ASSESSING PEAK-TO-AVERAGE POWER RATIOS FOR COMMUNICATIONS APPLICATIONS

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ABSTRACT

High peak-to-average ratio (PAR) is a significant drawback for orthogonal frequency division multiplexing (OFDM). Accordingly, many authors have proposed different PAR reduction methods that rely on different trade-offs. It is important to quantify the effectiveness of an individual PAR reduction method. The PAR at a certain probability level (10^{-4}, 10^{-5}, etc) has proven to be a popular but inconsistent metric. We propose to use the expected PAR, E[PAR], as a metric to compare the different PAR reduction methods. In this paper we derive a closed form expression for the E[PAR] for the selected mapping (SLM) approach. We also evaluate SLM in terms of its computational cost as characterized by the expected number of different phase mappings to be tried. [1]

1. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) has become a popular modulation method in high-speed wireless networks. By partitioning a wideband fading channel into flat narrowband channels, OFDM is able to mitigate the detrimental effects of multipath fading.

The drawback is that OFDM signals can have large peak-to-average power ratios (PARs). A high PAR will require a large power back off in the transmitting amplifier, which translates to low power efficiency. This is a critical issue in portable wireless devices where power is at a premium. Several schemes have been proposed to reduce the PAR of OFDM signals. Some, like companding and hard clipping, are simple signal transformations. Others, including selected mapping (SLM), partial transmit sequence, piecewise-scales transform, tone injection, tone reservation and block coding approaches are algorithmic in nature; see [5], [7], [8], [9], [10], [11], [12] and references therein. In each of these schemes, there is a tradeoff between PAR reduction and a host of other variables, including bit-error rate, complexity, and data rate.

With all of these different methods it is important to create metrics to quantify the performance of a PAR reduction scheme. The PAR reducing capability of an algorithm is often quoted at a specific (low) CCDF (complementary cumulative distribution function) value (10^{-3} in [2], 10^{-4} in [4], 10^{-5} in [7], 10^{-6} in [3]). However, as argued in [13], such metric can be misleading. Based on simulation and measurement results, the authors of [13] showed that filtered offset QPSK (OQPSK) exhibited higher spectral regrowth than filtered QPSK although the former has a lower PAR than the latter when measured at the 10^{-6} CCDF level. Such investigation points to the importance of using appropriate metrics in the evaluation of PAR.

In this paper, we propose an expected-value analysis of PAR reduction schemes as another metric. The expectation of a reduction scheme's PAR can be used as an easily interpretable and comparable metric. We also carry out an expectation analysis for the number of independent phase mappings required for SLM to guarantee a certain PAR.

2. SYSTEM MODEL

For our analysis we are only interested in examining the PAR of various OFDM signals. Therefore, the system model used in this paper is a simplified version of the practical OFDM model. Specifically, we ignore

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the guard interval because it does not contribute to the PAR [7], assume that any pulse shaping in the transmitter is flat over all of the subcarriers, and deal only with the PAR of the baseband signal.

With these assumptions, the OFDM signal is simply the inverse Fourier transform of a series of N points from the signal constellation, where N is the number of subcarriers. For the continuous-time case we have

$$x(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k \, e^{j\frac{2\pi kt}{T_s}}, \quad 0 \leq t \leq T_s, \tag{1}$$

where T_s is the symbol period and X_k is the symbol at the k^{th} subcarrier. The discrete-time case is

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k \, e^{j2\pi kn/N}. \tag{2}$$

The PAR for a Nyquist rate sampled OFDM symbol is defined as

$$\mathrm{PAR}\{x[n]\} = \frac{\max_{0\leq n\leq N-1} |x[n]|^2}{E[|x[n]|^2]}. \tag{3}$$

3. EXPECTATION OF PAR IN SLM

Selected mapping (SLM), first introduced in [5], has emerged as a promising way to reduce the PAR of OFDM signals. SLM takes advantage of the fact that the PAR of an OFDM signal is very sensitive to phase shifts in the frequency-domain data. PAR reduction is achieved by multiplying independent phase sequences to the original data and determining the PAR of each phase sequence/data combination. The combination with the lowest PAR is transmitted.

