Error Vector Magnitude Optimization for OFDM Systems with a Deterministic Peak-to-Average Power Ratio Constraint

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***Abstract*—Orthogonal Frequency Division Multiplexing (OFDM) systems often have to conform to error vector magnitude (EVM) and spectral mask constraints as specified by communications standards. On the other hand, peak-to-average power ratio (PAR) of the signal waveform is also an important metric that influences the system power efficiency and system performance in the presence of nonlinearities. An interesting PAR reduction method was proposed recently that aims at minimizing the symbol-wise PAR subject to symbol-wise EVM and spectral mask constraints. In this paper, we re-formulate the optimization problem and strive to minimize the symbol-wise EVM subject to deterministic PAR and spectral mask constraints. The PAR threshold is chosen such that the root mean-squared (RMS) EVM meets the standard's requirement. To solve this new optimization problem, a low complexity interior-point method (IPM) is developed. The new optimization framework may be more suitable for practical systems. We provide numerical examples to demonstrate the performance of the proposed technique.**

***Index Terms*—Error vector magnitude (EVM), orthogonal frequency division multiplexing (OFDM), peak-to-average power ratio (PAR)**

I. Introduction

Due to its high spectral efficiency and robustness against frequency-selective fading channels, orthogonal frequency division multiplexing (OFDM) has been adopted by many wireless communication standards [1]–[3]. However, one of the primary disadvantages of OFDM is that time-domain OFDM waveforms exhibit a large peak-to-average power ratio (PAR). In order to avoid severe in-band and out-of-band nonlinear distortions, a large back-off is often required for the power amplifier (PA) to operate, which results in power-*in*efficient transmissions [4].

Various distortion-based algorithms have been developed to reduce the PAR of OFDM signals; some by constraining the distortion energy on data subcarriers [5]–[7] or by projecting the distortion energy onto "free" or "reserved" subcarriers [4]. These algorithms are attractive because they achieve significant PAR reduction, are compatible with current standards, and do not require receiver-side modifications. In the recent literature [8] [9], PAR reduction has been cast as a convex optimization problem where the symbol-wise PAR is minimized subject to symbol-wise error vector magnitude (EVM) and spectral mask constraints. By exploiting the fast Fourier transform (FFT) structure of OFDM, an interior-point method (IPM) can be efficiently customized for such convex optimization problem to provide good PAR reduction performance with relatively low complexity.

EVM is frequently used in communication standards to quantify the amount of in-band distortion that occur in the communication system. EVM contributes directly to the bit error rate (BER). EVM can be caused by any number of non-ideal components in the transmit chain including the power amplifier (PA), the DAC, the mixer, etc. If a PAR reduction algorithm works by transforming the signal at the transmitter side and the transformation is not to be "un-done" at the receiver side, then any EVM increase due to the PAR reduction algorithm will need to be accounted for in the EVM budget as well. We assume that there is sufficient EVM "headroom" left from the analog devices to allow for a distortion-based PAR reduction algorithm.

In [8] and [9], symbol-wise EVM constraints are used. In communications standards however, a root mean-squared (RMS) EVM constraint is typically given, which means the symbol-wise EVM can fluctuate and does not have to be as tightly constrained as in [8] and [9]. Our strategy here is to take advantage of this degree of freedom afforded by the symbol-wise EVM to boost the PAR reduction performance. Another distinction from previous work is that we utilize a deterministic PAR constraint in our optimization set up. Since the PAR of the transmitted signal is guaranteed to never exceed a prescribed PAR threshold, we do not need to worry about EVM and spectral distortions due to further (even occasional) clipping. Specifically, we propose that the symbol-wise EVM be minimized subject to a carefully chosen deterministic PAR constraint and the usual spectral mask constraint. Although the symbol-wise EVM on any one optimized symbol may still exceed the RMS EVM threshold, we will optimally set the deterministic PAR constraint such that the RMS EVM of the optimized symbol blocks meets the standard's requirement. A low complexity interior-point method will be derived to solve the EVM optimization problem.

