140e+01 -2.49720e+01 -3.01450e+01 -3.49220e+01 2.25000e+00 1.77800e+00 2.75000e+00 2.22200e+00 1.85500e+00 1.39200e+00 1.48300e+00 1.83700e+00 5.91700e+00 4.00000e+00 8.08300e+00 5.77800e+00 1.25000e-01 1.85000e-01 1.81000e-01 1.85000e-01 3.08500e+00 2.64200e+00 2.52200e+00 2.50300e+00 -1.45000e-01 8.06000e-01 -2.15000e-01 8.75000e-01 -2.46000e-01 8.99000e-01 -2.04000e-01 8.50000e-01 2.30000e-01 3.18000e-01 1.35000e-01 2.29000e-01 7.57000e-01 8.29000e-01 6.83000e-01 7.62000e-01 3.36000e-01 3.93000e-01 2.76000e-01 3.37000e-01 6.54000e-01 7.07000e-01 5.99000e-01 6.57000e-01 3.76000e-01 5.62000e-01 1.61000e-01 3.65000e-01 1.67000e-01 3.33000e-01 2.50000e-01 2.22000e-01 5.75000e+00 5.88900e+00 6.08300e+00 6.11100e+00 2.35500e+00 2.11300e+00 1.55500e+00 1.83700e+00 1.91700e+00 -3.68700e+00 8.22900e+00 2.11100e+00 -4.52500e+00 1.08070e+01 1.87000e+00 -4.22000e+00 1.04530e+01 1.55000e+00 -3.53000e+00 8.96200e+00 5.62000e-01 2.87700e+00 8.74000e-01 3.88100e+00 1.58900e+00 3.17800e+00 1.82000e+00 3.28700e+00]

Conditional PMF

- We saw in module 1 that the conditional probability of event **A**, given the event **B**, can be calculated using:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A,B)}{P(B)}$$

- For a discrete random variable ***X*** and event, ***A***, the **conditional PMF** of ***X*** given ***A*** is defined as

$$P_{X|A}(x_i) = \frac{P\big((X = x_i) \cap A\big)}{P(A)} = \frac{P\big((X = x_i), A\big)}{P(A)}; \; x_i \in R_X$$

Conditional CDF

- Similarly, we define the conditional CDF of X given an event A as

$$F_{\boldsymbol{X}|A}(x) = P(X \leq x|A)$$

Exercício 4.4

- Em um experimento randômico equiprovável, tem-se uma caixa com 10 bolas numeradas de 0 (zero) a 9 (nove).
- Sequencialmente são sorteadas três bolas. Depois de cada sorteio a bola é reposta no caixa.
- Considere como uma variável randômica os dígitos produzidos pelos sorteio das três bolas, produzindo o evento $A=\{X < 400\}$ e o evento $B=\{750 \leq X \leq 999\}$.
- a) Determine a PMF de ***X*** dado que o evento ***A*** ocorreu.
- b) Determine a CDF de ***X*** dado que o evento ***A*** ocorreu.
- c) Repita os procedimentos dos itens (a) e (b) quando o evento **B** ocorre.

Exercício 4.4: solução

- a) Determine a PMF do evento A.

$$P_{\boldsymbol{X}|A}(x_i) = \frac{P\big((X = x_i) \cap A\big)}{P(A)}$$

$$P(A) = \frac{|A|}{|S|} = \frac{400}{1000} = \frac{2}{5}$$

$$P\big((X = x_i) \cap A\big) = \frac{|\big((X = x_i) \cap A\big)|}{|S|} = \frac{1}{1000}, \qquad 0 \leq x_i \leq 399$$

$$P\big((X = x_i) \cap A\big) = 0, \qquad 400 \leq x_i \leq 999$$

Exercício 4.4: solução

- Portanto:

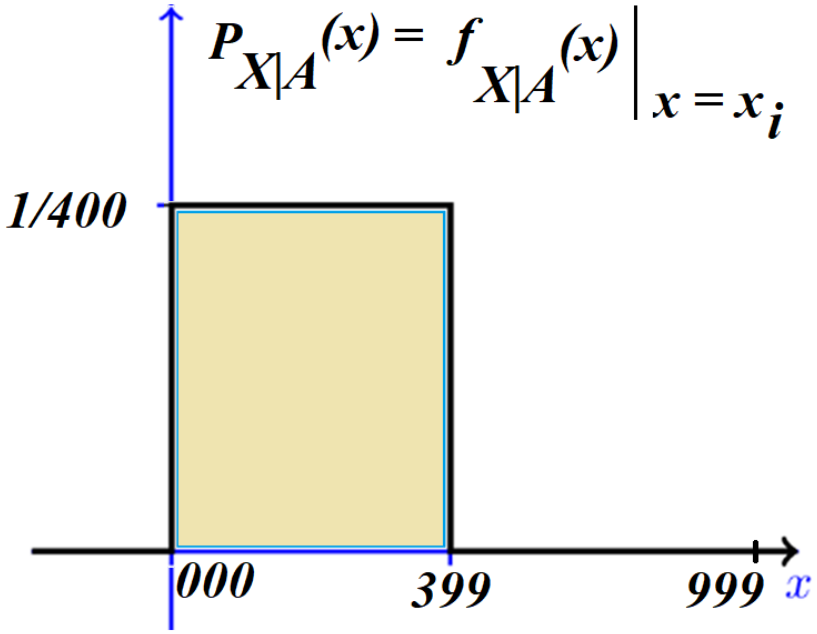
$$P_{\boldsymbol{X}|A}(x_i) = \frac{P\big((X = x_i) \cap A\big)}{P(A)} = \frac{\frac{1}{1000}}{\frac{2}{5}} = \frac{5}{2000} = \frac{1}{400}, 0 \leq x_i \leq 399$$

$$P_{\boldsymbol{X}|A}(x_i) = 0, \qquad 400 \leq x_i \leq 999$$

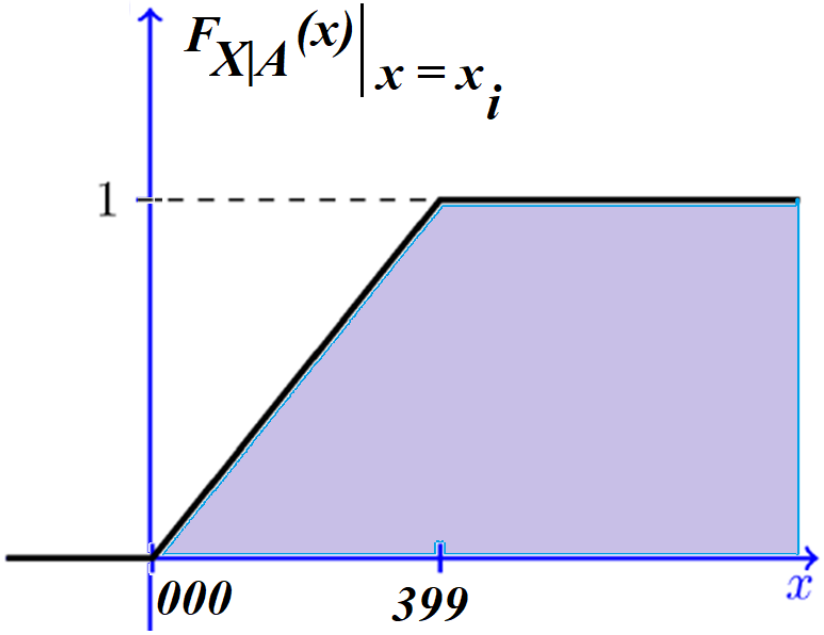
Exercício 4.4: solução

- b) Determine a CDF de $\boldsymbol{X}$ dado que o evento $\boldsymbol{A}$ ocorreu.

$$F_{\boldsymbol{X}|A}(x) = P(X \leq x|A)$$



- PMF condicional de X dado A.



- CDF condicional de X dado A.

The **conditional PMFs** of ***X*** given ***Y*** and ***Y*** given ***X***

- For discrete random variables ***X*** and ***Y***, the **conditional PMFs** of ***X*** given ***Y*** and vice versa are defined as

$$P_{X|Y}(x_i|y_j) = \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)},$$

$$P_{Y|X}(y_j|x_i) = \frac{P_{XY}(x_i, y_j)}{P_X(x_i)}$$

for any $x_i \in R_X$ and $y_j \in R_Y$.

Exercício 4.5

- Tendo como base as grandezas encontradas no exercício 4.3.
- Determine:

$a)\ P_{X|Y}\left(x_i = 0 \middle| y_j = 3\right)$

$b)\ P_{X|Y}\left(x_i = 1 \middle| y_j = 3\right)$

$c) P_{X|Y}\left(x_i = 0 \middle| 4 \leq y_j \leq 5\right)$

$d)\ P_{X|Y}\left(x_i \leq 1 \middle| 4 \leq y_j \leq 5\right)$

$e)\ P_{Y|X}\left(y_j = 5 \middle| x_i \leq 1\right)$

f) $P_{Y|X}\left(y_j \leq 3 \middle| x_i = 0\right)$

g) $P_{Y|X}\left(1 < y_j \leq 4 \middle| x_i = 1\right)$

Exercício 4.5: solução

- Reapresentando os resultados do exercício 4.3:
- $P_{XY}(x, y)$ – PMF conjunta

X/Y	1	2	3	4	5	6
0	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{2}{24}$
1	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{6}{24}$	$\frac{1}{24}$

- $P_X(x)$ – PMF Marginal da variável X

X	$P_X(x)$
0	$\frac{11}{24}$
1	$\frac{13}{24}$

- $P_Y(y)$ – PMF Marginal da variável Y

Y	1	2	3	4	5	6
$P_Y(y)$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{9}{24}$	$\frac{3}{24}$

Exercício 4.5: solução

- Usando a expressão da PMF condicional de ***X*** dado ***Y=3*** obtém-se:

$$a)\ P_{X|Y}\left(x_i = 0 \middle| y_j = 3\right) = \frac{P(x_i \cap y_j)}{P(y_j)} = \frac{P(x_i = 0_i, y_j = 3)}{P(y_j = 3)} = \frac{\frac{2}{24}}{\frac{3}{24}} = \frac{2}{3}$$

$$b)\ P_{X|Y}\left(x_i = 1 \middle| y_j = 3\right) = \frac{P(x_i = 1, y_j = 3)}{P(y_j = 3)}$$

$$= \frac{\frac{1}{24}}{\frac{3}{24}} = \frac{1}{3}$$

X|Y=3	$\mathbf{P_{X\|Y}(x)}$	$\mathbf{F_{X\|Y}(x)}$
0	$\frac{16}{24}$	$\frac{16}{24}$
1	$\frac{8}{24}$	$\frac{24}{24}$

- PMF/CDF condicional

Exercício 4.5: solução

- (Solução 2) Interpretando a partir da $P_{XY}(x, y)$ – PMF conjunta

$a)\ P_{X|Y}(x_i = 0|y_j = 3) = \frac{2}{3}$

$b)\ P_{X|Y}(x_i = 1|y_j = 3) = \frac{1}{3}$

X/Y	1	2	3	4	5	6
0	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{2}{24}$
1	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{6}{24}$	$\frac{1}{24}$

Exercício 4.5: solução

08/05/2024

- (Solução 3) Olhando para a tabela com os resultados do experimento randômicos.
- Considerando o evento $\boldsymbol{A}$ como a ocorrência de $y_i = 3$:

$$a)\ P_{X|Y}\left(x_i = 0 \middle| y_j = 3\right) = P(x_i = 0|\mathrm{A}) = \frac{|x_i = 0|}{|A|} = \frac{2}{3}$$

$$b)\ P_{X|Y}\left(x_i = 1 \middle| y_j = 3\right) = P(x_i = 1|A) = \frac{|x_i = 1|}{|A|} = \frac{1}{3}$$

índice	X	Y
1	1	1
2	1	4
3	0	1
4	1	5
5	0	3
6	1	1
7	0	6
8	0	2
9	1	2
10	0	5
11	0	2
12	1	5
13	1	3
14	0	5
15	1	5
16	1	5
17	0	6
18	0	3
19	0	4
20	1	5
21	1	5
22	0	5
23	1	4
24	1	6

Exercício 4.5: solução

08/05/2024

- $P_{XY}(x, y)$ – PMF conjunta

X/Y	1	2	3	4	5	6
0	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{2}{24}$
1	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{6}{24}$	$\frac{1}{24}$

$$c) P_{X|Y}\left(x_i = 0 \middle| 4 \leq y_j \leq 5\right)$$

$$= \frac{P(x_i = 0, 4 \leq y_j \leq 5)}{P(4 \leq y_j \leq 5)}$$

$$= \frac{\frac{1}{24} + \frac{3}{24}}{\frac{3}{24} + \frac{9}{24}} = \frac{4}{12} = \frac{1}{3}$$

- $P_Y(y)$ – PMF Marginal da variável Y

Y	1	2	3	4	5	6
$\mathbf{P_Y(y)}$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{9}{24}$	$\frac{3}{24}$

Exercício 4.5: solução

08/05/2024

- (Solução 2) Olhando para a tabela com os resultados do experimento randômicos.
- Considerando o evento ***A*** como a ocorrência da variável Y tal que $4 \leq y_i \leq 5$:

$$a)\ P_{X|Y}\left(x_i = 0 \middle| 4 \leq y_j \leq 5\right) = P(x_i = 0|\mathrm{A}) = \frac{|x_i=0|}{|A|} = \frac{4}{12} = \frac{1}{3}$$

Observe que também é imediato do cálculo de $P_{X|Y}\left(x_i = 1 \middle| 4 \leq y_j \leq 5\right)$:

$$P_{X|Y}\left(x_i = 1 \middle| 4 \leq y_j \leq 5\right) = 1 - P(x_i = 0|\mathrm{A}) = \frac{|x_i=1|}{|A|} = \frac{8}{12} = \frac{2}{3}$$

índice	X	Y
1	1	1
2	1	4
3	0	1
4	1	5
5	0	3
6	1	1
7	0	6
8	0	2
9	1	2
10	0	5
11	0	2
12	1	5
13	1	3
14	0	5
15	1	5
16	1	5
17	0	6
18	0	3
19	0	4
20	1	5
21	1	5
22	0	5
23	1	4
24	1	6

Exercício 4.5: solução

- $P_{XY}(x, y)$ – PMF conjunta

X/Y	1	2	3	4	5	6
0	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{2}{24}$
1	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{6}{24}$	$\frac{1}{24}$

$d)\ P_{X|Y}\left(x_i \leq 1 \middle| 4 \leq y_j \leq 5\right)$

$$= \frac{P(x_i \leq 1_i, 4 \leq y_j \leq 5)}{P(4 \leq y_j \leq 5)}$$

$$= \frac{\frac{1}{24} + \frac{2}{24} + \frac{3}{24} + \frac{6}{24}}{\frac{3}{24} + \frac{9}{24}}$$

$$= \frac{12}{12} = 1$$

- $P_Y(y)$ – PMF Marginal da variável Y

Y	1	2	3	4	5	6
$P_Y(y)$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{9}{24}$	$\frac{3}{24}$

Exercício 4.5: solução

- $P_{XY}(x, y)$ – PMF conjunta

X/Y	1	2	3	4	5	6
0	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{2}{24}$
1	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{6}{24}$	$\frac{1}{24}$

- $P_X(x)$ – PMF Marginal da variável X

X	$P_X(x)$
0	$\frac{11}{24}$
1	$\frac{13}{24}$

$$e)\ P_{Y|X}\left(y_j = 5 \middle| x_i \leq 1\right) = \frac{P(y_i \cap\ x_j)}{P(x_j)} = \frac{P(y_i, x_j)}{P(x_j)}$$

$$= \frac{P(y_j = 5, x_i \leq 1)}{P(x_i \leq 1)} = \frac{\frac{3}{24} + \frac{6}{24}}{\frac{11}{24} + \frac{13}{24}} = \frac{9}{24} = \frac{3}{8}$$

Independent Random Variables

- Two discrete random variables ***X*** and ***Y*** are independent if

$$P_{XY}(x,y) = P_X(x)P_Y(y), \qquad \text{for all } x, y.$$

- Equivalently, ***X*** and ***Y*** are independent if

$$F_{XY}(x,y) = F_X(x)F_Y(y), \qquad \text{for all } x, y.$$

Exercício 4.6

- Considere agora o experimento randômico no qual se joga uma moeda e depois se joga um dado. Tanto a moeda quanto o dados apresentam probabilidades equiprováveis.
- a) Construa a PMF conjunta que represente o experimento realizado.
- b) Determine a CDF conjunta.
- c) Encontre a PMF marginal $P_X(x)$ e $Py(y)$
- d) Encontre a CDF marginal $F_X(x)$ e $F_Y(y)$
- e) Verifique se $\boldsymbol{X}$ e $\boldsymbol{Y}$ são independentes.

Exercício 4.6: solução

a) Construa a PMF conjunta que represente o experimento realizado.

b) Determine a CDF conjunta.

- PMF conjunta

X/Y	1	2	3	4	5	6
0	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{2}{24}$
1	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{2}{24}$

- CDF conjunta

X/Y	1	2	3	4	5	6
0	$\frac{2}{24}$	$\frac{4}{24}$	$\frac{6}{24}$	$\frac{8}{24}$	$\frac{10}{24}$	$\frac{12}{24}$
1	$\frac{4}{24}$	$\frac{8}{24}$	$\frac{12}{24}$	$\frac{16}{24}$	$\frac{20}{24}$	$\frac{24}{24}$

Exercício 4.6: solução

- c) Encontre a PMF marginal $P_X(x)$ e $Py(y)$

- PMF marginal $P_X(x)$

X	$P_X(x)$
0	$\frac{12}{24}$
1	$\frac{12}{24}$

- PMF marginal $Py\ (y)$

Y	$P_Y(y)$
1	$\frac{4}{24}$
2	$\frac{4}{24}$
3	$\frac{4}{24}$
4	$\frac{4}{24}$
5	$\frac{4}{24}$
6	$\frac{4}{24}$

- d) Encontre a CDF marginal $F_X(x)$ e $F_Y\ (y)$

- CDF marginal $P_X(x)$

X	$F_X(x)$
0	$\frac{12}{24}$
1	$\frac{24}{24}$

- CDF marginal $Py(y)$

Y	$F_Y(y)$
1	$\frac{4}{24}$
2	$\frac{8}{24}$
3	$\frac{12}{24}$
4	$\frac{16}{24}$
5	$\frac{20}{24}$
6	$\frac{24}{24}$

Exercício 4.6: Solução

- e) Verifique se ***X*** e ***Y*** são independentes?
- Dois eventos A and B que fazem parte do espaço amostral S são independents se:

$$P(X \cap Y) = \mathrm{P(X,Y)} = P(X)P(Y).$$

- O que leva a: $P(Y|X) = \frac{P(X,Y)}{P(X)} = \frac{P(Y)P(X)}{P(X)} = \mathrm{P(Y)}$
- Podemos concluir que **X e Y são independentes**, por exemplo:

$$P(Y=1|X=0) = \frac{P(X=0,Y=1)}{P(X=0)} = \frac{\frac{2}{24}}{\frac{12}{24}} = \frac{2}{12} = \frac{1}{6} = P(Y=1) = \frac{4}{24} = \frac{1}{6}$$

Conditional Expectation

- If we know that event A has occurred, we can compute the conditional expectation of a random variable X, $E[X|A]$

$$E[X|A] = \sum_{x_i \in R_X} x_i P_{X|A}(x_i)$$

- Similarly, given that we have observed the value of random variable $\boldsymbol{Y}$, we can compute the conditional expectation of $\boldsymbol{X}$.