In the discrete-time case, where only the Nyquist sampled analog signal is examined, the cumulative distribution function (CDF) can be easily derived if certain assumptions are made. First we assume that N, the number of subcarriers, is large enough so that the discrete-time domain signal has an approximate complex Gaussian distribution [1]. It then follows that the instantaneous power of the discrete-time domain samples is Chi-Squared distributed. According to Theorem 4.4.1 of [1], after the IFFT each discrete time sample can be treated as independent of all other samples. With these two approximations, the probability that the power of at least one $x[n]$ out of N samples is above a given level, γ, is

$$\Pr(\max_{0\leq n\leq N} |x[n]|^2 > \gamma) = 1 - (1 - e^{-\gamma})^N. \tag{4}$$

If $E[|x[n]|^2]$ is set to one, then eq. (4) is also an expression of $\Pr(\mathrm{PAR} > \gamma)$.

For SLM we assume that each mapping is statistically independent of all other mappings. Thus, the probability that the PAR is above a level γ for a SLM signal is $(\Pr(PAR_{x[n]} > \gamma))^D$ or

$$\Pr(PAR_{SLM} > \gamma) = \left(1 - (1 - e^{-\gamma})^N\right)^D. \tag{5}$$

Finally, the CDF of the PAR of a discrete-time SLM signal is given by

$$F_{\mathrm{PAR}}(x) = 1 - \left(1 - (1 - e^{-x})^N\right)^D. \tag{6}$$

The probability density function (PDF) is simply the derivative of the CDF,

$$\begin{aligned} f_{\mathrm{PAR}}(x) = & \left[-D(1 - (1 - e^{-x})^N)^{D-1} \right] \\ & \cdot \left[-N(1 - e^{-x})^{N-1} \right] e^{-x}. \end{aligned} \tag{7}$$

Since PAR $\geq$ 1, the expected value of the PAR is given by

$$E[\mathrm{PAR}] = \int_1^\infty x \cdot f_{\mathrm{PAR}}(x) dx. \tag{8}$$

In the appendix, we show that

$$E[\mathrm{PAR}] = \frac{(D-1)!}{N} \sum_{k=1}^{N} \prod_{p=0}^{D-1} \frac{1}{(\frac{k}{N} + p)}. \tag{9}$$

Equation (9) shows that there is a simple closed form expression for the expectation of the PAR of a SLM signal for any N and D. The $D = 1$ case corresponds to expectation of the PAR of a discrete-time OFDM signal without SLM and simplifies to

$$E[\mathrm{PAR}]\Big|_{D=1} = \sum_{k=1}^{N} \frac{1}{k}. \tag{10}$$

	E[PAR] (dB)			
N	D = 1 (no SLM)	D = 2	D = 10	D = 30
64	6.76	6.08	5.04	4.55
128	7.35	6.76	5.88	5.48
256	7.87	7.35	6.59	6.25
512	8.34	7.87	7.20	6.91
1024	8.76	8.34	7.74	7.48

Table 1. The PAR value corresponding to various Ds for different Ns.

This is an interesting result when taken in light of the commonly-cited worst-case PAR for a given N, which assumes all of the subcarriers are the same symbol. In that case $PAR = N$. Equation (10) reveals that the E[PAR] increases much slower than linearly; in fact, even for $N = 2^{27}$, E[PAR] is only 12.8 dB. Table 1 gives several examples of E[PAR].

Figure 1 is a plot of the sample average PAR ($\hat{\mu}$) versus the theoretical E[PAR] for OFDM signals where the frequency domain symbols were taken from the 16PSK or 16QAM constellations. It shows that the theoretical E[PAR] is very close to the sample estimates even for small N. This result is important as a way of verifying that the errors associated with the commonly made assumptions are very small.

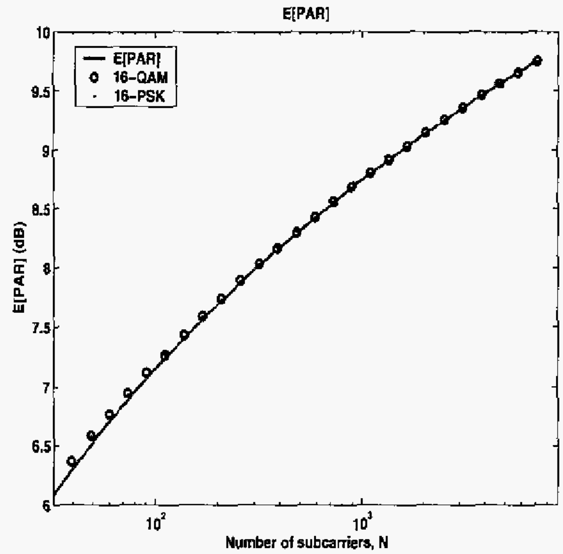

Fig. 1. The sample means versus the theoretical E[PAR] when X_k is taken from 16PSK or 16QAM. 30,000 OFDM blocks were used to obtain the sample mean estimates.

