Floating Point Numbers

- C has three floating point types
  - **`float`** ... typically 32-bit (lower precision, narrower range)
  - **`double`** ... typically 64-bit (higher precision, wider range)
  - **`long double`** ... typically 128-bits (but maybe only 80 bits used)
- Floating point constants, e.g : **`3.14159`** **`1.0e-9`** are **`double`**
- Reminder: division of 2 ints in C yields an int.
  - but division of double and int in C yields a double.

Floating Point Number - Output

```
double d = 4/7.0;
// prints in decimal with (default) 6 decimal places
printf("%lf\n", d);         // prints 0.571429
// prints in scientific notation
printf("%le\n", d);        // prints 5.714286e-01
// picks best of decimal and scientific notation
printf("%lg\n", d);        // prints 0.571429
//  prints in decimal with 9 decimal places
printf("%.9lf\n", d);     // prints 0.571428571
//  prints in decimal with 1 decimal place and field width of 5
printf("%10.1lf\n", d);  // prints        0.6
```

source code for float_output.c

Floating Point Numbers

- if we represent floating point numbers with a fixed small number of bits
    - there are only a finite number of bit patterns
    - can only represent a finite subset of reals
- almost all real values will have no exact representation
- value of arithmetic operations may be real with no exactly representation
- we must use closest value which can be exactly represented
- this approximation introduces an error into our calculations
- often does not matter
- sometime can be disasterous

Fixed Point Representation

- can have fractional numbers in other bases, e.g.: $110.101_2 == 6.625_{10}$
- could represent fractional numbers similarly to integers by assuming decimal point is in ***fixed*** position
- for example with 32 bits:
  - 16 bits could be used for integer part
  - 16 bits could be used for the fraction
  - equivalent to storing values as integers after multiplying (***scaling***) by 2^{16}
  - major limitation is only small range of values can be represented
    - minimum $2_{-16} \approx 0.000015$
    - maximum $2_{15} \approx 32768$
- usable for some problems, but not ideal
- used on small embedded processors without silicon floating point

floating_types.c - print characteristics of floating point types

```c
float f;
double d;
long double l;
printf("float       %2lu bytes  min=%-12g  max=%g\n", sizeof f, FLT
printf("double      %2lu bytes  min=%-12g  max=%g\n", sizeof d, DBL
printf("long double %2lu bytes  min=%-12Lg  max=%Lg\n", sizeof l, L
```

source code for floating_types.c

```
$ ./floating_types
float        4 bytes  min=1.17549e-38   max=3.40282e+38
double       8 bytes  min=2.22507e-308  max=1.79769e+308
long double 16 bytes  min=3.3621e-4932  max=1.18973e+4932
```

IEEE 754 standard

- C floats almost always IEEE 754 single precision (binary32)
- C double almost always IEEE 754 double precision (binary64)
- C long double might be IEEE 754 (binary128)
- IEEE 754 representation has 3 parts: *sign*, *fraction* and *exponent*
- numbers have form *sign fraction* $\times$ $2^{exponent}$, where *sign* is +/-
- ***fraction*** always has 1 digit before decimal point (***normalized***)
    - as a consequence only 1 representation for any value
- ***exponent*** is stored as positive number by adding constant value (***bias***)
- numbers close to zero have higher precision (more accurate)

Floating Point Numbers

Example of normalising the fraction part in binary:

- 1010.1011 is normalized as 1.0101011×2^{011}
- $1010.1011 = 10 + 11/16 = 10.6875$
- $1.0101011 \times 2^{011} = (1 + 43/128) \times 2^3 = 1.3359375 \times 8 = 10.6875$

