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Topics in the theory of enriched accessible categories

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Abstract

The aim of this thesis is to further develop the theory of accessible categories in the enriched context. We study and compare the two notions of *accessible* and *conically accessible* $\mathcal{V}$-categories, both arising as free cocompletions of small $\mathcal{V}$-categories: the former under flat-weighted colimits and the latter under filtered colimits. These two notions are not the same in general, however we show that they coincide for many significant bases of enrichment such as **Cat** and **SSet**, and differ just by Cauchy completeness for many algebraic examples including **Ab**, R-**Mod** and **GAb**. We then provide new characterization theorems for these by considering some notions of *virtual orthogonality* and *virtual reflectivity* which generalize the usual reflectivity and orthogonality conditions for locally presentable categories. The word virtual refers to the fact that the reflectivity and orthogonality conditions are given in the free completion of the $\mathcal{V}$-category involved under small limits, instead of the $\mathcal{V}$-category itself. We then prove that the 2-category of accessible $\mathcal{V}$-categories, accessible $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations has all flexible limits. In the final chapters we study, characterize, and provide duality theorems in the setting of accessible $\mathcal{V}$-categories with limits of a specified class Ψ; in this context, instead of the free completion under small limits, we consider "free completions" under a specific type of colimits $\mathfrak{C}$ for which, in particular, $\mathfrak{C}$-colimits commute in $\mathcal{V}$ with Ψ-limits. This allows us to capture the theories of weakly locally presentable, locally multipresentable, locally polypresentable, and accessible categories as instances of the same general framework.

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Statement of Originality

This work has not previously been submitted for a degree or diploma at any university. To the best of my knowledge and belief, this thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.

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Introduction

Accessible categories were first introduced in 1981 by Lair [67] as a generalization of the locally presentable categories of Gabriel and Ulmer. While the latter provide an intrinsic characterization of the categories of models of limit sketches, accessible categories characterize the categories which arise as models of general limit/colimit sketches in the sense of Ehresmann [38]; for this reason they were initially named *catégories modelables*. For a few years the theory remained confined to the French school, and was brought forward especially by Lair and Guitart [48]. In 1983 Rosický came up with the same definition independently in his doctoral work [90], describing accessible categories as categories of models of (some notion of) infinitary first order logic. However, the theory started to get more widely known only in the late 80s with the monograph [77] by Makkai and Paré, where they rediscovered the theory (for the third time) independently from the work of Lair and Rosický; it was in this book that these were called *accessible categories* for the first time. After that the importance of the theory was widely recognized and another monograph was written in 1994 by Adámek and Rosický [1]. We direct the reader to [77] and [1] for a detailed account of the history.

Since their introduction, accessible categories have been presented in many different ways. Intrinsically, they can be described as categories freely generated under α-filtered colimits by a small category, for some regular cardinal α. Moreover they arise as categories of models of limit/colimit sketches; as mentioned above, this is the reason why they were first considered by Lair. Rosický and Makkai-Paré also showed that accessible categories can be presented, from a logical point of view, as categories of models of infinitary first order theories. Finally, they can also be described as full subcategories of presheaf categories closed under α-filtered colimits, for some α, and whose inclusion functor satisfies the solution-set condition.

Unlike in the locally presentable case, where the theory works very smoothly and one can follow a very formal approach, with the theory of accessibility things become rather more complicated. In fact, the use of regular cardinals, which worked so well for local presentability, became somewhat an obstacle in the characterization theorems for accessible categories. It is not true that if a category $\mathcal{A}$ is α-accessible and $\beta > \alpha$, then $\mathcal{A}$ is also β-accessible (however, the same statement holds in the locally presentable case).

Similarly it is not true that sketches with any colimit and just finite-limit specifications classify finitely accessible categories (instead, finite-limit sketches classify locally finitely presentable categories). Moreover, the orthogonality and reflectivity conditions used in the locally presentable case need to be replaced by the less satisfying ones of injectivity and weak reflectivity.

Because of this, many of the proofs of the characterization theorems for accessible categories are technically involved and very **Set**-based. One of our goals, within the thesis, is to provide a more formal approach to accessibility; thus obtaining a clearer understanding of the theory as well as useful ways of recognizing accessible categories. Moreover, with a formal approach, we are also able to prove all of our results in the more general context of enriched category theory. In fact, while some of the results we present here are new even in the ordinary context, the main goal of this project is to further develop the theory of accessibility in the enriched setting.

Enrichment is nowadays a standard tool in category theory; its range of applications is so vast that it reaches very different areas of mathematics, such as algebra [53, 78, 97], homotopy theory [31, 75, 89], computer science [12, 51, 85], and functional analysis [74, 84]. Even though additive and abelian categories were introduced earlier and can be understood as some (easy) examples of enrichment, it was only in the 60s, after the development of differentially graded categories, that people started to think about a general framework for dealing with categories whose homs have a much richer structure than that of a set.

The first to, independently, envisage the potentials of such a theory were Mac Lane [69] and Bénabou [26], as well as Linton in [73]. However, Eilenberg and Kelly were the ones that actually developed a theory of enrichment in their paper [39]. After their work, the theory started to get studied and many results from ordinary category theory were transferred into this richer setting, sometimes with effort and some other times very easily and elegantly. Later, the theory evolved in new directions by introducing enrichment over bicategories [11, 103]; however that will not be the framework of this thesis, where we consider only enrichment over symmetric monoidal closed categories. An account of all the results we need is given in Kelly's book [56], which will be for us a standard reference on matters about enrichment.

Goals

Locally presentable categories found their way into enriched category theory fairly early thanks to Kelly's paper [57], which appeared almost contemporarily to his monograph on enriched categories. With this work, Kelly extended most of the results about locally presentable categories to an enriched framework, with the only request of the base of enrichment being local presentability. On the contrary, the theory of enriched accessible categories, first introduced in the late 1990s [17], is much less developed. We plan to address this gap in the literature by focusing on the following goals:

(a) There is not a unique way of defining what an accessible $\mathcal{V}$-category is; in fact two different notions have already been considered in the literature. Those that we simply call *accessible* arise as free cocompletions of small $\mathcal{V}$-categories under α-flat colimits for some α; the others, that we call *conically accessible*, arise instead as free cocompletions of small $\mathcal{V}$-categories under (conical) α-filtered colimits for some α. We shall compare these two notions and prove that, for some instances of enrichment, they actually coincide.

(b) The theory of accessible categories is much harder to address than that of locally presentable ones; even in the ordinary setting where many of the proofs are very

Set-based and thus not suitable for an enriched generalization. We shall introduce new notions that will provide a more formal approach to the theory and allow a characterization of the enriched accessible categories in terms of some orthogonality and reflectivity conditions.

(c) Weaker notions of local presentability have been introduced before by considering those accessible categories that only have limits of some class. The existence of these limits, for all the examples considered in the literature, turned out to be equivalent to asking the accessible categories to satisfy some "weakened" cocompleteness conditions. We shall introduce the notion of companion $\mathfrak{C}$ for a class of weights Ψ, and characterize the accessible $\mathcal{V}$-categories with Ψ-limits in terms of some cocompleteness involving the companion $\mathfrak{C}$. This will allow us to recover the standard theorems for locally presentable, locally multipresentable, and locally polypresentable categories as instances of the same general framework.

(d) For each of the weaker notions of local presentability discussed above there is a corresponding duality theorem. This, given a class of weights Ψ, establishes a duality between the 2-categories of α-accessible categories with Ψ-limits and the 2-category of those α-complete categories which arise as free cocompletions of a small category under colimits of a specific type. Examples are the Gabriel-Ulmer duality for locally presentable categories, and Diers duality for the locally multipresentable ones. We shall obtain these dualities, as well as new ones, using the theory of companions that has been introduced in the previous point.

While these goals can be considered as the main objectives of the thesis, along the way we will also prove some minor results that, although not being central in the theory of accessibility, are strictly related to the notions we introduce. We discuss all these objectives in more detail below.

Two notions of accessibility

Let us focus on point (a) and try to understand why there are two notions (while for ordinary categories we had only one) and what properties each of the two satisfy.

Recall that an ordinary functor $M\colon \mathcal{C}^{op} \to \mathbf{Set}$ is called *flat* if its left Kan extension $\mathrm{Lan}_Y M\colon [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$ along the Yoneda embedding preserves finite limits. Equivalently M is flat if and only if its category of elements is filtered; or even: M is flat if and only if it is a filtered colimit of representable functors. This outlines a deep connection between flatness and filtered colimits.

This connection plays a key role in the theory of accessible categories: a category is finitely accessible by definition if it is the free cocompletion of a small category $\mathcal{C}$ under filtered colimits; by the observation above this is the same as saying that $\mathcal{A}$ is equivalent to the category $\mathrm{Flat}(\mathcal{C}^{op}, \mathbf{Set})$ of flat presheaves on a small category $\mathcal{C}$. This is a fundamental step in the characterization of accessible categories as models of sketches and of first order theories.

The situation becomes rather more complicated when we move to enriched category theory. Let us fix a base of enrichment $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ which is symmetric monoidal closed and locally finitely presentable as a closed category. In this setting a weighted notion of finite limit has been introduced by Kelly [57]; then conical filtered colimits commute in $\mathcal{V}$ with these finite weighted limits. However conical colimits are not generally enough when enrichment is involved; this means that there might be a wider class of weighted colimits which commute with finite weighted limits in $\mathcal{V}$. That is exactly where the notion of flat $\mathcal{V}$-functor comes into play:

Definition ([57])**.** We say that a $\mathcal{V}$-functor $M\colon \mathcal{C}^{op} \to \mathcal{V}$ is *flat* if $\mathrm{Lan}_Y M\colon [\mathcal{C},\mathcal{V}] \to \mathcal{V}$ preserves all finite weighted limits.

Equivalently, M is flat if and only if M-weighted colimits commute with finite limits in $\mathcal{V}$. If $\mathcal{V} = \mathbf{Ab}$ and $\mathcal{C}$ is a one object $\mathbf{Ab}$-category, then $R := \mathcal{C}(*,*) \in \mathbf{Ab}$ is a ring and a $\mathcal{V}$-functor $M\colon \mathcal{C}^{op} \to \mathbf{Ab}$ is just an R-module M, so that $[\mathcal{C},\mathbf{Ab}] \cong R$-Mod. Moreover $\mathrm{Lan}_Y M \cong M \otimes -$; since right exact additive functors between abelian categories are left exact if and only if they preserve monomorphisms, we recover the algebraic notion of flatness introduced by Serre in [94].

We can now talk about flat-weighted colimits for $\mathcal{V}$-categories; these include, but do not reduce to, the conical filtered ones. For instance every absolute weighted colimit is flat but need not be filtered. An explicit example can be given for $\mathcal{V} = \mathbf{Ab}$: finite direct sums are absolute, and hence flat, but they are not filtered. Therefore, depending on which class one decides to work with, two different notions of accessibility can be introduced. Historically, the first to be considered was based on flat weights.

Definition ([17])**.** A $\mathcal{V}$-category $\mathcal{A}$ is called *finitely accessible* if it is the free cocompletion of a small $\mathcal{V}$-category under flat-weighted colimits.

On the bright side, this captures many of the characterization theorems from the ordinary setting. For instance a $\mathcal{V}$-category $\mathcal{A}$ is finitely accessible if and only if it is equivalent to $\mathrm{Flat}(\mathcal{C}^{op},\mathcal{V})$, the full subcategory of $[\mathcal{C}^{op},\mathcal{V}]$ spanned by the flat $\mathcal{V}$-functors, for some small $\mathcal{C}$; while, as we will see, a $\mathcal{V}$-category is α-accessible for some α if and only if it is the $\mathcal{V}$-category of models of a $\mathcal{V}$-sketch. The problem with this notion is that flat $\mathcal{V}$-functors can be hard to describe, and so it can be difficult to tell whether or not an enriched category is accessible in this sense.

More recently, however, various authors have used a different notion of enriched accessibility. An early example, involving the additive case $\mathcal{V} = \mathbf{Ab}$, was the work of Prest on the model theory of modules, as in [86]. There followed various homotopical examples [20, 23, 63], involving $\mathcal{V} = \mathbf{Cat}$ and $\mathcal{V} = \mathbf{SSet}$ as bases of enrichment. Each of these cases was based on filtered colimits rather than flat ones, and implicitly or explicitly relied on the following notion.

Definition. A $\mathcal{V}$-category $\mathcal{A}$ is *conically finitely accessible* if it is the free cocompletion of a small $\mathcal{V}$-category under conical filtered colimits.

This is more straightforward to work with but lacks the connection with sketches which was after all the original motivation for the notion of accessibility.

In Section 2.1.3 we will see that every accessible $\mathcal{V}$-category is also conically accessible, but the index of accessibility might need to be raised. However, the converse is not true in general since a conically accessible $\mathcal{V}$-category may not have flat colimits, as stated in the case $\mathcal{V} = \mathbf{Ab}$ above. Thus a natural question to ask is whether these notions coincide for some instances of enrichment. If they were in fact the same, or if at least we could understand well the relationship between them, then we would in some sense have the best of both worlds: both the relationship with sketches and other good theoretical properties, as well as the use of the simpler α-filtered colimits rather than α-flat ones.

We address these problems in Chapter 3 by giving an explicit description of α-flat colimits for certain classes of base of enrichment.

More specifically, we will prove that the two notions of accessibility coincide whenever the $\mathcal{V}$-functor $\mathcal{V}(I,-)\colon \mathcal{V}_0 \to \mathbf{Set}$, obtained by homming out of the unit of $\mathcal{V}$, is (weakly) strong monoidal and (weakly) cocontinuous. This includes many examples such as the symmetric monoidal closed categories **Set** of sets, **Cat** of small categories, **SSet** of simplicial sets, $\mathbb{2}$ of the free-living arrow, **Gpd** of groupoids, and $\mathcal{V}$-**Cat** of small $\mathcal{V}$-categories for any locally presentable $\mathcal{V}$.

On the other hand, for various instances of enrichment with an algebraic flavour, we show that accessibility and conical accessibility differ only by Cauchy completeness. Examples include the the symmetric monoidal categories **CMon** of commutative monoids, **Ab** of abelian groups, **GAb** of graded abelian groups, and R-**Mod** of R-modules for any commutative ring R. In all these cases the Cauchy colimits that need to be added to make the two notions coincide are finite direct sums and copowers by dualizable objects.

A characterization of accessible categories

Now that we have introduced the main objects of study we can focus on point (b). Here we consider mostly the accessible $\mathcal{V}$-categories; nonetheless we compare these with (and give results on) the conically accessible ones as well.

The characterization theorem we prove seems to be new even in the ordinary setting, although some of the machinery we use was already considered, for $\mathcal{V} = \mathbf{Set}$, by Guitart and Lair [48]. The idea is to generalize the notion of orthogonality and reflectivity (central in the characterization of locally presentable categories) to those of *virtual orthogonality* and *virtual reflectivity* as described below. Here the word "virtual" refers to the fact that the objects and morphisms we are considering lie not in a given category $\mathcal{A}$, but rather in its free completion $\mathcal{P}^\dagger\mathcal{A}$ under limits. The $\mathcal{V}$-category $\mathcal{P}^\dagger\mathcal{A}$ can be identified with the opposite of the full subcategory of $[\mathcal{A},\mathcal{V}]$ consisting of the *small* $\mathcal{V}$-functors; namely those that are small limits of representables.

There are virtual notions of: left adjoint, reflective subcategory, cocomplete, and orthogonality class. To introduce the notion of virtual left adjoint, recall that a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{K}$ has a left adjoint if $\mathcal{K}(X, F-)$ is representable for any $X \in \mathcal{K}$; we say instead that F has a *virtual left adjoint* if $\mathcal{K}(X, F-)$ is a small $\mathcal{V}$-functor for any $X \in \mathcal{K}$, that is if $\mathcal{K}(X, F-) \in \mathcal{P}^\dagger\mathcal{A}$ for all $X \in \mathcal{K}$. If F is fully faithful we then say that $\mathcal{A}$ is *virtually reflective* in $\mathcal{K}$. Clearly every reflective subcategory is virtually reflective, but the converse is not true. When $\mathcal{V} = \mathbf{Set}$ and $\mathcal{A}$ and $\mathcal{K}$ are both accessible, then virtual reflectivity is equivalent to the inclusion functor satisfying the solution-set condition (Corollary 2.2.46).

Regarding virtual colimits, we know that, given a weight $M\colon \mathcal{C}^{op} \to \mathcal{V}$ and a $\mathcal{V}$-functor $H\colon \mathcal{C} \to \mathcal{A}$, the colimit $M * H$ exists in $\mathcal{A}$ if the $\mathcal{V}$-functor $[\mathcal{C}^{op},\mathcal{V}](M, \mathcal{A}(H,-))$ is representable. Relaxing that condition, we say that the *virtual colimit* of H weighted by M exists in $\mathcal{A}$ if $[\mathcal{C}^{op},\mathcal{V}](M, \mathcal{A}(H,-))$ is a small $\mathcal{V}$-functor. Since every representable $\mathcal{V}$-functor is small, any cocomplete $\mathcal{V}$-category has virtual colimits; less trivial is the fact that also every accessible $\mathcal{V}$-category has them. When $\mathcal{V} = \mathbf{Set}$ a category $\mathcal{A}$ has virtual colimits if and only if it is pre-cocomplete in the sense of Freyd [42] (see Proposition 2.2.42).

An object A of a $\mathcal{V}$-category $\mathcal{K}$ is said to be orthogonal with respect to a morphism $f\colon X \to Y$ in $\mathcal{K}$ if the map $\mathcal{K}(f, A)$ is an isomorphism in $\mathcal{V}$. Then the notion of *virtual orthogonality* arises exactly in the same way with the only difference being that the morphism f now can be chosen to be of the form $f\colon ZX \to Y$ with $X \in \mathcal{K}$ and $Y \in \mathcal{P}^\dagger\mathcal{K}$, where $Z\colon \mathcal{K} \hookrightarrow \mathcal{P}^\dagger\mathcal{K}$ is the inclusion. Thus we will say that $A \in \mathcal{K}$ is orthogonal with respect to a morphism $f\colon ZX \to Y$ in $\mathcal{P}^\dagger\mathcal{K}$ if the map $\mathcal{P}^\dagger\mathcal{K}(f, ZA)$ is an isomorphism in $\mathcal{V}$. We call *virtual orthogonality class* a full subcategory of $\mathcal{K}$ which arises as a collection of objects which are virtually orthogonal with respect to a small set of morphisms as above.

Then our main theorem of Chapter 2 goes as follows:

Theorem 2.2.32. *The following are equivalent for an accessible $\mathcal{V}$-category $\mathcal{K}$ and a fully faithful inclusion $\mathcal{A} \hookrightarrow \mathcal{K}$:*

1. *$\mathcal{A}$ is accessible and accessibly embedded;*
2. *$\mathcal{A}$ is accessibly embedded and virtually reflective;*
3. *$\mathcal{A}$ is a virtual orthogonality class.*

Even though our proofs still rely on some inevitable **Set**-based conditions, such as the raising of the accessibility index, we believe they provide a more formal and cleaner approach to the theory with respect to some of the concepts studied in the past for ordinary accessible categories. We study the relationship between these and our notions in Section 2.2.6.

In the context of conically accessible $\mathcal{V}$-categories a similar characterization can be given. This shows that, in contrast to accessibility, conical accessibility of a subcategory can be recognized at the level of the underlying ordinary category.

Theorem 2.2.36. *The following are equivalent for a conically accessible $\mathcal{V}$-category $\mathcal{K}$ and a fully faithful inclusion $\mathcal{A} \hookrightarrow \mathcal{K}$:*

1. *$\mathcal{A}$ is conically accessible and conically accessibly embedded in $\mathcal{K}$;*
2. *$\mathcal{A}_0$ is accessible and accessibly embedded in $\mathcal{K}_0$;*
3. *$\mathcal{A}_0$ is accessibly embedded and virtually reflective in $\mathcal{K}_0$;*
4. *$\mathcal{A}_0$ is a virtual orthogonality class in $\mathcal{K}_0$.*

Accessible categories with limits

Next, as part of point (c), we study those accessible $\mathcal{V}$-categories that have limits of a specified class of weights Ψ. Recall that an accessible category is complete if and only if it is cocomplete, and in that case is locally presentable. It is often the case, however, that the accessible categories of interest only have some limits. A natural question is then how to characterize those accessible categories that have a particular class of limits in terms of some, generalized, cocompleteness conditions.

In the examples considered in the literature different authors proved characterizations theorems of the same style: in each case it was proved that a category is accessible with Ψ-limits, for a given class Ψ, if and only if it is accessible and satisfies some kind of cocompleteness conditions, if and only if it is equivalent to the category of models of a specific kind of sketch.

We will prove this theorem for accessible $\mathcal{V}$-categories with limits of a *weakly sound* class Ψ (see [4, 62] and Definition 1.3.4).

Theorem 5.3.16. *Let Ψ be a weakly sound class and Φ be the class of weights whose colimits commute with Ψ-limits in $\mathcal{V}$. The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:*

1. *$\mathcal{A}$ is accessible with Ψ-limits;*
2. *$\mathcal{A}$ is accessible and $\Phi^\dagger\mathcal{A}$ has colimits of objects from $\mathcal{A}$;*
3. *$\mathcal{A}$ is the $\mathcal{V}$-category of models of a limit/Φ-colimit sketch.*

The category $\Phi^{\dagger}\mathcal{A}$ here is the free completion of $\mathcal{A}$ under Φ-limits. When $\Psi = \emptyset$ then $\Phi = \mathcal{P}$ is the class of all weights, we recover the characterization of accessible categories as the sketchable ones [67, 18]. For the locally presentable case, one considers the weakly sound class $\Psi = \mathcal{P}$ of all small weights and the class $\Phi = \mathcal{Q}$ of all those that commute with them; that is, the class of Cauchy colimits. Since for an accessible category $\mathcal{A}$, its Cauchy completion is equivalent to $\mathcal{A}$ itself, being cocomplete for $\mathcal{A}$ is equivalent to the request that the Cauchy completion $\mathcal{Q}^{\dagger}\mathcal{A}$ has all colimits of diagrams landing in $\mathcal{A}$. Thus we recover the standard characterization of locally presentable $\mathcal{V}$-categories.

