Notes on equipartitions

June 26, 2023

1 Notation

- Given two vectors x, y of dimension d, their scalar product is $(x|y) := \sum_{i=1}^{d} x_i y_i$.
- $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ with elements ν_1, ν_2 and $\nu_1\nu_2 = \nu_2\nu_1$.
- Given a set of elements S in an object A (group elements in a group, vectors in a vector space, etc.), by $\langle S \rangle \subseteq A$ we denote the generated object (group, vector subspace, etc.).
- Given a normed vector space V, $S(V)$ will denote its sphere
- $\mathbb{S}_k$ is the group of permutations on k elements and $\mathbb{S}_k^{\pm} = (\mathbb{Z}_2)^k \rtimes \mathbb{S}_k$ the signed permutation group.
- G acts on $T^2 = S^1 \times S^1$ coordinate-wise, i.e.

$$\nu_1^T((\theta, \phi)) = (\theta + \pi, \phi)$$
$$\nu_2^T((\theta, \phi)) = (\theta, \phi + \pi)$$

- G acts on $\mathbb{R}^2$ linearly:

$$\nu_1(x, y) = (x, -y)$$
$$\nu_2(x, y) = (-x, -y)$$

- The previous action can be restricted to an action on $S^1 = S(\mathbb{R}^2)$:

$$\nu_1^S(\theta) = 2\pi - \theta$$
$$\nu_2^S(\theta) = \theta + \pi$$

- An arrangement of planes in $\mathbb{R}^3$ is a point $\mathcal{H} \in \left(S^3\right)^3$

$$\mathcal{H} = (\begin{pmatrix} h_1 \\ a_1 \end{pmatrix}, \begin{pmatrix} h_2 \\ a_2 \end{pmatrix}, \begin{pmatrix} h_3 \\ a_3 \end{pmatrix})$$

- Given $\omega \in (\mathbb{Z}_2)^3$ and an arrangement $\mathcal{H}$, denote by $\mathcal{O}(\mathcal{H}, \omega)$ the open orthant of $\mathbb{R}^3$ corresponding to the sign pattern ω; i.e.:

$$\mathcal{O}(\mathcal{H}, \omega) = \left\{ x \in \mathbb{R}^3 \mid (-1)^{\omega_i} (h_i|x) > (-1)^{\omega_i} a_i \right\}$$

- A measure on $\mathbb{R}^d$ is nice if it is a Borel probability measure that has a continuous density function with compact connected positive support.

- Given a nice measure μ and $\alpha \in (\mathbb{Z}_2)^3 \setminus \{0\}$, the alternating sum defined by α is the function $f_\alpha : (S^3)^3 \to \mathbb{R}$:

$$f_\alpha(\mathcal{H}) = \sum_{\beta \in (\mathbb{Z}_2)^3} (-1)^{(\alpha|\beta)} \mu(\mathcal{O}(\mathcal{H}, \beta))$$

- Fix $\alpha \in (\mathbb{Z}_2)^k \setminus \{0\}$, it is possible to define an action of $(\mathbb{Z}_2)^k$ on $\mathbb{R}$ by:

$$\omega(v) = (-1)^{(\omega|\alpha)} v$$

  Denote by V_α the resulting $(\mathbb{Z}_2)^k$ representation.

- The signed sum maps are $(\mathbb{Z}_2)^3$-equivariant maps. More precisely, given $\omega, \alpha \in (\mathbb{Z}_2)^3$ with $\alpha \neq 0$:

$$f_\alpha(\omega \mathcal{H}) = (-1)^{(\alpha|\omega)} f_\alpha(\mathcal{H})$$

The goal of this notes is to prove the following result:

Theorem 1. *Let μ be a nice measure on $\mathbb{R}^3$ and $\vec{x} \in S^2$ a direction. Then there exists a triple of affine planes $\mathcal{H} = (h_1, h_2, h_3) \in (S^3)^3$ and a vector $b \in \mathbb{R}^3$ such that:*

- *For all orthants $\mathcal{O}(\mathcal{H}, \omega)$, $\mu(\mathcal{O}(\mathcal{H}, \omega)) = \frac{1}{8}$*
- *$h_1 \cap h_2 = \mathbb{R}\vec{x} + b$*

2 Reduction to equivariant topology

The goal of this section is to show that a suitable Configuration Space/Test Map scheme for the problem is T^2 with a G-map $f : T^2 \to \mathbb{R}^2$. More precisely, we will first construct a test space E and then show that E is G-homeomorphic to the torus with the standard G-action.

