Directed/Weighted Graphs

- Generalising Graphs
- Directed Graphs (Digraphs)
- Digraph Representation
- Weighted Graphs
- Weighted Graph Representation
- Weighted Graph Implementation

❖ Generalising Graphs

Discussion so far has considered graphs as

- *V* = set of vertices,   *E* = set of edges

Real-world applications require more "precision"

- some edges are directional (e.g. one-way streets)
- some edges have a cost (e.g. distance, traffic)

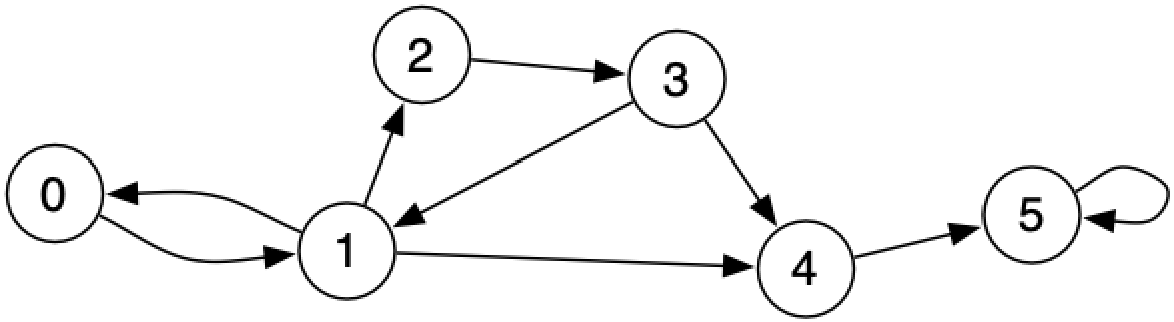
We need to consider directed graphs and weighted graphs

❖ Directed Graphs (Digraphs)

Directed graphs are ...

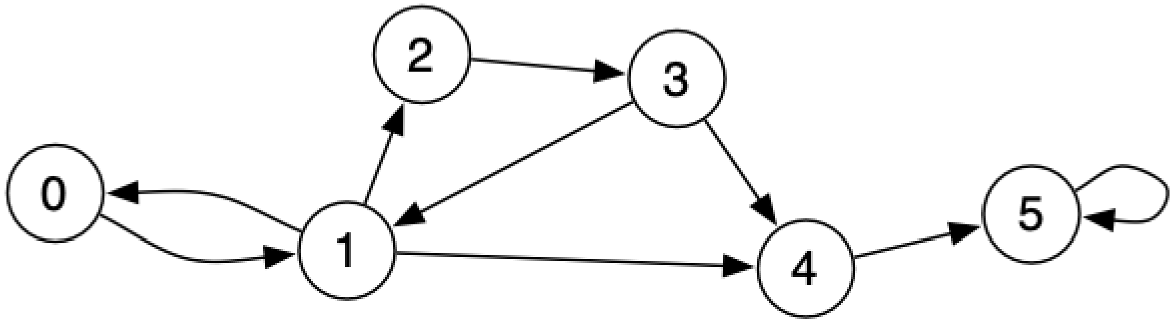
- graphs with *V* vertices, *E* edges *(v,w)*
- edge *(v,w)* has source *v* and destination *w*
- unlike undirected graphs, $v \rightarrow w \neq w \rightarrow v$

Example digraph:



❖ ... Directed Graphs (Digraphs)

Some properties of ...



- edges 1-2-3 form a cycle, edges 1-3-4 do *not* form a cycle
- vertex 5 has a self-referencing edge *(5,5)*
- vertices 0 and 1 reference each other, i.e. *(0,1)* and *(1,0)*
- there are no paths from 5 to any other nodes
- paths from 0→5: 0-1-2-3-4-5, 0-1-4-5, 0-1-2-3-1-4-5

❖ ... Directed Graphs (Digraphs)

Terminology for digraphs ...

