

# Toward the Effective 2-Topos

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# Motivation

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- In the topos  $\mathbf{Eff}$  all maps  $\mathbb{N} \rightarrow \mathbb{N}$  are **Computable**.
- $\mathbf{Eff}$  contains a **small**, full subcat  $\mathbf{Mod} \subseteq \mathbf{Eff}$  that's
  - internally **complete**,
  - , **not** a poset.
- The topos  $\mathbf{Eff}$  is **not** Grothendieck.
- A higher version  $\mathbf{Eff}^\infty$  would be an example of a (non-Grothendieck) **elementary  $\infty$ -topos**.
- **But** taking e.g.  $\mathbf{Eff}^\infty = (\mathbf{Eff}^{\Delta^{\text{op}}}, \text{kan})$  does **not** work...

# The Effective 1-Topos

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Recall the topos  $\mathcal{E}ff$  as an exact completion;

$PAsm \rightarrow Asm \rightarrow \mathcal{E}ff$

Lex

Regular

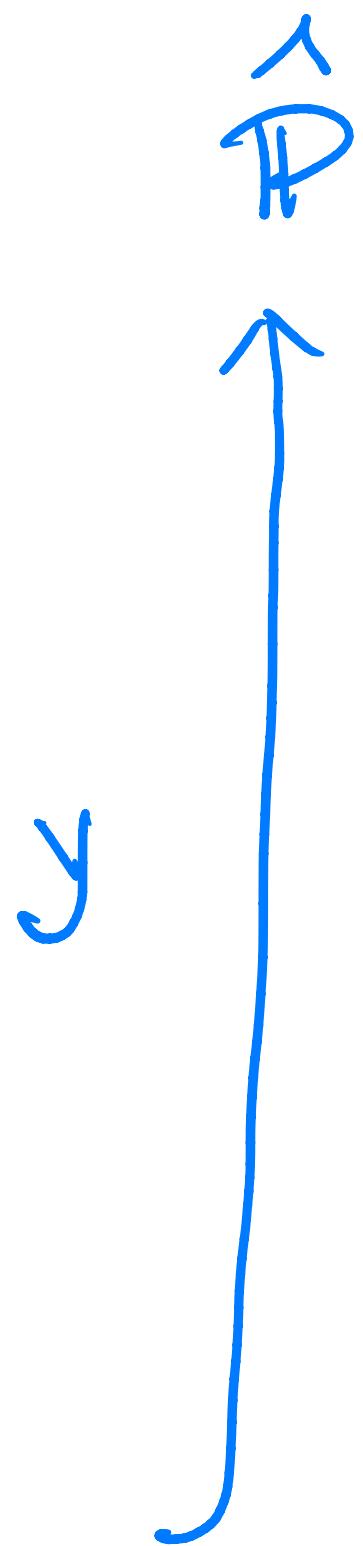
Exact

$Asm = \frac{\text{reg}}{\text{lex}}(PAsm)$  free completion

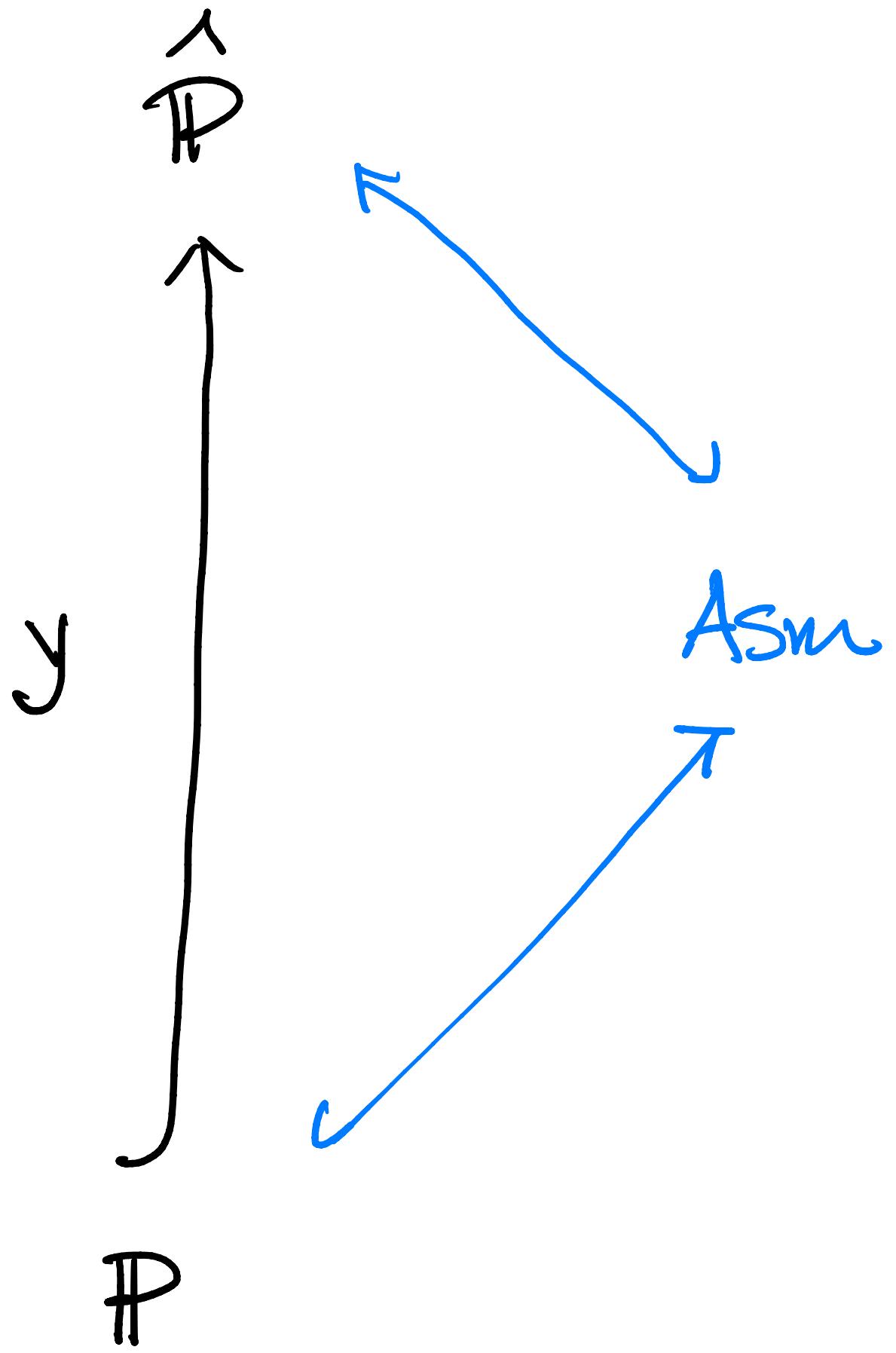
$\mathcal{E}ff = \frac{\text{ex}}{\text{reg}}(Asm)$  "