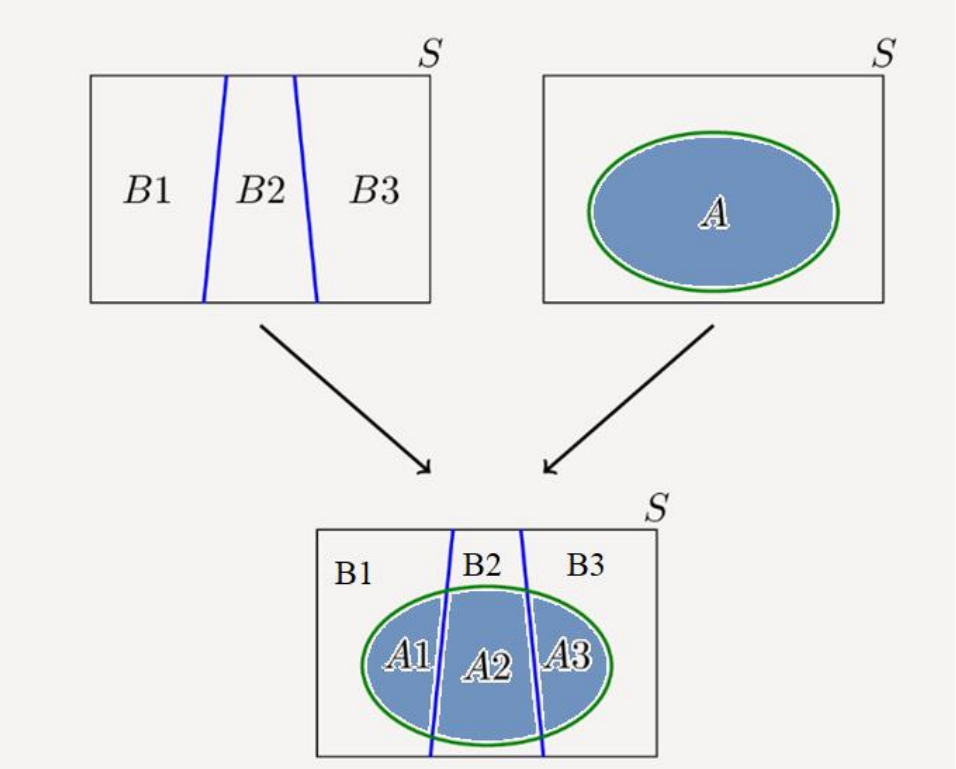
$$E[X|Y = y] = \sum_{x_i \in R_X} x_i P_{X|Y}(x_i|y)$$

Law of Total Probability

- Remember that: If B_1, B_2, B_3,$\cdots$ are a **partition** of the sample space S, then for any event A we have

$$P(A) = \sum_k P(A_k)$$

$$= \sum_k P(A, B_k) = \sum_k P(A|B_k)P(B_k)$$



$$P(X \in A) = \sum_{y_j \in R_Y} P(X \in A|Y = y_j)P_Y(y_j), \quad \text{for any set } A.$$

Law of Total Expectation

1. If $B_1, B_2, B_3, \ldots$ is a partition of the sample space S,

$$E[X] = \sum_i E[X|B_i]P(B_i)$$

2. For a random variable X and a discrete random variable Y,

$$E[X] = \sum_{y_j \in R_Y} E[X|Y = y_j]P_Y(y_j)$$

Functions of Two Random Variables

- Law of the unconscious statistician (LOTUS) for two discrete random variables

$$E[g(X,Y)] = \sum_{(x_i,y_j)\in R_{XY}} g(x_i,y_j)P_{XY}(x_i,y_j)$$

Conditional Expectation as a Function of a Random Variable

- Remember that the conditional expectation of X given that $Y = y$ is given by

$$E[X|Y=y] = \sum_{x_i \in R_X} x_i P_{X|Y}(x_i|y).$$

- Note that $\boldsymbol{E[X|Y=y]}$ depends on the value of y. In other words, by changing y, $\boldsymbol{E[X|Y=y]}$ can also change.

- Thus, we can say $E[X|Y=y]$ is a **function** of y, so let's write

$$g(y) = E[X|Y=y].$$

Conditional Expectation as a Function of a Random Variable

- Thus, we can think of $\boldsymbol{g(y) = E[X|Y = y]}$ as a function of the value of random variable Y . We then write

$$g(Y) = E[X|Y].$$

Conditional Expectation as a Function of a Random Variable

- We use this notation to indicate that $E[X|Y]$ is a **random variable** whose value equals $g(y) = E[X|Y = y]$ when $Y = y$.
- Thus, if Y is a random variable with range $R_Y = \{y_1, y_2, \cdots\}$, then **$E[X|Y]$ is also a random variable** with

$$E[X|Y] = \begin{cases} E[X|Y = y_1] & \text{with probability } P(Y = y_1) \\ E[X|Y = y_2] & \text{with probability } P(Y = y_2) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{cases}$$

Exercício 4.7

- Para os dados obtidos no Exercício 4.3:
- a) Encontre a PMF condicional $P_{X|Y}(x_i|y_j)$ e a $P_{Y|X}(y_i|x_j)$
- b) Encontre a CDF condicional $F_{X|Y}(x) = P_{X|Y}(X < x_i|y_j)$ e $F_{Y|X}(y) = P_{Y|X}(Y < y_i|x_j)$
- c) Encontre a expectância condicional de $\boldsymbol{Y}$ dado $\boldsymbol{X}$.
- d) Expresse $E[Y|X]$ como uma variável randômica cujo valor é representado por $\boldsymbol{g(x) = E[Y|X = x]}$ quando $X = x$

08/05/2024

Exercício 4.7: solução

- a) Encontre a PMF condicional $P_{X|Y}(x_i|y_j)$ e a $P_{Y|X}(y_i|x_j)$

- Reapresentando os resultados do exercício 4.3:

- $P_{XY}(x, y)$ – PMF conjunta

X/Y	1	2	3	4	5	6
0	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{2}{24}$
1	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{6}{24}$	$\frac{1}{24}$

- $P_X(x)$ – PMF Marginal da variável X

X	$P_X(x)$
0	$\frac{11}{24}$
1	$\frac{13}{24}$

- $P_Y(y)$ – PMF Marginal da variável Y

Y	1	2	3	4	5	6
$P_Y(y)$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{9}{24}$	$\frac{3}{24}$

Exercício 4.7: solução

$$P_{X|Y}(x_i|y_j) = \frac{P(x_i, y_j)}{P(y_j)}$$

- PMF condicional $P_{X|Y}(x_i|y_j)$:

$$P_{X|Y}(x_i = 0|y_j = 1) = \frac{P(x_i = 0, y_j = 1)}{P(y_j = 1)} = \frac{\frac{1}{24}}{\frac{3}{24}} = \frac{1}{3}$$

$$P_{X|Y}(x_i = 0|y_j = 2) = \frac{P(x_i = 0, y_j = 2)}{P(y_j = 2)} = \frac{\frac{2}{24}}{\frac{3}{24}} = \frac{2}{3}$$

$$P_{X|Y}(x_i = 0|y_j = 3) = \frac{P(x_i = 0, y_j = 3)}{P(y_j = 3)} = \frac{\frac{1}{24}}{\frac{3}{24}} = \frac{1}{3}$$

$$P_{X|Y}(x_i = 0|y_j = 4) = \frac{P(x_i = 0, y_j = 4)}{P(y_j = 4)} = \frac{\frac{1}{24}}{\frac{3}{24}} = \frac{1}{3}$$

$$P_{X|Y}(x_i = 0|y_j = 5) = \frac{P(x_i = 0, y_j = 5)}{P(y_j = 5)} = \frac{\frac{3}{24}}{\frac{9}{24}} = \frac{1}{3}$$

$$P_{X|Y}(x_i = 0|y_j = 6) = \frac{P(x_i = 0, y_j = 6)}{P(y_j = 6)} = \frac{\frac{2}{24}}{\frac{3}{24}} = \frac{2}{3}$$

$$P_{X|Y}(x_i = 1|y_j = 1) = \frac{2}{3}$$

$$P_{X|Y}(x_i = 1|y_j = 2) = \frac{1}{3}$$

$$P_{X|Y}(x_i = 1|y_j = 3) = \frac{2}{3}$$

$$P_{X|Y}(x_i = 1|y_j = 4) = \frac{2}{3}$$

$$P_{X|Y}(x_i = 1|y_j = 5) = \frac{2}{3}$$

$$P_{X|Y}(x_i = 1|y_j = 6) = \frac{1}{3}$$

Exercício 4.7: solução

$$P_{Y|X}(y_i|x_j) = \frac{P(x_i, y_j)}{P(x_j)}$$

- PMF condicional $P_{Y|X}(x_i|y_j)$:

$$P_{Y|X}(y_i = 1|x_j = 0) = \frac{P(y_i = 1, x_j = 0)}{P(x_j = 0)} = \frac{\frac{1}{24}}{\frac{11}{24}} = \frac{1}{11}$$

$$P_{Y|X}(y_i = 1|x_j = 1) = \frac{P(y_i = 1, x_j = 1)}{P(x_j = 1)} = \frac{\frac{2}{24}}{\frac{13}{24}} = \frac{2}{13}$$

$$P_{Y|X}(y_i = 2|x_j = 0) = \frac{P(y_i = 2, x_j = 0)}{P(x_j = 0)} = \frac{\frac{2}{24}}{\frac{11}{24}} = \frac{2}{11}$$

$$P_{Y|X}(y_i = 2|x_j = 1) = \frac{P(y_i = 2, x_j = 1)}{P(x_j = 1)} = \frac{\frac{1}{24}}{\frac{13}{24}} = \frac{1}{13}$$

$$P_{Y|X}(y_i = 3|x_j = 0) = \frac{P(y_i = 3, x_j = 0)}{P(x_j = 0)} = \frac{\frac{2}{24}}{\frac{11}{24}} = \frac{2}{11}$$

$$P_{Y|X}(y_i = 3|x_j = 1) = \frac{P(y_i = 3, x_j = 1)}{P(x_j = 1)} = \frac{\frac{1}{24}}{\frac{13}{24}} = \frac{1}{13}$$

$$P_{Y|X}(y_i = 4|x_j = 0) = \frac{1}{11}$$

$$P_{Y|X}(y_i = 4|x_j = 2) = \frac{2}{13}$$

$$P_{Y|X}(y_i = 5|x_j = 0) = \frac{3}{11}$$

$$P_{Y|X}(y_i = 5|x_j = 1) = \frac{6}{13}$$

$$P_{Y|X}(y_i = 6|x_j = 0) = \frac{2}{11}$$

$$P_{Y|X}(y_i = 6|x_j = 1) = \frac{1}{13}$$

Exercício 4.7: solução

PMF condicional $P_{X|Y}(x_i|y_j)$ e a $P_{Y|X}(y_i|x_j)$

$$P_{X|Y}(x_i|\boldsymbol{y_j})$$

X/Y	1	2	3	4	5	6
0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$

$$P_{Y|X}(y_i|\boldsymbol{x_j})$$

Y/X	0	1
1	$\frac{1}{11}$	$\frac{2}{13}$
2	$\frac{2}{11}$	$\frac{1}{13}$
3	$\frac{2}{11}$	$\frac{1}{13}$
4	$\frac{1}{11}$	$\frac{2}{13}$
5	$\frac{3}{11}$	$\frac{6}{13}$
6	$\frac{2}{11}$	$\frac{1}{13}$

Exercício 4.7: solução

b) Encontre a CDF condicional

$F_{X|Y}(x) = P_{X|Y}\left(X < x_i \middle| y_j\right)$ e

$F_{Y|X}(y) = P_{Y|X}\left(Y < y_i \middle| x_j\right)$

$F_{X|Y}(x)$

X/Y	1	2	3	4	5	6
0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
1	$\frac{3}{3}$	$\frac{3}{3}$	$\frac{3}{3}$	$\frac{3}{3}$	$\frac{3}{3}$	$\frac{3}{3}$

$F_{Y|X}(y)$

Y/X	0	1
1	$\frac{1}{11}$	$\frac{2}{13}$
2	$\frac{3}{11}$	$\frac{3}{13}$
3	$\frac{5}{11}$	$\frac{4}{13}$
4	$\frac{6}{11}$	$\frac{6}{13}$
5	$\frac{9}{11}$	$\frac{12}{13}$
6	$\frac{11}{11}$	$\frac{13}{13}$

Exercício 4.7: solução

- c) A expectância condicional de ***Y:***

$$E[Y|X=x] = \sum_{y_i \epsilon R_y} y_i P_{Y|X}(y_i|x_j)$$

$$E[Y|X=0] = \sum_{i=1}^{6} y_i P_{Y|X}(y_i|x_j = 0)$$

$$E[Y|X=1] = \sum_{i=1}^{6} y_i P_{Y|X}(y_i|x_j = 1)$$

Exercício 4.7: solução

$$E[Y|X=0] = \sum_{i=1}^{6} y_i P_{Y|X}(y_i|x_j = 0)$$

$$E[Y|X=0]$$
$$= 1.\frac{1}{11} + 2.\frac{2}{11} + 3.\frac{2}{11} + 4.\frac{1}{11} + 5.\frac{3}{11} + 6.\frac{2}{11}$$

$$= \frac{1+4+6+4+15+12}{11} = \frac{42}{11}$$

Y/X	0	1
1	$\frac{1}{11}$	$\frac{2}{13}$
2	$\frac{2}{11}$	$\frac{1}{13}$
3	$\frac{2}{11}$	$\frac{1}{13}$
4	$\frac{1}{11}$	$\frac{2}{13}$
5	$\frac{3}{11}$	$\frac{6}{13}$
6	$\frac{2}{11}$	$\frac{1}{13}$

Exercício 4.7: solução

08/05/2024

$$E[Y|X=1] = \sum_{i=1}^{6} y_i P_{Y|X}(y_i|x_j = 1)$$

$$E[Y|X=1]$$
$$= 1.\frac{2}{13} + 2.\frac{1}{13} + 3.\frac{1}{13} + 4.\frac{2}{13} + 5.\frac{6}{13} + 6.\frac{1}{13}$$
$$= \frac{2+2+3+8+30+6}{13} = \frac{51}{13}$$

Y/X	0	1
1	$\frac{1}{11}$	$\frac{2}{13}$
2	$\frac{2}{11}$	$\frac{1}{13}$
3	$\frac{2}{11}$	$\frac{1}{13}$
4	$\frac{1}{11}$	$\frac{2}{13}$
5	$\frac{3}{11}$	$\frac{6}{13}$
6	$\frac{2}{11}$	$\frac{1}{13}$

Exercício 4.7: solução

- d) Expresse $E[\boldsymbol{Y}|\boldsymbol{X}]$ como uma variável rendômica cujo valor é igual a $g(x) = E[\boldsymbol{Y}|\boldsymbol{X} = x]$ quando $X = x$.

$$\hat{\boldsymbol{X}} = \left\{\frac{42}{11}, \frac{51}{13}\right\}$$

Expectation for Independent Random Variables

- Note that if two random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent, then the conditional PMF of $\boldsymbol{X}$ given $\boldsymbol{Y}$ will be the same as the marginal PMF of X, i.e., for any $x \in R_{\boldsymbol{X}}$, we have

$$P_{X|Y}(x|y) = P_X(x)$$

Expectation for Independent Random Variables

- Thus, for independent random variables, we have

$$
\begin{aligned}
E[X|Y=y] &= \sum_{x \in R_X} x P_{X|Y}(x|y) \\
&= \sum_{x \in R_X} x P_X(x) \\
&= E[X]
\end{aligned}
$$

Expectation for Independent Random Variables

- More generally, if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent then any function of $\boldsymbol{X}$, say $g(\boldsymbol{X})$, and $\boldsymbol{Y}$ are independent, thus

$$E[g(X)|Y] = E[g(X)]$$

$E[XY]$ for Independent Random Variables

- If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent, then $E[\boldsymbol{XY}] = E[\boldsymbol{X}]E[\boldsymbol{Y}]$. Using LOTUS, we have

$$
\begin{aligned}
E[XY] &= \sum_{x \in R_x} \sum_{y \in R_y} xy P_{XY}(x,y) \\
&= \sum_{x \in R_x} \sum_{y \in R_y} xy P_X(x) P_Y(y) \\
&= \left(\sum_{x \in R_x} x P_X(x) \right) \left(\sum_{y \in R_y} y P_Y(y) \right) \\
&= E[X]E[Y]
\end{aligned}
$$

Expectation for Independent Random Variables - summary

- If ***X*** and ***Y*** are independent random variables, then

$$E[X|Y] = E[X]$$

$$E[g(X)|Y] = E[g(X)]$$

$$E[XY] = E[X]E[Y]$$

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Conditional Variance

- We can define the conditional variance of X, Var $(\boldsymbol{X}|\boldsymbol{Y} = y)$, which is the variance of $\boldsymbol{X}$ in the conditional space where we know $\boldsymbol{Y} = y$. If we let $\mu_{X|Y}(y) = E[\boldsymbol{X}|\boldsymbol{Y} = y]$, then

$$\begin{aligned}\mathrm{Var}(X|Y=y) &= E\left[(X-\mu_{X|Y}(y))^2|Y=y\right] \\ &= \sum_{x_i \in R_X} \left(x_i - \mu_{X|Y}(y)\right)^2 P_{X|Y}(x_i) \\ &= E\left[X^2|Y=y\right] - \mu_{X|Y}(y)^2\end{aligned}$$

Exercício 4.8

- Utilizando os dados e resultados obtidos no Exercício 4.7

- a) Encontre a variância condicional de $\boldsymbol{Y}$ dado $\boldsymbol{X}$.
- b) Expresse a variância condicional $\mathrm{Var}\,(\boldsymbol{Y}|\boldsymbol{X} = x)$ como uma variável randômica cujo valor é igual a $g(x) = \mathrm{Var}\,(\boldsymbol{Y}|\boldsymbol{X} = x)$ quando $X = x$.

Exercício 4.8: solução

- a) Encontre a variância condicional de ***Y*** dado ***X*.**

- A variância da variável randômica, Var (Y|X = x), pode ser expressa como:

$$\text{Var}\,(\boldsymbol{Y}|\boldsymbol{X}=x)=\text{E}\left[\left(\text{Y}-\mu_{\boldsymbol{Y}|\boldsymbol{X}}(x)\right)^2|X=x\right]=$$

$$\text{E}[Y^2|X=x]-\mu^2_{\boldsymbol{Y}|\boldsymbol{X}}(x)$$

- Como já temos calculado do Exercício 4.7 $\mu_{\boldsymbol{Y}|\boldsymbol{X}}(x=0)$ e $\mu_{\boldsymbol{Y}|\boldsymbol{X}}(x=1)$, vamos usar a segunda expressão:

$$\mu_{\boldsymbol{Y}|\boldsymbol{X}}(x=0)=\frac{42}{11} \qquad \mu_{\boldsymbol{Y}|\boldsymbol{X}}(x=1)=\frac{51}{13}$$

Exercício 4.8: solução

Cálculo da variância condicional de $\boldsymbol{Y}$ dado $\boldsymbol{X = x = 0}$

$$\mathrm{E}[Y^2|X = x] = \sum_{i=1}^{6} y_i^2 P_{Y|X}\left(y_i \middle| x_j = 0\right)$$

$$\mathrm{E}[Y^2|X = 0]$$

$$= (1^2)\frac{1}{11} + (2^2)\frac{2}{11} + (3^2)\frac{2}{11} + (4^2)\frac{1}{11} + (5^2)\frac{3}{11} + (6^2)\frac{2}{11}$$

$$= \frac{1 + 8 + 18 + 16 + 75 + 72}{11} = \frac{190}{11}$$

Y/X	0	1
1	$\frac{1}{11}$	$\frac{2}{13}$
2	$\frac{2}{11}$	$\frac{1}{13}$
3	$\frac{2}{11}$	$\frac{1}{13}$
4	$\frac{1}{11}$	$\frac{2}{13}$
5	$\frac{3}{11}$	$\frac{6}{13}$
6	$\frac{2}{11}$	$\frac{1}{13}$

Exercício 4.8: solução

08/05/2024

Y/X	0	1
1	$\frac{1}{11}$	$\frac{2}{13}$
2	$\frac{2}{11}$	$\frac{1}{13}$
3	$\frac{2}{11}$	$\frac{1}{13}$
4	$\frac{1}{11}$	$\frac{2}{13}$
5	$\frac{3}{11}$	$\frac{6}{13}$
6	$\frac{2}{11}$	$\frac{1}{13}$

