Splitting lines in $\mathbb{R}^3$

Notation

- $B_r(p)$ is the (closed) ball centered in p of radius r.
- $X := S^3 \times S^3 \times S^3$ as a subset of polynomials $\mathbb{R}[t_1, t_2, t_3]$ (product of three affine polynomials).
- X is a metric space with the distance:

$$d(x, y) = \max_{i \in I} |x_i - y_i|$$

  i.e. the distance between two polynomials is the biggest difference in their coefficients (it is always finite since the set of possible multi-indices is finite).
- Given $x \in X$, $Z(x) := \{p \in \mathbb{R}^3 | x(p) = 0\}$ is the plane configuration induced by x
- Given $x \in X$ and $\omega \in G$, the orthant $\mathcal{O}(x, \omega)$ is the (potentially empty) open set in $\mathbb{R}^3 \setminus Z(x)$ defined as:

$$\mathcal{O}(x, \omega) := \{p \in \mathbb{R}^3 | (-1)^{\omega_i} x_i(p) > 0 \ \forall i\}$$

- The degenerate set in X is $A := \{x \in X | \text{at least one orthant is empty}\}$
- $G_\pm := (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{S}_3$ where $\mathbb{S}_3$ is the permutation group on three elements
- $G := \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
- Γ is a collection of n different (affine) lines in $\mathbb{R}^3$
- Given $\varepsilon > 0$, the cut-off function $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$ is:

$$\eta_\varepsilon(x) = \begin{cases} 0 & \text{if } x \leq \varepsilon \\ \frac{1}{\varepsilon}x - 1 & \text{if } \varepsilon \leq x \leq 2\varepsilon \\ 1 & \text{if } x \geq 2\varepsilon \end{cases}$$

1 Preamble: $G_\pm$-representations

Before starting it is useful to clarify how $G_\pm$ acts on the target space (either $\mathbb{R}^G$ or $\mathbb{R}^{G\setminus 0}$).

The group acts on the two vector spaces differently. On the first space $V := \mathbb{R}^G$, the action is a permutation of the coordinates according to the following rule:

$$((\alpha \rtimes \sigma) \cdot v)_\omega = v_{\alpha + \sigma^{-1} \cdot \omega}$$

On the other end, the action on the space $W := \mathbb{R}^{G\setminus 0}$ is given by:

$$((\alpha \rtimes \sigma) \cdot v)_\omega = (-1)^{\left(\alpha | \sigma^{-1} \cdot \omega\right)} v_{\sigma^{-1} \cdot \omega}$$

There is a $G_\pm$-linear map between the two representations, the alternate sum map, $T : V \to W$ defined as:

$$(Tv)_\omega = \sum_{\alpha \in G} (-1)^{(\alpha|\omega)} v_\alpha$$

It is an easy computation to see that $\ker(T)$ is the linear space generated by $\mathbb{1}$ (the vector of all 1s).

2 Proof

The goal of this notes is to prove the following fact:

Theorem 1. *Suppose Γ is not degenerate (i.e. $\nexists x \in X$ such that $\bigcup \Gamma \subseteq Z(x)$). Then $\exists x \in X$ such that every orthant intersect at most $\frac{n}{2}$ lines in Γ.*

Recall that the Guth function of parameter $\delta > 0$ is defined as $I_\delta : X \to V$:

$$(I_\delta(x))_\omega = \sum_{\gamma \in \Gamma} \eta_\varepsilon \left(\int_{N_\delta \gamma \cap \mathcal{O}(x,\omega) \cap B_R} \eta_\varepsilon(|x(p)|) \delta^{-3} dp \right)$$

Where $\varepsilon := \varepsilon(\delta)$ and $R := R(\delta)$ are suitable functions (see [**GUTH2015**]) Since the Guth functions point-wise converge to the counting function for intersections, if we show that, for all sufficiently small $\delta > 0$, $T \circ I_\delta$ has a zero we obtain that $\exists x \in X$ for which the counting function is multiple (up to integer rounding) of $\mathbb{1}$; by intersection counting the correct multiple is $\frac{n}{2}$.

The first part of the proof will be dedicated to show that for a small enough δ the Guth function I_δ is $G_\pm$-homotopic to the induced function f for a measure, hence TI_δ and Tf are $G_\pm$-homotopic. What is more, we will show that this holds on a manifold with boundary $E \subseteq X \setminus A$ of dimension 7; since E avoids A, the action of $G_\pm$ is free on E.

