Graph Traversal

❖ Problems on Graphs

What kind of problems do we want to solve on graphs?

- is the graph fully-connected?
- can we remove an edge and keep it fully-connected?
- is one vertex reachable starting from some other vertex?
- what is the cheapest cost path from v to w?
- which vertices are reachable from v? (transitive closure)
- is there a cycle that passes through all vertices? (circuit)
- is there a tree that links all vertices? (spanning tree)
- what is the minimum spanning tree?

While these problems are expressed as problems on graphs, they are interesting because many real world problems can be mapped onto graphs, and the solutions to the above could then be applied in solving them.

❖ Graph Traversal

Many of the above problems can be solved by

- systematic exploration of a graph, via the edges

Algorithms for this typically require us to remember

- what vertices we have already visited
- the path we followed while visiting them

Since many graph search algorithms are recursive

- above information needs to be stored globally
- and updated by individual calls to the recursive function

Systematic exploration like this is called traversal or search.

❖ ... Graph Traversal

Consider two related problems on graphs ...

- is there a path between two given vertices (*src*,*dest*)?
- what is the sequence of vertices from *src* to *dest*?

An approach to solving this problem:

- examine vertices adjacent to *src*
- if any of them is *dest*, then done
- otherwise try vertices two edges from *src*
- repeat looking further and further from *src*

The above summarises one form of graph traversal.

❖ ... Graph Traversal

There are two strategies for graph traversal/search ...

Depth-first search (DFS)

- favours following path rather than neighbours
- can be implemented recursively or iteratively (via stack)
- full traversal produces a depth-first spanning tree

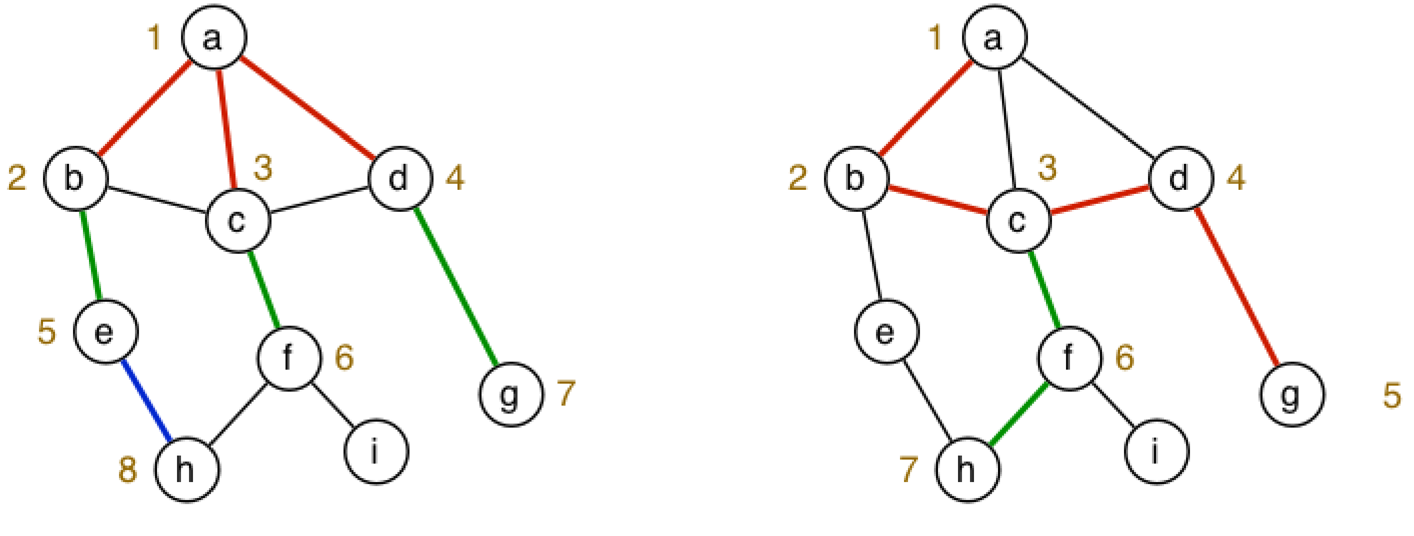
Breadth-first search (BFS)

- favours neighbours rather than path following
- can be implemented iteratively (via queue)
- full traversal produces a breadth-first spanning tree

The method on the previous slide is effectively breadth-first traversal.