When $\mathcal{V} = \mathbf{Set}$ and Ψ is the class for connected limits, the class Φ corresponds to that of discrete categories (namely, the shapes for products and coproducts). Recall that a category $\mathcal{A}$ is multicocomplete if its free completion under products $\mathrm{Fam}^{\dagger}\mathcal{A}$ has all colimits of objects from $\mathcal{A}$. Then we obtain the characterization theorem for locally multipresentable categories, due to Diers [35].

Even though the theorem above is already quite general and captures some classes of limits that were not considered before (Example 5.3.6), it does not cover two important examples: the weakly locally presentable categories (Ψ is the class for products) and the locally polypresentable categories (Ψ is the class of wide pullbacks). The problem being that in those two cases the classes of limits in question are not weakly sound.

In the case of locally polypresentable categories the known characterization theorem reads as follows:

Theorem. *The following are equivalent for a category $\mathcal{A}$:*

1. *$\mathcal{A}$ is accessible with wide pullbacks;*
2. *$\mathcal{A}$ is accessible and polycocomplete;*
3. *$\mathcal{A}$ is the category of models of a galoisian sketch.*

The history behind the proof of this theorem is complicated: its origins are in Lamarche's doctoral thesis [68], with further work by Taylor [98] and Hu and Tholen [50]. The notion of galoisian sketch and the equivalence of (3) to the other conditions is due to Ageron [6].

Unlike the previous cases, polycolimits in $\mathcal{A}$ are not computed in a free completion of $\mathcal{A}$, at least not in the usual sense. In fact, Hu and Tholen [50] prove that a category $\mathcal{A}$ has polycolimits if and only if the "free completion" of $\mathcal{A}$ under limits of *free groupoid actions* has colimits of objects from $\mathcal{A}$. This is not a free completion in the usual sense since the diagrams defining free groupoid actions are not just functors out of an indexing category, but need to satisfy some additional properties. Similarly, in [6] Ageron considers colimits of free groupoid actions to define what the cocone specifications look like on a galoisian sketch.

To capture this and the sound case under the same theory we shall prove the following theorem, and introduce the various notions on which it relies.

Theorem. *Let Ψ be a class of weights, and $\mathfrak{C}$ be a companion for Ψ. The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:*

1. *$\mathcal{A}$ is accessible with Ψ-limits;*
2. *$\mathcal{A}$ is accessible and $\mathfrak{C}_1^{\dagger}\mathcal{A}$ has colimits of representables.*
3. *$\mathcal{A}$ is the $\mathcal{V}$-category of $\mathfrak{C}$-models of a sketch.*

In particular in Chapter 5 we explain:

- what it means for $\mathfrak{C}$ to be a companion (Definition 5.2.11);
- what $\mathfrak{C}_1^\dagger\mathcal{A}$ is (Definition 5.2.8);
- what is a $\mathfrak{C}$-model of a sketch (Definition 5.2.19).

If Ψ is weakly sound this specializes to Theorem 5.3.16. If $\mathcal{V} = \mathbf{Set}$ and Ψ is the class for wide pullbacks we recover the characterization for locally polypresentable categories. In the context of weakly locally presentable categories, we do obtain a theorem but it does not exactly match the characterization theorems of [1, Chapter 4] involving weak reflections and weak cocompleteness; those will be recovered separately in Section 5.4.

Dualities

The last of our goals, point (d), concerns the study of dualities for finitely accessible $\mathcal{V}$-categories with Ψ-limits and their infinitary generalizations.

When considering the finitely accessible and complete categories, we obtain the well known Gabriel-Ulmer duality for locally finitely presentable categories [44]. This establishes a biequivalence between the 2-category of locally finitely presentable categories, right adjoint and finitary functors, and natural transformation and the opposite of the 2-category of *finite limit theories*; that is, the 2-category of small finitely complete categories, finitely continuous functors, and natural transformations.

In the case where the finitely accessible categories are not assumed to have limits, a duality theorem was established in [77, Proposition 4.2.1]. It is shown that the 2-category of finitely accessible categories, finitary functors, and natural transformations is dual to the 2-category of presheaf categories, cocontinuous and finite-limit preserving functors, and natural transformations. Similarly Diers proved a duality theorem in the context of locally finitely multipresentable categories [35].

Using the notion of companion $\mathfrak{C}$ for Ψ, in Chapter 6 we will give sufficient conditions on $\mathfrak{C}$ to induce the duality theorem below characterizing the α-accessible $\mathcal{V}$-categories with Ψ-limits.

Theorem 6.2.10. *The 2-functors*

$$\alpha\text{-}\mathrm{Acc}_\Psi(-,\mathcal{V})\colon \alpha\text{-}\mathbf{Acc}_\Psi \rightleftarrows \mathfrak{C}\text{-}\mathbf{Ex}_\alpha^{op} : \mathfrak{C}\text{-}\mathrm{Reg}_\alpha(-,\mathcal{V})$$

form a biequivalence of 2-categories.

Here, α-$\mathbf{Acc}_\Psi$ is the 2-category of α-accessible $\mathcal{V}$-categories with Ψ-limits, Ψ-continuous and α-flat colimit preserving $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. On the other hand, $\mathfrak{C}$-$\mathbf{Ex}_\alpha$ can be described as the 2-category with objects the α-complete $\mathcal{V}$-categories that are free $\mathfrak{C}$-cocompletions of small $\mathcal{V}$-categories, α-continuous and $\mathfrak{C}$-cocontinuous $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. All these notions will be introduced and explained in Section 6.2.

When $\Psi = \mathcal{P}$ is the class of all small weights, we recover the Gabriel-Ulmer duality between locally α-presentable $\mathcal{V}$-categories and the small α-cocomplete ones. If $\mathcal{V} = \mathbf{Set}$ and Ψ is the class for connected limits we recover Diers duality for locally finitely multipresentable categories [35]. More generally we obtain a duality theorem for α-accessible $\mathcal{V}$-categories with limits of a weakly sound class Ψ; on the opposite side we find the 2-category whose objects are the α-complete $\mathcal{V}$-categories which arise as free cocompletions of a small $\mathcal{V}$-category under Ψ-flat colimits.

When $\mathcal{V} = \mathbf{Set}$ and Ψ is the class for wide pullbacks we obtain the duality of Hu and Tholen [50] between locally α-polypresentable categories, and (what they call) α-complete quasi-based categories. In the case of weakly locally finitely presentable categories (Ψ being the class for products) the duality we obtain first appeared in Hu's PhD thesis [49]; we generalize that to the context of categories enriched over finitary varieties.

Additional results

Beside the main goals of the thesis described above, in Chapter 4 we study the accessible $\mathcal{V}$-functors $F\colon \mathcal{K} \to \mathcal{L}$ between locally presentable categories and show that such a functor F preserves all small limits if and only if it preserves γ-small limits, for some regular cardinal γ depending only on $\mathcal{K}$ (Theorem 4.1.6). This result can also be interpreted as a new adjoint functor theorem for α-accessible functors out of a locally α-presentable category. The "ur-adjoint functor theorem" says that if a category $\mathcal{K}$ has all (possibly large) limits and $U\colon \mathcal{K} \to \mathcal{L}$ preserves them, then U has a left adjoint. When $\mathcal{K}$ only has small limits (as usually happens), then one invokes Freyd's general adjoint functor theorem [43], which requires U to be continuous and to satisfy in addition the solution set condition; this is the case in particular when $\mathcal{K}$ and $\mathcal{L}$ are locally presentable, and U is a continuous and accessible functor. Then our Theorem 4.1.8 says that the condition on continuity can be weakened to γ-continuity (for some γ) when we restrict to the locally α-presentable case.

In Appendix A we prove some further results that, despite not being central to development of the theory of enriched accessibility, provide some useful insights on the notions we used and direct applications of results from previous chapters. In Section A.1 we compare the notions of saturated and pre-saturated classes of indexing categories and weights; we prove that every saturated class is pre-saturated and give conditions for the other implication to hold. In Section A.2 we consider an enriched notion of *pettiness* and extend the contents of Section 2.2.6, comparing virtual and cone reflectivity, to the enriched setting. By applying the results of Chapter 5 to the companion for the class of products, in Section A.3 we give another characterization of the definable categories of Prest [86]. Finally, in Section A.4 we extend the duality involving the 2-category of finitely accessible $\mathcal{V}$-categories (obtained in Chapter 6) to an adjunction between the 2-category of accessible $\mathcal{V}$-categories with filtered colimits and that of $\mathcal{V}$-topoi.

CHAPTER

1

Background notions

In this chapter we recall some standard results that will be needed throughout the thesis. These concern ordinary accessible categories (Section 1.1), enriched category theory (Section 1.2), and the notion of sound and weakly sound class of weights (Section 1.3).

1.1 Results on ordinary accessible categories

In this section we recall the only result on ordinary accessible categories that will be used throughout this paper; that is about raising the index of accessibility:

Definition 1.1.1. Given two regular cardinals α and β, we say that α is *sharply less* than β, and write $\alpha \lhd \beta$, if $\alpha < \beta$ and for every α-filtered category $\mathcal{C}$ and any β-small $\mathcal{D} \subseteq \mathcal{C}$ there exists a β-small and α-filtered $\mathcal{E}$ with $\mathcal{D} \subseteq \mathcal{E} \subseteq \mathcal{C}$.

This is one of many equivalent set-theoretic definitions of the sharply less than relation. Equivalently, one could consider in the definition above the case when $\mathcal{C}$ is just an α-directed poset (this notion was considered in [1, Theorem 2.11(iv)]). Another set-theoretic characterization is as follows: $\alpha \lhd \beta$ if and only if $\alpha < \beta$ and for every set X of cardinality less than β, the partially ordered set $\mathcal{P}_\alpha(X)$, of all subsets of X with cardinality less than α, has a final subset of cardinality less than β. This is how the sharply less relation was originally defined in [77, 2.3.1].

Remark 1.1.2. For any small set of regular cardinals $\{\alpha_i\}_{i\in I}$ there are arbitrarily large regular cardinals β for which $\beta \rhd \alpha_i$ holds for all $i \in I$ [1, Example 2.13(6)].

Before stating the next theorem let us fix some notation. For a regular cardinal α and a category $\mathcal{C}$ we denote by α-Ind$(\mathcal{C})$ the free cocompletion of $\mathcal{C}$ under α-filtered colimits; this can be described as the full subcategory of $[\mathcal{C}^{op}, \mathbf{Set}]$ spanned by the α-flat functors. For given regular cardinals $\alpha < \beta$ we denote by $\mathcal{C}_{\beta/\alpha}$ the free cocompletion of $\mathcal{C}$ under β-small α-filtered colimits; this is a one-step completion by [30, Corollary 4.13]. Let $J\colon \mathcal{C} \hookrightarrow \mathcal{C}_{\beta/\alpha}$ be the inclusion; then since $\mathcal{C}^{op}_{\beta/\alpha}$ is the free completion of $\mathcal{C}^{op}$ under β-small α-cofiltered limits, it follows that

$$(- \circ J)\colon {}^{\beta}/\alpha\text{-Cont}(\mathcal{C}^{op}_{\beta/\alpha}, \mathbf{Set}) \to [\mathcal{C}^{op}, \mathbf{Set}]$$

is an equivalence of categories with inverse $\mathrm{Ran}_{J^{op}}$, where ${}^{\beta}/\alpha\text{-Cont}[\mathcal{C}^{op}_{\beta/\alpha}, \mathbf{Set}]$ denotes the full subcategory of $[\mathcal{C}^{op}_{\beta/\alpha}, \mathbf{Set}]$ spanned by those functors that preserve β-small α-filtered limits. Consider now a representable $\mathcal{C}_{\beta/\alpha}(-, C)$, since C is a J-absolute β-small α-filtered colimit of elements from $\mathcal{C}$, it follows that $\mathcal{C}_{\beta/\alpha}(J-, C) \cong \operatorname*{colim}_i \mathcal{C}(-, C_i)$ is a β-small α-filtered colimit of representables, and in particular α-flat. Moreover, since pre-composition by J preserves β-filtered colimits (being cocontinuous) and these are also α-filtered, it follows that we have an induced functor

$$(- \circ J)\colon \beta\text{-Flat}(\mathcal{C}^{op}_{\beta/\alpha}, \mathbf{Set}) \hookrightarrow \alpha\text{-Flat}(\mathcal{C}^{op}, \mathbf{Set})$$

which is fully faithful since every β-flat functor is ${}^{\beta}/\alpha$-continuous (it preserves all existing β-small limits).

We are now ready to state and prove the following theorem, which can be seen as an expansion of [1, Theorem 2.11].

Theorem 1.1.3. *Let $\alpha < \beta$ be two regular cardinals and $J\colon \mathcal{C} \hookrightarrow \mathcal{C}_{\beta/\alpha}$ be as above; the following are equivalent:*

1. *$\alpha \lhd \beta$;*
2. *if $\mathcal{C}$ is an α-filtered category then $\mathcal{C}_{\beta/\alpha}$ is β-filtered;*
3. *if $M\colon \mathcal{C}^{op} \to \mathbf{Set}$ is an α-flat functor then $\mathrm{Ran}_{J^{op}}M\colon \mathcal{C}^{op}_{\beta/\alpha} \to \mathbf{Set}$ is β-flat;*
4. *each α-flat $M\colon \mathcal{C}^{op} \to \mathbf{Set}$ is the restriction of some β-flat $N\colon \mathcal{C}^{op}_{\beta/\alpha} \to \mathbf{Set}$;*
5. *$\alpha\text{-Ind}(\mathcal{C}) \simeq \beta\text{-Ind}(\mathcal{C}_{\beta/\alpha})$ for any small $\mathcal{C}$;*
6. *every α-accessible category is β-accessible.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{C}$ be an α-filtered category and $\{X_i\}_{i\in I}$ be a β-small family of objects in $\mathcal{C}_{\beta/\alpha}$. For each $i \in I$ fix a β-small diagram $H_i\colon \mathcal{D}_i \to \mathcal{C}$ whose colimit in $\mathcal{C}_{\beta/\alpha}$ is X_i. By (1) we can consider a β-small and α-filtered $\mathcal{E} \subseteq \mathcal{C}$ which contains the images of all the H_i's. Let X be the colimit of the inclusion of $\mathcal{E}$ in $\mathcal{C}_{\beta/\alpha}$; then by construction we have an induced arrow $X_i \to X$ for any $i \in I$, as desired.

Consider now a β-small family of parallel arrows $\{f_i\colon X \to Y\}_{i\in I}$ in $\mathcal{C}_{\beta/\alpha}$. Since X and Y are β-small colimits of objects of $\mathcal{C}$, each arrow f_i can be expressed as a β-small colimit of objects in $\mathcal{C}^{\mathbf{2}}$; thus we can find β-small categories $\mathcal{D}_i$ and diagrams $H_i\colon \mathcal{D}_i \to \mathcal{C}^{\mathbf{2}}$ whose colimits in $(\mathcal{C}_{\beta/\alpha})^{\mathbf{2}}$ are the f_i's. Again, by (1) we can find a β-small and α-filtered $\mathcal{E}$ in $\mathcal{C}$ which contains the images of all the H_i's. It follows then that the colimit of $\mathcal{E}$ in $\mathcal{C}_{\beta/\alpha}$ comes with a cocone for the family $\{f_i\}_{i\in I}$.

(2) $\Rightarrow$ (3). Consider an α-flat $M\colon \mathcal{C}^{op} \to \mathbf{Set}$ and its right Kan extension $N\colon \mathcal{C}^{op}_{\beta/\alpha} \to \mathbf{Set}$ along J^{op}. Using the fact that $\mathcal{C}_{\beta/\alpha}$ is the free cocompletion of $\mathcal{C}$ under β-small α-filtered colimits, one can show that $\mathrm{El}(N)$ is the free cocompletion of $\mathrm{El}(M)$ under the same kind

of colimits, so that $\mathrm{El}(N) = \mathrm{El}(M)_{\beta/\alpha}$. Now, since M is α-flat, then $\mathrm{El}(M)$ is α-filtered, and thus $\mathrm{El}(N)$ is β-filtered by (2).

(3) $\Rightarrow$ (4). This is trivial assuming (3) since M is always the restriction of its right Kan extension along J^{op}.

(4) $\Rightarrow$ (5). Thanks to (4) and the comments above the Theorem, $(- \circ J)$ induces an equivalence between the α-flat functors out of $\mathcal{C}^{op}$ and the β-flat functors out of $\mathcal{C}^{op}_{\beta/\alpha}$. Thus (5) follows at once.

(5) $\Rightarrow$ (6) is trivial, while (6) $\Rightarrow$ (1) can be shown as in the proof of [1, Theorem 2.11]. $\square$

A direct consequence of the equivalence between (1) and (6) is that the sharply less than relation is transitive.

Remark 1.1.4. Point (5) of the Theorem can be restated as follows. Given the pseudomonads $P = \beta\text{-Ind}(-)$, that freely adds β-filtered colimits, and $T = (-)_{\beta/\alpha}$, that freely adds β-small α-filtered colimits, the composite PT is still a monad and coincides with $\alpha\text{-Ind}(-)$. This results in a distributive law from T to P.

Moreover:

Corollary 1.1.5. *[77, Proposition 2.3.11] For an α-accessible category $\mathcal{A}$ and regular cardinals $\alpha \lhd \beta$, there is an equivalence $\mathcal{A}_\beta \simeq (\mathcal{A}_\alpha)_{\beta/\alpha}$ and hence every β-presentable object of $\mathcal{A}$ is a β-small α-filtered colimit of α-presentable objects.*

Proof. The fact that $\mathcal{A}_\beta \simeq (\mathcal{A}_\alpha)_{\beta/\alpha}$ is a direct consequence of condition (5) from the Theorem above and of the fact that $\alpha\text{-Ind}(\mathcal{C})_\alpha \simeq \mathcal{C}$ for any α and any Cauchy complete $\mathcal{C}$. The last assertion is a consequence of Remark A.1.10. $\square$

A direct consequence of these results is the following:

Corollary 1.1.6. *[1, Theorem 2.19] Given an accessible functor $F\colon \mathcal{A} \to \mathcal{B}$ between ordinary accessible categories, there exists α such that F preserves the β-presentable objects for each $\beta \rhd \alpha$.*

Proof. Let α_0 be such that $\mathcal{A}$, $\mathcal{B}$, and F are α_0-accessible and let $\alpha \rhd \alpha_0$ be such that $F(\mathcal{A}_{\alpha_0}) \subseteq \mathcal{B}_\alpha$. Consider now $\beta \rhd \alpha$; by transitivity of the sharply less than relation and Corollary 1.1.5 each object of $\mathcal{A}_\beta$ is a β-small α_0-filtered colimit of objects from $\mathcal{A}_{\alpha_0}$; since F preserves α_0-filtered colimits it follows that each object of $F(\mathcal{A}_\beta)$ is a β-small (α_0-filtered) colimit of objects from $\mathcal{B}_\alpha$, and thus is still β-presentable in $\mathcal{B}$. $\square$

1.2 Background on enriched categories

We now fix a base of enrichment $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ which is a complete, cocomplete, and symmetric monoidal closed category.

For matters concerning enrichment we follow the notations of [56], with the only change that "indexed" colimits are here called "weighted", as is now standard. In particular, given a $\mathcal{V}$-category $\mathcal{A}$ we denote by $\mathcal{A}_0$ its underlying ordinary category; similarly if $F\colon \mathcal{A} \to \mathcal{B}$ is a $\mathcal{V}$-functor we denote by $F_0\colon \mathcal{A}_0 \to \mathcal{B}_0$ the corresponding ordinary functor underlying F. For any ordinary small category $\mathcal{K}$ we denote by $\mathcal{K}_\mathcal{V}$ the free $\mathcal{V}$-category over $\mathcal{K}$. Our $\mathcal{V}$-categories are allowed to have a large set of objects, unless specified otherwise.

We call *weight* a $\mathcal{V}$-functor $M\colon \mathcal{C}^{op} \to \mathcal{V}$ with small domain. Given such a weight M and a $\mathcal{V}$-functor $H\colon \mathcal{C} \to \mathcal{A}$, we denote by $M * H$ (if it exists) the colimit of H weighted by M; this is determined by the object $M * H \in \mathcal{A}$ and a $\mathcal{V}$-natural transformation $M \to \mathcal{A}(H-, M * H)$ which induces an isomorphism

$$\mathcal{A}(M * H, A) \cong [\mathcal{C}^{op}, \mathcal{V}](M, \mathcal{A}(H-, A))$$

in $\mathcal{V}$, for any $A \in \mathcal{A}$. Dually, given a weight $N\colon \mathcal{C} \to \mathcal{V}$ and a $\mathcal{V}$-functor $K\colon \mathcal{C} \to \mathcal{A}$, the weighted limit of K by N is denoted by $\{N, K\}$. Conical limits and colimits are special cases of weighted ones; they coincide with those weighted by $\Delta I\colon \mathcal{B}_{\mathcal{V}}^{op} \to \mathcal{V}$ for some ordinary category $\mathcal{B}$. The conical colimit of a $\mathcal{V}$-functor $T_{\mathcal{V}}\colon \mathcal{B}_{\mathcal{V}} \to \mathcal{A}$, if it exists, will also be the ordinary colimit of the transpose $T\colon \mathcal{B} \to \mathcal{A}_0$ in $\mathcal{A}_0$ (but the converse is not generally true).

Since every weight is assumed to have a small domain, all the limits and colimits that we consider here will be *small.* Therefore, when talking about limits and colimits of some (maybe large) class of weights, these will always be weighted by a $\mathcal{V}$-functor with small domain.

We assume now that $\mathcal{V}$ is locally α-presentable as a closed category, meaning that it is locally α-presentable and the α-presentable objects contain the unit and are closed under tensor product [57]. Then Kelly introduced a notion of α-small weight which will be relevant throughout the thesis.

Definition 1.2.1 ([57])**.** We say that a weight $M\colon \mathcal{C}^{op} \to \mathcal{V}$ is *α-small* if $\mathcal{C}$ has less than α objects, $\mathcal{C}(C, D) \in \mathcal{V}_\alpha$ for any $C, D \in \mathcal{C}$, and $M(C) \in \mathcal{V}_\alpha$ for any $C \in \mathcal{C}$. An α-small (weighted) limit is one taken along an α-small weight. We say that a $\mathcal{V}$-category $\mathcal{C}$ is α-complete if it has all α-small limits; we say that a $\mathcal{V}$-functor is α-continuous if it preserves all α-small limits.

