

Enriched categories: theory and examples

Lecture notes from a postgraduate minicourse

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Introduction

Enrichment is nowadays a standard tool in category theory; its range of applications is so vast that it reaches very different areas of mathematics, such as algebra [7, 14, 18], homotopy theory [4, 12, 17], computer science [3, 6, 16], and functional analysis [11, 15].

Even though additive and abelian categories were introduced earlier and can be understood as some (easy) examples of enrichment, it was only in the 60s, after the development of differentially graded categories, that people started to think about a general framework for dealing with categories whose homs have a much richer structure than that of a set.

The first to, independently, envisage the potentials of such a theory were Mac Lane [9] and Bénabou [1], as well as Linton [10] and Maranda [13]. However, Eilenberg and Kelly were the ones that actually developed a theory of enrichment in their monograph [5]. Afterwards, the theory started to get studied and many results from ordinary category theory were transferred into this richer setting, sometimes with effort and some other times very easily and elegantly.

Later, the theory evolved in new directions by introducing enrichment over bicategories [2, 19]; however that will not be the framework of this course, where we consider only enrichment over symmetric monoidal closed categories. A more complete account of all the results we discuss is given in Kelly's book [8].

CHAPTER

1

Week 1

1.1 Monoidal categories

In this section we introduce those categories which, equipped with additional structure, will be our bases of enrichment.

Given an ordinary category \mathcal{C} , we can describe the composition operation as a family of functions

$$\mathcal{C}(B, C) \times \mathcal{C}(A, B) \xrightarrow{- \circ -} \mathcal{C}(A, C)$$

for any $A, B, C \in \mathcal{C}$, satisfying certain axioms. This suggests that we could define a notion of category enriched over any given category with binary products. However, that would be very restrictive as, while we would capture many examples (2-dimensional, simplicial, etc) we would also miss many other important ones (for instance, additive and DG-categories).

To capture all these in the same framework we shall need a category endowed with a “tensor product”:

Definition 1.1.1. A *monoidal category* $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$ is the data of a category \mathcal{V}_0 together with:

- (1) an object $I \in \mathcal{V}_0$ called *unit*;
- (2) a functor $- \otimes -: \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathcal{V}_0$, called *tensor product*;
- (3) natural isomorphisms:
 - (a) $\alpha_{A,B,C}: A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C$ for any $A, B, C \in \mathcal{V}_0$;
 - (b) $\lambda_A: I \otimes A \rightarrow A$ and $\rho_A: A \otimes I \rightarrow A$ for any $A \in \mathcal{V}_0$.

These are subject to the following coherence axioms:

- (1) for any A, B, C and D in \mathcal{V}_0 the diagram below commutes;

$$\begin{array}{ccc}
 & A \otimes (B \otimes (C \otimes D)) & \\
 1 \otimes \alpha \swarrow & & \searrow \alpha \\
 A \otimes ((B \otimes C) \otimes D) & & (A \otimes B) \otimes (C \otimes D) \\
 \alpha \searrow & & \swarrow \alpha \\
 (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha \otimes 1} & ((A \otimes B) \otimes C) \otimes D
 \end{array}$$

- (2) for any A and B in \mathcal{V}_0 the diagram below commutes.

$$\begin{array}{ccc}
 A \otimes (I \otimes B) & \xrightarrow{\alpha} & (A \otimes I) \otimes B \\
 1 \otimes \lambda \searrow & & \swarrow \rho \otimes 1 \\
 & A \otimes B &
 \end{array}$$

Given a monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$, we usually write $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ and assume that α, λ and ρ are understood. Let us see some examples:

Example 1.1.2.

- (1) If $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$ is a monoidal category, then

$$\mathcal{V}^{\text{op}} = (\mathcal{V}_0^{\text{op}}, \otimes^{\text{op}}, I, \alpha^{-1}, \lambda^{-1}, \rho^{-1})$$

is a monoidal category.