❖ ... Graph Traversal

Comparison of BFS/DFS search for checking **`hasPath(a,h)`**?



Breadth-first Search *Depth-first Search*

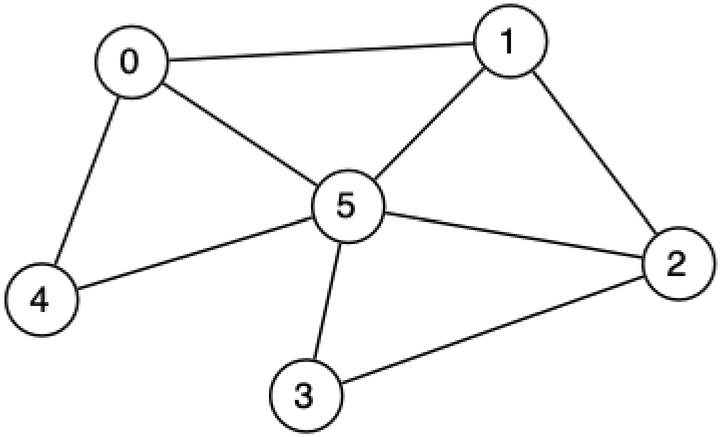
Both approaches ignore some edges by remembering previously visited vertices.

❖ ... Graph Traversal

A spanning tree of a graph

- includes all vertices, using a subset of edges, without cycles

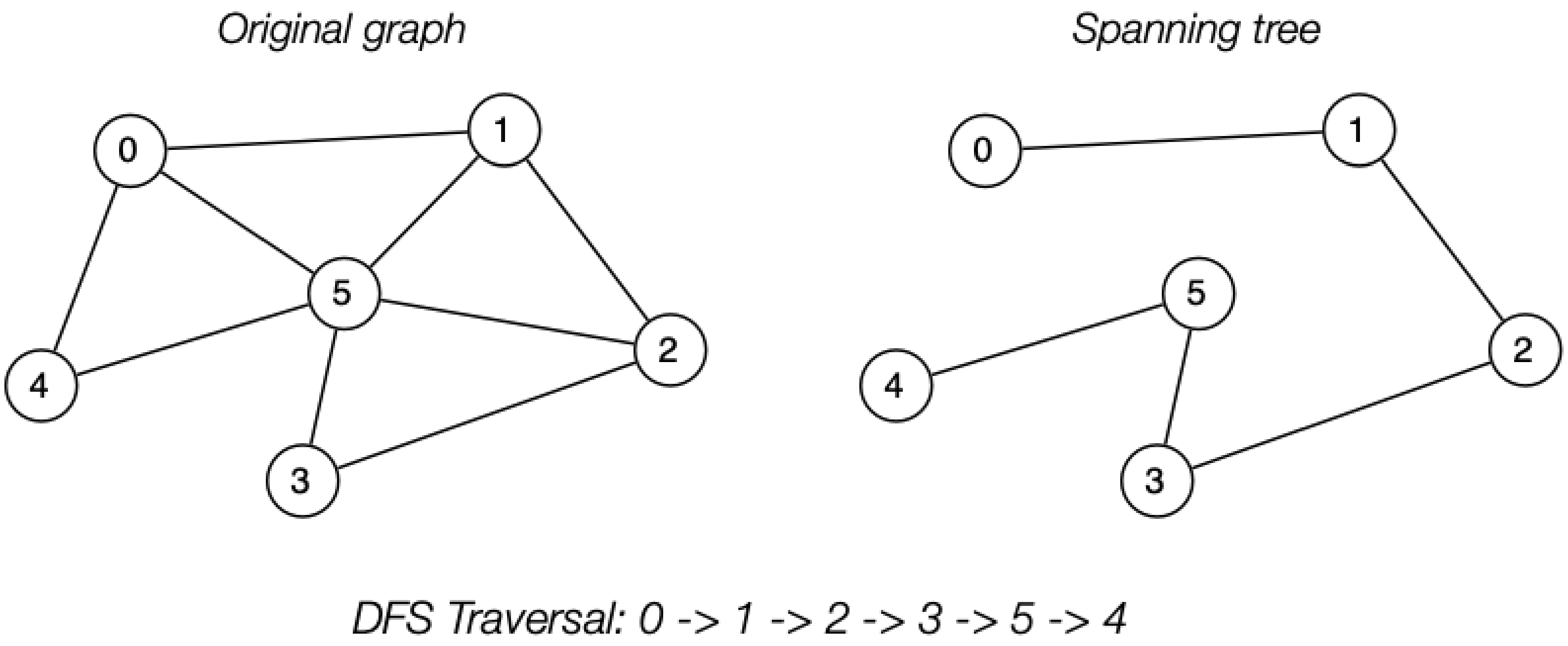
Consider the following graph:



