Optimal Subcarrier Power Allocation for OFDM in Peak-Power-Limited Channels

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***Abstract*—A major disadvantage of the orthogonal frequency division multiplexing (OFDM) system is the large dynamic range of its time-domain waveforms, making OFDM vulnerable to nonlinearities (including clipping effects) of the power amplifier (PA). The effects are nonlinear distortions as well as low DC to RF power conversion efficiency. The peak-to-average power ratio (PAR) is commonly used to characterize a signal's dynamic range. In the recent literature, the distribution of the PAR of OFDM signals is revealed to be dependent on the power allocation among subcarriers. In this paper, the effect of different power allocation schemes on the power efficiency of peak-power constrained OFDM transmitters is investigated. Optimization problems are formulated to determine the optimal power allocation with respect to the power efficiency, power consumption and system throughput performance in AWGN channels.**

***Index Terms*—Orthogonal frequency division multiplexing (OFDM), peak-to-average power ratio (PAR), power allocation**

I. Introduction

Due to its high spectral efficiency and robustness against frequency-selective fading effects, orthogonal frequency division multiplexing (OFDM) modulation has been adopted by many wireless communication standards [1], [2]. However, one of the primary disadvantages of OFDM is that time-domain OFDM waveforms exhibit large peak-to-average power ratios (PARs) [3]. To avoid severe nonlinear distortions both in-band and out-of-band, power amplifiers (PAs) are often operated with a large input back-off (IBO), resulting in poor DC to RF power conversion efficiency [4].

Many PAR reduction methods have been proposed for OFDM systems and they modify the OFDM signal in some symbol-based fashion [3]. For instance, distortion-based PAR reduction methods, such as clipping and filtering [5], [6], active constellation extension [7], and waveform optimization methods [8], [9], can improve the power efficiency by introducing deliberate but constrained distortions to each OFDM symbol. In the selected mapping [10], [11] and partial transmit sequence [12] methods, multiple time-domain representations are generated for each OFDM symbol and the representation with the smallest PAR is transmitted. The existing methods generally require symbol-wise processing to improve the power efficiency of the OFDM transmitter, and involve tradeoffs in terms of distortion, complexity, spectral efficiency, and possible receiver-side modification.

In the recent literature, the effect of the average subcarrier power allocation on the PAR distribution is studied [13]. Here, we exploit the relationship between the power allocation and the dynamic range for the piecewise linear scaling (PWLS) OFDM system where distortions are avoided for an ideal peak-power-limited PA [5]. The optimal power allocations with respect to the ensemble power efficiency and system throughput will be investigated for AWGN channels. Unlike existing methods, power allocation does not require extra symbol-wise processing beyond PWLS and does not require receiver-side modification. The optimal power allocation is obtained off-line and thus does not result in complexity increase on a real-time basis.

The organization of this paper is as follows. In Section II, the OFDM system and the PA model are introduced. Performance tradeoffs relevant to the subcarrier power allocation in the piecewise linear scaling OFDM system are discussed in Section III, followed by the power allocation optimizations for different scenarios in Section IV. Simulation results are shown in Section V and conclusions are drawn in Section VI.

II. System Model

In the OFDM system, information bits are mapped onto N orthogonal subcarriers. A length-N frequency-domain OFDM symbol is denoted as $\mathbf{X} = [X_{-N/2}, \cdots, X_{N/2-1}]^T$, where $\mathcal{N} \triangleq \{-N/2, \cdots, N/2-1\}$ stands for the set of subcarrier indices. Independent data are assumed to be modulated on different subcarriers, i.e., X_k and X_l ($\forall k \neq l$) are mutually independent. A symmetric constellation is usually used so that $\Re(X_k)$ and $\Im(X_k)$ are mutually independent and have the same symmetric distribution, thus $E[\Re(X_k)] = E[\Im(X_k)] = 0$, $E[\Re(X_k)\Im(X_k)] = 0$ and $E[\Re(X_k)^2] = E[\Im(X_k)^2] = P_k/2$ ($\forall k \in \mathcal{N}$). $\Re(\cdot)$ and $\Im(\cdot)$ denote the real and the imaginary parts, respectively. $P_k = E[|X_k|^2]$ denotes the average power allocated to the kth subcarrier. Accordingly, the normalized power allocation is defined as $P_k^\dagger = P_k / \sum_{l\in\mathcal{N}} P_l$, which is subject to design in this paper.

