Dualities for Accessible Categories

Giacomo Tendas

Joint work with:
Steve Lack

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Gabriel-Ulmer duality

$$\mathrm{LFP}(-,\mathbf{Set}) : \mathbf{LFP} \rightleftarrows \mathbf{Lex}^{op} : \mathrm{Lex}(-,\mathbf{Set})$$

Gabriel-Ulmer duality

$$\mathrm{LFP}(-, \mathbf{Set}) : \mathbf{LFP} \rightleftarrows \mathbf{Lex}^{op} : \mathrm{Lex}(-, \mathbf{Set})$$

Lex:

- $\mathcal{C}$ small with finite limits;
- $F : \mathcal{C} \to \mathcal{C}'$ preserving
  - finite limits;
- natural transformations.

Gabriel-Ulmer duality

$\mathrm{Lex}(K_f^{op}, \underline{Set}) \simeq K \longmapsto \mathrm{LFP}(K, \underline{Set}) \simeq K_f^{op}$

$$\mathrm{LFP}(-, \mathbf{Set}) : \mathbf{LFP} \underset{\simeq}{\rightleftarrows} \mathbf{Lex}^{op} : \mathrm{Lex}(-, \mathbf{Set})$$

$\mathcal{C}^{op} \hookrightarrow \mathrm{Lex}(\mathcal{C}, \underline{Set}) \longleftarrow\!\shortmid\; \mathcal{C}$

$\underline{Cat}$

fin. rank $\underline{sSet}$, $\underline{Set}$, $\underline{Ab}$

LFP:

- $\mathcal{K}$ complete + freely generated by a small category under filtered colimits; = finitely accessible
- $F : \mathcal{K} \to \mathcal{K}'$ preserving
  - limits
  - filtered colimits;
- natural transformations.

$K_f \hookleftarrow$

$\cap$

K

Lex:

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Diers duality

$$\mathrm{LMP}(-,\mathbf{Set}) : \mathbf{LMP} \rightleftarrows \mathbf{Fam\text{-}Lex}^{op} : \mathrm{Fam\text{-}Lex}(-,\mathbf{Set})$$

LOrd, Hilb, Field

LMP:

- $\mathcal{K}$ finitely accessible with connected limits;
- $F : \mathcal{K} \to \mathcal{K}'$ preserving
  - connected limits
  - filtered colimits;
- natural transformations.

Diers duality

$$\mathcal{K} \longmapsto \mathrm{LMP}(\mathcal{K}, \underline{\mathrm{Set}}) \simeq \mathrm{Fam}(\mathcal{K}_f^{op})$$

$$\mathrm{LMP}(-, \mathbf{Set}) : \mathbf{LMP} \rightleftarrows \mathbf{Fam\text{-}Lex}^{op} : \mathrm{Fam\text{-}Lex}(-, \mathbf{Set})$$

$$\mathcal{C}^{op} \rightsquigarrow \mathrm{Mult}(\mathcal{C}, \underline{\mathrm{Set}}) \simeq \mathrm{FamLex}(\mathcal{E}, \underline{\mathrm{Set}}) \longleftarrow\!\!| \ \mathcal{E}$$

LMP:

- $\mathcal{K}$ finitely accessible with connected limits;
- $F : \mathcal{K} \to \mathcal{K}'$ preserving
  - connected limits
  - filtered colimits;
- natural transformations.

commute in $\underline{\mathrm{Set}}$

Fam-Lex:

- $\mathcal{E} \simeq \mathrm{Fam}(\mathcal{C})$ with finite limits, $\mathcal{C}$ small; ↑ coproduct cocompletion
- $F : \mathcal{E} \to \mathcal{E}'$ preserving
  - coproducts
  - finite limits;
- natural transformations.

Examples

Ψ-*limits commuting with* $\mathscr{D}$-*colimits in* **Set**

Examples

Ψ-limits commuting with $\mathscr{D}$-colimits in **Set**

$+ \ldots$

good

1. $\Psi = \{\text{small categories}\}$ $\rightarrow$ $\mathscr{D} = \emptyset$; (Gabriel-Ulmer)
2. $\Psi = \{\text{connected diagrams}\}$ $\rightarrow$ $\mathscr{D} = \{\text{discrete categories}\}$; (Diers)
3. $\Psi = \emptyset$ $\rightarrow$ $\mathscr{D} = \{\text{small categories}\}$; ✓
4. $\Psi = \{\text{finite categories}\}$ $\rightarrow$ $\mathscr{D} = \{\text{filtered categories}\}$; ✓
5. $\Psi = \{\text{finite discrete categories}\}$ $\rightarrow$ $\mathscr{D} = \{\text{sifted categories}\}$;
6. $\Psi = \{\emptyset\}$ $\rightarrow$ $\mathscr{D} = \{\text{connected categories}\}$;
7. $\Psi = \{\text{discrete categories}\}$ $\rightarrow$ $\mathscr{D} = ??$
8. $\Psi = \{\text{wide pullbacks}\}$ $\rightarrow$ $\mathscr{D} = ??$

Classes of Diagrams

A class of diagrams $\mathscr{D}$ is the data of a full subcategory $\mathscr{D}_\mathcal{C} \subseteq [\mathcal{C}, \mathbf{Set}]$ for each small category $\mathcal{C}$.

