FREE SUBCARRIER OPTIMIZATION FOR PEAK-TO-AVERAGE POWER RATIO MINIMIZATION IN OFDM SYSTEMS

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ABSTRACT

Peak-to-average power ratio (PAR) reduction techniques are often employed to increase the power efficiency of orthogonal frequency division multiplexing (OFDM) systems. A recently proposed PAR optimization method demonstrates how the PAR can be minimized when free subcarriers and a certain distortion allowance on the error vector magnitude (EVM) are available. In this paper, we derive the lower bound on the capacity for such a system and investigate the capacity-maximizing number of free subcarriers that should be used.

***Index Terms*—** Orthogonal frequency division multiplexing (OFDM), peak-to-average power ratio (PAR), signal-to-noise-and-distortion ratio (SNDR)

1. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) is a widely used modulation method that has high spectral efficiency and exhibits robustness against frequency-selective fading channels. However, a significant drawback of OFDM is its low power efficiency due to the large peak-to-average power ratios (PARs) of its time-domain waveforms.

Many methods have been proposed to reduce the PAR of OFDM signals by constraining the distortion energy [1-3] or by projecting the distortion energy onto "free" subcarriers [4]. In [5], PAR reduction was cast as a convex optimization problem. In that work, PAR is minimized by exploiting the allowed distortion on the data subcarriers and by utilizing the "free" subcarriers. By exploiting the IFFT/FFT structure of OFDM, the interior-point method (IPM) can be customized for this optimization problem to provide good performance with relatively low complexity. Additionally, in [6] the same authors analyzed the signal-to-noise ratio (SNR) of the PAR-minimized signals to derive the optimal error vector magnitude (EVM) thresholds for transmitter power consumption minimization and channel capacity maximization.

However, in the presence of nonlinear distortions, the channel capacity was found in [7] to be a function of the signal-to-noise-and-distortion ratio (SNDR), as opposed to the SNR. The optimization scheme in [5] is a highly nonlinear operation, thus the SNDR analysis is pertinent. Moreover, the problem of finding the optimal number of free subcarriers, which significantly impacts the performance of the optimization algorithm, was not solved. Although increasing the number of free subcarriers to reduce the PAR can improve the SNDR and the capacity per channel, the total system capacity is not necessarily improved because more free subcarriers mean fewer subcarriers for data transmission. In fact, we will show that the total system capacity is concave in the number of free subcarriers and thus the capacity-maximizing number of free subcarriers can be found.

In this paper, we build upon the work of [6] and analyze the SNDR for the PAR-minimized OFDM signals. Using this SNDR formulation, the relationship between the number of free subcarriers and the system capacity will be established. Finally, we will investigate the optimal number of free subcarriers for different channel noise levels and different EVM thresholds.

2. SYSTEM MODEL

The frequency-domain OFDM symbol can be decomposed into two non-overlapping sets: data subcarriers and free subcarriers denoted by sets of indices $\mathcal{K}_d$ and $\mathcal{K}_f$, respectively. By denoting their cardinalities as $|\mathcal{K}_d| = d$ and $|\mathcal{K}_f| = f$, we have $d + f = N$ where N is the total number of subcarriers. For simplicity, pilot subcarriers are not considered in this paper, but the results can be easily extended to systems with pilot signals.

The frequency-domain OFDM symbol can be denoted as $\mathbf{X} = [X_0, X_1, \cdots, X_{N-1}]^T$. It consists of encoded data $X_{\mathcal{K}_d} \in \Omega$, where Ω is an ideal constellation, and free subcarriers that can take on any complex value $X_{\mathcal{K}_f} \in \mathbb{C}$ subject to the spectral mask constraints. Prior to cyclic extension, which does not impact the PAR [4], the L-times oversampled baseband OFDM signal can be expressed as

$$x[n] = \text{IFFT}_L(\mathbf{X})[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{LN-1} X'_k e^{\frac{j2\pi kn}{LN}}, \qquad (1)$$

where $\mathbf{X}' = [X_0, \cdots, X_{\frac{N}{2}-1}, 0, \cdots, 0, X_{\frac{N}{2}}, \cdots, X_{N-1}]^T$ is generated by zero padding $\mathbf{X}$ with $(L-1)N$ zeros; corre-

This work was supported in part by the U. S. Army Research Laboratory under the Collaborative Technology Alliance Program, Cooperative Agreement DAAD19-01-2-0011 and in part by the National Science Foundation Graduate Research Fellowship Program.