Suppose that the baseband OFDM signal is transmitted over $[-B/2, B/2]$, where B is the bandwidth. Prior to cyclic extension (which does not impact the signal dynamic range [4]), the time-domain OFDM signal is

$$x(t) = \frac{1}{\sqrt{N}} \sum_{k\in\mathcal{N}} X_k e^{j\omega_k t}, \ t \in [0, T), \tag{1}$$

where $\omega_k = (2k+1)\,\pi/T$ is the center frequency of the kth subcarrier, $T = NT_c$ is the symbol duration and $T_c = 2\pi/B$.

The average power of $x(t)$ can be readily found as

$$P_{\rm av} = \frac{1}{T}\int_0^T E[|x(t)|^2]dt = \frac{1}{N}\sum_{k\in\mathcal{N}} P_k. \tag{2}$$

When N is large, the OFDM signal in Eq. (1) can be approximately modeled by a (truncated) zero-mean Gaussian random process, which has large envelope fluctuations [4]. Peak-to-average power ratio (PAR) is a simple metric to quantify envelope excursions of a signal and is defined as

$$\mathrm{PAR} = \frac{\max_{0\le t<T}|x(t)|^2}{P_{\rm av}}. \tag{3}$$

It can be shown that $1 \le \mathrm{PAR} \le N$ when $x(t)$ is a time-domain OFDM signal with N subcarriers [4].

To transmit $x(t)$, a PA with output peak-power limit $P_{\rm peak}$ is used. In this paper, we consider a class-A PA with an ideal soft-limiter characteristic, whose output signal $y(t) = g(x(t))$ is characterized by

$$y(t) = \begin{cases} x(t), & |x(t)|^2 \le P_{\rm peak} \quad (4)\\ \sqrt{P_{\rm peak}}e^{j\angle x(t)}, & |x(t)|^2 > P_{\rm peak}, \quad (5)\end{cases}$$

where $\angle x(t)$ denotes the phase of $x(t)$. For simplicity, unit gain is assumed in the PA linear region. With this model, saturation happens when $|x(t)|^2 > P_{\rm peak}$ and may result in error floor, spectral broadening and other deleterious nonlinear effects [4].

III. Piecewise Linear Scaling and Performance Tradeoffs

To discern the effect on the power efficiency that is attributed to power allocation, distortionless transmission is considered in this paper, which can be realized via the block-by-block piecewise linear scaling (PWLS) method [5]. In PWLS, each OFDM symbol is multiplied by a symbol-wise gain G_s such that the PA input signal becomes

$$\tilde{x}(t) = G_s x(t) = \frac{\sqrt{P_{\rm peak}}}{\max\limits_{0\le s<T}|x(s)|}x(t), \tag{6}$$

which leads to $\max\limits_{0\le t<T}|\tilde{x}(t)| = \sqrt{P_{\rm peak}}$. At the same time, the frequency-domain symbol becomes $\tilde{\mathbf{X}} = G_s\mathbf{X}$. The symbol-wise gain is effectively a part of the communication channel and does not affect the receiver performance with ideal channel estimation [5]. In addition, G_s does not change the normalized power allocation $P_k^\dagger$.

The following performance metrics are pertinent to the OFDM system with PWLS:

1. Power consumption: The DC power consumed by the class-A PA is twice of the necessary peak-power limit, i.e., $P_{\rm dc} = 2P_{\rm peak}$.

2. Power efficiency: PWLS avoids clipping at the soft-limiter PA so that $y(t) = g(\tilde{x}(t)) = \tilde{x}(t)$ and the power efficiency can be found as

$$\rho = \frac{E[|y(t)|^2]}{P_{\rm dc}} = 0.5(\mathrm{IBO})^{-1}, \tag{7}$$

where the input back-off (IBO) is given by the harmonic mean of the PAR of $x(t)$, i.e., $1 \le \mathrm{IBO} = (E[\mathrm{PAR}^{-1}])^{-1} \le N$. The average output power is thus

$$\tilde{P}_{\rm av} = \frac{1}{T}\int_0^T E[|\tilde{x}(t)|^2]dt = \rho\cdot P_{\rm dc} = \frac{P_{\rm peak}}{\mathrm{IBO}}. \tag{8}$$

