

RECEIVER NONLINEARITY OPTIMIZATION IN CLIPPING CHANNELS

Robert J. Baxley[†] and G. Tong Zhou[‡]

[†]Georgia Tech Research Institute, Atlanta, GA 30332-0817, USA

[‡]School of Electrical and Computer Engineering, Georgia Tech, Atlanta, GA 30332-0250, USA

ABSTRACT

In this paper the effects of receiver nonlinearities are examined for clipping channels. The objective of this work is to determine the optimal receiver functions for additive noise clipping channels. In this case the optimal receiver function will be the one that maximizes the signal-to-noise-plus-distortion ratio (SNDR) between the transmitted and received variables. To solve the problem we utilize functional analysis to find a necessary condition for the SNDR-maximizing receiver function. The results are general and can be applied for any noise and signal distribution. Furthermore, the results show that for the example given, the linear receiver is not SNDR-optimal.

Index Terms— Orthogonal frequency division multiplexing, selected mapping, crest factor reduction, peak-to-average power ratio

1. INTRODUCTION

While it is common for communications channels to be assumed linear, there is almost always a nonlinear component to physical channels. The most obvious nonlinear characteristic of physical channels is their peak-limited nature [1]. Because it is impossible to drive a power amplifier (PA) with an infinite amount of power, there will be some limit to the peak power allowed by the channel.

The obvious question is how a transmitter should be designed when a peak limitation is imposed but the input signal is not limited to the same peak. This question is implicitly answered by the vast body of papers that discuss the “pre-distortion” of PA nonlinearities [2–4]. Typically, the term “pre-distortion” implies that an expanding nonlinear function is applied to signals before they reach the PA. The goal of predistortion is to have the concatenation of the predistortion function and the PA characteristic function be linear up to the saturation power of the PA; such a peak-limited linear function is known as a soft limiter. Implicit in all of this work is that the soft limiter is the most desirable transmit function.

However, until [5] was published in 2005, it was not clear that the soft limiter was optimal in any sense. In [5] it was demonstrated that the soft limiter *with gain* is optimal in terms of the signal-to-noise-plus-distortion ratio (SNDR) when the gain is chosen correctly. SNDR optimality is an important goal because SNDR has been shown to be directly related to the bit error rate (BER) and capacity [4–8].

In this work we seek to determine the SNDR-optimal receiver-side functions in the presence of peak-limited channels. To accomplish this, we must first find an expression for the SNDR of additive

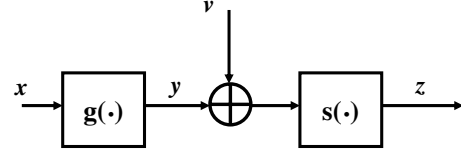


Fig. 1. System diagram.

noise channels with both transmitter and receiver memoryless nonlinear functions. Next, we follow some functional analysis methods [9] to maximize the SNDR w.r.t to the receiver function.

Other work has considered channels where a receiver-side nonlinearity is used to compensate for a transmitter nonlinearity. Frequently these schemes are found in PAR reduction literature under the name of companding, which is a combination of the words **compress** and **expand** [10–12]. In these schemes the receiver-side function is typically chosen to be the inverse of the transmitter function. The idea is that the signal will be compressed at the transmitter so as to avoid PA distortion and then expanded at the receiver with the inverse function to “undo” the compressing function. While this idea is intuitive, it was shown in [13] that using an inverse function pair is necessarily sub-optimal in terms of SNDR. In light of this, the obvious question is what are the SNDR-optimal receiver functions? The objective of this paper is to seek an answer to this question.

2. SNDR FORMULATION

SNDR is defined as the ratio of signal power to uncorrelated noise power. In [5], SNDR is derived for any transmitter non-linear memoryless function. In this section we will extend the SNDR definition to systems that have a nonlinear function both before and after noise is added. That is, where nonlinearities exist on both the transmitter (Tx) and receiver (Rx) sides of the channel.

