OPTIMIZING FREE SUBCARRIER INDEX TO MINIMIZE PEAK-TO-AVERAGE POWER RATIO FOR OFDM SYSTEMS

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ABSTRACT

Optimizing the indices and values of free subcarriers (FSs) can significantly reduce the peak-to-average power ratio (PAR) of orthogonal frequency division multiplexing (OFDM) signals. In this paper, a numerical method is proposed to analyze the average PAR reduction performance of the FS optimization method. Different PAR reduction performance is revealed for different FS index assignments and the optimal single FS index can be determined by the proposed approach.

***Index Terms*—** Free subcarrier optimization, orthogonal frequency division multiplexing (OFDM), peak-to-average power ratio (PAR).

1. INTRODUCTION

The orthogonal frequency division multiplexing (OFDM) technique has been adopted by many wireless communication standards due to its high spectral efficiency and low complexity over frequency-selective fading channels [1]. However, one of the primary disadvantages of OFDM is that time-domain OFDM waveforms exhibit large peak-to-average power ratios (PARs), rendering the power amplifier (PA) as power inefficient [2].

Many PAR reduction methods have been proposed for OFDM systems (see [3] and references herein). In particular, the selected mapping (SLM) method can achieve significant PAR reduction without introducing distortions [4, 5]. The basic idea of SLM is to produce multiple time-domain representations for each OFDM symbol by different phase rotations, and transmitting the representation with the smallest PAR. In order to decode, the receiver must know the phase sequence adopted by the transmitter and un-do the phase rotations. Researchers have proposed to reserve certain subcarriers for the transmission of side information. Error in the side information can significantly degrade the symbol decoding performance [5].

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Instead of transmitting the side information, the signals on the reserved subcarriers can be optimized so that the PAR of each OFDM symbol is minimized [6–9]. Such PAR reduction methods are commonly referred as tone reservation, or, free subcarrier (FS) optimization methods [6, 7]. The merit of FS optimization is that no receiver-side cooperation or side information is needed if the receiver and the transmitter agree on the set of FSs offline. Receivers can just discard the signals on the FSs and the data subcarriers can be decoded as usual.

In communication standards, FSs are usually located at the edge of the band [1, 6]. Based on this assignment, the optimal number of FSs has been determined for the FS optimization method [7]. However, the optimal FS index assignment is still an open problem. Although a random index assignment is usually claimed to be optimal based on empirical simulation results [2], to the best of our knowledge, no work has analyzed the effects of the FS index on the performance of the FS optimization method. In this paper, we analyze the PAR reduction performance of the FS optimization method with a single FS but different index assignments. A numerical method is proposed to determine the optimal single FS index with respect to the average PAR reduction performance. The analysis is corroborated by numerical simulations.

2. SYSTEM MODEL

Let $\mathbf{X} = [X_{-N/2}, \cdots, X_{N/2-1}]^T$ denote the frequency-domain OFDM symbol with N subcarriers, where ^{T} stands for transpose. The OFDM subcarriers $\mathcal{K} = \{-\frac{N}{2}, \cdots, \frac{N}{2} - 1\}$ can be split into two non-overlapping sets: data subcarriers and free subcarriers, represented by sets of indices $\mathcal{K}_d$ and $\mathcal{K}_f$, respectively, with $\mathcal{K}_d \cap \mathcal{K}_f = \emptyset$ and $\mathcal{K}_d \cup \mathcal{K}_f = \mathcal{K}$. On the data subcarriers, the information bits are mapped into an ideal constellation Ω of the specified modulation schemes, namely $X_k \in \Omega$ ($k \in \mathcal{K}_d$). In contrast, the free subcarriers can be used to improve the system performance, e.g., optimize the waveform and/or transmit side information in order to reduce the signal dynamic range [2, 4, 6]. In the following, we use $\mathbf{X}^{(r)}$ to denote the unmodified reference OFDM symbol where no FS energy is present, i.e., $X_k^{(r)} = 0$ ($\forall k \in \mathcal{K}_f$).