$= \frac{\text{ex}}{\text{lex}}(PAsm)$  "

free  
Colin



$$P_{\text{Asm}} = \hat{P}$$

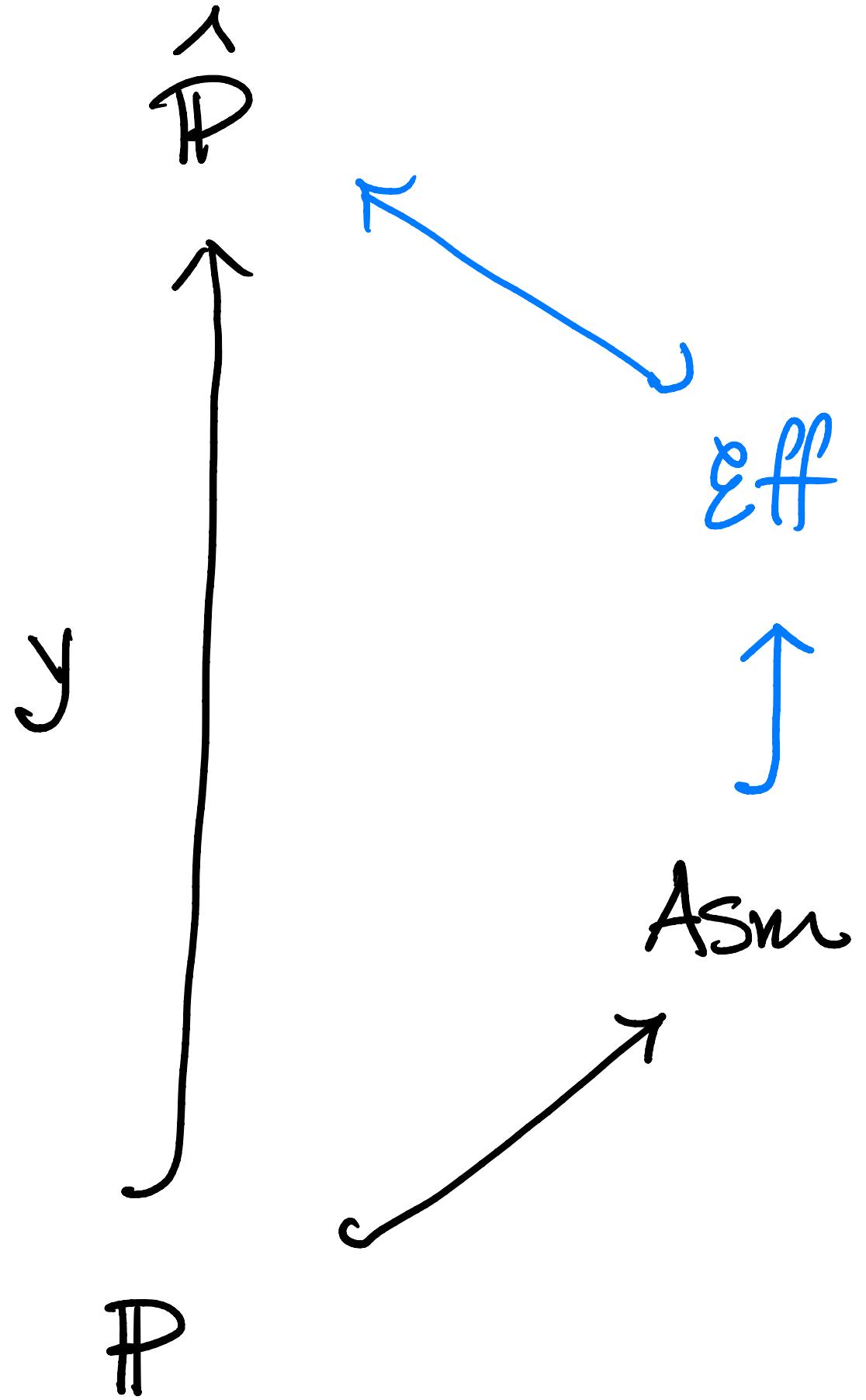


free kernel quotients

$$k \rightarrowtail P \xrightarrow{\quad} Q$$

$$\downarrow \pi$$

