Peak Reduction for Multiplexed Asynchronous OFDM-FDMA

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***Abstract*—Peak-to-Average Power Ratio (PAR) reduction algorithms have been thoroughly studied for single system multicarrier signals that tend to have low power efficiency. However, an emerging application for PAR reduction, namely asynchronous multiplexed signals, has not been examined. This paper frames the problem arising in modern base stations where multiple frequency-division OFDM waveforms are multiplexed so that the combined signal is transmitted through a single PA. PAR reduction on each constituent signal is not sufficient to reduce the multiplexed signal dynamic range. Thus, we propose a symbol-wise phase optimization procedure, multi-channel partial transmit sequences (MCPTS), to reduce the multiplexed signal dynamic range.**

***Index Terms*—Orthogonal frequency division multiplexing (OFDM), OFDM-based frequency-division multiple access (OFDM-FDMA), instantaneous-to-average power ratio (IAR)**

I. Introduction

Orthogonal frequency division multiplexing (OFDM) has been widely adopted by various modern communication standards [1]–[3]. A big advantage of OFDM is its robustness in the presence of multipath signal propagations. Additionally, OFDM-based frequency-division multiple access (OFDM-FDMA) systems were defined in several communication standards to support multiuser communications. In OFDM-FDMA, the frequency spectrum is channelized into adjacent fragments on each of which a (or multiple) separate OFDM user(s) can communicate. For instance, 5-20MHz and 1.25-20MHz channelization schemes were specified in the IEEE 802.11a and IEEE 802.16 standards, respectively [1], [2]. OFDM-FDMA is a promising technique for high data rate communications in future cellular systems. It is currently a strong candidate for the downlink multiple access scheme in the Long Term Evolution of cellular systems under consideration by the Third Generation Partnership Project (3GPP) [4]. OFDM-FDMA is also of intensive interests in user cooperation schemes where multiple OFDM symbols should be sent on different frequency bands from each transmitter [5].

However, OFDM as well as OFDM-FDMA waveforms exhibit very pronounced envelope fluctuations. They pose strict hardware restrictions for the transmitters whose RF front-end components, including the power amplifier (PA), should possess large linear dynamic ranges. To achieve this linearity, the amplifiers have to operate with a large back-off from their peak power, yielding low power efficiency and low effectively transmitted signal power. For OFDM systems, several methods have been proposed to improve the power efficiency and other pertinent measures of merits in peak power limited transmitters; see [6]–[8] for an overview. To improve the performance in OFDM-FDMA, a straightforward idea is to apply these methods and assign one PA to each channel individually, i.e. multiple PAs for every transmitter. However, this is not practical as it will considerably raise the cost and the size of transmitters when more than a few user waveforms have to be transmitted. Instead, a single wideband PA transmitting the signals from all channels, i.e. a single-PA transmitter, is preferred. In this case, reducing the dynamic range of the combined OFDM-FDMA waveform is the primary concern.

The nonlinear distortions introduced by a general power amplifier in the OFDM-FDMA systems have been studied in [9]. In [10], a solution to the power efficiency problem has been brought forward for the single-PA OFDM-FDMA transmitters in which the OFDM symbols are synchronously transmitted. The multi-channel partial transmit sequences (MCPTS) method was proposed for the synchronous OFDM-FDMA system and shown to achieve a significant peak-to-average power ratio (PAR) reduction by optimizing over the phases of all channels [10]. To carry on the discussion, a more general OFDM-FDMA model will be dealt with in this paper. In the generalized case, the OFDM symbols can be asynchronously transmitted and idle periods may occur between symbols. The dynamic range of the generalized OFDM-FDMA signals will be analyzed at first. Then, the MCPTS method will be extended to reduce the dynamic range and improve the power efficiency performance for the generalized OFDM-FDMA transmitters.

It is worthwhile to point out that, with a few appropriate modifications, the discussions in this paper can be further generalized to heterogeneous multiple access systems where various modulation schemes are used simultaneously. This situation is of particular practical interest in modern cellular systems where a single base station must support multiple heterogeneous waveforms from legacy to 4G from a single site.