- (2) If $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$ is a monoidal category, then

$$\mathcal{V}^{\text{rev}} = (\mathcal{V}_0, \hat{\otimes}, I, \hat{\alpha}, \hat{\lambda}, \hat{\rho})$$

is a monoidal category where:

- (a) $A \hat{\otimes} B := B \otimes A$;
 - (b) $\hat{\alpha}_{A,B,C} := \alpha_{C,B,A}^{-1}$;
 - (c) $\hat{\lambda}_A := \rho_A$ and $\hat{\rho}_A := \lambda_A$.
- (3) Let \mathcal{C} be any category with finite products; then $(\mathcal{C}, \times, 1)$ is a monoidal category where α, ρ and λ are induced by the universal property of the products; this structure is called the *cartesian* structure in \mathcal{C} . By (1) it follows that if \mathcal{C} has finite coproducts, then $(\mathcal{C}, +, 0)$ is a monoidal category; this monoidal structure is called *cocartesian*.
- (4) The category **Ab** of abelian groups and group homomorphisms, together with the tensor product \otimes of abelian groups, and the unit \mathbb{Z} , forms a monoidal category. More generally, the category $R\text{-Mod}$ of modules over a commutative ring R is monoidal with tensor product given by \otimes_R and unit R .

- (5) The category **DGAb** of differentially graded abelian groups (that is, chain complexes) is monoidal with the standard tensor product of chain complexes:

$$(A \otimes B)_n := \bigoplus_{i+j=n} A_i \otimes B_j$$

with differential generated by the rule

$$d_{A \otimes B}^n(a \otimes b) = d_A^i a \otimes b + (-1)^i a \otimes d_B^j b$$

where $a \in A_i$ and $b \in B_j$, with $i + j = n$. The unit I given by the chain complex with \mathbb{Z} on degree 0 and (0) otherwise.

- (6) Let (G, \cdot, e) be a group (a monoid is enough) and G_0 be the discrete category on the underlying set of G . Then $\mathcal{G} := (G_0, \cdot, e)$ is a (strict) monoidal category where the tensor product of two objects is defined by multiplication in G .

To define what a category enriched over \mathcal{V} we shall need only a monoidal category; however, to do certain constructions (like the opposite \mathcal{V} -category) we shall need the monoidal category \mathcal{V} to also be symmetric:

Definition 1.1.3. A *symmetric monoidal category* $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is the data of a monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$ together with a natural isomorphism $\sigma_{A,B}: A \otimes B \rightarrow B \otimes A$ for any $A, B \in \mathcal{V}_0$, subject to the following coherences:

- (1) for any $A, B \in \mathcal{V}_0$ we have $\sigma_{B,A} \circ \sigma_{A,B} = 1_{A \otimes B}$ and $\lambda_A \circ \sigma_{A,I} = \rho_A$;
- (2) for any $A, B, C \in \mathcal{V}_0$ the diagram below commutes.

$$\begin{array}{ccccc}
 & & (A \otimes B) \otimes C & & \\
 & \swarrow \sigma \otimes 1 & & \searrow \alpha & \\
 (B \otimes A) \otimes C & & & & A \otimes (B \otimes C) \\
 \downarrow \alpha & & & & \downarrow \sigma \\
 B \otimes (A \otimes C) & & & & (B \otimes C) \otimes A \\
 & \searrow 1 \otimes \sigma & & \swarrow \alpha & \\
 & B \otimes (C \otimes A) & & &
 \end{array}$$

Again, we will write simply $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ for a symmetric monoidal category \mathcal{V} where all the structure maps are understood. Often, we will say that a monoidal category \mathcal{V} is *symmetric* to mean that \mathcal{V} comes equipped with a chosen symmetry (note, a monoidal category might have more than one symmetry — see Exercise 1.3.4).

Example 1.1.4.

- (1) If $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is symmetric monoidal, then

$$\mathcal{V}^{\text{op}} = (\mathcal{V}_0^{\text{op}}, \otimes^{\text{op}}, I, \alpha^{-1}, \lambda^{-1}, \rho^{-1}, \sigma^{-1})$$

is symmetric monoidal.

- (2) If $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is symmetric monoidal, then

$$\mathcal{V}^{\text{rev}} = (\mathcal{V}_0, \hat{\otimes}, I, \hat{\alpha}, \hat{\lambda}, \hat{\rho}, \hat{\sigma})$$

is symmetric monoidal where $\hat{\sigma}_{A,B} = \sigma_{B,A}$.

- (3) Let \mathcal{C} be any category with finite products; then $(\mathcal{C}, \times, 1)$ is symmetric monoidal where σ is induced by the universal property of the products. Similarly, if \mathcal{C} has finite coproducts, then $(\mathcal{C}, +, 0)$ is symmetric monoidal.
- (4) The monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$ is symmetric with symmetry

$$\sigma_{A,B}: A \otimes B \rightarrow B \otimes A$$

defined by sending a generator $a \otimes b \in A \otimes B$ to $\sigma_{A,B}(a \otimes b) := b \otimes a \in B \otimes A$. Similarly, the category $R\text{-Mod}$ of modules over a commutative ring R is symmetric monoidal.