2.1. SNDR of Transmitter Functions

To start, we review the SNDR formulation for transmitter nonlinearities, which is the SNDR between output $y + v$ and input x in Fig. 1. By writing y in terms of x and an uncorrelated distortion term, d so that

$$y = g(x) = \alpha x + d, \quad (1)$$

where x is the input random variable to the transmitter nonlinearity, $g(\cdot)$, we can express the SNDR by

$$SNDR_{x,y+v} = \frac{|\alpha|^2 \sigma_x^2}{\sigma_d^2 + \sigma_v^2}. \quad (2)$$

In (1) α is chosen so that $E[dx^*] = 0$, where d is the distortion term. It is important to include α in the formulation so that the distortion

This work was supported in part by the U. S. Army Research Laboratory under the Collaborative Technology Alliance Program, Cooperative Agreement DAAD19-01-2-0011 and by the National Science Foundation Graduate Research Fellowship Program.

term will be uncorrelated with the useful signal, x . Referring to [5], we can compute SNDR according to

$$SNDR_{x,y+v} = \frac{|E[yx^*]|^2}{\sigma_x^2 E[|y|^2] - |E[yx^*]|^2 + \sigma_x^2 \sigma_v^2}. \quad (3)$$

2.2. SNDR of Transmitter and Receiver Functions

The SNDR analysis in the previous subsection applies to all memoryless non-linear functions. Specifically, it is possible to use the Bussgang decomposition even when the function involves additive random variables. Thus, because the overall Tx-noise-Rx system seen in Fig. 1 can be viewed as a memoryless non-linear function, we can use the same formulation as was used for the Rx nonlinearity by simply replacing all instances of y with z , so that

$$SNDR_{x,z} = \frac{|E[zx^*]|^2}{\sigma_x^2 E[|z|^2] - |E[zx^*]|^2}. \quad (4)$$

Here, $z = s(g(x) + v)$. We still need to calculate $E[zx^*]$ and $E[|z|^2]$ to determine the SNDR. However, since the non-linear function is in terms of the random variable v , we need to take the expectations over both v and x . To do this, we only need the joint density $f_{x,v}(x, v)$. If we can assume that x and v are independent then it will be possible to further simplify the joint pdf to a product of the individual pdfs, $f_{x,v}(x, v) = f_x(x)f_v(v)$. This assumption is made for all of the following analysis. Thus,

$$E[zx^*] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(g(x) + v) x^* f_x(x) f_v(v) dx dv \quad (5)$$

$$E[|z|^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |s(g(x) + v)|^2 f_x(x) f_v(v) dx dv, \quad (6)$$

which should be simple to calculate numerically for any pair of functions, $g(\cdot)$ and $s(\cdot)$ and any pair of distributions, $f_x(x)$ and $f_v(v)$. In [5], it was possible to simply use the magnitude pdf to calculate these expectations. Here, that may not be possible because v and $g(x)$ will not necessarily add in-phase.

For verification, set $s(x) = x$ which results in

$$E[zx^*] = E[(g(x) + v)x^*] = E[g(x)x^*] \quad (7)$$

$$E[|z|^2] = E[|g(x) + v|^2] = E[|g(x)|^2] + \sigma_v^2. \quad (8)$$

Plugging these into (4) and the original SNDR expression (3) follows.

In summary, the problem is to find the functions $g(\cdot)$ and $s(\cdot)$ that maximize

$$SNDR_{x,z} = \frac{|E[zx^*]|^2}{\sigma_x^2 E[|z|^2] - |E[zx^*]|^2}, \quad (9)$$

where $E[zx^*]$ and $E[|z|^2]$ are defined in (5) and (6). Or, to be precise,

$$\underset{g(\cdot), s(\cdot)}{\text{maximize}} \quad SNDR_{x,z}[g(\cdot), s(\cdot)] \quad (10)$$

When no constraints are placed on these functions the solution is straight forward: $g(x) = ax$ where $a \rightarrow \infty$ and $s(x) = x$; i.e. both functions are linear. Furthermore, when an average power constraint is placed on $g(x)$, such as $E[|g(x)|^2] \leq a$, then again the solution is two linear function, but with the gain of $g(\cdot)$ chosen so that the constraint is satisfied. As we will show, the problem is more

complicated when a peak power constraint is placed on one or both of the functions, such as

$$\begin{aligned} &\underset{g(\cdot), s(\cdot)}{\text{maximize}} \quad SNDR_{x,z}[g(\cdot), s(\cdot)] \\ &\text{subject to} \quad \max_x |g(x)|^2 \leq 1. \end{aligned} \quad (11)$$

In addition to jointly optimizing both functions, two possible sub problems can also be considered. One where the receiver function is known and the transmitter needs to be derived and the other where the transmitter function is known and the receiver is to be derived.