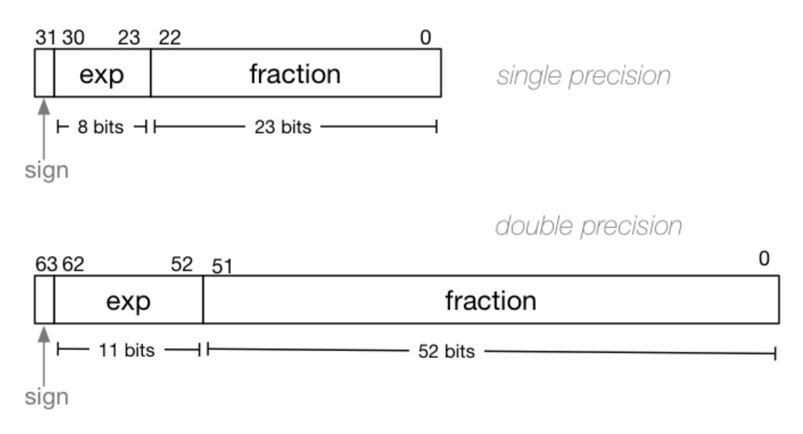
The normalised fraction part always has 1 before the decimal point.

Example of determining the exponent in binary:

- if exponent is 8-bits, then the bias $= 2^{8-1} - 1 = 127$
- valid bit patterns for exponent `00000001 .. 11111110`
- correspond to B exponent values -126 .. 127

Floating Point Numbers

Internal structure of floating point values



IEEE-754 Single Precision example: 0.15625

```
0.15625 is represented in IEEE-754 single-precision by these bits:
00111110001000000000000000000000
sign | exponent | fraction
   0 | 01111100 | 01000000000000000000000
sign bit = 0
sign = +
raw exponent    = 01111100 binary
                = 124 decimal
actual exponent = 124 - exponent_bias
                = 124 - 127
                = -3
number = +1.01000000000000000000000 binary * 2**-3
       = 1.25 decimal * 2**-3
       = 1.25 * 0.125
       = 0.15625
```

source code for explain_float_representation.c

IEEE-754 Single Precision example: -0.125

```
$ ./explain_float_representation -0.125
-0.125 is represented as a float (IEEE-754 single-precision) by the
10111110000000000000000000000000
sign | exponent | fraction
   1 | 01111100 | 00000000000000000000000
sign bit = 1
sign = -
raw exponent    = 01111100 binary
                = 124 decimal
actual exponent = 124 - exponent_bias
                = 124 - 127
                = -3
number = -1.00000000000000000000000 binary * 2**-3
       = -1 decimal * 2**-3
       = -1 * 0.125
       = -0.125
```

IEEE-754 Single Precision example: 150.75

```
$ ./explain_float_representation 150.75
150.75 is represented in IEEE-754 single-precision by these bits:
01000011000101101100000000000000
sign | exponent | fraction
   0 | 10000110 | 00101101100000000000000
sign bit = 0
sign = +
raw exponent    = 10000110 binary
                = 134 decimal
actual exponent = 134 - exponent_bias
                = 134 - 127
                = 7
number = +1.00101101100000000000000 binary * 2**7
       = 1.17773 decimal * 2**7
       = 1.17773 * 128
       = 150.75
```

IEEE-754 Single Precision example: -96.125

```
$ ./explain_float_representation -96.125
-96.125 is represented in IEEE-754 single-precision by these bits:
11000010110000000100000000000000
sign | exponent | fraction
   1 | 10000101 | 10000000100000000000000
sign bit = 1
sign = -
raw exponent    = 10000101 binary
                = 133 decimal
actual exponent = 133 - exponent_bias
                = 133 - 127
                = 6
number = -1.10000000100000000000000 binary * 2**6
       = -1.50195 decimal * 2**6
       = -1.50195 * 64
       = -96.125
```

IEEE-754 Single Precision exploring bit patterns #1

```
$ ./explain_float_representation 00111101110011001100110011001101
sign bit = 0
sign = +
raw exponent    = 01111011 binary
                = 123 decimal
actual exponent = 123 - exponent_bias
                = 123 - 127
                = -4
number = +1.10011001100110011001101 binary * 2**-4
       = 1.6 decimal * 2**-4
       = 1.6 * 0.0625
       = 0.1
```