Prior to cyclic extension which does not impact the signal dynamic range [2], the L-times oversampled time-domain signal can be obtained from the LN-point inverse fast Fourier transform (IFFT) operation, i.e.,

$$x[n] = \frac{1}{\sqrt{LN}} \sum_{k=-N/2}^{N/2-1} X_k e^{j2\pi \frac{kn}{LN}},\ n \in \mathcal{N}, \tag{1}$$

where $\mathcal{N} = \{0, \cdots, LN-1\}$. For notational simplicity, the time-domain waveform can be denoted as $\mathbf{x} = [x[0], \cdots, x[LN-1]]^T \triangleq \mathrm{IFFT}[\mathbf{X}]$ for each OFDM symbol. Correspondingly, $\mathbf{x}^{(r)} = \mathrm{IFFT}[\mathbf{X}^{(r)}]$.

To transmit the signal, a peak-power-limited PA with the output peak-power limit P_{peak} is used. Here we assume a class-A PA with an ideal soft-limiter characteristic, which implies that $P_{\text{peak}} = P_{\text{DC}}/2$ and the time-domain output signal $y[n] = g(x[n])$ is characterized by [2, 10]

$$y[n] = \begin{cases} x[n], & |x[n]|^2 \le P_{\text{peak}} \quad (2) \\ \sqrt{P_{\text{peak}}} e^{j\angle x[n]}, & |x[n]|^2 > P_{\text{peak}}, \quad (3) \end{cases}$$

where $\angle x$ denotes the phase of a complex variable x. For simplicity, a unit gain is assumed in the PA linear region. With this model, saturation happens when $|x[n]|^2 > P_{\text{peak}}$ and may result in error floor, spectral broadening and other deleterious nonlinear effects [2].

To completely avoid saturation, the piece-wise linear scaling (PWLS) method can be used [11]. With PWLS, the PA output signal is $\mathbf{y} = G(\mathbf{x}) \cdot \mathbf{x}$ where $G(\mathbf{x}) = \sqrt{P_{\text{peak}}}/\|\mathbf{x}\|_\infty$ is a symbol-wise gain and $\|\mathbf{x}\|_\ell$ denotes the ℓth norm of $\mathbf{x}$.

Within the free subcarrier optimization framework, we define the peak-to-average power ratio (PAR) as

$$\gamma(\mathbf{X}) = \frac{\|\mathbf{x}\|_\infty^2}{\frac{1}{LN}\|\mathbf{x}^{(r)}\|_2^2}. \tag{4}$$

In contrast to a PAR definition where the peak power is normalized by $\|\mathbf{x}\|_2^2/(LN)$, which also includes the power consumed on the free subcarriers, the PAR in (4) reflects the ratio between the peak power and the effective data power. With PWLS, the effective power efficiency of the class-A PA can be readily shown as $\eta = \gamma_h/2$, where γ_h is the harmonic mean of $\gamma(\mathbf{X})$ [7].

3. THE OPTIMAL INDEX OF FREE SUBCARRIER

By optimizing X_k ($k \in \mathcal{K}_f$), the PAR can be minimized so that the PWLS gain is increased to improve the error performance on data subcarriers [2, 6]. Because $\mathbf{x}^{(r)}$ is fixed, PAR minimization is also equivalent to minimizing the peak power of $\mathbf{x}$.

When only one FS is used in the FS optimization method, i.e., $\mathcal{K}_f = \{f\}$ and $f \in \mathcal{K}$, the optimized OFDM symbol can be denoted as

$$\tilde{\mathbf{X}}_f = [X_{-\frac{N}{2}}, \cdots, X_{f-1}, \tilde{X}_f, X_{f+1}, \cdots, X_{\frac{N}{2}-1}]^T. \tag{5}$$

For a given index f, the optimal FS value $\tilde{X}_f$ can be found by (c.f. [8])

$$\tilde{X}_f = \arg\min_{X_f} \left(\max_{n \in \mathcal{N}} |X_f - B_f[n]| \right), \tag{6}$$

where

$$B_f[n] = -\sum_{k \neq f, k \in \mathcal{K}} X_k e^{j2\pi \frac{(k-f)n}{LN}},\ n \in \mathcal{N}. \tag{7}$$

In other words, the FS value optimization is equivalent to determining the X_f that has the minimax Euclidean distance to the LN points $\{B_f[n]\}$ on the complex plane. In operations research, it is also called the minimum spanning circle (or single-facility location) problem [12]. Based on the Chrystal-Peirce algorithm [12], the minimum spanning circle method (MSCM) has been proposed to solve (6) with linear complexity $\mathcal{O}(LN)$ [8].