3. SNDR OPTIMIZATION

Using functional analysis [9] to optimize the SDNR w.r.t. either $s(\cdot)$ or $g(\cdot)$, we need to solve

$$\frac{\partial}{\partial g(x_0)} SNDR_{x,z}[g(\cdot), s(\cdot)] = 0 \quad (12)$$

and

$$\frac{\partial}{\partial s(x_0)} SNDR_{x,z}[g(\cdot), s(\cdot)] = 0 \quad (13)$$

simultaneously. Computing the partial w.r.t. $s(x_0)$, we find

$$\begin{aligned} \frac{\partial}{\partial s(x_0)} SNDR_{x,z}[g(\cdot), s(\cdot)] = \\ \frac{\frac{\partial N}{\partial s(x_0)} D - \frac{\partial D}{\partial s(x_0)} N}{(\sigma_x^2 E[|z|^2] - |E[zx^*]|^2)^2}, \end{aligned} \quad (14)$$

where N and D are the numerator and denominator of the SNDR expression, respectively. To solve (13), we need the numerator of (14) to be zero. Assuming real variables, and simplifying, we have

$$2 \frac{\partial E[xz]}{\partial s(x_0)} E[z^2] = \frac{\partial E[z^2]}{\partial s(x_0)} E[xz]. \quad (15)$$

This same simplification holds for the partial w.r.t. $g(x_0)$.

Finally, it is necessary to solve for $\frac{\partial E[xz]}{\partial s(x_0)}$, $\frac{\partial E[z^2]}{\partial g(x_0)}$, and $\frac{\partial E[z^2]}{\partial s(x_0)}$, which are

$$\begin{aligned} \frac{\partial E[xz]}{\partial s(x_0)} = \\ \int_{x_0 - \mathcal{R}_g} g^{-1}(x_0 - v) f_x(g^{-1}(x_0 - v)) f_v(v) dv, \end{aligned} \quad (16)$$

where \mathcal{R}_g is the set of values in the range of $g(x) \forall x \in \mathbb{R}$,

$$\frac{\partial E[z^2]}{\partial s(x_0)} = 2s(x_0) \int_{x_0 - \mathcal{R}_g} f_x(g^{-1}(x_0 - v)) f_v(v) dv, \quad (17)$$

$$\frac{\partial E[xz]}{\partial g(x_0)} = \int_{-\infty}^{\infty} s'(g(x_0) + v) f_x(x_0) f_v(v) dv, \quad (18)$$

and

$$\begin{aligned} \frac{\partial E[z^2]}{\partial g(x_0)} = \\ 2 \int_{-\infty}^{\infty} s(g(x) + v) s'(g(x_0) + v) f_x(x_0) f_v(v) dv. \end{aligned} \quad (19)$$

From here, it is difficult to further simplify the problem but insight can be gained by analyzing some specific scenarios of interest. However, using these computations (16)-(19), it is possible to find functions that satisfy (15) and its counterpart where the partials are w.r.t. $g(x_o)$. Notice that this is a necessary but not sufficient condition for the resulting functions to be SNDR optimal. It is possible that SNDR is not convex in the functional space in which case local maxima, minima and saddle points could result in solving (15) and its counterpart where the partials are w.r.t. $g(x_o)$. In this case, to find the optimal pair, all possible solutions would have to be enumerated to determine which maximized the SNDR.