With the normalized power allocation $P_k^\dagger$ unchanged, the transmit power on the kth subcarrier is

$$\tilde{P}_k = P_k^\dagger \tilde{P}_{\rm av}. \tag{9}$$

3. Average throughput: The average throughput characterizes the maximum achievable transmission rate by the use of random Gaussian codebooks with an infinite code length. In the OFDM system, the throughput is determined by the power on each subcarrier and given for AWGN channels as

$$\begin{aligned} C &= \frac{1}{N}\sum_{k\in\mathcal{N}} E\left[\log_2\left(1+\frac{P_{\rm peak}}{\mathrm{PAR}}\cdot\frac{P_k^\dagger}{\sigma_w^2}\right)\right] \mathrm{bps/Hz} \quad (10)\\ &= \frac{1}{N}\sum_{k\in\mathcal{N}} E\left[\log_2\left(1+\frac{\mathrm{IBO}}{\mathrm{PAR}}\cdot\frac{\tilde{P}_k}{\sigma_w^2}\right)\right] \mathrm{bps/Hz} \quad (11)\end{aligned}$$

where σ_w^2 denotes the channel noise power in each subcarrier and the expectation is taken over the PAR.

The PAR distribution is affected by the specific power allocation profile [13]. The complementary cumulative distribution function (CCDF) of the PAR can be well approximated by

$$\Pr\{\mathrm{PAR} > \eta\} \approx 1 - \exp\left\{-e^{-\eta}T\sqrt{\frac{\lambda}{\pi}\ln T}\right\}, \text{ for } \eta > 0, \tag{12}$$

where λ is the second-order central moment of the subcarrier center frequencies weighted by the normalized power allocation $P_k^\dagger$, i.e.,

$$\lambda = \sum_{k\in\mathcal{N}} P_k^\dagger \omega_k^2 - \left(\sum_{k\in\mathcal{N}} P_k^\dagger \omega_k\right)^2. \tag{13}$$

Therefore, $P_k^\dagger$ ($\forall k \in \mathcal{N}$) does not only explicitly affect the throughput as in Eq. (10), but also the power efficiency implicitly through the PAR distribution and the IBO.

The performance tradeoff can be illustrated by examples. According to Eq. (12), the smaller the λ, the smaller the PAR on average and the better the power efficiency. It follows that it is optimal to allocate all power to one single subcarrier, i.e., the single-carrier transmission that $P_l^\dagger = 1$ and $P_k^\dagger = 0$ ($\forall k \ne l \in \mathcal{N}$) which has $\mathrm{IBO} = 1$ and $\rho = 50\%$. When the average throughput is concerned, however, the single-carrier solution is not optimal anymore. As shown in Eq. (10), although PAR may be reduced on average by varying $P_k^\dagger$, the throughput is not necessarily increasing. In fact, with a fixed power consumption $P_{\rm dc}$, the throughput C_e of the equal power allocation $P_k^\dagger =$

N^{-1} $(\forall k \in \mathcal{N})$ can be bounded by Jensen's inequality as

$$
\begin{aligned}
C_e &\geq \frac{1}{N}\sum_{k\in\mathcal{N}}\log_2\left(1+\frac{\tilde{P}_k}{\sigma_w^2}\right) && (14)\\
&= \log_2\left(1+\frac{P_{\mathrm{dc}}}{2\sigma_w^2\mathrm{IBO}_e}\right) && (15)\\
&\geq \frac{1}{N}\log_2\left(1+\frac{P_{\mathrm{dc}}}{2\sigma_w^2}\right) = C_s, && (16)
\end{aligned}
$$

where IBO_e denotes the IBO of the equal power allocation and C_s is the throughput achieved by the single-carrier power allocation. The same inequality can be easily found for a fixed average power constraint.

Therefore, power consumption, power efficiency and average throughput are pairs of tradeoffs affected by the normalized power allocation. Although λ suffices to characterize the relationship between $P_k^\dagger$ and the PAR distribution, no closed-form solution is found for the general optimal power allocation.