The zero set of the homotopy is going to be a free $G_{\pm}$-manifold of dimension 1 with (non-empty) boundary in $E \times I$ and we will show that it has to avoid $\partial E \times I$. By choosing a clever measure we get that there has to be exactly one orbit of zeros on the final level of the homotopy. By the fact that $G_{\pm}$-homotopies can change the number of zeros only by multiples of $|G_{\pm}|$, the function TI_{δ} has to have at least one zero on E as desired.

2.1 Find the δ

The goal of this section is to find the suitable $\delta > 0$ as previously mentioned.

Definition 1. *Given $x \in X$ and $\gamma \in \Gamma$, $\delta > 0$ is acceptable for x with witness $p \in \mathbb{R}^3$ if:*

- *$p \in \gamma$ and $(1+\delta)\|p\| < R(\delta)$ (i.e. $B_{\delta}(p) \subseteq int(B_R)$)*
- $\min_{q \in B_{\delta}(p)} |x(q)| > 2\varepsilon$

The first useful remark is that $\forall x \in X$ there is $\delta_x > 0$ admissible. In fact, fix $x \in X$, then there is $\gamma \in \Gamma$ such that γ is not contained in $Z(x)$ thus it is possible to pick $p \in \gamma \setminus Z(x)$. Since $R \to \infty$, $\varepsilon \to 0$ and $\min_{q \in B_{\delta}(p)} |x(q)| \to |x(p)| > 0$ with $\delta \to 0$ eventually there will be a $\delta_x > 0$ for which all the conditions will be satisfied simultaneously $\Rightarrow \delta_x$ is acceptable.

Lemma 1. *There is $\tilde{\delta} > 0$ such that $\tilde{\delta}$ is acceptable $\forall x \in X$.*

Proof. $\forall \delta > 0$ define $U(\delta) := \{x \in X | \delta \text{ is acceptable for } x\}$. Since $X = \bigcup_{\delta > 0} U_{\delta}$ and $U_{\alpha} \subseteq U_{\beta}$ whenever $\alpha \leq \beta$, in order to obtain the thesis it is sufficient by compactness to show that U_{δ} is open $\forall \delta$.

Fix $x \in U_{\delta}$ and p a witness. Then it is enough to show:
Claim: If y is close enough to x, δ is acceptable for y with witness p.

Proof: The only condition we need to verify is that $\min_{q \in B_{\delta}(p)} |y(q)| > 2\varepsilon$.
$\forall q \in B_{\delta}(p)$, we have that:

$$|y(q)| \geq ||x(q)| - |x(q) - y(q)||$$

Since $m : y \mapsto \max_{q \in B_{\delta}(p)} |x(q) - y(q)|$ is continuous (lemma 3), it is possible to pick y close enough to x so that $m(y) < \frac{1}{2}\left(\max_{q \in B_{\delta}(p)} |x(q)| - 2\varepsilon\right)$; thus $|y(q)| > 2\varepsilon \;\; \forall q \in B_{\delta}(p)$ as desired. ☐

From now on, fix δ to be a value $\tilde{\delta} \geq \delta > 0$.

2.2 Find the homotopy

Since the space of bounded Borel measures on $\mathbb{R}^3$ is convex, the convex combination of any two measure gives a $G_{\pm}$-homotopy between the two induced functions. What is more, such an homotopy is never 0 on the degenerate set A by construction; hence, if we can construct an homotopy between the Guth function and a measure avoiding zeros on A then we can do the same with any other measure.

The key observation is that the Guth function is already almost an induced function of a measure, the only difference is the cut-off function wrapping the integral and the function under the integral sign; hence the strategy is to construct homotopy for these two functions that respect the properties we are interested in at all times.

Define thus the two homotopies:

$$\alpha_t(x) = t + (1-t)\eta_\varepsilon(x)$$
$$\beta_t(x) = tx + (1-t)\eta_\varepsilon(x)$$

Finally, we can define the combined homotopy; that is, on the coordinate ω:

$$(H_t(x))_\omega := \sum_{\gamma \in \Gamma} \beta_t \left(\int_{N_\delta \gamma \cap \mathcal{O}(x,\omega) \cap B_R} \alpha_t(|x(p)|)\delta^{-3} dp \right)$$

This is clearly $G_\pm$-equivariant, $H_0 \equiv I_\delta$, $H_1 \equiv \int_{N_\delta \Gamma \cap \mathcal{O}(x,\omega) \cap B_R} \delta^{-3} dp$ (the induced function for the measure supported on $N_\delta \Gamma \cap B_R$); the only property left to check is that it is never a multiple of the $\mathbb{1}$ vector on degenerate configurations.