Both conical α-small limits and powers by α-presentable objects are examples of α-small limits and together are enough to generate all α-small weighted limits [57, Section 4]. Note that where here are called “powers” in [56] were referred to as “cotensors”.

Definition 1.2.2. We say that a weight $M\colon \mathcal{C}^{op} \to \mathcal{V}$ is α-flat if its left Kan extension $\mathrm{Lan}_Y M\colon [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ along the Yoneda embedding is α-continuous. With α-flat colimits we mean those weighted by an α-flat weight.

Note that every conical α-filtered colimit is α-flat and that the α-flat $\mathcal{V}$-functors are closed in $[\mathcal{C}^{op}, \mathcal{V}]$ under α-flat colimits.

Proposition 1.2.3 ([57])**.** *Let $M\colon \mathcal{C}^{op} \to \mathcal{V}$ be a weight; the following are equivalent:*

1. *M is α-flat;*
2. *$M * -\colon [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ preserves all α-small limits;*
3. *M is an α-flat colimit of representables.*

If $\mathcal{C}$ is α-cocomplete they are further equivalent to:

4 *M is α-continuous;*

5 *M is a conical α-filtered colimit of representables.*

In that case the following isomorphism holds

$$M \cong \operatorname{colim}(\operatorname{El}(M_I)_{\mathcal{V}} \xrightarrow{\pi_{\mathcal{V}}} \mathcal{C} \xrightarrow{Y} [\mathcal{C}^{op}, \mathcal{V}])$$

where $M_I = \mathcal{V}_0(I, M_0-)$ *and* $\operatorname{El}(M_I)$ *is* α*-filtered.*

1.3 Soundness

Consider again a complete and cocomplete symmetric monoidal closed category $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ as the base for enrichment.

Definition 1.3.1 ([33])**.** Let Φ be a class of weights; a weight $M\colon \mathcal{C}^{op} \to \mathcal{V}$ is called Φ-flat if its left Kan extension $\operatorname{Lan}_Y M\colon [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ along the Yoneda embedding is Φ-continuous. We denote by $\Phi\text{-Flat}(\mathcal{C}^{op}, \mathcal{V})$ the full subcategory of $[\mathcal{C}^{op}, \mathcal{V}]$ spanned by the Φ-flat weights.

Equivalently, Φ-flat are those weights whose weighted colimits commute with Φ-limits in $\mathcal{V}$. When Φ is the class of α-small weights this is the usual notion of α-flat functor.

Lemma 1.3.2. *Let* Φ *be a class of weights,* $J\colon \mathcal{B} \to \mathcal{C}$ *be a* $\mathcal{V}$*-functor between small* $\mathcal{V}$*-categories, and* $M\colon \mathcal{B}^{op} \to \mathcal{V}$ *a weight; then:*

1. *if* M *is* Φ*-flat then* $\operatorname{Lan}_{J^{op}} M$ *is;*
2. *if* J *is fully faithful and* $\operatorname{Lan}_{J^{op}} M$ *is* Φ*-flat then* M *is* Φ*-flat as well.*

Proof. By definition a weight M is Φ-flat if it is small and its left Kan extension along the Yoneda embedding, which is the functor $M * -\colon [\mathcal{B}, \mathcal{V}] \to \mathcal{V}$, is Φ-continuous. Note that the following triangle commutes:

$$\begin{tikzcd} & {[\mathcal{B}, \mathcal{V}]} \arrow[dr, "M * -"] & \\ {[\mathcal{C}, \mathcal{V}]} \arrow[ur, "{[J, \mathcal{V}]}"] \arrow[rr, "\operatorname{Lan}_{J^{op}} M * -"'] & & \mathcal{V} \end{tikzcd}$$

indeed $\operatorname{Lan}_{J^{op}} M * F \cong M * FJ$ by [56, 4.19]. As a consequence if M is Φ-flat then so is $\operatorname{Lan}_{J^{op}} M$ since $[J^{op}, \mathcal{V}]$ is continuous. Conversely assume that J is fully faithful and $\operatorname{Lan}_{J^{op}} M$ is Φ-flat, then

$$\begin{aligned} M * - &\cong (M * -) \circ \mathrm{id}_{[\mathcal{B}, \mathcal{V}]} \\ &\cong (M * -) \circ [J, \mathcal{V}] \circ \operatorname{Ran}_{J^{op}} \\ &\cong (\operatorname{Lan}_{J^{op}} M * -) \circ \operatorname{Ran}_J \end{aligned}$$

where $\mathrm{id}_{[\mathcal{B}, \mathcal{V}]} \cong [J, \mathcal{V}] \circ \operatorname{Ran}_J$ since J is fully faithful. It follows that $M * -$ is Φ-continuous because $\operatorname{Lan}_{J^{op}} M * -$ is and Ran_J is continuous. □

Remark 1.3.3. In more familiar terms, this generalizes an easy-to-check fact about filtered categories:

- if $J\colon \mathcal{B} \to \mathcal{A}$ is final and $\mathcal{B}$ is filtered, then $\mathcal{A}$ is filtered as well;
- if $J\colon \mathcal{B} \to \mathcal{A}$ is fully faithful and final, and $\mathcal{A}$ is filtered, then $\mathcal{B}$ is filtered as well.

When $\mathcal{V} = \mathbf{Set}$, this can be seen as a consequence of the Lemma above since a category $\mathcal{B}$ is filtered if and only if the weight $\Delta 1_{\mathcal{B}} \colon \mathcal{B}^{op} \to \mathbf{Set}$ is flat, and a functor $J \colon \mathcal{B} \to \mathcal{A}$ is final if and only if $\Delta 1_{\mathcal{A}} \cong \mathrm{Lan}_{J^{op}}(\Delta 1_{\mathcal{B}})$.

In the second point, we cannot drop the assumption that J is fully faithful as the following example shows. Take the inclusion of the free-living pair into the free-living split pair; then the codomain is filtered (it is actually absolute) and the inclusion is final, but coequalizers are not filtered colimits.

Recall from [59] that a class of weights Φ is called *locally small* if for every small $\mathcal{V}$-category $\mathcal{C}$ the free cocompletion $\Phi\mathcal{C}$ is still a small $\mathcal{V}$-category.

Definition 1.3.4. We say that a locally small class of weights Φ is *weakly sound* if every Φ-continuous $\mathcal{V}$-functor $M \colon \mathcal{C}^{op} \to \mathcal{V}$ (from a small Φ-cocomplete $\mathcal{C}$) is Φ-flat.
We say that a locally small class of weights Φ is *sound* if, for any $M \colon \mathcal{C}^{op} \to \mathcal{V}$ with small domain, whenever $M * -$ preserves Φ-limits of representables, then M is Φ-flat.

In the ordinary context sound classes of weights were first considered by Adámek, Borceux, Lack, and Rosický in [4]. The relationship between their definition and ours is explained in [4, Remark 2.6]; there the authors also mention the notion of weakly sound class, but do not make use of that in their paper. In the enriched setting weakly sound classes were introduced in [62], where they were simply called "sound classes".

Of course if Φ is sound then it is weakly sound, but the converse does not always hold: see again [4, Remark 2.6]. However, as we are going to see below, the converse does hold when the class of weights Φ is *pre-saturated*, meaning that for any small $\mathcal{C}$ the free cocompletion $\Phi\mathcal{C}$ of $\mathcal{C}$ under Φ-colimits is a one-step closure in $[\mathcal{C}^{op}, \mathcal{V}]$. In other words Φ is pre-saturated if, for any $\mathcal{V}$-category $\mathcal{C}$, every object of $\Phi\mathcal{C}$ is a Φ-colimit of objects from $\mathcal{C}$. The relationship between pre-saturated and saturated classes is discussed in Section A.1.

Proposition 1.3.5. *Let Φ be a pre-saturated class of weights. Then Φ is weakly sound if and only if it is sound.*

Proof. One direction is clear. Suppose then that Φ is pre-saturated and weakly sound, and let $M \colon \mathcal{C}^{op} \to \mathcal{V}$ be such that $M * - \colon [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ preserves Φ-limits of representables; we need to prove that M is Φ-flat.

Let $\Phi^\dagger(\mathcal{C}^{op})$ be the free completion of $\mathcal{C}^{op}$ under Φ-limits and $J \colon \mathcal{C}^{op} \hookrightarrow \Phi^\dagger(\mathcal{C}^{op})$ be the inclusion, note that equivalently $\Phi^\dagger(\mathcal{C}^{op}) = \Phi(\mathcal{C})^{op}$ is the opposite of the free cocompletion of $\mathcal{C}$ under Φ-colimits. Consider $M' := \mathrm{Lan}_J M$; by Lemma 1.3.2 it follows that M is Φ-flat if and only if M' is Φ-flat. Moreover, since Φ is weakly sound, M' is Φ-flat if and only if it is Φ-continuous.

Thus it will suffice to prove that M' is Φ-continuous. Note first that M' preserves Φ-limits of diagrams landing in $\mathcal{C}^{op}$: take $N \colon \mathcal{D} \to \mathcal{V}$ in Φ and $H \colon \mathcal{D} \to \mathcal{C}^{op}$ then

$$\begin{aligned}
M'\{N, JH\} &\cong (\mathrm{Lan}_J M)\{N, JH\} \\
&\cong M - * \, \Phi^\dagger(\mathcal{C}^{op})(J-, \{N, JH\}) \\
&\cong M - * \{N\Box, \mathcal{C}^{op}(-, H\Box)\} \\
&\cong \{N\Box, M - * \mathcal{C}^{op}(-, H\Box)\} \qquad (1.1) \\
&\cong \{N, M \circ H\} \\
&\cong \{N, M' \circ JH\}
\end{aligned}$$

where (1.1) follows from the fact that $M * -$ preserves Φ-limits of representables.
Now, since Φ is pre-saturated, every object of $\Phi^\dagger(\mathcal{C}^{op})$ is a J-absolute Φ-limit of a diagram

landing in $\mathcal{C}^{op}$ (that is, every object of $\Phi(\mathcal{C})$ is a J-absolute Φ-colimit of a diagram landing in $\mathcal{C}$ in the sense of [56]); therefore M' preserves all the J-absolute limits of a chosen codensity presentation of J. By [32, Proposition 2.2] we then have $M' \cong \mathrm{Ran}_J M$ (so that left and right Kan extensions coincide). But $\Phi^\dagger(\mathcal{C}^{op})$ is the free completion of $\mathcal{C}^{op}$ under Φ-limits, therefore the functor $\mathrm{Ran}_J M$ is Φ-continuous by the universal property of such completion. This means that $M' \cong \mathrm{Ran}_J M$ is Φ-continuous, and hence Φ-flat. $\square$

Examples 1.3.6. The following are examples of locally small weakly sound classes. In some examples the class Φ is described as a class of indexing categories, these should be understood as the corresponding classes of conical weights.

1. $\Phi = \emptyset$. Then any weight is Φ-flat, and thus Φ is trivially sound.
2. $\mathcal{V}$ locally α-presentable as a closed category, Φ the class of α-small weights. Then Φ-flat weights are the usual α-flat $\mathcal{V}$-functors. Note that Φ is pre-saturated since every α-small colimit in a free cocompletion $\Phi\mathcal{C}$ can be written as a conical α-small colimit of α-small copowers of objects of $\mathcal{C}$ (see [57]), and that can be seen as an α-small weighted colimit of objects from $\mathcal{C}$. The class Φ is weakly sound by Proposition 1.2.3, and hence sound being pre-saturated.
3. $\mathcal{V}$ symmetric monoidal closed finitary quasivariety [64], Φ the class of weights for finite products and finitely presentable projective powers. This is weakly sound by [62, Theorem 5.8] applied to the saturation of Φ.
4. $\mathcal{V}$ cartesian closed, Φ the class for finite products. Then a weight M is Φ-flat if and only if $\mathrm{Lan}_\Delta M \cong M \times M$, where $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the diagonal and $M \times M\colon \mathcal{C} \times \mathcal{C} \to \mathcal{V}$ is defined by $(M \times M)(A, B) = MA \times MB$. This is weakly sound thanks to [54, Lemma 2.3], and hence, being pre-saturated, is also sound.
5. $\mathcal{V} = \mathbf{Set}$, $\Phi = \{\emptyset\}$. Then Φ-flat colimits are generated by connected colimits. Soundness is discussed in [4].
6. $\mathcal{V} = \mathbf{Set}$, Φ the class of finite connected categories. Then Φ-flat colimits are generated by coproducts and filtered colimits. Soundness is discussed in [4].
7. $\mathcal{V} = \mathbf{Set}$, Φ the class of finite non empty categories. Then a functor is Φ-flat if and only if its category of elements is empty or filtered. It follows easily from this that the class is sound.
8. $\mathcal{V} = \mathbf{Set}$, Φ the class of finite discrete non empty categories. Then a functor is Φ-flat if and only if its category of elements is empty or sifted. Soundness follows as above by replacing filtered with sifted.
9. $\mathcal{V} = \mathbf{Cat}$, Φ the class for finite connected 2-limits, meaning the class generated by finite connected conical limits and powers by finite connected categories. Then Φ-flat colimits contain filtered colimits and coproducts. The proof of [62, Theorem 5.8] easily adapts to this setting showing that the class Φ is weakly sound.
10. $\mathcal{V} = [\mathcal{C}^{op}, \mathbf{Set}]$ with any symmetric monoidal closed structure for which the representables contain the unit and are closed under tensor product; thus the symmetric monoidal structure arises via Day convolution from a symmetric monoidal structure on $\mathcal{C}$. Φ is the class for powers by representables. Then conical colimits are Φ-flat

and every Φ-continuous $\mathcal{V}$-functor can be written as a conical colimit of representables (arguing as in the case of α-continuous $\mathcal{V}$-functors). It follows that Φ is weakly sound.

Remark 1.3.7. Note that if Φ and Ψ are weakly sound then also $\Phi \cup \Psi$ is.

CHAPTER

2

Virtual concepts in the theory of accessible categories

The theory of accessible categories involves aspects of category theory, universal algebra, logic, and model theory. It has also been heavily used in abstract homotopy theory, for example in the context of Smith's theorem [9] or Dugger's work on presentations for model categories [36, 37], and also via its generalization to ∞-categories [75]. In the enriched context, the importance and usefulness of the theory has been recognised recently by many authors in different areas: for instance in the 2-categorical context [20], in the additive [86] and simplicial [23] ones, and in the world of ∞-cosmoi [22].

The theory of enriched locally presentable categories was developed in 1982 by Kelly [56], but the theory of enriched accessible categories, first introduced in the late 1990s [17], is much less developed. The purpose of this chapter is precisely to address this problem via the introduction of the new concepts of virtual left adjoint, virtual colimit, and virtual orthogonality that have been mentioned in the introduction to this thesis.

In Section 2.1, following the ideas of [4], we introduce the notion of Φ-accessible $\mathcal{V}$-categories for a small and weakly sound class of weights Φ and prove some basic results which generalize those of the accessible categories of [18]. Then we recall some basic facts about conically accessible $\mathcal{V}$-categories. Finally we compare the two notions, for a general base $\mathcal{V}$, by giving sufficient conditions for a conically accessible $\mathcal{V}$-category to be accessible, and by showing that if $\mathcal{A}$ is α-accessible then it is also conically β-accessible for any β sharply greater than α.

In Section 2.2 we prove the main results of this chapter. We begin by giving a new proof of the fact that a $\mathcal{V}$-category is accessible if and only if it is sketchable (which was first shown in [18]) and then we prove the characterization theorems. In the last subsection we compare the virtual concepts with those of cone-reflectivity and cone-injectivity, obtaining some of the results in [1] as a consequence of our theorem.

To conclude the chapter, in Section 2.3 we prove that the 2-category of accessible $\mathcal{V}$-categories, accessible $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations is closed in $\mathcal{V}$-**CAT** under all small flexible limits, and it therefore has all pseudo and bilimits as well. The same holds if we replace accessibility by conical accessibility.

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2.1 Accessible $\mathcal{V}$-categories

In this chapter, as well as in the following ones, we always consider a base of enrichment $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ which is symmetric monoidal closed and locally presentable. When considering a regular cardinal α, we always take it to be greater than or equal to a fixed α_0 for which $\mathcal{V}$ is locally α_0-presentable as a closed category (this exists by [58, Proposition 2.4]).

2.1.1 Φ-accessible $\mathcal{V}$-categories

In this section we introduce the main notion of accessibility that we consider in the present paper, this can be seen as a generalization of that of [18] to the "sound" context of [4]. Most of this section will depend on the assumption below.

Assumption 2.1.1. From now on we fix a locally small and weakly sound class Φ.

Where, recall from [59] that a class of weights Φ is called *essentially small* if for every small $\mathcal{V}$-category $\mathcal{C}$ the free cocompletion $\Phi\mathcal{C}$ is still small.

Definition 2.1.2. Let $\mathcal{A}$ be a $\mathcal{V}$-category with Φ-flat colimits. A $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ is called Φ-*accessible* if it preserves Φ-flat colimits; an object A of $\mathcal{A}$ is called Φ-*presentable* if $\mathcal{A}(A, -)\colon \mathcal{A} \to \mathcal{V}$ is Φ-accessible. We denote by $\mathcal{A}_\Phi$ the full subcategory of $\mathcal{A}$ spanned by the Φ-presentable objects.

Definition 2.1.3. We say that $\mathcal{A}$ is Φ-*accessible* if it has Φ-flat colimits and there exists a small $\mathcal{C} \subseteq \mathcal{A}_\Phi$ such that every object of $\mathcal{A}$ can be written as a Φ-flat colimit of objects from $\mathcal{C}$.

When $\Phi = \Phi_\alpha$ is the class of the α-small weights, for a given α, we simply say that $\mathcal{A}$ is α-*accessible* instead of Φ_α-accessible, and if this is so for some α, we say that $\mathcal{A}$ is *accessible*. This agrees with the definition in [18].

The first results we can prove about Φ-accessibility are a standard generalization of the ordinary ones.

Proposition 2.1.4. *Let $\mathcal{A}$ be a Φ-accessible $\mathcal{V}$-category and $H\colon \mathcal{A}_\Phi \hookrightarrow \mathcal{A}$ be the inclusion; then:*

1. *$\mathcal{A}_\Phi$ is Cauchy complete and closed in $\mathcal{A}$ under all existing Φ-colimits;*
2. *$\mathcal{A}_\Phi$ is (essentially) small;*
3. *every $A \in \mathcal{A}$ can be expressed as the Φ-flat colimit:*

$$A \cong \mathcal{A}(H-, A) * H.$$

Proof. (1). If $A \in \mathcal{A}$ is a Φ-colimit of α-presentable objects in $\mathcal{A}$, then $\mathcal{A}(A, -)$ is a Φ-limit of $\mathcal{V}$-functors preserving Φ-flat colimits. Since these colimits commute in $\mathcal{V}$ with Φ-limits, $\mathcal{A}(A, -)$ still preserves Φ-flat colimits and hence $A \in \mathcal{A}_\Phi$. The same argument applies to Cauchy colimits.

(2). Let $H \colon \mathcal{C} \hookrightarrow \mathcal{A}$ be a small full subcategory of Φ-presentable objects witnessing the fact that $\mathcal{A}$ is Φ-accessible. We shall show that $\mathcal{A}_\Phi$ is the Cauchy completion $\mathcal{Q}(\mathcal{C})$ of $\mathcal{C}$, and so is essentially small by [52]. Since $\mathcal{C}$ is contained in $\mathcal{A}_\Phi$ and $\mathcal{A}_\Phi$ is Cauchy complete, $\mathcal{Q}(\mathcal{C})$ is contained in $\mathcal{A}_\Phi$. The opposite inclusion is given by [59, Proposition 7.5].

(3). Given $A \in \mathcal{A}$, by hypothesis we can write $A \cong M * HF$ for some Φ-flat $M \colon \mathcal{C}^{op} \to \mathcal{V}$ and $F \colon \mathcal{C} \to \mathcal{A}_\Phi$. Then $\mathcal{A}(H-, A) \cong M\Box * \mathcal{A}_\Phi(-, F\Box)$ is a Φ-flat colimit of representables, and hence Φ-flat; moreover

$$\begin{aligned} \mathcal{A}(H-, A) * H &\cong M\Box * (\mathcal{A}_\Phi(-, F\Box) * H-) \\ &\cong M * F \\ &\cong A \end{aligned}$$

as desired. □

Proposition 2.1.5. *The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:*

1. *$\mathcal{A}$ is Φ-accessible;*
2. *$\mathcal{A}$ is the free cocompletion of a small $\mathcal{V}$-category $\mathcal{C}$ under Φ-flat colimits;*
3. *$\mathcal{A} \simeq \Phi\text{-Flat}(\mathcal{C}^{op}, \mathcal{V})$ for some small $\mathcal{C}$.*

In both (2) *and* (3) *the $\mathcal{V}$-category $\mathcal{C}$ can be chosen to be $\mathcal{A}_\Phi$; moreover, if $\mathcal{C}$ is Cauchy complete, then $\Phi\text{-Flat}(\mathcal{C}^{op}, \mathcal{V})_\Phi \simeq \mathcal{C}$.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) and the fact that $\mathcal{C}$ can be chosen to be $\mathcal{A}_\Phi$ are a direct consequence of [59, Proposition 4.3].

(2) $\Rightarrow$ (3). Let $H \colon \mathcal{C} \to \mathcal{A}$ be the inclusion. By [59, Proposition 4.3] $\mathcal{C}$ is made of Φ-presentable objects in $\mathcal{A}$ and is small; therefore the induced $\mathcal{V}$-functor $J := \mathcal{A}(H, 1) \colon \mathcal{A} \to [\mathcal{C}^{op}, \mathcal{V}]$ is fully faithful and preserves Φ-flat colimits. Given any $A \in \mathcal{A}$ we can write it as a Φ-flat colimit $A \cong M * HK$ of objects from $\mathcal{C}$; thus $JA \cong M * JHK \cong M * YK$ is a Φ-flat colimit of representables, and hence a Φ-flat $\mathcal{V}$-functor. Vice versa, given a Φ-flat $M \colon \mathcal{C}^{op} \to \mathcal{V}$ we know that $J(M * H) \cong M * Y \cong M$. As a consequence $\mathcal{A} \simeq \Phi\text{-Flat}(\mathcal{C}^{op}, \mathcal{V})$ as claimed and $\mathcal{C}$ can be chosen to be $\mathcal{A}_\Phi$ (since that was true for the second point).