IV. Optimal Power Allocation in AWGN Channels

By optimizing the power allocation among subcarriers, one of the aforementioned performance metrics can be improved with explicit constraints on the others. In this section, these optimization problems are framed and discussed for AWGN channels.

A. Maximize power efficiency

It is straightforward to show that, when the average power is fixed that $\tilde{P}_{\mathrm{av}} = \sigma^2$, the maximum throughput is obtained when power is equally allocated among subcarriers, i.e., $\tilde{P}_k = \sigma^2$ $(\forall k \in \mathcal{N})$, and it can be denoted as $C_e(\sigma^2)$. By sacrificing the throughput, however, unequal power allocation can be implemented so that the signal dynamic range is reduced. Reducing the dynamic range and maximizing the power efficiency can be accomplished, as shown in Eq. (7), by minimizing λ.

To illustrate the effect of power allocation optimizations on reducing the signal dynamic range, the following optimization framework is used

$$
\begin{aligned}
\underset{P_k^\dagger}{\text{minimize}} \quad & \lambda && (17)\\
\text{subject to} \quad & \tilde{P}_{\mathrm{av}} = \sigma^2 && (18)\\
& C \geq \beta C_e(\sigma^2), && (19)
\end{aligned}
$$

where $\beta \leq 1$ denotes the normalized throughput constraint. As special cases, when $\beta = 1$, equal power allocation will be necessary to satisfy (19) and yield the largest λ; when $\beta = 0$, the single-carrier transmission as discussed in Section III will be optimal.

To represent the AWGN channel noise level, signal-to-noise ratio (SNR) can be defined as the ratio between equally allocated subcarrier power and the channel noise power. Therefore, $\mathrm{SNR} = \sigma^2/\sigma_w^2$ for the power efficiency maximization in (17)-(19).

B. Minimize power consumption

For the peak-power-limited system with a fixed transmission rate, it is intuitive and practical to minimize the necessary power consumption. Denoting the throughput requirement as C_{req}, the power consumption minimization problem is given as

$$
\begin{aligned}
\underset{P_k^\dagger}{\text{minimize}} \quad & P_{\mathrm{dc}} = 2\tilde{P}_{\mathrm{av}}\mathrm{IBO} && (20)\\
\text{subject to} \quad & \lambda = \sum_{k\in\mathcal{N}} P_k^\dagger\omega_k^2 - \left(\sum_{k\in\mathcal{N}} P_k^\dagger\omega_k\right)^2 && (21)\\
& \mathrm{IBO} = \left(E\left[\mathrm{PAR}^{-1}\middle|\lambda\right]\right)^{-1} && (22)\\
& C = \frac{1}{N}\sum_{k\in\mathcal{N}} && (23)\\
& E\left[\log_2\left(1+\frac{\mathrm{IBO}}{\mathrm{PAR}}\cdot\frac{P_k^\dagger\tilde{P}_{\mathrm{av}}}{\sigma_w^2}\right)\middle|\lambda\right] \geq C_{\mathrm{req}}.
\end{aligned}
$$

First, the IBO in Eq. (22) is the harmonic mean of the PAR whose distribution is determined by λ of the normalized power allocation as in Eq. (21). On the other hand, the average power $\tilde{P}_{\mathrm{av}}$ also has to satisfy the throughput constraint as in (23), where the expectation over PAR further makes (20)-(23) a complicated problem. No closed-form solution is found, but numerical solutions will be shown in Section V.

As a benchmark, the power consumption of the equal power allocation that achieves the throughput requirement behaves as an upper bound to the solution of (20)-(23). Specifically, plugging $P_k^\dagger = N^{-1}$ $(\forall k \in \mathcal{N})$ into Eqs. (12) and (13), we can obtain the PAR distribution of the equal power allocation and its input back-off IBO_e. Based on the PAR distribution, $\tilde{P}_k = P_e$ $(\forall k \in \mathcal{N})$ can be found for the throughput C in Eq. (11) to satisfy (23), and gives the upper bound $2P_e\mathrm{IBO}_e$ for the solution. The SNR in this case becomes $\mathrm{SNR} = P_e/\sigma_w^2$.

