A COMPARISON OF SNDR MAXIMIZATION TECHNIQUES FOR OFDM

Robert J. Baxley and G. Tong Zhou

School of Electrical and Computer Engineering, Georgia Tech, Atlanta, GA 30332-0250, USA

ABSTRACT

When an orthogonal frequency division multiplexing (OFDM) signal is transmitted through a peak-power limited device, system designers must choose from several options to deal with the large dynamic range problem of OFDM. In this paper, we are interested in comparing i) clipping with gain, ii) modified piece-wise linear scaling (MPWLS) and iii) piecewise optimized clipping (PWOC) techniques in terms of the signal-to-noise-plus-distortion ratio (SNDR) metric. Existing work has shown that on a *per-sample* basis, clipping with a judiciously chosen gain and clipping ratio, dubbed uniform optimized clipping (UOC), can maximize the SNDR. Here, we are interested in comparing the performance of UOC with MPWLS and PWOC, two methods that allow for *symbol-wise* SNDR maximization. Through comparison, we show that the symbol-wise methods provide a slightly higher SNDR compared to UOC. However, this increase in performance comes at the expense of higher complexity.

***Index Terms*—** Orthogonal frequency division multiplexing (OFDM), clipping, crest factor reduction, peak-to-average power ratio, signal-to-noise-plus-distortion ratio (SNDR).

1. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) is an attractive multi-carrier transmission method because of its simple equalizer structure. It has been adopted by several communications standards, such as digital audio broadcasting, digital video broadcasting, wireless LAN and wireless MAN. However, one major problem associated with OFDM is its high peak-to-average power ratio (PAR) or crest factor (CF). When a high-PAR signal, such as OFDM, passes through a power amplifier (PA), the PA may be pushed to saturation, causing both in-band and out-of-band distortion. Hence, it is desirable to reduce the PAR of the input signal in order to maintain a reasonable level of power efficiency or to avoid excessive nonlinear distortion.

In this paper, three distortion-based PAR-reduction methods will be compared. Clipping is a simple method and has been well studied in the literature. Three previous works are particularly pertinent to the work in this paper as they present analysis on optimizing clipping in terms of the signal-to-noise-plus-distortion ratio (SNDR) [1–3]. In fact, in [2] it was proven that the soft clipping transfer function with carefully chosen gain and clipping ratio is SNDR-optimal among all peak-limited functions, when signals are operated on *one sample at a time*. Throughout the remainder of the paper we refer to this scheme as uniform optimized clipping (UOC).

In this work we will propose two novel distortion-based PAR-reduction methods and compare them with UOC. These methods have an additional degree of freedom compared with UOC in that they operate on signals from block-transmission schemes (e.g. OFDM), *one block at a time*. The first method presented is a generalization of piecewise linear scaling (PWLS) [4] that allows the SNDR to be optimized for any peak-signal-to-noise ratio (PSNR). The second method, dubbed piecewise optimized clipping (PWOC), preforms block-wise optimization to determine the highest SNDR transmitted signal.

2. SYSTEM MODEL

In OFDM, individual subcarriers in the frequency-domain are modulated with constellation points, transformed to the time-domain and transmitted with a cyclic prefix. For PAR analysis, the cyclic prefix can be ignored since it has no effect on the symbol PAR. Let the frequency-domain vector of constellation points be $\mathbf{x} = [x_1, x_2, ..., x_{N-1}, x_N]^T$, where x_k is drawn from a finite constellation. Using the inverse discrete Fourier transform, the time-domain symbol is

$$\mathbf{y} = \mathbf{Q}\mathbf{x}, \tag{1}$$

where $\mathbf{Q}$ is the inverse discrete Fourier transform matrix. For Nyquist sampling, $[\mathbf{Q}]_{k,n} = N^{-1/2} \exp(j2\pi(n-1)(k-1)/N)$. In general, this matrix model can be extended to the oversampling case by properly choosing the columns of $\mathbf{Q}$ to represent the baseband oversampled frequencies. In fact, the analysis in this paper can be extended to any block transmission scheme by choosing the appropriate matrix $\mathbf{Q}$ (e.g. code division multiple access (CDMA) can be analyzed by setting $\mathbf{Q}$ to the Hadamard matrix).