Consider how DFS and BFS could produce its spanning tree

❖ ... Graph Traversal

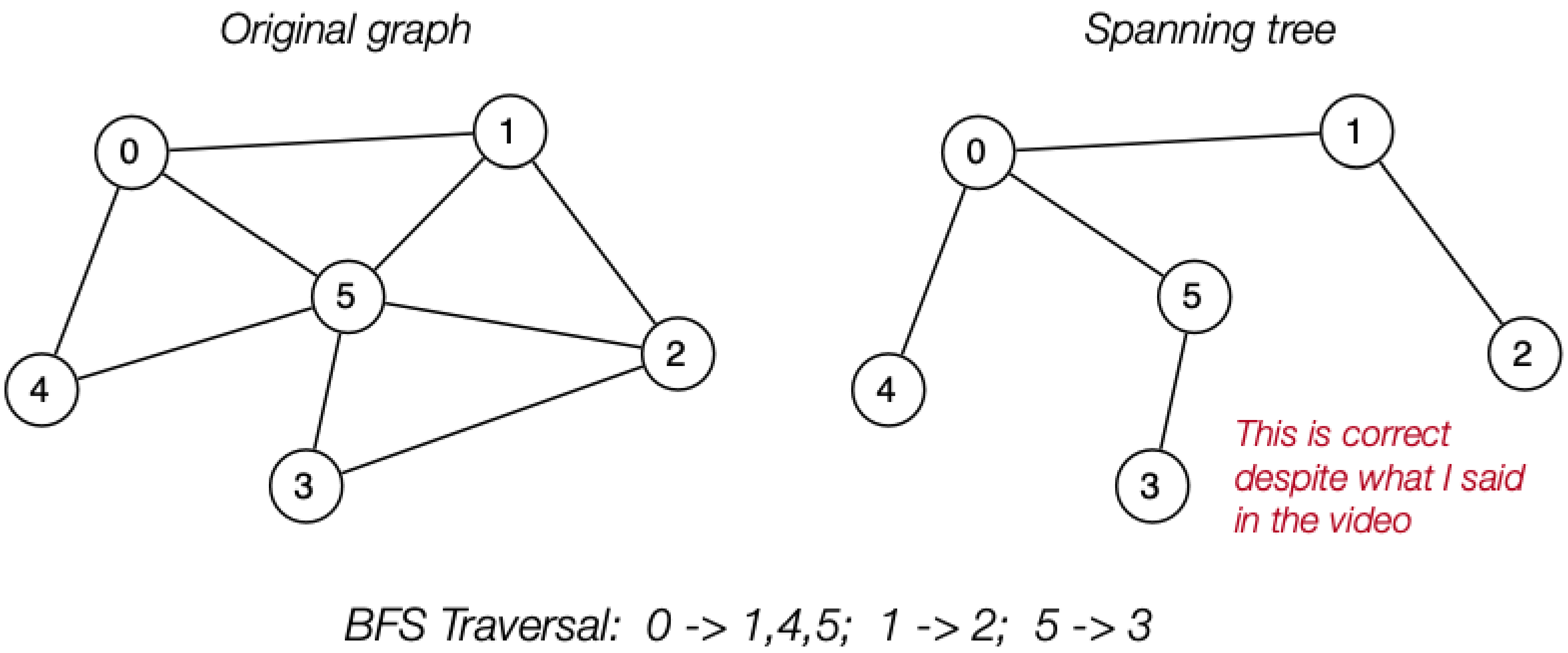
Spanning tree resulting from DFS ...