$$P/k \rightarrowtail P \xrightarrow{\quad} Q$$



free exact quotients

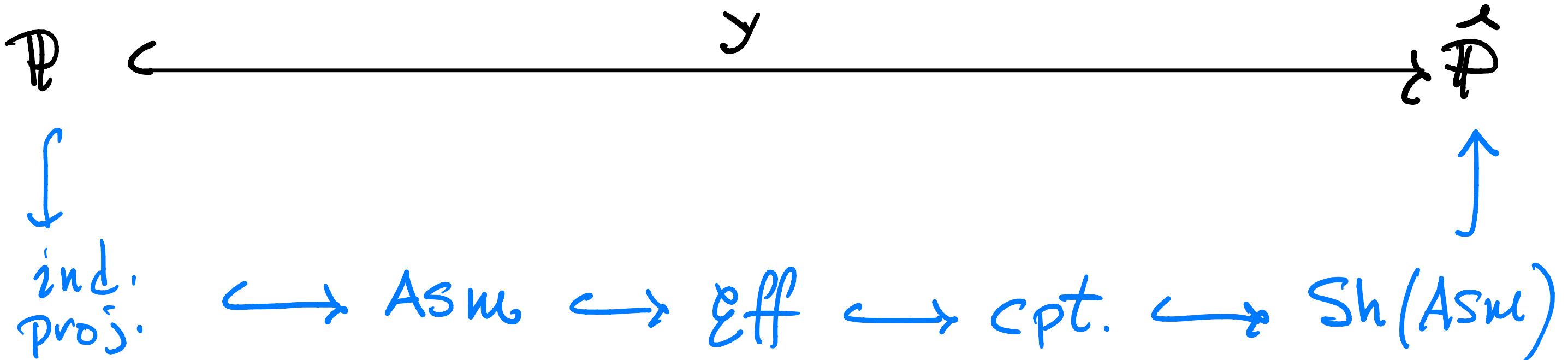
$$E \rightarrowtail P \twoheadrightarrow P/E$$

free Kernel quotients

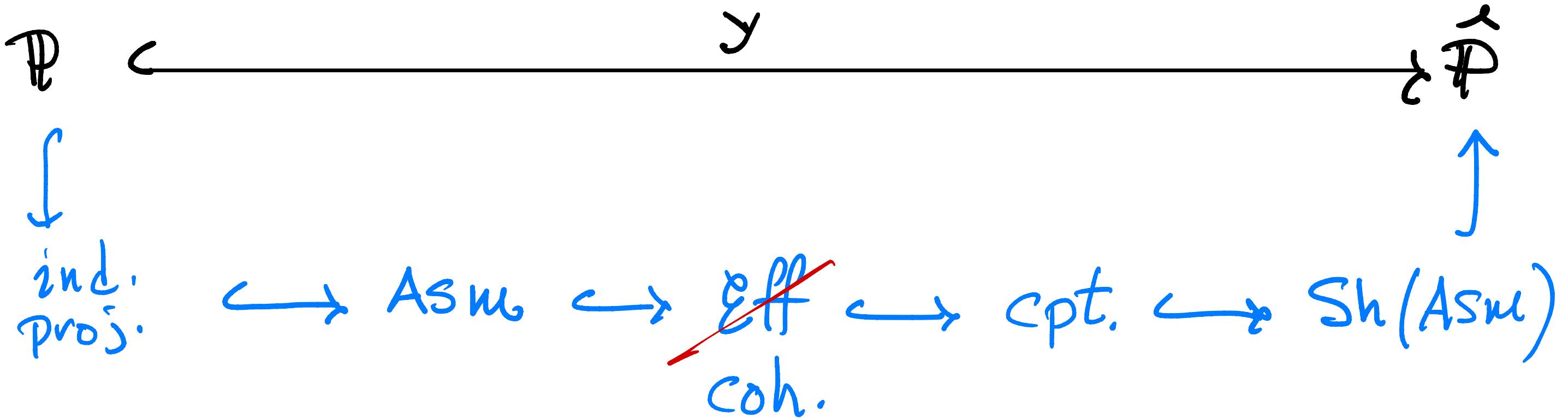
$$K \rightarrowtail P \longrightarrow Q$$