II. System Model

The structure diagram of the generalized OFDM-FDMA transmitter is shown in Fig. 1a. The whole available frequency spectrum is fragmented into M adjacent channels on each of which a (or multiple) separate user(s) can communicate. The spectrum diagram is plotted in Fig. 1b. The center frequency f_m and assigned bandwidth B_m of the mth channel ($m \in \{1, \cdots, M\}$) should satisfy non-overlapping condition that $f_{l+1} - f_l \geq \frac{1}{2}(B_{l+1} + B_l)$ ($l \in \{1, \cdots, M-1\}$).

In each channel, OFDM is the presumed modulation scheme. For the mth OFDM channel, N_m subcarriers are

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assumed being adopted and $T_m = \frac{2\pi N_m}{B_m}$ denotes the symbol duration. For notational simplicity, the OFDM cyclic prefix (CP) is omitted. With respect to the dynamic range issue concerned in this paper considering the CP is not necessary, but appropriate modifications can be applied to include it without affecting the following discussions. Using n to index samples, the L-times oversampled OFDM waveform can be represented as

$$x_m[n] = \sum_{k=0}^{N_m-1} X_m(k) \exp\left\{ j2\pi(n - n_m(s)) \frac{k}{LN_m} \right\},$$
$$n \in \{n_m(s), \cdots, n_m(s) + LN_m - 1\}, \quad (1)$$

where $n_m(s)$ is the starting sample index of the sth OFDM symbol in the mth channel. The frequency-domain OFDM symbol $\mathbf{X}_m = [X_m(0), \cdots, X_m(N_m - 1)]^T$ keeps unchanged in every symbol duration $[n_m(s), n_m(s) + LN_m - 1]$. In contrast to the synchronous model where $n_m(s) = n_l(s)$ and $n_m(s) = s \cdot LN_m$ ($\forall m, l \in \{1, \cdots, M\}$ and $s \in \mathbb{N}$), in the generalized model, the OFDM symbols may not be continuously transmitted. In other words, the idle period between two symbols $\tau_m \triangleq n_m(s + 1) - n_m(s) - LN_m + 1$ can be randomly distributed with the support of $\tau_m \in \mathbb{N}$. Moreover, the starting and the ending moments of the symbols in the mth channel are not necessarily aligned with those in other channels, i.e. $n_m(s)$ can be different from $n_l(s)$ ($\forall m \neq l \in \{1, \cdots, M\}$). Therefore, though the signals are sampled and represented by the same sampling clock, the generalized model is asynchronous on the symbol level.

The generalized asynchronous model is flexible and fits the modeling needs of a broad range of services, e.g. rate-variable video and audio via OFDM-FDMA systems. While, the arrival distribution of OFDM symbols depends on higher-layer modeling, channel usage modeling is not the focus of this paper. For simplicity, the following assumptions are taken in the analysis:

1) Independent, adjacent and isometric channels are assumed, i.e. $f_{l+1} - f_l = B$ ($l \in \{1, \cdots, M - 1\}$), $B_m = B$ and $N_m = N$ ($\forall m \in \{1, \cdots, M\}$);
2) The idle periods between symbols are assumed integers in units of samples and have $\tau_m = \lfloor e_m(\lambda_m) \rfloor$ where $\lfloor x \rfloor$ denotes the largest integer not greater than x and $e_m(\lambda_m)$ is an exponentially distributed random variable with the rate parameter λ_m, i.e. the probability density function (PDF) of e_m is
$$f(e_m; \lambda_m) = \lambda_m \exp\{-\lambda_m e_m\},$$
for $e_m \geq 0$;
3) In addition, more complicated assumptions can be adopted to model practical applications. For example, we can model the transmission of symbols that are transmitted in bursts, where the number of symbols transmitted in each burst or "packet" is p_m ($p_m \geq 1$ and $p_m \in \mathbb{Z}$).