Although the FS value optimization problem can be solved by the existing methods [6, 8], how to assign the index for the FS (i.e., f) to achieve the maximum PAR improvement on average is still an open question. It is not clear that whether different FS index assignments can lead to different average PAR reduction performance.

In this section, we propose a numerical method to solve this problem. Following (6) and (7), the optimal FS index $\tilde{f}$ can be found as

$$\tilde{f} = \arg\min_f E\left[\max_{n \in \mathcal{N}} \left| \tilde{X}_f - B_f[n] \right| \right], \tag{8}$$

where both $\tilde{X}_f$ and $\{B_f[n]\}$ are functions of the FS index f and data symbol $\mathbf{X}^{(r)}$, and the expectation is taken over all OFDM symbols. Numerically evaluating (8) would be computationally prohibitive.

Intuitively, (8) indicates that for the optimal FS index $\tilde{f}$, the LN points $\{B_{\tilde{f}}[n]\}$ should be closer on average over all reference OFDM symbols $\mathbf{X}^{(r)}$, i.e., the average radius of the minimum spanning circles of the sequence $\{B_{\tilde{f}}[n]\}$ is the smallest. For uncoded OFDM systems with mutually independent data X_k ($k \in \mathcal{K}_d$), due to the Central Limit Theorem, $B_f[n]$ ($n \in \mathcal{N}$) can be approximated as an identically distributed Gaussian random variable with zero mean and f-independent variance $\sigma^2 = E[|B_f[n]|^2] = N - 1$. For each given OFDM symbol $\mathbf{X}^{(r)}$ and the FS index f, let p denote the index of the corresponding $B_f[n]$ with the maximum magnitude, i.e.,

$$p = \arg\max_{n \in \mathcal{N}} |B_f[n]|. \tag{9}$$

Then, (8) can be expressed as

$$\begin{aligned} \tilde{f} &= \arg\min_f E\left[\max_{n \in \mathcal{N}} \left| \frac{B_f^*[p]}{|B_f[p]|} \left(\tilde{X}_f - B_f[n] \right) \right| \right] \quad (10) \\ &= \arg\min_f E\left[\frac{1}{|B_f[p]|} \max_{n \in \mathcal{N}} \left| \hat{X}_f - B_f^*[p] B_f[n] \right| \right], \end{aligned}$$

where $\hat{X}_f = B_f^*[p]\tilde{X}_f$. The distributions of $|B_f[p]|$ can be assumed approximately the same for different f's. Thus, the average "closeness" of the LN points $\{B_f[n]\}$ is determined by the radius of the minimum spanning circle of $E\left[B_f^*[p]B_f[n]\right]$ ($n \in \mathcal{N}$). In fact, the cross correlation function $\rho_f(\cdot)$ of the static random sequence $\{B_f[n]\}$ can be used to obtain $E\left[B_f^*[p]B_f[n]\right]$, i.e., [c.f. (7)]

$$\begin{aligned} \rho_f(n-p) &= \frac{E\left[B_f^*[p]B_f[n]\right]}{\sigma^2} & (11)\\ &= \frac{1}{N-1}\sum_{k\neq f, k\in\mathcal{K}} e^{j2\pi\frac{(k-f)(n-p)}{LN}}. & (12)\end{aligned}$$

The cross correlation $\rho_f(n-p)$ gives the average locations of the random points $B_f[n]$ ($n \in \mathcal{N}$) relative to a given point $B_f[p]$ on the complex plane. Since the index p is also random, for the LN-point sequence $\{B_f[n]\}$, $2LN-1$ cross correlations are of interests, i.e., $\{\rho_f(m)\}$ with $m \in \mathcal{M} \triangleq \{-LN+1, \cdots, LN-1\}$. The "closeness" of these $2LN-1$ relative positions is measured by the radius of their minimum spanning circle

$$r_f = \max_{m\in\mathcal{M}} \left|\tilde{C}_f - \rho_f(m)\right|, \qquad (13)$$

where $\tilde{C}_f$ is the center of the minimum spanning circle. By replacing $B_f[n]$ with $\rho_f(m)$ in (6), $\tilde{C}_f$ and r_f can be determined by the MSCM algorithms in [8].

