

# Splitting a measure along a fixed direction

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## Notation

- Given two vectors  $x, y$  of dimension  $d$ , their scalar product is  $(x|y) := \sum_{i=1}^d x_i y_i$ .
- A measure  $\mu$  on  $\mathbb{R}^d$  is nice if it is Borel,  $\mu(\mathbb{R}^d) = 1$ ,  $\mu(H) = 0$  for all affine hyperplanes and it has connected support.
- Given an arrangement  $\mathcal{H}$  of  $k$  oriented affine hyperplanes and a sign pattern  $\alpha \in (\mathbb{Z}_2)^k$ , the orthant  $\mathcal{O}(\mathcal{H}, \alpha)$  is defined as

$$\mathcal{O}(\mathcal{H}, \alpha) := \{x \in \mathbb{R}^d | (-1)^{\alpha_i} ((h_i|x) - a_i) > 0\}$$

Intuitively,  $\mathcal{O}(\mathcal{H}, \alpha)$  is the set of point lying on the side of  $H_i$  determined by  $\alpha_i$  (positive if  $\alpha_i = 0$ , negative otherwise).

- $G = \mathbb{Z}_4 \times \mathbb{Z}_2$  with generators  $g_1 = (1, 0)$  and  $g_2 = (0, 1)$ .
- If  $v \in \mathbb{R}^d$  is a non-zero vector,  $\bar{v}$  is its normalization  $\bar{v} = \frac{v}{\|v\|}$ .
- An oriented affine hyperplane  $H = \{p \in \mathbb{R}^d | (h|p) - a = 0\}$  defines a point  $\frac{(h,a)}{\|(h,a)\|}$  in  $S^d$ ; likewise, a point on  $x = (\tilde{x}, a) \in S^d$  defines an oriented affine hyperplane  $H(x) = \{p \in \mathbb{R}^d | (\tilde{x}|p) - a = 0\}$ .
- Given a nice measure  $\mu$  in  $\mathbb{R}^3$ , define the standard test map  $F : (S^3)^3 \times \mathbb{Z}_2^3 \rightarrow \mathbb{R}$ :

$$F(x, \omega) = \mu(\mathcal{O}(x, \omega)) - \frac{1}{8}$$

$x$  is an equipartition  $\iff F(x, \alpha) = 0 \ \forall \alpha$ .

- Given a test map  $F$ , the alternating sum functions are the discrete Fourier transform of the test map. More precisely, if  $\alpha \in (\mathbb{Z}_2)^3$   $\alpha \neq 0$ , then the alternating sum with parameter  $\alpha$  is:

$$F_\alpha(x) = \sum_{\omega \in \mathbb{Z}_2^3} (-1)^{(\alpha|\omega)} F(x, \omega)$$

$x$  is an equipartition  $\iff F_\alpha(x) = 0 \ \forall \alpha \neq 0$

- The group  $\mathbb{Z}_2^3$  acts on  $(S^3)^3$  by coordinate-wise antipodality and, given  $\alpha, g \in \mathbb{Z}_2^3$  with  $\alpha \neq 0$  and a test map  $F$ :

$$F_\alpha(g \cdot x) = (-1)^{(\alpha|g)} F_\alpha(x)$$

## The Result

Our goal is to prove the following result:

**Theorem 1.** *Given a nice measure  $\mu$  and a direction  $p \in S^2$  it is possible to find 3 affine oriented planes that equipartition the measure and the first two have the prescribed oriented intersection  $p$ . Formally,  $\exists \mathcal{H} = ((h_1, a_1), (h_2, a_2), (h_3, a_3))$  configuration of oriented planes such that:*

- $\forall \alpha \in (\mathbb{Z}_2)^k, \mu(\mathcal{O}(\mathcal{H}, \alpha)) = \frac{1}{8}$ .
- $\bar{h}_1 \wedge \bar{h}_2 = p$ .

*Proof.* Without loss of generality, let  $z = (0, 0, 1)$ . The proof will be split in two sections: the first deals with constructing a map  $\Phi : S^1 \times S^3 \rightarrow \mathbb{R}^4$  in  $\mathbb{R}^4$  whose zeros codify equipartitions of the mass. The second step will be to show that the map will respect a suitable action of  $G$  on the two space and that such equivariance forces the existence of a zero.

### Step 1

The key step in constructing the map is to show that we can parametrize pair of planes that have intersection direction  $z$  and split the mass in 4 equal parts with vector in  $S^1$ .