Fix $x \in A$. Then, there is $\omega_x \in G$ such that $\mathcal{O}(x, \omega_x)$ is empty $\Rightarrow H_t(x)_{\omega_x} \equiv 0$ at every t.

Claim: There is an element ξ and ϵ such that $H_t(x)_\xi \geq \epsilon > 0$ at all times.

Since the argument of β_t is always positive, H_t is a sum of positive functions hence it is enough to prove that $\exists \gamma \in \Gamma$ and $\xi \in G$ such that

$$\beta_t \left(\int_{N_\delta \gamma \cap \mathcal{O}(x,\xi) \cap B_R} \alpha_t(|x(p)|)\delta^{-3} dp \right) \geq \epsilon > 0$$

Let $\tilde{p} \in \gamma$ ($\gamma \in \Gamma$) be a witness for the acceptability of δ and let ξ be the index of the orthant containing $\tilde{p}$. Then, $\forall t \in [0,1]$:

$$\min_{p \in B_\delta(\tilde{p})} |x(p)| > 2\varepsilon \Rightarrow \alpha_t(|x(p)|) = 1$$

Hence:

$$\int_{N_\delta \gamma \cap \mathcal{O}(x,\xi) \cap B_R} \alpha_t(|x(p)|)\delta^{-3} dp \geq \int_{B_\delta(\tilde{p})} \delta^{-3} dp = \frac{4\pi}{3}$$

Since $\beta_t(x)$ is monotone increasing in t if $x \geq 1$

$$\beta_t(\int_{N_\delta \gamma \cap \mathcal{O}(x,\xi) \cap B_R} \alpha_t(|x(p)|)\delta^{-3} dp) \geq \eta_\varepsilon \left(\frac{4\pi}{3} \right) = 1$$

As a result, the map $T \circ H_t$ is a $G_\pm$-homotopy between the test maps that doesn't have any zeros on A. Without loss of generality we can assume that $T \circ H_t$ is an homotopy between I_δ and the induced function of the target measure defined in lemma 4.

2.3 Find E - Conclusion

By compactness of $X \times I$, there is an $\varepsilon > 0$ such that $\forall \alpha \in G\backslash 0$, $(T \circ H_t|_A)_\alpha > \varepsilon$. It is thus possible to choose a small $G_\pm$-invariant tubolar neighborhood N_A such that $(T \circ H_t|_{N_A})_\alpha > \frac{2\varepsilon}{3}$. For convenience, choose N_A to be closed and denote by U_A its interior.

Let $P := \{x \in X|$ one of the planes is parallel to the plane $t_1 = 0\}$. By construction $P = \left(S^1 \times S^3 \times S^3\right) \times \left(S^3 \times S^1 \times S^3\right) \times \left(S^3 \times S^3 \times S^1\right)$ and thus it is not a manifold.

The corner points are contained in A, hence $E = P \setminus U_A$ is a 7-dimensional manifold with boundary on which the action of $G_\pm$ is free.

By choosing suitable small compatible $G_\pm$-triangulations for X, N_A, $[0,1]$ and E, we there is a $G_\pm$-map $f : X \times [0,1] \to W$ that is:

- f is affine on every simplex of $X \times [0,1]$
- $G_\pm$-homotopic to $T \circ H$
- very close to $T \circ H$ (e.g. $\max_{x \in X, t \in I} \|f(x,t) - TH_t(x)\| \leq \frac{\varepsilon}{3}$)
- There is a unique orbit $G_\pm x$ in X such that $f(x,1) = 0$ and $x \neq A$
- $f|_E$ is generic (i.e. $f^{-1}(0)$ intersect only faces of dimension at least $7 = |G| - 1$)
- $\|f|_{\partial E}\| \geq \frac{\varepsilon}{3}$

(see lemma 5 for a proof).

As a result, $Z := f^{-1}(0) \cap E \times I$ is a 1-dimensional PL-manifold with boundary that is $G_\pm$ invariant.

If we choose a connected component starting from one of the point in $Z \cap E \times \{1\}$, this is an interval with exactly one endpoint on $E \times \{1\}$ and does not intersect $\partial E \times I$. It follows that the other endpoint has to be on $E \times \{0\}$ and thus we showed that $|Z \cap E \times \{0\}| = 1 \mod |G_\pm|$, hence non zero. Since this quantity is preserved under $G_\pm$-homotopies, the same has to be true for $T \circ H_0 = T \circ I_\delta$ as desired.