Cálculo da variância condicional de ***Y*** dado ***X*** = ***x*** = ***1***

$$\mathrm{E}[Y^2|X=x] = \sum_{i=1}^{6} y_i^2 P_{Y|X}\left(y_i \middle| x_j = 1\right)$$

$$\mathrm{E}[Y^2|X=1]$$

$$= (1^2)\frac{2}{13} + (2^2)\frac{1}{13} + (3^2)\frac{1}{13} + (4^2)\frac{2}{13} + (5^2)\frac{6}{13} + (6^2)\frac{1}{13}$$

$$= \frac{2+4+9+32+150+36}{13} = \frac{233}{13}$$

Exercício 4.8: solução

- A variância condicional Var ($\boldsymbol{Y}|\boldsymbol{X} = 0$) pode ser obtida por meio de

$$\text{Var}\,(\boldsymbol{Y}|\boldsymbol{X} = x) = \text{E}[Y^2|X = x] - \mu^2_{\boldsymbol{Y}|\boldsymbol{X}}(x)$$

$$\text{E}[Y^2|X = 0] = \frac{190}{11}, \qquad \mu_{\boldsymbol{Y}|\boldsymbol{X}}(x = 0) = \frac{42}{11},$$

$$\text{Var}\,(\boldsymbol{Y}|\boldsymbol{X} = 0) = \text{E}[Y^2|X = x] - \mu^2_{\boldsymbol{Y}|\boldsymbol{X}}(x) = \frac{190}{11} - \left(\frac{42}{11}\right)^2 = \frac{326}{121}$$

Exercício 4.8: solução

- Analogamente, a variância condicional Var ($\boldsymbol{Y}|\boldsymbol{X} = 1$) pode computada por:

$$\text{Var}\,(\boldsymbol{Y}|\boldsymbol{X} = x) = \mathsf{E}[Y^2|X = x] - \mu^2_{\boldsymbol{Y}|\boldsymbol{X}}(x)$$

$$\mathrm{E}[Y^2|X = 1] = \frac{233}{13},\ \mu_{\boldsymbol{Y}|\boldsymbol{X}}(x = 1) = \frac{51}{13}$$

$$\text{Var}\,(\boldsymbol{Y}|\boldsymbol{X} = 1) = \mathsf{E}[Y^2|X = x] - \mu^2_{\boldsymbol{Y}|\boldsymbol{X}}(x) = \frac{233}{13} - \left(\frac{51}{13}\right)^2 = \frac{428}{169}$$

Exercício 4.8: solução

- b) Expresse a variância condicional Var ($\boldsymbol{Y}|\boldsymbol{X} = x$) como uma variável randômica cujo valor é igual a $\boldsymbol{g(x) = \mathrm{Var}\,(Y|X = x)}$ quando $X = x$.

$$\bar{\bar{\boldsymbol{X}}} = \left\{\frac{326}{121}, \frac{428}{169}\right\}$$

08/05/2024

Two-Continuous Random Variables

Continuous random variables

- Joint probability density function (PDF)
- Marginal PDF
- Joint cumulative distribution function (CDF)
- Marginal CDF
- Conditional PDF
- Conditional CDF
- Continuous random independent variables

Continuous random variables

- Conditional expectation
- Law of Total Probability
- Law of total expectation
- Law of Total Variance
- Law of the unconscious statistician (LOTUS) for joint variables
- Conditional expectation as a function of a random variable
- Expectation for independent random variables
- Conditional variance
- Conditional variance as a function of a random variable
- The Method of Transformations

Two-Dimension Random Variables

- Covariance – discrete and continuous variables
- The Inner Product
- Pearson Correlation Coefficient
- Properties of the Pearson correlation coefficient
- Uncorrelated variables
- Evaluation for Gaussian Joint Probability Density Function (PDF)

Joint Probability Density Function (PDF)

Two random variables X and Y are **jointly continuous** if there exists a nonnegative function $f_{XY} : \mathbb{R}^2 \to \mathbb{R}$, such that, for any set $A \in \mathbb{R}^2$, we have

$$P\big((X,Y) \in A\big) = \iint_A f_{XY}(x,y)dxdy$$

The function $f_{XY}(x,y)$ is called the **joint probability density function (PDF)** of X and Y.

Joint Probability Density Function (PDF)

$$R_{XY} = \{(x, y) | f_{XY}(x, y) > 0\}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1$$

Joint Probability Density Function (PDF)

- One-dimensional Probability Density Function (PDF) approaching

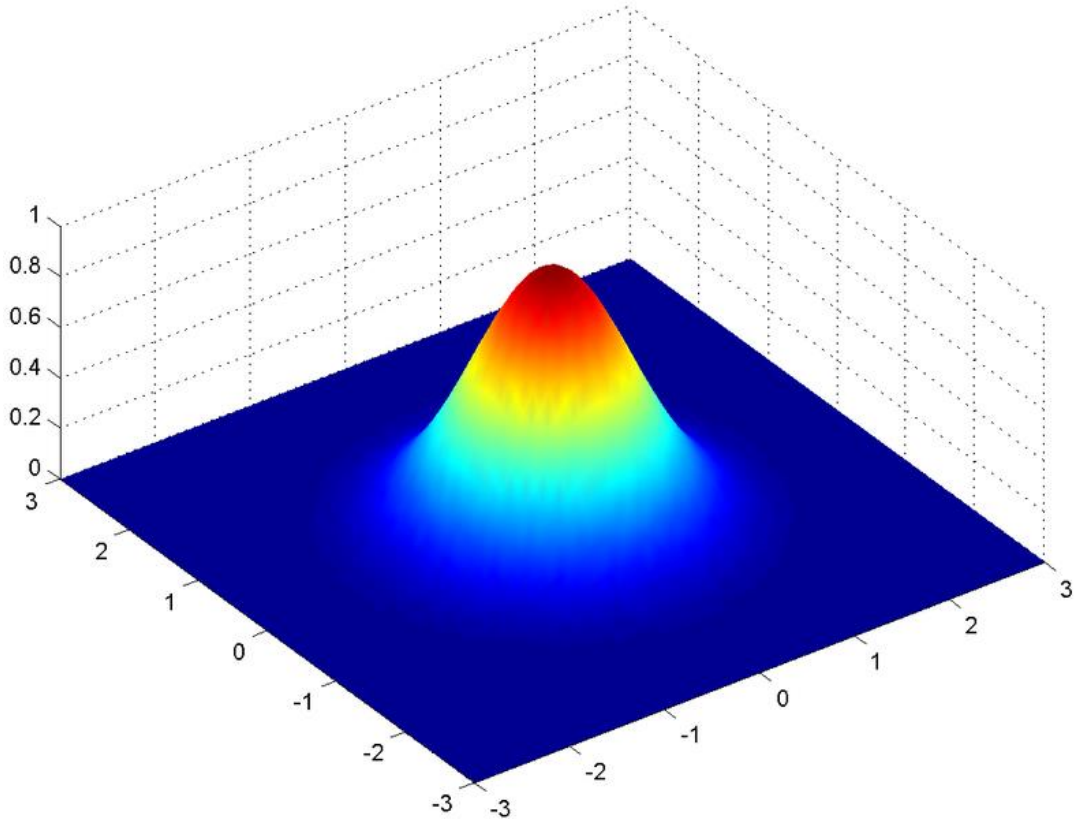
$$P(x < X \leq x + \delta) \approx f_X(x)\delta$$

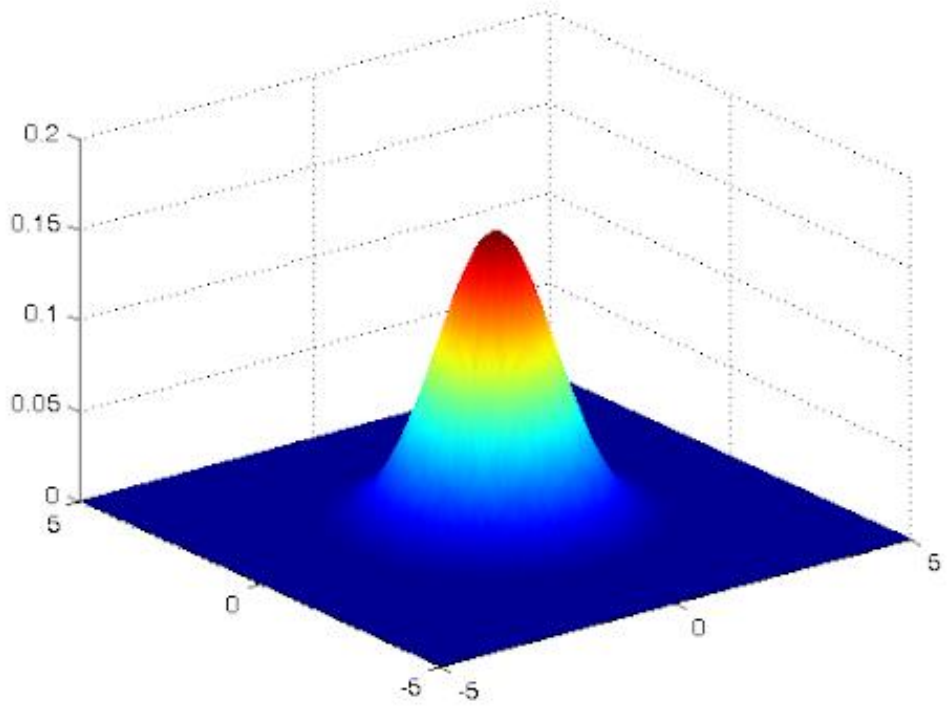
- Two-dimensional Joint Probability Density Function (PDF) approaching

$$P(x < X \leq x + \delta_x, y \leq Y \leq y + \delta_y) \approx f_{XY}(x, y)\delta_x\delta_y$$

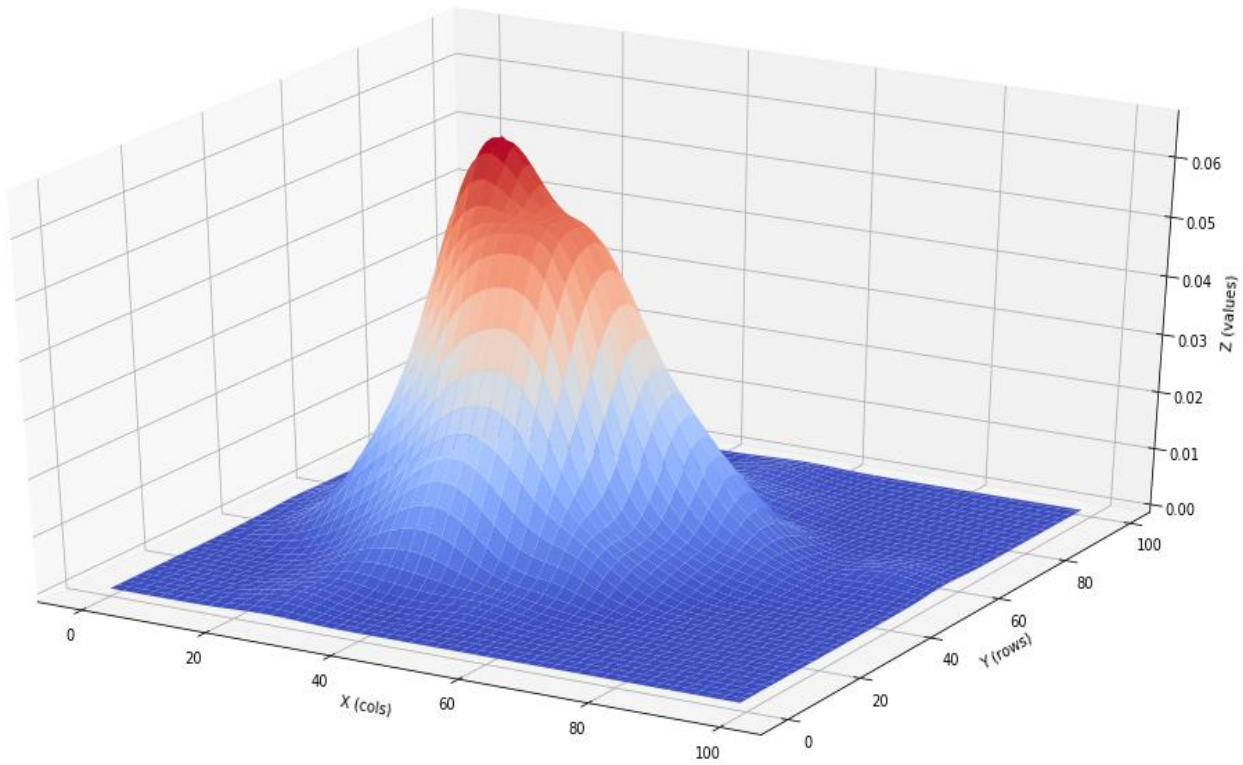
Joint Probability Density Function (PDF)

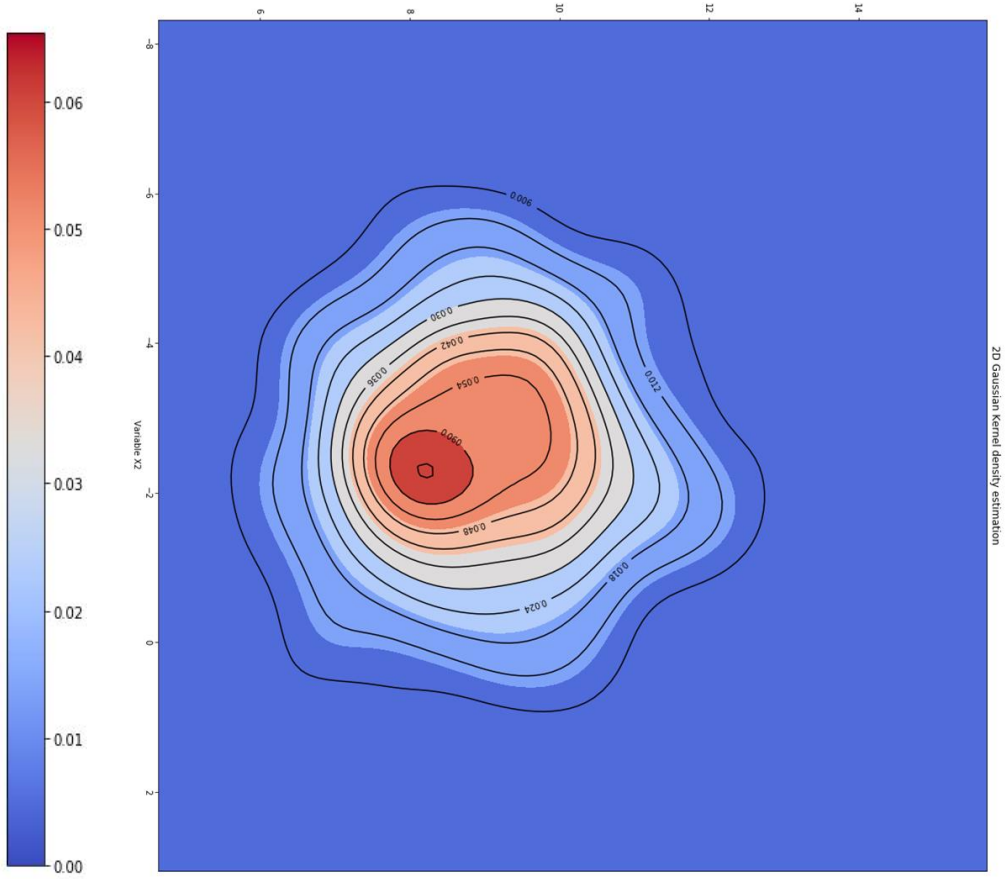
- Examples: Gaussian probability density function.



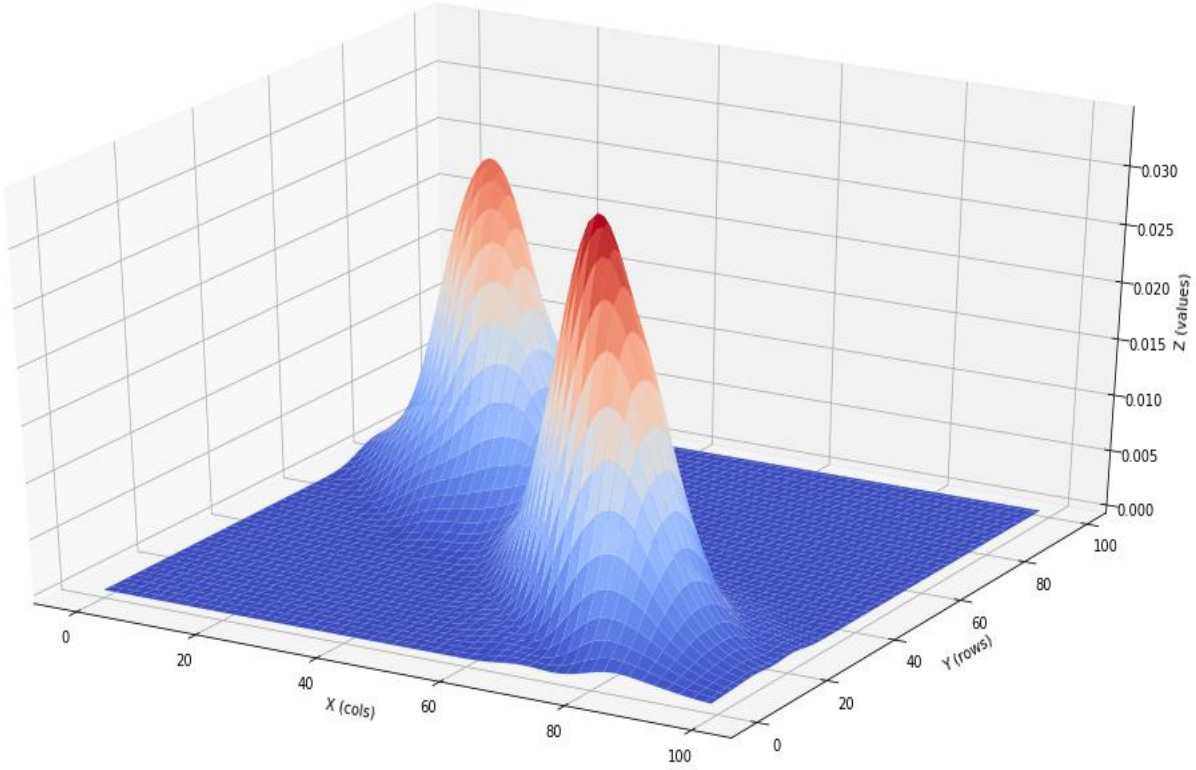


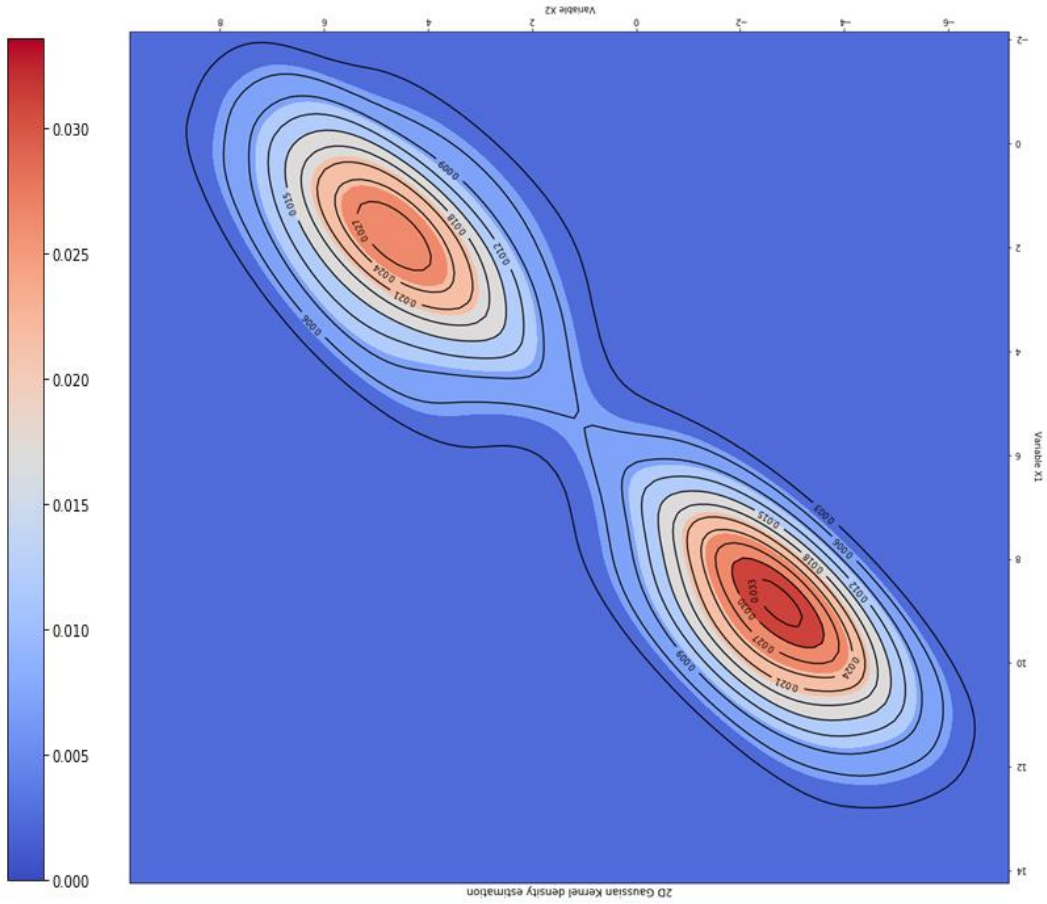
Joint Probability Density Function (PDF)