Definition

Let Ψ be a class of indexing categories; a class of diagrams $\mathscr{D}$ is a companion for Ψ if:

- Ψ-limits commute with $\mathscr{D}$-colimits in **Set**: for each $\mathcal{C}, \mathcal{B} \in \Psi$ and

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\forall H"] \arrow[d, dashed] & {[\mathcal{C}, \mathbf{Set}]} \\ \mathscr{D}_\mathcal{C} \arrow[ur] & \end{tikzcd} \qquad \operatorname{colim}_{c\in\mathcal{C}} \lim_{b\in\mathcal{B}} H(b,c) \cong \lim_{b\in\mathcal{B}} \operatorname{colim}_{c\in\mathcal{C}} H(b,c)$$

- Let $\mathcal{K}$ be finitely accessible with Ψ-limits and $F : \mathcal{K} \to \mathbf{Set}$ preserve Ψ-limits and filtered colimits. Then

$$F \cong \operatorname{colim}_i \mathcal{K}(X_i, -)$$

is a $\mathscr{D}$-colimit of representables with $X_i \in \mathcal{K}_f$.

< Examples 1-6 >

- . . . (technical)

Back to the examples

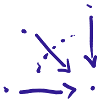
"products"

1. $\Psi = \{\text{discrete categories}\}$ - $\mathscr{D} = \{\text{pseudo equivalence relations}\}$

$$\begin{tikzcd} A \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, dashed, two heads, "\text{epi}"'] & & B \quad \text{in } \underline{\mathrm{Set}} \\ & C \arrow[ur, dashed, shift left] \arrow[ur, dashed, shift right, "\text{eq.relation}"'] & \end{tikzcd}$$

$\mathcal{D} \subseteq [\cdot \rightrightarrows \cdot, \underline{\mathrm{Set}}]$

2. $\Psi = \{\text{wide pullbacks}\}$ - $\mathscr{D} = \{\text{free groupoid actions}\}$. (Hu-Tholen ...)



$\hookrightarrow \; \mathcal{D}_{\mathcal{G}} \subseteq [\mathcal{G}, \underline{\mathrm{Set}}]$ free groupoid actions
($\mathcal{G}$ groupoid)

The Duality

Theorem

Let $\mathscr{D}$ be a companion for Ψ. The following is a biequivalence of 2-categories

$$\mathrm{fAcc}_\Psi(-,\mathbf{Set}) : \mathbf{fAcc}_\Psi \rightleftarrows \mathscr{D}\text{-}\mathbf{Lex}^{op} : \mathscr{D}\text{-}\mathrm{Lex}(-,\mathbf{Set})$$

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$$\mathrm{fAcc}_\Psi(-,\mathbf{Set}) : \mathbf{fAcc}_\Psi \rightleftarrows \mathscr{D}\text{-}\mathbf{Lex}^{op} : \mathscr{D}\text{-}\mathrm{Lex}(-,\mathbf{Set})$$

$\mathbf{fAcc}_\Psi$:

- $\mathcal{K}$ finitely accessible with Ψ-limits;
- $F : \mathcal{K} \to \mathcal{K}'$ preserving
  - Ψ-limits
  - filtered colimits;
- natural transformations.

The Duality

Theorem

Let $\mathscr{D}$ be a companion for Ψ. The following is a biequivalence of 2-categories

$$K \longmapsto \mathrm{fAcc}_{\Psi}(K, \mathbf{Set}) \simeq \mathcal{D}(K_f^{op})$$

$$\mathrm{fAcc}_{\Psi}(-, \mathbf{Set}) : \mathbf{fAcc}_{\Psi} \rightleftarrows \mathscr{D}\text{-}\mathbf{Lex}^{op} : \mathscr{D}\text{-}\mathrm{Lex}(-, \mathbf{Set})$$

$\mathbf{fAcc}_{\Psi}$:

- $\mathcal{K}$ finitely accessible with Ψ-limits;
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$\mathscr{D}$-Lex:

- $\mathcal{E} \simeq \mathscr{D}(\mathcal{C})$ with finite limits, $\mathcal{C}$ small;
- $F : \mathcal{E} \to \mathcal{E}'$ preserving
  - $\mathscr{D}$-colimits
  - finite limits;
- natural transformations.

