Splitting a measure along a fixed direction

June 14, 2023

Notation

- A measure μ on $\mathbb{R}^d$ is nice if it is Borel, $\mu(\mathbb{R}^d) = 1$, $\mu(H) = 0$ for all affine hyperplanes and it has connected support.
- Given an arrangement $\mathcal{H}$ of k oriented affine hyperplanes and a sign pattern $\alpha \in (\mathbb{Z}_2)^k$, the orthant $\mathcal{O}(\mathcal{H}, \alpha)$ is defined as

  $$\mathcal{O}(\mathcal{H}, \alpha) := \{x \in \mathbb{R}^d | (-1)^{\alpha_i}((h_i|x) - a_i) > 0\}$$

  Intuitively, $\mathcal{O}(\mathcal{H}, \alpha)$ is the set of point lying on the side of H_i determined by α_i (positive if $\alpha_i = 0$, negative otherwise).
- $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ with generators $g_1 = (1, 0)$ and $g_2 = (0, 1)$.
- If $v \in \mathbb{R}^d$ is a non-zero vector, $\overline{v}$ is its normalization $\overline{v} = \frac{v}{\|v\|}$.
- An oriented affine hyperplane $H = \{x \in \mathbb{R}^d | \, (h|x) - a = 0\}$ defines a point $\frac{(h,a)}{\|(h,a)\|}$ in S^d; likewise, a point on $x = (\bar{x}, a) \in S^d$ defines an oriented affine hyperplane $H(x) = \{p \in \mathbb{R}^d | \, (\bar{x}|p) - a = 0\}$.
- Given a nice measure μ in $\mathbb{R}^3$, define the standard test map $F : \left(S^3\right)^3 \times \mathbb{Z}_2^3 \to \mathbb{R}$:

  $$F(x, \omega) = \mu(\mathcal{O}(x, \omega)) - \frac{1}{8}$$

  x is an equipartition $\iff$ $F(x, \alpha) = 0 \; \forall \alpha$.
- Given a test map F, the alternating sum functions are the discrete Fourier transform of the test map. More precisely, if $\alpha \in (\mathbb{Z}_2)^3$ $\alpha \neq 0$, then the alternating sum with parameter α is:

  $$F_\alpha(x) = \sum_{\omega \in \mathbb{Z}_2^3} (-1)^{(\alpha|\omega)} F(x, \omega)$$

  x is an equipartition $\iff$ $F_\alpha(x) = 0 \; \forall \alpha \neq 0$
- The group $\mathbb{Z}_2^3$ acts on $\left(S^3\right)^3$ by coordinate-wise antipodality and, given $\alpha, g \in \mathbb{Z}_2^3$ with $\alpha \neq 0$ and a test map F:

  $$F_\alpha(g \cdot x) = (-1)^{(\alpha|g)} F_\alpha(x)$$

The Result

Our goal is to prove the following result:

Theorem 1. *Given a nice measure μ and a direction $p \in S^2$ it is possible to find 3 affine oriented planes that equipartition the measure and the first two have the prescribed oriented intersection p. Formally, $\exists \mathcal{H} = ((h_1, a_1), (h_2, a_2), (h_3, a_3))$ configuration of oriented planes such that:*

- $\forall \alpha \in (\mathbb{Z}_2)^k, \ \ \mu(\mathcal{O}(\mathcal{H}, \alpha)) = \frac{1}{8}$.
- $\overline{h}_1 \wedge \overline{h}_2 = p$.

Proof. Without loss of generality, let $z = (0, 0, 1)$. The proof will be split in two sections: the first deals with constructing a map $\Phi : S^1 \times S^3 \to \mathbb{R}^4$ in $\mathbb{R}^4$ whose zeros codify equipartitions of the mass. The second step will be to show that the map will respect a suitable action of G on the two space and that such equivariance forces the existence of a zero.

Step 1

The key step in constructing the map is to show that we can parametrize pair of planes that have intersection direction z and split the mass in 4 equal parts with vector in S^1.

