DSE 210: Worksheet #4 - Random Variables, Expectation, and Variance

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Problem 2

Let X be the number of rolls until a 6 is seen. For any single roll, $p(6) = \frac{1}{6}$.

If p is the probability of the event of interest, then

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \times p(X = k) = \sum_{k=1}^{\infty} k \times p(1-p)^{k-1} = \frac{1}{p}$$

Since in this case, $p = p(6) = \frac{1}{6}$, $\mathbb{E}[X] = \boxed{6}$

Problem 4

(a) Let X be the number of people who get out on the i^{th} floor.

There are 10 floors, so the probability that any one person gets out on the i^{th} floor is $\frac{1}{10}$.

There are n people, so the sample space is $\Omega = \{0, 1, 2, 3, \ldots, n\}$.

We can model X as a *binomial*(n, p) random variable, with $p = \frac{1}{10}$, so

$$p(X = 1) = \binom{n}{1} p(1-p)^{n-1} = np(1-p)^{n-1} = \boxed{n\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{n-1}}$$

(b) Define random variable X_i such that

$$X_i = \begin{cases} 1, & \text{if 1 person gets out on floor } i. \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathbb{E}[X_i] = n\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{n-1}$, as shown in question 4(a).

Let Y be the number of floors in which exactly 1 person gets out.

$$\mathbb{E}[Y] = \sum_{i=1}^{10} \mathbb{E}[X_i] = 10 \times n\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{n-1} = \boxed{n\frac{9}{10}^{n-1}}$$

Problem 6

Let X_i be the event that the i^{th} student ends up in the correct bed. $p(X_i) = \frac{1}{n}$.

Define random variable X_i such that

$$X_i = \begin{cases} 1, & \text{if } i^{th} \text{ student gets in correct bed.} \\ 0, & \text{otherwise.} \end{cases}$$

$\mathbb{E}[X_i] = 1 \times \frac{1}{n} + 0 \times \left(1 - \frac{1}{n}\right) = \frac{1}{n}$.

Now, let X be the number of students in the correct bed. Then

$$X = \sum_{i=1}^{n} X_i, \text{ and } \mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \frac{1}{n} = \boxed{1}$$

Problem 8

Given : $p(1) = p(2) = p(3) = p(4) = \frac{1}{8}$, and $p(5) = p(6) = \frac{1}{4}$.

(a) Let Z be the outcome of a die roll using a die with the above given probabilities.

$$\begin{aligned}
\mathbb{E}[Z] &= \sum_{k=1}^{8} k \times p(k) \\
&= 1 \times p(1) + 2 \times p(2) + 3 \times p(3) + 4 \times p(4) + 5 \times p(5) + 6 \times p(6) \\
&= 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + 3 \times \frac{1}{8} + 4 \times \frac{1}{8} + 5 \times \frac{1}{4} + 6 \times \frac{1}{4} \\
&= \frac{1+2+3+4}{8} + \frac{5+6}{4} = \frac{32}{8} = \boxed{4}
\end{aligned}$$

$$\begin{aligned}
var[Z] &= \mathbb{E}[Z^2] - \mathbb{E}[Z]^2, \text{ and} \\
\mathbb{E}[Z^2] &= \frac{1^2+2^2+3^2+4^2}{8} + \frac{5^2+6^2}{4} = 19, \text{ and} \\
\mathbb{E}[Z]^2 &= 4^2 = 16, \text{ so} \\
var[Z] &= 19 - 16 = \boxed{3}
\end{aligned}$$

(b) Let X be the sum of 10 die rolls. Then X = 10 × Z, where Z is a single die roll as defined in 8(a), and

$\mathbb{E}[X] = 10 \times \mathbb{E}[Z] = 10 \times 4 = \boxed{40}$.

Because each die roll is independent, we can apply the variance rule for independent X_i's:

$var(X_1 + \ldots + X_k) = var(X_1) + \ldots + var(X_k)$, so

$var[X] = var[10 \times Z] = 10 \times var[Z] = 10 \times 3 = \boxed{30}$

Problem 8 continued on next page...

(c) Let A be the average of n die rolls. Then $A = \frac{1}{n}\sum_{i=1}^{n} Z_i$, where Z_i is the outcome of the i^{th} die roll modeled by random variable Z, defined in 8(a).

$$\mathbb{E}[A] = \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n} Z_i] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[Z_i] = \frac{1}{n} \times n \times \mathbb{E}[Z_i] = \mathbb{E}[Z_i] = \boxed{4}$$

$$var[A] = var[\frac{1}{n}\sum_{i=1}^{n} Z_i] = \left(\frac{1}{n}\right)^2 \times var[\sum_{i=1}^{n} Z_i] = \frac{1}{n^2} \times n \times var[Z_i] = \boxed{\frac{3}{n}}$$

Problem 10

(a) We can model X_i as $binomial(m, \frac{1}{n})$, so $p(X_i = 0) = \binom{m}{0}\left(\frac{1}{n}\right)^0\left(\frac{n-1}{n}\right)^m = \boxed{\left(\frac{n-1}{n}\right)^m}$

(b) $X_i \sim binomial(m, \frac{1}{n})$, so $p(X_i = 1) = \boxed{\binom{m}{1}\left(\frac{1}{n}\right)^1\left(\frac{n-1}{n}\right)^{m-1}}$

(c) $X_i \sim binomial(m, \frac{1}{n})$, so $\mathbb{E}[X_i] = m \times \frac{1}{n} = \boxed{\frac{m}{n}}$

(d) $X_i \sim binomial(m, \frac{1}{n})$, so $var[X_i] = \left(m \times \frac{1}{n}\right)\left(1 - \frac{1}{n}\right) = \left(\frac{m}{n}\right)\left(\frac{n-1}{n}\right) = \boxed{\frac{m\ (n-1)}{n^2}}$

Problem 12

Let X be the number of coin tosses required to see the same result twice in a row.

Then $p(X = 1) = 0$, and $p(X = k) = \left(\frac{1}{2}\right)^{k-1}$ for $k > 1$. Thus,

$$\begin{aligned}
\mathbb{E}[X] &= 1 \times p(X = 1) + 2 \times p(X = 2) + 3 \times p(X = 3) + \ldots \\
&= 1 \times 0 + 2 \times \left(\frac{1}{2}\right)^{2-1} + 3 \times \left(\frac{1}{2}\right)^{3-1} + \ldots \\
&= 1 \times 0 + 2 \times \left(\frac{1}{2}\right)^{1} + 3 \times \left(\frac{1}{2}\right)^{2} + \ldots \\
&= 1 \times 0 + 2 \times \frac{1}{2} + 3 \times \frac{1}{4} + \ldots \\
&= \sum_{k=2}^{\infty} k\left(\frac{1}{2}\right)^{k-1} = \boxed{3}
\end{aligned}$$