Without loss of generality we can assume the direction we want the two planes to intersect is the $\vec{z}$ axis. Project the mass on the xy plane to obtain a nice measure $\mu^\#$ on $\mathbb{R}^2$. The following bisecting lemma applies:

Lemma 1 (Bisecting)**.** *Let $\mu^\#$ be a nice measure on $\mathbb{R}^2$ and $v \in S^1$ a direction. Then there exists two oriented affine lines $l_0 = \mathbb{R}\vec{l_0} + a_0$ and $l_1 = \mathbb{R}\vec{l_1} + a_1$ in $\mathbb{R}^2$ such that:*

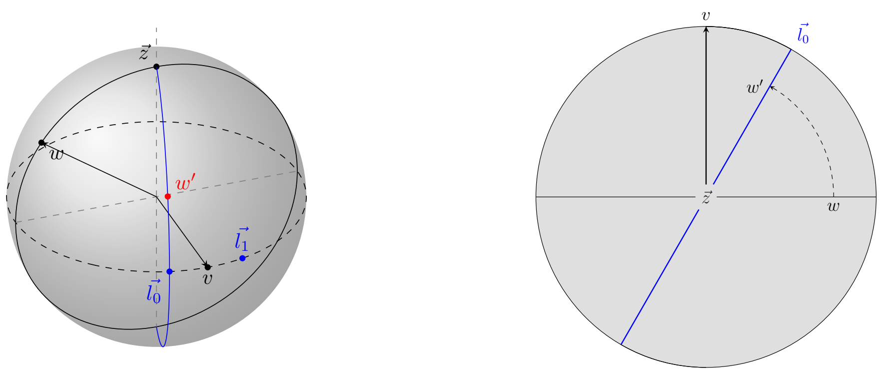
- *l_0 and l_1 equipartition $\mu^{\#}$*
- *v bisects the angle between l_0 and l_1*

What is more, we can choose consistently the direction $\vec{l_0}$ (e.g. fix $\vec{l_0}$ to be the first direction clockwise, while the first one contraclockwise is $\vec{l_1}$). Once this choice is made, $\vec{l_0}$ and $\vec{l_1}$ are unique and depend continuously on v.

The lemma guarantees that, once we fix a direction $v \in S^1 \subseteq S^2$ (seen as a vector on the horizontal equator in S^2) there are two affine lines in the xy plane $l_0 = \mathbb{R}\vec{l_0}(v) + a_0(v))$ and $l_1 = \mathbb{R}\vec{l_1}(v) + a_1(v))$ that bisect the projected measure $\mu^{\#}$.

Define $h_i(v)$ to be the affine (oriented) span of $l_i(v)$ and $\vec{z}$, the two planes now equipartition the measure μ and have the desired intersection. What is more, by our choice of direction on l_i we have that $h_i(-v) = -h_i(v)$ (i.e. the plane is the same, but it swaps orientation).

Now, fix $w \in S^2$ such that $(v|w) = 0$. There exist a unique rotation R_v of S^2 that fixes $\vec{z}$, maps the meridian orthogonal to v (and thus containing w) to the meridian containing $\vec{l_0}$ and has rotation angle smaller than $\frac{\pi}{2}$, thus the point $w' = R_v(w)$ is well defined (see figure 1).



(a) Procedure on S^2.

(b) Projection on the xy-plane.

Figure 1: Construction of h_3.

The direction w' defines an oriented linear plane $P = \left\{p \in \mathbb{R}^3 | \, (w'|v) = 0\right\}$ that is orthogonal to h_0 by construction, thus the projection of μ on p is bisected by $p \cap h_0$. By Ham-Sandwich theorem there is a (unique) line γ that splits the two halves simultaneously, define h_2 to be the affine span of w' and γ.