Project the mass on the xy plane to obtain a nice measure $\mu^{\#}$ on $\mathbb{R}^2$. The following bisecting lemma applies:

Lemma 1 (Bisecting). *Let $\mu^{\#}$ be a nice measure on $\mathbb{R}^2$ and $v \in S^1$ a direction. Then there exists two oriented affine lines $l_0 = \mathbb{R}\vec{l_0} + a_0$ and $l_1 = \mathbb{R}\vec{l_1} + a_1$ in $\mathbb{R}^2$ such that:*

- *l_0 and l_1 equipartition $\mu^{\#}$*
- *v bisects the angle between l_0 and l_1*

What is more, we can choose consistently the direction $\vec{l_0}$ (e.g. fix $\vec{l_0}$ to be the first direction clockwise, while the first one contraclockwise is $\vec{l_1}$). Once this choice is made, $\vec{l_0}$ and $\vec{l_1}$ are unique and depend continuously on v.

The lemma guarantees that, once we fix a direction $v \in S^1 \subseteq S^2$ (inclusion as the horizontal equator in S^2) there are two affine lines in the xy plane $l_0 = \mathbb{R}\vec{l_0}(v) + a_0(v))$ and $l_1 = \mathbb{R}\vec{l_1}(v) + a_1(v))$ that bisect the projected measure $\mu^{\#}$.

Define $H_i(v) = (h_i(v), a_i(v))$ to be the affine (oriented) span of $l_i(v)$ and z, the two planes now equipartition the measure μ and have the desired intersection.

Let now $\mathbb{Z}_4 = \langle g_1 \rangle$ act on S^1 by $\frac{\pi}{2}$ rotation counter clockwise. Then, by construction we have that:

$$H_0(g_1 \cdot v) = H_1(v)$$

$$H_1(g_1 \cdot v) = -H_0(v)$$

In other words, if we consider the planar problem with the bisecting vector rotated by $\frac{\pi}{2}$, the new first line is the previous second while the new second is is the old first with opposite orientation.

We can now define a map $S^1 \times S^3 \to (S^3)^3$, $(v, w) \mapsto (H_0(v), H_1(v), w)$.

To check now that a configuration of 3 planes equipartition the measure, it is equivalent to check that the alternating sum functions F_α are 0 for all signed patterns $\alpha \in \mathbb{Z}_2^3 \setminus 0$. However, by construction the first two planes split equally the mass, thus it is sufficient to check the 4 patterns involving the last one; i.e. $\alpha = (0,0,1), (1,0,1), (0,1,1)$ and $(1,1,1)$.

What is more, it is straightforward how the alternating sum functions behave under the G-action. Explicitly, $\forall (v,w) \in S^1 \times S^3$ the following equalities hold:

$$
\begin{aligned}
F_{(0,0,1)}(g_1 \cdot (v,w)) &= F_{(0,0,1)}(v,w) \\
F_{(0,1,1)}(g_1 \cdot (v,w)) &= -F_{(1,0,1)}(v,w) \\
F_{(1,0,1)}(g_1 \cdot (v,w)) &= F_{(0,1,1)}(v,w) \\
F_{(1,1,1)}(g_1 \cdot (v,w)) &= -F_{(1,1,1)}(v,w)
\end{aligned}
$$

$$
\begin{aligned}
F_{(0,0,1)}(g_2 \cdot (v,w)) &= -F_{(0,0,1)}(v,w) \\
F_{(0,1,1)}(g_2 \cdot (v,w)) &= -F_{(1,0,1)}(v,w) \\
F_{(1,0,1)}(g_2 \cdot (v,w)) &= -F_{(0,1,1)}(v,w) \\
F_{(1,1,1)}(g_2 \cdot (v,w)) &= -F_{(1,1,1)}(v,w)
\end{aligned}
$$

Additionally, we can choose a linear G-action on $\mathbb{R}^4$ that is consistent with the previous equations. In particular, if we define:

$$
\begin{aligned}
g_1 \cdot (x,y,z,u) &= (x, -z, y, -u) \\
g_2 \cdot (x,y,z,u) &= (-x, -y, -z, -u)
\end{aligned}
$$

Then it is easy to check that the alternating sum map $\Psi : S^1 \times S^3 \to \mathbb{R}^4$

$$
(v,w) \mapsto \big(F_{(0,0,1)}(v,w), F_{(0,1,1)}(v,w), F_{(1,0,1)}(v,w), F_{(1,1,1)}(v,w)\big)
$$

is actually a G-equivariant map whose zeros are exactly the configurations of planes that equipartition the measure and have the desired intersection property.

