DSE 210: Worksheet #6 - Generative Models 2

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Problem 1

Given:

$$\pi(\text{happy}) = \tfrac{3}{4} \qquad \pi(\text{sad}) = \tfrac{1}{4}$$
$$p(\text{talks a lot} \mid \text{happy}) = \tfrac{2}{3} \qquad p(\text{talks a lot} \mid \text{sad}) = \tfrac{1}{6}$$
$$p(\text{talks a little} \mid \text{happy}) = \tfrac{1}{6} \qquad p(\text{talks a little} \mid \text{sad}) = \tfrac{1}{6}$$
$$p(\text{silent} \mid \text{happy}) = \tfrac{1}{6} \qquad p(\text{silent} \mid \text{sad}) = \tfrac{2}{3}$$

(a) To calculate the most likely mood, we determine the probabilities of each mood given the fact that he is only talking a little:

$$\begin{aligned}
p(\text{happy} \mid \text{talks a little}) &= \frac{p(\text{talks a little} \mid \text{happy}) \times p(\text{happy})}{p(\text{talks a little})} \\
&= \frac{p(\text{talks a little} \mid \text{happy}) \times p(\text{happy})}{p(\text{talks a little} \mid \text{happy}) \times p(\text{happy}) + p(\text{talks a little} \mid \text{sad}) \times p(\text{sad})} \\
&= \frac{\frac{1}{6} \times \frac{3}{4}}{\frac{1}{6} \times \frac{3}{4} + \frac{1}{6} \times \frac{1}{4}} = \frac{\frac{3}{24}}{\frac{3}{24} + \frac{1}{24}} = \frac{\frac{3}{24}}{\frac{4}{24}} = \frac{3}{4}
\end{aligned}$$

$$\begin{aligned}
p(\text{sad} \mid \text{talks a little}) &= \frac{p(\text{talks a little} \mid \text{sad}) \times p(\text{sad})}{p(\text{talks a little})} \\
&= \frac{p(\text{talks a little} \mid \text{sad}) \times p(\text{sad})}{p(\text{talks a little} \mid \text{sad}) \times p(\text{sad}) + p(\text{talks a little} \mid \text{sad}) \times p(\text{sad})} \\
&= \frac{\frac{1}{6} \times \frac{1}{4}}{\frac{1}{6} \times \frac{1}{4} + \frac{1}{6} \times \frac{3}{4}} = \frac{\frac{1}{24}}{\frac{1}{24} + \frac{3}{24}} = \frac{\frac{1}{24}}{\frac{4}{24}} = \frac{1}{4}
\end{aligned}$$

Since $p(\text{happy} \mid \text{talks a little}) > p(\text{sad} \mid \text{talks a little})$, the most likely mood is $\boxed{\text{happy}}$

(b) The probability of the prediction in 1(a) being incorrect is equal to $p(\text{sad} \mid \text{talks a little})$, which was calculated above and is equal to $\boxed{\frac{1}{4}}$

Problem 2

To find the optimal classifier h^*, we need to find $h^*(x) = \arg\max_j \pi_j P_j(x)$.

$$\pi_1 P_1(x) = \begin{cases} \frac{1}{3} \times \frac{7}{8} = \frac{7}{24}, & \text{if } -1 \le x < 0 \\ \frac{1}{3} \times \frac{1}{8} = \frac{1}{24}, & \text{if } \quad 0 \le x \le 1 \end{cases},$$

$$\pi_2 P_2(x) = \begin{cases} \frac{1}{6} \times 0 = 0, & \text{if } -1 \le x < 0 \\ \frac{1}{6} \times 1 = \frac{1}{6}, & \text{if } \quad 0 \le x \le 1 \end{cases},$$

$$\pi_3 P_3(x) = \begin{cases} \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}, & \text{if } -1 \le x < 0 \\ \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}, & \text{if } \quad 0 \le x \le 1 \end{cases},$$

$$max\ \pi_j P_j(x) = \begin{cases} \pi_1 P_1(x) = \frac{1}{3} \times \frac{7}{8} = \frac{7}{24}, & \text{if } -1 \le x < 0 \\ \pi_3 P_3(x) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}, & \text{if } \quad 0 \le x \le 1 \end{cases},$$

$$\therefore h^*(x) = \begin{cases} 1, & \text{if } -1 \le x < 0 \\ 3, & \text{if } \quad 0 \le x \le 1 \end{cases}$$

Problem 3

(a) positively correlated

(b) positively correlated

(c) negatively correlated

Problem 4

The elements are perfectly correlated, i.e. correlation = $\boxed{1}$.