Project the mass on the  $xy$  plane to obtain a nice measure  $\mu^\#$  on  $\mathbb{R}^2$ . The following bisecting lemma applies:

**Lemma 1** (Bisecting). *Let  $\mu^\#$  be a nice measure on  $\mathbb{R}^2$  and  $v \in S^1$  a direction. Then there exists two oriented affine lines  $l_0 = \mathbb{R}\vec{l}_0 + a_0$  and  $l_1 = \mathbb{R}\vec{l}_1 + a_1$  in  $\mathbb{R}^2$  such that:*

- $l_0$  and  $l_1$  equipartition  $\mu^\#$
- $v$  bisects the angle between  $l_0$  and  $l_1$

*What is more, we can choose consistently the direction  $\vec{l}_0$  (e.g. fix  $\vec{l}_0$  to be the first direction clockwise, while the first one contraclockwise is  $\vec{l}_1$ ). Once this choice is made,  $\vec{l}_0$  and  $\vec{l}_1$  are unique and depend continuously on  $v$ .*

The lemma guarantees that, once we fix a direction  $v \in S^1 \subseteq S^2$  (inclusion as the horizontal equator in  $S^2$ ) there are two affine lines in the  $xy$  plane  $l_0 = \mathbb{R}\vec{l}_0(v) + a_0(v)$  and  $l_1 = \mathbb{R}\vec{l}_1(v) + a_1(v)$  that bisect the projected measure  $\mu^\#$ .

Define  $H_i(v) = (h_i(v), a_i(v))$  to be the affine (oriented) span of  $l_i(v)$  and  $z$ , the two planes now equipartition the measure  $\mu$  and have the desired intersection.

Let now  $\mathbb{Z}_4 = \langle g_1 \rangle$  act on  $S^1$  by  $\frac{\pi}{2}$  rotation counter clockwise. Then, by construction we have that:

$$H_0(g_1 \cdot v) = H_1(v)$$

$$H_1(g_1 \cdot v) = -H_0(v)$$

In other words, if we consider the planar problem with the bisecting vector rotated by  $\frac{\pi}{2}$ , the new first line is the previous second while the new second is the old first with opposite orientation.

We can now define a map  $S^1 \times S^3 \rightarrow (S^3)^3$ ,  $(v, w) \mapsto (H_0(v), H_1(v), w)$ .

To check now that a configuration of 3 planes equipartition the measure, it is equivalent to check that the alternating sum functions  $F_\alpha$  are 0 for all signed patterns  $\alpha \in \mathbb{Z}_2^3 \setminus 0$ . However, by construction the first two planes split equally the mass, thus it is sufficient to check the 4 patterns involving the last one; i.e.  $\alpha = (0, 0, 1), (1, 0, 1), (0, 1, 1)$  and  $(1, 1, 1)$ .

What is more, it is straightforward how the alternating sum functions behave under the  $G$ -action. Explicitly,  $\forall (v, w) \in S^1 \times S^3$  the following equalities hold:

$$\begin{aligned} F_{(0,0,1)}(g_1 \cdot (v, w)) &= F_{(0,0,1)}(v, w) \\ F_{(0,1,1)}(g_1 \cdot (v, w)) &= -F_{(1,0,1)}(v, w) \\ F_{(1,0,1)}(g_1 \cdot (v, w)) &= F_{(0,1,1)}(v, w) \\ F_{(1,1,1)}(g_1 \cdot (v, w)) &= -F_{(1,1,1)}(v, w) \end{aligned}$$

$$\begin{aligned} F_{(0,0,1)}(g_2 \cdot (v, w)) &= -F_{(0,0,1)}(v, w) \\ F_{(0,1,1)}(g_2 \cdot (v, w)) &= -F_{(1,0,1)}(v, w) \\ F_{(1,0,1)}(g_2 \cdot (v, w)) &= -F_{(0,1,1)}(v, w) \\ F_{(1,1,1)}(g_2 \cdot (v, w)) &= -F_{(1,1,1)}(v, w) \end{aligned}$$

Additionally, we can choose a linear  $G$ -action on  $\mathbb{R}^4$  that is consistent with the previous equations. In particular, if we define:

$$\begin{aligned} g_1 \cdot (x, y, z, u) &= (x, -z, y, -u) \\ g_2 \cdot (x, y, z, u) &= (-x, -y, -z, -u) \end{aligned}$$

Then it is easy to check that the alternating sum map  $\Psi : S^1 \times S^3 \rightarrow \mathbb{R}^4$

$$(v, w) \mapsto (F_{(0,0,1)}(v, w), F_{(0,1,1)}(v, w), F_{(1,0,1)}(v, w), F_{(1,1,1)}(v, w))$$

is actually a  $G$ -equivariant map whose zeros are exactly the configurations of planes that equipartition the measure and have the desired intersection property.