In particular, we have constructed a continuous map $\phi : E \to \left(S^3\right)^3$:

$$(v, w) \mapsto (h_0(v), h_1(v), h_2(v, w))$$

with E the S^1-bundle over S^1 defined as

$$E = \left\{(v, w) \in S^2 \times S^2 \mid v_3 = 0 \text{ and } (v|w) = 0\right\}.$$

It is possible to define a G action on E as follows:

$$\nu_1((\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix},\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}))=(\begin{pmatrix}-v_1\\-v_2\\-v_3\end{pmatrix},\begin{pmatrix}-w_1\\-w_2\\w_3\end{pmatrix})$$
$$\nu_2((\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix},\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}))=(\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix},\begin{pmatrix}-w_1\\-w_2\\-w_3\end{pmatrix})$$

It is straightforward to verify that, if we fix the action of G on $\left(S^3\right)^3$ to be the one induced by the subgroup of $(\mathbb{Z}_2)^3 = \langle \eta_1, \eta_2, \eta_3 \rangle$ generated by $\eta_1 + \eta_2$ and η_3, then the map ϕ is G-equivariant.

Any plane arrangement $\mathcal{H}$ is an equipartition if and only if all the signed sums f_α are 0, however, all the signed sums beside $f_{(1,1,1)}$ and $f_{(0,1,1)}$ are 0 for any arrangement obtained through the previously outlined construction.

Thus, $\mathcal{H}$ equipartitions μ if and only if the G-equivariant map in $\mathbb{R}^2$

$$f(\mathcal{H}) = \left(f_{(1,1,1)}(\mathcal{H}), f_{(0,1,1)}(\mathcal{H})\right)$$

has a 0.

The last thing left to check is that the configuration space is indeed G-homeomorphic to the standard torus $T^2 = S^1 \times S^1$.

However, it is possible to define an explicit G-isomorphism $\psi : T^2 \to E$:

$$(\phi,\theta) \mapsto (\begin{pmatrix}\cos(\phi)\\ \sin(\phi)\\ 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\sin(\phi)\\ -\sin(\theta)\cos(\phi)\\ \cos(\theta)\end{pmatrix})$$

The G-map is continuous, injective, surjective and closed $\Rightarrow$ it is a G-homeomorphism.

3 Equivariant topology

Theorem 2. *Let $f : T^2 \to_G \mathbb{R}^2$. Then f has a zero.*

First Proof: fundamental group obstruction

Proof. By contradiction, suppose $\exists f : T^2 \to_G \mathbb{R}^2 \setminus \{0\} \sim S^1$. Then f induces a map on the level of fundamental groups:

$$f_* : \pi_1(T^2) \simeq \mathbb{Z}^2 \to \pi_1(S^1) \simeq \mathbb{Z}$$

Denote by μ, λ the usual generators for $\pi_1(T^2)$ (see figure 2) and by γ the class of the identity in $\pi_1(S^1)$.

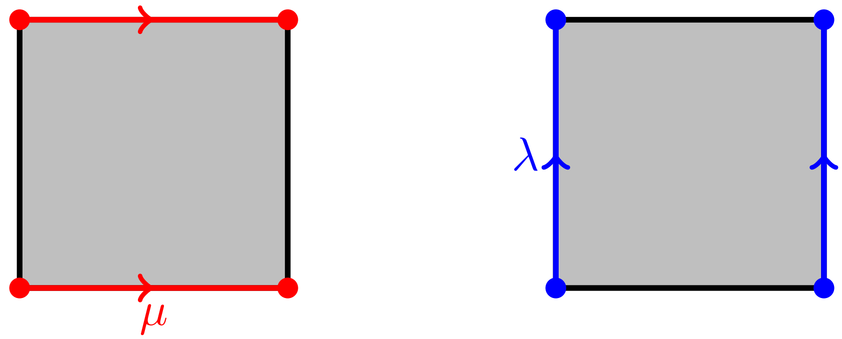


Figure 2: Representatives for the generators.

Using the generators, we can explicitly compute the action of G on both the fundamental groups:

Action on $\pi_1(T^2)$	Action on $\pi_1(S^1)$
Trivial, i.e. $\nu_1^T = \nu_2^T = Id$	$\nu_1^S(\gamma) = -\gamma$ $\nu_2^S(\gamma) = \gamma$

The group homomorphism $f_* : \mathbb{Z}^2 \to \mathbb{Z}$ is completely determined by the images $f_*(\lambda) = n\gamma$ and $f_*(\mu) = m\gamma$.