In a block communication system, minimizing the symbol-wise PAR does not automatically imply power efficiency improvements, unless one implements adaptive biasing or adaptive scaling to "harvest" the PAR reduction. It would be simpler to just set a deterministic PAR threshold to reflect the dynamic range of the system or the back-off that can be afforded and not have to rely on block-by-block adaptive biasing or adaptive scaling. We believe that our proposed EVM optimization problem is better suited for the objectives and requirements laid out in the communications standards.

The organization of this paper is as follows. The OFDM PAR problem is outlined in Section II. We formulate the optimization problem in Section III and develop an interior-point method in Section IV. Good PAR and EVM performance will be shown in Section V and conclusions are drawn in Section VI.

II. System model

At the OFDM transmitter, an inverse FFT (IFFT) operation is performed to generate the time-domain signal, which is amplified by the PA and transmitted through the channel, i.e.,

$$x[n] = \mathrm{IFFT}_L(\mathbf{X})[n] = \frac{1}{\sqrt{LN}} \sum_{k=0}^{LN-1} X'_k e^{\frac{j2\pi kn}{LN}}, \tag{1}$$

where $\mathbf{X} = [X_0, \cdots, X_{N-1}]^T$ denotes the frequency-domain OFDM symbol with N subcarriers.

$$\mathbf{X}' = [X_0, \cdots, X_{\frac{N}{2}-1}, 0, \cdots, 0, X_{\frac{N}{2}}, \cdots, X_{N-1}]^T \tag{2}$$

is generated by zero-padding $\mathbf{X}$ by $(L-1)N$ zeros. Correspondingly, IFFT_L designates the L-times oversampled IFFT operation (LN-point IFFT) prior to cyclic extension. We denote the time-domain quantity as $\mathbf{x} = [x[0], \cdots, x[LN-1]]^T$. The PAR of the time-domain waveform is defined as

$$\mathrm{PAR} = \frac{\max\limits_{n\in\{0,\cdots,LN-1\}} |x[n]|^2}{\frac{1}{LN}\|\mathbf{x}\|_2^2}, \tag{3}$$

which is insensitive to the cyclic extension.

According to the Central Limit Theorem, the time-domain signal $\mathbf{x}$ exhibits a complex Gaussian distribution. As a result, the time-domain symbol can have a large PAR with non-negligible probability. To avoid nonlinear distortion caused by the PA, OFDM signals are usually backed off so that the PA is rarely saturated. When there is potential for large PAR values to occur, the back-off needs to be made large enough, the downside of which is low power efficiency. Therefore, it is desirable to have an algorithm that guarantees with probability one that the PAR of the signal will not exceed a certain threshold.

In this paper, we adopt a deterministic (as opposed to probabilistic) PAR constraint

$$\mathrm{PAR} \leq \gamma. \tag{4}$$

The PAR threshold γ is guided by the back-off factor necessary to ensure a certain level of linearity and power efficiency from the PA, as well as a desirable RMS EVM performance. A small γ means a small back-off required of the PA and thus a high power efficiency. In the following sections, an optimization technique will be developed to implement the deterministic PAR constraint in (4) as well as the spectral mask requirement, while minimizing the symbol-wise EVM. We also relate the symbol-wise EVM minimization to the RMS EVM constraint.

III. The EVM optimization problem framework

According to the standards, OFDM subcarriers can be categorized into two non-overlapping sets: data subcarriers and free subcarriers denoted by sets of indices $\mathcal{K}_d$ and $\mathcal{K}_f$, respectively. They have the cardinality $|\mathcal{K}_d| = d$ and $|\mathcal{K}_f| = f$ so that $d + f = N$. For simplicity, pilot subcarriers are not considered in this paper. However, the results of this paper can be extended to systems with pilot signals.

On the data subcarriers, encoded data $X_{\mathcal{K}_d} \in \Omega$ are transmitted, where Ω is an ideal constellation of the modulation schemes specified in the standards. On the free subcarriers, any complex valued $X_{\mathcal{K}_f} \in \mathcal{C}$ can be transmitted subject to the spectral mask constraint specified in the standards,

$$E[|X_k|^2] \leq \mathcal{M}_k, \; k \in \mathcal{K}_f, \tag{5}$$

where $\mathcal{M}_k$ represents the spectral mask.

