# Splitting a measure along a fixed direction

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## Notation

- Given two vectors $x, y$ of dimension $d$, their scalar product is $(x|y) := \sum_{i=1}^{d} x_i y_i$.
- A measure $\mu$ on $\mathbb{R}^d$ is nice if it is Borel, $\mu(\mathbb{R}^d) = 1$, $\mu(H) = 0$ for all affine hyperplanes and it has connected support.
- Given an arrangement $\mathcal{H}$ of $k$ oriented affine hyperplanes and a sign pattern $\alpha \in (\mathbb{Z}_2)^k$, the orthant $\mathcal{O}(\mathcal{H}, \alpha)$ is defined as

  $$\mathcal{O}(\mathcal{H}, \alpha) := \{x \in \mathbb{R}^d | (-1)^{\alpha_i}((h_i|x) - a_i) > 0\}$$

  Intuitively, $\mathcal{O}(\mathcal{H}, \alpha)$ is the set of point lying on the side of $H_i$ determined by $\alpha_i$ (positive if $\alpha_i = 0$, negative otherwise).
- $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ with generators $g_1 = (1, 0)$ and $g_2 = (0, 1)$.
- If $v \in \mathbb{R}^d$ is a non-zero vector, $\overline{v}$ is its normalization $\overline{v} = \frac{v}{\|v\|}$.
- An oriented affine hyperplane $H = \{p \in \mathbb{R}^d | \, (h|p) - a = 0\}$ defines a point $\frac{(h,a)}{\|(h,a)\|}$ in $S^d$; likewise, a point on $x = (\tilde{x}, a) \in S^d$ defines an oriented affine hyperplane $H(x) = \{p \in \mathbb{R}^d | \, (\tilde{x}|p) - a = 0\}$.
- Given a nice measure $\mu$ in $\mathbb{R}^3$, define the standard test map $F : \left(S^3\right)^3 \times \mathbb{Z}_2^3 \to \mathbb{R}$:

  $$F(x, \omega) = \mu(\mathcal{O}(x, \omega)) - \frac{1}{8}$$

  $x$ is an equipartition $\iff F(x, \alpha) = 0 \ \forall \alpha$.
- Given a test map $F$, the alternating sum functions are the discrete Fourier transform of the test map. More precisely, if $\alpha \in (\mathbb{Z}_2)^3$ $\alpha \neq 0$, then the alternating sum with parameter $\alpha$ is:

  $$F_\alpha(x) = \sum_{\omega \in \mathbb{Z}_2^3} (-1)^{(\alpha|\omega)} F(x, \omega)$$

  $x$ is an equipartition $\iff F_\alpha(x) = 0 \ \forall \alpha \neq 0$

- The group $\mathbb{Z}_2^3$ acts on $\left(S^3\right)^3$ by coordinate-wise antipodality and, given $\alpha, g \in \mathbb{Z}_2^3$ with $\alpha \neq 0$ and a test map $F$:

$$F_\alpha(g \cdot x) = (-1)^{(\alpha|g)} F_\alpha(x)$$

# The Result

Our goal is to prove the following result:

**Theorem 1.** *Given a nice measure $\mu$ and a direction $p \in S^2$ it is possible to find 3 affine oriented planes that equipartition the measure and the first two have the prescribed oriented intersection $p$. Formally, $\exists \mathcal{H} = ((h_1, a_1), (h_2, a_2), (h_3, a_3))$ configuration of oriented planes such that:*

- $\forall \alpha \in (\mathbb{Z}_2)^k, \ \ \mu(\mathcal{O}(\mathcal{H}, \alpha)) = \frac{1}{8}$.
- $\overline{h}_1 \wedge \overline{h}_2 = p$.

*Proof.* Without loss of generality, let $z = (0, 0, 1)$. The proof will be split in two sections: the first deals with constructing a map $\Phi : S^1 \times S^3 \to \mathbb{R}^4$ in $\mathbb{R}^4$ whose zeros codify equipartitions of the mass. The second step will be to show that the map will respect a suitable action of $G$ on the two space and that such equivariance forces the existence of a zero.

**Step 1**

The key step in constructing the map is to show that we can parametrize pair of planes that have intersection direction $z$ and split the mass in 4 equal parts with vector in $S^1$.