infinity.c - exploring infinity

- C (IEEE-754) has a representation for +/- infinity
- propagates sensibly through calculations

```
double x = 1.0/0.0;
printf("%lf\n", x); //prints inf
printf("%lf\n", -x); //prints -inf
printf("%lf\n", x - 1); // prints inf
printf("%lf\n", 2 * atan(x)); // prints 3.141593
printf("%d\n", 42 < x); // prints 1 (true)
printf("%d\n", x == INFINITY); // prints 1 (true)
```

source code for infinity.c

nan.c - handling errors robustly

- C (IEEE-754) has a representation for invalid results:
  - NaN (not a number)
- ensures errors propagates sensibly through calculations

```c
double x = 0.0/0.0;
printf("%lf\n", x); //prints nan
printf("%lf\n", x - 1); // prints nan
printf("%d\n", x == x); // prints 0 (false)
printf("%d\n", isnan(x)); // prints 1 (true)
```

source code for nan.c

IEEE-754 Single Precision example: inf

```
$ ./explain_float_representation inf
inf is represented in IEEE-754 single-precision by these bits:
01111111100000000000000000000000
sign | exponent | fraction
   0 | 11111111 | 00000000000000000000000
sign bit = 0
sign = +
raw exponent    = 11111111 binary
                = 255 decimal
number = +inf
```

IEEE-754 Single Precision exploring bit patterns #2

```
$ ./explain_float_representation 01111111110000000000000000000000
sign bit = 0
sign = +
raw exponent    = 11111111 binary
                = 255 decimal
number = NaN
```

source code for explain_float_representation.c

Consequences of most reals not having exact representations

```
double a, b;
a = 0.1;
b = 1 - (a + a + a + a + a + a + a + a + a + a);
if (b != 0) {  // better would be fabs(b) > 0.000001
    printf("1 != 0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1\n");
}
printf("b = %g\n", b); // prints 1.11022e-16
```

source code for double_imprecision.c

- do not use `==` and `!=` with floating point values
- instead check if values are close

Consequences of most reals not having exact representations

```c
double x = 0.000000011;
double y = (1 - cos(x)) / (x * x);
// correct answer y = ~0.5
// prints y = 0.917540
printf("y = %lf\n", y);
// division of similar approximate value
// produces large error
// sometimes called catastrophic cancellation
printf("%g\n", 1 - cos(x)); // prints  1.11022e-16
printf("%g\n", x * x); // prints 1.21e-16
```

source code for double_catastrophe.c

Another reason not to use == with floating point values

```c
if (d == d) {
    printf("d == d is true\n");
} else {
    // will be executed if d is a NaN
    printf("d == d is not true\n");
}
if (d == d + 1) {
    // may be executed if d is large
    // because closest possible representation for d + 1
    // is also closest possible representation for d
    printf("d == d + 1 is true\n");
} else {
    printf("d == d + 1 is false\n");
}
```

source code for double_not_always.c

Another reason not to use == with floating point values

```
$ dcc double_not_always.c -o double_not_always
$ ./double_not_always 42.3
d = 42.3
d == d is true
d == d + 1 is false
$  ./double_not_always 4200000000000000000
d = 4.2e+18
d == d is true
d == d + 1 is true
$ ./double_not_always NaN
d = nan
d == d is not true
d == d + 1 is false
```

because closest possible representation for d + 1 is also closest possible representation for d

source code for double_not_always.c

Consequences of most reals not having exact representations

```
double d = 9007199254740992;
// loop never terminates
while (d < 9007199254740999) {
    printf("%lf\n", d); // always prints 9007199254740992.000000
    // 9007199254740993 can not be represented as a double
    // closest double is 9007199254740992.0
    // so 9007199254740992.0 + 1 = 9007199254740992.0
    d = d + 1;
}
```

source code for double_disaster.c

- 9007199254740993 is $2^{53} + 1$
  it is smallest integer which can not be represented exactly as a double
- The closest double to 9007199254740993 is 9007199254740992.0
- aside: 9007199254740993 can not be represented by a int32_t
  it can be represented by int64_t

Exercise: Floating point → Decimal

Convert the following floating point numbers to decimal.

Assume that they are in IEEE 754 single-precision format.

`0 10000000 11000000000000000000000`

`1 01111110 10000000000000000000000`