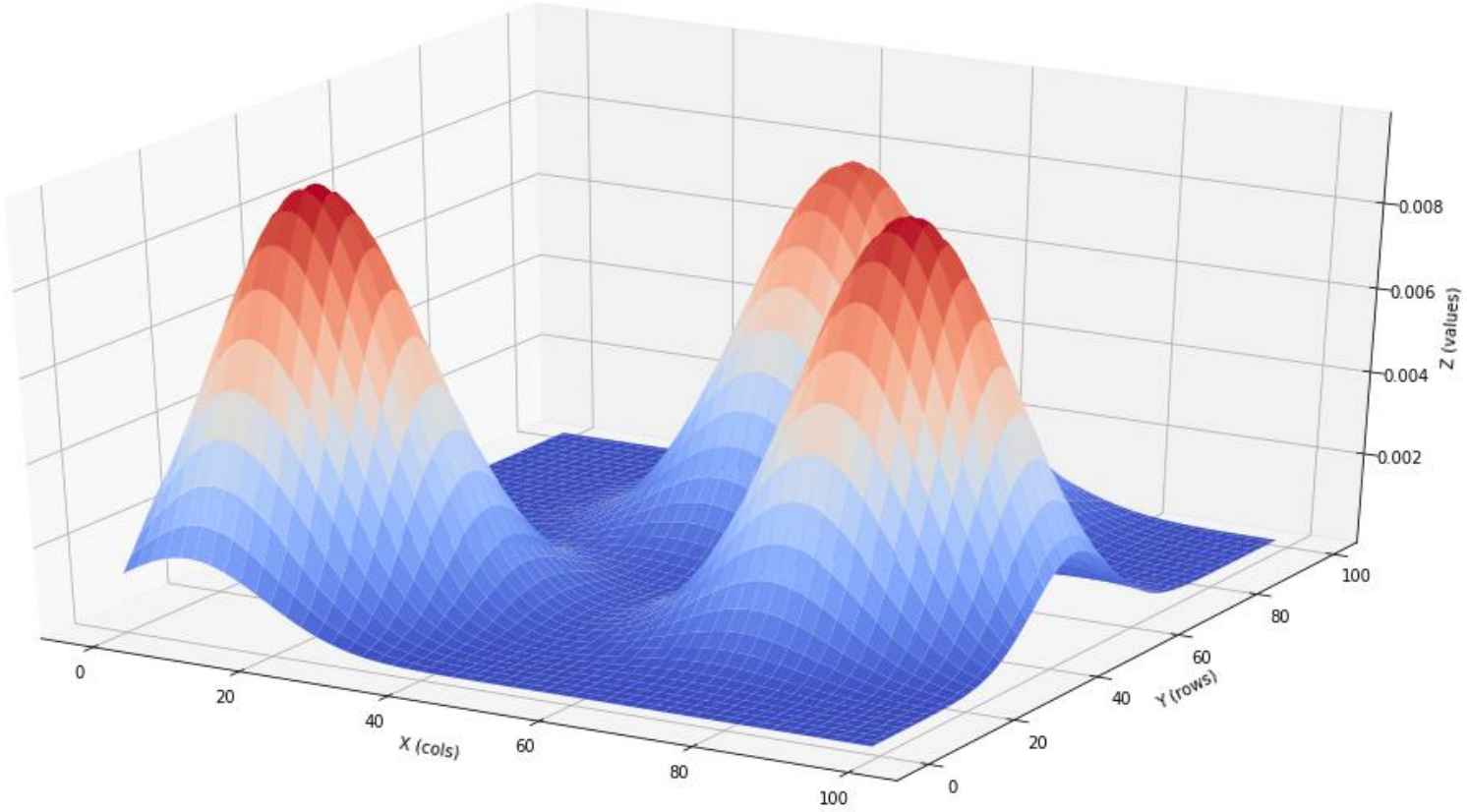


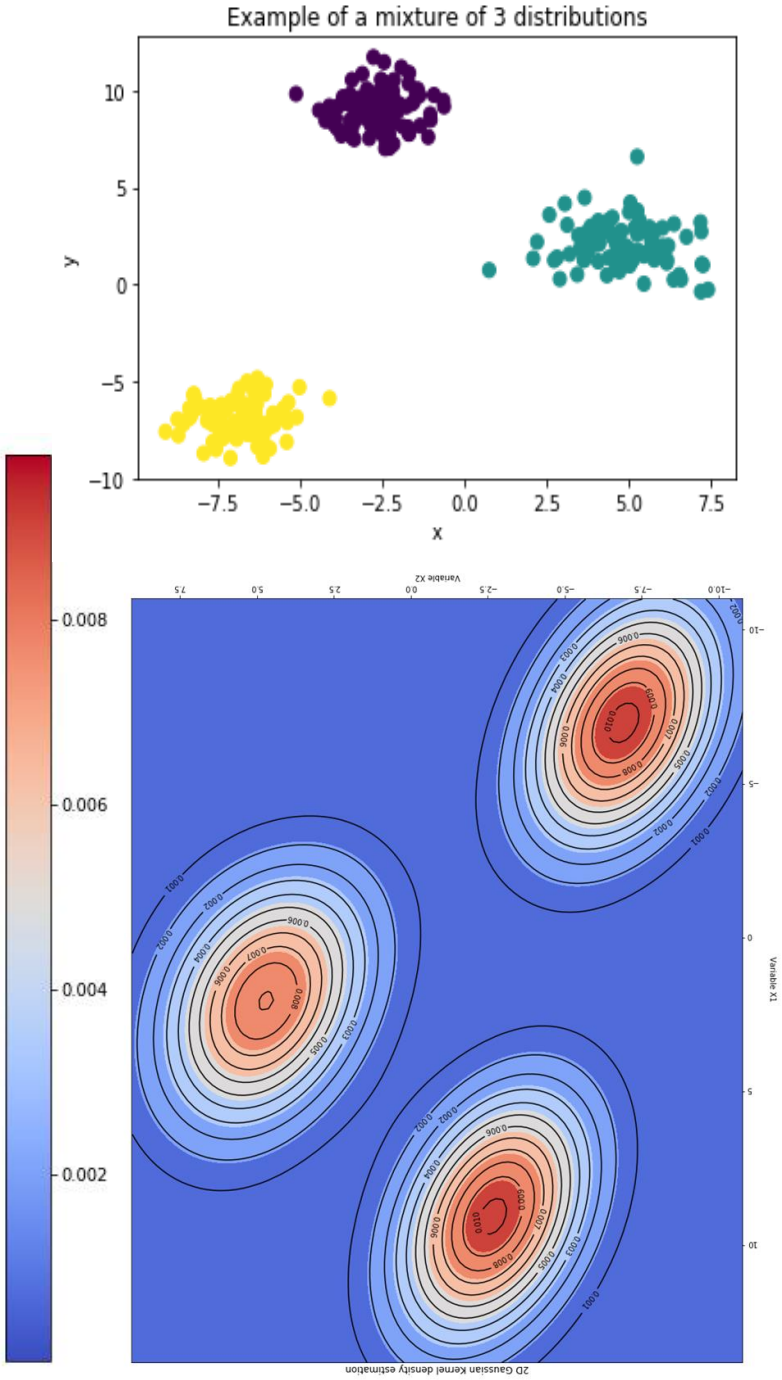
Joint Probability Density Function (PDF) - Gaussian mixture



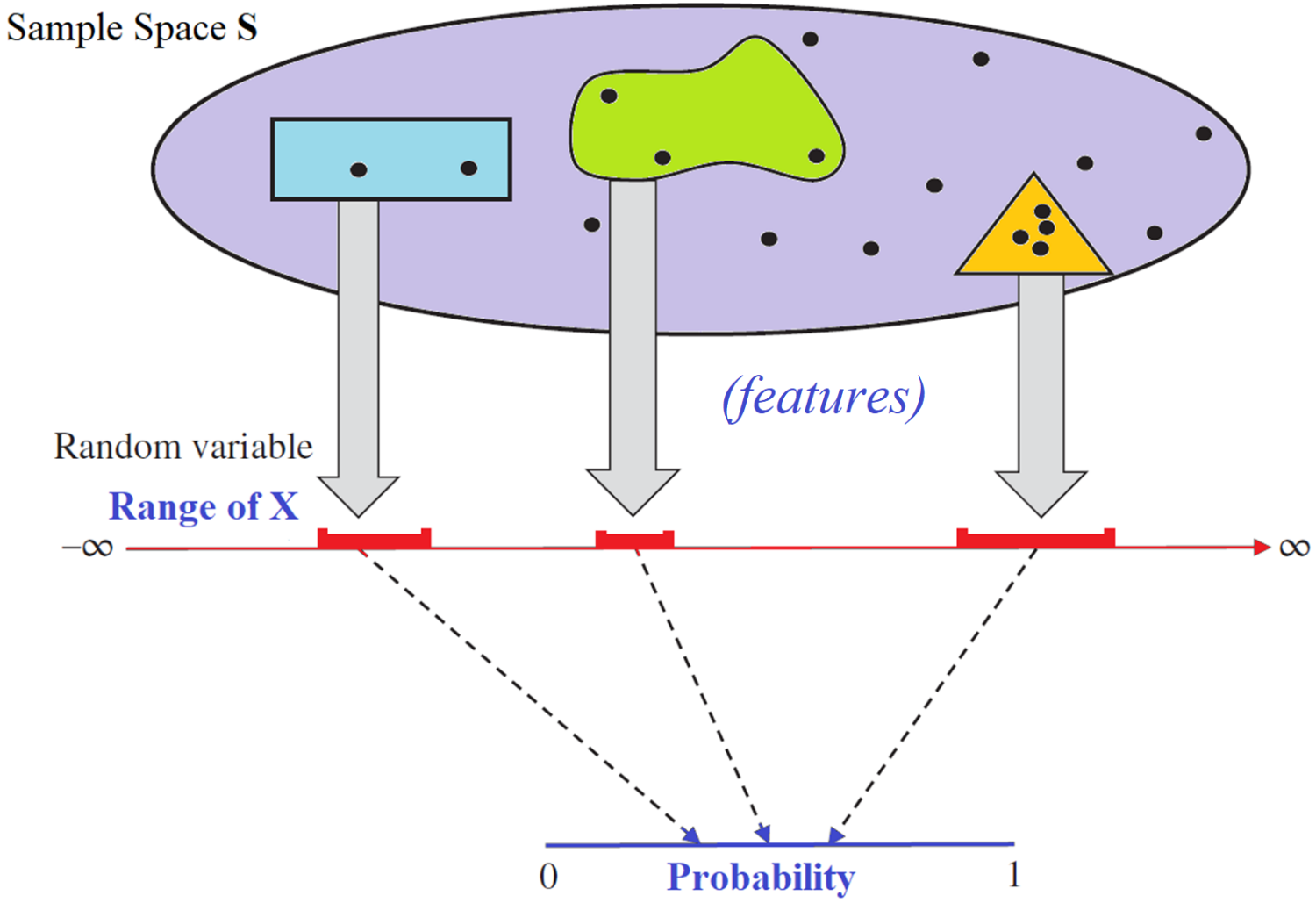


Joint Probability Density Function (PDF) - Gaussian mixture

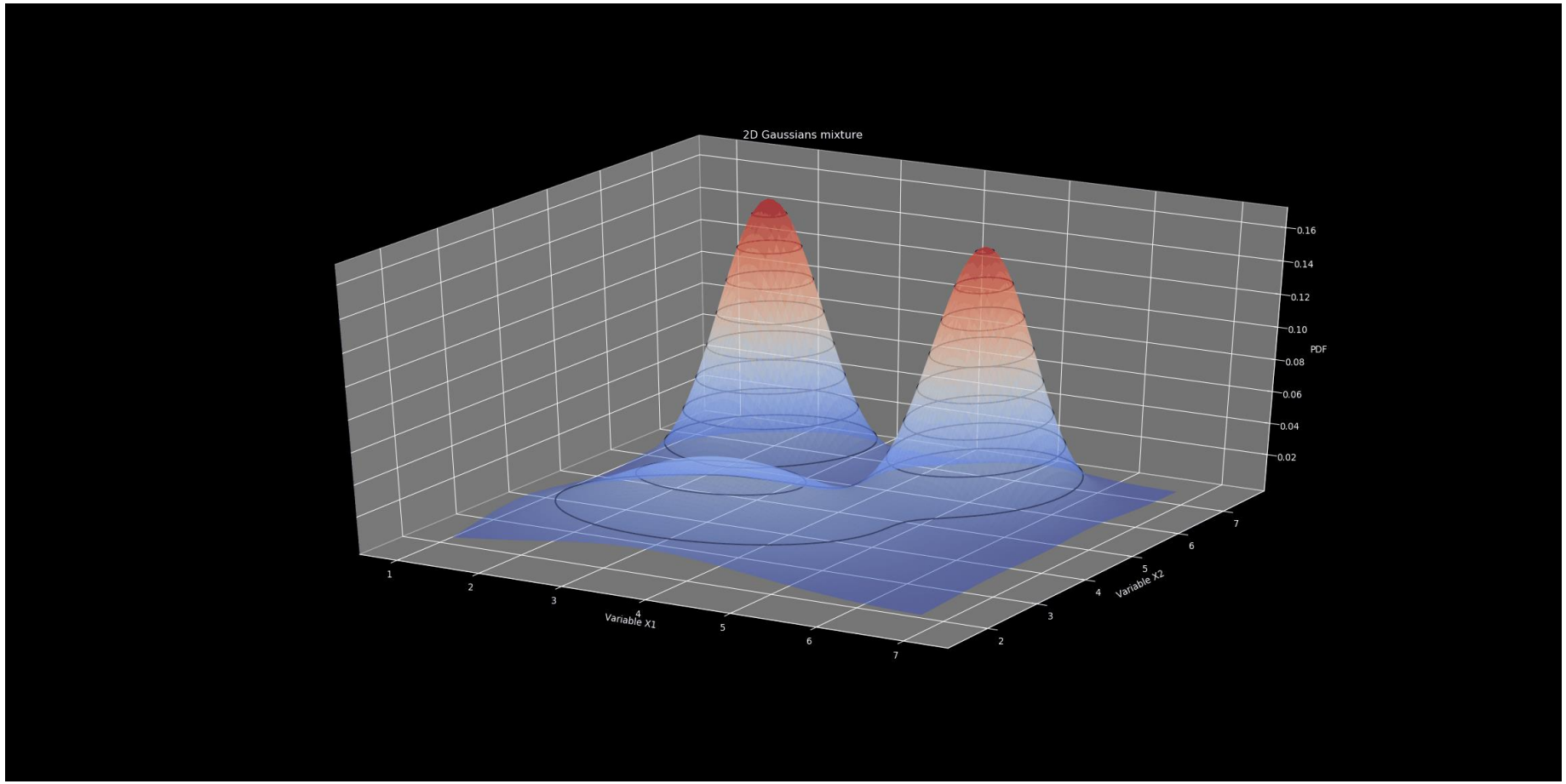




Look again at the context of the stochastic world: the relationship among sample space, random variable, and probability.



Joint Probability Density Function (PDF) - Gaussian mixture machine learninsg algorithm



08/05/2024

Exercise 4.9

- Let the random variable $\boldsymbol{X}$ denote the time until a computer server connects to your machine (in milliseconds), and let $\boldsymbol{Y}$ denote the time until the server authorizes you as a valid user (in milliseconds).
- Each of these random variables measures the wait from a common starting time and $\boldsymbol{X} < \boldsymbol{Y}$.
- Assume that the joint probability density function (PDF) for $\boldsymbol{X}$ and $\boldsymbol{Y}$ is

$$f_{XY}(x, y) = 6.0 \text{ x } 10^{-6}\, e^{-0.001x - 0.002y} \text{ for } x > 0 \text{ and } x < y$$

Exercise 4.9

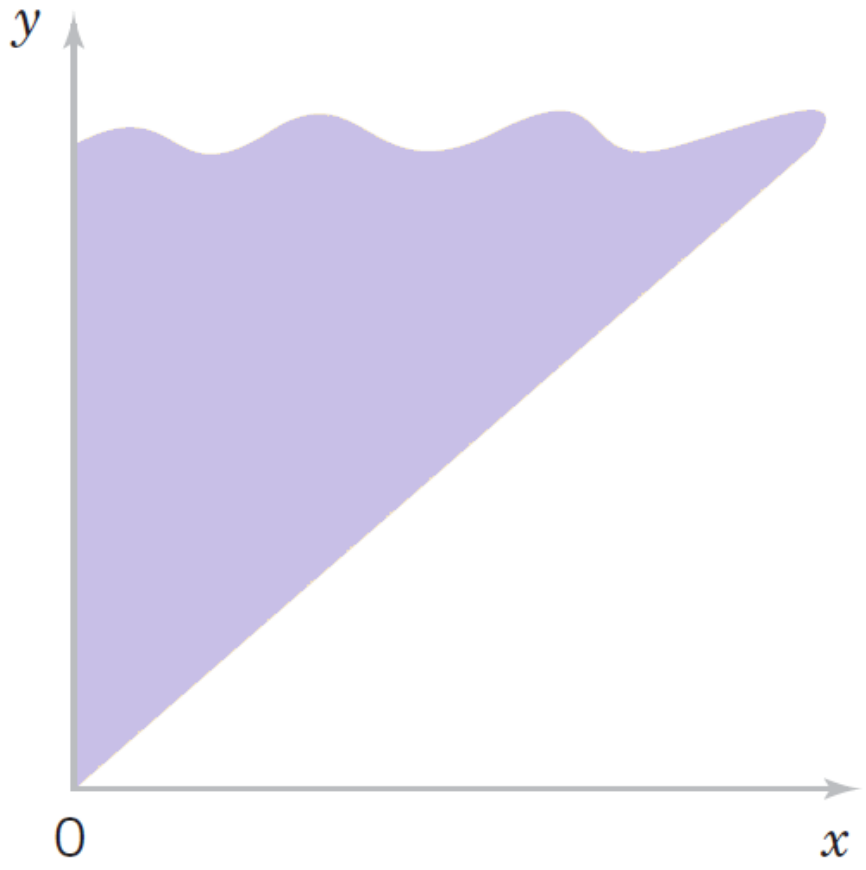
- a) Show that f_{XY} is a PDF.

- b) Find $P(\boldsymbol{X} \leq 1000, \boldsymbol{Y} \leq 2000)$ (remember that this is the same as saying:

$$P(\boldsymbol{X} \leq 1000 \; and \; \boldsymbol{Y} \leq 2000)\,)$$

Exercise 4.9: Solution

- The sample space can be shown in the figure on the right.