Examples

(1) $\Psi = \{$small categories$\}$ - $\mathscr{D} = \emptyset$ $\to$ Gabriel-Ulmer duality;

(2) $\Psi = \{$connected diagrams$\}$ - $\mathscr{D} = \{$discrete categories$\}$ $\to$ Diers duality;

Examples

(1) $\Psi = \{\text{small categories}\}$ - $\mathscr{D} = \emptyset$ $\rightarrow$ Gabriel-Ulmer duality;

(2) $\Psi = \{\text{connected diagrams}\}$ - $\mathscr{D} = \{\text{discrete categories}\}$ $\rightarrow$ Diers duality;

(3) $\Psi = \emptyset$ - $\mathcal{P} = \{\text{small categories}\}$ $\rightarrow$

$$\mathrm{fAcc}(-,\mathbf{Set}) : \underline{\mathbf{fAcc}} \underset{\simeq}{\rightleftarrows} \mathcal{P}\text{-}\mathbf{Lex}^{op} : \mathcal{P}\text{-Lex}(-,\mathbf{Set})$$

$\cong$

$\mathcal{G}\mathrm{Top}^{op}$

$[\mathcal{C}, \underline{\mathrm{Set}}]$ ← dog

$\downarrow$ lex + cod ← wor

$[\mathcal{D}, \underline{\mathrm{Set}}]$

Examples

(1) $\Psi = \{$small categories$\}$ - $\mathscr{D} = \emptyset$ $\rightarrow$ Gabriel-Ulmer duality;

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(3) $\Psi = \emptyset$ - $\mathcal{P} = \{$small categories$\}$ $\rightarrow$

$$\mathrm{fAcc}(-,\mathbf{Set}) : \mathbf{fAcc} \rightleftarrows \mathcal{P}\text{-}\mathbf{Lex}^{op} : \mathcal{P}\text{-}\mathrm{Lex}(-,\mathbf{Set})$$

(4) $\Psi = \{$finite categories$\}$ - $\mathscr{D} = \{$filtered categories$\}$ $\rightarrow$ $\mathbf{fAcc}_\Psi = \mathscr{D}\text{-}\mathbf{Lex}$, so

$$\mathbf{fAcc}_\Psi \simeq \mathbf{fAcc}_\Psi^{op}.$$

The weakly locally presentable case

" products "

(5) $\Psi = \{\text{discrete categories}\}$ - $\mathscr{D} = \{\text{pseudo equivalence relations}\}$

$$\mathrm{wLFP}(-,\mathbf{Set}) : \mathbf{wLFP} \rightleftarrows \mathbf{p\text{-}Ex}^{op} : \mathrm{Reg}(-,\mathbf{Set})$$

The weakly locally presentable case

(5) $\Psi = \{\text{discrete categories}\}$ - $\mathscr{D} = \{\text{pseudo equivalence relations}\}$

wLFP:

- $\mathcal{K}$ finitely accessible with products;
- $F : \mathcal{K} \to \mathcal{K}'$ preserving
  - products
  - filtered colimits;
- natural transformations.

$$\mathrm{wLFP}(-, \mathbf{Set}) : \mathbf{wLFP} \rightleftarrows \mathbf{p\text{-}Ex}^{op} : \mathrm{Reg}(-, \mathbf{Set})$$

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p-Ex:

- $\mathcal{E}$ small exact with enough projectives;
- $F : \mathcal{E} \to \mathcal{E}'$ preserving
  - regular epimorphisms
  - finite limits;
- natural transformations.

$$\mathrm{wLFP}(-, \mathbf{Set}) : \mathbf{wLFP} \rightleftarrows \mathbf{p\text{-}Ex}^{op} : \mathrm{Reg}(-, \mathbf{Set})$$

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$$\begin{tikzcd}
\mathrm{wLFP}(-,\mathbf{Set}) : \mathbf{wLFP} \arrow[r, shift left] \arrow[r, phantom, "\simeq"] \arrow[d, hook] & \mathbf{p\text{-}Ex}^{op} : \mathrm{Reg}(-,\mathbf{Set}) \arrow[l, shift left] \arrow[d, hook] \\
\mathrm{Def}(-,\mathbf{Set}) : \mathbf{DEF} \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & \mathbf{Ex}^{op} : \mathrm{Reg}(-,\mathbf{Set}) \arrow[l, shift left]
\end{tikzcd}$$

Models of Regular Theories

Thank You