Note: choose neighbours in ascending order

❖ ... Graph Traversal

Spanning tree resulting from BFS ...



Note: choose neighbours in ascending order

❖ Depth-first Search

Depth-first search can be described recursively as

```
visited = {}

depthFirst(G,v):
|  visited = visited ∪ {v}
|  for all (v,w) ∈ edges(G) do
|  |  if w ∉ visited then
|  |  |  depthFirst(G,w)
|  |  end if
|  end for
```

The recursion induces backtracking

❖ ... Depth-first Search

Recursive DFS path checking

```
visited = {}

hasPath(G,src,dest):
|  Input  graph G, vertices src,dest
|  Output true if there is a path from src to dest,
|         false otherwise
|
|  return dfsPathCheck(G,src,dest)
```

Requires wrapper around recursive function **dfsPathCheck()**

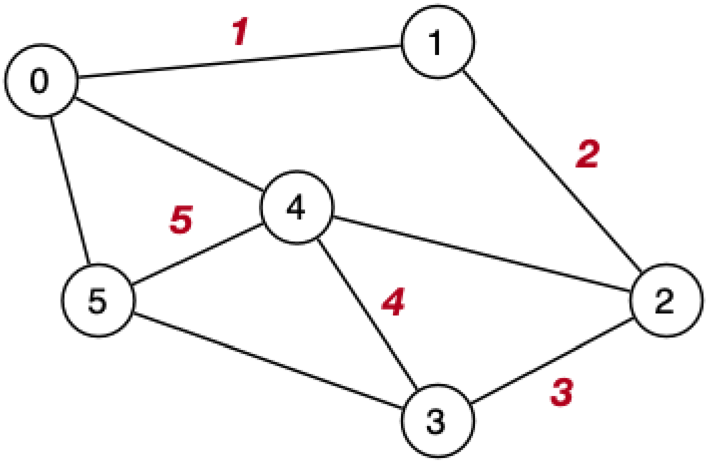
❖ ... Depth-first Search

Recursive function for path checking

```
dfsPathCheck(G,v,dest):
|  visited = visited ∪ {v}
|  for all (v,w) ∈ edges(G) do
|  |  if w=dest then   // found edge to dest
|  |     return true
|  |  else if w ∉ visited then
|  |     if dfsPathCheck(G,w,dest) then
|  |        return true  // found path via w to dest
|  |     end if
|  |  end if
|  end for
|  return false  // no path from v to dest
```

❖ Depth-first Traversal Example

Tracing the execution of **`dfsPathCheck(G,0,5)`** on:



Reminder: we consider neighbours in ascending order

Clearly does not find the shortest path

❖ DFS Cost Analysis

Cost analysis:

- each vertex visited at most once ⇒ cost = *O(V)*
- visit all edges incident on visited vertices ⇒ cost = *O(E)*
  - assuming an adjacency list representation

Time complexity of DFS: *O(V+E)* (adjacency list representation)

❖ Path Finding

Knowing whether a path exists can be useful

Knowing what the path is, is even more useful

Strategy:

- record the previously visited node as we search
- so that we can then trace path (backwards) through graph

Requires a global array (not a set):