Problem 5

(a) $\mu_x = 2, \qquad \sigma_x = 1,$
$\mu_y = 2, \qquad \sigma_y = 0.5,$
$corr(x, y) = -0.5,$

Mean vector $\begin{bmatrix}\mu_x \\ \mu_y\end{bmatrix} = \begin{bmatrix}2 \\ 2\end{bmatrix}$, and covariance matrix $\Sigma = \begin{bmatrix}\Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy}\end{bmatrix}$,

where:

$$\Sigma_{xx} = var(x) = \sigma_x^2 = 1^2 = 1$$
$$\Sigma_{yy} = var(y) = \sigma_y^2 = 0.5^2 = 0.25$$
$$\Sigma_{xy} = \Sigma_{yx} = cov(x, y) = corr(x, y) \times \sigma_x \times \sigma_y = -0.5 \times 1 \times 0.5 = -0.25$$

$$\therefore \boxed{\mu = \begin{bmatrix}2 \\ 2\end{bmatrix} \text{ and } \Sigma = \begin{bmatrix}1 & -0.25 \\ -0.25 & 0.25\end{bmatrix}}$$

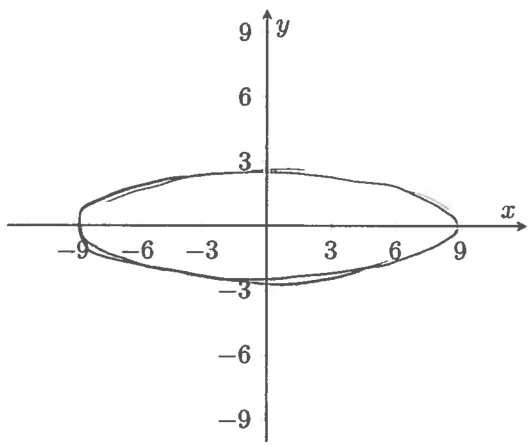
(b) $\mu_x = 1, \qquad \sigma_x = 1,$
$\mu_y = 1, \qquad \sigma_y = 1,$
$corr(x, y) = 1,$

$$\therefore \boxed{\mu = \begin{bmatrix}1 \\ 1\end{bmatrix} \text{ and } \Sigma = \begin{bmatrix}1 & 1 \\ 1 & 1\end{bmatrix}}$$

Problem 6

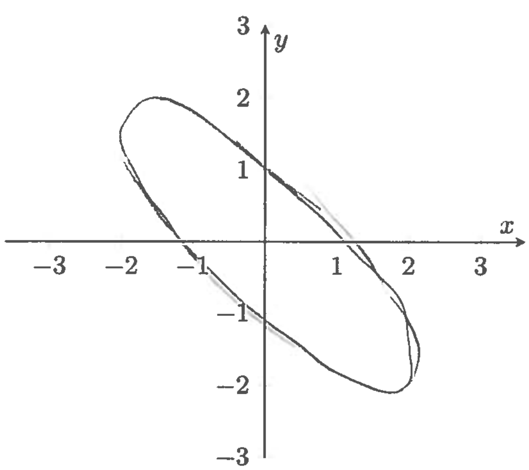
(a) $\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix},$

Gaussian contour sketch:



(b) $\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma = \begin{bmatrix} 1 & -0.75 \\ -0.75 & 1 \end{bmatrix},$

Gaussian contour sketch:



Problem 7

See Worksheet6_Problem7.ipynb notebook.