Before passing to the power amplifier and transmitting antenna, the signals of M channels are summed together. The passband waveform samples passed through the PA are thus

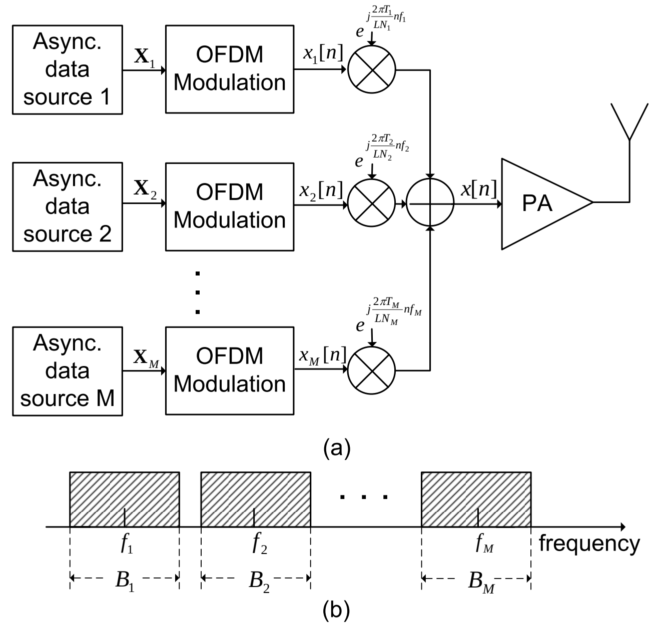

Fig. 1. The (a) structure and (b) frequency spectrum diagram of the generalized OFDM-FDMA transmitter.

$$x[n] = \sum_{m=1}^{M} x_m[n] \exp\left\{ j2\pi f_m \frac{nT_m}{LN_m} \right\}, \; n \in \mathbb{N}. \quad (2)$$

For OFDM and synchronous OFDM-FDMA systems, peak-to-average power ratio (PAR) is usually used to characterize the dynamic range of the waveform. However, in the asynchronous OFDM-FDMA system, the signals in Eq. (2) can no longer be symbol-wise segmented because OFDM symbols from multiple users will have some probability of overlapping in time but not lining up along a symbol period.

Therefore, PAR, a symbol-wise metric, does not apply for this case. To measure the dynamic range of the asynchronous OFDM-FDMA signal, the instantaneous-to-average power ratio (IAR) should be used instead. The IAR is a sample-wise metric defined as

$$\mathrm{IAR} = \frac{|x[n]|^2}{\sigma_x^2}, \quad (3)$$

where $\sigma_x^2 = E[|x[n]|^2]$ is the average power of $x[n]$. In peak power limited PAs, the power efficiency can be defined as

$$\rho = \frac{\sigma_x^2}{P_{\mathrm{dc}}}$$

where P_{dc} is the power drawn from the DC source. Specifically, for a class-A PA with the peak power limit P_{peak}, the input back-off (IBO) is defined as $\mathrm{IBO} = \frac{P_{\mathrm{peak}}}{\sigma_x^2}$ and the power efficiency becomes $\rho = \frac{\sigma_x^2}{2P_{\mathrm{peak}}} = \frac{1}{2\mathrm{IBO}}$.

III. The IAR distribution of asynchronous OFDM-FDMA signals

Based on the preceding discussion, the IAR distribution of the asynchronous OFDM-FDMA signals is of concern and can be theoretically analyzed. According to the Central Limit Theorem, the OFDM waveform exhibits an approximate complex Gaussian distribution when N is reasonably large [6], [11]. Composed of the sum of multiple OFDM signals, the OFDM-FDMA signal also converges to the complex Gaussian distribution. Therefore, its IAR, which is proportional to the instantaneous power of the samples, follows an approximate exponential distribution. However, at different sampling moments, some channels may be transmitting signals (i.e. active) while others are inactive (i.e. in the idle periods). Hence, the IAR is not a stationary random variable and has a mean value that may vary sample by sample. The IAR distribution is thus dependent on the sample-based channel usage state (CUS) and the average IAR distribution should be evaluated by averaging over CUS.