Step 2

Suppose now by contradiction that Ψ does not have a zero. This means that $\overline{\Psi} : S^1 \times S^3 \to S^3$ is a well defined G-equivariant map.

Denote by Ψ_a, $a \in S^1$, the map $\Psi_a : S^3 \to S^3$, $\Psi_a(p) = \overline{\Psi}(a,p)$; this function have two key properties:

1. $\forall a \in S^1$, Ψ_a is antipodal
2. $\forall a, b \in S^1$, Ψ_a and Ψ_b are homotopic

However, the map induced by g_1 on the sphere has degree -1 and thus we have:

$$[\Psi_a] = [\Psi_{g_1 \cdot a}] = [g_1 \cdot \Psi_a] = -[\Psi_a]$$

Thus Ψ_a is null-homotopic, but Borsuk-Ulam implies that an antipodal map of S^3 can't be. $\lightning$

$\square$

Topology facts needed in the proof

Theorem 2. *If $A \in O(n)$ is an orthogonal matrix, then the induced continuous map $A : S^{n-1} \to S^{n-1}$ has degree* $\deg(A) = \det(A)$.

Proof. Assume $\det(A) = 1$ (i.e. $A \in SO(n)$).

Let P an invertible matrix that puts A in Jordan normal form (R_θ denotes the 2×2 matrix of the rotation by θ):

$$A = P^{-1} \begin{pmatrix} R_{\theta_1} & & & & \\ & R_{\theta_2} & & & \\ & & \dots & & \\ & & & R_{\theta_k} & \\ & & & & Id_{n-2k} \end{pmatrix} P$$

Then, there is a path $\gamma : Id \rightsquigarrow A$ defined as:

$$\gamma(t) = P^{-1} \begin{pmatrix} R_{t\theta_1} & & & & \\ & R_{t\theta_2} & & & \\ & & \dots & & \\ & & & R_{t\theta_k} & \\ & & & & Id_{n-2k} \end{pmatrix} P$$

As a result, the map $A : S^{n-1} \to S^{n-1}$ is homotopic to the identity through the homotopy $H : S^{n-1} \times [0, 1] \to S^{n-1}$; $H(x, t) = \gamma(1 - t)x$.

Hence $deg(A) = deg(Id) = 1$.

Let now $\det(A) = -1$, then $QA \in SO(n)$ where

$$Q := \begin{pmatrix} Id & \\ & -1 \end{pmatrix}$$

this means that $1 = deg(QA) = deg(Q)deg(A) = -deg(A)$. $\square$

Theorem 3. *Let $f : S^{n-1} \to S^{n-1}$ be an antipodal map. Then* $\deg(f) \neq 0$.

Proof. Suppose by contradiction $\exists f : S^{n-1} \to S^{n-1}$ antipodal and $\deg(f) = 0$.

Then f can be extended to a map $F : D^n \to S^{n-1}$; using this map it is possible to construct $\tilde{F} : S^n \to S^{n-1}$:

$$\tilde{F}(x_1, \ldots, x_{n+1}) = \begin{cases} F(x_1, \ldots, x_n) & \text{if } x_{n+1} \geq 0 \\ -F(-x_1, \ldots, -x_n) & \text{if } x_{n+1} \leq 0 \end{cases}$$

It is well defined because on the intersection of the two pathes (i.e. the horizontal equator) both sides coincide with f; what is more, $\tilde{F}$ is antipodal:

$$\tilde{F}(-(x, x_{n+1})) = -F(-(-x)) = -F(x) = -\tilde{F}((x, x_{n+1}))$$

$$\tilde{F}(-(x, x_{n+1})) = F(-x) = -(-F(-x)) = -\tilde{F}((x, x_{n+1}))$$

Thus $\tilde{F}$ violates Borsuk-Ulam.

$\square$