- **`visited[v]`** contains vertex **`w`** from which we reached **`v`**

❖ ... Path Finding

Function to find path src→dest and print it

```
visited[] // store previously visited node
          // for each vertex 0..nV-1

findPath(G,src,dest):
|  Input graph G, vertices src,dest
|
|  for all vertices v∈G do
|     visited[v]=-1
|  end for
|  visited[src]=src  // starting node of the path
|  if dfsPathCheck(G,src,dest) then
|  |  // show path in dest..src order
|  |  v=dest
|  |  while v≠src do
|  |     print v"-"
|  |     v=visited[v]
|  |  end while
|  |  print src
|  end if
```

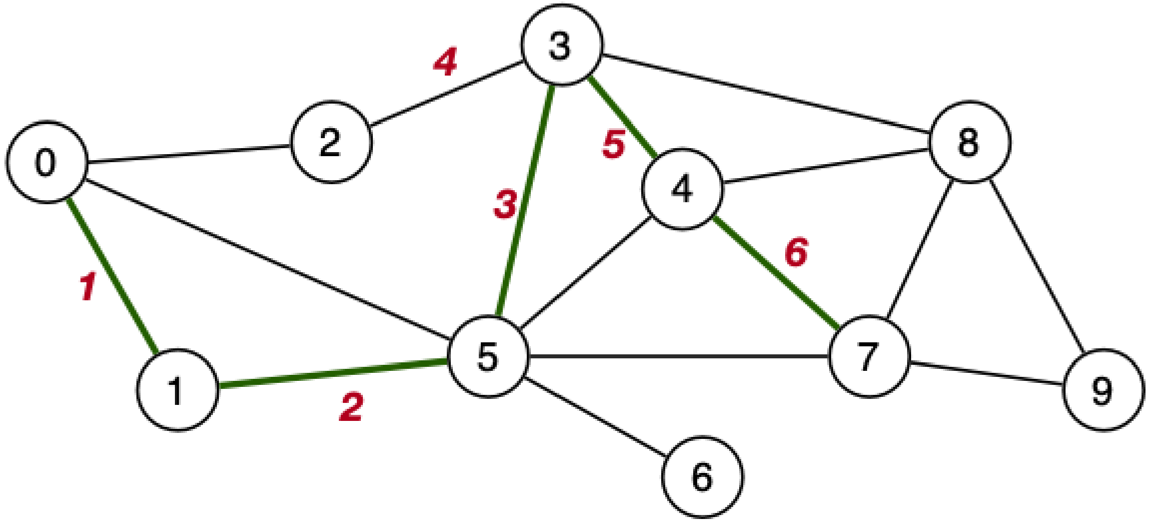
❖ ... Path Finding

Recursive function to build path in **visited[]**

```
dfsPathCheck(G,v,dest):
|  for all (v,w) ∈ edges(G) do
|  |  if visited[w] = -1 then
|  |     visited[w] = v
|  |     if w = dest then  // found edge from v to dest
|  |        return true
|  |     else if dfsPathCheck(G,w,dest) then
|  |        return true  // found path via w to dest
|  |     end if
|  |  end if
|  end for
|  return false  // no path from v to dest
```

❖ ... Path Finding

The **`visited[]`** array after **`dfsPathCheck(G,0,7)`** succeeds



	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
visited	0	0	3	5	3	1	5	4	-1	-1

❖ ... Path Finding

DFS can also be described non-recursively (via a stack):

```
visited[] // store previously visited node
          // for each vertex 0..nV-1

findPathDFS(G,src,dest):
|  Input graph G, vertices src,dest
|
|  for all vertices v∈G do
|     visited[v]=-1
|  end for
|  found=false
|  visited[src]=src
|  push src onto new stack S
|  while not found ∧ S is not empty do
|  |  pop v from S
|  |  if v=dest then
|  |     found=true
|  |  else
|  |  |  for each (v,w)∈edges(G) with visited[w]=-1 do
|  |  |     visited[w]=v
|  |  |     push w onto S
```

```
|   |   |  end for
|   |  end if
|   end while
|   if found then
|      display path in dest..src order
|   end if
```