C. Maximize average throughput

Conversely, when the power consumption is not adjustable, the objective becomes to efficiently use the given power and maximize the throughput. For a given P_{dc}, the optimization problem can be formulated as

$$
\begin{aligned}
\underset{P_k^\dagger}{\text{maximize}} \quad & C && (24)\\
\text{subject to} \quad & \tilde{P}_{\mathrm{av}}\mathrm{IBO} \leq P_{\mathrm{peak}} = \frac{1}{2}P_{\mathrm{dc}}. && (25)
\end{aligned}
$$

Again, a lower bound is provided by the equal power allocation, which can be obtained by plugging $\tilde{P}_k = P_{\mathrm{dc}}/(2\mathrm{IBO}_e)$ into Eq. (11). Accordingly, $\mathrm{SNR} = P_{\mathrm{dc}}/(2\mathrm{IBO}_e\sigma_w^2)$.

The constructed problems in this section are all constrained nonlinear programming problems and can be numerically solved by standard nonlinear optimization algorithms, e.g., the particle swarm optimization (PSO) method [14]. Although solving these problems may be computationally intensive, it is not a major concern because the optimal power allocations can be determined off-line and remain the same for all symbols.

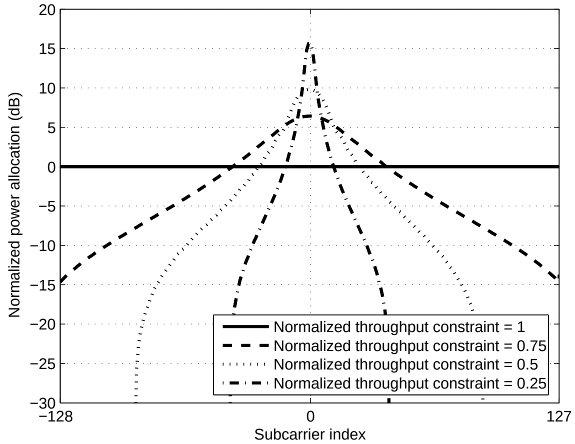

Fig. 1. The optimal normalized power allocations for maximizing power efficiency ρ with different normalized throughput constraints; SNR=10dB.

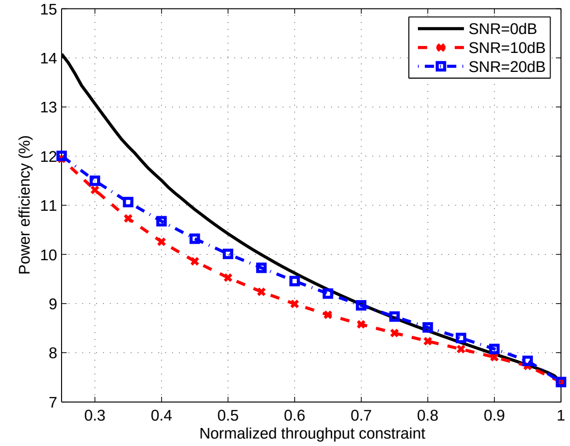

Fig. 2. The power efficiency improvements for different SNR values and normalized throughput constraints.

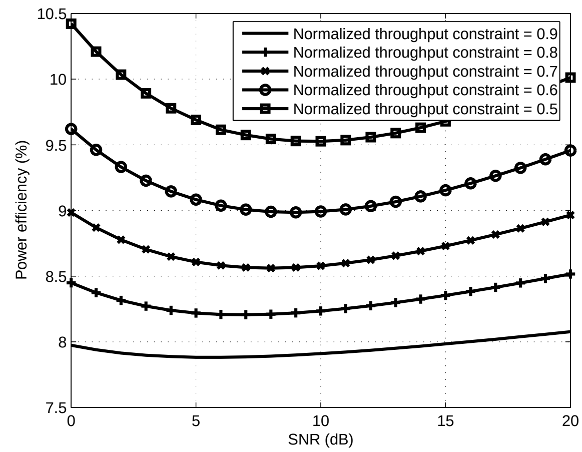

Fig. 3. The power efficiency versus SNR curves for different normalized throughput constraints.

V. Simulation Results

In this section, a few numerical solutions to the power allocation optimization problems in Section IV are illustrated. The peak-power-limited OFDM system with $N = 256$ subcarriers and a soft-limiter PA were assumed.