The PAR of the signal $\mathbf{y}$ is defined by

$$PAR(\mathbf{y}) = \frac{\|\mathbf{y}\|_\infty^2}{\sigma_y^2}, \tag{2}$$

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where σ_y^2 is the power in each element of $\mathbf{y}$ (the elements of $\mathbf{y}$ are assumed to have the same variance). For transmission through a peak-power limited device it is desirable to carry out crest factor reduction for high PAR (or crest factor) signals.

The device in this paper is assumed memoryless and peak-power limited to a maximum power of A. The transfer function of the peak-limited device is

$$c(z) = \begin{cases} \sqrt{A}z, & |z| \leq 1 \\ \sqrt{A}\, e^{j\angle z}, & |z| > 1 \end{cases}, \tag{3}$$

where z is the signal before the peak-limiting process, and $\angle z$ is its angle. Thus, $|c(z)|^2 \leq A$. In the linear region, the power gain of this device is A. By generalizing $c(\cdot)$ to a vector function $c(\mathbf{z}) : \mathbb{C}^{N\times 1} \to \mathbb{C}^{N\times 1}$, the overall channel becomes $c(\mathbf{z})+\mathbf{w}$, where $\mathbf{w}$ is additive white Gaussian noise (AWGN).

Three schemes for generating $\mathbf{z} = g(\mathbf{y})$ ($g(\mathbf{y}) : \mathbb{C}^{N\times 1} \to \mathbb{C}^{N\times 1}$), where $\mathbf{y}$ is the time-domain OFDM symbol given in (1), are compared in this paper. The resulting $\mathbf{z}$ is then passed through $c(\mathbf{z})$ defined in (3). The metric used to quantify the performance of each scheme is the signal-to-noise-plus- distortion ratio (SNDR). Notice that $\mathbf{r} = c(\mathbf{z})$ can be written as the sum of a scaled version of $\mathbf{y}$ and an uncorrelated distortion term, i.e. $\mathbf{r} = c \circ g(\mathbf{y}) = \alpha\mathbf{y} + \mathbf{d}$, where α is chosen so that $\mathrm{E}[\mathbf{y}^{\mathcal{H}}\mathbf{d}] = 0$ [2]. Assuming that the AWGN channel noise $\mathbf{w}$ is uncorrelated with $\mathbf{y}$ and $\mathbf{d}$, i.e. $\mathrm{E}[\mathbf{y}^{\mathcal{H}}\mathbf{w}] = \mathrm{E}[\mathbf{d}^{\mathcal{H}}\mathbf{w}] = 0$, then the SNDR is defined as

$$SNDR = \frac{|\alpha|^2\mathrm{E}[\mathbf{y}^{\mathcal{H}}\mathbf{y}]}{\mathrm{E}[\mathbf{d}^{\mathcal{H}}\mathbf{d}] + \mathrm{E}[\mathbf{w}^{\mathcal{H}}\mathbf{w}]}. \tag{4}$$

By inferring that $\alpha = \mathrm{E}[\mathbf{y}^{\mathcal{H}}\mathbf{r}]/(N\sigma_y^2)$ and that $\mathrm{E}[\mathbf{d}^{\mathcal{H}}\mathbf{d}] = \mathrm{E}[\mathbf{r}^{\mathcal{H}}\mathbf{r}] - |\mathrm{E}[\mathbf{y}^{\mathcal{H}}\mathbf{r}]|^2/(N\sigma_y^2)$, (4) can be rewritten as

$$SNDR = \frac{|\mathrm{E}[\mathbf{y}^{\mathcal{H}}\mathbf{r}]|^2}{\mathrm{E}[\mathbf{r}^{\mathcal{H}}\mathbf{r}]N\sigma_y^2 - |\mathrm{E}[\mathbf{y}^{\mathcal{H}}\mathbf{r}]|^2 + N^2\sigma_w^2\sigma_y^2}, \tag{5}$$

where σ_w^2 is the noise power. The objective is to design the function $g(\mathbf{y})$ so that after the peak-limiting device, the resulting process, $c \circ g(\mathbf{y}) + \mathbf{w}$ has a higher SNDR than $c(\mathbf{y}) + \mathbf{w}$.