The smaller the r_f, the closer the points $\{B_f[n]\}$ are on average. Thus, to determine the optimal FS index, the following numerical method can be used:

1. Calculate $\rho_f(m)$ ($\forall m \in \mathcal{M}, f \in \mathcal{K}$) with (12);
2. Determine the minimum spanning circle radius r_f ($\forall f \in \mathcal{K}$) by solving (13) with the MSCM algorithm;
3. The optimal FS index is $\tilde{f} = \arg\min_f r_f$.

Two more remarks are now in order: (i) If Nyquist sampling (i.e., $L = 1$) is used to generate the OFDM waveform, the average PAR reduction performance will exhibit no difference over the FS index f. The reason is, with $L = 1$, the points $\{\rho_f(m)\}$ given by (12) are identical (though in different order) for all $f \in \mathcal{K}$. Nevertheless, oversampling is necessary to accurately estimate the continuous OFDM waveform [2] and should be used in FS optimizations. Otherwise, Nyquist-sampled signals after digital-to-analog converters and pulse shaping will suffer peak regrowth, and the optimized FS will no longer be optimal; (ii) The optimal FS index $\tilde{f}$ can be predetermined offline and acknowledged *a priori* by both the transmitter and the receiver. Neither side information nor adaptive receiver-side cooperation is needed.

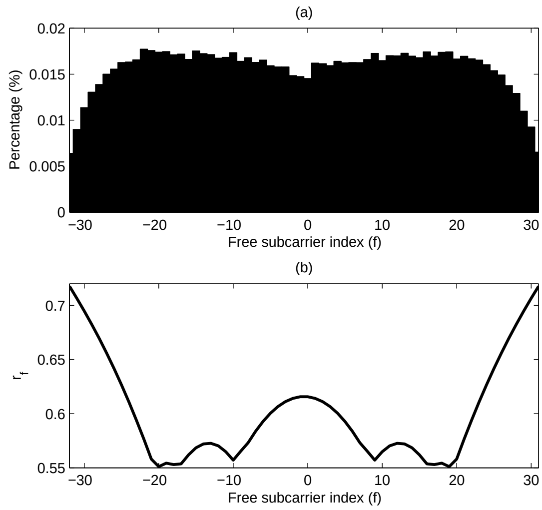

Fig. 1. (a) The numerical test result of the optimal FS index; (b) the radius r_f of the minimum spanning circles of the cross correlation functions.

4. SIMULATION RESULTS

A few numerical examples are presented to justify the proposed method and illustrate the performance of the FS optimization method. In the simulations, $L = 4$, $N = 64$, $|\mathcal{K}_f| = 1$ free subcarrier and $|\mathcal{K}_d| = 63$ data subcarriers with the QPSK constellation were used.

To illustrate the average PAR reduction performance of different FS index assignments, the following numerical test was conducted. For each symbol with the random data vector $\mathbf{s} \in \Omega^{N-1}$, N FS optimization problems given by (6) are solved for different FS index assignments $f \in \mathcal{K}$ with the corresponding data subcarriers $[X_{-\frac{N}{2}}, \cdots, X_{f-1}, X_{f+1}, \cdots, X_{\frac{N}{2}-1}]^T = \mathbf{s}$. The optimizations yield N PAR-minimized symbols $\tilde{\mathbf{X}}_f$ as in (5) and the minimized PAR values $\gamma_f = \gamma(\tilde{\mathbf{X}}_f)$. The FS index with the smallest γ_f has the best PAR reduction performance and is said to be chosen for the specific OFDM symbol. For multiple data symbols, the more frequently an FS index is chosen, the better the average PAR reduction performance is for this FS index assignment.