spondingly, IFFT_L designates the L-times oversampled waveform in the time domain that maintains the same average power with $\mathbf{X}$ as $\frac{1}{LN}\|\mathbf{x}\|_2^2 = \frac{1}{N}\|\mathbf{X}\|_2^2$. Define the PAR of an OFDM symbol as

$$\text{PAR} = \frac{\max\limits_{n\in\{0,\cdots,LN-1\}} |x[n]|^2}{E[|x[n]|^2]}. \tag{2}$$

Assume that a linear or linearized Class-A power amplifier (PA) is used in the transmitter. For a given DC power P_{dc}, the peak output power is limited to $P_{\max} = P_{\text{dc}}/2$. In order to achieve the maximum average power efficiency, symbol-wise linear scaling should be applied such that the peak power of each output OFDM symbol achieves $P_{\max}$ [8]. At the receiver, the scaling factors can be regarded as a part of the channel and compensated for by equalization. Therefore, in flat-fading channels, the instantaneous peak-signal-to-noise power ratio (PSNR) for all symbols is

$$\text{PSNR} = \frac{h \cdot P_{\max}}{N_0}, \tag{3}$$

where h is the channel gain and N_0 is the power of the additive white Gaussian noise per subcarrier.

3. PAR MINIMIZATION

Let $\mathbf{X}^o = [X_0^o, X_1^o, \cdots, X_{N-1}^o]^T$ denote the PAR-minimized OFDM symbol for a given set of constraints. One of the constraints is the EVM which is defined as

$$\epsilon(\mathbf{X}, \mathbf{X}^o) = \sqrt{\frac{\frac{1}{d}\sum_{k\in\mathcal{K}_d} |X_k - X_k^o|^2}{P_0}}, \tag{4}$$

where P_0 is the average power of the constellation Ω [9]. In addition to the EVM constraint, the free subcarriers should satisfy

$$E[|X_k^o|^2] \le \mathcal{M}_k,\ k \in \mathcal{K}_f, \tag{5}$$

where $\mathcal{M}_k$ is the spectral mask requirement.

The convex PAR minimization problem can be cast as the minimization of the time-domain waveform peak while keeping the average data power bounded from below, i.e.

$$\begin{aligned}
&\underset{\mathbf{X}^o}{\text{Minimize}} && p && (6)\\
&\text{Subject to} && \mathbf{x}^o = \text{IFFT}_L(\mathbf{X}^o) && (7)\\
& && |x^o[n]| \le p,\ n \in \{0,\cdots,LN-1\} && (8)\\
& && \epsilon(\mathbf{X}, \mathbf{X}^o) \le e && (9)\\
& && |X_k^o|^2 \le \mathcal{M}_k,\ k \in \mathcal{K}_f && (10)\\
& && \frac{1}{d}\sum_{k\in\mathcal{K}_d} \Re[X_k^*(X_k^o - X_k)] \ge -P_0\frac{e^2}{2}, && (11)
\end{aligned}$$

where e is the EVM threshold, (11) seeks to maintain the average data power [5] and (10) provides a stricter but solvable constraint than (5). This optimization problem can be solved by the IPM. After optimization, the ratio of the power in the data subcarriers to the total power becomes a function of the EVM threshold e and the number of free subcarriers f, i.e.,

$$r_{\text{data}}(e,f) = E\left[\frac{\sum_{k\in\mathcal{K}_d} |X_k^o|^2}{\sum_{k\in\mathcal{K}_d\cup\mathcal{K}_f} |X_k^o|^2}\right]. \tag{12}$$

Let us denote the peak power and the symbol-wise PAR of the ith PAR-minimized symbol by

$$\begin{aligned}
P_{\text{peak}}(e,f) &= \max_{n\in\{0,\cdots,LN-1\}} |x_i^o[n]|^2, && (13)\\
\text{PAR}_s(e,f) &= \frac{P_{\text{peak}}(e,f)}{\frac{1}{LN}\|\mathbf{x}_i^o\|_2^2}, && (14)
\end{aligned}$$

respectively. The above quantities can be determined through simulations.