$$\downarrow P/K$$

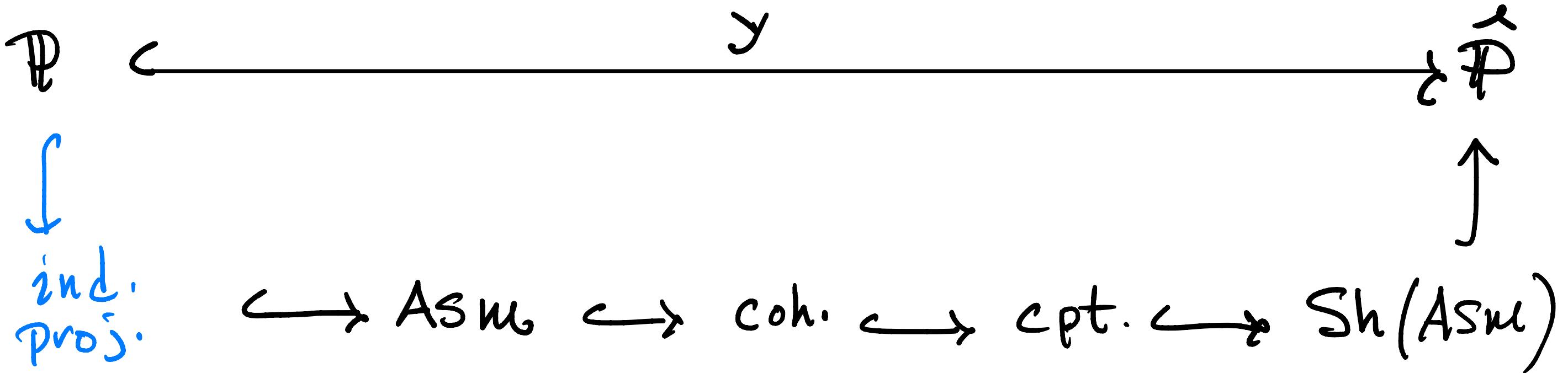
## Factorization of Yoneda



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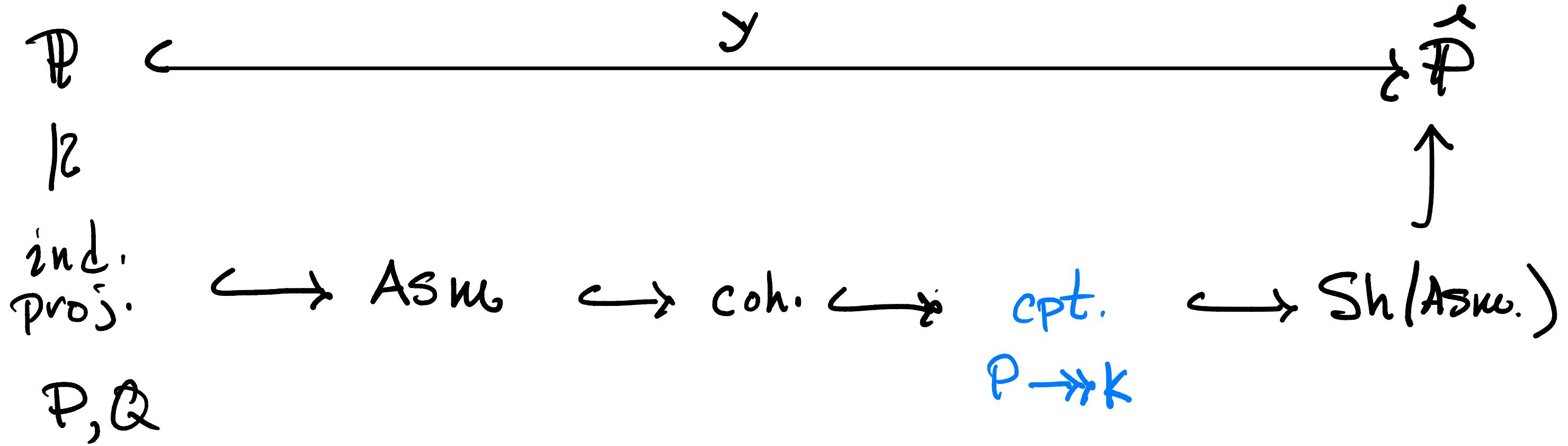
# Factorization of Yoneda



indecomposable projectives:

- $I = X + Y \Rightarrow I = X$       or       $I = Y$       }       $\Leftrightarrow$       "representable"
- $P \xrightarrow{\sim} X$        $E$
- $P = YP$   
f. some  $P \in P$ .

# Factorization of Yoneda



K (Super)Compact :

