

DSE 210: Worksheet #6 - Generative Models 2

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Problem 1

Given:

$$\begin{aligned}\pi(\text{happy}) &= \frac{3}{4} & \pi(\text{sad}) &= \frac{1}{4} \\ p(\text{talks a lot} \mid \text{happy}) &= \frac{2}{3} & p(\text{talks a lot} \mid \text{sad}) &= \frac{1}{6} \\ p(\text{talks a little} \mid \text{happy}) &= \frac{1}{6} & p(\text{talks a little} \mid \text{sad}) &= \frac{1}{6} \\ p(\text{silent} \mid \text{happy}) &= \frac{1}{6} & p(\text{silent} \mid \text{sad}) &= \frac{2}{3}\end{aligned}$$

- (a) To calculate the most likely mood, we determine the probabilities of each mood given the fact that he is only talking a little:

$$\begin{aligned}p(\text{happy} \mid \text{talks a little}) &= \frac{p(\text{talks a little} \mid \text{happy}) \times p(\text{happy})}{p(\text{talks a little})} \\ &= \frac{p(\text{talks a little} \mid \text{happy}) \times p(\text{happy})}{p(\text{talks a little} \mid \text{happy}) \times p(\text{happy}) + p(\text{talks a little} \mid \text{sad}) \times p(\text{sad})} \\ &= \frac{\frac{1}{6} \times \frac{3}{4}}{\frac{1}{6} \times \frac{3}{4} + \frac{1}{6} \times \frac{1}{4}} = \frac{\frac{3}{24}}{\frac{3}{24} + \frac{1}{24}} = \frac{\frac{3}{24}}{\frac{4}{24}} = \frac{3}{4}\end{aligned}$$

$$\begin{aligned}p(\text{sad} \mid \text{talks a little}) &= \frac{p(\text{talks a little} \mid \text{sad}) \times p(\text{sad})}{p(\text{talks a little})} \\ &= \frac{p(\text{talks a little} \mid \text{sad}) \times p(\text{sad})}{p(\text{talks a little} \mid \text{sad}) \times p(\text{sad}) + p(\text{talks a little} \mid \text{happy}) \times p(\text{happy})} \\ &= \frac{\frac{1}{6} \times \frac{1}{4}}{\frac{1}{6} \times \frac{1}{4} + \frac{1}{6} \times \frac{3}{4}} = \frac{\frac{1}{24}}{\frac{1}{24} + \frac{3}{24}} = \frac{\frac{1}{24}}{\frac{4}{24}} = \frac{1}{4}\end{aligned}$$

Since $p(\text{happy} \mid \text{talks a little}) > p(\text{sad} \mid \text{talks a little})$, the most likely mood is happy

- (b) The probability of the prediction in 1(a) being incorrect is equal to $p(\text{sad} \mid \text{talks a little})$, which was calculated above and is equal to $\frac{1}{4}$

Problem 2

To find the optimal classifier h^* , we need to find $h^*(x) = \arg \max_j \pi_j P_j(x)$.

$$\begin{aligned}\pi_1 P_1(x) &= \begin{cases} \frac{1}{3} \times \frac{7}{8} = \frac{7}{24}, & \text{if } -1 \leq x < 0 \\ \frac{1}{3} \times \frac{1}{8} = \frac{1}{24}, & \text{if } 0 \leq x \leq 1 \end{cases}, \\ \pi_2 P_2(x) &= \begin{cases} \frac{1}{6} \times 0 = 0, & \text{if } -1 \leq x < 0 \\ \frac{1}{6} \times 1 = \frac{1}{6}, & \text{if } 0 \leq x \leq 1 \end{cases}, \\ \pi_3 P_3(x) &= \begin{cases} \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}, & \text{if } -1 \leq x < 0 \\ \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}, & \text{if } 0 \leq x \leq 1 \end{cases}, \\ \max \pi_j P_j(x) &= \begin{cases} \pi_1 P_1(x) = \frac{1}{3} \times \frac{7}{8} = \frac{7}{24}, & \text{if } -1 \leq x < 0 \\ \pi_3 P_3(x) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}, & \text{if } 0 \leq x \leq 1 \end{cases}, \\ \therefore h^*(x) &= \begin{cases} 1, & \text{if } -1 \leq x < 0 \\ 3, & \text{if } 0 \leq x \leq 1 \end{cases}\end{aligned}$$

Problem 3

- (a) positively correlated (although this probably varies by make, model, and type of vehicle; for example, some sports cars engineered to be very light would be expensive)
- (b) positively correlated
- (c) negatively correlated

Problem 4

The elements are perfectly correlated, i.e. correlation = $\boxed{1}$.