3 Technical Lemmas

Lemma 2 (convergence in X implies global ptwise convergence)**.** *Let $x_n \to x_\infty$ a converging sequence in X with the distance previously defined. Then $\forall p \in \mathbb{R}^3$, $x_n(p) \to x_\infty(p)$.*

Proof. Fix $\varepsilon > 0$ and $p \in \mathbb{R}^3$ and denote by y^i the homogeneous component of degree i of a polynomial $y \in X$. If $p = 0$ then, for n big enough:

$$|x_n(0) - x_\infty(0)| = |x_n^0 - x_\infty^0| \leq d(x_n, x_\infty) \leq \varepsilon$$

Thus we can assume $\|p\| \neq 0$ and denote by $\hat{p} := \frac{p}{\|p\|}$.

If $0 < \|p\| \leq 1$; for n big enough we have:

$$|x_n(p) - x_\infty(p)| \leq \sum_{i=0}^{3} \|p\|^i |x_n^i(\hat{p}) - x_\infty^i(\hat{p})| \leq \sum_{i=0}^{3} c_i d(x_n, x_\infty) \leq C d(x_n, x_\infty) \leq \varepsilon$$

for some constants c_i, C.

Analogously, if $1 \leq \|p\|$, then:

$$|x_n(p) - x_\infty(p)| \leq \sum_{i=0}^{3} \|p\|^i |x_n^i(\hat{p}) - x_\infty^i(\hat{p})| \leq \sum_{i=0}^{3} \|p\|^i c_i d(x_n, x_\infty) \leq \|p\|^3 C d(x_n, x_\infty)$$

However, since p is fixed, we can choose n big enough such that

$$d(x_n, x_\infty) \leq \frac{\varepsilon}{\|p\|^3 C}$$

hence the sequence $x_n(p)$ converges to $x_\infty(p)$ as desired. $\square$

Lemma 3 (max function is continuous)**.** *Fix $x \in X$, $\delta > 0$ and $p \in \mathbb{R}^3$, then the function $m : X \to \mathbb{R}$, $m(y) = \max_{q \in B_\delta(p)} |y(q) - x(q)|$ is continuous.*

Proof. It is enough to prove sequential continuity. Let $y_n \to y_\infty$ be a converging sequence in $X \Rightarrow y_n(p) \to y_\infty(p)$ (lemma 2).

Let q_n be a point that realizes $m(y_n)$ (i.e. $m(y_n) = |y_n(q_n) - x(q_n)|$), up to taking a sub-sequence we can assume q_n converges to some point $q_\infty \in B_\delta(p)$.

What is more, the family $\{y_n\}$ is equicontinuous as functions $y_n : B_\delta(p) \to \mathbb{R}$ (they are differentiable and have bounded derivative) and thus they converge uniformly on $B_\delta(p)$.

CLAIM: q_∞ realizes $m(y_\infty)$.

Assuming the claim, we get that for n big enough:

$$\begin{aligned}
|m(y_n) - m(y_\infty)| &= |y_n(q_n) - x(q_n) - y_\infty(q_\infty) + x(q_\infty)| \\
&\leq |y_n(q_n) - y_n(q_\infty)| + |y_n(q_\infty) - y_\infty(q_\infty)| + |x(q_\infty) - x(q_n)| \\
&\leq \varepsilon + \varepsilon + \varepsilon
\end{aligned}$$

where the last inequality holds by equicontinuity (first term), pointwise convergence (second term) and continuity of x (third term).

The only thing left to prove is the claim.

$\square$

Lemma 4 (Target Measure)**.** *Let μ the probability measure with support on $S = \{(t, t^2, t^3) \in \mathbb{R}^3 | t \in [-1, 1]\}$ and uniform density. Then there is a unique $G_\pm$-orbit of points in P that equipartitions μ.*

Equivalently, up to order and signs there is a unique triple of planes such that every orthant has the same measure and the first one is parallel to the horizontal plane $\{p \in \mathbb{R}^3 | p_1 = 0\}$.

Lemma 5 (Generic Homotopy). *Let $TH_t : X \to W$ the homotopy constructed in section 2.2 and $\varepsilon > 0$ small enough. Then there are compatible $G_\pm$-triangulation for X, E, $I := [0,1]$, $X \times I$ and a $G_\pm$-function $f : X \times I \to W$ with the following properties:*

1. $\max_{(x,t)\in X\times I} \|f(x,t) - TH_t(x)\| < \varepsilon$
2. *f is $G_\pm$-homotopic to TH*
3. *f is affine on every simplex in $X \times I$*
4. *f is generic: i.e. $f^{-1}(0)$ intersects only simplices of dimension at least $7 = \dim W$*
5. *$f(x,1) = 0$ on exactly one orbit in E*