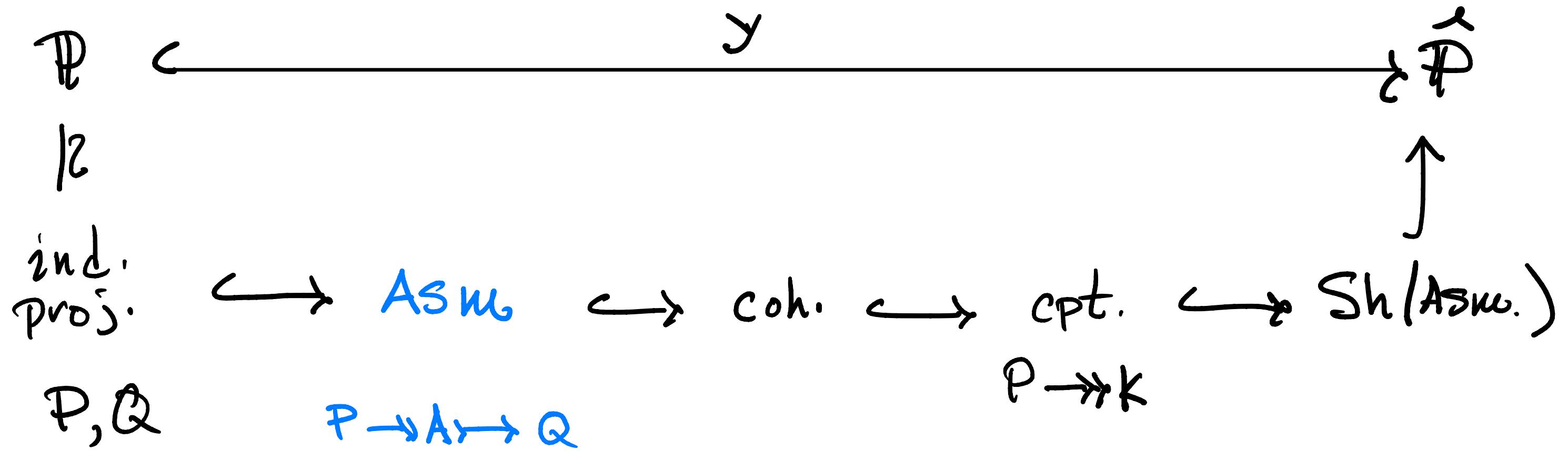
$(X_i \rightarrow K)_i$  covers

$\nexists X_k \twoheadrightarrow K$  f. some  $k$

$\Leftrightarrow$

$P \twoheadrightarrow K$   
f. some  $P$

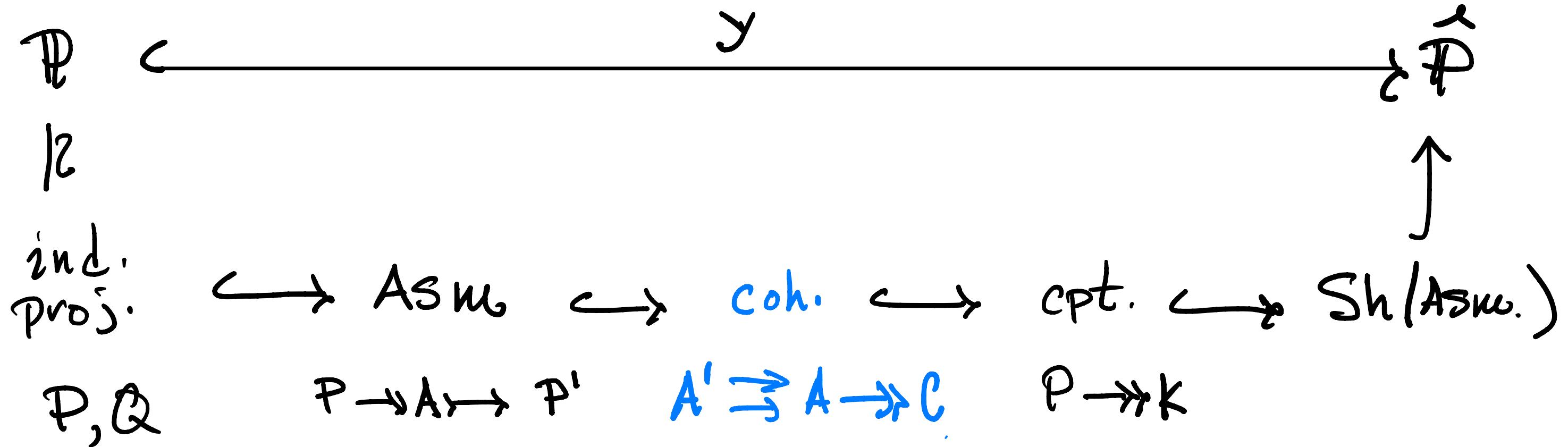
# Factorization of Yoneda



Assemblies:

$$\begin{array}{c}
 P \rightarrowtail A \rightarrowtail Q \\
 \text{for ind. proj. } P, Q
 \end{array}
 \quad \left\{ \quad \rightleftharpoons \quad
 \begin{array}{c}
 A \text{ cpt. \&} \\
 A \rightarrowtail Q
 \end{array}$$

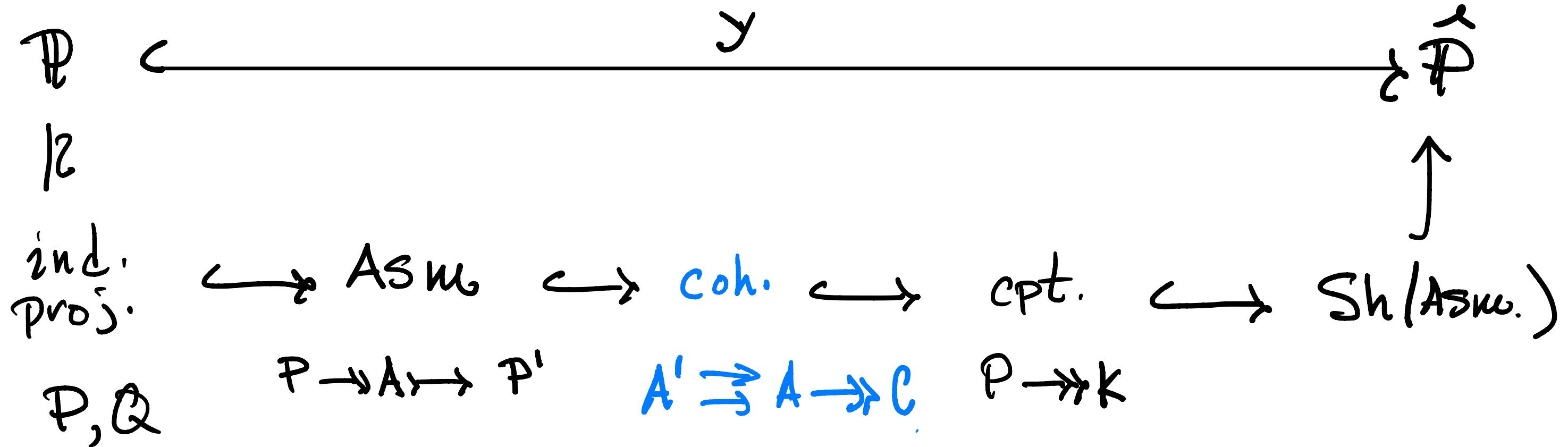
# Factorization of Yoneda



C Coherent :

