Problem 2.10 Functions $x(t)$ and $h(t)$ are both rectangular pulses, as shown in Fig. P2.10. Apply graphical convolution to determine $y(t) = x(t) * h(t)$ for:

(a) $A = 1,\ B = 1,\ T_1 = 2$ s, $T_2 = 4$ s

(b) $A = 2,\ B = 1,\ T_1 = 4$ s, $T_2 = 2$ s

(c) $A = 1,\ B = 2,\ T_1 = 4$ s, $T_2 = 2$ s.

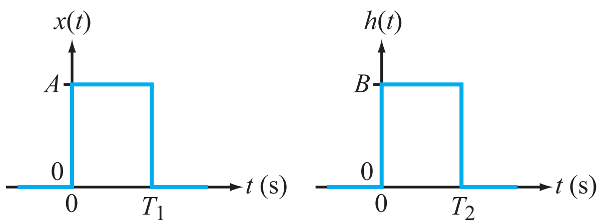
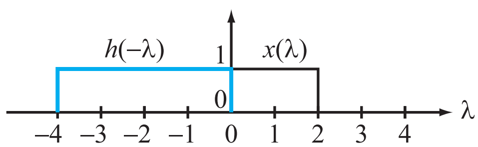
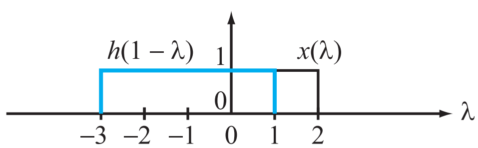


Figure P2.10: Waveforms of $x(t)$ and $h(t)$.

Solution: (a)

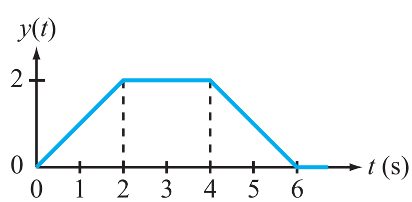


$t = 0$



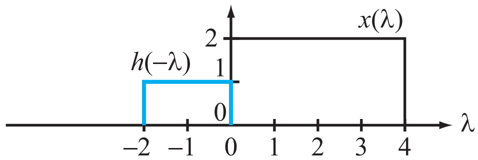
At $t = 1$ s, overlap = 1 unit

Progressive sliding of $h(-\lambda)$ to the right leads to:

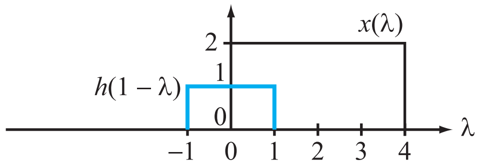


$y(t)$

(b)

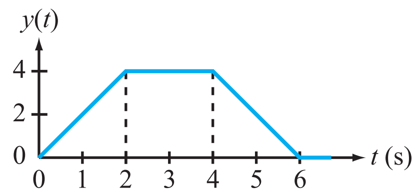


$t = 0$



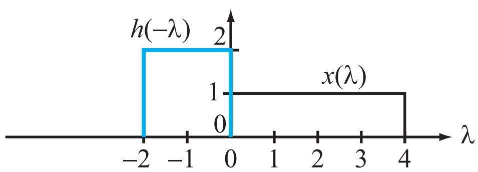
At $t = 1$ s, overlap = 2 units

Progressive sliding of $h(-\lambda)$ to the right leads to:

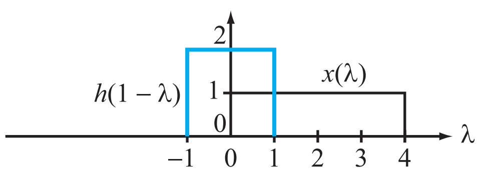


$y(t)$

(c)

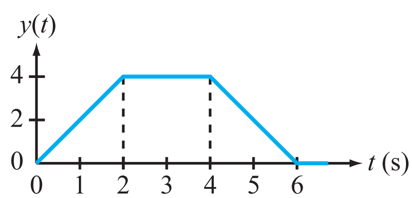


$t = 0$



At $t = 1$ s, overlap = 2 units

Progressive sliding of $h(-\lambda)$ to the right leads to:



$y(t)$

Problem 2.13 Functions $x(t)$ and $h(t)$ have the waveforms shown in Fig. P2.13. Determine and plot $y(t) = x(t) * h(t)$ by:

(a) integrating the convolution analytically, and

(b) integrating the convolution graphically.