(3) $\Rightarrow$ (1). Let $\mathcal{A} = \Phi\text{-Flat}(\mathcal{C}^{op}, \mathcal{V})$, $J \colon \mathcal{A} \to [\mathcal{C}^{op}, \mathcal{V}]$ be the inclusion, and $H \colon \mathcal{C} \hookrightarrow \mathcal{A}$ be the full subcategory of $\mathcal{A}$ spanned by the representables, so that $JH = Y$ is the Yoneda embedding. Then

$$\mathcal{A}(HC, -) \cong [\mathcal{C}^{op}, \mathcal{V}](YC, J-) \cong \mathrm{ev}_C \circ J$$

for any $C \in \mathcal{C}$; thus $\mathcal{A}(HC, -)$ preserves Φ-flat colimits since J does and ev_C is cocontinuous. It follows that the representable functors in $\mathcal{A}$ are Φ-presentable objects. Moreover we can write every $M \in \mathcal{A}$ as a Φ-flat colimit of representables as $M \cong M * Y$. This shows that $\mathcal{A}$ is Φ-accessible.

Regarding the last statement, as a consequence of the proof of Proposition 2.1.4, we know that $\Phi\text{-Flat}(\mathcal{C}^{op}, \mathcal{V})_\Phi \simeq \mathcal{Q}(\mathcal{C})$. Thus, if $\mathcal{C}$ is Cauchy complete then $\mathcal{C} = \mathcal{Q}(\mathcal{C})$, and thus $\Phi\text{-Flat}(\mathcal{C}^{op}, \mathcal{V})_\Phi \simeq \mathcal{C}$. □

Remark 2.1.6. It follows, from the universal property of free cocompletions, that a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ out of a Φ-accessible $\mathcal{V}$-category $\mathcal{A}$ is Φ-accessible if and only if it is the left Kan extension of its restriction to $\mathcal{A}_\Phi$.

We will show in Section 2.1.3 that, in the context of α-accessible categories, the index of accessibility can again be raised with the sharply less than relation, as in the ordinary context. Moreover we will see that the underlying ordinary category of an accessible $\mathcal{V}$-category is again accessible.

2.1.2 Conically accessible $\mathcal{V}$-categories

In this section we consider a different notion of enriched accessibility which involves only (conical) α-filtered colimits. This goes as follows:

Definition 2.1.7. Let α be a regular cardinal and $\mathcal{A}$ be a $\mathcal{V}$-category with α-filtered colimits. A $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ is called *conically α-accessible* if it preserves α-filtered colimits; an object A of $\mathcal{A}$ is called *conically α-presentable* if $\mathcal{A}(A,-)\colon \mathcal{A} \to \mathcal{V}$ is conically α-accessible. We denote by $\mathcal{A}_\alpha^c$ the full subcategory of $\mathcal{A}$ spanned by the conically α-presentable objects.

Definition 2.1.8. We say that $\mathcal{A}$ is *conically α-accessible* if it has α-filtered colimits and there exists $\mathcal{C} \subseteq \mathcal{A}_\alpha^c$ small such that every object of $\mathcal{A}$ can be written as an α-filtered colimit of objects from $\mathcal{C}$.

The next results are then immediate consequences of the definitions:

Proposition 2.1.9. *Let $\mathcal{A}$ be a conically α-accessible $\mathcal{V}$-category and $H\colon \mathcal{A}_\alpha^c \hookrightarrow \mathcal{A}$ be the inclusion; then:*

1. *$\mathcal{A}_\alpha^c$ is closed in $\mathcal{A}$ under existing α-small colimits;*
2. *$\mathcal{A}_\alpha^c$ is (essentially) small;*
3. *every $A \in \mathcal{A}$ can be expressed as the α-filtered colimit:*
$$A \cong \operatorname{colim}\left((\mathcal{A}_\alpha^c)_0/A \xrightarrow{\pi} \mathcal{A}_\alpha \longrightarrow \mathcal{A}\right);$$
4. *$\mathcal{A}_0$ is α-accessible and $(\mathcal{A}_\alpha^c)_0 = (\mathcal{A}_0)_\alpha$.*

Proof. (1). Let $A = \operatorname{colim} A_i$ be an α-small colimit of conically α-presentable objects; then $\mathcal{A}(A,-) \cong \lim \mathcal{A}(A_i,-)$ is an α-small limit of functors which preserve α-filtered colimits. Since α-small limits commute with α-filtered colimits, $\mathcal{A}(A,-)$ preserves α-filtered colimits as well and $A \in \mathcal{A}_\alpha^c$.

(2). Let $\mathcal{C} \subseteq \mathcal{A}_\alpha^c$ be small and generate $\mathcal{A}$ under α-filtered colimits, and let $A \in \mathcal{A}_\alpha^c$. As a consequence $A \cong \operatorname{colim} C_i$ is an α-filtered colimit of elements from $\mathcal{C}$. Since by hypothesis $\mathcal{A}(A,-)$ preserves α-filtered colimits, it follows that the identity map Id_A factors through some C_i. Therefore A is a split subobject of some object of $\mathcal{C}$ and hence $(\mathcal{A}_\alpha^c)_0$ is the Cauchy completion of $\mathcal{C}_0$. Since $\mathcal{C}$ is small, $\mathcal{A}_\alpha^c$ is as well.

(3). Let $\mathcal{C}$ be as in (2) above and A be any object of A. Then $A \cong \operatorname{colim} HF$ where $F\colon \mathcal{D} \to \mathcal{A}_\alpha^c$ is a diagram with α-filtered domain. Then the colimit cocone of A induces a final functor $K\colon \mathcal{D} \to (\mathcal{A}_\alpha)_0/A$ such that $\pi \circ K = H$. It follows then that $(\mathcal{A}_\alpha)_0/A$ is α-filtered (this is a general fact about final functors with α-filtered domain, see [65, Remark 2.8] for instance) and the colimit of its projection on $\mathcal{A}$ is A.

(4). Given any object $A \in \mathcal{A}$, since the unit of $\mathcal{V}$ is α-presentable, if $\mathcal{A}(A,-)$ preserves α-filtered colimits then so does $\mathcal{A}_0(A,-) = \mathcal{V}_0(I, \mathcal{A}(A,-)_0)$; therefore $(\mathcal{A}^c_\alpha)_0 \subseteq (\mathcal{A}_0)_\alpha$. As a consequence $(\mathcal{A}^c_\alpha)_0$ is a small full subcategory of $\mathcal{A}_0$ made of α-presentable objects, and generates $\mathcal{A}$ under α-filtered colimits. This implies that $\mathcal{A}_0$ is α-accessible as an ordinary category. Finally, given $A \in (\mathcal{A}_0)_\alpha$, arguing as in (2) we can write A as a split subobject of some $B \in (\mathcal{A}_\alpha)_0$; since $(\mathcal{A}_\alpha)_0$ is closed under split subobjects in $\mathcal{A}_0$ it follows that $(\mathcal{A}^c_\alpha)_0 = (\mathcal{A}_0)_\alpha$.

□

Proposition 2.1.10. *The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:*

1. *$\mathcal{A}$ is conically α-accessible;*
2. *$\mathcal{A}$ is the free cocompletion of a small category under α-filtered colimits.*

Proof. This is a direct consequence of [59, Proposition 4.3]. □

Remark 2.1.11. In the same spirit of Remark 2.1.6, it follows that a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ out of a conically α-accessible $\mathcal{V}$-category $\mathcal{A}$ is conically α-accessible if and only if it is the left Kan extension of its restriction to $\mathcal{A}^c_\alpha$.

As we can raise the index of accessibility of an ordinary accessible category, we can do the same with conical accessibility:

Corollary 2.1.12. *Given any α-accessible $\mathcal{V}$-category $\mathcal{A}$ and a regular cardinal $\beta \rhd \alpha$, then $\mathcal{A}$ is conically β-accessible.*

Proof. By Proposition 2.1.9 $\mathcal{A}_0$ is α-accessible and $(\mathcal{A}^c_\alpha)_0 = (\mathcal{A}_0)_\alpha$; moreover by Theorem 1.1.3 $\mathcal{A}_0$ is β-accessible and $(\mathcal{A}_0)_\beta$ is given by the closure of $(\mathcal{A}_0)_\alpha$ in $\mathcal{A}$ under β-small α-filtered colimits. It follows that $(\mathcal{A}_0)_\beta \subseteq (\mathcal{A}^c_\beta)_0$ (since the latter is closed under existing β-small colimits). As a consequence every element of $\mathcal{A}$ is a β-filtered colimit of objects from $\mathcal{A}^c_\beta$ (since that is true in $\mathcal{A}_0$). This is enough to imply that $\mathcal{A}$ is conically β-accessible. □

Corollary 2.1.13. *For any accessible $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ between conically accessible $\mathcal{V}$-categories, there exists an α such that F preserves the conically β-presentable objects for each $\beta \rhd \alpha$.*

Proof. Direct consequence of Proposition 2.1.9(4) and Corollary 1.1.6. □

2.1.3 Accessible vs. conically accessible

The aim of this section is to compare the two notions of accessibility just introduced. In general, for a $\mathcal{V}$-category $\mathcal{A}$ with α-flat colimits, we only have the inclusion $\mathcal{A}_\alpha \subseteq \mathcal{A}^c_\alpha$ (since every α-filtered colimit is α-flat). This inclusion is not an equality in general and moreover conically accessibility does not imply accessibility (since some α-flat colimits may not be α-filtered, see [65]). However for many significant base of enrichment the two notions do coincide, or differ only by Cauchy completeness [65, Section 3-4]. In the remainder of this section we give conditions on when a conically accessible $\mathcal{V}$-category is accessible, and prove that every α-accessible $\mathcal{V}$-category is conically α^+-accessible.

Definition 2.1.14. We say that a $\mathcal{V}$-category $\mathcal{A}$ is *accessible* if it is α-accessible for some α; we say that it is *conically accessible* if it is conically α-accessible for some α. We say that a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{K}$ (not necessarily between accessible $\mathcal{V}$-categories) is *accessible* if $\mathcal{A}$ has and F preserves α-flat colimits for some α; we say that it is *conically accessible* if $\mathcal{A}$ has and F preserves α-filtered colimits for some α. If F is fully faithful we say that $\mathcal{A}$ is respectively *accessibly embedded* and *conically accessibly embedded*.

For this section we will not be considering Φ-accessible $\mathcal{V}$-categories for a general weakly sound class Φ, these will come into play again in Section 2.2.1.

In the first part of this section we give conditions for conical accessibility to imply accessibility. As mentioned above, it is not true in general because some flat-weighted colimits might be missing in the $\mathcal{V}$-category in question; things change if the $\mathcal{V}$-category is complete or cocomplete:

Proposition 2.1.15. *Let $\mathcal{A}$ be a complete or α-cocomplete $\mathcal{V}$-category; then $\mathcal{A}$ has α-flat colimits if and only if it has α-filtered colimits. A $\mathcal{V}$-functor from such an $\mathcal{A}$ preserves α-flat colimits if and only if it preserves α-filtered colimits.*

Proof. Assume first that $\mathcal{A}$ is α-cocomplete and consider an α-flat weight $M\colon \mathcal{C}^{op} \to \mathcal{V}$ together with a diagram $H\colon \mathcal{C} \to \mathcal{A}$. Let $J\colon \mathcal{C} \hookrightarrow \mathcal{D}$ be the inclusion of $\mathcal{C}$ into its free cocompletion under α-small colimits. Since $\mathcal{A}$ has them we can consider $H' := \mathrm{Lan}_J H$, while on the weighted side we take $M' := \mathrm{Lan}_{J^{op}} M$. By Lemma 1.3.2 the weight M' is still α-flat and its domain is α-complete. Therefore by Proposition 1.2.3 we can write $M' \cong \mathrm{colim} YF$ as an α-filtered colimit of representables; here $Y\colon \mathcal{D} \to [\mathcal{D}^{op}, \mathcal{V}]$ is the Yoneda embedding and $F\colon \mathcal{E}_{\mathcal{V}} \to \mathcal{D}$ is a functor with α-filtered domain. As a consequence we obtain (each side existing if the other does):

$$\begin{aligned} M * H &\cong M * H'J \cong M' * H' \\ &\cong (\mathrm{colim} YF) * H' \\ &\cong \mathrm{colim}(YF * H') \\ &\cong \mathrm{colim}(H'F). \end{aligned}$$

Thus the existence and preservation of the α-flat colimit $M * H$ is equivalent to that of the α-filtered colimit $\mathrm{colim} H'F$.

The case when $\mathcal{A}$ is complete goes exactly as above with the only difference that we consider $H' := \mathrm{Ran}_J H$ (instead of $\mathrm{Lan}_J H$): this exists because $\mathcal{A}$ is assumed to be complete. The other arguments apply identically since the isomorphism $M * H \cong M' * H'$ still holds. □

A direct consequence is the following:

Corollary 2.1.16. *The following are equivalent for a complete or α-cocomplete $\mathcal{V}$-category $\mathcal{A}$:*

1. *$\mathcal{A}$ is α-accessible;*
2. *$\mathcal{A}$ is conically α-accessible;*

In this case $\mathcal{A}_\alpha = \mathcal{A}_\alpha^c$ and $\mathcal{A}$ is locally α-presentable in the sense of [57].

Let us now see how the accessibility of the underlying category relates to that of the $\mathcal{V}$-category.

Proposition 2.1.17. *Let $\mathcal{A}$ be a $\mathcal{V}$-category for which $\mathcal{A}_0$ is accessible; then:*

1. *$\mathcal{A}$ is conically accessible if and only if it has conical α-filtered colimits for some α and every object is conically presentable;*
2. *$\mathcal{A}$ is accessible if and only if it has α-flat colimits for some α and every object is presentable.*

Proof. (1). If $\mathcal{A}$ is conically α-accessible then it has conical α-filtered colimits by definition; moreover every object is conically presentable being a small colimit of α-presentable objects.
Conversely, let $\beta \geq \alpha$ be such that $\mathcal{A}_0$ is β-accessible and let $\gamma \rhd \beta$ be such that $\mathcal{A}$ has conical γ-filtered colimits and

$$(\mathcal{A}_0)_\beta \subseteq (\mathcal{A}^c_\gamma)_0;$$

this exists since $(\mathcal{A}_0)_\beta$ is small and each object of $\mathcal{A}$ is conically presentable. It follows that

$$(\mathcal{A}^c_\gamma)_0 = (\mathcal{A}_0)_\gamma.$$

Indeed, the inclusion $(\mathcal{A}^c_\gamma)_0 \subseteq (\mathcal{A}_0)_\gamma$ is always true (since the unit I of $\mathcal{V}$ is γ-presentable); on the other hand every $X \in (\mathcal{A}_0)_\gamma$ is a γ-small β-filtered colimit of objects from $(\mathcal{A}_0)_\beta$; since $\mathcal{A}$ has γ-filtered colimits, the colimit expressing X is actually enriched; hence X is a γ-small colimit of conically β-presentable objects in $\mathcal{A}$, and this makes X a conically γ-presentable object of $\mathcal{A}$.
Given the equality above, and the fact that $\mathcal{A}_0$ is γ-accessible, it follows that $\mathcal{A}^c_\gamma$ generates $\mathcal{A}$ under γ-filtered colimits; therefore $\mathcal{A}$ is conically accessible.
(2). The proof is essentially the same as above, one just has to replace conical presentability with actual presentability. □

Corollary 2.1.18. *Let $\mathcal{K}$ be conically accessible, $J\colon \mathcal{A} \hookrightarrow \mathcal{K}$ be conically accessibly embedded, and $\mathcal{A}_0$ be accessible; then $\mathcal{A}$ is conically accessible. If moreover $\mathcal{K}$ is accessible and $\mathcal{A}$ is accessibly embedded, then it is accessible.*

Proof. This is a direct consequence of the previous Proposition since each object of $\mathcal{A}$ will be conically presentable in (1) and presentable in (2). □

We now turn to proving that every accessible $\mathcal{V}$-category is also conically accessible; the next Lemma will be an important step.

Lemma 2.1.19. *Let $\mathcal{K}$ be a locally α-presentable $\mathcal{V}$-category, and $\mathcal{A}$ and $\mathcal{B}$ be two conically α-accessible full subcategories of $\mathcal{K}$ for which the inclusions preserve the conically α-presentable objects. Let $\mathcal{C} := \mathcal{A} \cap \mathcal{B}$ and $\beta \rhd \alpha$:*

1. *if $\mathcal{A}$ and $\mathcal{B}$ are closed under α-filtered colimits in $\mathcal{K}$, then $\mathcal{C}$ is conically β-accessible and closed under β-filtered colimits in $\mathcal{K}$.*
2. *if $\mathcal{A}$ and $\mathcal{B}$ are closed under α-flat colimits in $\mathcal{K}$, then $\mathcal{C}$ is β-accessible and closed under β-flat colimits in $\mathcal{K}$; moreover $\mathcal{C}_\beta = \mathcal{C}^c_\beta$.*

Proof. (1). Consider $\beta \rhd \alpha$; then $\mathcal{A}$ and $\mathcal{B}$ are conically β-accessible, the inclusions in $\mathcal{K}$ still preserve β-filtered colimits and conically β-presentable objects (since these are β-small α-filtered colimits of the conically α-presentable ones). Note first that $\mathcal{C}$ is closed in $\mathcal{A}, \mathcal{B}$, and $\mathcal{K}$ under β-filtered colimits since both $\mathcal{A}$ and $\mathcal{B}$ are so in $\mathcal{K}$.

Let $\mathcal{C}' \subseteq \mathcal{C}$ be the intersection $\mathcal{C}' = \mathcal{A}^c_\beta \cap \mathcal{B}^c_\beta$; then $\mathcal{C}' \subseteq \mathcal{C}^c_\beta$ and to prove (1) it is enough to show that $\mathcal{C}'$ generates $\mathcal{C}$ under β-filtered colimits. For any $X \in \mathcal{C}$ consider the slice $\mathcal{C}'_0/X$ and the inclusion $J\colon \mathcal{C}'_0/X \to (\mathcal{A}^c_\beta)_0/X$; we wish to prove that J is final. This will suffice since then $\mathcal{C}'_0/X$ will be β-filtered (because $(\mathcal{A}^c_\beta)_0/X$ is and the inclusion is fully faithful) and the colimit of $\pi_X\colon \mathcal{C}'_0/X \to \mathcal{C}$ will be X (because the colimit of $(\mathcal{A}^c_\beta)_0/X \to \mathcal{A}$ is X and $\mathcal{C}$ is closed in $\mathcal{A}$ under β-filtered colimits).

So we are reduced to proving that $J\colon \mathcal{C}'_0/X \to (\mathcal{A}^c_\beta)_0/X$ is final; which is as saying that every map $A \to X$ with $A \in \mathcal{A}^c_\beta$ factors through some $C \in \mathcal{C}'$ (the fact that any two such factorizations are connected will follow from this plus the filteredness of $(\mathcal{A}^c_\beta)_0/X$ and fully faithfulness of the inclusion). Fix then a map $f\colon A \to X$ with $A \in \mathcal{A}^c_\beta$, we regard this as a morphism in $\mathcal{K}$ and construct a β-small chain $(d_{i,j}\colon K_i \to K_j)_{i<j<\alpha}$ of conically β-presentable objects in $\mathcal{K}$ together with a cocone $(c_i\colon K_i \to X)_{i<\alpha}$. Set $K_0 = A$ and $c_0 = f$, then we alternate elements of $\mathcal{A}^c_\beta$ and $\mathcal{B}^c_\beta$ as follows (taking colimits in $\mathcal{K}$ at the limit steps). Assume to have K_i and c_i for $i = \lambda + 2n$ with λ limit; then $c_i\colon K_i \to X$ factors through some $B \in \mathcal{B}^c_\beta$ since K_i is conically β-presentable (remember that $\mathcal{A}^c_\beta, \mathcal{B}^c_\beta \subseteq \mathcal{K}^c_\beta$) and X, being in $\mathcal{B}$, is a β-filtered colimit of objects from $\mathcal{B}^c_\beta$. Let then $K_{i+1} = B$ with $d_{i,i+1}$ and c_{i+1} given by the factorization. If $i = \lambda + 2n + 1$ we argue as above but inverting the roles of $\mathcal{A}$ and $\mathcal{B}$. Finally if $i = \lambda$ is limit, we take K_i to be the colimit of the chain $(K_j)_{j<i}$ in $\mathcal{K}$ and consider the induced factorizations. Let $C := \operatorname{colim}_{i<\alpha} K_i$ in $\mathcal{K}$; then by construction we have a factorization of f through C. Moreover the sub-chains of $(K_i)_{i<\alpha}$ spanned by the objects in $\mathcal{A}^c_\beta$ and $\mathcal{B}^c_\beta$ are final; thus C is both in $\mathcal{A}$ and in $\mathcal{B}$ and hence in $\mathcal{C}$. Finally, since the chain involved was β-small, C is actually an object of $\mathcal{A}^c_\beta \cap \mathcal{B}^c_\beta = \mathcal{C}'$ as required. Note that, since in $\mathcal{C}'$ idempotents split, this also implies that $\mathcal{C}^c_\beta = \mathcal{A}^c_\beta \cap \mathcal{B}^c_\beta$.