- $C$  is cpt.
- $C \xrightarrow{\Delta} C \times C$  cpt.

# Factorization of Yoneda



$C$  coherent :

- $C$  is cpt.
- $C \xrightarrow{\Delta} C \times C$  cpt.

where

$f: X \rightarrow Y$  is cpt :

$$\begin{array}{ccc} \mathcal{K}' & \xrightarrow{\quad} & X \\ \downarrow & \lrcorner & \downarrow f \\ \mathcal{K} & \xrightarrow{\quad} & Y \end{array}$$

# Factorization of Yoneda

$$\begin{array}{ccccc} & & \gamma & & \\ P & \leftarrow & & & \rightarrow \hat{P} \\ |z & & & & \uparrow \end{array}$$

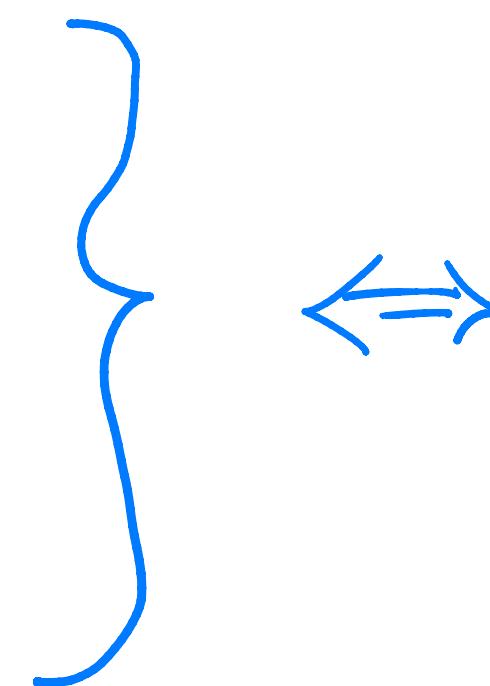
$\stackrel{\text{ind.}}{\text{proj.}}$ :  $\hookrightarrow \text{ASm} \hookrightarrow \text{coh.} \hookrightarrow \text{cpt.} \hookrightarrow \text{Sh}(\text{ASm.})$

$$P, Q \quad P \rightarrow A \rightarrow P' \quad A' \xrightarrow{\exists} A \rightarrow C \quad P \rightarrow \rightarrow K$$

$C$  Coherent :

- $C$  is cpt.

- $C \xrightarrow{\Delta} C \times C$  cpt.



$$A' \xrightarrow{\exists} A \rightarrow \rightarrow C$$

exact

# Factorization of Yoneda

$$\begin{array}{ccccc}
 & & y & & \\
 P & \leftarrow & & & \rightarrow \hat{P} \\
 |z & & & & \uparrow \\
 \text{ind: proj.} & \hookrightarrow & \text{ASm} & \hookrightarrow \text{coh.} & \hookrightarrow \text{cpt.} \hookrightarrow \text{Sh(ASm.)}
 \end{array}$$

$P, Q$        $P \rightarrow A \rightarrow P'$      $A' \xrightarrow{\exists} A \rightarrow C$      $P \rightarrowtail K$

$F$  sheaf on ASm  
 (for the reg. epi. top.)

$\Leftrightarrow$

$$\begin{array}{ccc}
 A \times_A & \xrightarrow{\quad} & A \xrightarrow{\quad} F \\
 \downarrow & & \downarrow \\
 B & \dashrightarrow & B
 \end{array}$$

# Factorization of Yoneda

$$\begin{array}{ccccc} \mathcal{P} & \xleftarrow{\quad} & \mathbf{y} & \xrightarrow{\quad} & \widehat{\mathcal{P}} \\ |z & & & & |z \end{array}$$

ind.  
proj.  $\hookrightarrow \text{ASm} \hookrightarrow \text{coh.} \hookrightarrow \text{cpt.} \hookrightarrow \text{Sh}(\text{ASm.})$

$$P, Q \quad P \rightarrow A \rightarrow P' \quad A' \xrightarrow{\exists} A \rightarrow C \quad P \rightarrow K$$

F sheaf on ASm  $\Leftrightarrow$   $\begin{array}{ccc} A \times_A A & \xrightarrow{\quad} & A \\ \downarrow B & & \downarrow B \\ A & \xrightarrow{\quad} & F \end{array}$

(for the reg. epis. top.)

$\Leftrightarrow F \in \widehat{\mathcal{P}} = [\mathcal{P}^{\text{op}}, \text{Set}]$ .