Project the mass on the $xy$ plane to obtain a nice measure $\mu^{\#}$ on $\mathbb{R}^2$. The following bisecting lemma applies:

**Lemma 1** (Bisecting)**.** *Let $\mu^{\#}$ be a nice measure on $\mathbb{R}^2$ and $v \in S^1$ a direction. Then there exists two oriented affine lines $l_0 = \mathbb{R}\vec{l_0} + a_0$ and $l_1 = \mathbb{R}\vec{l_1} + a_1$ in $\mathbb{R}^2$ such that:*

- *$l_0$ and $l_1$ equipartition $\mu^{\#}$*
- *$v$ bisects the angle between $l_0$ and $l_1$*

*What is more, we can choose consistently the direction $\vec{l_0}$ (e.g. fix $\vec{l_0}$ to be the first direction clockwise, while the first one contraclockwise is $\vec{l_1}$). Once this choice is made, $\vec{l_0}$ and $\vec{l_1}$ are unique and depend continuously on $v$.*

The lemma guarantees that, once we fix a direction $v \in S^1 \subseteq S^2$ (inclusion as the horizontal equator in $S^2$) there are two affine lines in the $xy$ plane $l_0 = \mathbb{R}\vec{l_0}(v) + a_0(v))$ and $l_1 = \mathbb{R}\vec{l_1}(v) + a_1(v))$ that bisect the projected measure $\mu^{\#}$.

Define $H_i(v) = (h_i(v), a_i(v))$ to be the affine (oriented) span of $l_i(v)$ and $z$, the two planes now equipartition the measure $\mu$ and have the desired intersection.

Let now $\mathbb{Z}_4 = \langle g_1 \rangle$ act on $S^1$ by $\frac{\pi}{2}$ rotation counter clockwise. Then, by construction we have that:

$$H_0(g_1 \cdot v) = H_1(v)$$
$$H_1(g_1 \cdot v) = -H_0(v)$$

In other words, if we consider the planar problem with the bisecting vector rotated by $\frac{\pi}{2}$, the new first line is the previous second while the new second is is the old first with opposite orientation.

We can now define a map $S^1 \times S^3 \to (S^3)^3$, $(v, w) \mapsto (H_0(v), H_1(v), w)$.

To check now that a configuration of 3 planes equipartition the measure, it is equivalent to check that the alternating sum functions $F_\alpha$ are 0 for all signed patterns $\alpha \in \mathbb{Z}_2^3 \setminus 0$. However, by construction the first two planes split equally the mass, thus it is sufficient to check the 4 patterns involving the last one; i.e. $\alpha = (0,0,1), (1,0,1), (0,1,1)$ and $(1,1,1)$.

What is more, it is straightforward how the alternating sum functions behave under the $G$-action. Explicitly, $\forall (v, w) \in S^1 \times S^3$ the following equalities hold:

$$\begin{aligned}
F_{(0,0,1)}(g_1 \cdot (v,w)) &= F_{(0,0,1)}(v,w) \\
F_{(0,1,1)}(g_1 \cdot (v,w)) &= -F_{(1,0,1)}(v,w) \\
F_{(1,0,1)}(g_1 \cdot (v,w)) &= F_{(0,1,1)}(v,w) \\
F_{(1,1,1)}(g_1 \cdot (v,w)) &= -F_{(1,1,1)}(v,w)
\end{aligned}$$

$$\begin{aligned}
F_{(0,0,1)}(g_2 \cdot (v,w)) &= -F_{(0,0,1)}(v,w) \\
F_{(0,1,1)}(g_2 \cdot (v,w)) &= -F_{(1,0,1)}(v,w) \\
F_{(1,0,1)}(g_2 \cdot (v,w)) &= -F_{(0,1,1)}(v,w) \\
F_{(1,1,1)}(g_2 \cdot (v,w)) &= -F_{(1,1,1)}(v,w)
\end{aligned}$$

Additionally, we can choose a linear $G$-action on $\mathbb{R}^4$ that is consistent with the previous equations. In particular, if we define:

$$\begin{aligned}
g_1 \cdot (x,y,z,u) &= (x, -z, y, -u) \\
g_2 \cdot (x,y,z,u) &= (-x, -y, -z, -u)
\end{aligned}$$

Then it is easy to check that the alternating sum map $\Psi : S^1 \times S^3 \to \mathbb{R}^4$

$$(v,w) \mapsto \big(F_{(0,0,1)}(v,w), F_{(0,1,1)}(v,w), F_{(1,0,1)}(v,w), F_{(1,1,1)}(v,w)\big)$$

is actually a $G$-equivariant map whose zeros are exactly the configurations of planes that equipartition the measure and have the desired intersection property.