- (5) The monoidal category $\mathbf{DGA}b$ of differentially graded abelian groups, with the standard tensor product, is symmetric. The symmetry is defined on the generators as

$$\sigma_{A,B}(a \otimes b) := (-1)^{ij} b \otimes a,$$

where $a \in A_i$ and $b \in B_j$.

- (6) Let (G, \cdot, e) be a group. Then $\mathcal{G} = (G_0, \cdot, e)$ is a symmetric monoidal if and only if the multiplication in G is commutative; that is, if and only if G is abelian.

Example 1.1.5. The category \mathbf{Grp} of groups and group homomorphisms, with the monoidal closed structure given by the standard tensor product of groups is not symmetric.

The last concept, that will be needed to consider for instance enriched categories of functors, is that of symmetric closed monoidal category.

Definition 1.1.6. A *symmetric monoidal closed category* $\mathcal{V} = (\mathcal{V}_0, \otimes, [-, -])$ is the data of a symmetric monoidal category together with functor

$$[-, -]: \mathcal{V}_0^{\text{op}} \times \mathcal{V}_0 \rightarrow \mathcal{V}_0,$$

called *internal hom*, such that $(-) \otimes B \dashv [B, -]$ for any $B \in \mathcal{V}_0$. In other words, we have an isomorphism

$$\mathcal{V}_0(A \otimes B, C) \cong \mathcal{V}_0(A, [B, C])$$

natural in A, B , and C in \mathcal{V}_0 .

Since the internal-hom functor $[-, -]$, when it exists, is uniquely determined (up to isomorphism) from the tensor product, we will still denote a symmetric monoidal closed category by just $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$.

Remark 1.1.7. Fixed a symmetric monoidal closed category \mathcal{V} , we will often use the tensor-hom adjunction to “transpose” maps out of a tensor products to maps into an internal-hom. More precisely, to give a morphism

$$f: A \otimes B \longrightarrow C$$

is the same as to give a map

$$f^t: A \rightarrow [B, C]$$

which is the transpose of f under $(-) \otimes B \dashv [B, -]$. Since \mathcal{V} is also symmetric, the map f also corresponds to $f\sigma: B \otimes A \rightarrow C$ (up to isomorphism). Therefore to give f is also the same as to give

$${}^t f: B \rightarrow [A, C].$$

We will use these natural bijections very often when constructing certain \mathcal{V} -categories. The notation with f , f^t , and ${}^t f$ will not always be consistent with the one used above; for instance, if we start with $g: B \rightarrow [A, C]$ we will use the transpose to denote the other two maps.

The following examples will provide interesting instances of enrichment.

Example 1.1.8.

- (1) The singleton $1 = \{*\}$, seen as a one-object discrete category, is (trivially) symmetric monoidal closed.
- (2) The category **Set** of sets and functions with its cartesian structure is symmetric monoidal closed (also called, cartesian closed). The internal hom is simply given by

$$[A, B] := \mathbf{Set}(A, B).$$

Indeed, for any sets A, B , and C we have a natural isomorphism

$$\mathbf{Set}(A \times B, C) \cong \mathbf{Set}(A, \mathbf{Set}(B, C))$$

given by sending $f: A \times B \rightarrow C$ to $\hat{f}: A \rightarrow \mathbf{Set}(B, C)$ defined by $\hat{f}(a)(b) := f(a, b)$.

- (3) Every (elementary) topos with its cartesian structure is symmetric monoidal closed by definition. The internal hom is usually denoted by $[A, B] := B^A$.
- (4) The category **Cat** of categories is cartesian closed; the internal hom $[A, B]$ is the category of functors $A \rightarrow B$ and natural transformations between them.
- (5) The monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$ is symmetric monoidal closed with internal hom $[A, B] := \mathbf{Ab}(A, B)$ endowed with the pointwise addition which makes it an abelian group. Similarly, the category $R\text{-Mod}$ of modules over a commutative ring R is symmetric monoidal closed.
- (6) The monoidal category **DGAb** of differentially graded abelian groups, with the standard tensor product, is symmetric monoidal closed. The internal hom is given by

$$[A, B]_n := \prod_{i \in \mathbb{Z}} \mathbf{Ab}(A_i, B_{i+n})$$

and with differential $d(f) = d_B \circ f - (-1)^n f \circ d_A$, where $f \in [A, B]_n$.