4. EXPECTATION OF D

Another application of the first order statistic in SLM is in determining the expectation of the number, D, of independent phase sequences needed to guarantee that a given ODFM symbol has a PAR less than some level γ. This is important in practical applications because the transmission PA is peak-power limited, which translates to a fixed clipping level; therefore, there is no reason to further reduce the PAR of an OFDM symbol if its PAR is already low enough to avoid clipping [6]. Accordingly, any practical implementation of SLM would have some threshold PAR, γ, where once the signal PAR is below this level, the signal is sent without any further phase mapping. Thus, we are interested in an expression for the expected number of phase mappings necessary to guarantee a certain PAR.

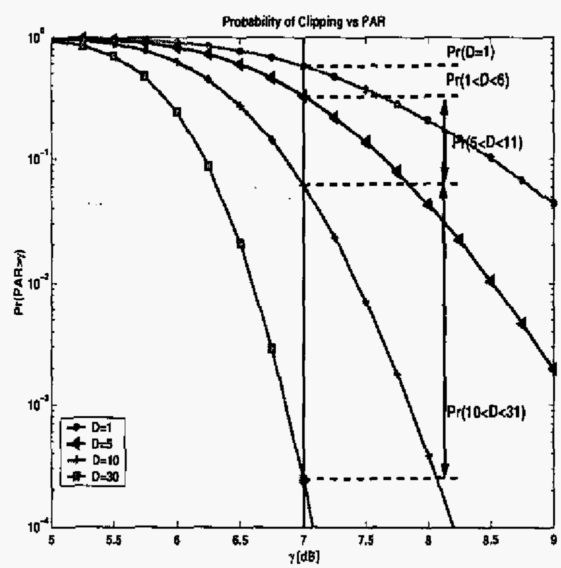

Fig. 2. CCDF for $N = 128$ for various Ds. The vertical line represents the PAR value at which we guarantee no clipping occurs.

Figure 2 shows the PAR CCDF for $D \in$

[1 5 10 30]. The plot shows that for a guarantee level (GL) of 7dB there is a 70% chance that it will be necessary to use SLM to avoid clipping. Accordingly, there is a 30% chance that we will need $D > 5$. With this understanding the expectation comes easily. It is simply D multiplied by the probability of needing that D. The probability that SLM with D mappings can avoid a clipping level, γ, is

$$\begin{aligned}&\Pr(D = d|Pr(\text{PAR} > \gamma)) \\ &= Pr(\text{PAR} > \gamma)^{d-1} - Pr(\text{PAR} > \gamma)^{d}.\end{aligned} \tag{11}$$

Thus, $E[D]$ for a general SLM scheme is

$$\begin{aligned}E[D] &= \sum_{k=1}^{\infty} k\left[Pr(\text{PAR} > \gamma)^{k-1} - Pr(\text{PAR} > \gamma)^{k}\right] \\ &= 1 + \frac{Pr(\text{PAR} > \gamma)}{1 - Pr(\text{PAR} > \gamma)}.\end{aligned} \tag{12}$$

If we substitute for the probability of clipping in (5), the expectation becomes

$$E[D] = (1 - e^{-\gamma})^{-N}. \tag{13}$$

Obviously a practical system would not be able to evaluate for an infinite number of Ds. Instead a practical system would give up after $D = D_{max}$, where D_{max} is that maximum possible number of sequences tested. In this case, the expectation in (12) is an upper bound on $E[D]$ for $1 \leq D \leq D_{max}$.

5. CONCLUSIONS

In this paper we have obtained two important expectations involved in OFDM selected mapping systems. The first was the expected PAR of a SLM OFDM signal. This expectation provided insight into the validity of the assumption made in deriving the commonly-cited CCDF for the critically-sampled SLM OFDM signal.

The second expectation is on the SLM parameter, D, which is the number of phase mappings needed in an SLM scheme. In a transmission system the peak power is fixed, thus the clipping level can be considered fixed. Also, each mapping tested requires, among other computational and power resources, an IFFT. In this paper we derived an expression for $E[D]$, which is the expected value of D such that clipping of a given level γ is completely avoided. This is a useful measure for a peak-power limited transmission system because it indicates the expected amount of computation that will be necessary to guarantee that no symbols are clipped.