On the data subcarriers, $X_{\mathcal{K}_d}$, a certain amount of in-band distortion is allowed. First define the symbol-wise error vector magnitude (EVM)

$$\epsilon(\mathbf{X}, \mathbf{X}^\dagger) = \sqrt{\frac{\frac{1}{d}\sum_{k\in\mathcal{K}_d} |X_k - X_k^\dagger|^2}{P_0}}, \tag{6}$$

where $\mathbf{X}^\dagger$ denotes the modified signal whose time-domain counterpart $\mathbf{x}^\dagger$ has a lower PAR than the original signal $\mathbf{x}$. P_0 is the average power of the data subcarriers or the constellation itself [2]. Such in-band distortion arises as the result of the PAR reduction. The RMS EVM must stay below some threshold ε, which is a constraint set forth in the standards, i.e.,

$$\mathrm{RMS\ EVM} = \sqrt{E[\epsilon(\mathbf{X}, \mathbf{X}^\dagger)^2]} \leq \varepsilon. \tag{7}$$

In this paper, we propose an optimization technique to minimize the symbol-wise EVM while simultaneously satisfying the deterministic PAR constraint in (4) and the spectral mask constraint in (5). For a given OFDM symbol, the optimized symbol-wise EVM value may be higher or lower than the RMS EVM threshold ε. In general, the higher the PAR threshold γ, the smaller the optimized symbol-wise EVM tends to be. The PAR threshold γ is a user-defined parameter. Certainly γ is (much) less than the dynamic range of the device. Thus, under the deterministic constraint in (4), clipping never occurs to the modified signal $\mathbf{x}^\dagger$. We can determine, by off-line Monte Carlo simulations, the lowest possible γ whose corresponding RMS EVM performance (measured over a large number of OFDM symbols) and spectral characteristics meet the standard's requirements.

We formulate the symbol-wise EVM optimization problem as follows:

$$\underset{\mathbf{X}^\dagger}{\text{Minimize}} \quad e \tag{8}$$
$$\text{Subject to} \quad \sqrt{dP_0}\cdot\epsilon(\mathbf{X},\mathbf{X}^\dagger) \le e \tag{9}$$
$$|X_k^\dagger|^2 \le \mathcal{M}_k,\ k\in\mathcal{K}_f \tag{10}$$
$$\sum_{k\in\mathcal{K}_d}\Re(X_k^*(X_k^\dagger - X_k)) \ge -\frac{e^2}{2} \tag{11}$$
$$\mathbf{x}^\dagger = \text{IFFT}_L(\mathbf{X}^\dagger) \tag{12}$$
$$|x^\dagger[n]| \le \frac{\sqrt{\gamma}}{\sqrt{LN}}\|\mathbf{x}\|_2,\ n\in[0, LN-1], \tag{13}$$

where $\Re(x)$ denotes the real part of x. In particular, although (10) provides a stricter constraint than (5), it is easier to solve symbol-wise. Additionally, since directly constraining the PAR leads to a complicated non-convex problem, we follow the derivation in [8] and separately restrict the peak power as in (13) and the average power on the data subcarriers according to (11). Eqs. (11) and (13) guarantee that the PAR of the optimized signal $\mathbf{x}^\dagger$ will not exceed the threshold γ.

IV. Customized interior-point method

Let us use an $N\times N$ matrix $\mathbf{S}$ to indicate the locations of the data subcarriers, whose (m,n)-th element is given by

$$S_{m,n} = \begin{cases} 1, & m=n\in\mathcal{K}_d, \\ 0, & m\in\mathcal{K}_f,\ \text{or}\ n\in\mathcal{K}_f,\ \text{or}\ m\ne n. \end{cases} \tag{14}$$

Thus, $\mathbf{S}$ consists of block identity and zero matrices. The symbol-wise EVM in (6) can be re-written as

$$\epsilon(\mathbf{X},\mathbf{X}^\dagger) = \frac{1}{\sqrt{dP_0}}\|\mathbf{S}(\mathbf{X}-\mathbf{X}^\dagger)\|_2. \tag{15}$$

Using the indicator matrix notation $\mathbf{S}$, the constraint in (9) becomes

$$\|\mathbf{S}(\mathbf{X}-\mathbf{X}^\dagger)\|_2 \le e. \tag{16}$$

Moreover, the average power constraint (11) can also be expressed as

$$\Re(\mathbf{X}^H\mathbf{S}(\mathbf{X}^\dagger-\mathbf{X})) \ge -\frac{e^2}{2}. \tag{17}$$