The sample-based CUS can be defined as $\mathcal{C} = (c_1, \cdots, c_M)$ where c_m is a binary variable indicating the state of the mth channel at the given sampling moment. $c_m = 0$ if the mth channel is inactive while $c_m = 1$, otherwise. Consequently, the cardinality of $\mathcal{C}$ is $|\mathcal{C}| = 2^M$ and its stationary probability can be denoted as $\pi(\mathcal{C}) \triangleq \Pr\{\mathcal{C}\}$. With the assumption of independent users,

$$\pi(\mathcal{C}) = \prod_{m=1}^{M} \Pr\{c_m\}$$

follows. For instance, with the aforementioned channel usage model, the stationary probability of CUS can be approximated as

$$\pi(\mathcal{C}) \approx \prod_{m=1}^{M} \frac{(\lambda_m T E[p_m])^{c_m}}{(1+\lambda_m T E[p_m])^2}, \qquad (4)$$

where $E[p_m]$ is the average number of symbols in each packet. The approximation in Eq. (4) is attributed to the discretization of an exponential distribution for τ_m and is generally accurate for $\lambda_m \leq 0.1$.

Based on the CUS definition, the average power σ_x^2 and the IAR distribution can be expressed explicitly. Denote the power allocated to the channels in active states as

$$P_m = E\left[|x_m[n]|^2 \middle| c_m = 1\right], \ \forall m \in \{1, \cdots, M\}. \qquad (5)$$

The average power of $x[n]$ is thus

$$\begin{aligned} \sigma_x^2 &= \sum_{\mathcal{C}} E\left[|x[n]|^2 \middle| \mathcal{C}\right] \cdot \pi(\mathcal{C}) \\ &= \sum_{\mathcal{C}} \pi(\mathcal{C}) \sum_{m=1}^{M} c_m P_m, \end{aligned} \qquad (6)$$

which becomes $\sigma_x^2 = \sum_{m=1}^{M} P_m \cdot \Pr\{c_m = 1\}$ in the case of independent channels. Therefore, PDF of the average IAR can be found as

$$\begin{aligned} f_{\mathrm{IAR}}(\eta) &= \sum_{\mathcal{C}} f_{\mathrm{IAR}}\left(\eta \middle| \mathcal{C}\right) \cdot \pi(\mathcal{C}) \\ &= \sum_{\mathcal{C}} f_{|x[n]|^2}\left(\sigma_x^2 \cdot \eta \middle| \mathcal{C}\right) \cdot \pi(\mathcal{C}) \\ &= \sum_{\mathcal{C}} \frac{\pi(\mathcal{C})}{\mathcal{P}(\mathcal{C})} e^{-\frac{\sigma_x^2 \cdot \eta}{\mathcal{P}(\mathcal{C})}}, \end{aligned} \qquad (7)$$

where $f_x(v)$ denotes the PDF for the variable x valued at v. $\mathcal{P}(\mathcal{C})$ is the average power given the CUS of $\mathcal{C}$ and has $\mathcal{P}(\mathcal{C}) = \sum_{m=1}^{M} c_m P_m$.

For given power allocation and channel usage model, the average power in Eq. (6) and the IAR distribution of Eq. (7) can be numerically assessed.

IV. The MCPTS IAR reduction method

To reduce the IAR values and thus the dynamic range of the generalized OFDM-FDMA signals, the multi-channel partial transmit sequences (MCPTS) method in [10] can be further modified and adopted. From the perspective of transmission performance, the power allocation P_m for the active channels is assumed to be prescribed. It could be determined offline according to the requirements of the quality of services, e.g. average bit error rate or throughput. In this case, the objective of MCPTS is to reduce the occurrence of clipping in the peak power limited transmitters by optimizing over the phases of the OFDM symbols in different channels.