3. UNIFORM OPTIMIZED CLIPPING (UOC)

In the UOC scheme

$$\mathbf{z} = g(\mathbf{y}) = \frac{\mathbf{y}}{\eta\sigma_y}, \tag{6}$$

where η is a scaling factor that is sample-independent (i.e. uniform for all samples). In [2], the authors provided a proof that the composition $c \circ g(\cdot)$, with $c(\cdot)$ from (3) and an η properly chosen for the $g(\cdot)$ in (6), maximizes the SNDR among all memoryless nonlinear mappings with a peak-power limit A. For Gaussian samples $\mathbf{y}$, the SNDR-optimal η can be calculated using $\eta^\star = T^{-1}(A/\sigma_w^2)$, where

$$T(\eta) = \frac{2\eta}{\sqrt{\pi}\mathrm{erfc}(\eta)}. \tag{7}$$

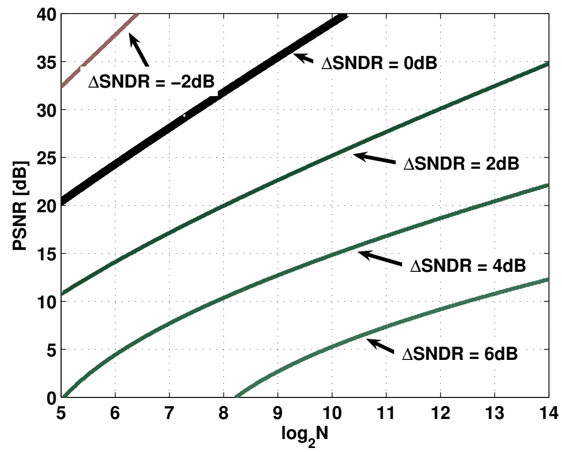

Fig. 1. Level curves of the SNDR differential between UOC and PWLS. To the top left of the bold line, PWLS outperforms UOC, to the bottom right of the bold line, UOC outperforms PWLS.

The above claim from [2] assumes that every sample of $\mathbf{y}$ is scaled by the same factor $1/(\eta\sigma_y)$. However, in block transmission schemes like OFDM, there is some flexibility w.r.t. the scaling factor. Since each block is equalized separately and inter-symbol interference is avoided with the use of a cyclic prefix, it is possible to use different scaling factors (or clipping levels) for different blocks. The following two sections present two alternative block-scaling methods that take advantage of this extra degree of freedom that block transmission affords.

4. (MODIFIED) PIECEWISE LINEAR SCALING ((M)PWLS)

PWLS was presented and analyzed in [4]. The idea behind PWLS, is to scale the signal prior to transmission so that no part of the signal would be clipped. Thus, the signal is scaled by a factor of $1/\|\mathbf{y}\|_\infty$ so that

$$\mathbf{z} = g(\mathbf{y}) = \frac{\mathbf{y}}{\|\mathbf{y}\|_\infty}. \tag{8}$$

Here $|\mathbf{z}|^2 \leq 1$ so $\mathbf{z}$ is scaled linearly without clipping when passing through the $c(\cdot)$ in (3). Unlike UOC, the scaling factor in PWLS changes every block based on the peak amplitude of $\mathbf{y}$.

Fig. 1 is a plot of the level curves for ΔSNDR, which is the difference (in dB) between the UOC SNDR and the PWLS SNDR. The plot demonstrates that, despite UOC being SNDR-optimal on a per-sample basis, PWLS can outperform clipping for certain values of N and PSNR, where $PSNR = A/\sigma_w^2$. Specifically, when N is small or when the PSNR is high, PWLS is preferable.