(2). Since $\mathcal{A}$ and $\mathcal{B}$ are closed in $\mathcal{K}$ under β-flat colimits, also $\mathcal{C}$ is. Moreover, thanks to point (1), to prove that $\mathcal{C}$ is β-accessible it is enough to show that $\mathcal{C}^c_\beta \subseteq \mathcal{C}_\beta$. Let $X \in \mathcal{C}^c_\beta$ and denote by J the inclusion of $\mathcal{C}$ in $\mathcal{K}$; since X, seen as an object of $\mathcal{A}$, is conically β-presentable and the inclusion of $\mathcal{A}$ in $\mathcal{K}$ preserves conically β-presentable objects, it follows that $JX \in \mathcal{K}^c_\beta$. But $\mathcal{K}$ is locally β-presentable and thus $\mathcal{K}^c_\beta = \mathcal{K}_\beta$; as a consequence $\mathcal{C}(X, -) \cong \mathcal{K}(JX, J-)$ preserves β-flat colimits and hence $X \in \mathcal{C}_\beta$. This proves that $\mathcal{C}^c_\beta \subseteq \mathcal{C}_\beta$; since the other inclusion always holds, it follows that $\mathcal{C}^c_\beta = \mathcal{C}^\beta$. □

As promised, the next result says that we can raise the index of accessibility of an accessible $\mathcal{V}$-category and at the same time make the $\mathcal{V}$-category conically accessible. This can be seen as a sharpening of [18, Theorem 7.10]; in fact, in [18] the choice of β depends on the $\mathcal{V}$-category $\mathcal{A}$ taken into consideration, while in our result it depends only on α and the *sharply less* relation.

Theorem 2.1.20. *If $\mathcal{A}$ is an α-accessible $\mathcal{V}$-category, then for any $\beta \rhd \alpha$ the following hold:*

1. *$\mathcal{A}$ is β-accessible;*
2. *$\mathcal{A}$ is conically β-accessible;*
3. *$\mathcal{A}_0$ is an ordinary β-accessible category;*
4. *$(\mathcal{A}_\beta)_0 = (\mathcal{A}^c_\beta)_0 = (\mathcal{A}_0)_\beta$.*

Proof. Let $\mathcal{C} = \mathcal{A}^{op}_\alpha$; since $\mathcal{A}$ is α-accessible we can write it as $\mathcal{A} \simeq \alpha\text{-Flat}(\mathcal{C}, \mathcal{V})$, with inclusion $J\colon \mathcal{A} \hookrightarrow [\mathcal{C}, \mathcal{V}]$. Now consider $H\colon \mathcal{C} \hookrightarrow \mathcal{D}$ to be the free completion of $\mathcal{C}$ under α-small limits; we show first that $\mathcal{A}$ can be identified with the intersection

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook, "J"] \arrow[d, hook, "J"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{C},\mathcal{V}]} \arrow[d, hook, "\mathrm{Lan}_H"] \\
{[\mathcal{C},\mathcal{V}]} \arrow[r, hook, "\mathrm{Ran}_H"'] & {[\mathcal{D},\mathcal{V}]}
\end{tikzcd}$$

where we are embedding $[\mathcal{C},\mathcal{V}]$ in $[\mathcal{D},\mathcal{V}]$ in two different ways. To prove that, it is enough to show that a $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{V}$ is α-flat if and only if $\mathrm{Lan}_H F \cong \mathrm{Ran}_H F$. If F is α-flat then $\mathrm{Lan}_H F$ is α-flat as well by Lemma 1.3.2 and therefore is α-continuous; since $\mathcal{D}$ is the free completion of $\mathcal{C}$ under α-small limits and $(\mathrm{Lan}_H F)J \cong F$ this means exactly that $\mathrm{Lan}_H F \cong \mathrm{Ran}_H F$. Vice versa, if $\mathrm{Lan}_H F \cong \mathrm{Ran}_H F$ then $\mathrm{Lan}_H F$ is α-continuous and hence α-flat; thus F is α-flat itself again by Lemma 1.3.2.

To conclude the proof of $(1),(2)$, and (3) it is now enough to show that, for this intersection, the hypotheses of Lemma 2.1.19 are satisfied. The $\mathcal{V}$-categories $[\mathcal{C},\mathcal{V}]$ and $[\mathcal{D},\mathcal{V}]$ are locally α-presentable (for any α) and hence conically α-accessible; moreover their α-presentable and conically α-presentable objects coincide. Now $\mathrm{Lan}_H\colon [\mathcal{C},\mathcal{V}] \hookrightarrow [\mathcal{D},\mathcal{V}]$ is cocontinuous and sends representable functors to representables; therefore it preserves all α-flat colimits and the α-presentable objects (since these coincide with the α-small colimits of representables). It remains to consider $\mathrm{Ran}_H\colon [\mathcal{C},\mathcal{V}] \hookrightarrow [\mathcal{D},\mathcal{V}]$; this identifies $[\mathcal{C},\mathcal{V}]$ with the full subcategory of $[\mathcal{D},\mathcal{V}]$ spanned by the α-continuous functors. Since these are closed under α-flat colimits it follows at once that Ran_H preserves α-flat colimits and we are only left to prove that it preserves the α-presentable objects. Under the identification just described, the α-presentable objects of $[\mathcal{C},\mathcal{V}]$ correspond to the representables in α-$\mathrm{Cont}(\mathcal{D},\mathcal{V})$ by the enriched Gabriel-Ulmer duality; therefore Ran_H sends the α-presentable objects to the representables in $[\mathcal{D},\mathcal{V}]$, and these are α-presentable. In conclusion, we can apply Lemma 2.1.19 to obtain $(1),(2)$, and (3). Point (4) is now a consequence of Proposition 2.1.9. $\square$

Remark 2.1.21. The Theorem above implies in particular that if $\mathcal{A}$ is α-accessible then it is also conically α^+-accessible and (conically) β-accessible for arbitrarily large regular cardinals β. Here α^+ is the cardinal successor of α, this is sharply greater than α by [1, Example 2.13(2)]. We do not know yet whether every α-accessible $\mathcal{V}$-category is also conically α-accessible or not.

Corollary 2.1.22. *Given an accessible $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ between accessible $\mathcal{V}$-categories, there exists α such that F preserves the β-presentable objects for each $\beta \rhd \alpha$.*

Proof. Direct consequence of Corollary 2.1.20 above and Corollary 1.1.6. $\square$

Proposition 2.1.23. *A $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ between accessible $\mathcal{V}$-categories is accessible if and only if it is conically accessible.*

Proof. Every accessible functor is conically accessible. Conversely let $F\colon \mathcal{A} \to \mathcal{B}$ be conically α-accessible for some α and consider $\beta \rhd \alpha$, by Theorem 2.1.20 it follows that $\mathcal{A}$ is conically β-accessible. Since F is also conically β-accessible, it is the left Kan extension of its restriction to $\mathcal{A}^c_\beta = \mathcal{A}_\beta$, which is made of β-presentable objects; thus F preserves β-flat colimits as well. $\square$

2.2 The main results

The aim of this section is to introduce and work with the virtual notions discussed in the introduction. We prove the main results in Section 2.2.3, 2.2.4, and 2.2.5, and then compare these with those already known in the literature in Section 2.2.6. In the first subsection we establish once more the connection between accessible $\mathcal{V}$-categories and sketches.

2.2.1 Accessibility and sketches

The relationship between accessible $\mathcal{V}$-categories and sketches already appeared in [18]; however their proof relies on some (non trivial) results on ordinary accessible categories; here we give a proof that is only based on Section 2.1 and on some standard results about locally presentable $\mathcal{V}$-categories.

First we need to recall the notion of sketch, which in a general enriched context was already considered in [18]:

Definition 2.2.1. A *sketch* is the data of a triple $\mathcal{S} = (\mathcal{B}, \mathbb{L}, \mathbb{C})$ where:

- $\mathcal{B}$ is a small $\mathcal{V}$-category;
- $\mathbb{L}$ is a set of cylinders in $\mathcal{B}$: $\mathcal{V}$-natural transformations $c\colon N \to \mathcal{B}(B, H-)$, where $N\colon \mathcal{D} \to \mathcal{V}$ is a weight, $H\colon \mathcal{D} \to \mathcal{B}$ is a $\mathcal{V}$-functor, and B is an object of $\mathcal{B}$;
- $\mathbb{C}$ is a set of cocylinders in $\mathcal{B}$: $\mathcal{V}$-natural transformations $d\colon M \to \mathcal{B}(K-, C)$, where $M\colon \mathcal{E}^{op} \to \mathcal{V}$ is a weight, $K\colon \mathcal{E} \to \mathcal{B}$ is a $\mathcal{V}$-functor, and C is an object of $\mathcal{B}$.

A sketch for which $\mathbb{C}$ is empty is called a *limit sketch*.

Definition 2.2.2. A model of a sketch $\mathcal{S} = (\mathcal{B}, \mathbb{L}, \mathbb{C})$ is a $\mathcal{V}$-functor $F\colon \mathcal{B} \to \mathcal{V}$ which transforms each cylinder of $\mathbb{L}$ into a limit cylinder in $\mathcal{V}$, and each cocylinder of $\mathbb{C}$ into a colimit cocylinder in $\mathcal{V}$. We denote by $\mathrm{Mod}(\mathcal{S})$ the full subcategory of $[\mathcal{B}, \mathcal{V}]$ spanned by the models of $\mathcal{S}$.

The $\mathcal{V}$-categories of models of limit sketches characterize locally presentable $\mathcal{V}$-categories (see [57, Section 10] or [18, Corollary 7.4]).

Proposition 2.2.3. *Let Φ be a locally small and weakly sound class of weights. Any Φ-accessible $\mathcal{V}$-category $\mathcal{A}$ is equivalent to the $\mathcal{V}$-category of models of a sketch involving colimits and Φ-limits.*

Proof. Let $\mathcal{A}$ be Φ-accessible; then by Proposition 2.1.5 we can write $\mathcal{A} \simeq \Phi\text{-Flat}(\mathcal{C}, \mathcal{V})$ for some small $\mathcal{C}$; thus it is enough to prove that $\Phi\text{-Flat}(\mathcal{C}, \mathcal{V})$ is the $\mathcal{V}$-category of models of a suitable sketch. Let $\mathcal{D}$ be the closure of $\mathcal{C}$ in $[\mathcal{C}^{op}, \mathcal{V}]$ under Φ-limits; then by left Kan extending along the inclusion of $\mathcal{C}$ in $\mathcal{D}$, the $\mathcal{V}$-category $\Phi\text{-Flat}(\mathcal{C}, \mathcal{V})$ becomes equivalent to the full subcategory of $[\mathcal{D}, \mathcal{V}]$ spanned by those Φ-continuous $\mathcal{V}$-functors $F\colon \mathcal{D} \to \mathcal{V}$ which preserve some specified weighted colimits (those that exhibit each object of $\mathcal{D}$, seen in $[\mathcal{C}^{op}, \mathcal{V}]$, as a weighted colimit of representables). This is clearly the $\mathcal{V}$-category of models of a sketch on $\mathcal{D}$ involving Φ-limits and colimits. □

The converse does not hold even when Φ is the class of α-small limits, see for instance [1, Remark 2.59].

In the proof of the theorem below we'll make use of the enriched notion of orthogonality class, which can be found for example in [56, Section 6.2].

The equivalence between (1) and (3) below already appeared in [18, Theorems 7.6/7.8].

Theorem 2.2.4. *The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:*

1. *$\mathcal{A}$ is accessible (that is, α-accessible for some α);*
2. *$\mathcal{A}$ is Φ-accessible for some locally small and weakly sound class Φ;*
3. *$\mathcal{A}$ is sketchable.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial and (2) $\Rightarrow$ (3) is given by Proposition 2.2.3 above.

(3) $\Rightarrow$ (1). Let $\mathcal{A} = \mathrm{Mod}(\mathcal{S})$ be the $\mathcal{V}$-category of models of a sketch $\mathcal{S} = (\mathcal{B}, \mathbb{L}, \mathbb{C})$; then $\mathcal{A} = \mathrm{Mod}(\mathcal{B}, \mathbb{L}) \cap \mathrm{Mod}(\mathcal{B}, \mathbb{C})$ can be seen as the intersection of the limit part and the colimit part. Let us focus on $\mathrm{Mod}(\mathcal{B}, \mathbb{C})$ as a full subcategory of $[\mathcal{B}, \mathcal{V}]$. Let $\mathcal{C}$ be the $\mathcal{V}$-category obtained from $\mathcal{B}$ freely adding the colimits of each diagram (M, H) appearing in $\mathbb{C}$, denote by $J \colon \mathcal{B} \hookrightarrow \mathcal{C}$ the inclusion. Then $\mathrm{Lan}_J \colon [\mathcal{B}, \mathcal{V}] \to [\mathcal{C}, \mathcal{V}]$ is fully faithful and, for any $F \colon \mathcal{B} \to \mathcal{V}$, its left Kan extension $\mathrm{Lan}_J F$ preserves the specified colimits. Consider now the cylinders $c \colon M \to \mathcal{B}(H-, B)$ appearing in $\mathbb{C}$, by construction these correspond to maps $\bar{c} \colon M * JH \to JB$ in $\mathcal{C}$. Denote by $\mathcal{M}$ the family of morphisms in $[\mathcal{C}, \mathcal{V}]$ given by $\mathcal{C}(\bar{c}, -)$ for each $c \in \mathbb{C}$; then the square below is a pullback.

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{B}, \mathbb{C}) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, very near start, "\lrcorner"] & {[\mathcal{B}, \mathcal{V}]} \arrow[d, hook, "\mathrm{Lan}_J"] \\
\mathcal{M}^{\perp} \arrow[r, hook] & {[\mathcal{C}, \mathcal{V}]}
\end{tikzcd}$$

Indeed, $F \colon \mathcal{B} \to \mathcal{V}$ is a model of $(\mathcal{B}, \mathbb{C})$ if and only if for each $c \in \mathbb{C}$ as above the induced map $\tilde{c} \colon M * FH \to FB$ is an isomorphism; but $FB \cong (\mathrm{Lan}_J F)JB$, while $M * FH \cong M * (\mathrm{Lan}_J F)JH \cong \mathrm{Lan}_J F(M * JH)$, and $\tilde{c}$ corresponds under these isomorphisms to $\mathrm{Lan}_J F(\bar{c})$. It follows that F is a model of the colimit sketch if and only if $\mathrm{Lan}_J F(\bar{c})$ is an isomorphism for each c, which is equivalent to $\mathrm{Lan}_J F$ being orthogonal to $\mathcal{C}(\bar{c}, -)$ for each c.

In conclusion we can express the $\mathcal{V}$-category $\mathcal{A}$ as the intersection in $[\mathcal{C}, \mathcal{V}]$ of the accessibly embedded subcategories $\mathcal{M}^{\perp}$ and $\mathrm{Mod}(\mathcal{B}, \mathbb{L})$. Now it is enough to observe that $\mathrm{Mod}(\mathcal{B}, \mathbb{L})$ and $\mathcal{M}^{\perp}$ are locally presentable; the first being the $\mathcal{V}$-category of models of a limit sketch, and the latter being an accessibly embedded and reflective subcategory of $[\mathcal{C}, \mathcal{V}]$ by [56, Theorem 6.5]. As a consequence $\mathcal{A}$ is accessible by Corollary 2.1.13 and Lemma 2.1.19. $\square$

2.2.2 Accessibility and the free completion

Here we collect a few results about the free completion under small limits of an accessible $\mathcal{V}$-category; this will serve as an introduction to the virtual concepts we consider in Section 2.2.3.

Given a $\mathcal{V}$-category $\mathcal{A}$ we denote by $\mathcal{P}^{\dagger}\mathcal{A}$ its free completion under (small) limits; this can be seen as the full subcategory of $[\mathcal{A}, \mathcal{V}]^{op}$ spanned by the small limits of representables. More common is the free cocompletion under colimits, denoted by $\mathcal{P}\mathcal{A}$; this has been studied in [33] and is related to the free completion under limits through the duality $\mathcal{P}^{\dagger}\mathcal{A} = \mathcal{P}(\mathcal{A}^{op})^{op}$. Note moreover that, when $\mathcal{C}$ is a small $\mathcal{V}$-category, $\mathcal{P}^{\dagger}\mathcal{C} = [\mathcal{C}, \mathcal{V}]^{op}$.

Definition 2.2.5. Given a $\mathcal{V}$-category $\mathcal{A}$, we say that a small full subcategory $H\colon \mathcal{G} \hookrightarrow \mathcal{A}$ is *closed under virtual* Φ*-colimits* in $\mathcal{A}$ if $\mathcal{P}^\dagger H\colon \mathcal{P}^\dagger\mathcal{G} \hookrightarrow \mathcal{P}^\dagger\mathcal{A}$ is Φ-cocontinuous.

Notice that, since $\mathcal{G}$ is small, $\mathcal{P}^\dagger\mathcal{G} = [\mathcal{G},\mathcal{V}]^{op}$ is cocomplete and therefore $\mathcal{P}^\dagger H$ will be a genuine Φ-cocontinuous $\mathcal{V}$-functor. The fact $\mathcal{P}^\dagger\mathcal{A}$ has enough colimits is not needed to prove the result below; however that will be a consequence of Proposition 2.2.18 where we prove that $\mathcal{P}^\dagger\mathcal{A}$ is cocomplete whenever $\mathcal{A}$ is accessible.

Recall that $H\colon \mathcal{G} \hookrightarrow \mathcal{A}$ is a strong generator if the $\mathcal{V}$-functor $\mathcal{A}(H,1)\colon \mathcal{A} \to [\mathcal{G}^{op},\mathcal{V}]$ is conservative. The following is an equivalent way of characterizing Φ-accessible $\mathcal{V}$-categories.

Proposition 2.2.6. *Let Φ be a locally small weakly sound class and $\mathcal{A}$ be a $\mathcal{V}$-category with Φ-flat colimits. The following are equivalent for $\mathcal{G} \subseteq \mathcal{A}_\Phi$:*

1. *$\mathcal{G}$ exhibits $\mathcal{A}$ as a Φ-accessible $\mathcal{V}$-category;*
2. *$\mathcal{G}$ is a small strong generator of $\mathcal{A}$ that is closed under virtual Φ-colimits.*

Proof. Let $Y\colon \mathcal{G}^{op} \to [\mathcal{G},\mathcal{V}] = \mathcal{P}^\dagger(\mathcal{G})^{op}$ be the Yoneda embedding and $A \in \mathcal{A}$; then

$$\mathrm{Lan}_Y \mathcal{A}(H-,A) \cong \mathrm{ev}_A \circ (\mathcal{P}^\dagger H)^{op}$$

for any $\mathcal{A}$ in $\mathcal{A}$, indeed this follows from the fact that the $\mathcal{V}$-functor on the right-hand-side is cocontinuous and restricts to $\mathcal{A}(H-,A)$.

Assume now that $\mathcal{A}$ is Φ-accessible and let $\mathcal{G} = \mathcal{A}_\Phi$; the $\mathcal{V}$-functor $\mathcal{A}(H-,A)$ is Φ-flat by Proposition 2.1.4; therefore $\mathrm{Lan}_Y\mathcal{A}(H-,A)$ preserves all Φ-limits. By the isomorphism above it follows that $\mathrm{ev}_A^{op} \circ \mathcal{P}^\dagger H$ is Φ-cocontinuous; thus, since Φ-colimits in $\mathcal{P}^\dagger\mathcal{A}$ (when they exist) are computed pointwise, $\mathcal{P}^\dagger H$ is Φ-cocontinuous too. It follows that $\mathcal{G}$ is a small strong generator (being dense) and is closed in $\mathcal{A}$ under virtual Φ-colimits.

Conversely, assume that there exists $H\colon \mathcal{G} \hookrightarrow \mathcal{A}$ as in (2). Consider the $\mathcal{V}$-functor $W\colon \mathcal{A} \to [\mathcal{G}^{op},\mathcal{V}]$ defined by $W = \mathcal{A}(H,1)$; since $\mathcal{G}$ is a strong generator made of Φ-presentable objects it follows that W is conservative and preserves Φ-flat colimits. Moreover, since $\mathcal{P}^\dagger H$ is Φ-cocontinuous and ev_A is continuous for any $A \in \mathcal{A}$, it follows that $\mathrm{Lan}_Y\mathcal{A}(H-,A)$ is Φ-continuous, and hence $WA = \mathcal{A}(H-,A)$ is Φ-flat. As a consequence, given any $A \in \mathcal{A}$, we have that

$$WA \cong WA * Y \cong WA * WH \cong W(WA * H);$$

therefore $A \cong WA * H$ (by conservativeness of W) is a Φ-flat colimit of elements of $\mathcal{G}$, showing that $\mathcal{A}$ is Φ-accessible ☐

Thanks to this, one obtains easily the standard characterization theorem of locally presentable $\mathcal{V}$-categories:

Corollary 2.2.7. *Let Φ be a locally small weakly sound class. The following are equivalent for a cocomplete $\mathcal{V}$-category $\mathcal{A}$:*

1. *$\mathcal{A}$ is Φ-accessible;*
2. *$\mathcal{A}$ has a small strong generator made of Φ-presentable objects.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. For (2) $\Rightarrow$ (1) consider a small strong generator $\mathcal{G} \subseteq \mathcal{A}_\Phi$ and take its closure $\mathcal{G}'$ in $\mathcal{A}$ under Φ-colimits. Then $\mathcal{G}'$ is still strongly generating and is also closed in $\mathcal{A}$ under virtual Φ-colimits by the dual of [33, Remark 6.6]. It follows by the proposition above that $\mathcal{A}$ is Φ-accessible. ☐

For a locally α-presentable $\mathcal{V}$-category $\mathcal{K}$ we know that the inclusion $\mathcal{K}_\alpha \to \mathcal{K}$ of the α-presentable objects in $\mathcal{K}$ is α-cocontnuous; that is, α-small colimits exist in $\mathcal{K}_\alpha$ and are preserved by the inclusion. This cannot be said in a general α-accessible category since the colimits in question may not exist; however, by taking $\mathcal{G} = \mathcal{A}_\Phi$ in Proposition 2.2.6 above, we obtain:

Proposition 2.2.8. *Let Φ be a locally small weakly sound class of weights. For any Φ-accessible $\mathcal{V}$-category $\mathcal{A}$ the full subcategory $\mathcal{A}_\Phi$ is closed in $\mathcal{A}$ under virtual Φ-colimits. The same holds for a conical α-accessible $\mathcal{A}$ with $\mathcal{A}_\alpha^c$ in place of $\mathcal{A}_\Phi$.*

Proof. For the first part simply take $\mathcal{G} = \mathcal{A}_\Phi$ in the proposition above. For the latter apply the same proof by noticing that $\mathcal{A}(H-, A)$ will still be α-flat even when H is the inclusion of $\mathcal{A}_\alpha^c$ in $\mathcal{A}$. $\square$

If we see $\mathcal{P}^\dagger\mathcal{A}$ as a full subcategory of $[\mathcal{A}, \mathcal{V}]^{op}$ then we have a nice way of describing its elements in the case where $\mathcal{A}$ is accessible:

Proposition 2.2.9. *For any accessible $\mathcal{V}$-category $\mathcal{A}$ the free completion $\mathcal{P}^\dagger\mathcal{A}$ consists exactly of the accessible functors from $\mathcal{A}$ to $\mathcal{V}$. For any conically accessible $\mathcal{V}$-category $\mathcal{A}$ the free completion $\mathcal{P}^\dagger\mathcal{A}$ consists exactly of the conically accessible functors from $\mathcal{A}$ to $\mathcal{V}$.*

Proof. Let $F\colon \mathcal{A} \to \mathcal{V}$ be an object of $\mathcal{P}^\dagger\mathcal{A}$; then $F = \mathrm{Lan}_H FH$ for some small $H\colon \mathcal{C} \hookrightarrow \mathcal{A}$. Since $\mathcal{C}$ is small, we can now consider α for which $\mathcal{C} \subseteq \mathcal{A}_\alpha$; it follows that F is also the left Kan extension of its restriction to $\mathcal{A}_\alpha$, and hence it preserves α-flat colimits. Conversely, every $F\colon \mathcal{A} \to \mathcal{V}$ which preserves α-flat colimits (for an α for which $\mathcal{A}$ is α-accessible) is the left Kan extension of its restriction to $\mathcal{A}_\alpha$; hence it is a small presheaf. The same argument applies to the conical case. $\square$

Corollary 2.2.10. *For any accessible $\mathcal{V}$-category $\mathcal{A}$ there is an adjunction:*

$$\begin{tikzcd} \mathcal{P}^\dagger(\mathcal{A}_0) \arrow[r, "R"'] \arrow[r, phantom, "\perp", shift left=1ex] & (\mathcal{P}^\dagger\mathcal{A})_0 \arrow[l, bend right, "L"'] \end{tikzcd}$$

where R is the unique continuous functor induced by the universal property of $\mathcal{P}^\dagger(\mathcal{A}_0)$ applied to the underlying functor of the inclusion $Z\colon \mathcal{A} \to \mathcal{P}^\dagger\mathcal{A}$, and L is given pointwise by $LF = \mathcal{V}_0(I, F_0-)$. In fact it suffices that $\mathcal{A}$ be conically accessible.