Unlike the symbol-based MCPTS method where the optimal channel phases are determined naturally at the starting moments of the synchronous OFDM symbols [10], in the generalized OFDM-FDMA transmitters, the MCPTS method depends on the channel usage states of every sampling moment. It should be carried out only at the sampling moments when at least one new symbol begins being transmitted on the channels. Additionally, instead of PAR, IAR is used to measure the dynamic range of the generalized OFDM-FDMA signal. The optimal channel phases hereby are determined to minimize the maximum IAR (the worst-case dynamic range) of the following samples in this symbol period. Therefore, the MCPTS IAR reduction method can be modeled as follows. Denote $n^{(u)} \in \mathbb{N}$ as the uth sampling moment when there are new symbols to start transmitting on channels with the index set $\mathcal{M}^{(u)} \subseteq \{1, \cdots, M\}$. The channel phase optimization is undertaken to find

$$\theta_m^\star = \arg\min_{\theta_m} \max_{n} \frac{\left| x_{\mathrm{ex}}[n] + \sum_{l \in \mathcal{M}^{(u)}} e^{j\theta_l} x_l[n] \right|^2}{\sigma_x^2}, \qquad (8)$$

where $m \in \mathcal{M}^{(u)}$ and $n \in \{n^{(u)}, \cdots, n^{(u)} + LN - 1\}$. $\mathbf{x}_{\mathrm{ex}}^{(u)} \triangleq [x_{\mathrm{ex}}[n^{(u)}], \cdots, x_{\mathrm{ex}}[n^{(u)} + LN - 1]]^T$ denotes the existing samples in channels other than $\mathcal{M}^{(u)}$ at $n^{(u)}$, and $\mathbf{x}_m^{(u)} \triangleq [x_m[n^{(u)}], \cdots, x_m[n^{(u)} + LN - 1]]^T$ is the new OFDM symbol in the channel $m \in \mathcal{M}^{(u)}$. Having the optimal phase $\theta_m^\star$, the actually transmitted OFDM symbol on the mth channel becomes $e^{j\theta_m^\star}\mathbf{x}_m^{(u)}$ and the following samples passed through PA are

$$\mathbf{x} = \mathbf{x}_{\text{ex}}^{(u)} + \sum_{m\in\mathcal{M}^{(u)}} e^{j\theta_m^\star}\mathbf{x}_m^{(u)}, \tag{9}$$

unless $n^{(u+1)}$ occurs during this symbol period and the same procedure will be taken again for the samples after $n^{(u+1)}$. The phase rotation of each OFDM symbol is essentially part of the channel and can be recovered by channel estimations at receivers. Therefore, the MCPTS method does not require extra modifications at the receiver side as long as the receiver estimates the channel every symbol. In addition, since the phase rotations do not affect the average power σ_x^2, the IAR minimization (8) is equivalent to

$$\theta_m^\star = \arg\min_{\theta_m} \left\| \mathbf{x}_{\text{ex}}^{(u)} + \sum_{l\in\mathcal{M}^{(u)}} e^{j\theta_l}\mathbf{x}_l^{(u)} \right\|_\infty^2, \tag{10}$$

for $m \in \mathcal{M}^{(u)}$ and $n \in \{n^{(u)}, \cdots, n^{(u)} + LN - 1\}$.

By extending the MCPTS method to the generalized OFDM-FDMA system, the merits of MCPTS are inherited, e.g. no need for side information and receiver-side modification. Moreover, nonlinear optimization techniques, such as the particle swarm optimization (PSO) method, are available to solve Eq. (10).