- (7) Let (G, \cdot, e) be an abelian group. Then $\mathcal{G} = (G_0, \cdot, e)$ is symmetric monoidal closed with internal hom

$$[g, h] := k \cdot h^{-1}$$

for any $g, h \in G$.

- (8) Consider the arrow category $\mathbf{2} = \{0 \rightarrow 1\}$, then $(\mathbf{2}, \times, 1)$ is symmetric monoidal closed (it is cartesian closed as a full subcategory of **Set**).

- (9) Consider the category \mathbf{Set}_* of pointed sets: objects are pairs $(X \in \mathbf{Set}, x \in X)$, and morphisms $(X, y) \rightarrow (Y, y)$ are functions $f: X \rightarrow Y$ sending x to y . Between any two objects $A = (X, x)$ and $B = (Y, y)$ we can define a tensor product

$$A \wedge B := \left(\frac{X \times Y}{A \vee B}, [A \vee B] \right)$$

where $A \vee B := (\{x\} \times Y) \cup (X \times \{y\}) \subseteq X \times Y$, and $[A \vee B]$ is the single point defined by the equivalence class of $A \vee B$ in the quotient. The unit is defined as $I := (\{0, 1\}, 0)$. It is easy to define the structure maps that make $(\mathbf{Set}_*, \wedge, I)$ into a symmetric monoidal category. This is in addition monoidal closed with internal hom defined as

$$[A, B] := (\mathbf{Set}_*(A, B), \Delta y)$$

where $B = (Y, y)$ and Δy is the constant function at $y \in Y$.

- (10) Consider the poset $\overline{\mathbb{R}}_+ = ([0, \infty], \geq)$ as a category. Then $(\overline{\mathbb{R}}_+, +, 0)$ forms a symmetric monoidal closed category with internal hom

$$[a, b] = \begin{cases} b - a & \text{if } b \geq a \\ 0 & \text{otherwise} \end{cases}$$

for any $a, b \in \overline{\mathbb{R}}_+$.

- (11) Consider the category \mathbf{Met} whose objects are generalized metric spaces (we allow the distance to take value in $[0, \infty]$, everything else is as usual), and whose morphisms $f: (X, d_X) \rightarrow (Y, d_Y)$ are functions $f: X \rightarrow Y$ such that $d_Y(fx, fx') \leq d_X(x, x')$. On \mathbf{Met} we have a symmetric monoidal closed structure with unit given by the singleton 1, tensor product

$$(X, d_X) \otimes (Y, d_Y) := (X \times Y, d_{X \otimes Y}((x, y), (x', y')) := d_X(x, x') + d_Y(y, y')),$$

and internal hom

$$[(X, d_X), (Y, d_Y)] := (\mathbf{Met}((X, d_X), (Y, d_Y)), d(f, g) := \sup\{d_Y(fx, gy) \mid x \in X\})$$

for any $(X, d_X), (Y, d_Y) \in \mathbf{Met}$.

1.2 Enriched categories

From now on $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ is assumed to be a monoidal category.

Definition 1.2.1. A \mathcal{V} -category \mathcal{C} is the data of:

- (1) a class $\text{Ob}(\mathcal{C})$ whose elements are called the *objects* of \mathcal{C} ;
- (2) for any $A, B \in \text{Ob}(\mathcal{C})$ an object $\mathcal{C}(A, B) \in \mathcal{V}$ called the *hom-object* of morphisms from A to B ;
- (3) for any $A \in \text{Ob}(\mathcal{C})$ an *identity map* $\text{Id}_A: I \rightarrow \mathcal{C}(A, A)$ in \mathcal{V} ;

(4) for any $A, B, C \in \text{Ob}(\mathcal{C})$ a *composition map*

$$\circ_{A,B,C}: \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$$

in \mathcal{V} .