**Step 2**

Suppose now by contradiction that $\Psi$ does not have a zero. This means that $\overline{\Psi} : S^1 \times S^3 \to S^3$ is a well defined $G$-equivariant map.

Denote by $\Psi_a$, $a \in S^1$, the map $\Psi_a : S^3 \to S^3$, $\Psi_a(p) = \overline{\Psi}(a, p)$; this function have two key properties:

1. $\forall a \in S^1$, $\Psi_a$ is antipodal
2. $\forall a, b \in S^1$, $\Psi_a$ and $\Psi_b$ are homotopic

However, the map induced by $g_1$ on the sphere has degree $-1$ and thus we have:

$$[\Psi_a] = [\Psi_{g_1 \cdot a}] = [g_1 \cdot \Psi_a] = -[\Psi_a]$$

Thus $\Psi_a$ is null-homotopic, but Borsuk-Ulam implies that an antipodal map of $S^3$ can't be. $\lightning$

$\square$

# Topology facts needed in the proof

**Theorem 2.** *If $A \in O(n)$ is an orthogonal matrix, then the induced continuous map $A : S^{n-1} \to S^{n-1}$ has degree* $\deg(A) = \det(A)$.

*Proof.* Assume $\det(A) = 1$ (i.e. $A \in SO(n)$).

Let $P$ an invertible matrix that puts $A$ in Jordan normal form ($R_\theta$ denotes the $2 \times 2$ matrix of the rotation by $\theta$):

$$A = P^{-1} \begin{pmatrix} R_{\theta_1} & & & & \\ & R_{\theta_2} & & & \\ & & \dots & & \\ & & & R_{\theta_k} & \\ & & & & Id_{n-2k} \end{pmatrix} P$$

Then, there is a path $\gamma : Id \rightsquigarrow A$ defined as:

$$\gamma(t) = P^{-1} \begin{pmatrix} R_{t\theta_1} & & & & \\ & R_{t\theta_2} & & & \\ & & \dots & & \\ & & & R_{t\theta_k} & \\ & & & & Id_{n-2k} \end{pmatrix} P$$

As a result, the map $A : S^{n-1} \to S^{n-1}$ is homotopic to the identity through the homotopy $H : S^{n-1} \times [0, 1] \to S^{n-1}$; $H(x, t) = \gamma(1 - t)x$.

Hence $deg(A) = deg(Id) = 1$.

Let now $\det(A) = -1$, then $QA \in SO(n)$ where

$$Q := \begin{pmatrix} Id & \\ & -1 \end{pmatrix}$$

this means that $1 = deg(QA) = deg(Q)deg(A) = -deg(A)$. $\square$

**Theorem 3.** *Let $f : S^{n-1} \to S^{n-1}$ be an antipodal map. Then* $\deg(f) \neq 0$.

*Proof.* Suppose by contradiction $\exists f : S^{n-1} \to S^{n-1}$ antipodal and $\deg(f) = 0$.

Then $f$ can be extended to a map $F : D^n \to S^{n-1}$; using this map it is possible to construct $\tilde{F} : S^n \to S^{n-1}$:

$$\tilde{F}(x_1, \ldots, x_{n+1}) = \begin{cases} F(x_1, \ldots, x_n) & \text{if } x_{n+1} \geq 0 \\ -F(-x_1, \ldots, -x_n) & \text{if } x_{n+1} \leq 0 \end{cases}$$

It is well defined because on the intersection of the two pathes (i.e. the horizontal equator) both sides coincide with $f$; what is more, $\tilde{F}$ is antipodal:

$$\tilde{F}(-(x, x_{n+1})) = -F(-(-x)) = -F(x) = -\tilde{F}((x, x_{n+1}))$$

$$\tilde{F}(-(x, x_{n+1})) = F(-x) = -(-F(-x)) = -\tilde{F}((x, x_{n+1}))$$

Thus $\tilde{F}$ violates Borsuk-Ulam.

$\square$