It is interesting to point out that, in operations research, a similar problem has been thoroughly investigated, named the weighted minimum spanning circle (or single-facility location) problem [12]. Assuming no more than one new symbol at any $n^{(u)}$ (i.e. $|\mathcal{M}^{(u)}| = 1$), our phase optimization can be expressed in terms of the weighted minimum spanning circle problem as

$$\min_{\theta_m\in[0,2\pi)} \max_n \left| \frac{x_{\text{ex}}[n]}{x_m[n]} + e^{j\theta_m} \right| \cdot |x_m[n]|, \tag{11}$$

where $n \in \{n^{(u)}, \cdots, n^{(u)} + LN - 1\}$. In the language of operations research, the goal is to locate a single center on the unit circle using a *minmax* criterion with respect to the weighted Euclidean distances to existing points in the complex plane. The existing points are $-x_{\text{ex}}[n]/x_m[n]$ and the corresponding weights are $|x_m[n]|$ in our specific case. Efficient algorithms exist to solve the spanning circle problem, such as the weighted Elzinga-Hearn algorithm [12]. However, the addition of the unit circle (only changing the phase) constraint reduces the support of the optimization variable, which makes the spanning circle methods unnecessary and the PSO method favored in the MCPTS problem.

With prescribed power allocation, the MCPTS method can help reduce the dynamic range of the generalized OFDM-FDMA signal. However, for a fixed IBO, it cannot guarantee the avoidance of clipping distortion. Thus, the possible distortions should be considered and the signal-to-noise-and-distortion ratio (SNDR) has to be used as the measure of merit for transmission performance [13]. Similar to the IAR analysis in section III, by averaging the clipping effect on spectrum [14] over the channel usage states, the average SNDR can be evaluated for any given power allocation in the generalized OFDM-FDMA system [9]. Then, according to the performance (thus SNDR) requirements, the power allocation could be determined offline.

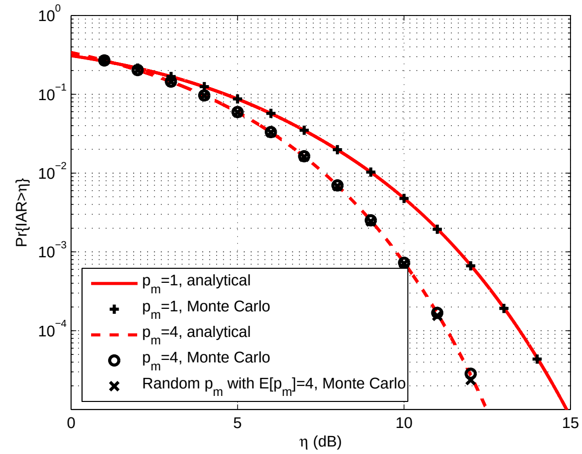

Fig. 2. The CCDF curves for the IAR of asynchronous OFDM-FDMA signals; 64QAM, $\lambda_m = 0.001$, $L = M = 4$ and $N = 64$.

V. Simulation Results

Simulation results of the IAR distribution and extended MCPTS IAR reduction method are shown in this section. The aforementioned channel usage model was adopted. Unless otherwise indicated, $\lambda_m = 0.001$ ($\forall m \in \{1, \cdots, M\}$) was assumed and the OFDM modulation with 64QAM and $N = 64$ subcarriers was used in the following simulations. Without loss of generality, equal power allocations were assumed in the OFDM-FDMA system, i.e. $P_m = P$ for $m \in \{1, \cdots, M\}$, which is valid when all users have the same quality of service requirements and the channel responses are independent and identically distributed.

In Fig. 2, the numerical results of the complimentary cumulative distribution function (CCDF) of IAR are compared with the Monte Carlo simulation results. About the number of symbols per packet, two fixed cases of $p_m = 1$ and $p_m = 4$ as well as a random p_m with the discretized exponential distribution and $E[p_m] = 4$ were adopted as examples. The analytical results are shown to agree well with the simulations. In addition, the example of $p_m = 1$ is shown to have greater IAR values than that of $p_m = 4$ mainly because the average power σ_x^2 is smaller for $p_m = 1$ than $p_m = 4$. These results explicitly illustrate the large dynamic range that the asynchronous OFDM-FDMA signals could have. For instance, if the input back-off is set at $\text{IBO} = 10\text{dB}$ (power efficiency $\rho = 5\%$), about 5×10^{-3} and 7×10^{-4} samples will be clipped for $p_m = 1$ and $p_m = 4$ in the peak power limited transmitter, respectively.