These are subject to the following axioms:

(1) *Identity*: for any $A, B \in \text{Ob}(\mathcal{C})$ the diagram below commutes.

$$\begin{array}{ccccc} I \otimes \mathcal{C}(A, B) & & & & \mathcal{C}(A, B) \otimes I \\ \text{Id}_B \otimes 1 \downarrow & \searrow \lambda & & \swarrow \rho & \downarrow 1 \otimes \text{Id}_A \\ \mathcal{C}(B, B) \otimes \mathcal{C}(A, B) & & \mathcal{C}(A, B) & & \mathcal{C}(A, A) \otimes \mathcal{C}(A, B) \\ & \nearrow \circ_{A,B,B} & & \nwarrow \circ_{A,A,B} & \end{array}$$

(2) *Associativity*: for any $A, B, C, D \in \text{Ob}(\mathcal{C})$ the following diagram

$$\begin{array}{ccc} \mathcal{C}(C, D) \otimes (\mathcal{C}(B, C) \otimes \mathcal{C}(A, B)) & \xrightarrow{\alpha} & (\mathcal{C}(C, D) \otimes \mathcal{C}(B, C)) \otimes \mathcal{C}(A, B) \\ 1 \otimes \circ_{A,B,C} \downarrow & & \downarrow \circ_{B,C,D} \otimes 1 \\ \mathcal{C}(C, D) \otimes \mathcal{C}(A, C) & & \mathcal{C}(B, D) \otimes \mathcal{C}(A, B) \\ & \searrow \circ_{A,C,D} \quad \swarrow \circ_{A,B,D} & \\ & \mathcal{C}(A, D) & \end{array}$$

commutes.

The \mathcal{V} -category \mathcal{C} is called (*essentially*) *small* if the class of objects $\text{Ob}(\mathcal{C})$ forms (up to isomorphism) a small set.

From now on we will write $A \in \mathcal{C}$ to mean that $A \in \text{Ob}(\mathcal{C})$. When it is clear from the context, we will simply write \circ instead of $\circ_{A,B,C}$ for the composition map.

Given a \mathcal{V} -category \mathcal{C} we can always construct an ordinary category \mathcal{C}_0 called the *underlying category of \mathcal{C}* . This has the same objects as \mathcal{C} and hom-sets

$$\mathcal{C}_0(A, B) := \mathcal{V}_0(I, \mathcal{C}(A, B));$$

identities are the same as in \mathcal{C} and the composition map is given by

$$\mathcal{C}_0(B, C) \times \mathcal{C}_0(A, B) \xrightarrow{\beta} \mathcal{V}_0(I, \mathcal{C}(B, C) \otimes \mathcal{C}(A, B)) \xrightarrow{\mathcal{V}_0(I, \circ_{A,B,C})} \mathcal{C}_0(A, C)$$

where $\beta(f, g) := (f \otimes g)\lambda_I^{-1}$. It is easy to see that the identity and associativity axioms still hold from \mathcal{C}_0 , making it a category.

Notation 1.2.2. For an enriched category \mathcal{C} we will always denote by \mathcal{C}_0 its underlying category as defined above. If we say that $f: A \rightarrow B$ is a morphism in \mathcal{C} , we actually mean that f is a morphism in \mathcal{C}_0 , and therefore a map $f: I \rightarrow \mathcal{C}(A, B)$.

Given morphism $f: A \rightarrow B$ in \mathcal{C} , for any $C \in \mathcal{C}$ we have induced morphisms

$$\mathcal{C}(C, f): \mathcal{C}(C, A) \longrightarrow \mathcal{C}(C, B)$$

and

$$\mathcal{C}(f, C): \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C)$$

in \mathcal{V} . The former is defined as the composite

$$\mathcal{C}(C, A) \xrightarrow{\lambda^{-1}} I \otimes \mathcal{C}(C, A) \xrightarrow{f \otimes 1} \mathcal{C}(A, B) \otimes \mathcal{C}(C, A) \xrightarrow{\circ} \mathcal{C}(A, C),$$

the latter is defined similarly using ρ^{-1} .

We now turn to the definition of \mathcal{V} -functor:

Definition 1.2.3. Given \mathcal{V} -categories \mathcal{C} and \mathcal{D} , a \mathcal{V} -functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is the data of:

- (1) a function $F: \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$;
- (2) for any $A, B \in \mathcal{C}$ a morphism

$$F_{A,B}: \mathcal{C}(A, B) \longrightarrow \mathcal{D}(FA, FB)$$

in \mathcal{V} .