Despite better performance in some regions of the N-PSNR plane, PWLS is at a disadvantage to UOC in other

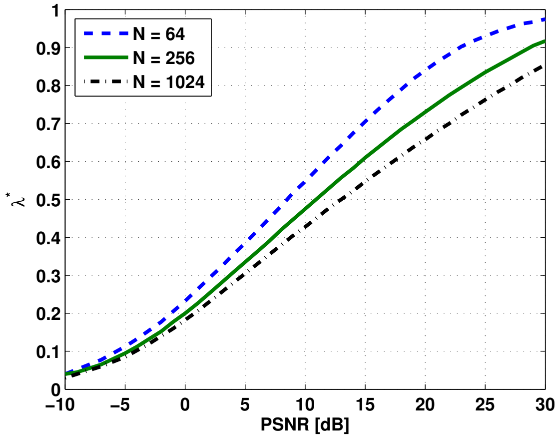

Fig. 2. SNDR-optimal values for λ.

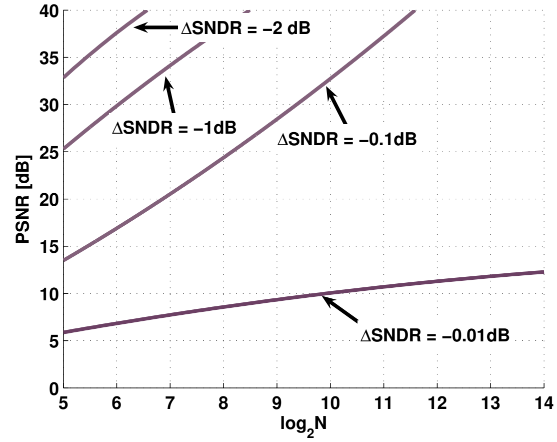

Fig. 3. Level curves of the SNDR differential between UOC and MPWLS. Negative values for ΔSNDR indicate that MPWLS outperforms UOC.

regions of the plane. This is attributed to PWLS being too "conservative" in trying to avoid any clipping and thus not delivering sufficient signal power. In [3], it was pointed out that deliberately introduced non-linearity can lead to SNDR improvements. Based on this observation, as a modification to improve the performance of PWLS, we propose that the scaling factor be parameterized with a PSNR-dependent variable λ so that

$$\mathbf{z} = g(\mathbf{y}) = \frac{\mathbf{y}}{\sqrt{\lambda}\|\mathbf{y}\|_\infty}. \tag{9}$$

We call this modified PWLS (MPWLS). Notice that $\lambda \in (0, 1]$. For the special case when $\lambda = 1$, MPLWS will be identical to PWLS. For $\lambda \in (0, 1)$, $\mathbf{z}$ in (9) will necessarily experience some clipping distortion when it is passed through $c(\cdot)$. However, some distortion maybe a desirable tradeoff for the increase in signal power so the end result may be a larger SNDR. Thus, it is necessary to optimize λ in terms of the PSNR and the number of subcarriers N.

Through Monte Carlo simulations we are able to determine the SNDR optimizing values for λ, $\lambda^\star$. Fig. 2 is a plot of $\lambda^\star$ versus the PSNR. Interestingly, the trend shows that for high PSNR, larger values of λ are required so that less clipping is incurred. Conversely, in the low-PSNR regime, λ is relatively smaller which results in relatively more clipping distortion but also more transmitted signal power. A similar trend is reported in the clipping scheme [2] for the optimal back-off values, η, from (6).

Fig. 3 is a plot of the level curves for ΔSNDR, which is the difference (in dB) between the UOC SNDR and the MPWLS SNDR. As expected, MPWLS always outperforms UOC. However, in certain regions of the N-PSNR plane the performance gap is relatively small. Comparing Fig. 3 and Fig. 1, we see that MPWLS is an improvement over PWLS.