As we discussed in Section 2, the PAR-minimized signal should be scaled symbol-wise so that the peak output power of all symbols stays at $P_{\max}$ to ensure maximum power efficiency. Thus, the signal transmitted through the channel is $x_i^o[n]\sqrt{P_{\max}/P_{\text{peak}}(e,f)}$. The average output power becomes

$$\begin{aligned}
P_{\text{out}}(e,f) &= E\left[\left|x_i^o[n]\sqrt{\frac{P_{\max}}{P_{\text{peak}}(e,f)}}\right|^2\right] && (15)\\
&= P_{\max} \cdot E\left[\frac{\frac{1}{LN}\|\mathbf{x}_i^o\|_2^2}{P_{\text{peak}}(e,f)}\right] && (16)\\
&= \frac{P_{\max}}{\text{PAR}_{\text{av}}(e,f)}, && (17)
\end{aligned}$$

where

$$\text{PAR}_{\text{av}}(e,f) = \left(E\left[\frac{1}{\text{PAR}_s(e,f)}\right]\right)^{-1}. \tag{18}$$

Thus, (17) and (18) imply that the harmonic mean of the random variable $\text{PAR}_s(e,f)$ is a meaningful statistic.

4. PARAMETER OPTIMIZATION

The PAR-minimized signal $\mathbf{X}^o$ is a highly nonlinear function of the original OFDM symbol $\mathbf{X}$. According to the Bussgang Theorem, $\mathbf{X}^o$ can be expressed in terms of $\mathbf{X}$ as [1]

$$X_k^o = \alpha X_k + Q_k,\ k \in \mathcal{K}_d, \tag{19}$$

where the scaling factor $\alpha = E[X_k^* X_k^o]/E[|X_k|^2]$ ($k \in \mathcal{K}_d$) is chosen so that the distortion term Q_k is uncorrelated with X_k. In the presence of Q_k, the capacity of the data subcarriers becomes a function of the SNDR [7] which is defined as

$$\text{SNDR} = \frac{h\alpha^2\sigma_x^2}{h\sigma_q^2 + N_0}, \tag{20}$$

where σ_x^2 and σ_q^2 are the variances of X_k and Q_k, respectively.

Since there is not a closed form expression linking $\mathbf{X}$ to $\mathbf{X}^o$, it is difficult to analyze α theoretically. However, we can use some simplifying assumptions in order to gain some insight regarding the SNDR. First, let us assume that the error vector $\mathbf{X} - \mathbf{X}^o$ is uncorrelated with the data vector $\mathbf{X}$. Next, when e is not too large, due to the fact that the EVM of the optimized signals is very close to the threshold $\epsilon(\mathbf{X}, \mathbf{X}^o) \approx e$ [5], the average distortion power on each data subcarrier can be approximated by $e^2\sigma_x^2$. Thus, the rest of the power sums up to $\alpha^2\sigma_x^2 \approx (1-e^2)\sigma_x^2$. These approximations have been verified by computer simulations, the results of which are omitted here due to space limitations.

Thus, the average SNDR for the data subcarriers is

$$\begin{aligned} \text{SNDR} &\approx \frac{(1-e^2) r_{\text{data}}(e,f) P_{\text{out}}(e,f)}{e^2 r_{\text{data}}(e,f) P_{\text{out}}(e,f) + \frac{d \cdot N_o}{N \cdot h}} \\ &= \frac{(1-e^2) r_{\text{data}}(e,f)}{e^2 r_{\text{data}}(e,f) + \frac{d}{N} \cdot \frac{\text{PAR}_{\text{av}}(e,f)}{\text{PSNR}}}. \end{aligned} \tag{21}$$

Notice that this formulation differs from the SNR expression in [6] in three important ways: i) the PAR in [6] was defined at the 10^{-2} probability level, whereas we have demonstrated that $\text{PAR}_{\text{av}}(e,f)$ as defined in (18) is more meaningful; ii) the degradation to the useful signal as quantified by the factor $1-e^2$ is taken into account in (21), which was not considered in [6]; iii) the free subcarriers are considered in terms of $r_{\text{data}}(e,f)$ in (21).

Because the distortion term Q_k in (19) is approximately Gaussian distributed [6], the lower bound of the capacity on each data subcarrier is $C_k = \log_2(1 + \text{SNDR})$ bits ($\forall k \in \mathcal{K}_d$)[7] and the total system capacity per symbol is

$$C = \sum_{k \in \mathcal{K}_d} C_k = d \cdot \log_2(1 + \text{SNDR}) \text{ bits}. \tag{22}$$

Finally, when using the PAR minimization algorithm of Section 3, the pair (e,f) should be selected so that the capacity is maximized. However, this two-variable optimization problem is difficult to solve and will be the subject of future research. In this paper, we will focus on optimizing the number of free subcarriers for a given EVM threshold e, i.e.

$$\underset{f}{\text{Maximize}} \quad C \tag{23}$$

$$\text{Subject to} \quad \text{PSNR},\ e,\ \mathcal{M}_k\ (k \in \mathcal{K}_f), \tag{24}$$

which can be solved by searching through all possible values of $f \in \{0, 1, ..., N\}$.