3.1. Example: Uniform Noise, Signal, Clipping Transmitter

As an example, let us assume that both the noise and the signal are uniformly distributed and that the transmitter is a soft limiter with gain of one. That is

$$x \sim U[-u_x, u_x], \quad (20)$$

$$v \sim U[-u_v, u_v], \quad (21)$$

$$u_x \geq 1, \quad (22)$$

$$g(x) = \begin{cases} -1, & x < -1 \\ x, & |x| \leq 1 \\ 1, & x > 1. \end{cases} \quad (23)$$

We can compute

$$\frac{\partial E[xz]}{\partial s(x_o)} = \frac{1}{4u_x u_v} \int_{-u_v}^{u_v} g^{-1}(x_o - v) U_x(g^{-1}(x_o - v)) U_v(v) dv, \quad (24)$$

where

$$U_x(x) = \begin{cases} 1, & x \in [-u_x, u_x] \\ 0, & \text{else} \end{cases} \quad (25)$$

and $U_v(v)$ is similarly defined. Next, define

$$q_{xz}(x_o) \triangleq \int_{-u_v}^{u_v} g^{-1}(x_o - v) U_x(g^{-1}(x_o - v)) U_v(v) dv$$

$$= \begin{cases} \int_{-u_v}^{x_o+1} x_o - v dv = \frac{1}{2}(-1 + u_v^2 + 2u_v x_o + x_o^2), & x_o \in [-u_v - 1, -u_v + 1] \cup u_v \geq 1 = \mathcal{S}_1 \\ \int_{x_o-1}^{x_o+1} x_o - v dv = 0, & x_o \in [-u_v + 1, u_v - 1] \cup u_v \geq 1 = \mathcal{S}_2 \\ \int_{x_o-1}^{u_v} x_o - v dv = \frac{1}{2}(1 - u_v^2 + 2u_v x_o + x_o^2), & x_o \in [u_v - 1, u_v + 1] \cup u_v \geq 1 = \mathcal{S}_3 \\ \int_{-u_v}^{u_v} x_o - v dv = 2u_v x_o, & x_o \in [u_v - 1, -u_v + 1] \cup u_v < 1 = \mathcal{S}_4. \end{cases} \quad (26)$$

The partial w.r.t. $s(x_o)$ is

$$\frac{\partial E[z^2]}{\partial s(x_o)} = \frac{2s(x_o)}{4u_x u_v} \int_{-u_v}^{u_v} U_x(g^{-1}(x_o - v)) U_v(v) dv. \quad (27)$$

Define

$$q_{zz}(x_o) \triangleq \int_{-u_v}^{u_v} U_x(g^{-1}(x_o - v)) U_v(v) dv$$

$$= \begin{cases} \int_{-u_v}^{x_o+1} dv = x_o + 1 + u_v, & (u_v, x_o) \in \mathcal{S}_1 \\ \int_{x_o-1}^{x_o+1} dv = 2, & (u_v, x_o) \in \mathcal{S}_2 \\ \int_{x_o-1}^{u_v} dv = -x_o + 1 + u_v, & (u_v, x_o) \in \mathcal{S}_3 \\ \int_{-u_v}^{u_v} dv = 2u_v, & (u_v, x_o) \in \mathcal{S}_4. \end{cases} \quad (28)$$

We need

$$2 \frac{\partial E[xz]}{\partial s(x_o)} E[z^2] = \frac{\partial E[z^2]}{\partial s(x_o)} E[xz]. \quad (29)$$

So,

$$2 \frac{1}{4u_x u_v} q_{xz}(x_o) E[z^2] = \frac{2s(x_o)}{4u_x u_v} q_{zz}(x_o) E[xz]$$

$$\frac{q_{xz}(x_o)}{q_{zz}(x_o)} \frac{E[z^2]}{E[xz]} = s(x_o), \quad (30)$$

where

$$\frac{q_{xz}(x_o)}{q_{zz}(x_o)} = \begin{cases} \frac{1}{2}(-1 + u_v + x_o), & (u_v, x_o) \in \mathcal{S}_1 \\ 0, & (u_v, x_o) \in \mathcal{S}_2 \\ \frac{1}{2}(1 - u_v + x_o), & (u_v, x_o) \in \mathcal{S}_3 \\ x_o, & (u_v, x_o) \in \mathcal{S}_4. \end{cases} \quad (31)$$