Uses standard stack operations ... Time complexity is still *O(V+E)*

❖ Breadth-first Search

Basic approach to breadth-first search (BFS):

- visit and mark current vertex
- visit all neighbours of current vertex
- then consider neighbours of neighbours

Notes:

- tricky to describe recursively
- but a minor variation on non-recursive DFS search works
  ⇒ switch the *stack* for a *queue*

❖ ... Breadth-first Search

BFS path finding algorithm:

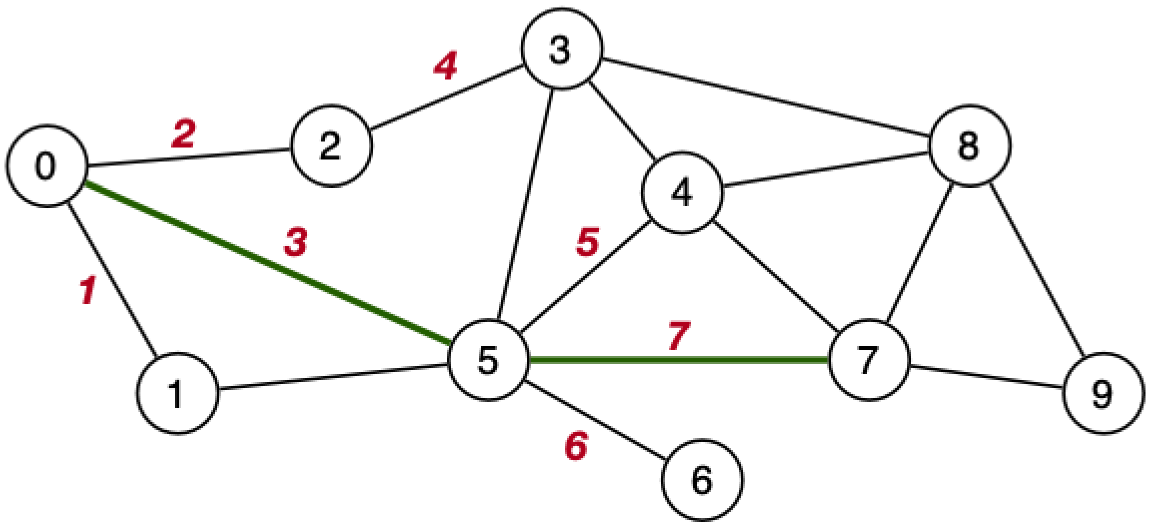
```
visited[] // store previously visited node
          // for each vertex 0..nV-1

findPathBFS(G,src,dest):
|  Input  graph G, vertices src,dest
|
|  for all vertices v∈G do
|     visited[v]=-1
|  end for
|  found=false
|  visited[src]=src
|  enqueue src into queue Q
|  while not found ∧ Q is not empty do
|  |  dequeue v from Q
|  |  if v=dest then
|  |     found=true
|  |  else
|  |  |  for each (v,w) ∈ edges(G) with visited[w]=-1 do
|  |  |     visited[w]=v
|  |  |     enqueue w into Q
```

```
|   |   |  end for
|   |  end if
|  end while
|  if found then
|     display path in dest..src order
|  end if
```

Uses standard queue operations (enqueue, dequeue, check if empty)

❖ ... Breadth-first Search

The **`visited[]`** array after **`findPathBFS(G,0,7)`** succeeds



	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
visited	0	0	0	2	5	0	5	5	-1	-1

❖ ... Breadth-first Search

Time complexity of BFS: *O(V+E)* (same as DFS)

BFS finds a "shortest" path

- based on minimum # edges between *src* and *dest*.
- stops with first-found path, if there are multiple ones

In many applications, edges are weighted and we want path

- based on minimum sum-of-weights along path *src* .. *dest*

We discuss weighted/directed graphs later.

Produced: 22 Jun 2020