

Splitting a measure along a fixed direction

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Notation

- A measure μ on \mathbb{R}^d is nice if it is Borel, $\mu(\mathbb{R}^d) = 1$, $\mu(H) = 0$ for all affine hyperplanes and it has connected support.
- Given an arrangement \mathcal{H} of k oriented affine hyperplanes and a sign pattern $\alpha \in (\mathbb{Z}_2)^k$, the orthant $\mathcal{O}(\mathcal{H}, \alpha)$ is defined as

$$\mathcal{O}(\mathcal{H}, \alpha) := \{x \in \mathbb{R}^d \mid (-1)^{\alpha_i}((h_i|x) - a_i) > 0\}$$

Intuitively, $\mathcal{O}(\mathcal{H}, \alpha)$ is the set of point lying on the side of H_i determined by α_i (positive if $\alpha_i = 0$, negative otherwise).

- $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ with generators $g_1 = (1, 0)$ and $g_2 = (0, 1)$.
- If $v \in \mathbb{R}^d$ is a non-zero vector, \bar{v} is its normalization $\bar{v} = \frac{v}{\|v\|}$.
- An oriented affine hyperplane $H = \{x \in \mathbb{R}^d \mid (h|x) - a = 0\}$ defines a point $\frac{(h,a)}{\|(h,a)\|}$ in S^d and vice versa.
- Given a nice measure μ in \mathbb{R}^3 , define the standard test map $F : (S^3)^3 \times \mathbb{Z}_2^3 \rightarrow \mathbb{R}$:

$$F(x, \omega) = \mu(\mathcal{O}(x, \omega)) - \frac{1}{8}$$

x is an equipartition $\iff F(x, \alpha) = 0 \ \forall \alpha$.

- Given a test map F , the alternating sum functions are the discrete Fourier transform of the test map. More precisely, if $\alpha \in (\mathbb{Z}_2)^3$ $\alpha \neq 0$, then the alternating sum with parameter α is:

$$F_\alpha(x) = \sum_{\omega \in \mathbb{Z}_2^3} (-1)^{(\alpha|\omega)} F(x, \omega)$$

x is an equipartition $\iff F_\alpha(x) = 0 \ \forall \alpha \neq 0$

- The group \mathbb{Z}_2^3 acts on $(S^3)^3$ by coordinate-wise antipodality and, given $\alpha, g \in \mathbb{Z}_2^3$ with $\alpha \neq 0$ and a test map F :

$$F_\alpha(g \cdot x) = (-1)^{(\alpha|g)} F_\alpha(x)$$

The Result

Our goal is to prove the following result:

Theorem 1. *Given a nice measure μ and a direction $p \in S^2$ it is possible to find 3 affine oriented planes that equipartition the measure and the first two have the prescribed oriented intersection p . Formally, $\exists \mathcal{H} = ((h_1, a_1), (h_2, a_2), (h_3, a_3))$ configuration of oriented planes such that:*

- $\forall \alpha \in (\mathbb{Z}_2)^k, \mu(\mathcal{O}(\mathcal{H}, \alpha)) = \frac{1}{8}$.
- $\bar{h}_1 \wedge \bar{h}_2 = p$.

Proof. Without loss of generality, let $z = (0, 0, 1)$. The proof will be split in two sections: the first deals with constructing a map $\Phi : S^1 \times S^3 \rightarrow \mathbb{R}^4$ in \mathbb{R}^4 whose zeros codify equipartitions of the mass. The second step will be to show that the map will respect a suitable action of G on the two space and that such equivariance forces the existence of a zero.

Step 1

The key step in constructing the map is to show that we can parametrize pair of planes that have intersection direction z and split the mass in 4 equal parts with vector in S^1 .

Project the mass on the xy plane to obtain a nice measure $\mu^\#$ on \mathbb{R}^2 . The following bisecting lemma applies:

Lemma 1 (Bisecting). *Let $\mu^\#$ be a nice measure on \mathbb{R}^2 and $v \in S^1$ a direction. Then there exists two oriented affine lines $l_0 = \mathbb{R}\vec{l}_0 + a_0$ and $l_1 = \mathbb{R}\vec{l}_1 + a_1$ in \mathbb{R}^2 such that:*

- l_0 and l_1 equipartition $\mu^\#$
- v bisects the angle between l_0 and l_1

What is more, we can choose consistently the direction \vec{l}_0 (e.g. fix \vec{l}_0 to be the first direction clockwise, while the first one contraclockwise is \vec{l}_1). Once this choice is made, \vec{l}_0 and \vec{l}_1 are unique and depend continuously on v .

The lemma guarantees that, once we fix a direction $v \in S^1 \subseteq S^2$ (inclusion as the horizontal equator in S^2) there are two affine lines in the xy plane $l_0 = \mathbb{R}\vec{l}_0(v) + a_0(v)$ and $l_1 = \mathbb{R}\vec{l}_1(v) + a_1(v)$ that bisect the projected measure $\mu^\#$.

Define $H_i(v) = (h_i(v), a_i(v))$ to be the affine (oriented) span of $l_i(v)$ and z , the two planes now equipartition the measure μ and have the desired intersection.

Let now $\mathbb{Z}_4 = \langle g_1 \rangle$ act on S^1 by $\frac{\pi}{2}$ rotation counter clockwise. Then, by construction we have that:

$$H_0(g_1 \cdot v) = H_1(v)$$

$$H_1(g_1 \cdot v) = -H_0(v)$$

In other words, if we consider the planar problem with the bisecting vector rotated by $\frac{\pi}{2}$, the new first line is the previous second while the new second is the old first with opposite orientation.