Thm (Lack)

For a reg. cat.  $\mathcal{R}$ ,

$$\mathcal{R}_{\text{ex/reg}} \subseteq \text{Sh}(\mathcal{R}, \text{reg epi})$$

$$= \langle E \mid yA' \xrightarrow{\quad} yA \rightarrowtail E \text{ ex.} \rangle$$

f.  $A', A \in \mathcal{R}$

Thm (Lack)

For a reg. cat.  $R$ ,

$$R_{\text{ex/reg}} \subseteq \text{Sh}(R, \text{reg epi})$$

$$= \langle E \mid yA' \xrightarrow{\exists} yA \rightarrowtail E \text{ ex.} \rangle$$

f.  $A', A \in R$

Cor.  $\text{Sh}(\text{Asm})_{\text{coh}} = \text{ex/reg}(\text{Asm}).$

Thm (Lack)

For a reg. cat.  $R$ ,

$$R_{\text{ex/reg}} \subseteq \text{Sh}(R, \text{reg epi})$$

$$= \langle E \mid yA' \xrightarrow{\exists} yA \rightarrowtail E \text{ ex.} \rangle$$

$$\text{f. } A, A' \in R$$

Cor.  $\text{Sh(Asm)}_{\text{coh}} = \text{ex/reg(Asm)} = \text{eff!}$

# Summary

$\text{PAsm} \subset \text{Asm} \subset \text{Eff} \subset \overset{\text{l}}{\text{PAsm}}$

l2 l2

CohSh(Asm) < Sh(Asm)

## Summary

$$\text{PAsm} \subset \text{Asm} \subset \mathcal{E}\text{ff} \subset \overset{\wedge}{\text{PAsm}}$$

Now in order to get a 2-topos we'll take  
internal groupoids in the Groth. topos  $\overset{\wedge}{\text{PAsm}}$ .

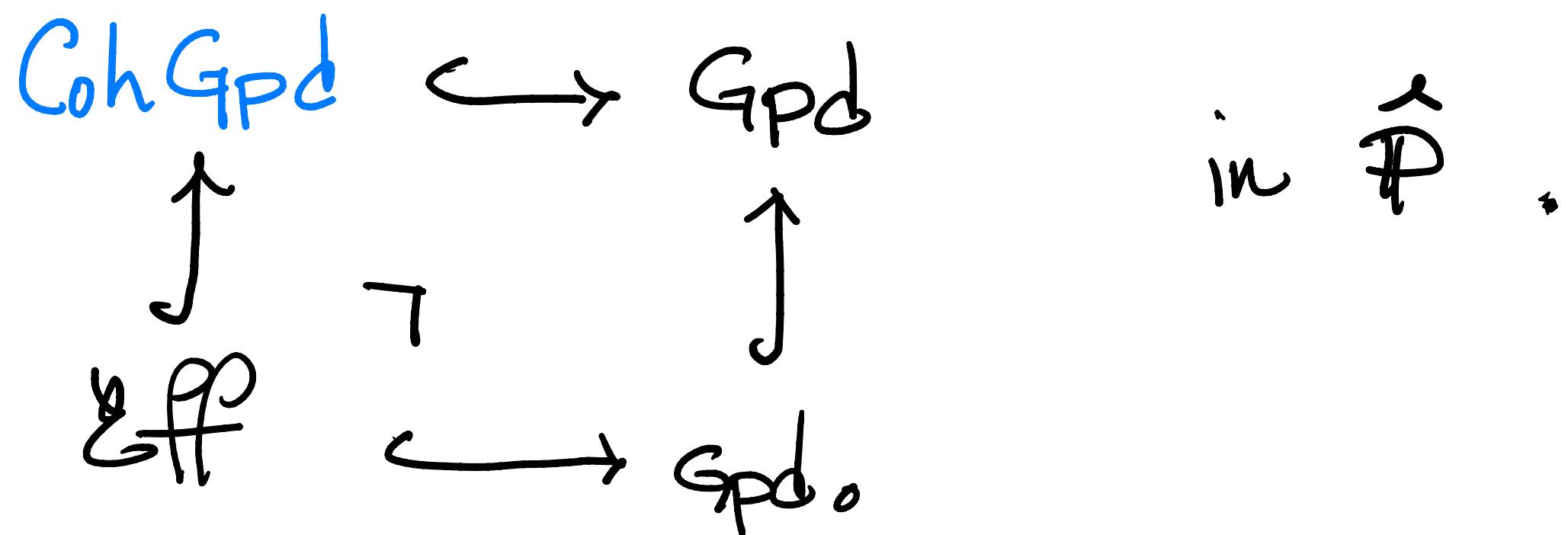
## Summary

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Now in order to get a 2-topos we'll take  
internal groupoids in the Groth. topos  $\overset{\wedge}{\text{PAsm}}$ ,

$$\text{Gpd}(\overset{\wedge}{\mathbb{P}}) = [\overset{\wedge}{\mathbb{P}}^\text{op}, \text{Gpd}] .$$

Finally, we shall restrict  $\text{Gpd}(\hat{\mathcal{P}})$  to a  
subcategory of "Coherent groupoids" such that



## QMS on $\text{Gpd}(\mathcal{E})$

There are different QMS on  $\text{Gpd}(\mathcal{E})$  for Groth. bpos  $\mathcal{E}$ . We use one that:

- (1) admits a model of H $\infty$ T $\tau$ ,
- (2) has Eff as the coherent  $\sigma$ -types:

$$\text{Eff} = \text{Coh}_{\text{Gpd}(\mathcal{E})} \subseteq \text{Gpd}(\mathcal{E}) \subseteq \text{Gpd}(\mathcal{E}).$$

For (1), use Shulman's thm. on  $\infty$ -toposes:

Thm. There's a type theoretic model structure  
on  $\text{Gpd}(\hat{\mathbb{P}})$  with :

- Weak equivalences = objectwise equivalences of cats,
- Cofibrations = objectwise injections on objects,
- Fibrations = algebraic "cobar" fibrations .