The inequality constrained optimization problem in (8)-(11) can be approximately formulated as an equality constrained problem with the inequality constraints implicit in the objective function [10, Ch. 11]:

$$\underset{\mathbf{X}^\dagger}{\text{Minimize}} \quad f_o(\mathbf{X},\mathcal{M}_k,e,\mathcal{K}_f,\mathbf{S}) \tag{18}$$
$$\text{Subject to} \quad \mathbf{x}^\dagger = \text{IFFT}_L(\mathbf{X}^\dagger), \tag{19}$$

where

$$f_o(\mathbf{X},\mathcal{M}_k,e,\mathcal{K}_f,\mathbf{S}) = e + \mathrm{I}_-[-\delta_e] + \sum_{n=0}^{LN-1}\mathrm{I}_-[-\delta_n] + \sum_{k\in\mathcal{K}_f}\mathrm{I}_-[-\phi_k] + \mathrm{I}_-[-\delta_a], \tag{20}$$

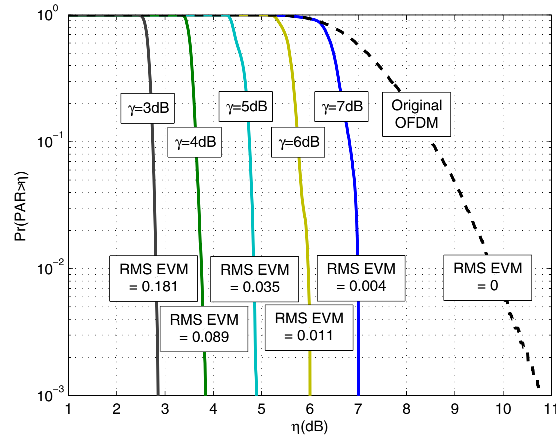

Fig. 1. CCDF of the PAR of the EVM-optimized signal $\mathbf{x}^\dagger$ for different PAR thresholds γ; the number of free subcarriers is $f=12$. CCDF of the PAR of the original OFDM signal is also shown.

$\mathrm{I}_-[\cdot]$ is the indicator function for the nonpositive reals,

$$\mathrm{I}_-[u] = \begin{cases} 0, & u\le 0, \\ \infty, & u>0. \end{cases} \tag{21}$$

Let us denote

$$\delta_e = e^2 - \|\mathbf{S}(\mathbf{X}-\mathbf{X}^\dagger)\|_2^2, \tag{22}$$
$$\delta_n = \frac{\gamma}{LN}\|\mathbf{x}\|_2^2 - |x^\dagger[n]|^2,\ n\in\{0,\cdots,LN-1\}, \tag{23}$$
$$\phi_k = \begin{cases} \mathcal{M}_k - |X_k^\dagger|^2, & k\in\mathcal{K}_f, \\ 0, & \text{otherwise}, \end{cases} \tag{24}$$
$$\delta_a = \frac{e^2}{2} + \Re(\mathbf{X}^H\mathbf{S}(\mathbf{X}^\dagger-\mathbf{X})), \tag{25}$$

all of which should be positive as required by the constraints.

By using the standard log-barrier interior-point method as described in [10, Ch. 11], an iterative algorithm can be constructed to efficiently solve the equivalent symbol-wise EVM optimization problem in (18)-(19). We will explain the details of the customized IPM algorithm in an expanded version of this paper. Next, we show PAR reduction and EVM minimization performance of the proposed optimization technique where the customized IPM is used.

V. Numerical results

In the following simulations, $L=4$, $N=64$ and the spectral mask as defined in the IEEE 802.11a standard was used [2]. The OFDM symbols were drawn from a QPSK constellation for which the RMS EVM threshold is $\varepsilon=0.1$.