5. PIECEWISE OPTIMIZED CLIPPING (PWOC)

In UOC, the scaling factor is block-independent and in MPWLS, the scaling factor is related to the block PAR, which can outperform UOC. However, MPWLS is not SNDR-optimal among all possible peak-limiting signal modifications that operate in a block-wise fashion. Instead, the global SNDR-optimal solution can be found by maximizing the instantaneous SNDR of each block defined by

$$\overline{SNDR}(\mathbf{y}, \mathbf{r}) = \frac{|\mathbf{y}^{\mathcal{H}}\mathbf{r}|^2}{\|\mathbf{y}\|_2^2\|\mathbf{r}\|_2^2 - |\mathbf{y}^{\mathcal{H}}\mathbf{r}|^2 + N\|\mathbf{y}\|_2^2\sigma_w^2}. \tag{10}$$

Notice that this definition is ambivalent to the sign of $\mathbf{r}$, which means that $\overline{SNDR}(\mathbf{y}, \mathbf{r}) = \overline{SNDR}(\mathbf{y}, -\mathbf{r})$. To alleviate this ambiguity and make the problem concave, the $\mathbf{r}$ chosen is the one most in the direction of $\mathbf{y}$. Thus, for each symbol an optimization problem needs to be solved in order to maximize the SNDR. Define $\mathbf{z} = g(\mathbf{y})$ for PWOC to be the function that solves

$$\begin{array}{ll} \underset{\mathbf{z}}{\text{maximize}} & \overline{SNDR}(\mathbf{y}, \sqrt{A}\mathbf{z}) \\ \text{subject to} & \|\mathbf{z}\|_\infty \le 1, \\ & \Re(\mathbf{y}^{\mathcal{H}}\mathbf{z}) \ge 0. \end{array} \tag{11}$$

Note that since $\|\mathbf{z}\|_\infty \le 1$, $\mathbf{r} = c(\mathbf{z}) = \sqrt{A}\mathbf{z}$. It is straightforward to show that this is a concave problem that can be solved using interior point methods (see [5]). In fact, the gradient of $\overline{SNDR}(\mathbf{y}, \sqrt{A}\mathbf{z})$ is

$$\nabla_{\mathbf{z}}\overline{SNDR}(\mathbf{y}, \sqrt{A}\mathbf{z}) = \mathbf{y}^{\mathcal{H}}\mathbf{z}\left[(\|\mathbf{z}\|_2^2 + N\sigma_w^2/A)\mathbf{y} - (\mathbf{z}^{\mathcal{H}}\mathbf{y})\mathbf{z}\right], \tag{12}$$

which can be used in conjunction with the boundary constraint, $\|\mathbf{z}\|_\infty \le 1$ to perform a gradient ascent search [5] for the SNDR-optimizing vector $\mathbf{z}$. The gradient ascent search (descent search for convex problems), is basically an iterative search technique that finds the optimizing vector by moving in the direction of the gradient in each successive iteration.

To simplify the problem, without loss of generality, assume that in the first iteration, $\mathbf{z}$ is initialized with $\mathbf{y}$. Even after applying the boundary constraints, which do not effect the phase of $\nabla_{\mathbf{z}}\overline{SNDR}$, both $\mathbf{y}^{\mathcal{H}}\mathbf{z}$ and, trivially, $\mathbf{z}^{\mathcal{H}}\mathbf{y}$ are real numbers. Thus, the gradient always searches in the direction of $\mathbf{y}$ subject to $\|\mathbf{z}\|_\infty^2 \le 1$. Based on this, the optimization problem in (11) can be simplified from an N-variate problem to a single-variate problem. Specifically, the optimization problem in (11) is equivalent to

$$\begin{aligned} &\underset{\rho}{\text{maximize}} && \overline{SNDR}(\mathbf{y}, c(\rho\mathbf{y})) \\ &\text{subject to} && \rho > 0, \end{aligned} \tag{13}$$

where $c(\cdot)$ is the function defined in (3), $c(\rho^\star\mathbf{y}) = \sqrt{A}\mathbf{z}$, and $\rho^\star$ is the optimizing value of ρ. Notice that the problem in (13) can be solved with a simple grid search, which has convergence that is geometric in the iteration number, i.e. $\|c(\rho^\star\mathbf{y}) - c(\rho^{(i)}\mathbf{y})\|_2^2 \propto 10^{-i}$, where $\rho^{(i)}$ is the value of ρ after i iterations [6]. Because PWOC maximizes the instantaneous symbol SNDR, $\overline{SNDR}$, it is obvious that the scheme also maximizes the SNDR, which is $E[\overline{SNDR}]$. Thus, the PWOC SNDR will be the upper-bound SNDR under the peak-limited channel constraint for block-wise signaling schemes.