Since f_* is G-equivariant we have:

$$n\gamma = f_*(\lambda) = f_*(\nu_1^T(\lambda)) = \nu_1^S(f_*(\lambda)) = -n\gamma \Rightarrow n = 0$$

On the other hand, by checking what happens along the diagonal $\lambda + \mu$ we have:

$$m\gamma = f_*((\nu_1^T \nu_2^T)(\lambda + \mu)) = (\nu_1^S \nu_2^S)(f_*(\lambda + \mu)) = -m\gamma \Rightarrow m = 0$$

However, we can map S^1 on the torus $g : S^1 \to S^1 \times S^1$, $g(\theta) = (0, \theta)$ such that g is an antipodal map, i.e. $g(\pi + \theta) = \nu_1^T(g(\theta))$. The composition $fg : S^1 \to S^1$ is an antipodal map on $S^1 \Rightarrow deg(fg)$ is odd, but $deg(fg) = 0$ since f_* is identically 0.

$\square$

Second Proof: Fadell-Husseini index

Proof. The target space $\mathbb{R}^2$ is G-isomorphic to the representation $V_{01} \oplus V_{11}$ and thus $S(\mathbb{R}^2) = S(V_{01}) * S(V_{11})$. Fixing $\mathbb{F}_2$ as coefficients we have that:

$$Ind_G(V_{(\alpha_0,\alpha_1)}) = (\alpha_0 t_0 + \alpha_1 t_1) \subseteq \mathbb{F}_2[t_0, t_1]$$
$$Ind_G(X) Ind_G(Y) \subseteq Ind_G(X * Y)$$

As a consequence $(t_0 t_1 + t_1^2) \subseteq Ind_G(S(\mathbb{R}^2))$; however:

$$t_0 t_1 + t_1^2 \notin (t_0^2, t_1^2) = Ind_G(S^1 \times S^1) \Rightarrow Ind_G(S(\mathbb{R}^2)) \nsubseteq Ind_G(S^1 \times S^1)$$

Thus, there is no G-equivariant map $S^1 \times S^1 \to S(\mathbb{R}^2)$. $\square$

3.1 Explicit construction of the test map

The purpose of this section is to explicitly define and check the symmetries of the map:

$$\Phi : E \to \left(S^3\right)^3$$

In order to simplify a bit the notation, we will denote by $\overline{x} = \frac{x}{\|x\|}$ the normalization of a non-zero vector x.

Construction of the map

Given a point $(v, w) \in E$, the bisecting lines for the projected mass on the horizontal plane are going to be defined as $l_0 = \mathbb{R}\vec{l_0}(v) + b_0(v)$ (with $\vec{l_0}, b_0 \in \mathbb{R}^3$). The vector $\vec{l_0}$ is defined as the unique vector of length one spanning the linear part of l_0 such that such that $z = \overline{(\vec{l_0} \wedge v)}$ (equivalently, $\vec{l_0}$ is the first one contraclockwise from v)[1].

In particular, the plane $H_0 = \langle \vec{l_0}, z \rangle + b_0$ has a natural orientation defined by the direction $z \wedge \vec{l_0}(v)$ and it induces a well defined point $h_0 \in S^3$:

$$h_0(v) = \overline{\begin{pmatrix} z \wedge \vec{l_0}(v) \\ \left(b_0(v) | (z \wedge \vec{l_0}(v))\right) \end{pmatrix}} \in S^3$$

The plane H_1 and the point h_1 are constructed in the same way using the line l_1.

Let now P be the linear oriented plane defined by $w' = R_v(w)$ (i.e. the plane orthogonal to w' with the induced orientation) and λ be the affine line corresponding to the projection of H_0 on P; by construction $\lambda = H_0 \cap P$, thus it has a natural orientation. Denote its direction by $\vec{\lambda}$.

By Ham-Sandwich theorem, there is a unique $\gamma = \mathbb{R}y(v, w) + c(v, w) \subseteq P$ line (with $y, c \in \mathbb{R}^3$) that, together with λ, split the projected mass in 4 equal

[1]Both v and $\vec{l_0}$ are always orthogonal to z by construction so the only possible case where the definition doesn't make sense is where $\vec{l_0} = \pm v$; however, if this was the case, the lines l_0 and l_1 would coincide and thus wouldn't be bisecting the projected problem.

pieces. The vector y is the unique vector of length 1 such that $\vec{\overline{\lambda}} \wedge y = w'$ and y spans the linear part of γ.