The joint probability density function of X and Y is nonzero over the shaded region.

$$f_{XY}(x, y) = 6.0 \text{ x } 10^{-6}\, e^{-0.001x-0.002y} \text{ for } x < y$$

Exercise 4.9: Solution

- a) Show that $f_{XY}(x,y)$ is a PDF.
- The property that this joint probability density function integrates to 1 can be verified by the integral of $f_{XY}(x, y)$ over this region as follows:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dydx =$$

$$= \int_{0}^{\infty}\left(\int_{x}^{\infty} 6.0 \text{ x } 10^{-6}\, e^{-0.001x-0.002y}dy\right)dx$$

$$= 6.0 \text{ x } 10^{-6} \int_{0}^{\infty}\left(\int_{x}^{\infty} e^{-0.002y}dy\right)e^{-0.001x}dx$$

Exercise 4.9: Solution

- Continuation 4.9.a

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dydx =$$

$$= 6.0 \text{ x } 10^{-6} \int_{0}^{\infty} \left(-\frac{e^{-0.002y}}{0.002}\Bigg|_{x}^{\infty} \right) e^{-0.001x} dx$$

$$= 6.0 \text{ x } 10^{-6} \int_{0}^{\infty} \left(\frac{e^{-0.002x}}{0.002} \right) e^{-0.001x} dx$$

Exercise 4.9: Solution

- Continuation 4.9.a

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dydx =$$

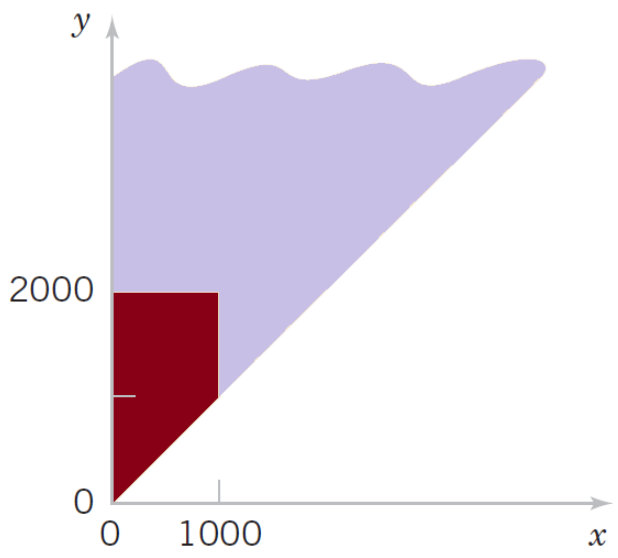
$$= 0.003\int_{0}^{\infty} e^{-0.003x}dx = 0.003\left(-\frac{e^{-0.003x}}{0.003}\Bigg|_{0}^{\infty}\right)$$

$$= 0.003\frac{1}{0.003} = 1$$

- $f_{XY}(x,y)$ is a PDF.

Exercise 4.9: Solution

- b) Find $P(\boldsymbol{X} \leq 1000, \boldsymbol{Y} \leq 2000)$
- The probability that $P(\boldsymbol{X} \leq 1000, \boldsymbol{Y} \leq 2000)$ is determined as the integral over the darkly shaded region in figure below



Region of integration for the probability that $X < 1000$ and $Y < 2000$ is darkly shaded.

Exercise 4.9: Solution

- Therefore, it can be done

$$P(\boldsymbol{X} \leq 1000, \boldsymbol{Y} \leq 2000) = \int_0^{1000} \int_x^{2000} f_{XY}(x, y) dy dx =$$

$$= 6.0 \text{ x } 10^{-6} \int_0^{1000} \left(\int_x^{2000} e^{-0.002y} dy \right) e^{-0.001x} dx$$

Exercise 4.9: Solution

- Continuation 4.9.b

$$P(\boldsymbol{X} \leq 1000, \boldsymbol{Y} \leq 2000) = \int_0^{1000} \int_x^{2000} f_{XY}(x,y)\,dy\,dx$$

$$= 6.0 \text{ x } 10^{-6} \int_0^{1000} \left(-\frac{e^{-0.002y}}{0.002} \Bigg|_x^{2000} \right) e^{-0.001x} dx$$

$$= 6.0 \text{ x } 10^{-6} \int_0^{1000} \left(\frac{e^{-0.002x} - e^{-4}}{0.002} \right) e^{-0.001x} dx$$

Exercise 4.9: Solution

- Continuation 4.9.b

$$P(\boldsymbol{X} \leq 1000, \boldsymbol{Y} \leq 2000) = 0.003 \int_0^{1000} (e^{-0.003x} - e^{-4} e^{-0.001x})\, dx$$

$$= 0.003 \left\{ \left(-\frac{e^{-0.003x}}{0.003} \Bigg|_0^{1000} \right) - e^{-4} \left(-\frac{e^{-0.001x}}{0.001} \Bigg|_0^{1000} \right) \right\}$$

$$= 0.003 \left\{ \left(\frac{1 - e^{-3}}{0.003} \right) - e^{-4} \left(\frac{1 - e^{-1}}{0.001} \right) \right\}$$

Exercise 4.9: Solution

- Finally,

$$P(\boldsymbol{X} \leq 1000, \boldsymbol{Y} \leq 2000) = 0.003\left\{\left(\frac{1-\ e^{-3}}{0.003}\right) - e^{-4}\left(\frac{1-e^{-1}}{0.001}\right)\right\}$$

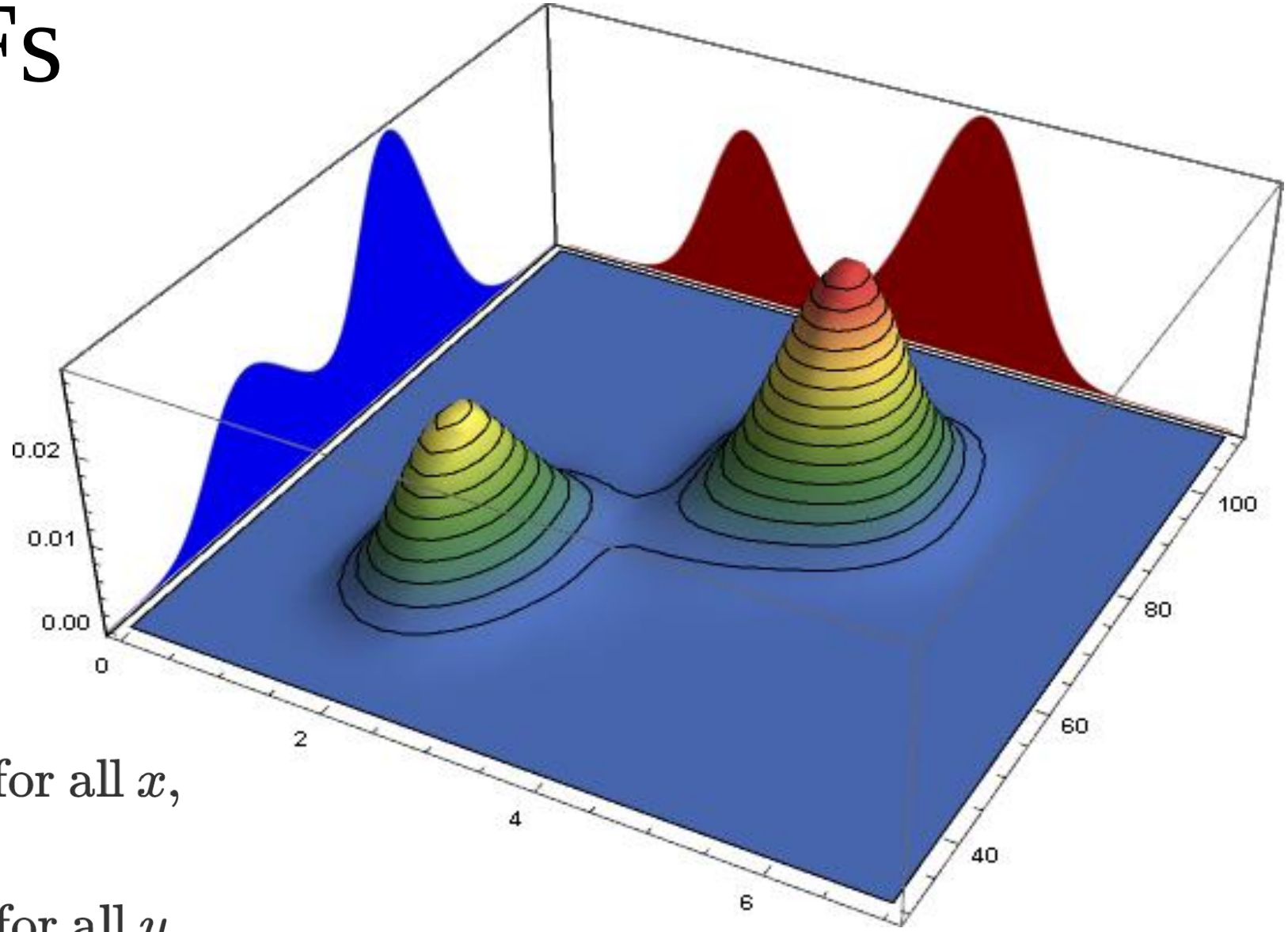
$$P(\boldsymbol{X} \leq 1000, \boldsymbol{Y} \leq 2000) = 0.003(316.738 - 11.578\ ) = 0.915$$

Marginal PDFs

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy, \qquad \text{for all } x,$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx, \qquad \text{for all } y.$$

Marginal PDFs

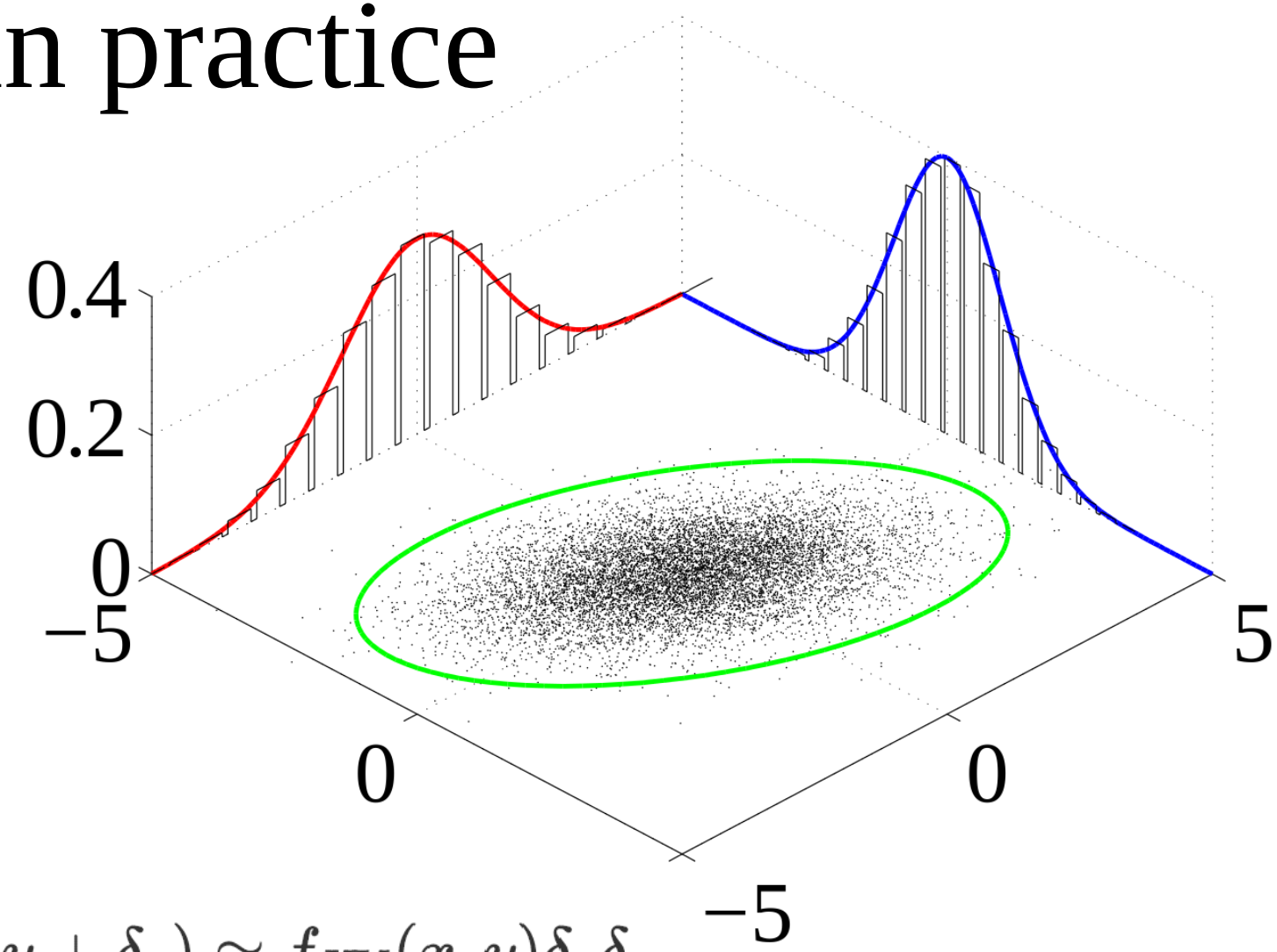


$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy, \qquad \text{for all } x,$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx, \qquad \text{for all } y.$$

Marginal PDFs in practice

- The PDF calculation is done using the point cloud.
- It is assumed incremental intervals of the joint random variables.



$$P(x < X \leq x + \delta_x, y \leq Y \leq y + \delta_y) \approx f_{XY}(x, y)\delta_x\delta_y$$

Exercise 4.10

- For the random variables that denote times in Exercise 4.9,
- a) Find the PDF marginal $f_Y(y)$
- b) calculate the probability that $\boldsymbol{Y}$ exceeds 2000 milliseconds.

- The PDF of Exercise 4.9 is given by

$$f_{XY}(x, y) = 6.0 \text{ x } 10^{-6}\, e^{-0.001x - 0.002y} \text{ for } x < y$$

Exercise 4.10: Solution

- a) Find the PDF marginal $f_Y(y)$: the probability the marginal probability distribution of Y can be computed as follows.

$$f_Y(y) = \int f_{XY}(x, y)dx, \text{for all } y$$

- For $y > 0$,

$$f_Y(y) = \int_0^y 6.0 \text{ x } 10^{-6}\, e^{-0.001x-0.002y} dx$$

$$= 6.0 \text{ x } 10^{-6} e^{-0.002y} \int_0^y e^{-0.001x} dx$$

Exercise 4.10: Solution

- We have obtained the marginal probability density function of Y.

$$f_Y(y) = 6.0 \text{ x } 10^{-6} e^{-0.002y} \left(-\frac{e^{-0.001x}}{0.001} \Bigg|_0^y \right)$$

$$f_Y(y) = 0.006 e^{-0.002y} \left(1 - e^{-0.001y} \right), \text{for } y > 0$$

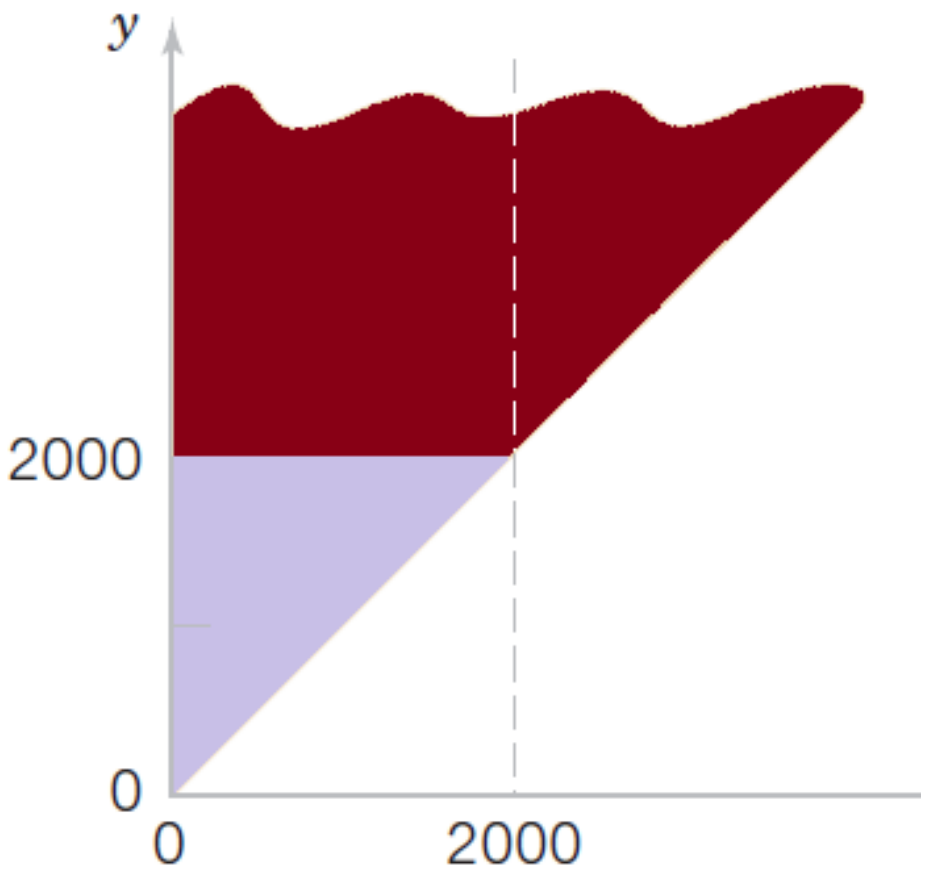
08/05/2024

Exercise 4.10: Solution

- b) calculate the probability that $\boldsymbol{Y}$ exceeds 2000 milliseconds.

- $P(\boldsymbol{Y} > 2000)$: This probability is determined as the integral of $f_{XY}(x, y)$ over the darkly red shaded region in the figure on the right side.

- The region is partitioned into two parts (a rectangle and a triangle) and different limits of integration are determined for each part.



Exercise 4.10: Solution

- (Solution 1) Using the joint PDF: $P(\boldsymbol{Y} > 2000)$ can be expressed as
  - The area under the rectangle.

$$P(\boldsymbol{Y} > 2000) = \int_0^{2000} \left(\int_{2000}^{\infty} 6.0 \text{ x } 10^{-6}\, e^{-0.001x - 0.002y} dy \right) dx$$

  - The area under the triangle.

$$+ \int_{2000}^{\infty} \left(\int_{x}^{\infty} 6.0 \text{ x } 10^{-6}\, e^{-0.001x - 0.002y} dy \right) dx$$

$$= A_R + A_T$$

Exercise 4.10: Solution

- The area under the rectangle

$$A_R = 6.0 \text{ x } 10^{-6} \int_0^{2000} \left(-\frac{e^{-0.002y}}{0.002} \Bigg|_{2000}^{\infty} \right) e^{-0.001x} dx$$

$$= 0.003e^{-4} \int_0^{2000} e^{-0.001x} dx = 0.003e^{-4} \left(-\frac{e^{-0.001x}}{0.001} \Bigg|_0^{2000} \right)$$

$$= 3.0e^{-4}(1 - e^{-2}) = 0.0475$$

Exercise 4.10: Solution

- The area under the triangle

$$A_T = \int_{2000}^{\infty} \left(\int_{x}^{\infty} 6.0 \text{ x } 10^{-6}\, e^{-0.002y} dy \right) e^{-0.001x} dx$$

$$= 6.0 \text{ x } 10^{-6} \int_{2000}^{\infty} \left(-\frac{e^{-0.002y}}{0.002} \Bigg|_{x}^{\infty} \right) e^{-0.001x} dx$$

$$= 0.003 \int_{2000}^{\infty} e^{-0.003x} dx = 0.003 \left(-\frac{e^{-0.003x}}{0.003} \Bigg|_{2000}^{\infty} \right)$$

$$= e^{-6} = 0.0025$$

Exercise 4.10: Solution

- Therefore,

$$P(\boldsymbol{Y} > 2000) = A_R + A_T = 0475 + 0.0025 = 0.05$$

Exercise 4.10: Solution

- (Solution 2) Using the marginal PDF: $P(\boldsymbol{Y} > 2000)$ for $x \epsilon R_X$ such that $x < y$: Alternatively, the probability $P(\boldsymbol{Y} > 2000)$ can be calculated from the marginal probability distribution of $\boldsymbol{Y}$.

$$f_Y(y) = \int f_{XY}(x, y)dx$$

$$= 0.006e^{-0.002y}\left(1 - e^{-0.001y}\right), \text{for } y > 0$$

Exercise 4.10: Solution

- Since we have obtained the marginal probability density function of $\boldsymbol{Y}$ we can easily compute $P(\boldsymbol{Y} > 2000)$.

$$P(\boldsymbol{Y} > 2000) = \int_{2000}^{\infty} 0.006e^{-0.002y}\left(1 - e^{-0.001y}\right) dy$$

Exercise 4.10: Solution

- Solving the integral,

$$P(\boldsymbol{Y} > 2000) = 0.006 \int_{2000}^{\infty} \left(e^{-0.002y} - e^{-0.003y} \right) dy$$