First, the PAR reduction performance through power allocations is illustrated by solving the power efficiency maximization problem in (17)-(19). In Fig. 1, the optimal normalized power allocations $P_k^{\dagger}$ are plotted for different normalized throughput constraints and $\mathrm{SNR} = \sigma^2/\sigma_w^2 = 10\mathrm{dB}$. As discussed in Section IV-A, equal power allocation achieves the maximum throughput $C_e(\sigma^2)$. Along with the decrease of the normalized throughput constraint β, more power can be allocated on the middle subcarriers such that λ is reduced. In Fig. 2, the optimized power efficiency is plotted for varying normalized throughput constraints. Additionally, for given normalized throughput constraints, the power efficiency versus SNR curves are summarized in Fig. 3. The results illustrate that, although the improvements may depend on the SNR value, optimally allocating power among subcarriers can significantly increase the power efficiency at the cost of reduced throughput.

For the peak-power-limited transmitter with a fixed throughput requirement C_{req}, solving (20)-(23) can provide the power allocation that minimizes the necessary PA power consumption. The optimal normalized power allocations have been found for different C_{req} requirements and plotted in Fig. 4. In contrast to Fig. 1, notice that the power allocation is plotted on a linear scale in Fig. 4. Thus, the optimal power allocation is shown to be close to the equal power allocation. Further, the numerical results of the optimal power allocation for $C_{\mathrm{req}} = 1\mathrm{bps/Hz}$ are summarized in Table I. By concentrating more power in the center subcarriers, the optimal power allocation improves the power efficiency and reduces the IBO. However, to guarantee the same throughput, more average power $\tilde{P}_{\mathrm{av}}$ is required in comparison with the equal power allocation. Overall, only a marginal power consumption reduction can be achieved.

The same observation is given for solving the throughput maximization problem in (24)-(25). For different SNR values, the comparisons between the average throughput of the equal and the optimal power allocations are shown in Table II. The optimal power allocation can only yield marginal throughput improvements relative to the equal power allocation.

VI. Conclusions

In this paper, the effects of subcarrier power allocation optimizations are investigated for distortionless peak-power-limited OFDM systems. Particular optimization problems are formulated to improve the power efficiency, power consumption and average throughput performance, respectively. Simulation results illustrate that power efficiency can be significantly increased by the optimal power allocation at the price

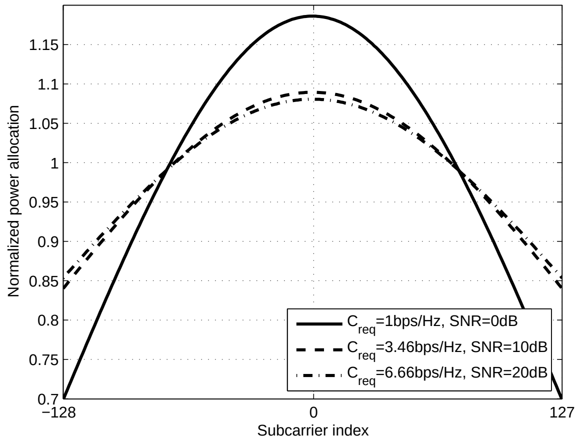

Fig. 4. The optimal power allocations for minimizing power consumption $P_{\rm dc}$ with different throughput requirements.

TABLE I

COMPARISON OF THE POWER CONSUMPTIONS OF THE EQUAL AND THE OPTIMAL POWER ALLOCATIONS; $C_{\rm req} = 1{\rm bps/Hz}$

	$\tilde{P}_{\rm av}$	IBO	$P_{\rm dc}$
Equal power allocation	1	6.75	13.50
Optimal power allocation	1.006	6.68	13.41

TABLE II

COMPARISON OF THE AVERAGE THROUGHPUT (IN ${\rm bps/Hz}$) OF THE EQUAL AND THE OPTIMAL POWER ALLOCATIONS WITH DIFFERENT SNR VALUES

SNR	0dB	10dB	20dB
Equal power allocation	1	3.459	6.658
Optimal power allocation	1.004	3.468	6.671

of reduced transmission rates. When power consumption and average throughput are concerned, the equal power allocation is close to the optimal power allocation and provides near-optimal performance.

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