5. SIMULATIONS

In all simulations, $N = 64$, $L = 4$ and the spectral mask as defined in the 802.11a standard were used [9]. The OFDM symbols were drawn from a QPSK constellation. Simulation results show that the distribution of the optimized PAR is independent of the choice of the constellation. The EVM threshold was $e = -16$dB for the results shown here, but the proposed method works for any other EVM values specified in the standard. Moreover, we assume that the free subcarriers are selected in pairs. We also assume that the free subcarriers utilized for PAR minimization are the edge-most ones since they usually have stricter power constraint in order to avoid spectral regrowth caused by the PA nonlinearity.

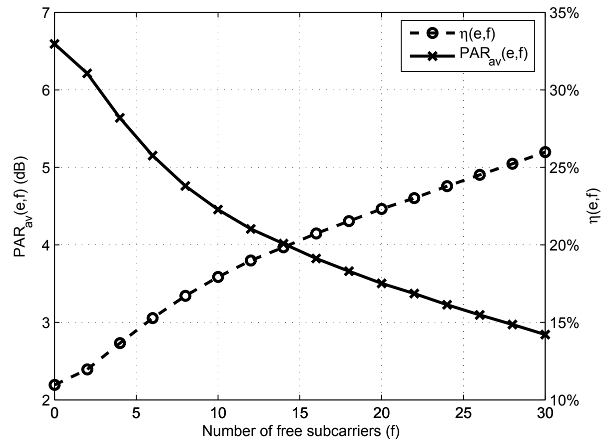

Fig. 1. $\text{PAR}_{\text{av}}(e,f)$ and average power efficiency $\eta(e,f)$ as a function of the number of free subcarriers; $e = -16$dB.

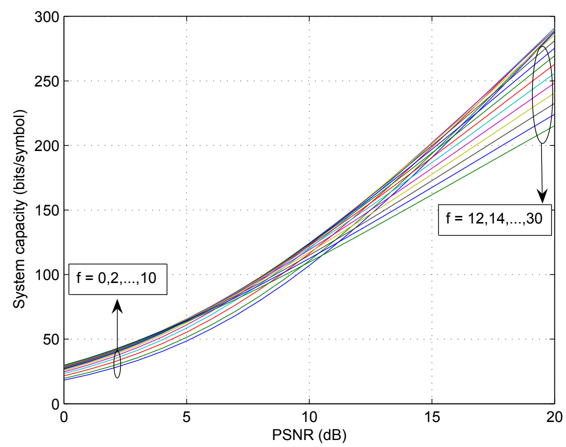

Fig. 2. System capacity as a function of PSNR for different numbers of free subcarriers $f = 0, 2, \cdots, 30$; $e = -16$dB.

Fig. 1 shows $\text{PAR}_{\text{av}}(e,f)$ and the corresponding average power efficiency $\eta(e,f) = 0.5/\text{PAR}_{\text{av}}(e,f)$. The power efficiency increases monotonically as f increases. For the Class-A PA with a fixed DC power, its power consumption is constant for all cases, thus, the higher the power efficiency, the larger the average output power $P_{\text{out}}(e,f)$.

The system capacity (22), in terms of the SNDR expres-

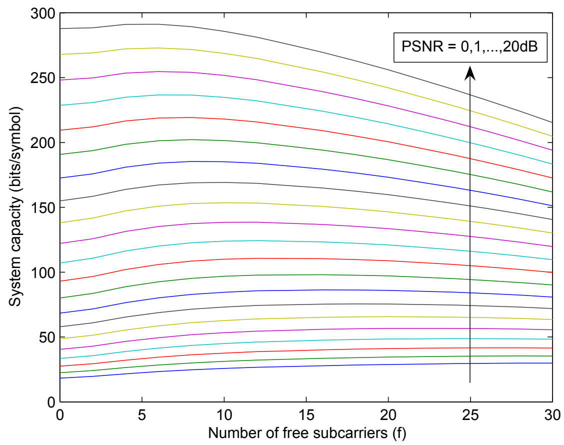

Fig. 3. System capacity as a function of the number of free subcarriers f for PSNR$= 0, 1, \cdots, 20$dB; $e = -16$dB.