Directed path: sequence of $n \ge 2$ vertices $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_n$

- where $(v_i, v_{i+1}) \in edges(G)$ for all v_i, v_{i+1} in sequence

If $v_1 = v_n$, we have a directed cycle

Degree of vertex: number of incident edges

- outdegree: $deg(v)$ = number of edges of the form $(v, \_)$
- indegree: $deg^{-1}(v)$ = number of edges of the form $(\_, v)$

❖ ... Directed Graphs (Digraphs)

More terminology for digraphs …

Reachability:

- w is reachable from v if $\exists$ directed path $v,\ldots,w$

Strong connectivity:

- every vertex is reachable from every other vertex

Directed acyclic graph (DAG):

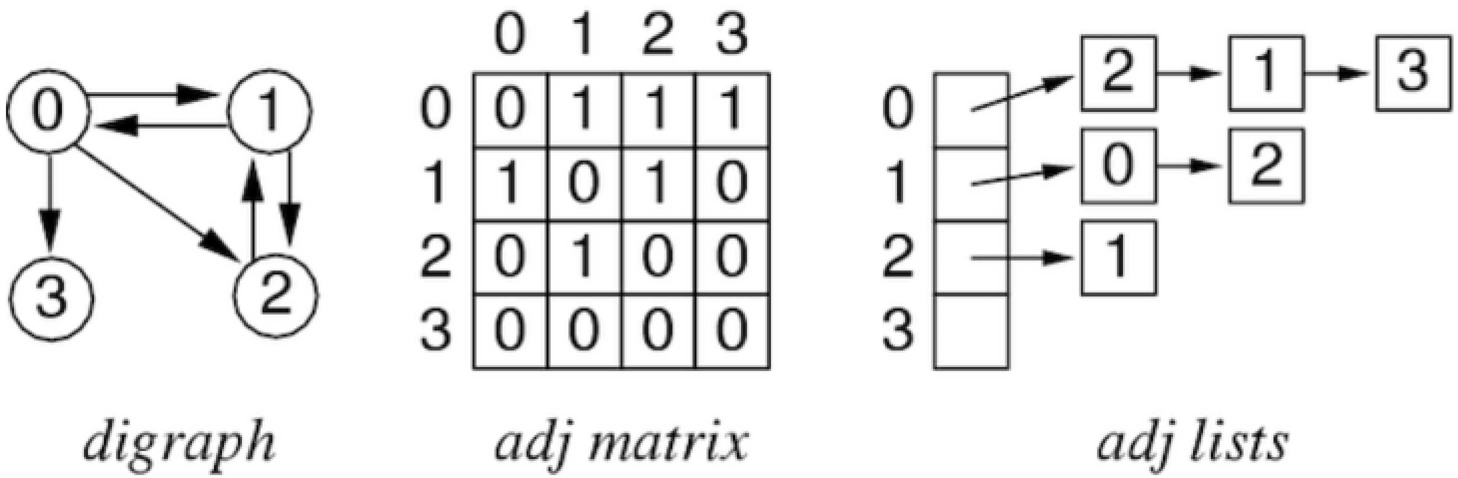
- contains no directed cycles

❖ Digraph Representation

Similar set of choices as for undirectional graphs:

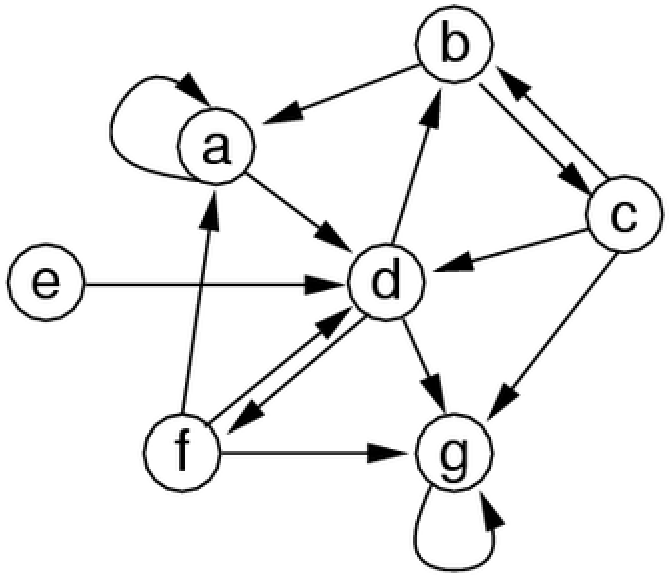
- array of edges (directed)
- vertex-indexed adjacency matrix (non-symmetric)
- vertex-indexed adjacency lists