Problem 5

(a) $\mu_x = 2, \quad \sigma_x = 1,$
 $\mu_y = 2, \quad \sigma_y = 0.5,$
 $\text{corr}(x, y) = -0.5,$

Mean vector $\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, and covariance matrix $\Sigma = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}$,

where:

$$\Sigma_{xx} = \text{var}(x) = \sigma_x^2 = 1^2 = 1$$

$$\Sigma_{yy} = \text{var}(y) = \sigma_y^2 = 0.5^2 = 0.25$$

$$\Sigma_{xy} = \Sigma_{yx} = \text{cov}(x, y) = \text{corr}(x, y) \times \sigma_x \times \sigma_y = -0.5 \times 1 \times 0.5 = -0.25$$

$$\therefore \mu = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 1 & -0.25 \\ -0.25 & 0.25 \end{bmatrix}$$

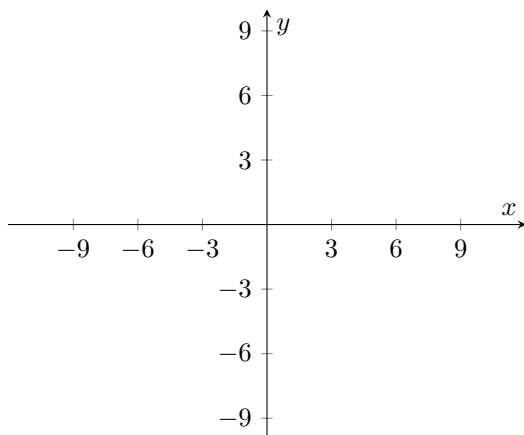
(b) $\mu_x = 1, \quad \sigma_x = 1,$
 $\mu_y = 1, \quad \sigma_y = 1,$
 $\text{corr}(x, y) = 1,$

$$\therefore \mu = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Problem 6

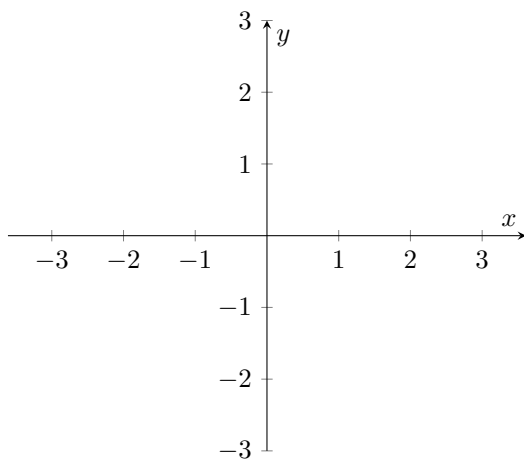
(a) $\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix},$

Gaussian contour sketch:



(b) $\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma = \begin{bmatrix} 1 & -0.75 \\ -0.75 & 1 \end{bmatrix},$

Gaussian contour sketch:



Problem 7

See Worksheet6_Problem7.ipynb notebook at <https://github.com/mas-dse/jsw037/tree/master/DSE210>.

Problem 8

Given: $\vec{w} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $\theta = 12$. Find the decision boundary in \mathbb{R}^2 .

The boundary will be orthogonal to \vec{w} , and the minimum distance between any point along the boundary and the origin will be $\frac{\theta}{\|\vec{w}\|}$.

$$\|\vec{w}\| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5, \text{ so } \frac{\theta}{\|\vec{w}\|} = \frac{12}{5}.$$

The unit vector in the direction of \vec{w} is $\vec{u}_w = \frac{\vec{w}}{\|\vec{w}\|} = \begin{bmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{bmatrix}$.

The intersection of \vec{w} and the boundary occurs at the point $\frac{\theta}{\|\vec{w}\|} \times \vec{u}_w = \frac{12}{5} \begin{bmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{-36}{25} \\ \frac{48}{25} \end{bmatrix}$.

If we define \vec{v} to be a vector parallel to the boundary, we know that $\vec{w} \bullet \vec{v} = 0$, since the boundary is orthogonal to \vec{w} .

$$\vec{w} \bullet \vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -3v_1 + 4v_2 = 0 \implies 4v_2 = 3v_1 \implies v_2 = \frac{3}{4}v_1, \text{ so the slope of } \vec{v} \text{ is } \frac{3}{4}.$$

Now, since we know a point on the boundary and the slope of the boundary, we can find the $y = mx + b$ slope-intercept equation for the boundary line by using $x = \frac{-36}{25}$, $y = \frac{48}{25}$, and $m = \frac{3}{4}$:

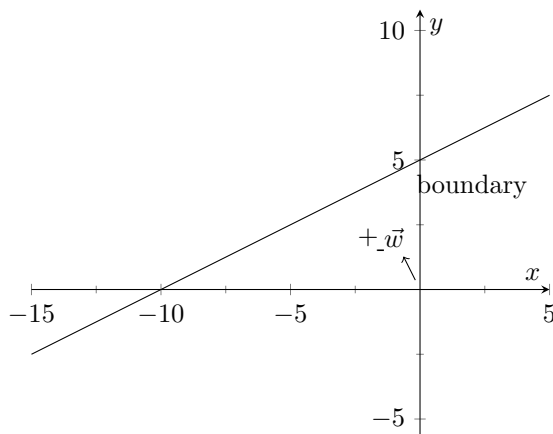
$$\frac{48}{25} = \frac{3}{4} \times \frac{-36}{25} + b \implies b = \frac{48}{25} - \frac{3}{4} \times \frac{-36}{25} = \frac{48}{25} + \frac{27}{25} = \frac{75}{25} = 3,$$

So the boundary line is defined by the equation $y = \frac{3}{4}x + 3$.

Now we can set each variable to 0 to solve for where the boundary intersects each coordinate axis.

If $y = 0$, then $x = -4$, and if $x = 0$, then $y = 3$.

The boundary intersects the coordinate axes at points $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 0 \end{bmatrix}$.



Problem 9

See Worksheet6_Problem9.ipynb notebook at <https://github.com/mas-dse/jsw037/tree/master/DSE210>.