6. APPENDIX: PROOF OF EQ. (9)

The expectation is given by

$$E[\text{PAR}] = \int_{1}^{\infty} x\, f_{\text{PAR}}(x)dx. \tag{14}$$

The integral above yields a messy result because the lower limit is one, which is due to the fact that $PAR \geq 1$. However, as $N \to \infty$ the lower limit on the expectation integral can be changed from one to zero. In practice, altering the limits of the integral, even for $N = 20$, only yields an error of 10^{-5}. And for practical values, where $N \geq 64$, the error is less than 10^{-11}. It is also important to note that the complex Gaussian assumption made in deriving the PDF in (7) is more likely to introduce error than the limit change. Thus, we have

$$E[\text{PAR}] \overset{N\to\infty}{=} \int_{0}^{\infty} x\, f_{\text{PAR}}(x)dx. \tag{15}$$

With a simple change of variables ($x \triangleq 1-(1-e^{-u})^{N}$) we have

$$E[\text{PAR}] = \int_{1}^{0} Dx^{D-1} \ln\left[1 - (1-x)^{\frac{1}{N}}\right]dx. \tag{16}$$

Note that with

$$\begin{aligned}&\int \ln[1 - (1-x)^{\frac{1}{N}}]dx \\ &= x[\ln(1 - (1-x)^{\frac{1}{N}})] + \sum_{k=1}^{N} \frac{1}{k}[(1-x)^{\frac{k}{N}} - 1]\end{aligned} \tag{17}$$

the expectation integral becomes an integration by parts problem. Let's define two functions, R and G in order to use simpler expressions:

$$\begin{aligned}R(x,N) &\triangleq x[\ln(1 - (1-x)^{\frac{1}{N}})] + \sum_{k=1}^{N} \frac{1}{k}[(1-x)^{\frac{k}{N}} - 1] \\ G(x,N) &\triangleq \sum_{k=1}^{N} \frac{1}{k}[(1-x)^{\frac{k}{N}} - 1].\end{aligned} \tag{18}$$

Now the indefinite expectation integral, (16), becomes

$$\int x^{D-1} \ln[1-(1-x)^{\frac{1}{N}}]dx$$
$$= x^D R(x,N) - (D-1)\int x^{D-1}[\ln(1-(1-x)^{\frac{1}{N}})]dx$$
$$- (D-1)\int x^{D-2} G(x,N)dx. \tag{19}$$

If we move the middle term on the right hand side of (19) to the left hand side, we have

$$D\int x^{D-1} \ln[1-(1-x)^{\frac{1}{N}}]dx$$
$$= x^D R(x,N) - (D-1)\int x^{D-2} G(x,N)dx. \tag{20}$$

The left over integral can also be done with integration by parts. The result is

$$D\int x^{D-1} \ln[1-(1-x)^{\frac{1}{N}}]dx$$
$$= x^D R(x,N) - \sum_{q=1}^{D-1}\Bigg\{(-1)^{q+1}x^{D-1-q}$$
$$\cdot \sum_{k=1}^{N}\left[\frac{(-1)^q (1-x)^{\frac{k}{N}+l}}{k\prod_{p=1}^{q}(\frac{k}{N}+p)} - \frac{x^q}{k\,q!}\right]\prod_{l=1}^{q}(D-l)\Bigg\}, \tag{21}$$

which we define as $\Upsilon(x,N,D)$. Therefore,

$$E[\text{PAR}] = \Upsilon(0,N,D) - \Upsilon(1,N,D). \tag{22}$$

After simplification, $\Upsilon(1,N,D)$ becomes

$$\Upsilon(1,N,D) = -\sum_{k=1}^{N}\frac{1}{k} - \sum_{q=1}^{D-1}\sum_{k=1}^{N}\frac{(-1)^q}{k\,q!}\prod_{l=1}^{q}(D-l). \tag{23}$$

It can be shown that $\frac{1}{q!}\prod_{l=1}^{q}(D-l) = \binom{D-1}{q}$. Thus, $\Upsilon(1,N,D)$ is the alternating sum of a binomial series which is $0 \;\forall\; N$ and D:

$$\Upsilon(1,N,D) = \sum_{q=0}^{D-1}\sum_{k=1}^{N}\frac{(-1)^{q+1}}{k}\binom{D-1}{q} = 0. \tag{24}$$

Therefore the expectation of the PAR is $\Upsilon(0,N,D)$. Observe that all terms of (21) in the summation over q go to zero except $q = D-1$; thus,

$$E[\text{PAR}] = \frac{(D-1)!}{N}\sum_{k=1}^{N}\prod_{p=0}^{D-1}\frac{1}{(\frac{k}{N}+p)}. \tag{25}$$

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