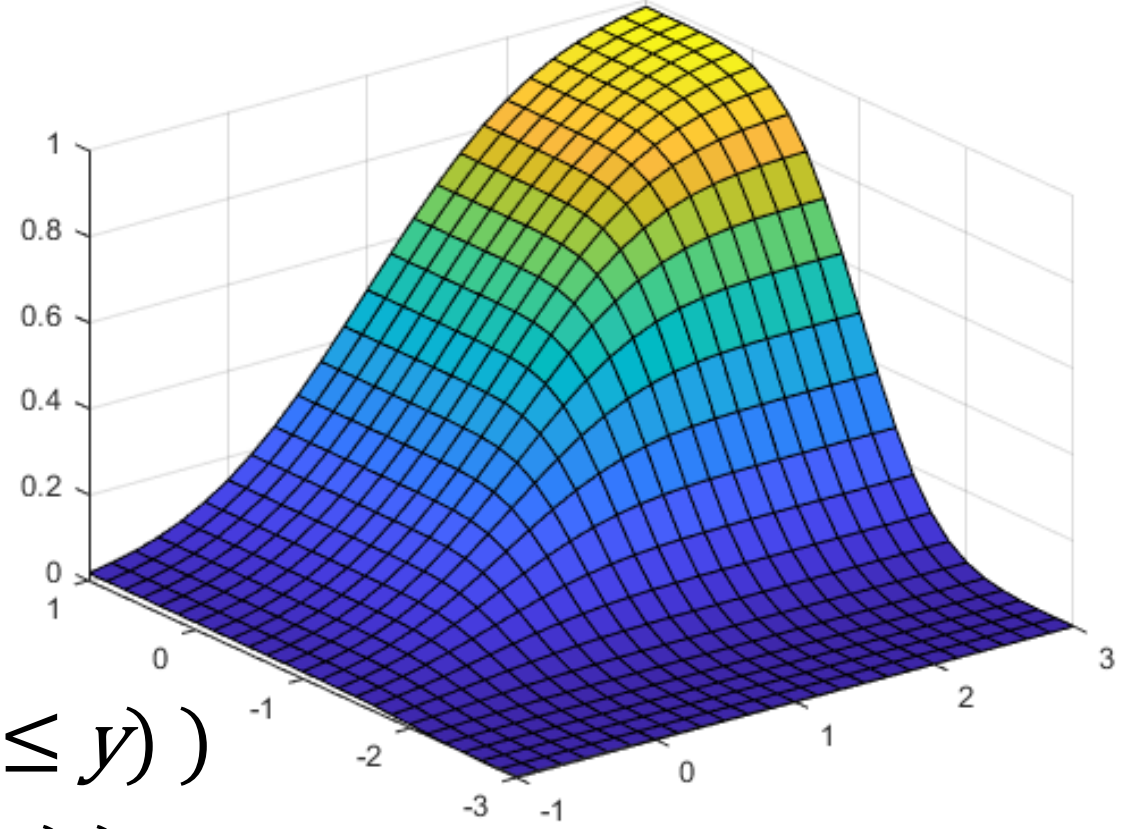
$$= 0.006 \left(-\frac{e^{-0.002y}}{0.002} \bigg|_{2000}^{\infty} + \frac{e^{-0.003y}}{0.003} \bigg|_{2000}^{\infty} \right)$$

$$= 3e^{-4} - 2e^{-6} = 0.05$$

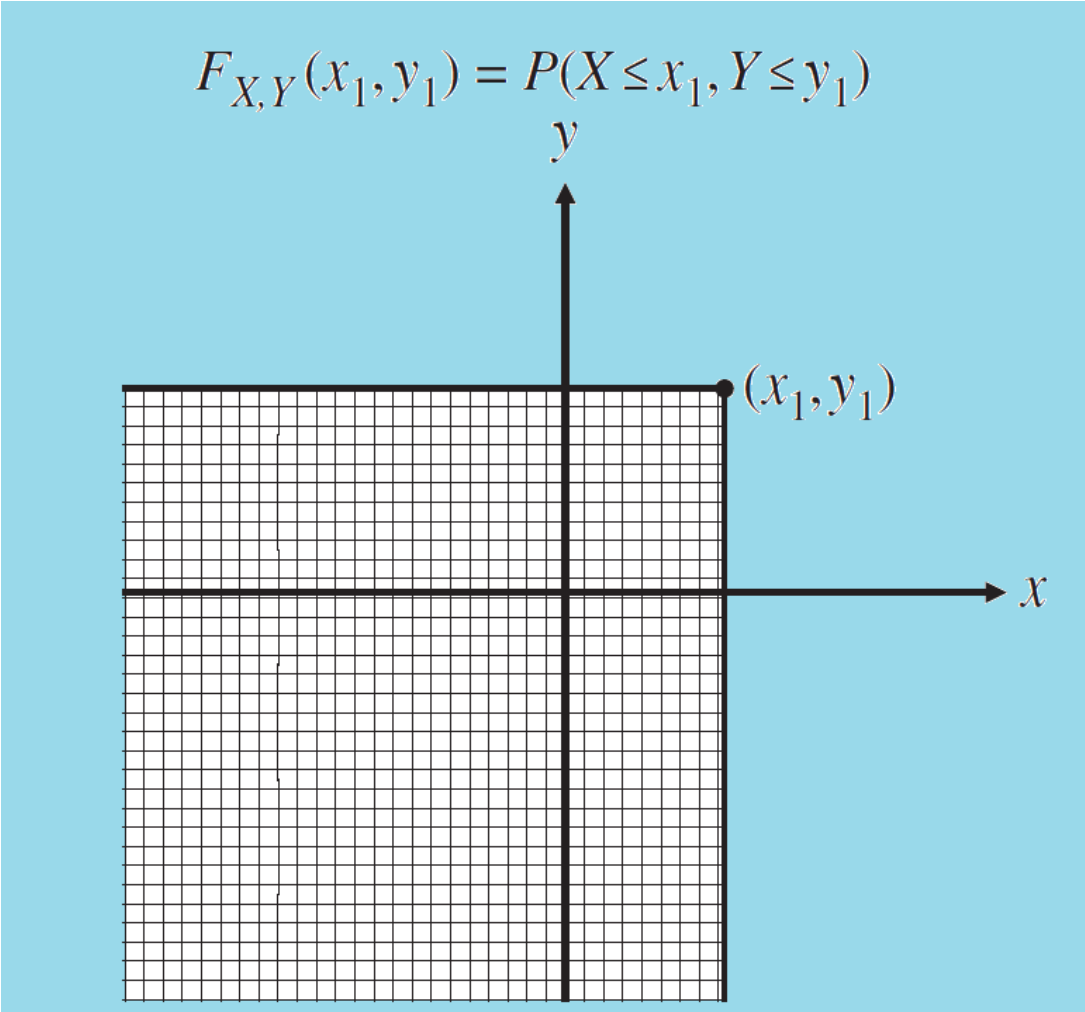
Joint Cumulative Distributive Function (CDF) of a continuous random variables

- The joint cumulative distribution function of two random variables X and Y is defined as

$$F_{XY}(x,y) = P(X \leq x, Y \leq y)$$
$$= P(\,(X \leq x) \text{ and } (Y \leq y)\,)$$
$$= P(\,(X \leq x) \cap (Y \leq y)\,)$$

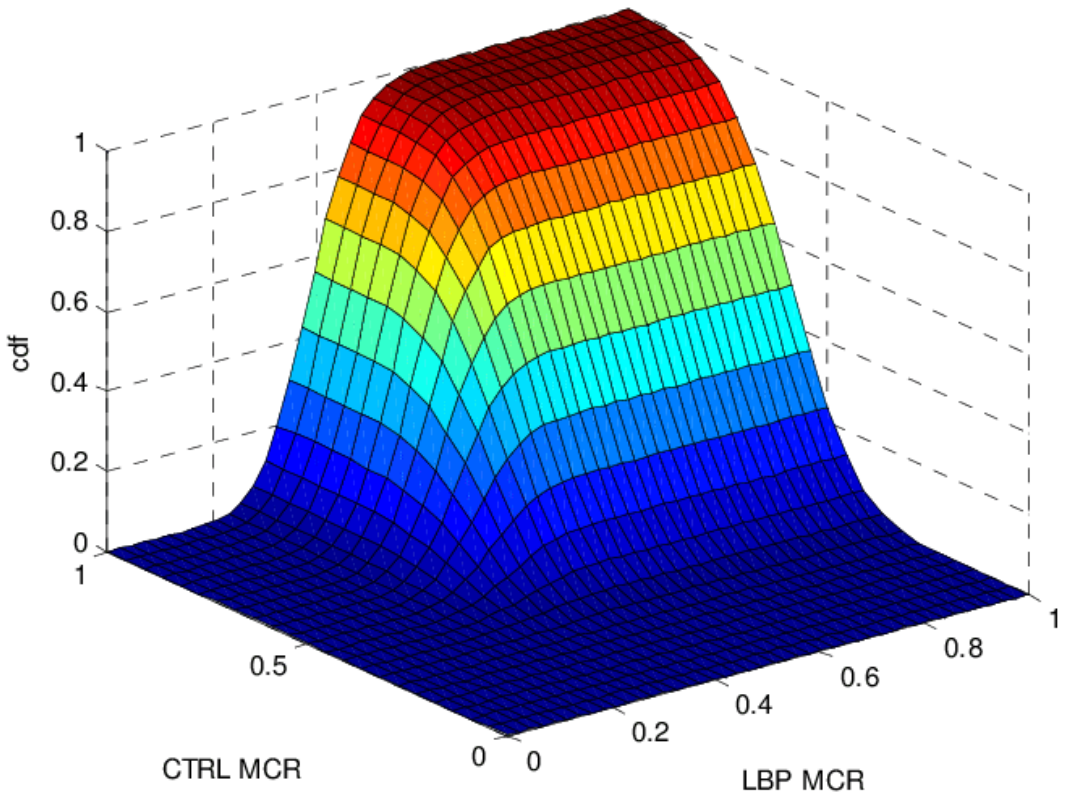


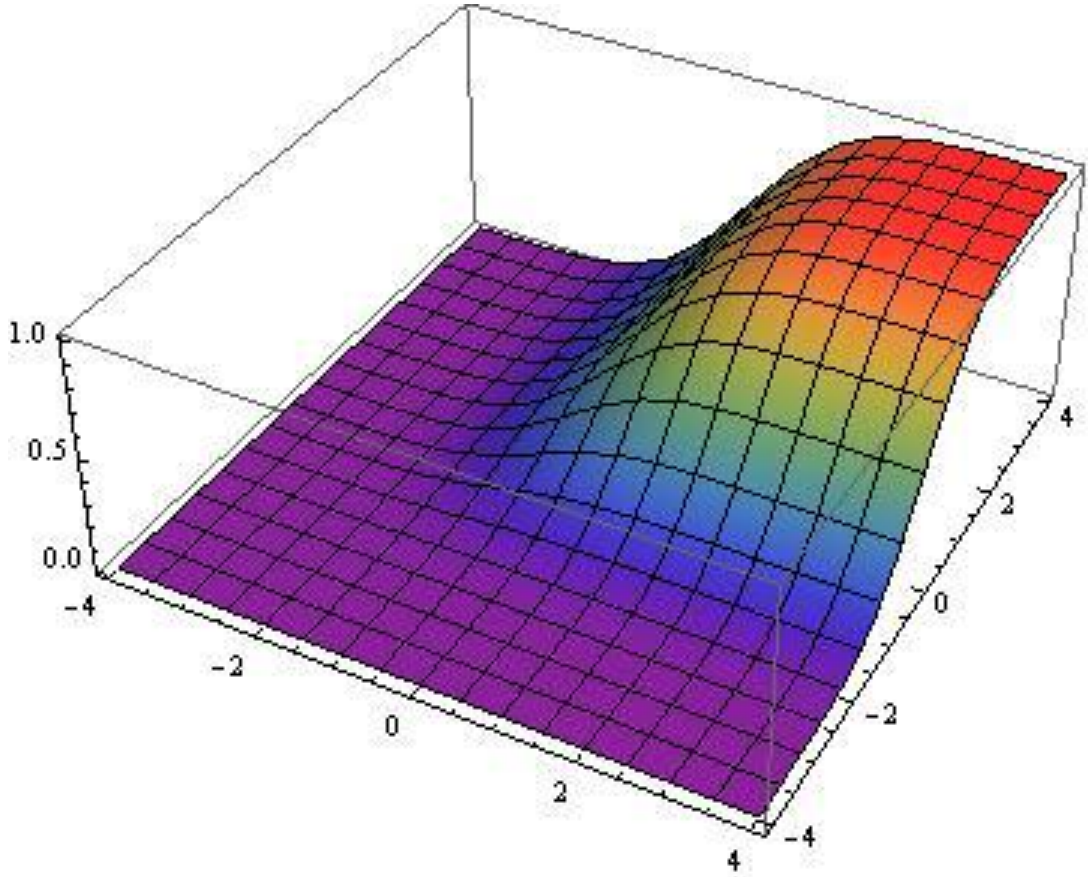
Joint CDF defined by the semi-infinite rectangle



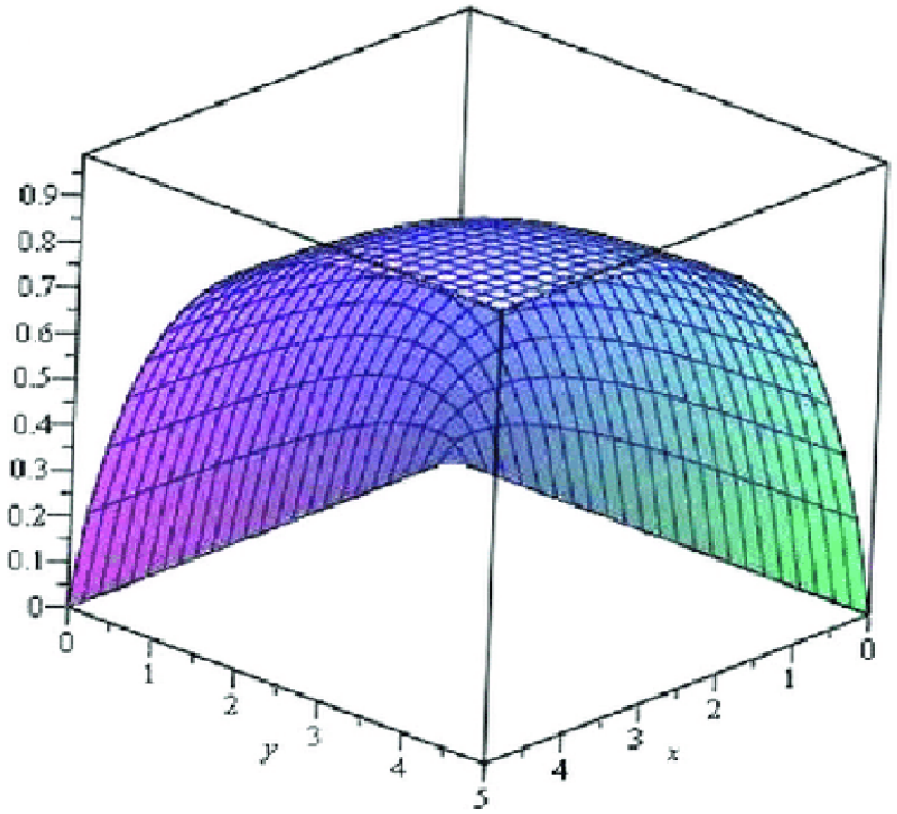
Examples of Joint Cumulative Distributive Function (CDF)

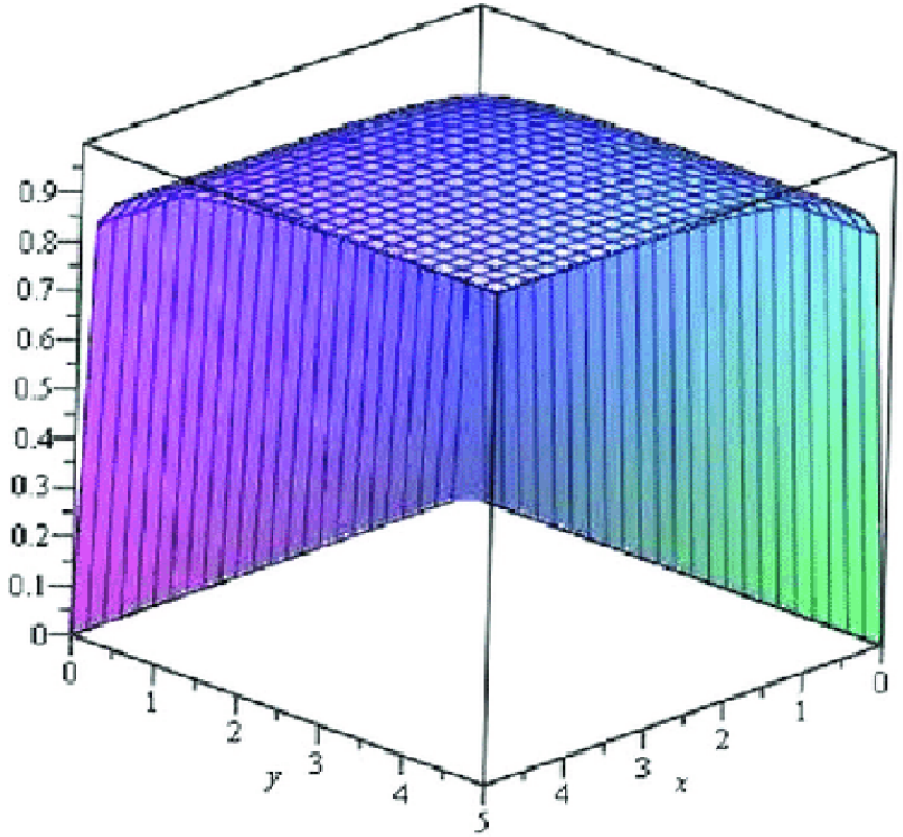
08/05/2024



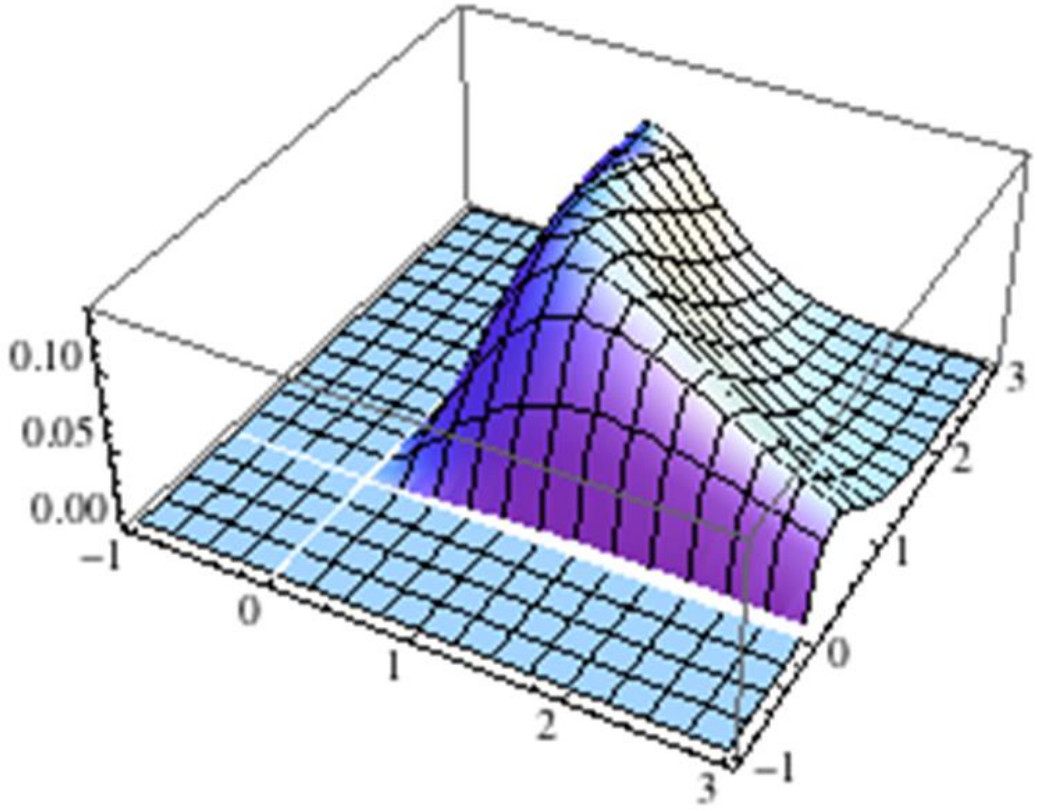


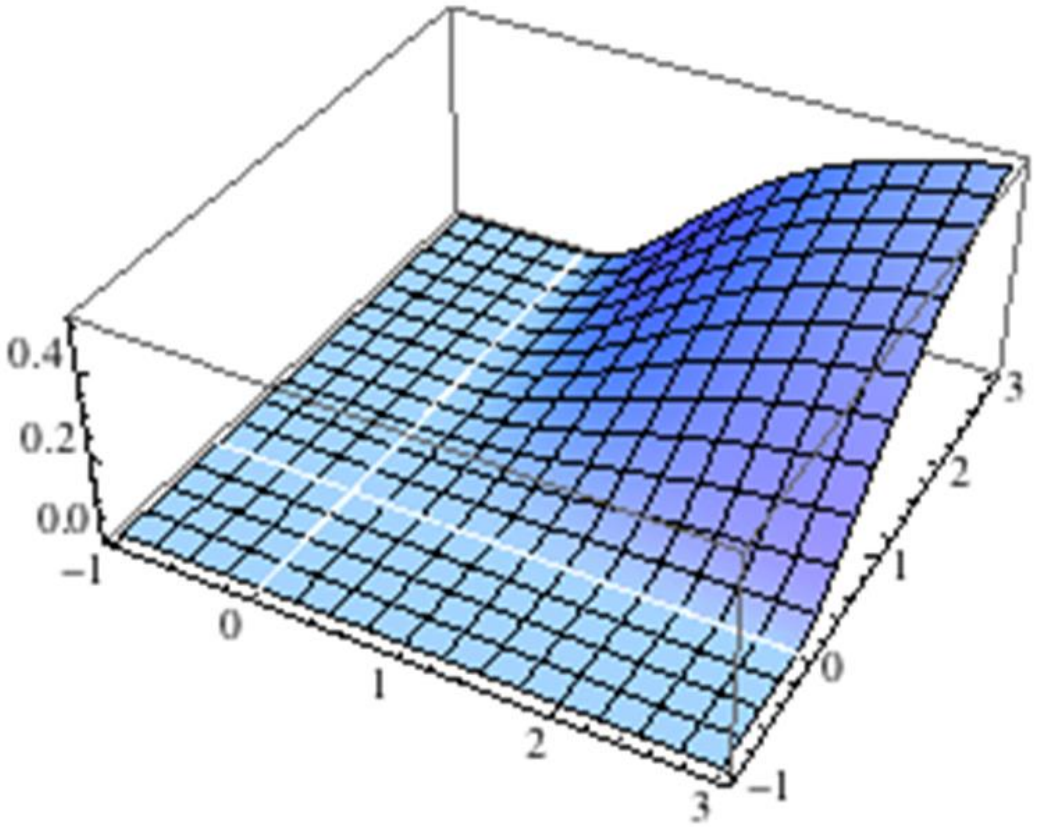
Examples of Joint Cumulative Distributive Function (CDF)