Proof. Let $\bar{Z}\colon \mathcal{A}_0 \hookrightarrow \mathcal{P}^\dagger(\mathcal{A}_0)$ be the inclusion of the free completion of $\mathcal{A}_0$, so that $R\bar{Z} \cong Z_0$. Then R has a left adjoint L if and only if for any G in $(\mathcal{P}^\dagger\mathcal{A})_0$ there exists some $LG \in \mathcal{P}^\dagger(\mathcal{A}_0)$ such that $\mathcal{P}^\dagger(\mathcal{A}_0)(LG, -) \cong (\mathcal{P}^\dagger\mathcal{A})_0(G, R-)$. By restricting the isomorphism to $\mathcal{A}_0$ through $\bar{Z}$, this says that $LG \cong (\mathcal{P}^\dagger\mathcal{A})_0(G, Z_0-)$; in particular it follows that a left adjoint exists if and only if $(\mathcal{P}^\dagger\mathcal{A})_0(G, Z_0-)$ is a small functor for any G in $(\mathcal{P}^\dagger\mathcal{A})_0$.
To conclude it is then enough to notice that $(\mathcal{P}^\dagger\mathcal{A})_0(G, Z_0-) \cong \mathcal{V}_0(I, G_0-)$ is small by Proposition 2.2.9 since both $\mathcal{A}$ and $\mathcal{A}_0$ are accessible. $\square$

2.2.3 Virtual left adjoints and virtual colimits

In this section we introduce the *virtual* concepts that will help us in finding new characterizations of accessible $\mathcal{V}$-categories. The word virtual here refers to something that "lives" in the free completion $\mathcal{P}^\dagger\mathcal{A}$ of a $\mathcal{V}$-category $\mathcal{A}$. The relation between these concepts and those used in [1] for ordinary accessible categories will be discussed in Section 2.2.6.

Recall that a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{K}$ has a left adjoint if and only if $\mathcal{K}(X, F-)$ is representable for each $X \in \mathcal{K}$; generalizing this we make the following definition:

Definition 2.2.11. We say that a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{K}$ has a *virtual left adjoint* if for each $X \in \mathcal{K}$ the $\mathcal{V}$-functor $\mathcal{K}(X, F-)$ is small. If F is moreover fully faithful we say that $\mathcal{A}$ is *virtually reflective* in $\mathcal{K}$.

From now on, given a $\mathcal{V}$-category $\mathcal{A}$ we denote by $Z_\mathcal{A}\colon \mathcal{A} \hookrightarrow \mathcal{P}^\dagger\mathcal{A}$ the inclusion into its free completion; we'll drop the subscript $(-)_\mathcal{A}$ whenever it is safe to do so.

Remark 2.2.12. Virtual left adjoints have been considered in the literature for different purposes. For instance, in [102, 3.4] a functor is a *right* $\mathfrak{D}$*-pro adjoint* (where $\mathfrak{D}$ is the class of all small categories) if and only if it has a virtual left adjoint (in our sense). While, in the context of KZ-doctrines, virtual left adjoints were considered as the dual of [25, Definition 1.1] for the KZ-doctrine given by freely adding all small colimits.

Remark 2.2.13. When $\mathcal{V} = \mathbf{Set}$, Guitart and Lair consider in [48, Section 5] the notion of "small locally free diagram" (*petit diagramme localement libre*). Given a fully faithful functor $J\colon \mathcal{A} \to \mathcal{K}$, they say that an object $X \in \mathcal{K}$ has a small locally free diagram over $\mathcal{A}$ if there exists a diagram $H\colon \mathcal{C} \to \mathcal{A}$ for which:

$$\mathcal{K}(X, JA) \cong \operatorname{colim} \mathcal{A}(H-, A)$$

naturally in $A \in \mathcal{A}$. Now, since $\operatorname{colim} \mathcal{A}(H-, A) \cong \mathcal{P}^\dagger\mathcal{A}(\lim ZH, ZA) \cong (\lim ZH)(A)$, this condition is simply saying that $\mathcal{K}(X, J-)$ is small. It follows that every element of $\mathcal{K}$ has a small locally free diagram over $\mathcal{A}$ if and only if J has a virtual left adjoint. In Theorem 1 of the same paper they show that, given a sketch $\mathcal{S}$ on $\mathcal{C}$, every functor $F\colon \mathcal{C} \to \mathbf{Set}$ has a small locally free diagram over $\mathrm{Mod}(\mathcal{S})$ (see also [46, Theorem 2.3]); in our terminology this says that $\mathrm{Mod}(\mathcal{S})$ is virtually reflective in $[\mathcal{C}, \mathbf{Set}]$. However they do not prove that every virtually reflective and accessibly embedded subcategory of $[\mathcal{C}, \mathbf{Set}]$ is accessible (or equivalently, sketchable); we do that in Proposition 2.2.23 for the more general enriched context.

Other ways to recognize when a $\mathcal{V}$-functor has a virtual left adjoint are given below.

Proposition 2.2.14. *Let $F\colon \mathcal{A} \to \mathcal{K}$ be a $\mathcal{V}$-functor. The following are equivalent:*

1. *F has a virtual left adjoint;*
2. *F has a relative left adjoint with respect to the inclusion $Z_\mathcal{A}\colon \mathcal{A} \hookrightarrow \mathcal{P}^\dagger\mathcal{A}$;*
3. *the induced continuous $\mathcal{V}$-functor $\mathcal{P}^\dagger F\colon \mathcal{P}^\dagger\mathcal{A} \to \mathcal{P}^\dagger\mathcal{K}$ has a left adjoint.*

Moreover the left adjoint $L\colon \mathcal{P}^\dagger\mathcal{K} \to \mathcal{P}^\dagger\mathcal{A}$ in (3) *is given by precomposition with F.*

Proof. (3) $\Rightarrow$ (2). Let $L\colon \mathcal{P}^\dagger\mathcal{K} \to \mathcal{P}^\dagger\mathcal{A}$ be the left adjoint to $\mathcal{P}^\dagger F$ and let $X \in \mathcal{P}^\dagger\mathcal{K}$; then

$$LX(A) \cong \mathcal{P}^\dagger\mathcal{A}(LX, Z_\mathcal{A}A) \cong \mathcal{P}^\dagger\mathcal{K}(X, \mathcal{P}^\dagger F Z_\mathcal{A}A) \cong \mathcal{P}^\dagger\mathcal{K}(X, Z_\mathcal{K}FA) \cong X(FA).$$

This proves that L, when it exists, is given by precomposition with F. Now the relative left adjoint to F is given by the composite $LZ_{\mathcal{K}}\colon \mathcal{K} \to \mathcal{P}^{\dagger}\mathcal{A}$.

(2) $\Rightarrow$ (3). Let $L'\colon \mathcal{K} \to \mathcal{P}^{\dagger}\mathcal{A}$ be a relative left adjoint to F with respect to $Z_{\mathcal{A}}$. Then $L\colon = \mathrm{Ran}_{Z_{\mathcal{K}}} L'\colon \mathcal{P}^{\dagger}\mathcal{K} \to \mathcal{P}^{\dagger}\mathcal{A}$ exists (since $\mathcal{P}^{\dagger}\mathcal{A}$ is complete) and is a left adjoint to $\mathcal{P}^{\dagger}F$.

(2) $\Leftrightarrow$ (1). The argument given above proves that the relative left adjoint $L'\colon \mathcal{K} \to \mathcal{P}^{\dagger}\mathcal{A}$, when it exists, is given by $L'X = \mathcal{K}(X, F-)$. Thus the equivalence between (1) and (2) follows at once by definitions of $\mathcal{P}^{\dagger}\mathcal{A}$ and virtual left adjoints. $\square$

Next we consider the notion of virtual colimit in a $\mathcal{V}$-category. Recall that, given a weight $M\colon \mathcal{C}^{op} \to \mathcal{V}$ and a $\mathcal{V}$-functor $H\colon \mathcal{C} \to \mathcal{A}$, we say that the colimit of H weighted by M exists in $\mathcal{A}$ if the $\mathcal{V}$-functor $[\mathcal{C}^{op}, \mathcal{V}](M, \mathcal{A}(H,-))\colon \mathcal{A} \to \mathcal{V}$ is representable. Similarly:

Definition 2.2.15. Given a $\mathcal{V}$-category $\mathcal{A}$, a weight $M\colon \mathcal{C}^{op} \to \mathcal{V}$ with small domain, and $H\colon \mathcal{C} \to \mathcal{A}$, we say that the *virtual colimit* of H weighted by M exists in $\mathcal{A}$ if $[\mathcal{C}^{op}, \mathcal{V}](M, \mathcal{A}(H,-))$ is a small $\mathcal{V}$-functor. We say that $\mathcal{A}$ is *virtually cocomplete* if it has all virtual colimits.

Proposition 2.2.16. *Given H and M as above, the virtual colimit of H weighted by M exists in $\mathcal{A}$ if and only if the colimit $M * ZH$ exists in $\mathcal{P}^{\dagger}\mathcal{A}$. In this case*

$$[\mathcal{C}^{op}, \mathcal{V}](M, \mathcal{A}(H,-)) \cong M * ZH.$$

Proof. Consider the $\mathcal{V}$-functor $X := [\mathcal{C}^{op}, \mathcal{V}](M, \mathcal{A}(H,-))\colon \mathcal{A} \to \mathcal{V}$, if the virtual colimit of H weighted by M exists in $\mathcal{A}$ then X is small (by definition) and

$$\begin{aligned} \mathcal{P}^{\dagger}\mathcal{A}(X, ZA) &\cong XA \\ &\cong [\mathcal{C}^{op}, \mathcal{V}](M, \mathcal{A}(H, A)) \\ &\cong [\mathcal{C}^{op}, \mathcal{V}](M, \mathcal{P}^{\dagger}\mathcal{A}(ZH, ZA)) \end{aligned}$$

for any $A \in \mathcal{A}$. Since the representables are codense in $\mathcal{P}^{\dagger}\mathcal{A}$ it follows that $X \cong M * ZH$ exists in $\mathcal{P}^{\dagger}\mathcal{A}$. Conversely, if $M * ZH$ exists in $\mathcal{P}^{\dagger}\mathcal{A}$ then the same chain of isomorphism above shows that it is isomorphic to $[\mathcal{C}^{op}, \mathcal{V}](M, \mathcal{A}(H,-))$, which is then small. Therefore the virtual colimit of H weighted by M exists in $\mathcal{A}$. $\square$

It follows that $\mathcal{A}$ is virtually cocomplete if and only if $\mathcal{P}^{\dagger}\mathcal{A}$ has all colimits of representables; this is equivalent to $\mathcal{P}^{\dagger}\mathcal{A}$ actually being cocomplete by [33, Theorem 3.8].

Proposition 2.2.17. *Let $\mathcal{A}$ be a $\mathcal{V}$-category and $Y\colon \mathcal{A} \to \mathcal{P}\mathcal{A}$ be the inclusion. Then $\mathcal{A}$ is virtually cocomplete if and only if for any small $F\colon \mathcal{A}^{op} \to \mathcal{V}$ — that is, for any $F \in \mathcal{P}\mathcal{A}$ — the $\mathcal{V}$-functor $\mathcal{P}\mathcal{A}(F, Y-)\colon \mathcal{A} \to \mathcal{V}$ is also small.*

Proof. Given a small $\mathcal{V}$-category $\mathcal{C}$ and $\mathcal{V}$-functors $M\colon \mathcal{C} \to \mathcal{V}$ and $H\colon \mathcal{C} \to \mathcal{A}$, then

$$[\mathcal{C}^{op}, \mathcal{V}](M, \mathcal{A}(H,-)) \cong \mathcal{P}\mathcal{A}(F, Y-)$$

where $F := \mathrm{Lan}_{H^{op}} M$. To conclude it is then enough to recall that the virtual colimit of H weighted by M exists in $\mathcal{A}$ if and only if $[\mathcal{C}^{op}, \mathcal{V}](M, \mathcal{A}(H,-))$ is small, and then note that a $\mathcal{V}$-functor $F\colon \mathcal{A}^{op} \to \mathcal{V}$ is small if and only if it is the left Kan extension of its restriction to some small full subcategory of $\mathcal{A}^{op}$. $\square$

Next we can prove the following (see also [33, Remark 3.5]):

Proposition 2.2.18. *A $\mathcal{V}$-category which is accessible or conically accessible is virtually cocomplete.*

Proof. Since every accessible $\mathcal{V}$-category is conically accessible, it is enough to prove the proposition for a conically accessible $\mathcal{V}$-category. If $\mathcal{A}$ is conically accessible then $\mathcal{P}^{\dagger}\mathcal{A}$, seen as a full subcategory of $[\mathcal{A},\mathcal{V}]^{op}$, consists of the conically accessible presheaves (Proposition 2.2.9). Since these are closed under small colimits in $[\mathcal{A},\mathcal{V}]^{op}$, it follows that $\mathcal{P}^{\dagger}\mathcal{A}$ is cocomplete. $\square$

Virtually cocomplete $\mathcal{V}$-categories behave well with respect to virtual left adjoints:

Proposition 2.2.19. *Let $\mathcal{A}$ be virtually cocomplete and $H\colon \mathcal{G}\hookrightarrow\mathcal{A}$ be a full subcategory; then the induced $\mathcal{V}$-functor $\mathcal{A}(H,1)\colon \mathcal{A}\to[\mathcal{G}^{op},\mathcal{V}]$ has a virtual left adjoint.*

Proof. Let $\mathcal{K}=[\mathcal{G}^{op},\mathcal{V}]$ and $F=\mathcal{A}(H,1)$; we need to prove that for each $X\in\mathcal{K}$ the functor $\mathcal{K}(X,F-)$ is small. If $X=\mathcal{G}(-,G)$ for some $G\in\mathcal{G}$ then

$$\begin{aligned}\mathcal{K}(X,F-)&\cong[\mathcal{G}^{op},\mathcal{V}](\mathcal{G}(\square,G),\mathcal{A}(H\square,-))\\&\cong\mathcal{A}(G,-);\end{aligned}$$

hence $\mathcal{K}(X,F-)\cong\mathcal{A}(G,-)$ is small. If X is any presheaf on $\mathcal{G}$ then we can write it as a weighted colimit of representables; thus $\mathcal{K}(X,F-)$ will be a limit of small functors by the argument above. In other words $\mathcal{K}(X,F-)$ is a colimit of elements in $\mathcal{P}^{\dagger}\mathcal{A}$ and hence, since $\mathcal{P}^{\dagger}\mathcal{A}$ is cocomplete by hypothesis, is small. It follows that F has a virtual left adjoint. $\square$

The following characterizes accessible $\mathcal{V}$-functors between accessible $\mathcal{V}$-categories.

Proposition 2.2.20. *Let $F\colon\mathcal{A}\to\mathcal{K}$ be a $\mathcal{V}$-functor between accessible $\mathcal{V}$-categories; the following are equivalent:*

1. *F has a virtual left adjoint;*
2. *F is accessible.*

Moreover if $\mathcal{A},\mathcal{K}$, and F are α-accessible, the virtual left adjoint restricts to the α-presentable objects: if $L\dashv\mathcal{P}^{\dagger}F$ then L restricts to $L_\alpha\colon\mathcal{P}^{\dagger}(\mathcal{K}_\alpha)\to\mathcal{P}^{\dagger}(\mathcal{A}_\alpha)$.

A corresponding statement holds in the case of conically accessible categories: $\mathcal{A},\mathcal{K}$, and F are assumed to be conically (α-)accessible instead of (α-)accessible and the full subcategories $\mathcal{A}_\alpha$ and $\mathcal{K}_\alpha$ are replaced by $\mathcal{A}^c_\alpha$ and $\mathcal{K}^c_\alpha$.

Proof. A virtual left adjoint to F exists if and only if $\mathcal{K}(X,F-)$ is small for each $X\in\mathcal{K}$, and this, by Proposition 2.2.9, is the same as saying that $\mathcal{K}(X,F-)$ is an accessible $\mathcal{V}$-functor for each X in $\mathcal{K}$.

Suppose that F is accessible, take then $X\in\mathcal{K}$, and consider α such that X is α-presentable and F preserves α-flat colimits. Then $\mathcal{K}(X,F-)$ preserves α-flat colimits as well and hence is small. Vice versa, assume that each $\mathcal{K}(X,F-)$ is accessible. Let α be such that $\mathcal{K}$ is α-accessible and consider $\beta\geq\alpha$ such that for each $X\in\mathcal{K}_\alpha$ the $\mathcal{V}$-functor $\mathcal{K}(X,F-)$ preserves all β-flat colimits. Since $\mathcal{K}_\alpha$ is a strong generator made of (α- and hence) β-presentable objects, it follows that F preserves β-flat colimits as well.

Regarding the assertion that if $\mathcal{A},\mathcal{K}$, and F are α-accessible the virtual left adjoint restricts to the α-presentable, it is enough to note that for any $X\in\mathcal{K}_\alpha$

$$\mathcal{K}(X,J-)\cong\mathrm{Lan}_H\mathcal{K}(X,JH-),$$

since $\mathcal{K}(X, J-)$ preserves α-flat colimits, where $H\colon \mathcal{A}_\alpha \hookrightarrow \mathcal{A}$ is the inclusion. This means exactly that the left adjoint $L\colon \mathcal{P}^\dagger(\mathcal{K}) \to \mathcal{P}^\dagger(\mathcal{A})$ restricts to $L_\alpha\colon \mathcal{P}^\dagger(\mathcal{K}_\alpha) \to \mathcal{P}^\dagger(\mathcal{A}_\alpha)$ as desired.

The same proof applies in the conically accessible case. □

Remark 2.2.21. Note that in the first part of the previous proposition it is enough to ask that each object of $\mathcal{K}$ is presentable, instead of $\mathcal{K}$ being accessible.

An immediate consequence is:

Corollary 2.2.22. *Let $\mathcal{A}$ be an accessible and accessibly embedded subcategory of an accessible $\mathcal{V}$-category $\mathcal{K}$; then $\mathcal{A}$ is virtually reflective in $\mathcal{K}$. In fact it suffices that $\mathcal{A}$ be conically accessible and conically accessibly embedded.*

Proof. Follows directly from Proposition 2.2.20 applied to the inclusion of $\mathcal{A}$ in $\mathcal{K}$. □

The next step is to prove the opposite direction:

Proposition 2.2.23. *Let $\mathcal{K}$ be a locally presentable $\mathcal{V}$-category and $J\colon \mathcal{A} \hookrightarrow \mathcal{K}$ be a virtually reflective and accessibly embedded subcategory; then $\mathcal{A}$ is accessible.*

Proof. Strategy. We shall choose small full subcategories $H\colon \mathcal{C} \hookrightarrow \mathcal{K}$ and $H'\colon \mathcal{D} \to \mathcal{A}$ such that:

(i) H is dense, so that $\mathcal{K}(H,1)\colon \mathcal{K} \hookrightarrow [\mathcal{C}^{op}, \mathcal{V}]$ is fully faithful;

(ii) $\mathcal{D} \subseteq \mathcal{C}$, with inclusion $J'\colon \mathcal{D} \hookrightarrow \mathcal{C}$, so that we have a fully faithful $\mathcal{V}$-functor $\mathrm{Lan}_{J'^{op}}\colon [\mathcal{D}^{op}, \mathcal{V}] \hookrightarrow [\mathcal{C}^{op}, \mathcal{V}]$;

(iii) the intersection of these full subcategories is $\mathcal{A}$, as in the diagram below.

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "\mathcal{A}(H',1)"] \arrow[d, hook, "J"'] & {[\mathcal{D}^{op}, \mathcal{V}]} \arrow[d, hook, "\mathrm{Lan}_{J'^{op}}"] \\ \mathcal{K} \arrow[r, hook, "\mathcal{K}(H,1)"'] & {[\mathcal{C}^{op}, \mathcal{V}]} \end{tikzcd}$$

Now $[\mathcal{D}^{op}, \mathcal{V}]$, $[\mathcal{C}^{op}, \mathcal{V}]$, and $\mathcal{K}$ are all accessible $\mathcal{V}$-categories (in fact locally presentable) while $\mathrm{Lan}_{J'^{op}}$ and $\mathcal{K}(H',1)$ are accessible embeddings; thus $\mathcal{A}$ is accessible by Lemma 2.1.19.