These are subject to the following axioms:

- (1) for any $A \in \mathcal{C}$ the triangle below commutes;

$$\begin{array}{ccc} & I & \\ \text{Id}_A^{\mathcal{C}} \swarrow & & \searrow \text{Id}_{FA}^{\mathcal{D}} \\ \mathcal{C}(A, A) & \xrightarrow{F_{A,A}} & \mathcal{D}(FA, FA) \end{array}$$

- (2) for any $A, BC \in \mathcal{C}$ the following square

$$\begin{array}{ccc} \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) & \xrightarrow{\circ^{\mathcal{C}}} & \mathcal{C}(A, C) \\ \downarrow F_{B,C} \otimes F_{A,B} & & \downarrow F_{A,C} \\ \mathcal{D}(FB, FC) \otimes \mathcal{D}(FA, FC) & \xrightarrow{\circ^{\mathcal{D}}} & \mathcal{D}(FA, FC) \end{array}$$

commutes.

Given a \mathcal{V} -functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between enriched categories, we can define an ordinary functor $F_0: \mathcal{C}_0 \rightarrow \mathcal{D}_0$, called *underlying functor of F* , between the corresponding underlying categories. The functor F_0 acts on objects as F and the action on the hom-sets is defined as

$$(F_0)_{A,B} := \mathcal{V}_0(I, F_{A,B}): \mathcal{C}_0(A, B) \longrightarrow \mathcal{D}_0(FA, FB).$$

It is easy to see that F_0 preserves identities and composition (since F does) and is therefore

a functor.

For any pair of \mathcal{V} -functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{E}$ we define the *composite \mathcal{V} -functor* $GF: \mathcal{C} \rightarrow \mathcal{E}$ by setting $GF(A) := G(F(A))$ and

$$(GF)_{A,B}: \mathcal{C}(A, B) \xrightarrow{F_{A,B}} \mathcal{D}(FA, FB) \xrightarrow{G_{FA, FB}} \mathcal{E}(GFA, GFB)$$

for any $A, B \in \mathcal{C}$. The fact that GF satisfies the axioms of a \mathcal{V} -functor follows by stacking together the commutativity conditions of F and G .

For any \mathcal{V} -category \mathcal{C} we have an identity \mathcal{V} -functor $1_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$ defined by $1_{\mathcal{C}}(A) = A$ and $(1_{\mathcal{C}})_{A,B} = 1_{\mathcal{C}(A,B)}$ in \mathcal{V} .

Notation 1.2.4. For an enriched functor F we will always denote by F_0 its underlying functor as defined above. Essentially small \mathcal{V} -categories, \mathcal{V} -functors, identity \mathcal{V} -functors, and composition of \mathcal{V} -functors, form a category $\mathcal{V}\text{-}\mathbf{Cat}$. The underlying category/functor construction defines a functor

$$(-)_0: \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}.$$

When the \mathcal{V} -categories are not assumed to be essentially small, we call the induced category $\mathcal{V}\text{-}\mathbf{CAT}$.

One-object \mathcal{V} -categories are important on their own:

Example 1.2.5. Let \mathcal{M} be a one object \mathcal{V} -category; then $M := \mathcal{M}(*, *)$ (where $*$ is the unique object of \mathcal{M}) is endowed with a monoid structure induced by the composition map

$$\circ: M \otimes M \rightarrow M$$

and the identity $\text{Id}_*: I \rightarrow M$ (the unitality and associativity axioms follow from those of a \mathcal{V} -category). Conversely, any monoid (M, μ, η) in \mathcal{V} induces a unique \mathcal{V} category \mathcal{M} with a single object $*$, hom-object $\mathcal{M}(*, *) := M$, composition given by μ , and identity given by η . Under this interpretation, a \mathcal{V} -functor between single-object \mathcal{V} -categories is just a monoid morphism.

We now turn to some more concrete examples:

Example 1.2.6.

- (1) Let $\mathcal{V} = (\mathbf{Set}, \times, 1)$. Then a \mathcal{V} -category is just an ordinary category and a \mathcal{V} -functor is just a functor; therefore $\mathcal{V}\text{-}\mathbf{Cat} \cong \mathbf{Cat}$.
- (2) Let $\mathcal{V} = 1$ be the discrete category with one object (with the trivial monoidal structure). Then a \mathcal{V} -category is completely determined by its objects (every hom-object is the same and composition and identities are forced to be the identities of 1). It follows that $\mathcal{V}\text{-}\mathbf{Cat} \cong \mathbf{Set}$.
- (3) Let $\mathcal{V} = (\mathbf{Cat}, \times, 1)$ be the cartesian closed category of small categories. Then a \mathcal{V} -category is commonly called a *2-category*. We will study these in more detail later.
- (4) Let $\mathcal{V} = (\mathbf{Ab}, \otimes, \mathbb{Z})$ be the symmetric monoidal closed category of abelian groups. These \mathcal{V} -categories are commonly called *pre-additive* categories. To give a pre-additive category \mathcal{C} is the same as giving an ordinary category whose hom-sets $\mathcal{C}(A, B)$ come with an abelian group structure, and such that composition $- \circ -$ is linear in both variables.