We can now define a map $S^1 \times S^3 \rightarrow (S^3)^3$, $(v, w) \mapsto (H_0(v), H_1(v), w)$.

To check now that a configuration of 3 planes equipartition the measure, it is equivalent to check that the alternating sum functions F_α are 0 for all signed patterns $\alpha \in \mathbb{Z}_2^3 \setminus 0$. However, by construction the first two planes split equally the mass, thus it is sufficient to check the 4 patterns involving the last one; i.e. $\alpha = (0, 0, 1), (1, 0, 1), (0, 1, 1)$ and $(1, 1, 1)$.

What is more, it is straightforward to verify how the alternating sum functions behave under the G -action. Explicitly, $\forall (v, w) \in S^1 \times S^3$ the following equalities hold:

$$\begin{aligned} F_{(0,0,1)}(g_1 \cdot (v, w)) &= F_{(0,0,1)}(v, w) \\ F_{(0,1,1)}(g_1 \cdot (v, w)) &= -F_{(1,0,1)}(v, w) \\ F_{(1,0,1)}(g_1 \cdot (v, w)) &= F_{(0,1,1)}(v, w) \\ F_{(1,1,1)}(g_1 \cdot (v, w)) &= -F_{(1,1,1)}(v, w) \end{aligned}$$

$$\begin{aligned} F_{(0,0,1)}(g_2 \cdot (v, w)) &= -F_{(0,0,1)}(v, w) \\ F_{(0,1,1)}(g_2 \cdot (v, w)) &= -F_{(1,0,1)}(v, w) \\ F_{(1,0,1)}(g_2 \cdot (v, w)) &= -F_{(0,1,1)}(v, w) \\ F_{(1,1,1)}(g_2 \cdot (v, w)) &= -F_{(1,1,1)}(v, w) \end{aligned}$$

Additionally, we can choose a linear G -action on \mathbb{R}^4 that is consistent with the previous equations. In particular, if we define:

$$\begin{aligned} g_1 \cdot (x, y, z, u) &= (x, -z, y, -u) \\ g_2 \cdot (x, y, z, u) &= (-x, -y, -z, -u) \end{aligned}$$

Then it is easy to check that the alternating sum map $\Psi : S^1 \times S^3 \rightarrow \mathbb{R}^4$

$$(v, w) \mapsto (F_{(0,0,1)}(v, w), F_{(0,1,1)}(v, w), F_{(1,0,1)}(v, w), F_{(1,1,1)}(v, w))$$

is actually a G -equivariant map whose zeros are exactly the configurations of planes that equipartition the measure and have the desired intersection property.

Step 2

Suppose now by contradiction that Ψ does not have a zero. This means that $\bar{\Psi} : S^1 \times S^3 \rightarrow S^3$ is a well defined G -equivariant map.

Denote by $\Psi_a, a \in S^1$, the map $\Psi_a : S^3 \rightarrow S^3, \Psi_a(p) = \overline{\Psi}(a, p)$; this function have two key properties:

1. $\forall a \in S^1, \Psi_a$ is antipodal
2. $\forall a, b \in S^1, \Psi_a$ and Ψ_b are homotopic

However, the map induced by g_1 on the sphere has degree -1 and thus we have:

$$[\Psi_a] = [\Psi_{g_1 \cdot a}] = [g_1 \cdot \Psi_a] = -[\Psi_a]$$

Thus Ψ_a is null-homotopic, but Borsuk-Ulam implies that an antipodal map of S^3 can't be. \nexists

□

Topology facts needed in the proof

Theorem 2. *If $A \in O(n)$ is an orthogonal matrix, then the induced continuous map $A : S^{n-1} \rightarrow S^{n-1}$ has degree $\deg(A) = \det(A)$.*

Proof. WLOG $\det(A) = 1$ (i.e. $A \in SO(n)$). Since $SO(n)$ is arc connected, there is a path $\gamma : A \rightsquigarrow I$ to the identity matrix.

As a result, the map $A : S^{n-1} \rightarrow S^{n-1}$ is homotopic to the identity through the homotopy $H : S^{n-1} \times [0, 1] \rightarrow S^{n-1}; H(x, t) = \gamma(t)x$.

□

Theorem 3. *Let $f : S^{n-1} \rightarrow S^{n-1}$ be an antipodal map. Then $\deg(f) \neq 0$.*

Proof. Suppose by contradiction $\exists f : S^{n-1} \rightarrow S^{n-1}$ antipodal and $\deg(f) = 0$.

Then f can be extended to a map $F : D^n \rightarrow S^{n-1}$; using this map it is possible to construct $\tilde{F} : S^n \rightarrow S^{n-1}$:

$$\tilde{F}(x_1, \dots, x_{n+1}) = \begin{cases} F(x_1, \dots, x_n) & \text{if } x_{n+1} \geq 0 \\ -F(-x_1, \dots, -x_n) & \text{if } x_{n+1} \leq 0 \end{cases}$$

It is well defined because on the horizontal equator both sides coincide with f since it is antipodal; what is more, \tilde{F} is antipodal:

$$\tilde{F}(-(x, x_{n+1})) = -F(-(-x)) = -F(x) = -\tilde{F}((x, x_{n+1}))$$

$$\tilde{F}(-(x, x_{n+1})) = F(-x) = -(-F(-x)) = -\tilde{F}((x, x_{n+1}))$$

Thus \tilde{F} violates Borsuk-Ulam. \nexists

□