DSE210 Worksheet 6

Problem 7

In [1]:
```
import numpy as np
```

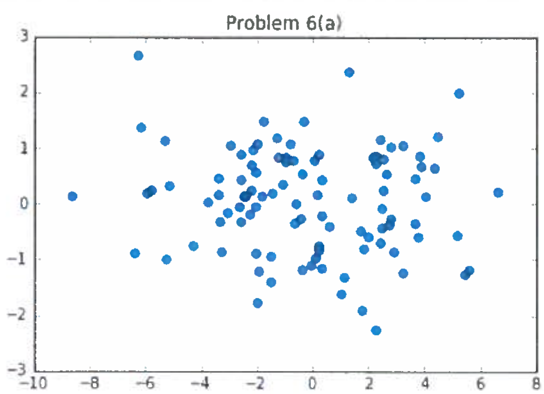
In [2]:
```
%pylab inline
import matplotlib.pyplot as plt
```

```
Populating the interactive namespace from numpy and matplotlib
```

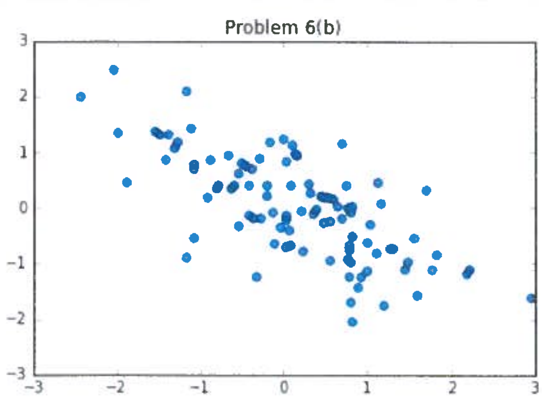
In [3]:
```
samples_a = np.random.multivariate_normal(mean = [0,0], cov = [[9,0],[
0,1]], size = 100)
x_a = [sample[0] for sample in samples_a]
y_a = [sample[1] for sample in samples_a]
```

In [5]:
```
samples_b = np.random.multivariate_normal(mean = [0,0], cov = [[1,-0.7
5],[-0.75,1]], size = 100)
x_b = [sample[0] for sample in samples_b]
y_b = [sample[1] for sample in samples_b]
```

In [36]:
```
plt.plot(x_a, y_a, 'o', alpha = 0.75)
plt.title('Problem 6(a)')
plt.show();
```



In [37]:
```
plt.plot(x_b, y_b, 'o', alpha = 0.75)
plt.title('Problem 6(b)')
plt.show();
```



In [ ]:

Problem 8

Given: $\vec{w} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}, \theta = 12$. Find the decision boundary in $\mathbb{R}^2$.

The boundary will be orthogonal to $\vec{w}$, and the minimum distance between any point along the boundary and the origin will be $\frac{\theta}{\|\vec{w}\|}$.

$\|\vec{w}\| = \sqrt{(-3)^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$, so $\frac{\theta}{\|\vec{w}\|} = \frac{12}{5}$.

The unit vector in the direction of $\vec{w}$ is $\vec{u}_w = \frac{\vec{w}}{\|\vec{w}\|} = \begin{bmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{bmatrix}$.

The intersection of $\vec{w}$ and the boundary occurs at the point $\frac{\theta}{\|\vec{w}\|} \times \vec{u}_w = \frac{12}{5} \begin{bmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{-36}{25} \\ \frac{48}{25} \end{bmatrix}$.

If we define $\vec{v}$ to be a vector parallel to the boundary, we know that $\vec{w} \bullet \vec{v} = 0$, since the boundary is orthogonal to $\vec{w}$.

$\vec{w} \bullet \vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -3v_1 + 4v_2 = 0 \implies 4v_2 = 3v_1 \implies v_2 = \frac{3}{4}v_1$, so the slope of $\vec{v}$ is $\frac{3}{4}$.

Now, since we know a point on the boundary and the slope of the boundary, we can find the $y = mx + b$ slope-intercept equation for the boundary line by using $x = \frac{-36}{25}, y = \frac{48}{25}$, and $m = \frac{3}{4}$:

$$\frac{48}{25} = \frac{3}{4} \times \frac{-36}{25} + b \implies b = \frac{48}{25} - \frac{3}{4} \times \frac{-36}{25} = \frac{48}{25} + \frac{27}{25} = \frac{75}{25} = 3,$$

So the boundary line is defined by the equation $y = \frac{3}{4}x + 3$.

Now we can set each variable to 0 to solve for where the boundary intersects each coordinate axis.

If $y = 0$, then $x = -4$, and if $x = 0$, then $y = 3$.

The boundary intersects the coordinate axes at points $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 0 \end{bmatrix}$.