Main Step. Let α be some regular cardinal such that $\mathcal{K}$ is an α-accessible $\mathcal{V}$-category and the inclusion J is an α-accessible $\mathcal{V}$-functor. Let $\mathcal{C}$ be any full subcategory of $\mathcal{K}$ containing $\mathcal{K}_\alpha$ and closed under α-small colimits, denote by $H\colon \mathcal{C} \hookrightarrow \mathcal{K}$ the inclusion. This satisfies (i). Commutativity of the square in (iii) says that the canonical maps

$$\mathcal{K}(HC, JH'-) * \mathcal{A}(H'-, A) \longrightarrow \mathcal{K}(HC, JA)$$

are invertible for all $C \in \mathcal{C}$, $A \in \mathcal{A}$: the colimit on the left gives the left Kan extension $\mathrm{Lan}_{J'^{op}}\mathcal{A}(H'-, A)$ evaluated at C. Now $\mathcal{K}(HC, J-)$ is the value at HC of the virtual reflection $L\colon \mathcal{K} \to \mathcal{P}^\dagger\mathcal{A}$, and invertibility of the above maps says that $\mathcal{K}(HC, J-)$ is the left Kan extension of its restriction $\mathcal{K}(HC, JH'-)$ to $\mathcal{D}$, or in other words that

(iii') L maps $\mathcal{C} \subseteq \mathcal{K}$ into $\mathcal{P}^\dagger\mathcal{D}$.

In fact we'll see that this implies the intersection property of the square in (iii). Suppose then that $K \in \mathcal{K}$, and $\mathcal{K}(H-,K)$ is in the image of $\mathrm{Lan}_{J'^{op}}$, so that in fact

$$\mathcal{K}(H-,K) \cong \mathrm{Lan}_{J'^{op}}\mathcal{K}(HJ'-,K).$$

Since $H\colon \mathcal{C} \hookrightarrow \mathcal{K}$ is α-cocontinuous, $\mathcal{K}(H-,K)$ is α-continuous and hence α-flat, and now by Lemma 1.3.2 also $\mathcal{K}(HJ'-,K)$ is α-flat. Thus we can form the colimit $\mathcal{K}(HJ'-,K) * H' \in \mathcal{A}$, and it will be preserved by J, giving

$$\begin{aligned} J(\mathcal{K}(HJ'-,K) * H')) &\cong \mathcal{K}(HJ'-,K) * JH' \\ &\cong \mathcal{K}(HJ'-,K) * HJ' \\ &\cong \mathrm{Lan}_{J'^{op}}\mathcal{K}(HJ'-,K) * H \\ &\cong \mathcal{K}(H-,K) * H \\ &\cong K, \end{aligned}$$

so $K \in \mathcal{A}$ as required.

Choosing $\mathcal{C}$ and $\mathcal{D}$. It remains to show that we can choose a small $\mathcal{K}_\alpha \subseteq \mathcal{C} \subseteq \mathcal{K}$ closed under α-small colimits and $\mathcal{D} \subseteq \mathcal{A} \cap \mathcal{C}$ such that L maps $\mathcal{C}$ into $\mathcal{P}^\dagger\mathcal{D}$. We do this by recursion on $0 < i < \alpha$. Define $\mathcal{C}_1 := \mathcal{K}_\alpha$, and $\mathcal{D}_1 \subseteq \mathcal{A}$ to be small and such that $L(\mathcal{C}_1) \subseteq \mathcal{P}^\dagger(\mathcal{D}_1)$ (this exists since $\mathcal{C}_1$ is small). Now, given $0 < i < \alpha$, and the small $\mathcal{V}$-categories $\mathcal{C}_i$ and $\mathcal{D}_i$, we define $\mathcal{C}_{i+1}$ to be the closure of $J\mathcal{D}_i$ in $\mathcal{K}$ under α-small colimits, and $\mathcal{D}_{i+1} \subseteq \mathcal{A}$ to be such that $\mathcal{D}_i \subseteq \mathcal{D}_{i+1}$ and $L(\mathcal{C}_{i+1}) \subseteq \mathcal{P}^\dagger(\mathcal{D}_{i+1})$. Take unions at the limit steps and then define $\mathcal{D} := \cup_{i<\alpha}\mathcal{D}_i$ and $\mathcal{C} := \cup_{i<\alpha}\mathcal{C}_i$. Then L maps $\mathcal{C}$ into $\mathcal{P}^\dagger\mathcal{D}$ by construction and, by regularity of α, each α-small diagram in $\mathcal{C}$ factors through some $\mathcal{C}_{i+1}$ which is closed in $\mathcal{K}$ under α-small colimits by construction; hence $\mathcal{C}$ is closed under them as well. □

The same holds if the ambient $\mathcal{V}$-category $\mathcal{K}$ is just accessible:

Corollary 2.2.24. *Let $\mathcal{K}$ be an accessible $\mathcal{V}$-category and $J\colon \mathcal{A} \hookrightarrow \mathcal{K}$ be virtually reflective and accessibly embedded; then $\mathcal{A}$ is accessible.*

Proof. Since $\mathcal{K}$ is accessible then we can find a small $\mathcal{C}$ and an accessible embedding $\mathcal{K} \hookrightarrow [\mathcal{C},\mathcal{V}]$ which has a virtual left adjoint by Corollary 2.2.22. Since virtual adjoints compose we can now apply Proposition 2.2.23 to conclude that $\mathcal{A}$ is accessible. □

It is not true in general that every α-accessibly embedded and virtually reflective subcategory of a locally α-presentable $\mathcal{V}$-category, is α-accessible. However this holds with a further assumption:

Corollary 2.2.25. *Let $\mathcal{K}$ be locally presentable and $J\colon \mathcal{A} \hookrightarrow \mathcal{K}$ be virtual reflective, α-accessibly embedded, and such that the virtual reflection $L\colon \mathcal{P}^\dagger\mathcal{K} \to \mathcal{P}^\dagger\mathcal{A}$ restricts to $L_{(\alpha)}\colon \mathcal{P}^\dagger(\mathcal{K}_\alpha) \to \mathcal{P}^\dagger(\mathcal{A}_{(\alpha)})$, where $\mathcal{A}_{(\alpha)} := \mathcal{A} \cap \mathcal{K}_\alpha$. Then $\mathcal{A}$ is α-accessible.*

Proof. In the proof of Proposition 2.2.23 we can choose $\mathcal{C} = \mathcal{K}_\alpha$ and $\mathcal{D} = \mathcal{A}_{(\alpha)} \subseteq \mathcal{A}_\alpha$. It follows by the intersection property in (iii) that $\mathcal{A}_{(\alpha)}$ is dense; moreover the weights $\mathcal{A}(H'-,A)$ are all α-flat, since $\mathrm{Lan}_{J'^{op}}\mathcal{A}(H'-,A) \cong \mathcal{K}(H-,JA)$ is α-continuous. Therefore it follows that $\mathcal{A}$ is α-accessible. □

2.2.4 Virtual orthogonality

Next we introduce the third and last virtual concept of this thesis; this is a generalization of the more common notion of orthogonality.

Definition 2.2.26. Let $\mathcal{K}$ be a $\mathcal{V}$-category, $Z\colon \mathcal{K} \hookrightarrow \mathcal{P}^\dagger\mathcal{K}$ be the inclusion, and $f\colon ZX \to P$ a morphism in $\mathcal{P}^\dagger\mathcal{K}$ with representable domain. We say that an object A of $\mathcal{K}$ is *orthogonal with respect to* f if

$$\mathcal{P}^\dagger\mathcal{K}(f, ZA)\colon \mathcal{P}^\dagger\mathcal{K}(P, ZA) \longrightarrow \mathcal{P}^\dagger\mathcal{K}(ZX, ZA)$$

is an isomorphism in $\mathcal{V}$; in other words if $ZA \in \mathcal{P}^\dagger\mathcal{K}$ is orthogonal with respect to f.

Let us unwind this definition. Given an object P in $\mathcal{P}^\dagger\mathcal{K}$, we can write it as a limit of representables $P \cong \{M, ZH\}$. Thus to give $f\colon ZX \to P$ is the same as giving a cylinder $\bar{f}\colon M \to \mathcal{K}(X, H-)$; moreover $\mathcal{P}^\dagger\mathcal{K}(ZX, ZA) \cong \mathcal{K}(X, A)$ and $\mathcal{P}^\dagger\mathcal{K}(P, ZA) \cong M * \mathcal{K}(H-, A)$. As a consequence, an object A of $\mathcal{K}$ is orthogonal with respect to $f\colon ZX \to P$ if and only if the map

$$M * \mathcal{K}(H-, A) \to \mathcal{K}(X, A)$$

induced by $\bar{f}\colon M \to \mathcal{K}(X, H-)$ is an isomorphism.

When $P = ZY$ is representable we recover the usual notion of orthogonality.

Definition 2.2.27. Let $\mathcal{K}$ be a $\mathcal{V}$-category and $\mathcal{M}$ be a small collection of morphisms in $\mathcal{P}^\dagger\mathcal{K}$ of the form $f\colon ZX \to P$. We denote by $\mathcal{M}^\perp$ the full subcategory of $\mathcal{K}$ spanned by the objects which are orthogonal with respect to each $f \in \mathcal{M}$. We call *virtual orthogonality class* any full subcategory of $\mathcal{K}$ which arises in this way.

Remark 2.2.28. In the ordinary context, an equivalent form of this notion was already considered by Guitart and Lair in [48, Section 4]. Given a cone $c\colon \Delta X \to H$ in a category $\mathcal{K}$, they say that an object $A \in \mathcal{K}$ "satisfies" the cone c if $\mathcal{K}(c, A)$ induces an isomorphism

$$\mathcal{K}(X, A) \cong \operatorname{colim} \mathcal{K}(H-, A).$$

It is easy to see that, considering $P := \lim YH \in \mathcal{P}^\dagger\mathcal{K}$ and the induced map $\bar{c}\colon ZX \to P$, an object A of $\mathcal{K}$ satisfies H if and only if it is orthogonal with respect to $\bar{c}$. In [48, Section 4] they prove then that each sketchable category $\mathrm{Mod}(\mathcal{S})$ is (in our terminology) a virtual orthogonality class in its ambient category $[\mathcal{C}, \mathbf{Set}]$, but not the vice versa.

The next results can be seen as the analogue of the relation between locally presentable categories and orthogonality classes.

Proposition 2.2.29. *Let $\mathcal{K}$ be accessible and $J\colon \mathcal{A} \hookrightarrow \mathcal{K}$ be accessible and accessibly embedded; then $\mathcal{A}$ is a virtual orthogonality class in $\mathcal{K}$.*

Proof. Consider a regular cardinal α for which $\mathcal{A}, \mathcal{K}$, and J are α-accessible and J preserves the α-presentable objects (Corollary 2.1.22); this implies in particular that $\mathcal{A}_\alpha = \mathcal{A} \cap \mathcal{K}_\alpha$. Denote the inclusions as below.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook, "J"'] & \mathcal{K} \\
\mathcal{A}_\alpha \arrow[u, hook, "H"] \arrow[r, hook, "J_\alpha"'] & \mathcal{K}_\alpha \arrow[u, hook, "H'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}^\dagger(\mathcal{A}) \arrow[r, hook, "\mathcal{P}^\dagger J"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathcal{P}^\dagger(\mathcal{K}) \arrow[l, bend right, "L"'] \\
\mathcal{A} \arrow[u, hook, "W"] \arrow[r, hook, "J"'] & \mathcal{K} \arrow[u, hook, "Z"']
\end{tikzcd}$$

We wish to show that $\mathcal{A}$ can be identified with the virtual orthogonality class defined by the set

$$\mathcal{M} := \{\eta_X\colon ZX \to (\mathcal{P}^\dagger J)LZX \mid X \in \mathcal{K}_\alpha\}$$

where each $\eta_X\colon ZX \to (\mathcal{P}^\dagger J)LZX$ is the component at X of the unit of the adjunction. On one hand, given any $A \in \mathcal{A}$, the object JA is orthogonal with respect to each η_X in $\mathcal{M}$; in fact the orthogonality condition holds with respect to η_X for any $X \in \mathcal{K}$:

$$\begin{aligned}
\mathcal{P}^\dagger\mathcal{K}(ZX, ZJA) &\cong \mathcal{P}^\dagger\mathcal{K}(ZX, (\mathcal{P}^\dagger J)WA) \\
&\cong \mathcal{P}^\dagger\mathcal{A}(LZX, WA) \\
&\cong \mathcal{P}^\dagger\mathcal{K}((\mathcal{P}^\dagger J)LZX, (\mathcal{P}^\dagger J)WA) \\
&\cong \mathcal{P}^\dagger\mathcal{K}((\mathcal{P}^\dagger J)LZX, ZJA).
\end{aligned}$$

Conversely suppose that $Y \in \mathcal{K}$ is orthogonal with respect to η_X for each $X \in \mathcal{K}_\alpha$. Note first that

$$\begin{aligned}
(\mathcal{P}^\dagger J)LZX &\cong \mathrm{Lan}_J\mathcal{K}(X, J-) \\
&\cong \mathrm{Lan}_J\mathrm{Lan}_H\mathcal{K}(X, JH-) \\
&\cong \mathrm{Lan}_{JH}\mathcal{K}(X, JH-) \\
&\cong \{\mathcal{K}(X, JH-), ZJH\},
\end{aligned}$$

where the second isomorphism follows from the fact that $\mathcal{K}(X, J-)$ preserves α-flat colimits and $\mathcal{A}$ is α-accessible. As a consequence we obtain that for each $X \in \mathcal{K}_\alpha$:

$$\begin{aligned}
\mathcal{K}(X, Y) &\cong \mathcal{P}^\dagger\mathcal{K}((\mathcal{P}^\dagger J)LZX, ZY) \qquad (2.1)\\
&\cong \mathcal{P}^\dagger\mathcal{K}(\{\mathcal{K}(X, JH-), ZJH\}, ZY) \\
&\cong \mathcal{K}(X, JH-) * \mathcal{K}(JH-, Y) \\
&\cong \mathcal{K}(X, H'J_\alpha-) * \mathcal{K}(H'J_\alpha-, Y) \\
&\cong \mathcal{K}_\alpha^{op}(J_\alpha-, X) * \mathcal{K}(H'J_\alpha-, Y) \\
&\cong \mathrm{Lan}_{J_\alpha^{op}}\mathcal{K}(H'J_\alpha-, Y)(X)
\end{aligned}$$

where (2.1) holds because Y is orthogonal with respect to η_X. It follows that

$$\mathcal{K}(H'-, Y) \cong \mathrm{Lan}_{J_\alpha^{op}}\mathcal{K}(H'J_\alpha-, Y),$$

but the weight $\mathcal{K}(H'-, Y)$ is α-flat; therefore $\mathcal{K}(JH-, Y) \cong \mathcal{K}(H'J_\alpha-, Y)$ is α-flat as well by Lemma 1.3.2. As a consequence, for each $X \in \mathcal{K}_\alpha$

$$\begin{aligned}\mathcal{K}(X,Y) &\cong \mathcal{K}(JH-,Y) * \mathcal{K}(X,JH-)\\ &\cong \mathcal{K}(X,\mathcal{K}(JH-,Y) * JH)\\ &\cong \mathcal{K}(X,J(\mathcal{K}(JH-,Y) * H));\end{aligned}$$

thus $Y \cong J(\mathcal{K}(JH-,Y) * H)$ lies in $\mathcal{A}$. $\square$

Conversely, the accessibility of virtual orthogonality classes can be obtained as a consequence of the following.

Proposition 2.2.30. *Each virtual orthogonality class $\mathcal{A}$ of a presheaf $\mathcal{V}$-category $\mathcal{K} = [\mathcal{C},\mathcal{V}]$ is equivalent to the category of models of a sketch. More precisely: there exists a fully faithful $J\colon \mathcal{C} \hookrightarrow \mathcal{B}$ and a sketch $\mathcal{S} = (\mathcal{B},\mathbb{L},\mathbb{C})$ on $\mathcal{B}$ such that* Ran_J *induces an equivalence $\mathcal{A} \simeq \mathrm{Mod}(\mathcal{S})$.*

Proof. Let $J\colon \mathcal{A} \hookrightarrow \mathcal{K}$ be a virtual orthogonality class in $\mathcal{K}$ defined by a set of morphisms $\mathcal{M}$. Without loss of generality we can assume that $\mathcal{M}$ consists of a single arrow $f\colon ZX \to P = \{M, ZH\}$ with $X \in \mathcal{K}$, $Z\colon \mathcal{K} \hookrightarrow \mathcal{P}^\dagger\mathcal{K}$ being the inclusion, $M\colon \mathcal{D} \to \mathcal{V}$ a weight, and $H\colon \mathcal{D} \to \mathcal{K}$ a diagram in $\mathcal{K}$. Consider now the closure $\mathcal{B}^{op}$ of $\mathcal{C}^{op} \hookrightarrow \mathcal{K} = [\mathcal{C},\mathcal{V}]$ under α-small colimits, where α is such that $\mathcal{B}$ contains X and the image of H; let $H'\colon \mathcal{D}^{op} \to \mathcal{B}$ be the induced map. In particular $\mathcal{B}$ is the free completion of $\mathcal{C}$ under α-small limits; so that right Kan extending along the inclusion induces an equivalence

$$W\colon [\mathcal{C},\mathcal{V}] \longrightarrow \alpha\text{-}\mathrm{Cont}[\mathcal{B},\mathcal{V}].$$

Note moreover that f corresponds to a cylinder $\bar{f}\colon M \to \mathcal{K}(X,H-) \cong \mathcal{B}(H'-,X)$.

Now, a $\mathcal{V}$-functor $A \in \mathcal{K}$ is orthogonal with respect to f if and only if

$$\mathcal{P}^\dagger\mathcal{K}(f,A)\colon \mathcal{P}^\dagger\mathcal{K}(P,ZA) \to \mathcal{P}^\dagger\mathcal{K}(ZX,ZA)$$

is an isomorphism; but on one hand we have

$$\begin{aligned}\mathcal{P}^\dagger\mathcal{K}(ZX,ZA) &\cong \mathcal{K}(X,A)\\ &\cong [\mathcal{B},\mathcal{V}](WX,WA)\\ &\cong [\mathcal{B},\mathcal{V}](\mathcal{B}(X,-),WA)\\ &\cong (WA)(X)\end{aligned}$$

on the other

$$\begin{aligned}\mathcal{P}^\dagger\mathcal{K}(P,ZA) &\cong M * \mathcal{K}(H-,A)\\ &\cong M * [\mathcal{B},\mathcal{V}](WH-,WA)\\ &\cong M * [\mathcal{B},\mathcal{V}](\mathcal{B}(H',-),WA)\\ &\cong M * (WA)H'.\end{aligned}$$

It follows then that A is orthogonal with respect to f if and only if WA sends the cylinder $\bar{f}$ to a colimiting cylinder. In conclusion $\mathcal{A}$ is equivalent to the full subcategory of $[\mathcal{B},\mathcal{V}]$ given by the α-continuous functors which send $\bar{f}$ to a colimiting cylinder, and this is the $\mathcal{V}$-category of models of a sketch on $\mathcal{B}$. $\square$

One can also show the converse: every $\mathcal{V}$-category of models of a sketch is a virtual orthogonality class in its ambient category. This can of course be seen as a consequence of the fact that sketchable implies accessible (Theorem 2.2.4) which in turn implies virtual orthogonality class (Proposition 2.2.29), but we can also provide a direct proof:

Proposition 2.2.31. *Let $\mathcal{S} = (\mathcal{B}, \mathbb{L}, \mathbb{C})$ be a sketch; then $\mathrm{Mod}(\mathcal{S})$ is a virtual orthogonality class in $[\mathcal{B}, \mathcal{V}]$.*

Proof. Let $\mathcal{K} := [\mathcal{B}, \mathcal{V}]$, $Y : \mathcal{B}^{op} \to \mathcal{K}$ be the Yoneda embedding, and $Z \colon \mathcal{K} \hookrightarrow \mathcal{P}^{\dagger}\mathcal{K}$ be the inclusion into the free completion. It is enough to show that each cylinder $c \in \mathbb{L}$ there exists a morphism $f_c \colon ZX \to P$ in $\mathcal{P}^{\dagger}\mathcal{K}$ for which a functor $F \colon \mathcal{B} \to \mathcal{V}$ sends c to a limiting cylinder if and only if F is orthogonal with respect to f_c; plus the colimit version of this for any $d \in \mathbb{C}$.

In the limit case, the virtual orthogonality notion coincides with standard orthogonality. Let $c \colon N \to \mathcal{B}(B, H-)$ be a cylinder in $\mathbb{L}$; then we can consider $X := N * YH^{op} \in \mathcal{K}$ and the map $f_c \colon X \to YB$ induced by c. Now, for each $F \colon \mathcal{B} \to \mathcal{V}$, since by construction $\mathcal{K}(X, F) \cong \{N, FH-\}$, it follows that F sends c to a limiting cylinder if and only if it is orthogonal with respect to f_c, and this is the same as virtual orthogonality with respect to Zf_c.

For the colimit case, consider $d \colon M \to \mathcal{B}(K-, C)$ in $\mathbb{C}$ and define $P := \{M, ZYK^{op}\}$ in $\mathcal{P}^{\dagger}\mathcal{K}$; then d induces a map $f_d \colon ZYC \to P$. Now note that, for each $F \colon \mathcal{B} \to \mathcal{V}$,

$$\mathcal{P}^{\dagger}\mathcal{K}(ZYC, ZF) \cong [\mathcal{B}, \mathcal{V}](YC, F) \cong FC$$

and on the other hand

$$\mathcal{P}^{\dagger}\mathcal{K}(P, ZF) \cong M * \mathcal{P}^{\dagger}\mathcal{K}(ZYK^{op}-, ZF) \cong M * [\mathcal{B}, \mathcal{V}](YK^{op}-, F) \cong M * FK.$$

Thus it follows again that F sends d to a colimiting cylinder if and only if it is orthogonal with respect to f_d. ☐

2.2.5 The characterization theorems

We can now sum up all the results above in the characterization Theorem below.