### Step 2

Suppose now by contradiction that  $\Psi$  does not have a zero. This means that  $\bar{\Psi} : S^1 \times S^3 \rightarrow S^3$  is a well defined  $G$ -equivariant map.

Denote by  $\Psi_a, a \in S^1$ , the map  $\Psi_a : S^3 \rightarrow S^3, \Psi_a(p) = \bar{\Psi}(a, p)$ ; this function have two key properties:

1.  $\forall a \in S^1, \Psi_a$  is antipodal
2.  $\forall a, b \in S^1, \Psi_a$  and  $\Psi_b$  are homotopic

However, the map induced by  $g_1$  on the sphere has degree  $-1$  and thus we have:

$$[\Psi_a] = [\Psi_{g_1 \cdot a}] = [g_1 \cdot \Psi_a] = -[\Psi_a]$$

Thus  $\Psi_a$  is null-homotopic, but Borsuk-Ulam implies that an antipodal map of  $S^3$  can't be.  $\nmid$

□

## Topology facts needed in the proof

**Theorem 2.** *If  $A \in O(n)$  is an orthogonal matrix, then the induced continuous map  $A : S^{n-1} \rightarrow S^{n-1}$  has degree  $\deg(A) = \det(A)$ .*

*Proof.* Assume  $\det(A) = 1$  (i.e.  $A \in SO(n)$ ).

Let  $P$  an invertible matrix that puts  $A$  in Jordan normal form ( $R_\theta$  denotes the  $2 \times 2$  matrix of the rotation by  $\theta$ ):

$$A = P^{-1} \begin{pmatrix} R_{\theta_1} & & & & \\ & R_{\theta_2} & & & \\ & & \dots & & \\ & & & R_{\theta_k} & \\ & & & & Id_{n-2k} \end{pmatrix} P$$

Then, there is a path  $\gamma : Id \rightsquigarrow A$  defined as:

$$\gamma(t) = P^{-1} \begin{pmatrix} R_{t\theta_1} & & & & \\ & R_{t\theta_2} & & & \\ & & \dots & & \\ & & & R_{t\theta_k} & \\ & & & & Id_{n-2k} \end{pmatrix} P$$

As a result, the map  $A : S^{n-1} \rightarrow S^{n-1}$  is homotopic to the identity through the homotopy  $H : S^{n-1} \times [0, 1] \rightarrow S^{n-1}; H(x, t) = \gamma(1 - t)x$ .

Hence  $\deg(A) = \deg(Id) = 1$ .

Let now  $\det(A) = -1$ , then  $QA \in SO(n)$  where

$$Q := \begin{pmatrix} Id & \\ & -1 \end{pmatrix}$$

this means that  $1 = \deg(QA) = \deg(Q)\deg(A) = -\deg(A)$ .  $\square$

**Theorem 3.** *Let  $f : S^{n-1} \rightarrow S^{n-1}$  be an antipodal map. Then  $\deg(f) \neq 0$ .*

*Proof.* Suppose by contradiction  $\exists f : S^{n-1} \rightarrow S^{n-1}$  antipodal and  $\deg(f) = 0$ .

Then  $f$  can be extended to a map  $F : D^n \rightarrow S^{n-1}$ ; using this map it is possible to construct  $\tilde{F} : S^n \rightarrow S^{n-1}$ :

$$\tilde{F}(x_1, \dots, x_{n+1}) = \begin{cases} F(x_1, \dots, x_n) & \text{if } x_{n+1} \geq 0 \\ -F(-x_1, \dots, -x_n) & \text{if } x_{n+1} \leq 0 \end{cases}$$

It is well defined because on the intersection of the two pathes (i.e. the horizontal equator) both sides coincide with  $f$ ; what is more,  $\tilde{F}$  is antipodal:

$$\tilde{F}(-(x, x_{n+1})) = -F(-(-x)) = -F(x) = -\tilde{F}((x, x_{n+1}))$$

$$\tilde{F}(-(x, x_{n+1})) = F(-x) = -(-F(-x)) = -\tilde{F}((x, x_{n+1}))$$

Thus  $\tilde{F}$  violates Borsuk-Ulam.  $\square$