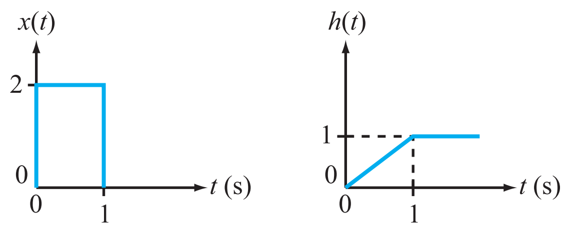


Figure P2.13: Waveforms for Problem 2.13.

Solution:

(a) Analytical Integration

$$x(t) = 2[u(t) - u(t-1)],$$
$$h(t) = t\,u(t) - (t-1)\,u(t-1).$$

Application of Eq. (2.52) gives

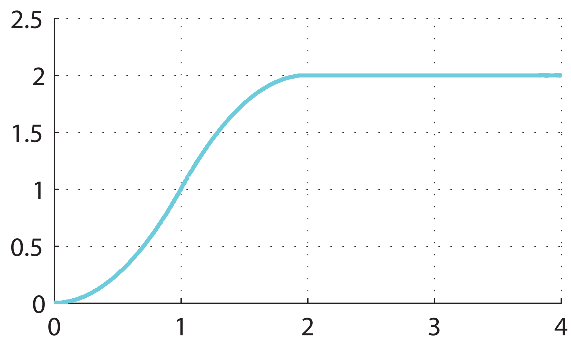
$$\begin{aligned}
y(t) &= x(t) * h(t) \\
&\int_0^t x(t-\lambda)\,h(\lambda)\,d\lambda \\
&= \int_0^t 2[u(t-\lambda) - u(t-1-\lambda)][\lambda\,u(\lambda) - (\lambda-1)\,u(\lambda-1)]\,d\lambda \\
&= 2\int_0^t \lambda\,u(t-\lambda)\,u(\lambda)\,d\lambda - 2\int_0^t (\lambda-1)\,u(t-\lambda)\,u(\lambda-1)\,d\lambda \\
&\quad - 2\int_0^t \lambda\,u(\lambda)\,u(t-1-\lambda)\,d\lambda + 2\int_0^t (\lambda-1)\,u(t-1-\lambda)\,u(\lambda-1)\,d\lambda.
\end{aligned}$$

In view of the relation expressed by Eq. (2.52),

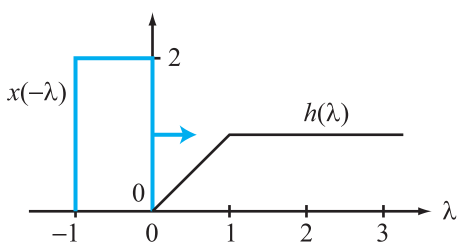
$$\begin{aligned}
y(t) = &\left[2\int_0^t \lambda\,d\lambda\right] u(t) \\
&- \left[2\int_1^t (\lambda-1)\,d\lambda\right] u(t-1) \\
&- \left[2\int_0^{t-1} \lambda\,d\lambda\right] u(t-1) \\
&+ \left[2\int_1^{t-1} (\lambda-1)\,d\lambda\right] u(t-2)
\end{aligned}$$

$$
\begin{aligned}
&= 2\frac{t^2}{2}\,u(t) - 2\left(\frac{\lambda^2}{2} - \lambda\right)\Bigg|_1^t\, u(t-1) \\
&\quad - 2\,\frac{\lambda^2}{2}\Bigg|_0^{t-1}\, u(t-1) \\
&\quad + 2\left(\frac{\lambda^2}{2} - \lambda\right)\Bigg|_1^{t-1}\, u(t-2) \\
&= t^2\,u(t) - (2t^2 - 4t + 2)\,u(t-1) + (t^2 - 4t + 4)\,u(t-2).
\end{aligned}
$$

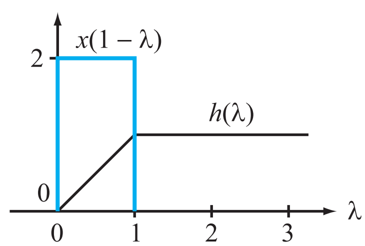
The figure below displays a plot of $y(t)$:



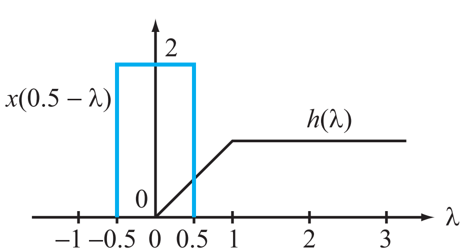
(b) Graphical Integration



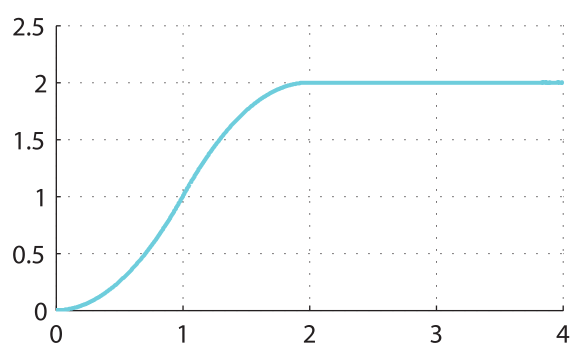
(1) At $t = 0$



(3) At $t = 1$ s, Overlap = 1



(2) At $t = 0.5$ s, Overlap = $\frac{1}{4}$



(4) $y(t)$

Problem 2.13. Method 2.

$x(t) = 2[u(t) - u(t-1)]$ $\quad h(t) = r(t) - r(t-1)$.

$u(t) * r(t) \longleftarrow$ Causal * Causal

$= \int_0^t u(\tau)\, r(t-\tau)\, d\tau \cdot u(t)$

$= \int_0^t 1 \cdot (t-\tau) \cdot d\tau \; u(t)$

$= \left[t \cdot \int_0^t d\tau - \int_0^t \tau\, d\tau\right] u(t)$

$= \frac{1}{2} t^2 u(t)$.

$x(t) * h(t)$

$= 2[u(t) - u(t-1)] * [r(t) - r(t-1)]$

$= 2\, u(t) * r(t) - 2\, u(t-1) * r(t) - 2\, u(t) * r(t-1) + 2\, u(t-1) * r(t-1)$

$= t^2 u(t) - (t-1)^2 u(t-1) - (t-1)^2 u(t-1) + (t-2)^2 u(t-2)$

$= t^2 u(t) - (2t^2 - 4t + 2)\, u(t-1) + (t^2 - 4t + 4)\, u(t-2)$

Problem 2.17 Compute the following convolutions:

(a) $e^{-t}\, u(t) * e^{-2t}\, u(t)$

(b) $e^{-2t}\, u(t) * e^{-3t}\, u(t)$

(c) $e^{-3t}\, u(t) * e^{-3t}\, u(t)$

Solution: The convolution of two causal signals is $y(t) = u(t) \int_0^t h(\tau)\, x(t-\tau)\, d\tau$.

(a)

$$\begin{aligned} e^{-t}\, u(t) * e^{-2t}\, u(t) &= u(t) \int_0^t e^{-\tau} e^{-2(t-\tau)}\, d\tau \\ &= e^{-2t}\, u(t) \int_0^t e^{\tau}\, d\tau \\ &= e^{-2t}\, u(t)[e^t - 1] \\ &= \boxed{e^{-t}\, u(t) - e^{-2t}\, u(t).} \end{aligned}$$

(b)

$$\begin{aligned} e^{-2t}\, u(t) * e^{-3t}\, u(t) &= u(t) \int_0^t e^{-2\tau} e^{-3(t-\tau)}\, d\tau \\ &= e^{-3t}\, u(t) \int_0^t e^{\tau}\, d\tau \\ &= e^{-3t}\, u(t)[e^t - 1] \\ &= \boxed{e^{-2t}\, u(t) - e^{-3t}\, u(t).} \end{aligned}$$

(c)

$$\begin{aligned} e^{-3t}\, u(t) * e^{-3t}\, u(t) &= u(t) \int_0^t e^{-3\tau} e^{-3(t-\tau)}\, d\tau \\ &= e^{-3t}\, u(t) \int_0^t d\tau \\ &= \boxed{e^{-3t} t\, u(t).} \end{aligned}$$