Examples of **PMF** and **CDF**





- PDF
- CDF

The **joint cumulative function** (**CDF**) and **marginal CDF** of two random variables X and Y is defined as

$$F_{XY}(x,y) = P(X \leq x, Y \leq y).$$

The joint CDF satisfies the following properties:

1. $F_X(x) = F_{XY}(x,\infty)$, for any x (marginal CDF of X);
2. $F_Y(y) = F_{XY}(\infty,y)$, for any y (marginal CDF of Y);
3. $F_{XY}(\infty,\infty) = 1$;
4. $F_{XY}(-\infty,y) = F_{XY}(x,-\infty) = 0$;
5. $P(x_1 < X \leq x_2,\ \ y_1 < Y \leq y_2) =$
$$F_{XY}(x_2,y_2) - F_{XY}(x_1,y_2) - F_{XY}(x_2,y_1) + F_{XY}(x_1,y_1)$$
6. if X and Y are independent, then $F_{XY}(x,y) = F_X(x)F_Y(y)$.

Relationship between **PDF** and **CDF**:

$$F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(u,v)\,du\,dv,$$

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

Exercise 4.11

- Using the information given in Exercise 4.9,

  - a) Compute the respective CDF $F_{XY}(x, y)$.
  - b) Find the marginal CDF $F_X(x)$.
  - c) Find the marginal CDF $F_Y(y)$

- The PDF reported in Exercise 4.9 was

$$f_{XY}(x, y) = 6.0 \text{ x } 10^{-6}\, e^{-0.001x - 0.002y} \text{ for } x < y$$

Exercise 4.11: Solution

- a) Compute the respective CDF $F_{XY}(x, y)$.
  - Therefore, it can be done

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)\,du\,dv = \int_{0}^{x}\int_{x}^{y} f_{XY}(u,v)\,du\,dv$$

$$= \int_{0}^{x}\left(\int_{u}^{y} 6.0 \text{ x } 10^{-6}\, e^{-0.001u-0.002v}\,dv\right)du$$

$$= 6.0 \text{ x } 10^{-6} \int_{0}^{x}\left(\int_{u}^{y} e^{-0.002v}\,dv\right)e^{-0.001u}\,du$$

Exercise 4.11: Solution

- Solving the integral

$$F_{XY}(x,y) = 6.0 \text{ x } 10^{-6} \int_0^x \left(-\frac{e^{-0.002v}}{0.002}\Bigg|_u^y \right) e^{-0.0001u} du$$

$$= 6.0 \text{ x } 10^{-6} \int_0^x \left(\frac{e^{-0.002u} - e^{-0.002y}}{0.002} \right) e^{-0.001u} du$$

$$= 0.003 \int_0^x \left(e^{-0.003u} - e^{-0.002y} e^{-0.001u} \right) du$$

Exercise 4.11: Solution

- Continuation 4.11.a

$$F_{XY}(x,y) = 0.003\left\{\left(-\frac{e^{-0.003u}}{0.003}\bigg|_0^x\right) - e^{-0.002y}\left(-\frac{e^{-0.001u}}{0.001}\bigg|_0^x\right)\right\}$$

- Finally,

$$F_{XY}(x,y) = 1 - e^{-0.003x} - 3e^{-0.002y}\left(1 - e^{-0.001x}\right),$$

for $x > 0$ and $x < y$

Exercise 4.11: Solution

- b) Find the marginal CDF $F_X(x)$.

$$F_{XY}(x,y) = 1 - e^{-0.003x} - 3e^{-0.002y}\left(1 - e^{-0.001x}\right),$$

for $x > 0$ and $x < y$

- where

$$F_X(x) = \lim_{y\to\infty} F_{XY}(x,y)$$

- Therefore, it leads to

$$F_X(x) = \lim_{y\to\infty}\left\{1 - e^{-0.003x} - 3e^{-0.002y}\left(1 - e^{-0.001x}\right)\right\}$$

$$F_X(x) = 1 - e^{-0.003x}$$

Exercise 4.11: Solution

- c) Find the marginal CDF $F_Y(y)$.

$$F_Y(y) = \lim_{x\to\infty} F_{XY}(x,y)$$

- Observe that when $x \to \infty \Longrightarrow y \to \infty$

$$F_Y(y) = \lim_{\substack{x\to\infty \\ y>x}} \left\{1 - e^{-0.003x} - 3e^{-0.002y}\left(1 - e^{-0.001x}\right)\right\} = 1\,???$$

- On the other hand, $F_Y(y)$ can also be computed by:

$$F_Y(y) = \int f_Y(y)dy = \int_0^y 0.006e^{-0.002u}\left(1 - e^{-0.001u}\right)du$$

$$= 1 - 3e^{-0.002y} + 2e^{-0.003y}$$

Conditioning

$$E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x) dx,$$

$$E[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x) dx,$$

$$\mathrm{Var}(X|A) = E[X^2|A] - (E[X|A])^2$$

Conditioning by Another Random Variable

- For two jointly continuous random variables ***X*** and ***Y***, we can define the following conditional concepts:

1. The conditional PDF of X given $Y = y$:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

2. The conditional probability that $X \in A$ given $Y = y$:

$$P(X \in A|Y = y) = \int_A f_{X|Y}(x|y)dx$$

3. The conditional CDF of X given $Y = y$:

$$F_{X|Y}(x|y) = P(X \leq x|Y = y) = \int_{-\infty}^{x} f_{X|Y}(x|y)dx$$

Conditioning by Another Random Variable

- Because the **conditional probability density function** $f_{X|Y}(x)$ (or $f_{Y|X}(\mathrm{y})$) is a **probability density function (PDF)** for all x (or all y) in R_X, the following properties are satisfied:

$$f_{X|Y}(x) = \frac{f_{XY}(x, y)}{f_Y(y)} \geq 0 \qquad f_{Y|X}(\mathrm{y}) = \frac{f_{XY}(x, y)}{f_X(x)} \geq 0$$

$$\int f_{Y|X}(\mathrm{y})dy = 1 \qquad \int f_{X|Y}(x)dx = 1$$

$$P(X \in A|Y = y) = \int_A f_{X|Y}(x)dx \qquad P(Y \in B|X = x) = \int_B f_{Y|X}(\mathrm{y})dy$$

Exercise 4.12

- For the random variables that denote times in Exercise 4.9,
  - a) Determine the conditional probability density function for $\boldsymbol{Y}$ given that $\boldsymbol{X} = x$.
  - b) Determine $P(\boldsymbol{Y} > 2000|\boldsymbol{X} = 1500)$

- The joint PDF is given

$$f_{XY}(x, y) = 6.0 \text{ x } 10^{-6}\, e^{-0.001x-0.002y} \text{ for } x < y$$

Exercise 4.12: Solution

- a) The conditional probability density function for $\boldsymbol{Y}$ given that $\boldsymbol{X} = x$ is computed by

$$f_{Y|X}(\mathrm{y}) = \frac{f_{XY}(x, y)}{f_X(x)}$$

- First, we need to compute the marginal density function of x, for $x > 0$,

$$f_X(x) = \int f_{XY}\,(x, y)dy$$

Exercise 4.12: Solution

- Substituting $f_{XY}(x, y)$ in its the equation

$$f_X(x) = \int_x^{\infty} 6.0 \text{ x } 10^{-6}\, e^{-0.001x - 0.002y} dy$$

$$= 6.0 \text{ x } 10^{-6} e^{-0.001x} \int_x^{\infty} e^{-0.002y} dy$$

$$= 6.0 \text{ x } 10^{-6} e^{-0.001x} \left(-\frac{e^{-0.002y}}{0.002} \Bigg|_x^{\infty} \right)$$

Exercise 4.12: Solution

- Finally

$$f_X(x) = 0.003e^{-0.003x}, \text{for } x > 0$$

- Therefore

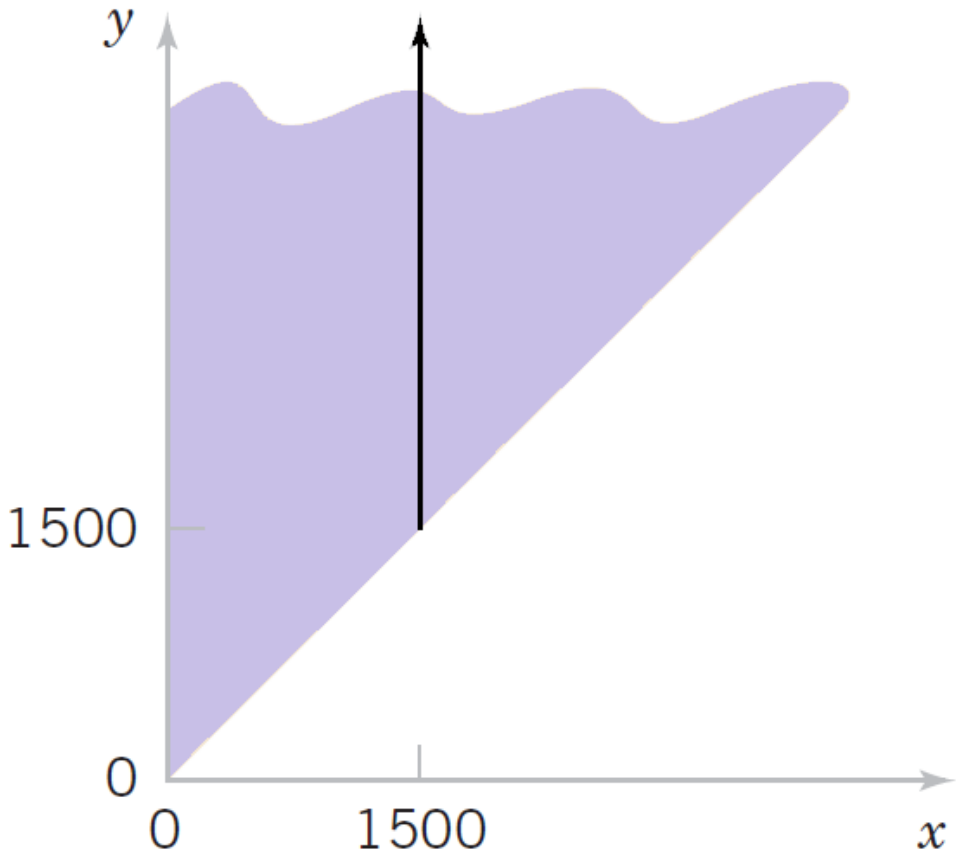
$$f_{Y|X}(\mathrm{y}) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{6.0 \text{ x } 10^{-6}\ e^{-0.001x-0.002y}}{0.003e^{-0.003x}}$$

$$= 0.002e^{0.002x-0.002y}, \text{for } x > 0 \text{ and } x < y$$

Exercise 4.12: Solution

- b) Determine $P(\boldsymbol{Y} > 2000|\boldsymbol{X} = 1500)$

- The conditional probability density function for $\boldsymbol{Y}$, given that x = 1500, is nonzero over the solid line.



Exercise 4.12: Solution

- Substituting $f_{X|Y}$ (x, y) in its the equation for x = 1500, it is obtained:

$$P(\boldsymbol{Y} > 2000|\boldsymbol{X} = 1500) = \int_{2000}^{\infty} f_{Y|X=1500}(\mathrm{y})dy$$

$$= \int_{2000}^{\infty} 0.002e^{0.002(1500)-0.002y}dy$$

Exercise 4.12: Solution

- Finally,

$$P(\boldsymbol{Y} > 2000|\boldsymbol{X} = 1500) = 0.002e^{3}\int_{2000}^{\infty} e^{-0.002y}dy$$

$$= 0.002e^{3}\left(-\frac{e^{-0.002x}}{0.002}\bigg|_{2000}^{\infty}\right)$$

$$= e^{3}e^{-4} = e^{-1} = 0.368$$

- Physical interpretation: If the connect time is 1500 ms, then the time probability to be authorized greater then 2000 ms is 36.8%.

Conditional mean

- The conditional mean of **X** given **Y** = y, denoted as $E[\boldsymbol{X}|\boldsymbol{Y}=y]$ or $\mu_{X|Y}$, is

$$E[\boldsymbol{X}|\boldsymbol{Y}=y]=\int x f_{X|Y}(x)dx$$

- Similarly, the conditional mean of ***Y*** given ***X*** = *x*, denoted as $E[\boldsymbol{Y}|\boldsymbol{X}=x]$ or $\mu_{Y|X}$, is

$$E[\boldsymbol{Y}|\boldsymbol{X}=x]=\int y f_{Y|X}(\mathrm{y})dy$$

Conditional variance

- The conditional variance of **X** given **Y** = y, denoted as $Var[\boldsymbol{X}|\boldsymbol{Y}=y]$ or $\sigma^2_{X|Y}$, is

$$Var[\boldsymbol{X}|\boldsymbol{Y}=y] = \int \left(x-\mu_{X|Y}\right)^2 f_{X|Y}(x)dx = \int x^2 f_{X|Y}(x)dx - \mu^2_{X|Y}$$

- Similarly, the conditional variance of ***Y*** given ***X*** = *x*, denoted as $Var[\boldsymbol{Y}|\boldsymbol{X}=x]$ or $\sigma^2_{Y|X}$, is

$$Var[\boldsymbol{Y}|\boldsymbol{X}=x] = \int \left(y-\mu_{Y|X}\right)^2 f_{Y|X}(\mathrm{y})dy = \int y^2 f_{Y|X}(y)dy - \mu^2_{Y|X}$$

Exercise 4.13

- For the random variables that denote times in Exercise 4.9:
  - a) Find the conditional mean for $\boldsymbol{Y}$ given that $x = 1500$.

Exercise 4.13: Solution

- a) Find the conditional mean for $\boldsymbol{Y}$ given that x = 1500.

- From Exercise 4.12 it is obtained

$$f_{Y|X}(\mathrm{y}) = \frac{f_{XY}(x, y)}{f_X(x)} = 0.002e^{0.002x - 0.002y}, \text{for } x > 0 \text{ and } x < y$$

Exercise 4.13: Solution

- The conditional mean for **Y** when **X** = 1500 can be computed by.

$$E[\boldsymbol{Y}|\boldsymbol{X} = x] = \int y f_{Y|X}(\mathrm{y}) dy$$

$$E[\boldsymbol{Y}|\boldsymbol{X} = 1500] = \int_{1500}^{\infty} y \left\{ 0.002 e^{0.002(1500) - 0.002y} \right\} dy$$

Exercise 4.13: Solution

- Moving the multiplicative constants outside the integral

$$E[\boldsymbol{Y}|\boldsymbol{X} = 1500] = 0.002e^{3}\int_{1500}^{\infty} ye^{-0.002y}dy$$

- integrating by parts: $\int uv' = uv - \int u'v \qquad u = y; v' = e^{-0.002y}$

$$E[\boldsymbol{Y}|\boldsymbol{X} = 1500] =$$

$$= 0.002e^{3}\left\{\left(-y\frac{e^{-0.002y}}{0.002}\Bigg|_{1500}^{\infty}\right) - \int_{1500}^{\infty}\left(-\frac{e^{-0.002y}}{0.002}\right)dy\right\}$$

Exercise 4.13: Solution

- Solving the integral

$$E[\boldsymbol{Y}|\boldsymbol{X} = 1500] = 1500 - 0.002e^{3}\left(\frac{e^{-0.002y}}{4 \text{ x } 10^{-6}}\bigg|_{1500}^{\infty}\right)$$

$$E[\boldsymbol{Y}|\boldsymbol{X} = 1500] = 1500 - 0.002e^{3}\left(-\frac{e^{-3}}{4 \text{ x } 10^{-6}}\right) = 1500 + 500$$

$$E[\boldsymbol{Y}|\boldsymbol{X} = 1500] = 2000 \text{ ms}$$

- Physical interpretation: If the connect time is 1500 ms, then the expected time to be authorized is 2000 ms.

Independent Continuous Random Variables

- For random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$, if any one of the following properties is true, the others are also true, and $\boldsymbol{X}$ and $\boldsymbol{Y}$ are **independent**.

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

$$f_{Y|X}(\mathrm{y}) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{f_X(x) f_Y(y)}{f_X(x)} = f_Y(y)$$

$$f_{X|X}(\mathrm{y}) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{f_X(x) f_Y(y)}{f_Y(y)} = f_X(x)$$

$P(\boldsymbol{X} \in A, \boldsymbol{Y} \in B) = P(\boldsymbol{X} \in A) P(\boldsymbol{Y} \in B)$, for any events $\boldsymbol{A}$ and $\boldsymbol{B}$ in the range of X and Y, respectively.

Exercise 4.14

- The statement of exercise 9 is repeated as follows: Let the random variable $\boldsymbol{X}$ denote the time until a computer server connects to your machine (in milliseconds), and let $\boldsymbol{Y}$ denote the time until the server authorizes you as a valid user (in milliseconds).
- Each of these random variables measures the wait from a common starting time and $\boldsymbol{X} < \boldsymbol{Y}$.
- Suppose that Exercise 4.9 **is modified** so that the joint probability density function of $\boldsymbol{X}$ and $\boldsymbol{Y}$ is

$$f_{XY}(x, y) = 2.0 \times 10^{-6}\, e^{-0.001x - 0.002y} \text{ for } x \geq 0,\ y \geq 0$$

  - a) Show that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent.
  - b) Find $P(\boldsymbol{X} > 1000, \boldsymbol{Y} < 1000)$.

