Enriched purity: towards enriched model theory

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Purity in logic

Let $\mathbb{L}$ be a language with function and relation symbols. A positive-primitive formula is one of the form

$$\psi(x) := \exists y \ \varphi(x, y)$$

where ψ is a conjunction of atomic formulas. E.g. $\varphi(x, y) = (t(x, y) = s(x, y)) \wedge R(x, y)$.

Definition

A monomorphisms of $\mathbb{L}$-structures $f \colon M \to L$ is called pure if for any pp-formula $\psi(x)$ and any $a \in M^n$ we have

$$M \models \psi(a) \quad \text{iff} \quad L \models \psi(fa)$$

$$\begin{tikzcd}
\| \psi \|_M \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M^n \arrow[d, "f^n"] \\
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Theorem (classical)

The following are equivalent for a full subcategory $\mathcal{H}$ of $\mathbf{Str}(\mathbb{L})$:

- $\mathcal{H} = \mathrm{Mod}(\mathbb{T})$ *for a regular $\mathbb{L}$-theory $\mathbb{T}$;*
- $\mathcal{H}$ *is closed under products, filtered colimits, and pure subobjects.*

Injectivity classes

Note: to say that A satisfies $\varphi(x) = \exists y\ \psi(x, y)$ is the same as requiring that the composite

$$\psi_A = \{(a, b) \mid A \models \psi(a, b)\} \xrightarrow{i} A \times A \xrightarrow{\pi_1} A$$

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The categorical analogue of regular theories are injectivity classes:

Definition

An object M is injective w.r.t. a morphism $f\colon A \to B$ in a category $\mathcal{K}$ if

$$\forall u\colon A \to M,\ \exists v\colon B \to M\ (vf = u).$$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\forall u"'] & & B \arrow[dl, dashed, "\exists v"] \\ & M & \end{tikzcd}$$

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Equivalently, if $\mathcal{K}(f,M)\colon \mathcal{K}(B,M) \longrightarrow \mathcal{K}(A,M)$ is surjective.
An injectivity class in $\mathcal{K}$ is a full subcategory of $\mathcal{K}$ spanned by the objects injective with respect to a set $\{f_i\colon A_i \to B_i\}_{i\in I}$.

Injectivity classes II

Theorem (Rosický-Adámek-Borceux)

TFAE for a full subcategory $\mathcal{H}$ of a locally finitely presentable category $\mathcal{K}$:

- *$\mathcal{H}$ is a (finite) injectivity class in $\mathcal{K}$;*
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What is purity in this context?

Definition

A morphism $f\colon M \to L$ in $\mathcal{K}$ is pure provided that in each commutative diagram on the right, where A and B are finitely presentable, there is a morphism $t\colon B \to M$ such that $tg = u$.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ M \arrow[r, "f"'] & L \end{tikzcd}$$

Note: $\mathcal{H}$ is a (finite) injectivity class in some locally finitely presentable category $\mathcal{K}$ if and only if $\mathcal{M} \simeq \mathbf{Mod}(\mathbb{T})$ for some regular theory $\mathbb{T}$ on a language $\mathbb{L}$.

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Enrichment

We fix:

- a symmetric monoidal closed category $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ which is locally presentable;
- a factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{V}$;

To keep in mind:

1. **Set** and **Ab** with (epi, mono);
2. **Met** with (dense, closed isometry);
3. **DGAb** with (regular epi, mono).

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Then there is an enriched notion of $\mathcal{E}$-injectivity:

Definition (Lack-Rosický)

An object M is $\mathcal{E}$-injective w.r.t. a morphism $f : A \to B$ in a $\mathcal{V}$-category $\mathcal{K}$ if the map

$$\mathcal{K}(f, M) \colon \mathcal{K}(B, M) \longrightarrow \mathcal{K}(A, M)$$

lies in $\mathcal{E}$. An $\mathcal{E}$-injectivity class in $\mathcal{K}$ is a full subcategory of $\mathcal{K}$ spanned by the objects $\mathcal{E}$-injective with respect to a set $\{f_i \colon A_i \to B_i\}_{i \in I}$.