As shown in Fig. 3, the dynamic range of the OFDM-FDMA signals can be significantly reduced by the MCPTS IAR reduction method. A 10-particle 1,000-iteration PSO method was used to solve Eq. (10). Different numbers of channels ($M = 4$ or 8) and channel usage models (fixed $p_m = 1$ or random p_m with the mean of 4) were compared in these

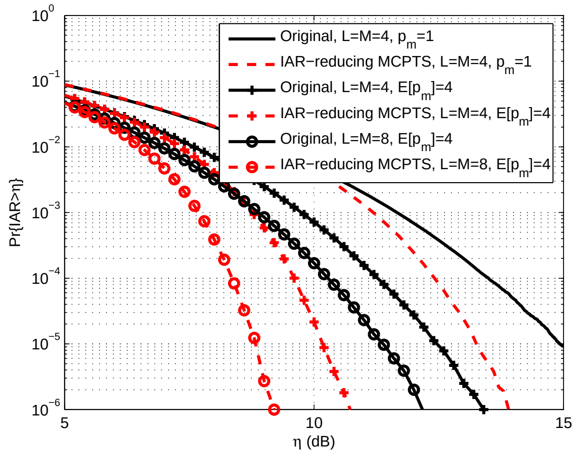

Fig. 3. With prescribed equal power allocation, the CCDF curves for the IAR of the MCPTS IAR reduction method as well as the original OFDM-FDMA signals; 64QAM, $\lambda_m = 0.001$, $N = 64$.

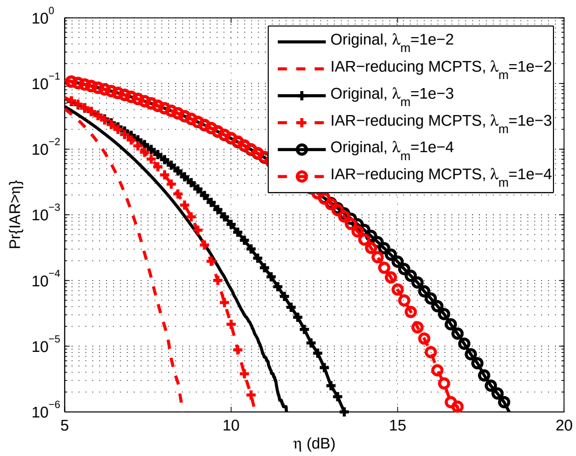

Fig. 4. With different λ_m's, the CCDF curves for the IAR of the MCPTS IAR reduction method as well as the original OFDM-FDMA signals; 64QAM, $L = M = 4$, $N = 64$, $E[p_m] = 4$.

simulations. In general, a 3dB IAR reduction performance can be steadily observed at the 10^{-6} CCDF level for all the examples in Fig. 3. In addition, the results show that the busier the channels, the more significant the IAR reduction performance. This can be further corroborated in Fig. 4, where only the arriving rate λ_m varies. Equivalently, the proposed MCPTS method tends to provide better performance for the aforementioned models with large M, λ_m and $E[p_m]$. In comparison, the synchronous OFDM-FDMA system [10] can be regarded as an extreme case where the channels are always active. Consequently, the MCPTS method for the synchronous OFDM-FDMA system can be expected to achieve the best dynamic range reduction performance.

VI. Conclusion

In this paper two important results were presented. First, we outlined the framework for the asynchronous multiplexed OFDM-FDMA peak reduction problem. As we demonstrated, there is no symbol paradigm for aggregated multiplexed signals, which means that the traditionally-used PAR metric can not be used to measure signal dynamic range. Instead, IAR is the pertinent metric. Accordingly, we derived the IAR distribution of asynchronous multiplexed OFDM-FDMA signals that have an exponential arrival distribution. The second result presented here is an IAR reduction scheme based on optimally phasing arriving signals so that the multiplexed aggregate signal has minimum IAR. Using PSO, we demonstrate significant IAR reduction with the proposed scheme.

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