In our simulations, the numerical test was conducted for 500,000 independent OFDM symbols. The results are summarized in Fig. 1, in which the percentage for each FS index to be chosen is plotted in Fig. 1(a). In addition, as the function of f, the radii (i.e., r_f's) of the minimum spanning circles of the cross correlation functions are also plotted in Fig. 1(b). The simulation and analysis results agree well that the larger the r_f is, the less often the FS index assignment is chosen and the worse the average PAR reduction performance is. Moreover, the results indicate that when the FS is assigned at the

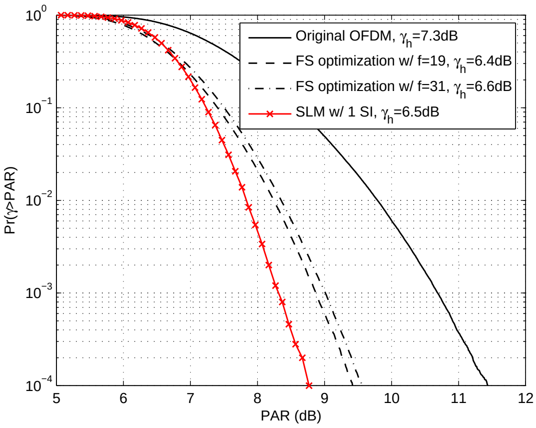

Fig. 2. CCDF curves for the PAR of the original OFDM, the OFDM systems with the FS optimization and the SLM methods.

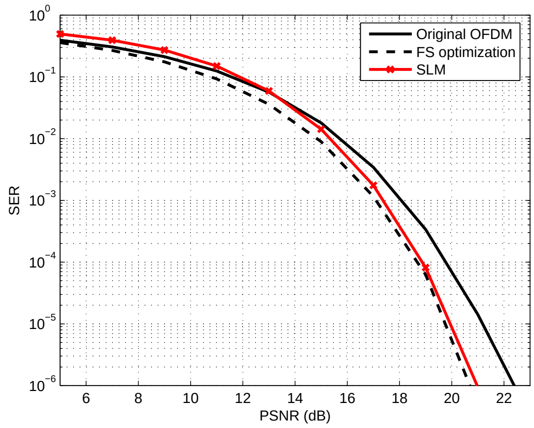

Fig. 3. SER curves for the original OFDM, the OFDM systems with the FS optimization and SLM methods; $f = 19$.

edge subcarriers as in general communication standards, the performance is actually the worst. The optimal FS index is neither in the center, but around $f = \pm 20$ when $N = 64$.

The complimentary cumulative distribution function (CCDF) curves are plotted for PAR values in Fig. 2. By showing that $f = 19$ outperforms $f = 31$ in the FS optimization method, it further corroborates the optimal FS index assignment given by Fig. 1. Additionally, for a given spectral efficiency cost, the FS optimization method is shown to have PAR performance similar to the SLM method which uses 4 independent phase sequences and thus needs 1 subcarrier to transmit the QPSK-modulated side information.

Though the PAR improvements are similar, the error performance of the FS optimization method is not affected by the side information transmission. Thus, it can fully take advantage of the improved power efficiency. In Fig. 3, the symbol error rate (SER) versus peak signal-to-noise ratio (PSNR) curves are plotted in an AWGN channel, where the PSNR is defined as $\mathrm{PSNR} = \frac{P_{\mathrm{peak}}}{N_0}$ to compare the channel noise variance N_0 with the power consumption of the peak-power limited PA. The FS optimization method is shown to be constantly better than the original OFDM. In contrast, although the power efficiency is increased by SLM, its SER performance may be significantly degraded by error decisions on the side information. Especially in the low PSNR region, its SER performance may be even worse than the original OFDM.

5. CONCLUSION

In this paper, a numerical method is proposed to analyze the PAR reduction performance of the FS optimization method with different FS index assignments. The proposed method can help determine the optimal subcarrier assignment scheme for the FS optimization PAR reduction method in OFDM systems. In the future, we intend to extend this work by finding the optimal set of FS indices for set sizes larger than one.

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