Finally, the plane H_2 spanned by γ and w' has a natural orientation, induced by the normal $y \wedge w'$ and thus it defines a point h_2 in S^3:

$$h_2(v,w) = \overline{\begin{pmatrix} y(v,w) \wedge w' \\ (c(v,w)|(y(v,w) \wedge w')) \end{pmatrix}}.$$

The map is thus the following:

$$(v,w) \mapsto (h_0(v), h_1(v), h_2(v,w))$$

All the constructed lines depend continuously on v, w and the mass μ and since the orientation can be chosen consistently, the total map is well defined and continuous.

Symmetries

The space E has the natural product G-action:

$$\nu_1^E(v,w) = (-v,w)$$
$$\nu_2^E(v,w) = (v,-w)$$

DANGER: This action is NOT the same as the action described in the previous section! This causes problems.

Lemma 2. *Suppose* $\Psi(v,w) = (h_0, h_1, h_2)$*, then:*

$$\Psi(\nu_1^E(v,w)) = (-h_0, -h_1, -h_2)$$
$$\Psi(\nu_2^E(v,w)) = (h_0, h_1, -h_2)$$

Proof. If we swap v with $-v$ in the construction, the vector $\vec{l_0}$ keeps the same direction but changes sign, i.e. $\vec{l_0}(-v) = -\vec{l_0}$. Since b_0 is the translation vector, this is not changed and so $h_0(-v) = -h_0(v)$ (the same reasoning works for h_1 as well).

Since both v and $\vec{l_0}$ change sign, the rotation R_v still maps the orthogonal meridian to the one containing $\vec{l_0}$, w' does not change; as a result, the plane P does not change orientation. However, h_1 has the opposite orientation and thus the line λ swaps orientation as well.

As a consequence, the vector y for the new problem changes sign (the sign of y is defined by the equation $\vec{\overline{\lambda}} \wedge y = w'$, where $\vec{\lambda}(-v) = -\vec{\lambda}(v)$ and $w'(-v) = w'(v)$).

Overall, we have that $\Psi((-v,w)) = (-h_0, -h_1, -h_2)$.

The other case is straightforward: h_0 and h_1 do not depend on w so they are not changed; on the other hand, $w'(v,-w) = R_v(-w) = -w'(v,w)$ and thus the plane P changes orientation. As a result, $\vec{lambda}$ changes sign and thus y doesn't.

Overall, since w' changes sign and y doesn't, we have that $\Psi((v,-w)) = (h_0, h_1, -h_2)$. □

4 Error in the proof

The construction is wrong: the action on the torus T that we need in order for it to be G-homeomorphic to E is the following:

$$\nu_1^T(\phi, \theta) = (\phi + \pi, 2\pi + \theta)$$
$$\nu_2^T(\phi, \theta) = (\phi, \pi + \theta)$$

The issue is that there is no homotopic obstruction to having a G-map from the torus to $\mathbb{R}^2$; in fact, there exist such map:

$$(\phi, \theta) \mapsto (\cos(\theta), \sin(\theta))$$

It is still worth noting that every isolated step of the previous proof is correct; in particular the construction of the affine planes with prescribed intersection outlined in section 2.

5 Order 4 symmetries

Denote by H the group $\mathbb{Z}_4 \times \mathbb{Z}_2$.

The goal of this section is to construct a different set up of configuration space - test map, where the objects involved are going to be H-equivariant.

5.1 Configuration space-Test map setup

Without loss of generality, we can assume the direction of the desired intersection to be $\vec{z}$. Fix now a vector $v \in S^1$, by the same reasoning of section 2, it is possible to construct two affine oriented planes H_0 and H_1 that split the measure equally, have the desired intersection and depend continuously on v (denote by h_i the corresponding points on S^3).