A corresponding notion of purity was missing.

Enriched purity

The result below is based on some assumptions on $\mathcal{E}$ and a class of objects $\mathcal{G} \subseteq \mathcal{V}$ such that powers by $\mathcal{G}$ satisfy a stability condition with respect to $\mathcal{E}$. Then we can prove:

Theorem

TFAE for a full subcategory $\mathcal{H}$ of a locally finitely presentable $\mathcal{V}$-category $\mathcal{K}$:

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What is an $\mathcal{E}$-pure morphism?

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What is an $\mathcal{E}$-pure morphism? Definition by examples:

Met:

$$d(fu, vg) \leq \varepsilon \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ K \arrow[r, "f"'] & L \end{tikzcd}$$

DGab:

$$du = dv = 0 \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ K \arrow[r, "f"'] & L \end{tikzcd}$$

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$dt \neq 0$ generally

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A logical interpretation

- The canonical language $\mathbb{L}$ on a locally finitely presentable $\mathcal{V}$-category $\mathcal{K}$ is has sorts and function symbols given by the objects and morphisms of $\mathcal{K}_f^{\mathrm{op}}$.

Every object M of $\mathcal{K}$ defines an $\mathbb{L}$-structure in $\mathcal{V}$:

- a sort A is assigned to $M_A := \mathcal{K}(A, M)$;
- a function symbol $f : (A, B)$ is assigned to $M_f := \mathcal{K}(f, M) : M_A \to M_B$.

pp-formulas:

$$\psi(x) \equiv \exists y \; \phi(x, y)$$

where $\phi(x, y)$ is a conjunction of

$$(f(x) = g(y))$$

for some $f : B \to A$, $g : B \to C$.

Interpretation:

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$$\| \phi \|_M \rightarrowtail M_A \times M_C \underset{M_g p_2}{\overset{M_f p_1}{\rightrightarrows}} M_B$$

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$$\begin{tikzcd}
\| \phi \|_M \arrow[r, tail] \arrow[dr, two heads, "\mathcal{E}"'] & M_A \times M_C \arrow[r, "p_1"] & M_A \\
& \| \psi \|_M \arrow[ur, tail, "\mathcal{M}"'] &
\end{tikzcd}$$

A logical interpretation II

Definition

Let $f : M \to L$ be a morphism in a locally finitely-presentable $\mathcal{V}$-category $\mathcal{K}$. We say that f is elementary with respect to a pp-formula ψ if the square below is a pullback.

$$\begin{tikzcd} \| \psi \|_M \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M_A \arrow[d, "f_A"] \\ \| \psi \|_L \arrow[r, tail] & L_A \end{tikzcd}$$

Theorem

Let $\mathcal{K}$ be a locally finitely presentable $\mathcal{V}$-category and $f : M \to L$ be a morphism in it. Then f is $\mathcal{E}$-pure if and only if it is elementary with respect to any pp-formula in the canonical language associated to $\mathcal{K}$.

Can we talk about languages, structures, terms, and formulas in general?

Towards enriched model theory

Introduce enriched languages, structures, and terms:

Definition

- A (single-sorted) enriched (operational) language $\mathbb{L}$ is the data of a set of operation symbols $f\colon (X, Y)$ whose arities X and Y are objects of $\mathcal{V}_f$.

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- The class of $\mathbb{L}$-terms is defined recursively as follows:
  1. Every morphism $f\colon Y \to X$ of $\mathcal{V}_f$ is an (X, Y)-ary term;
  2. Every operation symbol $f : (X, Y)$ of $\mathbb{L}$ is an (X, Y)-ary term;
  3. If t is a (X, Y)-ary term and Z is an arity, then t^Z is a $(Z \otimes X, Z \otimes Y)$-ary term;
  4. If t and s are (X, Y)-ary and (Y, W)-ary terms; then $s \circ t$ is a (X, W)-ary term.

Then one can define equational theories: to appear soon.
What about regular theories???

Thank You