Now it is necessary to find a function $s(x_o)$ so that (30) holds. There is no clear systematic method for finding such an $s(x_o)$ and the problem can be particularly difficult if there is a constraint on $s(x_o)$. However, by conjecture, suppose that one way to satisfy (30) is to introduce a variable multiplier so that

$$s(x_o) = a \frac{q_{xz}(x_o)}{q_{zz}(x_o)}. \quad (32)$$

Thus, we have to find the value for a that satisfies (30). Fortunately, $\frac{E[z^2]}{E[xz]}$ is a constant w.r.t. x_o and the SNDR is scale invariant w.r.t. $s(x_o)$, so it is not necessary to determine a . Thus, one candidate for the SNDR-maximizing $s(x_o)$ among all possible functions for the constraints in (20)-(23) is simply

$$s(x_o) = \frac{q_{xz}(x_o)}{q_{zz}(x_o)}. \quad (33)$$

Interestingly, this function has no dependence on u_x . This is because one of the constraints was that $u_x \geq 1$, which necessitates some clipping. For the case when $u_x < 1$, the transmitter appears linear and the optimal receiver is simply linear.

Solution Functions: Fig. 2 is a plot of the solution functions $s(x)$ for different values of u_v . It is not clear if these are SNDR-optimal among all possible $s(x)$, but we will show that they do outperform the linear receiver. The plot shows several interesting features. The first is that when the $u_v > 1$ which is the maximum output of $g(x)$, the solution receiver function actually zeros out part of the received signal. The second is that when $u_v < 1$, the solution function receiver is a piecewise function with different slopes in for different input values.

SNDR Results: Fig. 3 is a plot of the SNDR of the proposed $s(x)$ in (33) and the SNDR of a linear receiver. We can see that most of the difference occurs in the high noise regime, which is intuitive as this is where the solution receiver differs the most from the linear receiver. Although it is still not clear that this is the optimal $s(x)$, it is clear that this function is better than the linear receiver.

4. CONCLUSIONS

In this paper we have presented a necessary but not sufficient condition for determining the optimal transmitter and receiver function pairs in additive noise channels. To illustrate the concept, we used an example where the transmitter function is a soft clipper and found

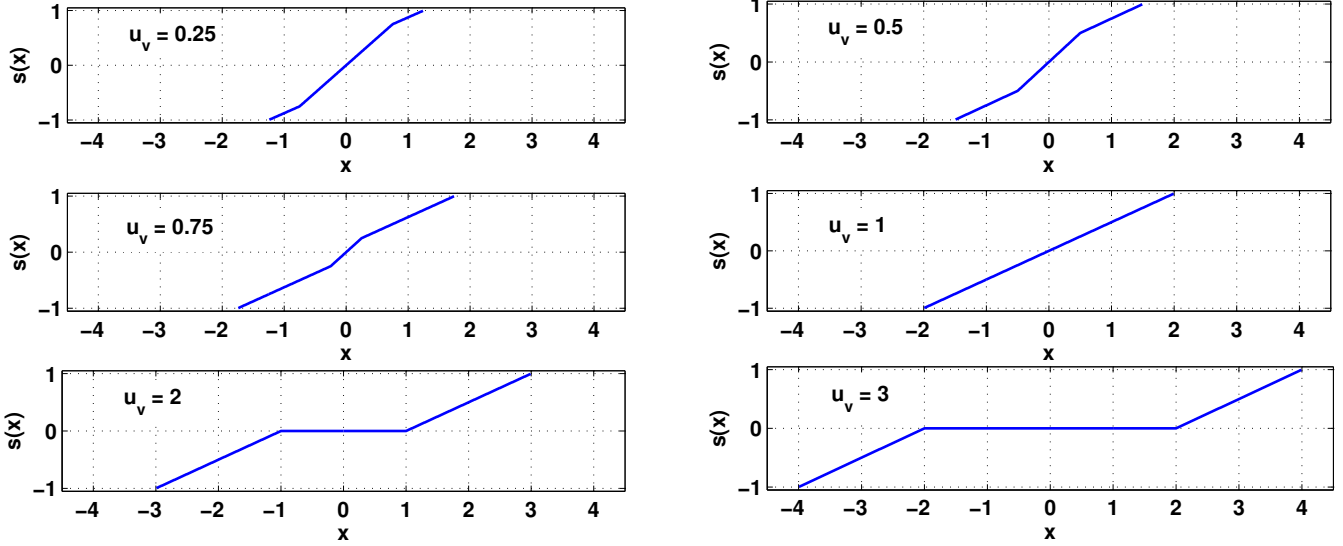


Fig. 2. Optimal receiver functions for different noise distribution supports.