V vertices identified by *0 .. V-1*



❖ ... Digraph Representation

Example digraph and adjacency matrix representation:



	a	b	c	d	e	f	g
a	1	0	0	1	0	0	0
b	1	0	1	0	0	0	0
c	0	1	0	1	0	0	1
d	0	1	0	0	0	1	1
e	0	0	0	1	0	0	0
f	1	0	0	1	0	0	1
g	0	0	0	0	0	0	1

Undirectional ⇒ symmetric matrix
Directional ⇒ non-symmetric matrix

Maximum #edges in a digraph with V vertices: V^2

❖ ... Digraph Representation

Costs of representations: (where degree *deg(v)* = #edges leaving *v*)

	array of edges	adjacency matrix	adjacency list
space usage	*E*	V^2	*V+E*
insert edge	*E*	*1*	*1*
exists edge *(v,w)*?	*E*	*1*	*deg(v)*
get edges leaving *v*	*E*	*V*	*deg(v)*

Overall, adjacency list representation is best

- real graphs tend to be sparse
  (large number of vertices, small average degree *deg(v)*)
- algorithms frequently iterate over edges from *v*

❖ Weighted Graphs

Graphs so far have considered

- edge = an association between two vertices/nodes
- may be a precedence in the association (directed)

Some applications require us to consider

- a cost or weight of an association
- modelled by assigning values to edges (e.g. positive reals)

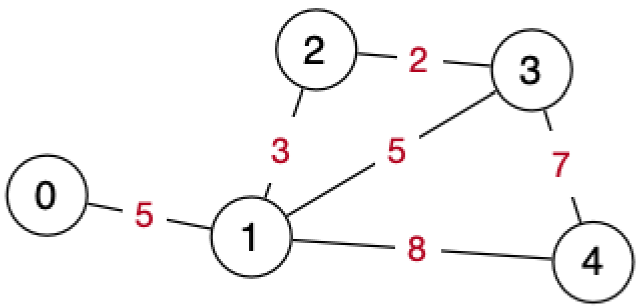
❖ ... Weighted Graphs

Weighted graphs are ...

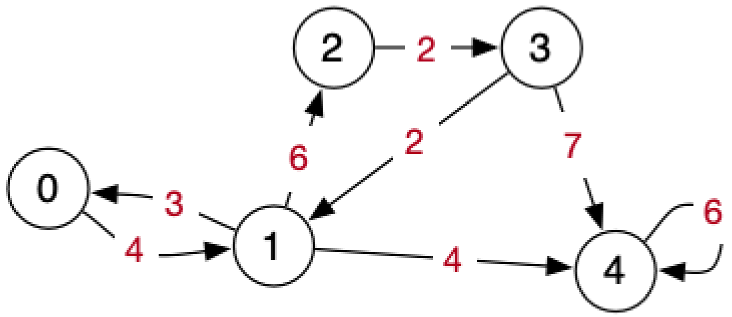
- graphs with *V* vertices, *E* edges *(s,t)*
- each edge *(s,t,w)* connects vertices *s* and *t* and has weight *w*

Weights can be used in both directed and undirected graphs.

Example weighted graphs:



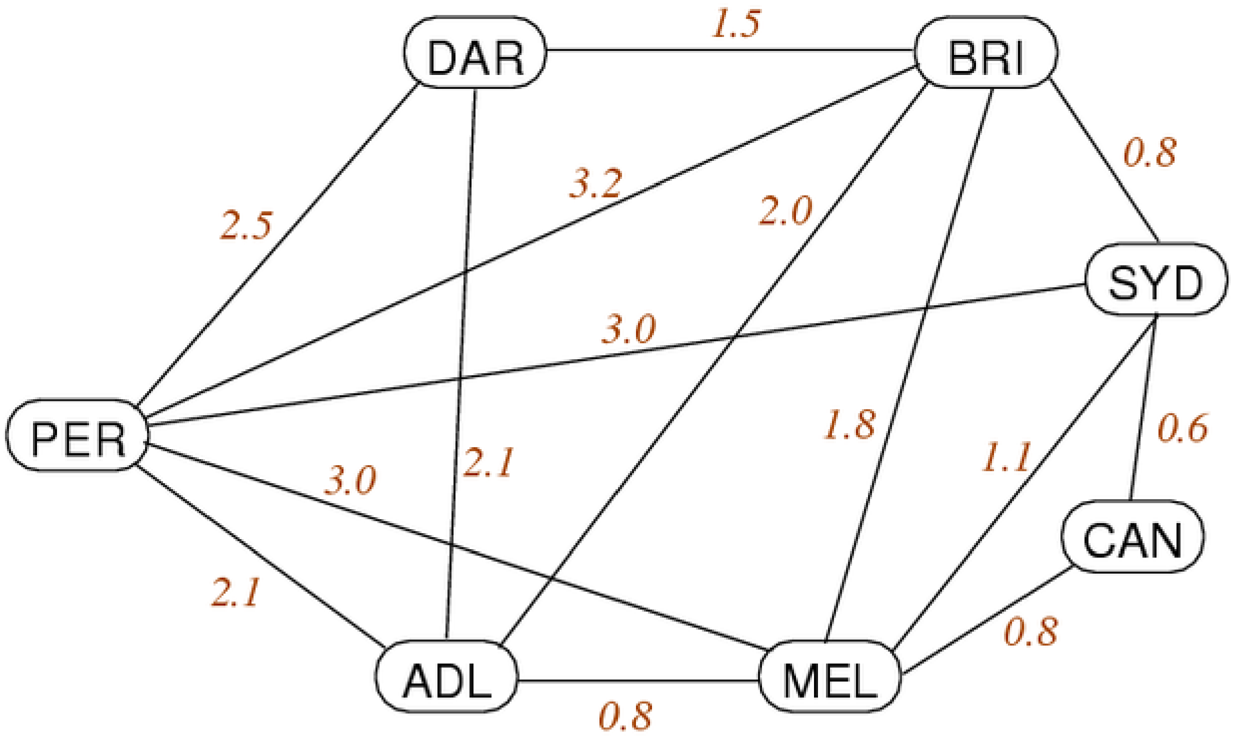
Weighted Graph



Directed Weighted Graph

❖ ... Weighted Graphs

Example: major airline flight routes in Australia



Representation: edge = direct flight; weight = approx flying time (hours)

❖ ... Weighted Graphs

Weights lead to minimisation-type questions, e.g.

1. Cheapest way to connect all vertices?

- a.k.a. minimum spanning tree problem
- assumes: edges are weighted and undirected

2. Cheapest way to get from *A* to *B*?

- a.k.a shortest path problem
- assumes: edge weights positive, directed or undirected

❖ Weighted Graph Representation

Weights can easily be added to:

- adjacency matrix representation (0/1 → int or float)
- adjacency lists representation (add int/float to list node)

The edge list representation changes to list of *(s,t,w)* triples

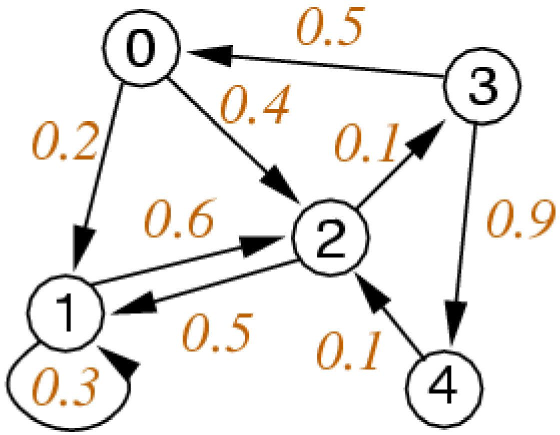
All representations can also work with directed edges

Weight values are determined by domain being modelled

- in some contexts weight could be zero or negative

❖ ... Weighted Graph Representation

Adjacency matrix representation with weights:



Weighted Digraph

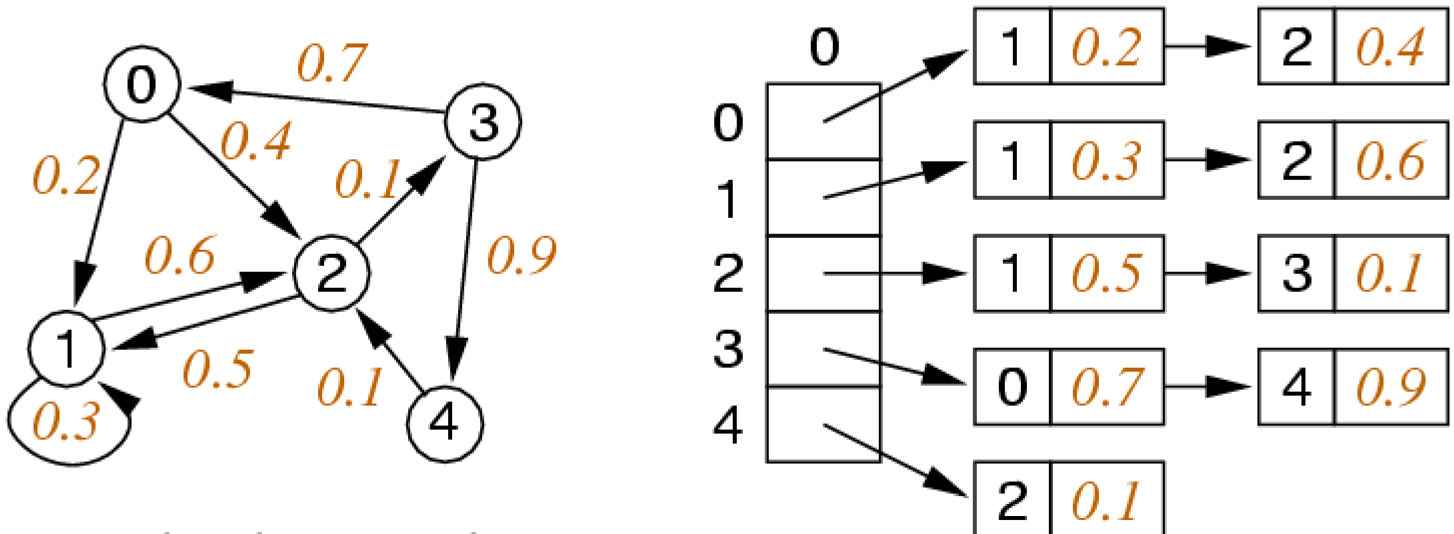
	0	1	2	3	4
0	*	0.2	0.4	*	*
1	*	0.3	0.6	*	*
2	*	0.5	*	0.1	*
3	0.5	*	*	*	0.9
4	*	*	0.1	*	*

Adjacency Matrix

Note: need distinguished value to indicate "no edge".

❖ ... Weighted Graph Representation

Adjacency lists representation with weights:

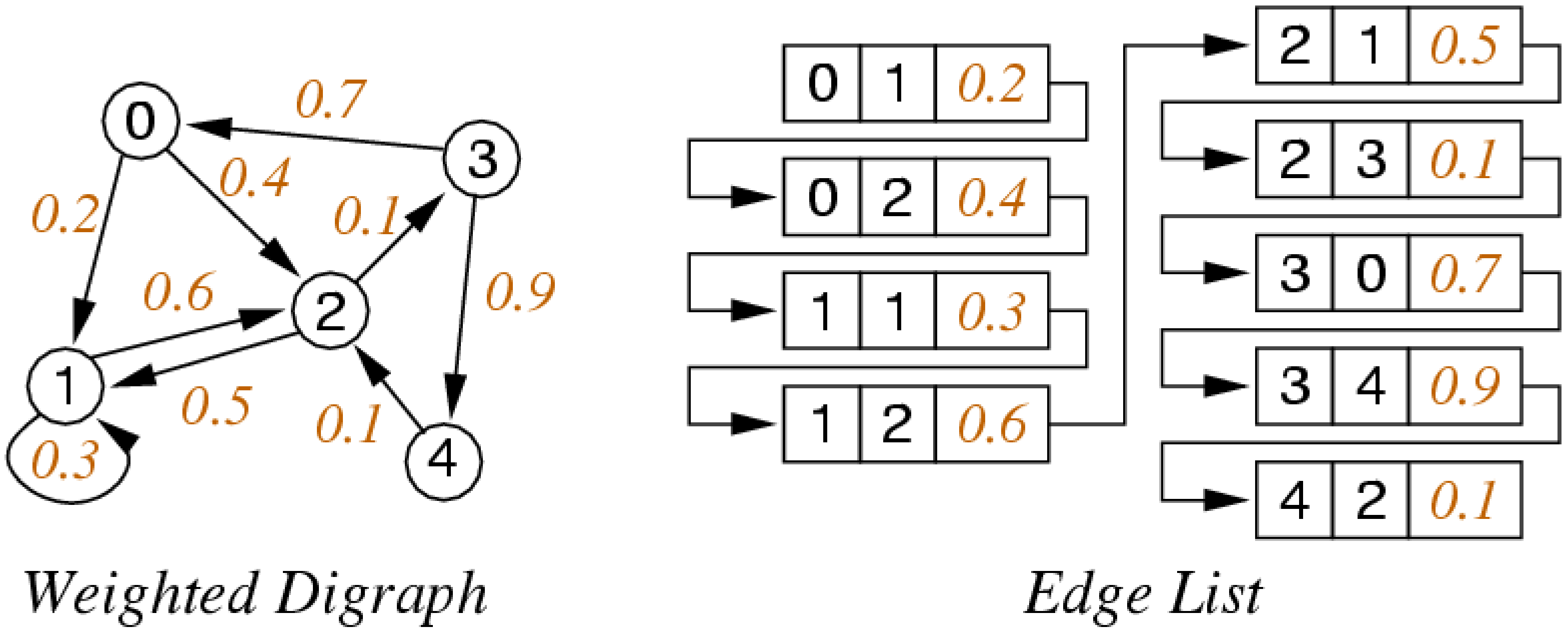


Weighted Digraph *Adjacency Lists*

Note: if undirected, each edge appears twice with same weight

❖ ... Weighted Graph Representation

Edge array / edge list representation with weights:



Note: not very efficient for use in processing algorithms, but does give a possible representation for min spanning trees or shortest paths

❖ Weighted Graph Implementation

Changes to preious grpah data structures to include weights:

WGraph.h

```
// edges are pairs of vertices (end-points) plus weight
typedef struct Edge {
   Vertex v;
   Vertex w;
   int    weight;
} Edge;

// returns weight, or 0 if vertices not adjacent
int adjacent(Graph, Vertex, Vertex);
```

Note: here, we assume all weights are positive, but not required

❖ ... Weighted Graph Implementation

`WGraph.c` (assuming adjacency matrix representation)

```
typedef struct GraphRep {
   int **edges;  // adjacency matrix storing weights
                 // 0 if nodes not adjacent
   int   nV;     // #vertices
   int   nE;     // #edges
} GraphRep;

bool adjacent(Graph g, Vertex v, Vertex w) {
   assert(valid graph, valid vertices)
   return (g->edges[v][w] != 0);
}
```

❖ ... Weighted Graph Implementation

More `WGraph.c`

```
void insertEdge(Graph g, Edge e) {
   assert(valid graph, valid edge)
   // edge e not already in graph
   if (g->edges[e.v][e.w] == 0) g->nE++;
   // may change weight of existing edge
   g->edges[e.v][e.w] = e.weight;
   g->edges[e.w][e.v] = e.weight;
}

void removeEdge(Graph g, Edge e) {
   assert(valid graph, valid edge)
   // edge e not in graph
   if (g->edges[e.v][e.w] == 0) return;
   g->edges[e.v][e.w] = 0;
   g->edges[e.w][e.v] = 0;
   g->nE--;
}
```

Produced: 28 Jun 2020