It is also worth noting that, if $g_1^{S^1}$ is a rotation by $\frac{\pi}{2}$ clockwise of S^1, then by uniqueness of the cuts on the plane we have:

$$h_0(g_1^{S^1}(v)) = -h_1(v)$$
$$h_1(g_1^{S^1}(v)) = h_0(v)$$

Now, fix $w \in S^2$; there exists a unique affine plane H_2 orthogonal to w (oriented consistently with w) that bisect the mass. Thus the point $h_2(w) \in S^3$ is well defined, depends continuously on w and is such that $h_2(-w) = -h_2(w)$.

By construction, all the planes split the mass in half and H_0, H_1 split the measure in equal parts. Thus, the plane arrangement defined by $\mathcal{H}(v, w) = (h_0(v), h_1(v), h_2(w))$ is an equipartition if and only if:

$$\begin{cases} f_{(1,1,1)}(\mathcal{H}) = 0 \\ f_{(0,1,1)}(\mathcal{H}) = 0 \\ f_{(1,0,1)}(\mathcal{H}) = 0 \end{cases}$$

Denote by $g_2^{S^2}$ the antipodal action on S^2; then the following relations hold:

$$\begin{aligned} f_{(1,1,1)}(\mathcal{H}(g_1^{S^1} v, w)) &= -f_{(1,1,1)}(\mathcal{H}(v, w)) \\ f_{(0,1,1)}(\mathcal{H}(g_1^{S^1} v, w)) &= f_{(1,0,1)}(\mathcal{H}(v, w)) \\ f_{(1,0,1)}(\mathcal{H}(g_1^{S^1} v, w)) &= -f_{(0,1,1)}(\mathcal{H}(v, w)) \end{aligned}$$

$$\begin{aligned} f_{(1,1,1)}(\mathcal{H}(v, g_2^{S^2} w)) &= -f_{(1,1,1)}(\mathcal{H}(v, w)) \\ f_{(0,1,1)}(\mathcal{H}(v, g_2^{S^2} w)) &= -f_{(0,1,1)}(\mathcal{H}(v, w)) \\ f_{(1,0,1)}(\mathcal{H}(v, g_2^{S^2} w)) &= -f_{(1,0,1)}(\mathcal{H}(v, w)) \end{aligned}$$

In particular, if we fix the following action of $H = \langle g_1, g_2 \rangle$ on $\mathbb{R}^3$:

$$\begin{aligned} g_1^{\mathbb{R}}(x, y, z) &= (-x, z, -y) \\ g_2^{\mathbb{R}}(x, y, z) &= (-x, -y, -z) \end{aligned}$$

and we define

$$F(v, w) = \begin{pmatrix} f_{(1,1,1)}(\mathcal{H}(v, w)) \\ f_{(0,1,1)}(\mathcal{H}(v, w)) \\ f_{(1,0,1)}(\mathcal{H}(v, w)) \end{pmatrix}$$

then, we have that we have constructed an H-map $F : S^1 \times S^2 \to \mathbb{R}^3$ whose zeros correspond to equipartition with the desired intersection.

5.2 Equivariant topology

The goal of this section is to show the following result:

Theorem 3. *Let $F : S^1 \times S^2 \to_H \mathbb{R}^3$. Then F has a zero.*

Proof. By contradiction, suppose there exists a map F that does not hit 0; up to normalizing, we have a map $F : S^1 \times S^2 \to_H S^2$.

Given a point $a \in S^1$, fixing the first coordinate $F_a \equiv F(a, \bullet)$ defines an element in $\pi_2(S^2)$ and any two such maps are homotopic[2]; as a result, let d be the degree of any such map.

[2]Given $a = e^{i\theta}$ and $b = e^{i\phi}$, the homotopy is given by $H(t, x) = F(e^{i(t\phi+(1-t)\theta)}, x)$.

Fix now a point $a \in S^1$ and let $b = g_1^{S^1}(a)$. Then, $F_b(x) = g_1^{\mathbb{R}}(F_a(x))$ by H-equivariance, but this implies that:

$$d[Id] = [F_b] = [g_1^{\mathbb{R}} \circ F_a] = \det(g_1^{\mathbb{R}})[F_a] = -[F_a] = -d[Id]$$

Thus $d = 0$ and F_a is nullhomotopic. However, by H-equivariance, F_a is an antipodal map from $S^2 \to S^2$ and thus it can not be nullhomotopic by Borsuk-Ulam. $\square$