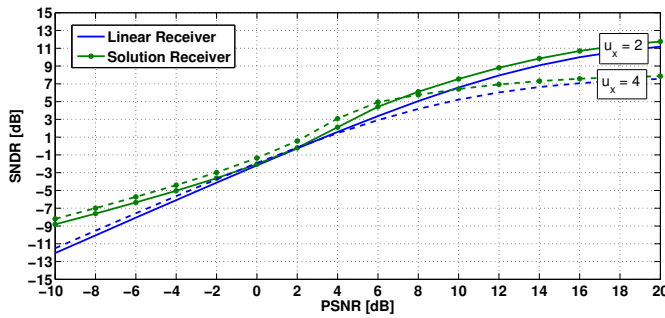


Fig. 3. SNDR of for a linear receiver and for the proposed receiver that satisfies (15).

the solution receiver function when both the noise and signal are uniformly distributed. In this example, we found that several dBs of SNDR improvement are possible. In the future it will be of interest to expand this work in three possible directions i) solve for other practical examples where the noise takes on different distributions; ii) reformulate the result for complex functions and variables; iii) jointly solve for both the Tx and Rx functions.

5. REFERENCES

- [1] S. A. Maas, *Nonlinear Microwave Circuits*. IEEE Press, 1997.
- [2] J. Tsimbinos and K. V. Lever, "Nonlinear system compensation based on orthogonal polynomial inverses," *IEEE Trans. on Circuits and Syst. I: Fundamental Theory and Applications*, vol. 48, no. 4, pp. 406–417, 2001.
- [3] K. Dogancay, "Blind compensation of nonlinear distortion for bandlimited signals," *IEEE Transactions on Circuits and Syst. I: Regular Papers*, vol. 52, no. 9, pp. 1872–1882, 2005.
- [4] R. Raich, *Nonlinear System Identification and Analysis with Applications to Power Amplifier Modeling and Power Amplifier Predistortion*. PhD thesis, Georgia Institute of Technology, May 2004.
- [5] R. Raich, H. Qian, and G. T. Zhou, "Optimization of SNDR for amplitude limited nonlinearities," *IEEE Transaction on Communications*, vol. 53, pp. 1964–1972, November 2005.
- [6] D. Dardari, V. Tralli, and A. Vaccari, "A theoretical characterization of nonlinear distortion effects in OFDM systems," *IEEE Trans. on Commun.*, vol. 48, no. 10, pp. 1755–1764, 2000.
- [7] H. Ochiai and H. Imai, "Performance of the deliberate clipping with adaptive symbol selection for strictly band-limited OFDM systems," *IEEE Journal on Selected Areas in Communications*, vol. 18, pp. 2270–2277, November 2000.
- [8] H. Ochiai and H. Imai, "Performance analysis of deliberately clipped OFDM signals," *IEEE Transactions on Communications*, vol. 50, pp. 89–101, January 2002.
- [9] I. M. Gelfand and S. V. Fomin, *Calculus of Variations*. Dover Publications, 2000.
- [10] X. Wang, T. T. Tjhung, and C. S. Ng, "Reduction of peak-to-average power ratio of OFDM system using a companding technique," *IEEE Transactions on Broadcasting*, vol. 45, pp. 303–307, September 1999.
- [11] X. Huang, J. Lu, J. Zheng, K. B. Letaief, and J. Gu, "Companding transform for reduction in peak-to-average power ratio of OFDM signals," *IEEE Transactions on Wireless Communications*, vol. 3, pp. 2030–2039, November 2004.
- [12] T. Jiang, Y. Yang, and Y.-H. Song, "Exponential companding technique for PAPR reduction in OFDM systems," *IEEE Transactions on Broadcasting*, vol. 51, pp. 244–248, June 2005.
- [13] C. Zhao, *Distortion-Based Crest Factor Reduction Algorithms In Multi-Carrier Transmission Systems*. PhD thesis, Georgia Institute of Technology, December 2007.