Note : Quillen equiv. to the J-T Strong stacks QMS.

# Coherent Groupoid

For (2):

Def. A groupoid  $\mathcal{G} = (G, \rightrightarrows G_0)$  in  $\hat{\mathcal{P}}$  is  
coherent if:

- $\mathcal{G}$  is cpt,
- $\Delta_1 : \mathcal{G} \longrightarrow \mathcal{G} \times \mathcal{G}$  is cpt,
- $\Delta_2 : \Delta_1 \longrightarrow \mathcal{G} \times \mathcal{G}$  is cpt.

# Cohesive Groupoid

For (2):

Def. A groupoid  $\mathcal{G} = (G, \rightrightarrows G_0)$  in  $\widehat{\mathbf{P}}$  is  
cohesive if:

- $\mathcal{G}$  is  $\overset{h}{\wedge}$ cpt,  $P \rightarrow \mathcal{G}$  eso
- $\Delta_1 : \mathcal{G} \longrightarrow \mathcal{G} \times \mathcal{G}$  is  $\overset{h}{\wedge}$ cpt,
- $\Delta_2 : \Delta_1 \longrightarrow \Delta_1 \times \Delta_1$  is  $\overset{h}{\wedge}$ cpt.

Prop.

Let  $\mathbb{G}$  be a coherent gpd and a 0-type  $\Delta_1 : \mathbb{G} \rightarrow \mathbb{G} \times \mathbb{G}$ . Then

- $\pi_0 \mathbb{G}$  is in  $\mathcal{E}\mathbf{ff}_s$ .
- $\mathbb{G} \xrightarrow{\text{w.e.}} \pi_0 \mathbb{G}$ .

Prop. If  $\mathbb{G}$  is a coherent 0-type, then  $\prod_0 \mathbb{G}$  is coherent.

Pf. Since  $\mathbb{G}$  is cpt it has a cover  $P \rightarrow \mathbb{G}$ .

Take the p.b.  $\mathbb{K}$  and its cover  $P'$ , since  $\Delta$  cpt.

$$\begin{array}{ccccc} P' & \xrightarrow{\quad} & \mathbb{K} & \longrightarrow & \mathbb{G}^I \\ & \searrow & \downarrow & & \downarrow \tilde{\alpha} \\ & & P \times P & \longrightarrow & \mathbb{G} \times \mathbb{G} \end{array}$$

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Since  $\mathbb{G}$  is a 0-type,

$\Delta: \mathbb{G} \rightarrow \mathbb{G} \times \mathbb{G}$  is monic.

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Hence so is  $K \rightarrow P \times P$ ,  
so  $K$  is an assembly.

Prop. If  $\mathbb{G}$  is a coherent 0-type, then  $\Pi_0 \mathbb{G}$  is coherent.

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Take the P.b.  $K$  and its cover  $P'$ .

Since  $\mathbb{G}$  is a 0-type

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Write  $\bar{\mathbb{G}} = \pi_0 \mathbb{G} = G_0/G_1$ .

Prop. If  $\mathbb{G}$  is a coherent 0-type, then  $\Pi_0 \mathbb{G}$  is coherent.

Pf. Since  $G_0$  is cpt it has a cover  $P \rightarrow \mathbb{G}$ .

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We then have  $K \xrightarrow{\quad} P \rightarrow \bar{\mathbb{G}}$  exact.

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$$\begin{array}{ccccc} P' & \xrightarrow{\quad} & K & \xrightarrow{\quad} & I \\ \searrow & & \downarrow & & \downarrow \\ & & P \times P & \xrightarrow{\quad} & \mathbb{G} \times \mathbb{G} \\ & & & & \xrightarrow{\quad} & \bar{\mathbb{G}} \times \bar{\mathbb{G}} \\ & & & & & \downarrow \\ & & & & & \bar{\mathbb{G}} \end{array}$$

Finally,  $\mathbb{G} \simeq \bar{\mathbb{G}}$  is a weak equivalence.  $\square$

Then Over  $\hat{P}$ , we have

$$\begin{array}{ccc} \mathcal{E}\mathbf{ff} & \cong & \text{CohGpd}_0 \longrightarrow \text{Gpd}_0 \\ & & \downarrow \\ & & \text{CohGpd} \longrightarrow \text{Gpd} \end{array} .$$

Then Over  $\hat{P}$ , we have

$$\mathcal{E}ff \cong \text{CohGpd}_0 \longrightarrow \text{Gpd}_0$$

$$\downarrow$$

$$\text{Take } \mathcal{E}ff^2 := \text{CohGpd} \longrightarrow \text{Gpd}.$$

to get an elementary  $(2,1)$ -topos with

$$\mathcal{E}ff_0 \cong \mathcal{E}ff.$$

THANKS !  
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