- (5) Let $\mathcal{V} = (\mathbf{DGA}b, \times, I)$ be the symmetric monoidal closed category of chain complexes. Then a \mathcal{V} -category is commonly called a *DG-category*. We will study these in more detail later.
- (6) Let $\mathcal{V} = (\mathbf{2}, \times, 1)$, where $\mathbf{2} = \{0 \rightarrow 1\}$ is the arrow category. We will show that to give a \mathcal{V} -category is the same as to give a preorder. Given a \mathcal{V} -category \mathcal{C} , for any pair of objects A, B , we have

$$\mathcal{C}(A, B) \in \{0, 1\}$$

and, the fact that we have maps $\text{Id}_A: 1 \rightarrow \mathcal{C}(A, A)$, says that $\mathcal{C}(A, A) = 1$ for each A (and that Id_A is the identity map in $\mathbf{2}$). The composition map

$$\circ_{A,B,C}: \mathcal{C}(B, C) \times \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$$

is uniquely determined when at least one between $\mathcal{C}(B, C)$ and $\mathcal{C}(A, B)$ is 0; when they are both 1 one necessarily has that also $\mathcal{C}(A, C) = 1$ (and hence the composition map is the identity). Finally, the identity and commutativity axioms do not add any information (every diagram commutes in $\mathbf{2}$). Thus, given \mathcal{C} , consider the set (or class) $\Sigma := \text{Ob}(\mathcal{C})$, and define the relation: $A \leq B$ if and only if $\mathcal{C}(A, B) = 1$. Because of the arguments above it follows that $A \leq A$ for any $A \in \Sigma$, and that whenever $A \leq B \leq C$ then also $A \leq C$ (composition law). Therefore (Σ, \leq) forms a preorder.

Conversely, if (Σ, \leq) is a preorder we can define a \mathcal{V} -category \mathcal{C} with $\text{Ob}(\mathcal{C}) := \Sigma$ and

$$\mathcal{C}(A, B) = \begin{cases} 1 & \text{if } A \leq B \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that this is a well defined \mathcal{V} -category. Similarly, one shows that to give a \mathcal{V} -functor, under this translation, is the same as giving an order-preserving function. It follows that $\mathcal{V}\text{-Cat} \cong \mathbf{Prd}$ is isomorphic to the category of preorders.

It is straightforward to see that a \mathcal{V} -category \mathcal{C} is skeletal (that is, the only isomorphisms are the identities) if and only if the corresponding preorder (Σ, \leq) satisfies the antisymmetry law ($A \leq B \leq A$ implies $A = B$), if and only if (Σ, \leq) is a poset. As a consequence, the full subcategory $\mathcal{V}\text{-Cat}_{\text{sk}}$ of $\mathcal{V}\text{-Cat}$ spanned by the skeletal \mathcal{V} -categories is isomorphic to \mathbf{Pos} , the category of posets and order preserving maps.

- (7) Let $(\overline{\mathbb{R}}_+, +, 0)$ be the monoidal category where $\overline{\mathbb{R}}_+ = ([0, \infty], \geq)$. Then a small $\overline{\mathbb{R}}_+$ -category is a *Lawvere metric space*: to give a small $\overline{\mathbb{R}}_+$ -category we need

- (a) a set X (the objects);
- (b) for any $x, y \in X$ a real number $d(x, y) \in [0, \infty]$ (the hom-objects);
- (c) (identity maps) for any $x \in X$ we have $0 \geq d(x, x)$;
- (d) (composition maps) for any $x, y, z \in X$ we have

$$d(y, z) + d(x, y) \geq d(x, z).$$

There are no axioms to satisfy since all diagrams commute in a poset. It follows that a $\overline{\mathbb{R}}_+$ -category is determined by a set X together with a metric $d: X \times X \rightarrow [0, \infty]$ which satisfies the triangle inequality and for which $d(x, x) = 0$ for any $x \in X$ (note that d may not be symmetric).

In such a $\overline{\mathbb{R}}_+$ -category, two points might be distinct and still have distance equal to 0. This can be avoided (arguing as in the previous example) by considering skeletal $\overline{\mathbb{R}}_+$ -categories.