In our EVM optimization formulation, since eq. (11) lower-bounds the average power, and eq. (13) upper-bounds the peak power, the optimized OFDM signal is expected to yield PAR values that are strictly less than the specified threshold γ as expressed by (4). Fig. 1 shows the complimentary cumulative distribution function (CCDF) curves of the resulting PAR values for various thresholds $\gamma=3,4,\cdots,7$dB. The corresponding RMS EVM values of the PAR-reduced

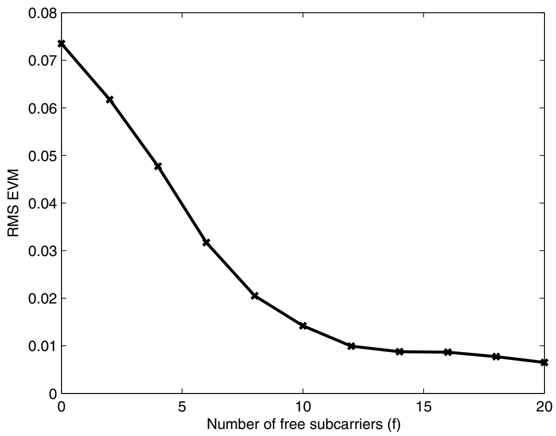

Fig. 2. RMS EVM of the EVM-optimized signal $\mathbf{x}^{\dagger}$ for different numbers of free subcarriers; the PAR threshold is $\gamma = 6$dB.

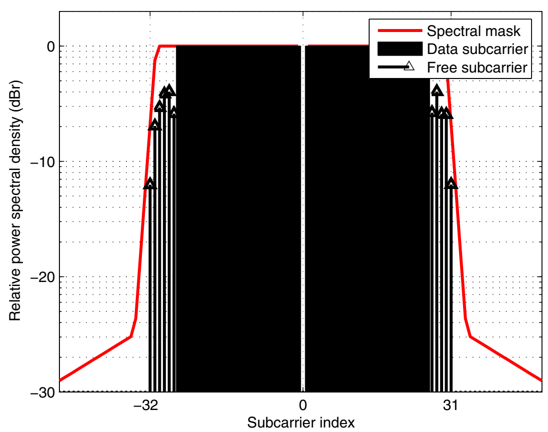

Fig. 3. One realization of the EVM-optimized power allocation for $\mathbf{X}^{\dagger}$; the PAR threshold is $\gamma = 6$dB.

waveforms are also indicated. In this example, $f = 12$ free subcarriers were used as allocated in IEEE 802.11a [2]. We observe that these curves do not go beyond the $\eta = \gamma$ lines, thus confirming that the customized IPM does implement the intended deterministic PAR constraint. In this example, the RMS EVM requirement of $\varepsilon = 0.1$ can be met by using a PAR threshold $\gamma = 4$dB. When comparing with the original OFDM signal, a PAR reduction of 7dB was readily achieved at the 10^{-3} CCDF level.

The RMS EVM value can be further reduced if one is allowed to use more free subcarriers. Fig. 2 shows the achievable RMS EVM as a function of f for a given PAR threshold $\gamma = 6$dB. The more the number of free subcarriers, the lower the RMS EVM. When $f = 12$ free subcarriers are used, RMS EVM ≈ 0.01 can be achieved. Comparing with the previous example, we see that the 10-fold decrease (from 0.1 to 0.01) in the RMS EVM is due to a much more relaxed PAR threshold γ (from 4dB to 6dB). In this particular example, there is a diminishing return in further RMS EVM reduction beyond $f = 12$.

One realization of the optimized power allocation for $\mathbf{X}^{\dagger}$ is shown in Fig. 3. The PAR threshold was $\gamma = 6$dB in this case. It also shows that the optimized signal $\mathbf{X}^{\dagger}$ meets the spectral mask constraint imposed by the standard.

VI. Conclusions

In this paper, we have developed a PAR reduction algorithm that can guarantee a low fixed power back-off level γ, does not violate the spectral mask constraint and leads to a minimal symbol-wise EVM. We formulated a convex optimization approach, which experiences no local minima and can be efficiently implemented by a customized interior-point method. The PAR threshold γ is chosen such that the resulting RMS EVM value meets the standard's requirement. The benefits of the proposed algorithm are reduced sensitivity to system nonlinearities and improved power efficiency.

Acknowledgment

This work was supported in part by the U. S. Army Research Laboratory under the Collaborative Technology Alliance Program, Cooperative Agreement DAAD19-01-2-0011 and in part by the National Science Foundation Graduate Research Fellowship Program.

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