6. COMPARISONS

To re-cap, we have considered two methods that will incur clipping distortion: UOC and MPWLS. We emphasize that the clipping notion that we are referring to here differs from the simple slipping technique in [7–9], where the clipped signal is

$$\tilde{c}(y) = \begin{cases} y, & |y| \le \sqrt{A} \\ \sqrt{A}\, e^{j\angle y}, & |y| > \sqrt{A} \end{cases} . \tag{14}$$

No gain is applied in (14), and (14) is not SNDR-optimal even on a per-sample basis. In UOC and MPWLS, $\mathbf{y}$ is judiciously scaled to $\mathbf{z}$ via $\mathbf{z} = g(\mathbf{y})$ and the $c(\cdot)$ function in (3) is then applied to $\mathbf{z}$. The key to designing $g(\cdot)$ is to ensure that the average power increase due to scaling dominates any increase in the clipping distortion power.

For the two other methods discussed in this paper, namely PWLS and PWOC, the $g(\cdot)$ functions are designed such that $|\mathbf{z}| < 1$ and thus $c(\mathbf{z}) = \sqrt{A}\mathbf{z}$ and no clipping distortion is encountered. As we will see, PWLS is too "timid", whereas PWOC performs the best since it maximizes the instantaneous SNDR for each block.

In this section, we will evaluate the complexity and SNDR performance of each of the discussed schemes: UOC, MPWLS and PWOC. First, in terms of complexity, UOC is the simplest scheme to implement with a complexity of $O(N)$ as only a multiplication with $1/\eta$ is necessary for each sample. Next, MPWLS requires determining the peak values of each symbol and the multiplication of each sample by $1/\sqrt{\lambda}$, which is more than twice as complex. However, the complexity is still linear in N so MPWLS also has complexity $O(N)$. Finally, using the modified objective function in (13), each of the iterations of the grid search has a complexity linear in N so for I iterations, the complexity is $O(NI)$.

Fig. 4 is a plot of the SNDR for the three schemes versus the relevant parameters. The plot demonstrates the importance of proper parameter selection in both UOC and MPWLS. Notice that for $\lambda = 1$, MPWLS becomes PWLS. The plot shows that for the PSNRs plotted, the $\lambda = 1$ case has a significant SNDR disadvantage compared to the optimal λ case. The plot also demonstrates that MPWLS has an SNDR that is near the upper-bound PWOC SNDR.

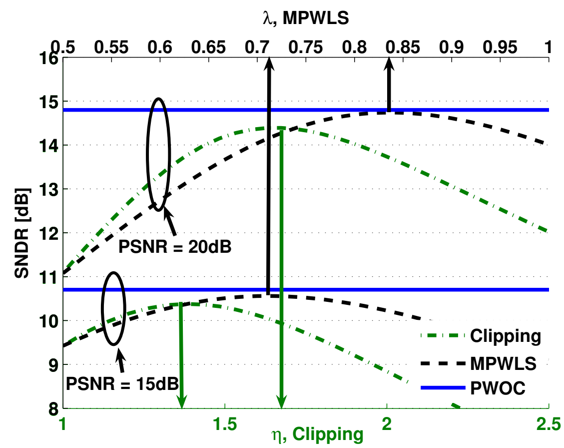

Fig. 4. Plot of the three methods versus η for UOC on the bottom x-axis and versus λ for MPWLS corresponding to the top x-axis; $N = 64$. In the plot vertical arrows indicate which axis each line is read to.