1.3 Exercises

Exercise 1.3.1. Prove that the category **Ab** of abelian groups is not cartesian closed.

Exercise 1.3.2. Let **Rng** be the category of rings with unit and ring homomorphisms. Show that there is no symmetric monoidal closed structure on **Rng** with unit \mathbb{Z} .

Exercise 1.3.3. Show that the category **Met**, of generalized metric spaces and non-increasing maps, is not cartesian closed.

Exercise 1.3.4. Prove that the monoidal category **GA**b of graded abelian groups (with unit and tensor product defined as for **DGA**b forgetting about the differential) has exactly two different symmetries given by:

$$\sigma_{A,B}(a \otimes b) := b \otimes a \quad \text{and} \quad \tau_{A,B}(a \otimes b) := (-1)^{ij} b \otimes a$$

where $a \in A_i$ and $b \in B_j$.

Exercise 1.3.5. Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ be symmetric monoidal closed and consider the functor

$$(-)^* := [-, I]: \mathcal{V}_0^{\text{op}} \longrightarrow \mathcal{V}_0.$$

Exhibit a natural transformation $(-) \Rightarrow (-)^{**}$ and show that it is not in general an isomorphism.

Exercise 1.3.6. Assume that \mathcal{V} is symmetric monoidal closed. Rewrite the definition of \mathcal{V} -category and \mathcal{V} -functor in an equivalent form that only uses the unit I and the internal hom $[-, -]$, but not the tensor product \otimes .

Exercise 1.3.7. Assume that \mathcal{V} is symmetric monoidal closed. Show that the tensor product and internal hom induce \mathcal{V} -functors

$$X \otimes -, [X, -]: \mathcal{V} \longrightarrow \mathcal{V}$$

for any $X \in \mathcal{V}$.

Exercise 1.3.8. Assume that I is a generator of \mathcal{V}_0 ; that is, $\mathcal{V}_0(I, -): \mathcal{V}_0 \rightarrow \mathbf{Set}$ is faithful. Show that to give a \mathcal{V} -natural transformation $F \Rightarrow G$ between \mathcal{V} -functors is the same as giving a natural transformation $F_0 \Rightarrow G_0$ between the underlying functors.

Exercise 1.3.9. Let $\mathcal{V}\text{-Mon}$ be the full subcategory of $\mathcal{V}\text{-Cat}$ spanned by the 1-object \mathcal{V} -categories; we know this to be isomorphic to the category of monoids in \mathcal{V} . Show that:

- (1) for $\mathcal{V} = \mathbf{Ab}$ we have $\mathbf{Ab}\text{-Mon} \cong \mathbf{Rng}$;
- (2) given any $R \in \mathbf{Rng}$ denote by \bar{R} the corresponding **Ab**-category, then to give an **Ab**-functor $M: \bar{R} \rightarrow \mathbf{Ab}$ is the same as to give an R -module;
- (3) for any $R \in \mathbf{Rng}$, to give an **Ab**-functor $\bar{R} \rightarrow \mathbf{Ab}$ is the same as to give a left R -module.

Exercise 1.3.10. Consider the functor $U: \mathcal{V}\text{-Cat} \rightarrow \mathbf{Set}$ which sends a \mathcal{V} -category \mathcal{C} to its set of objects. Give conditions on \mathcal{V} so that U has a right adjoint, and compute the adjoint explicitly.

Exercise 1.3.11. Let (G, \cdot, e) be a group. A G -torsor is the data of a set T together with a group action $a: G \times T \rightarrow T$ satisfying the following condition: for any $s, t \in T$ there is a unique $g \in G$ for which $a(g, s) = t$.

Consider the monoidal category $\mathcal{G} = (G_0, \cdot, e)$ induced from G as in Example 1.1.2. Show that every G -torsor can be interpreted as a \mathcal{G} -category. What condition do you need to impose on a \mathcal{G} -category so that it is induced by a G -torsor?

Exercise 1.3.12. A category \mathcal{C} with both a terminal and a initial object that are isomorphic, is called *pointed*. Show that:

- (1) \mathbf{Set}_* is pointed;
- (2) every pointed category is naturally enriched over \mathbf{Set}_* ;
- (3) if \mathcal{D} is a \mathbf{Set}_* -category, then: \mathcal{D}_0 has an initial and a terminal object if and only if it is pointed.

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