Proof. By equivariant simplicial approximation we can find small enough triangulations for the spaces and an affine map g that is $G_\pm$-homotopic to TH and $\max_{(x,t)\in X\times I} \|f(x,t) - TH_t(x)\| < \frac{\varepsilon}{2}$. By lemma 6, the property of having exactly one orbit of zeros is preserved under perturbations small enough (since 0 is a regular value of the function on E) hence the only condition we need to show is that we can find a perturbation that is generic. □

Lemma 6. *Fix μ the measure defined in lemma 4 and denote by $g : X \to V$ the function $g(x)_\omega = \int_{\mathcal{O}(x,\omega)} \mu$, then g is smooth and its critical values are away from the diagonal of V (i.e. 0 is a regular value for Tg).*

Proof. [Sketch / find a less convoluted proof for last point in lemma 5]

By lemma 4, $(Tg)^{-1}(0) = G_\pm \bar{x}$ where

$$\bar{x} = \left((1,0,0,0), (-\frac{5}{16}, \frac{1}{2}, 1, -\frac{3}{32}), (-\frac{5}{16}, -\frac{1}{2}, 1, \frac{3}{32}) \right)$$

Since $G_\pm$ acts on the differential as multiplication with an invertible matrix, 0 is a regular value for Tg if and only if Dg_x is full rank.

Since $E \subseteq (\mathbb{R}^4)^3$, the 0-set of the function $E \times \mathbb{R} \to \mathbb{R}$ defined as

$$\phi(x_1, x_2, x_3, t) = x_1(t, t^2, t^3) x_2(t, t^2, t^3) x_3(t, t^2, t^3)$$

gives a parametrization of the intersections of (the zero sets of) points of E with the moment curve in a small neighborhood of $\bar{x}$ by the implicit function theorem. Denote by z_i the implicit functions defined by ϕ in a neighborhood of $(\bar{x}, \frac{i}{4})$ for $|i| \leq 3$ integer.

By definition of the target measure, we have that (in a small neighborhood of $\bar{x}$) g can be expressed as:

$$\begin{aligned}
g_{(0,0,0)}(x) &= 1 - t_3(x)\\
g_{(0,0,1)}(x) &= t_1(x) - t_0(x)\\
g_{(0,1,0)}(x) &= t_2(x) - t_3(x)\\
g_{(0,1,1)}(x) &= t_1(x) - t_2(x)\\
g_{(1,0,0)}(x) &= t_{-1}(x) - t_{-2}(x)\\
g_{(1,0,1)}(x) &= t_0(x) - t_{-1}(x)\\
g_{(1,1,0)}(x) &= t_{-2}(x) - t_{-3}(x)\\
g_{(1,1,1)}(x) &= t_{-3}(x) + 1
\end{aligned}$$

Since $(\partial_t \phi)(\bar{x}, t_i) \neq 0$ for all i, we can explicitly write the differential of g around $\bar{x}$ using the partial derivatives of the functions t_i.

The differential Dg_x is the matrix 8×12 matrix[1]:

$$(Dg_x)^t = \begin{pmatrix} \partial_0 g_{(0,0,0)} & \partial_0 g_{(0,0,1)} & \partial_0 g_{(0,1,0)} & \dots & \partial_0 g_{(1,1,0)} & \partial_0 g_{(1,1,1)} \\ & & & \ddots & & \\ \partial_{11} g_{(0,0,0)} & \partial_{11} g_{(0,0,1)} & \partial_{11} g_{(0,1,0)} & \dots & \partial_{11} g_{(1,1,0)} & \partial_{11} g_{(1,1,1)} \end{pmatrix}$$

Thus, if we denote by $v_i = (\partial_t \phi)(\bar{x}, t_i(\bar{x}))^{-1}$, the matrix $M = (Dg_{\bar{x}})^t$ be explicitly written as

$$M = \begin{pmatrix} v_3(\partial_0 \phi)(\bar{x}, -\frac{3}{4}) & \partial_0 g_{(0,0,1)} & \partial_0 g_{(0,1,0)} & \dots & \partial_0 g_{(1,1,0)} & \partial_0 g_{(1,1,1)} \\ & & & \ddots & & \\ \partial_{11} g_{(0,0,0)} & \partial_{11} g_{(0,0,1)} & \partial_{11} g_{(0,1,0)} & \dots & \partial_{11} g_{(1,1,0)} & \partial_{11} g_{(1,1,1)} \end{pmatrix}$$

[**guth˙polynomial˙2015**] □

[1]The entries will be indexed by G on the rows.