Fig. 5 is a plot of the difference between PSNR and SNDR, which confirms that the SNDR performance of MPWLS is very close to the PWOC SNDR bound. In the plot, the UOC and PWLS performances are also evaluated. UOC has an impressive performance in the low-PSNR region compared to the other schemes considering that it has the lowest complexity. Conversely, PWLS is meaningful only at high PSNR values. However, PWLS is the only scheme that does not require any transmitter-side knowledge about the channel noise level. Also, it should be noted that distortionless PAR-reduction techniques such as those discussed in [10] may improve the performance of PWLS and narrow its SNDR performance gap with the other schemes.

Fig. 6 demonstrates the SNDR gains that can be realized by using a PAR-reduction technique. For the plot, selected mapping (SLM) [11] was used with 16 phase mappings, which corresponds to 4 bits of side information. That

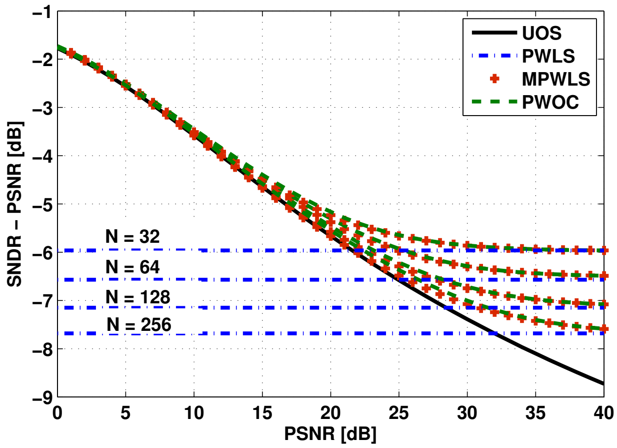

Fig. 5. Plot of the difference $SNDR - PSNR$ in dB scale.

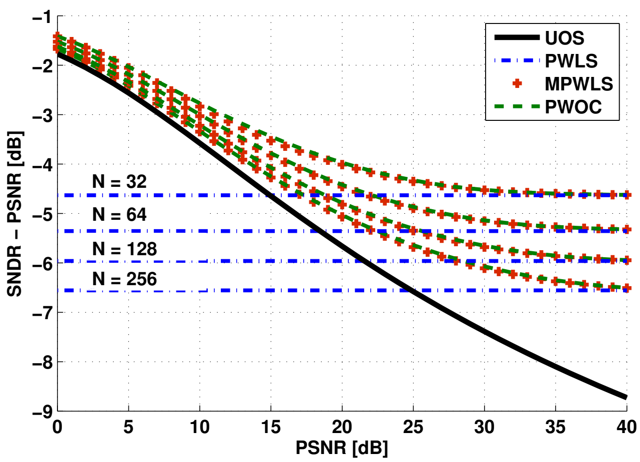

Fig. 6. Plot of the difference $SNDR - PSNR$ in dB scale, where 16 mappings are used in the selected mapping (SLM) PAR-reduction technique.

is, the signal $\mathbf{y}$ that is modified by the three schemes is the lowest-PAR signal among 16 alternative signal mappings (including the original OFDM signal). The SNDR savings relative to UOC (compared with Fig. 5) is evident across all the curves plotted.

7. CONCLUSIONS

In this paper we have presented a comparison of three SNDR optimization schemes: UOC, MPWLS and PWOC. Both MPWLS and PWOC are new schemes proposed in this paper that are designed to operate upon block-wise communications signals such as OFDM. Specifically, we demonstrated that PWOC will create signals that have the upper-bound SNDR. Using this upper bound, we also demonstrated that the significantly less complex technique, MPWLS, results in signals that have SNDR values very near the upper bound. Additionally, we showed that the proposed techniques can be combined with distortionless PAR-reduction algorithms such as SLM to increase the SNDR. Future research will focus on determining the SNDR-optimal scaling factor in UOC and MPWLS for the case when there is limited knowledge of the channel noise and the case where there is a spectral mask constraint.

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