Exercise 4.14: Solution

- a) Show that ***X*** and ***Y*** are independent.
- If variables ***X*** and ***Y*** are independent, then we have to show that

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

and

$$f_X(x) = \int f_{XY}(x, y) dy, \text{for all } x$$

$$f_Y(y) = \int f_{XY}(x, y) dx, \text{for all } y$$

Exercise 4.14: Solution

- Note that the range of positive probability is rectangular so that independence is possible but not yet demonstrated.
- The marginal probability density function of $\boldsymbol{X}$ is

$$f_X(x) = \int_0^\infty f_{XY}(x, y)dy$$

$$f_X(x) = \int_0^\infty 2.0 \text{ x } 10^{-6}\, e^{-0.001x-0.002y} dy$$

$$= 2.0 \text{ x } 10^{-6} e^{-0.001x} \left(-\frac{e^{-0.002y}}{0.002}\bigg|_0^\infty \right)$$

$$= 0.001e^{-0.001x}, \text{for } x > 0.$$

Exercise 4.14: Solution

- Similarlly, the marginal probability density function of **Y** is

$$f_Y(y) = \int_0^{\infty} f_{XY}(x, y)dx$$

$$f_Y(y) = \int_0^{\infty} 2.0 \text{ x } 10^{-6}\, e^{-0.001x - 0.002y} dx$$

$$= 2.0 \text{ x } 10^{-6} e^{-0.002y} \left(-\frac{e^{-0.001x}}{0.001}\Bigg|_0^{\infty} \right)$$

$$= 0.002e^{-0.002y}, \text{for } y > 0$$

Exercise 4.14: Solution

- Therefore,

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

$$= 2.0 \text{ x } 10^{-6}\, e^{-0.001x - 0.002y}$$

- for all $x \geq 0$ and $y \geq 0$, and $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent.

Exercise 4.14: Solution

- b) Find $P(\boldsymbol{X} > 1000, \boldsymbol{Y} < 1000)$.

$$P(\boldsymbol{X} > 1000, \boldsymbol{Y} < 1000) = P(\boldsymbol{X} > 1000)P(\boldsymbol{Y} < 1000)$$

$$P(\boldsymbol{X} > 1000, \boldsymbol{Y} < 1000) =$$

$$= \int_{1000}^{\infty} 0.001e^{-0.001x}dx \int_{0}^{1000} 0.002e^{-0.002y}dy$$

$$= 0.001\left(-\frac{e^{-0.001x}}{0.001}\Big|_{1000}^{\infty}\right) 0.002\left(-\frac{e^{-0.002y}}{0.002}\Big|_{0}^{1000}\right)$$

$$= e^{-1}(1 - e^{-2}) = 0.318$$

Summary: Independent Continuous Random Variables

Two continuous random variables X and Y are independent if

$$f_{XY}(x,y) = f_X(x)f_Y(y), \quad \text{for all } x, y.$$

Equivalently, X and Y are independent if

$$F_{XY}(x,y) = F_X(x)F_Y(y), \quad \text{for all } x, y.$$

If X and Y are independent, we have

$$E[XY] = E[X]E[Y]\,,$$
$$E[g(X)h(Y)] = E[g(X)]E[h(Y)].$$

Summary: two jointly continuous random variables X and Y: Conditional **expectation and variance**

1. Expected value of X given $Y = y$:

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

2. Conditional LOTUS:

$$E[g(X)|Y = y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y) dx$$

3. Conditional variance of X given $Y = y$:

$$Var(X|Y = y) = E[X^2|Y = y] - (E[X|Y = y])^2$$

The **Law Of The Unconscious Statistician (LOTUS)**

Total Probability

- Law of Total Probability:

$$P(A) = \int_{-\infty}^{\infty} P(A|X = x) f_X(x) \ dx$$

- Law of Total Expectation:

$$E[Y] = \int_{-\infty}^{\infty} E[Y|X = x] f_X(x) \ dx = E[E[Y|X]]$$

- Law of Total Variance:

$$\mathrm{Var}(Y) = E[\mathrm{Var}(Y|X)] + \mathrm{Var}(E[Y|X])$$

Functions of Two Continuous Random Variables

- LOTUS for two continuous random variables:

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)\ dxdy$$

The Method of Transformations

Theorem

Let X and Y be two jointly continuous random variables. Let $(Z,W) = g(X,Y) = (g_1(X,Y), g_2(X,Y))$, where $g : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a continuous one-to-one (invertible) function with continuous partial derivatives. Let $h = g^{-1}$, i.e., $(X,Y) = h(Z,W) = (h_1(Z,W), h_2(Z,W))$. Then Z and W are jointly continuous and their joint PDF, $f_{ZW}(z,w)$, for $(z,w) \in R_{ZW}$ is given by

$$f_{ZW}(z,w) = f_{XY}(h_1(z,w), h_2(z,w))|J|,$$

where J is the Jacobian of h defined by

$$J = \det \begin{bmatrix} \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \\ \\ \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial w} \end{bmatrix} = \frac{\partial h_1}{\partial z} \cdot \frac{\partial h_2}{\partial w} - \frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial w}.$$

Linear convolution

$$f_Z(z) = f_X(z) * f_Y(z)$$
$$= \int_{-\infty}^{\infty} f_X(w) f_Y(z-w) dw = \int_{-\infty}^{\infty} f_Y(w) f_X(z-w) dw$$

Linear convolution

If X and Y are two jointly continuous random variables and $Z = X + Y$, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(w, z-w)dw = \int_{-\infty}^{\infty} f_{XY}(z-w, w)dw.$$

If X and Y are also independent, then

$$\begin{aligned} f_Z(z) &= f_X(z) * f_Y(z) \\ &= \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)dw = \int_{-\infty}^{\infty} f_Y(w) f_X(z-w)dw. \end{aligned}$$

Inner Product

Notation: $< X.Y >$

- For continuous complex functions (continuous signals):

$$< x(t).y(t) > = \int_{-\infty}^{\infty} x(t)y(t)^{*}dt$$

- For discrete complex sequences (digital signals):

$$< x_i.y_i > = \sum_{i} x_i {y_i}^{*}$$

Covariance as the Inner Product (scalar product between two variables)

- If the random variables are continuous:

$$Cov(\boldsymbol{X},\boldsymbol{Y}) = <(\mathrm{X}-\mu_{\boldsymbol{X}}).(\mathrm{Y}-\mu_{\boldsymbol{Y}})> = \int_{-\infty}^{\infty}\{x(t)-\mu_{\boldsymbol{X}}\}\{\mathrm{y}(t)-\mu_{\boldsymbol{Y}}\}dt$$

- The random variables are discrete:

$$Cov(\boldsymbol{X},\boldsymbol{Y}) = <(\mathrm{X}-\mu_{\boldsymbol{X}}).(\mathrm{Y}-\mu_{\boldsymbol{Y}})> = \sum_{i}(x_i-\mu_{\boldsymbol{X}})(y_i-\mu_{\boldsymbol{Y}})$$

Covariance – discrete and continuous variables

- The **covariance** between $\boldsymbol{X}$ and $\boldsymbol{Y}$ is defined as

$$Cov(X,Y) = E[(X-\mu_{\boldsymbol{X}})(Y-\mu_{\boldsymbol{Y}})] = E[(X-E[X])(Y-E[Y])]$$

$$= E\big[XY - XE[Y] - YE[X] + E[X]E[Y]\big]$$

$$= E[XY] - E[X]E[Y] - E[Y]E[X] + E[X]E[Y]$$

$$= E[XY] - E[X]E[Y] = E[XY] - \mu_{\boldsymbol{X}}\mu_{\boldsymbol{Y}}$$

The covariance properties:

1. $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$;
2. if X and Y are independent then $\mathrm{Cov}(X, Y) = 0$;
3. $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$;
4. $\mathrm{Cov}(aX, Y) = a\mathrm{Cov}(X, Y)$;
5. $\mathrm{Cov}(X + c, Y) = \mathrm{Cov}(X, Y)$;
6. $\mathrm{Cov}(X + Y, Z) = \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)$;
7. more generally,

$$\mathrm{Cov}\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \mathrm{Cov}(X_i, Y_j).$$

Pearson Correlation Coefficient: ρ

$$\rho_{XY} = \rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$|\rho_{XY}| \leq 1$$

Properties of the correlation coefficient

1. $-1 \leq \rho(X,Y) \leq 1$;
2. if $\rho(X,Y) = 1$, then $Y = aX + b$, where $a > 0$;
3. if $\rho(X,Y) = -1$, then $Y = aX + b$, where $a < 0$;
4. $\rho(aX + b, cY + d) = \rho(X,Y)$ for $a, c > 0$.

Uncorrelated variables

If X and Y are uncorrelated, then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

More generally, if $X_1, X_2, \ldots, X_n$ are pairwise uncorrelated, i.e., $\rho(X_i, X_j) = 0$ when $i \neq j$, then

$$\mathrm{Var}(X_1 + X_2 + \ldots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n).$$

Evaluation for Gaussian Joint Probability Density Function **(PDF)**

- The two-dimensional Gaussian PDF model can be expressed as:

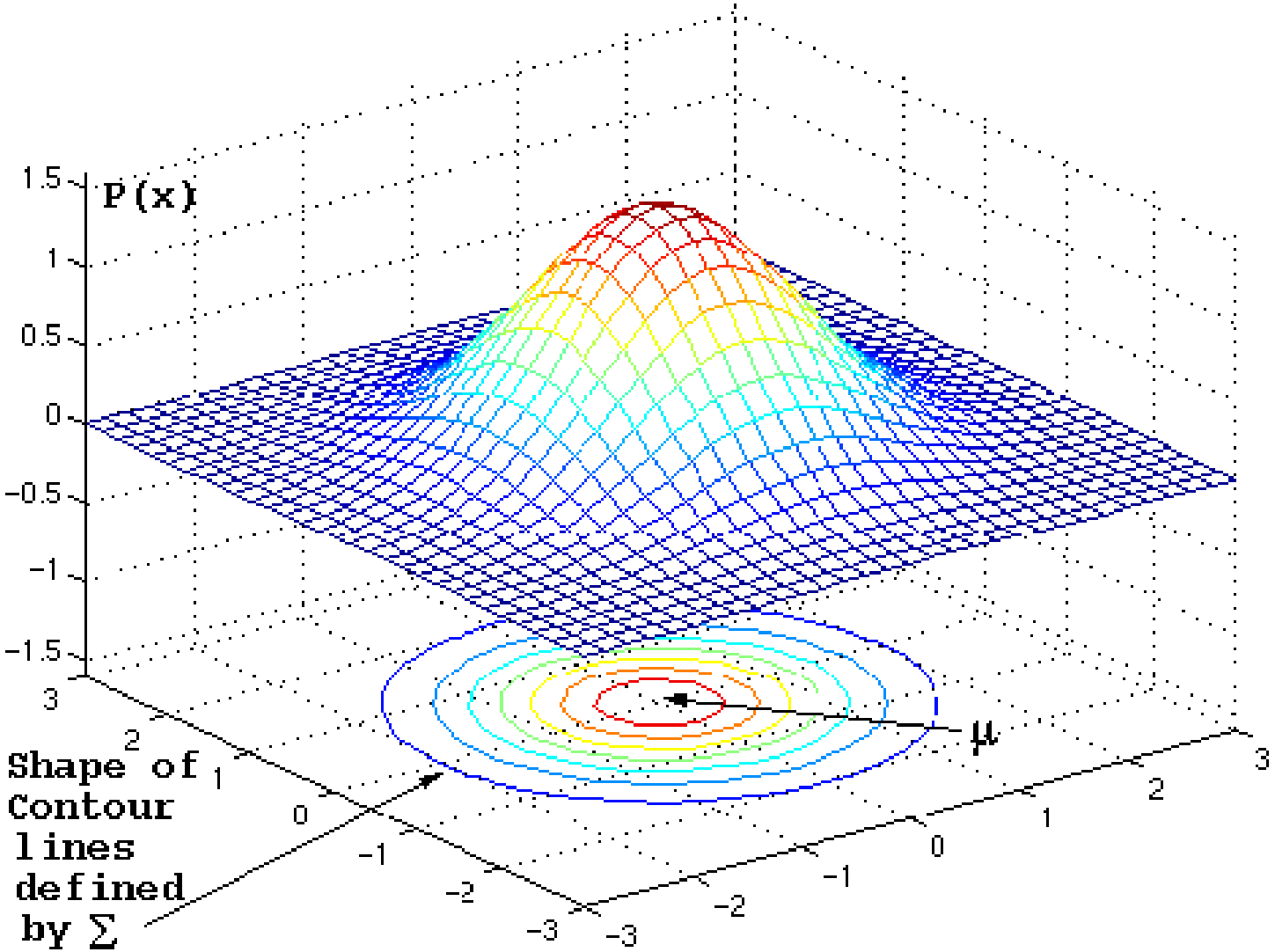
$$p(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\sqrt{(2\pi)^2 \det\{\Sigma\}}}\, e^{-\frac{1}{2}\left\{\left(\underline{Z}-\underline{\boldsymbol{\mu}}\right)^T \Sigma^{-1}\left(\underline{Z}-\underline{\boldsymbol{\mu}}\right)\right\}}$$

$$\underline{Z} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \underline{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}$$

- Covariance matrix

$$\Sigma = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{bmatrix} = \begin{bmatrix} \sigma_X^2 & \rho_{XY}\sigma_X\sigma_Y \\ \rho_{XY}\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

Gaussian Joint Probability Density Function **(PDF)**



- Covariance matrix

$$\Sigma = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{bmatrix}$$

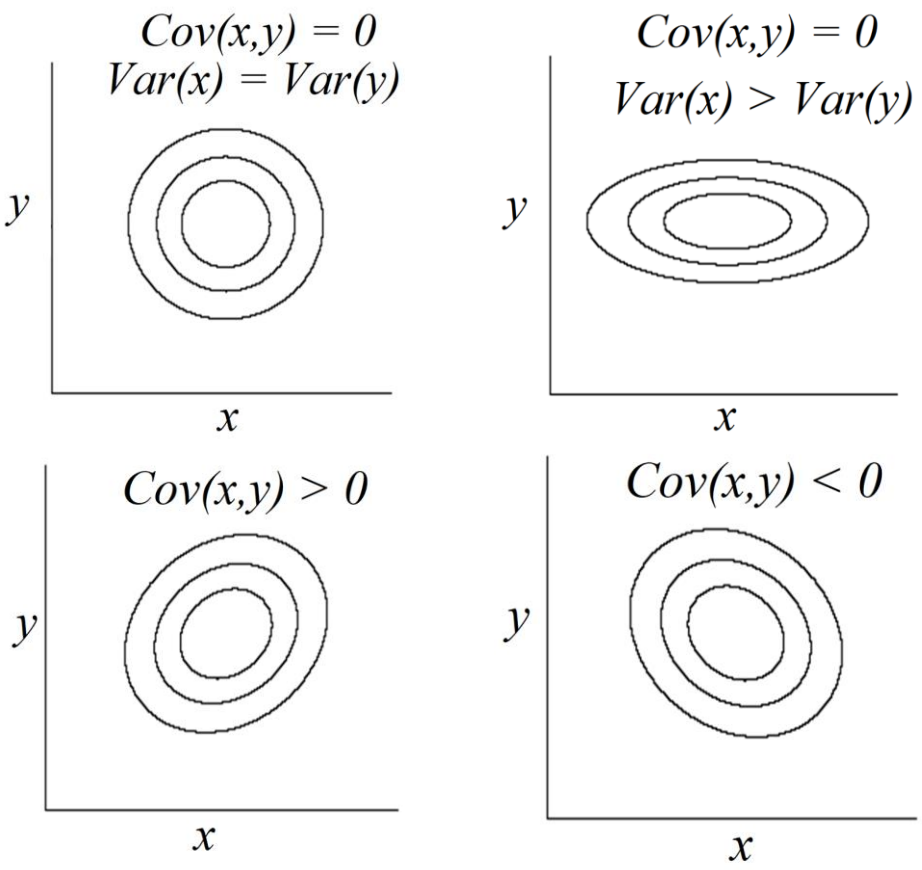
$$= \begin{bmatrix} \sigma_X^2 & \rho_{XY}\sigma_X\sigma_Y \\ \rho_{XY}\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

$$\underline{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}$$

- If $Cov(X,Y) = 0$ then $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independente.

Two random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ can be expressed bivariate Gaussian distribution

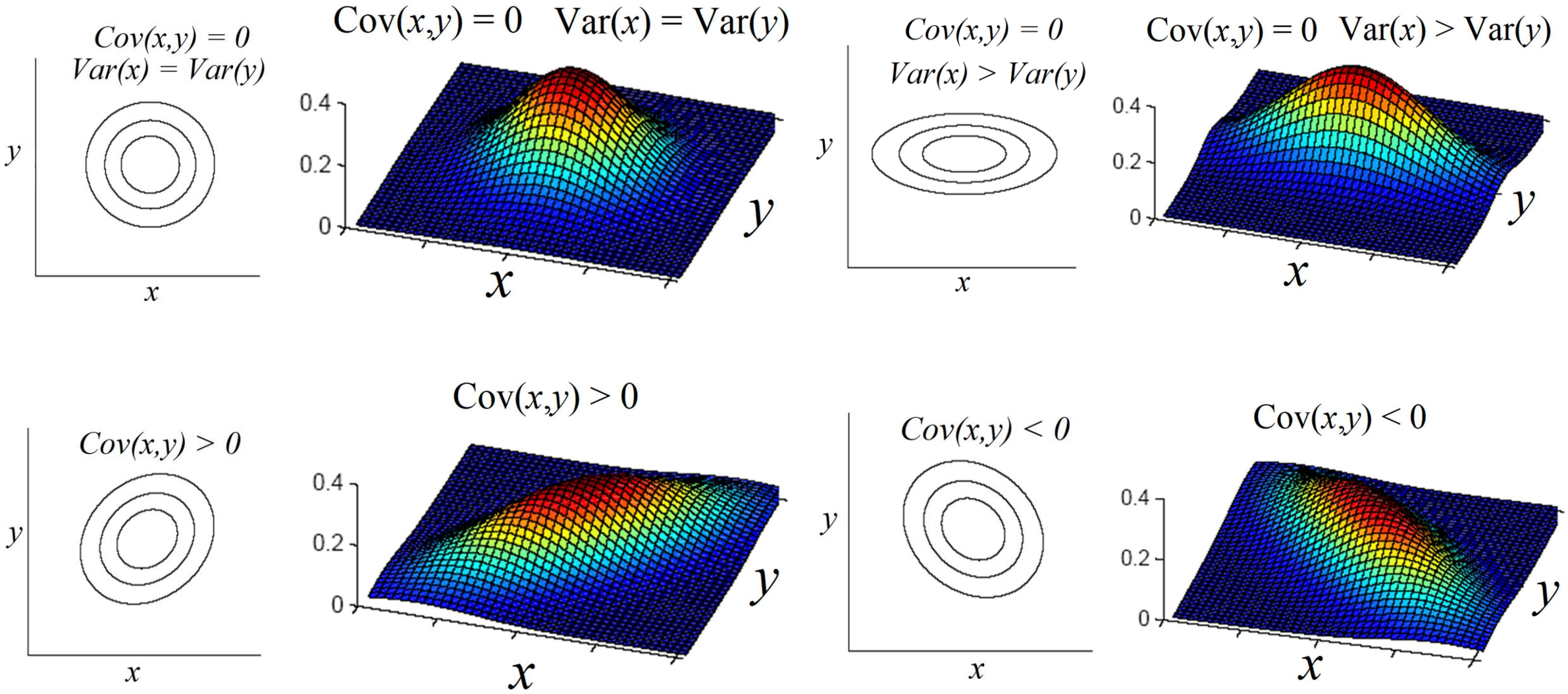
- $\boldsymbol{\rho}$ is the *Pearson* **correlation coeficient:** $\rho \in (-1, 1)$.



$$f_{XY}(X,Y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\{x^2-2\rho xy+y^2\}}$$

- For $|\rho| < 1$, $x^2 - 2\rho xy + y^2 = constant$ is the equation of an ellipse.
- If $\rho = 0$, then the two variables are independent and Var[$\boldsymbol{X}$]=Var[$\boldsymbol{Y}$]. $x^2 + y^2 = constant$ is the equation of a circle.
- If $\rho > 0$, the major axis of the ellipse has a positive slope.
- if $\rho < 0$, the major axis has a negative slope.

Gaussian Joint Probability Density Function **(PDF)**



The End of Module 4.

08/05/2024