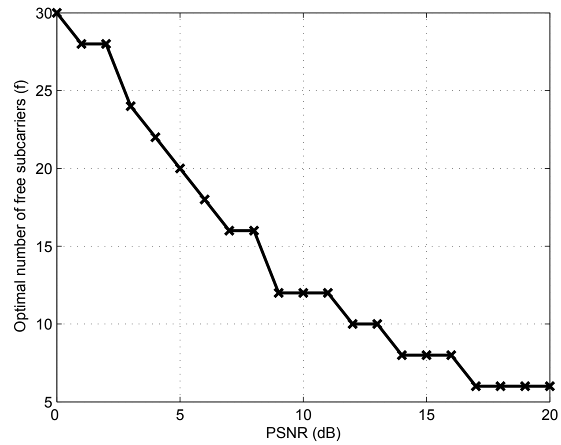

Fig. 4. The optimal number of free subcarriers f for PSNR$= 0, 1, \cdots, 20$dB; $e = -16$dB.

sion (21) and the number of data subcarriers d, can be calculated as a function of PSNR for different f values, as illustrated in Fig. 2. The intersection of these curves shows that the optimal f is dependent on the PSNR. Moreover, it proves that optimization over f is important: a poor choice of f can degrade the PSNR by as much as 4dB.

In Fig. 3, by plotting the system capacity as a function of f for different PSNR values, the results can be shown more clearly. Moreover, the optimal choice of f depends on the PSNR, as can be seen in Fig. 4. For large PSNR values, fewer free subcarriers are needed to achieve the maximum system capacity, which agrees with intuition. In this region, using more subcarriers for further PAR reduction in order to increase the average output power will not significantly improve the SNDR. It will, however, reduce the system capacity because fewer subcarriers are used for data transmission. Thus, transmitting data on more subcarriers may benefit the capacity despite causing a higher PAR and a lower power efficiency.

Finally, if no channel state information is available at the transmitter, the optimal number of free subcarriers can be determined offline by comparing the expectation of the system capacity over a practical PSNR range. Given an equiprobable PSNR ranging from 0dB to 20dB, for instance, our analysis shows that the optimal f is 10 for $e = -16$dB in the 802.11a standard [9].

6. CONCLUSIONS

In this paper, we proposed a novel method for determining the capacity-optimizing number of free subcarriers that should be used in the PAR minimization algorithm. The results demonstrated that the optimal number of free subcarriers is a monotonically decreasing function of the peak-signal-to-noise power ratio. Additionally, it was shown that improperly choosing the number of free subcarriers could lead to a significant amount of information loss. In our future research, we will investigate two-variable capacity maximization in both the EVM and the number of free subcarriers.

7. REFERENCES

[1] H. Ochiai and H. Imai, "Performance analysis of deliberately clipped OFDM signals," *IEEE Trans. on Commun.*, vol. 50, no. 1, pp. 89–101, Jan. 2002.

[2] B. S. Krongold and D. L. Jones, "PAR reduction in OFDM via active constellation extension," *IEEE Trans. on Broadcast.*, vol. 49, no. 3, pp. 258–268, Sept. 2003.

[3] R. J. Baxley, C. Zhao, and G. T. Zhou, "Constrained clipping for crest factor reduction in OFDM," *IEEE Trans. on Broadcast.*, vol. 52, no. 4, pp. 570–575, Dec. 2006.

[4] J. Tellado, *Multicarrier Modulation with Low PAR: Applications to DSL and Wireless*, Norwell, MA: Kluwer, Sept. 2000.

[5] A. Aggarwal and T. H. Meng, "Minimizing the peak-to-average power ratio of OFDM signals using convex optimization," *IEEE Trans. on Signal Process.*, vol. 54, no. 8, pp. 3099–3110, Aug. 2006.

[6] A. Aggarwal, E. R. Stauffer, and T. H. Meng, "Computing the optimal amount of constellation distortion in OFDM systems," in *Proc. IEEE ICC*, Glasgow, Scotland, June 2007, pp. 2918–2923.

[7] R. Raich, H. Qian, and G. T. Zhou, "Optimization of SNDR for amplitude-limited nonlinearities," *IEEE Trans. on Commun.*, vol. 53, no. 11, pp. 1964–1972, Nov. 2005.

[8] H. Ochiai, "Performance analysis of peak power and band-limited OFDM system with linear scaling," *IEEE Trans. on Wireless Commun.*, vol. 2, no. 5, pp. 1055–1065, Sept. 2003.

[9] *Wireless LAN Medium Access Control (MAC) and Physical Layer (PHY) Specifications: High-Speed Physical Layer in the 5 GHz Band*, IEEE Std. 802.11a, Sept. 1999.