Theorem 2.2.32. *For an accessible $\mathcal{V}$-category $\mathcal{K}$ and a fully faithful inclusion $\mathcal{A} \hookrightarrow \mathcal{K}$, the following are equivalent:*

1. *$\mathcal{A}$ is accessible and accessibly embedded;*
2. *$\mathcal{A}$ is accessibly embedded and virtually reflective;*
3. *$\mathcal{A}$ is a virtual orthogonality class.*

Proof. (1) $\Leftrightarrow$ (2) are given by the Corollaries 2.2.22 and 2.2.24. (1) $\Rightarrow$ (3) is a consequence of Proposition 2.2.29. For the implication (3) $\Rightarrow$ (1), it follows from Propositions 2.2.30 and 2.2.4 that $\mathcal{A}$ is accessible; moreover it is easily seen to be closed under α-flat colimits in $\mathcal{K}$, where α is such that all the morphisms in the family defining the virtual orthogonality class lie in $\mathcal{P}^{\dagger}(\mathcal{K}_\alpha)$. ☐

In general we obtain:

Theorem 2.2.33. *The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:*

1. *$\mathcal{A}$ is accessible;*

2. $\mathcal{A} \simeq \alpha\text{-Flat}(\mathcal{C}, \mathcal{V})$ *for some* α *and some small* $\mathcal{C}$*;*
3. $\mathcal{A}$ *is accessibly embedded and virtually reflective in* $[\mathcal{C}, \mathcal{V}]$ *for some small* $\mathcal{C}$*;*
4. $\mathcal{A}$ *is a virtual orthogonality class in* $[\mathcal{C}, \mathcal{V}]$ *for some small* $\mathcal{C}$*;*
5. $\mathcal{A}$ *is equivalent to the* $\mathcal{V}$*-category of models of a sketch.*

Proof. The equivalences (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) are a direct consequence of Theorem 2.2.32. The implication (1) $\Leftrightarrow$ (2) is given by Proposition 2.1.5, while (1) $\Leftrightarrow$ (5) is given by Theorem 2.2.4. □

A few consequences of these characterization theorems are:

Corollary 2.2.34. *If* $\mathcal{A}$ *is a* $\mathcal{V}$*-category with* α*-flat colimits, for some* α*, then* $\mathcal{A}$ *is accessible if and only if it is virtually cocomplete and has a dense generator made of presentable objects.*

Proof. By the previous theorem it is enough to prove that $\mathcal{A}$ is accessibly embedded and virtually reflective in some category of presheaves. Let $\mathcal{C} \subseteq \mathcal{A}$ be a dense presentable generator; then the inclusion $\mathcal{A} \hookrightarrow [\mathcal{C}^{op}, \mathcal{V}]$ is accessible (since every object of $\mathcal{C}$ is presentable) and virtually reflective by Proposition 2.2.19. □

Corollary 2.2.35. *If* $\mathcal{A}$ *is a* $\mathcal{V}$*-category with* α*-flat colimits, for some* α*, then* $\mathcal{A}$ *is accessible if and only if it has a dense generator and* $\mathcal{P}^{\dagger}\mathcal{A}$ *consists exactly of the accessible presheaves out of* $\mathcal{A}$*.*

Proof. If $\mathcal{A}$ is accessible then it has a dense generator by definition and $\mathcal{P}^{\dagger}\mathcal{A}$ consists exactly of the accessible presheaves by Proposition 2.2.9. Conversely, assume that $\mathcal{A}$ has a dense generator and $\mathcal{P}^{\dagger}\mathcal{A}$ consists exactly of the accessible presheaves out of $\mathcal{A}$. Then each object of $\mathcal{A}$ is presentable, since representable functors are small; moreover $\mathcal{P}^{\dagger}\mathcal{A}$ is cocomplete, since accessible functors are limit closed in $[\mathcal{A}, \mathcal{V}]$. Thus the result follows from the previous corollary. □

So far we have only given a characterization of the accessible $\mathcal{V}$-categories, but not of the conically accessible ones; this is what we can say in that context:

Theorem 2.2.36. *For a conically accessible* $\mathcal{V}$*-category* $\mathcal{K}$ *and a fully faithful inclusion* $\mathcal{A} \hookrightarrow \mathcal{K}$*, the following are equivalent:*

1. $\mathcal{A}$ *is conically accessible and conically accessibly embedded;*
2. $\mathcal{A}_0$ *is accessible and accessibly embedded in* $\mathcal{K}_0$*;*
3. $\mathcal{A}_0$ *is accessibly embedded and virtually reflective in* $\mathcal{K}_0$*;*
4. $\mathcal{A}_0$ *is a virtual orthogonality class in* $\mathcal{K}_0$*.*

Proof. Follows from Theorem 2.2.32 for $\mathcal{V} = \mathbf{Set}$ and Corollary 2.1.18. □

And as a consequence:

Theorem 2.2.37. *The following are equivalent for a* $\mathcal{V}$*-category* $\mathcal{A}$*:*

1. $\mathcal{A}$ *is conically accessible;*

2. *$\mathcal{A}$ is conically accessibly embedded in $[\mathcal{C},\mathcal{V}]$, for some small category $\mathcal{C}$, and $\mathcal{A}_0$ is virtually reflective in $[\mathcal{C},\mathcal{V}]_0$;*

3. *$\mathcal{A}$ is a full subcategory of $[\mathcal{C},\mathcal{V}]$, for some small category $\mathcal{C}$, and $\mathcal{A}_0$ accessible and accessibly embedded in $[\mathcal{C},\mathcal{V}]_0$;*

4. *$\mathcal{A}$ is a full subcategory of $[\mathcal{C},\mathcal{V}]$, for some small category $\mathcal{C}$, and $\mathcal{A}_0$ is a virtual orthogonality class in $[\mathcal{C},\mathcal{V}]_0$.*

We do not yet know whether a $\mathcal{V}$-category which is conically accessibly embedded and virtually reflective in some $[\mathcal{C},\mathcal{V}]$, is also conically accessible. The issue is that the fact that a $\mathcal{V}$-category $\mathcal{A}$ is virtually reflective in $\mathcal{K}$ is not known to imply that $\mathcal{A}_0$ is virtually reflective in $\mathcal{K}_0$.

2.2.6 Cone-reflectivity and cone-injectivity

In this section we go back to the ordinary setting ($\mathcal{V} = \mathbf{Set}$) and compare the virtual concepts introduced in Section 2.2.3 and 2.2.4 with those of cone-reflectivity and cone-injectivity class from [1]. For that we first need to recall the notions of petty and lucid functors introduced by Freyd:

Definition 2.2.38 ([41])**.** Let $\mathcal{A}$ be a category; a functor $P\colon \mathcal{A} \to \mathbf{Set}$ is called *petty* if there exists a family $(A_i)_{i\in I}$ in $\mathcal{A}$ and an epimorphism

$$\sum_{i\in I} \mathcal{A}(A_i, -) \twoheadrightarrow P.$$

Denote by $\mathrm{Pt}(\mathcal{A})$ the full subcategory of $[\mathcal{A}^{op},\mathbf{Set}]$ spanned by the petty functors.

Clearly every small functor is petty since every small colimit of representables is in particular a coequalizer of coproducts of them. Thus we have a fully faithful inclusion $\mathcal{P}\mathcal{A} \hookrightarrow \mathrm{Pt}(\mathcal{A})$ as full subcategories of $[\mathcal{A}^{op},\mathbf{Set}]$; moreover the category $\mathrm{Pt}(\mathcal{A})$ is locally small and, if we allow some colimits to be large, it can be seen as some kind of free cocompletion of $\mathcal{A}$:

Remark 2.2.39. Let us say that a category $\mathcal{L}$ is *well cocomplete* if it is cocomplete and has all (possibly large) cointersections of regular epimorphisms. A functor $F\colon \mathcal{L} \to \mathcal{K}$ is well cocontinuous if it is cocontinuous and preserves all the cointersections of regular epimorphisms. Then, for any category $\mathcal{A}$, it is easy to see that $\mathrm{Pt}(\mathcal{A})$ is well cocomplete and also the free *well cocompletion* of $\mathcal{A}$: for any well cocomplete category $\mathcal{B}$, precomposition with the inclusion $V\colon \mathcal{A} \hookrightarrow \mathrm{Pt}(\mathcal{A})$ induces an equivalence between $[\mathcal{A},\mathcal{B}]$ and the category of well cocontinuous functors $\mathrm{Pt}(\mathcal{A}) \to \mathcal{B}$.

Definition 2.2.40 ([41])**.** We say that a functor $L\colon \mathcal{A} \to \mathbf{Set}$ is *lucid* if it is petty and for any other petty P and $f,g\colon P \to L$, the equalizer of (f,g) is still petty. Denote by $\mathrm{Lcd}(\mathcal{A})$ the full subcategory of $\mathrm{Pt}(\mathcal{A})$ given by the lucid functors.

Note that, thanks to [41, Proposition 1.1], in the definition above we can assume P to be representable. As a consequence L is lucid if and only if it is petty and for any representable $A \in \mathcal{A}$ and $f,g\colon \mathcal{A}(A,-) \to L$, the equalizer of (f,g) is still petty.

Remark 2.2.41. In Section A.2 we define an enriched notion of pettiness and prove the corresponding versions of Propositions 2.2.42 and 2.2.46 below.

Every lucid functor is petty (by definition), however in general lucid and small functors are not comparable. This changes if the following conditions are satisfied:

Proposition 2.2.42. *The following are equivalent for a category $\mathcal{A}$:*

1. $\mathrm{Pt}(\mathcal{A})$ *has limits of representables (i.e. $\mathcal{A}$ is pre-complete);*
2. $\mathrm{Lcd}(\mathcal{A})$ *is complete and contains the representables;*
3. $\mathcal{P}\mathcal{A}$ *has limits of representables;*
4. $\mathcal{P}\mathcal{A}$ *is complete;*

and in that case $\mathcal{P}\mathcal{A} = \mathrm{Lcd}(\mathcal{A})$.

Proof. (1) $\Leftrightarrow$ (2) is [41, Theorem 1.(12)]. The fact that this implies $\mathcal{P}\mathcal{A} = \mathrm{Lcd}(\mathcal{A})$ is [91, Lemma 1] (see note below their proof), and (2) $\Rightarrow$ (3) can be seen as a consequence. (3) $\Leftrightarrow$ (4) is [33, Theorem 3.8] and finally (3) $\Rightarrow$ (1) is trivial since $\mathcal{P}\mathcal{A} \subseteq \mathrm{Pt}(\mathcal{A})$. $\square$

Taking the dual notions in the statement above, the proposition says in particular that a category is pre-cocomplete (1) if and only if it is virtually cocomplete (3).

In the next part of the section we keep working with the dual notions, $\mathrm{Pt}^\dagger(\mathcal{A})$ and $\mathcal{P}^\dagger\mathcal{A}$. As we considered virtual left adjoints (relatively to small functors), one could introduce an adjointness condition with respect to petty functors by imposing that, for $F\colon \mathcal{A} \to \mathcal{K}$, the functors $\mathcal{K}(X, F-)\colon \mathcal{A} \to \mathbf{Set}$ are petty for any $X \in \mathcal{K}$. In other words, this is saying that $F\colon \mathcal{A} \to \mathcal{K}$ has a relative left adjoint with respect to the inclusion $\mathcal{A} \hookrightarrow \mathrm{Pt}^\dagger\mathcal{A}$.

The condition above turns out to be the same as F satisfying the *solution-set condition*: indeed $\mathcal{K}(X, F-)$ is petty if and only if there exists an epimorphism $\sum_{i\in I}\mathcal{A}(A_i, -) \twoheadrightarrow \mathcal{K}(X, F-)$, for some $A_i \in \mathcal{A}$, if and only if there exists a cone $(h_i\colon X \to FA_i)_{i\in I}$ such that any map $h\colon X \to FA$ factors as $h = F(f) \circ h_i$ for some $i \in I$ and f in $\mathcal{A}$; this is exactly the solution-set condition for F.

Definition 2.2.43 ([1])**.** We say that a fully faithful inclusion $J\colon \mathcal{A} \hookrightarrow \mathcal{K}$ is *cone-reflective* if J satisfies the solution-set condition.

For fully faithful functors, having a virtual left adjoint or satisfying the solution-set condition is almost the same, at least in the virtually cocomplete context:

Proposition 2.2.44. *Given a virtually cocomplete category $\mathcal{K}$ and a fully faithful functor $J\colon \mathcal{A} \hookrightarrow \mathcal{K}$, the following are equivalent:*

1. *$\mathcal{A}$ is cone-reflective in $\mathcal{K}$;*
2. *$\mathcal{A}$ is virtually reflective in $\mathcal{K}$.*

Proof. (2) $\Rightarrow$ (1) is trivial since every small functor is petty.

(1) $\Rightarrow$ (2). Let J have a relative left adjoint $L\colon \mathcal{K} \to \mathrm{Pt}^\dagger(\mathcal{A})$, we want to prove that this actually lands in $\mathcal{P}^\dagger\mathcal{A}$. Note first that L extends to a left adjoint $-\circ J\colon \mathrm{Pt}^\dagger(\mathcal{K}) \to \mathrm{Pt}^\dagger(\mathcal{A})$ to the inclusion $\mathrm{Pt}^\dagger(J)\colon \mathrm{Pt}^\dagger(\mathcal{A}) \hookrightarrow \mathrm{Pt}^\dagger(\mathcal{K})$; indeed if $P\colon \mathcal{K} \to \mathbf{Set}$ is petty then PJ is covered by a family of functors of the form $\mathcal{K}(X, J-)$, but each of these is covered by a family of representables (from $\mathcal{A}$) by hypothesis, and hence PJ is petty as well.
Now, since $\mathcal{K}$ is virtually cocomplete, $\mathrm{Pt}^\dagger(\mathcal{K})$ has colimits of representables (Proposition 2.2.42); therefore $\mathrm{Pt}^\dagger(\mathcal{A})$ has colimits of representables as well, being reflective in $\mathrm{Pt}^\dagger(\mathcal{K})$ with the inclusion induced by J. It follows again by Proposition 2.2.42 that

$\mathcal{P}^\dagger\mathcal{A} = \mathrm{Lcd}^\dagger(\mathcal{A})$; therefore to prove the virtual reflectivity of $\mathcal{A}$ in $\mathcal{K}$ it is enough to show that for each $X \in \mathcal{K}$ the functor $LX = \mathcal{K}(X, J-)\colon \mathcal{A} \to \mathbf{Set}$ is lucid.
Given $X \in \mathcal{K}$ the functor $\mathcal{K}(X, J-)$ is petty by cone-reflectivity of $\mathcal{A}$; thus we only need to show that the equalizer of any pair $f, g\colon \mathcal{A}(A, -) \to \mathcal{K}(X, J-)$ is still petty (see just below the definition of lucidity). Such a pair corresponds to maps $h, k\colon X \to JA$ in $\mathcal{K}$ which in turn give a pair $f', g'\colon \mathcal{K}(JA, -) \to \mathcal{K}(X, -)$ between representables in $\mathrm{Pt}(\mathcal{K}^{op}) = \mathrm{Pt}^\dagger(\mathcal{K})^{op}$. Since $\mathrm{Pt}^\dagger(\mathcal{K})$ has colimits of representables we can consider the equalizer P of f', g' in $\mathrm{Pt}(\mathcal{K}^{op})$; it follows at once that PJ is the equalizer of f, g and this is petty because P was petty and $- \circ J$ preserves petty functors. □

Remark 2.2.45. One might think that the equivalent conditions above arise from a left adjoint to the inclusion $J\colon \mathcal{P}^\dagger\mathcal{A} \hookrightarrow \mathrm{Pt}^\dagger(\mathcal{A})$, but that (almost) never happens. In fact J has a left adjoint if and only if the two categories $\mathcal{P}^\dagger\mathcal{A}$ and $\mathrm{Pt}^\dagger(\mathcal{A})$ coincide (since they both contain the representables, such a left adjoint is forced to be the identity), and this does not hold even when $\mathcal{A}$ is locally presentable ($\mathcal{A} = \mathbf{Ab}$ is a counterexample, see [41]).

Corollary 2.2.46. *Given a fully faithful functors $J\colon \mathcal{A} \hookrightarrow \mathcal{K}$ between accessible categories, the following are equivalent:*

1. *$\mathcal{A}$ is accessibly embedded in $\mathcal{K}$;*
2. *$\mathcal{A}$ is cone-reflective in $\mathcal{K}$;*
3. *$\mathcal{A}$ is virtually reflective in $\mathcal{K}$.*

Proof. Put together Propositions 2.2.44 and 2.2.20. □

The equivalence between (1) and (2) was first proved in [93, Theorem 3.10]. Note however that the same equivalence cannot be proved in standard set theory when J is replaced by any (non necessarily fully faithful) functor between accessible categories; see [1, Theorem 6.30] and the Remark just below it. On the other hand, we have already shown that (1) $\Leftrightarrow$ (2) always holds (Proposition 2.2.20).
As a corollary we recover the characterization of accessibility given in [1]:

Corollary 2.2.47. *[1, Theorem 2.53] Let $\mathcal{K}$ be an accessible category and $\mathcal{A}$ be an accessibly embedded full subcategory of $\mathcal{K}$. Then $\mathcal{A}$ is accessible if and only if it is cone-reflective in $\mathcal{K}$.*

Proof. This is now a direct consequence of Theorem 2.2.32 and Proposition 2.2.44 above. □

Another way of characterizing accessible categories is via cone injectivity classes. We can recover this characterization using virtual orthogonality classes.

Definition 2.2.48 ([1])**.** Let $\mathcal{K}$ be a category and let $(f_i\colon X \to X_i)_{i\in I}$ be a cone in $\mathcal{K}$. We say that $A \in \mathcal{K}$ is *injective* with respect to the cone $(f_i)_{i\in I}$ if for any $h\colon X \to A$ there exists $i \in I$ for which h factorises through f_i.

Equivalently, $A \in \mathcal{K}$ is injective with respect to the cone $(f_i)_{i\in I}$ if and only if the map

$$\coprod_{i\in I} \mathcal{K}(X_i, A) \longrightarrow \mathcal{K}(X, A)$$

induced by the cone $(f_i)_{i\in I}$, is a surjection of sets.

Definition 2.2.49 ([1])**.** Let $\mathcal{K}$ be a category and $\mathcal{M}$ be a small collection of cones in $\mathcal{K}$. We denote by $\mathcal{M}$-inj the full subcategory of $\mathcal{K}$ spanned by the objects which are injective with respect to each cone in $\mathcal{M}$. We call *cone-injectivity class* any full subcategory of $\mathcal{K}$ which arises in this way.

Then cone-injectivity classes and virtual orthogonality classes turn out to be strictly related by the following:

Proposition 2.2.50. *The following hold for a given category $\mathcal{K}$:*

1. *every cone-injectivity class in $\mathcal{K}$ is a virtual orthogonality class;*
2. *if $\mathcal{K}$ has pushouts, every virtual orthogonality class in $\mathcal{K}$ is a cone-injectivity class.*

Proof. (1). It is enough to prove that injectivity with respect to a cone can be seen as orthogonality with respect to a suitable morphism in $\mathcal{P}^\dagger\mathcal{K}$; we are going to use the fact that a map is an epimorphism if and only if the co-diagonal out of its cokernel pair is an isomorphism. Let $(f_i\colon X \to X_i)_i$ be a cone in $\mathcal{K}$; consider the corresponding $f\colon ZX \to P := \prod_i ZX_i$ in $\mathcal{P}^\dagger\mathcal{K}$ and take the kernel pair of f with the corresponding diagonal map δ:

$$ZX \xrightarrow{\ \delta\ } P' \rightrightarrows ZX \xrightarrow{\ f\ } P.$$

This is sent to a cokernel pair through $\mathcal{P}^\dagger\mathcal{K}(-,ZA)$ for each $A \in \mathcal{K}$ (since $\mathcal{P}^\dagger\mathcal{K}(-,ZA) \cong \mathrm{ev}_A$ is cocontinuous), with co-diagonal $\mathcal{P}^\dagger\mathcal{K}(\delta, ZA)$. As a consequence the map

$$\textstyle\coprod_i \mathcal{K}(X_i, A) \cong \mathcal{P}^\dagger\mathcal{K}(P, ZA) \xrightarrow{\mathcal{P}^\dagger\mathcal{K}(f,ZA)} \mathcal{P}^\dagger\mathcal{K}(ZX, ZA) \cong \mathcal{K}(X, A)$$

is an epimorphism if and only if $\mathcal{P}^\dagger\mathcal{K}(\delta, ZA)$ is an isomorphism, which in turn means that A is injective with respect to $(f_i)_i$ if and only if it is orthogonal with respect to the map $\delta\colon ZX \to P'$.

(2). Similarly, it is enough to prove that orthogonality with respect to a map $f\colon ZX \to P$ in $\mathcal{P}^\dagger\mathcal{K}$ is the same as injectivity with respect to a set of cones. Since $P \in \mathcal{P}^\dagger\mathcal{K}$, we can find $H\colon \mathcal{I} \to \mathcal{K}$ such that $P \cong \operatorname{colim} ZH$. Then f corresponds to a cone $(f_i\colon X \to Hi)_{i\in\mathcal{I}}$ over H in $\mathcal{K}$. Now let $A \in \mathcal{K}$; then A is orthogonal with respect to f if and only if the map

$$\rho\colon \operatorname{colim} \mathcal{K}(H-, A) \longrightarrow \mathcal{K}(X, A)$$

induced by $(f_i)_{i\in\mathcal{I}}$, is bijective. Notice now that ρ is surjective if and only if A is injective with respect to the cone $(f_i)_{i\in\mathcal{I}}$; thus we only need to express the fact that ρ is a monomorphism in terms of injectivity. For this observe that ρ is injective if and only if, for any $h\colon X \to \mathcal{A}$ which factors as $h = h_i \circ f_i = h_j \circ f_j$, for some $h_i\colon Hi \to A$ and $h_j\colon Hj \to A$, there exists a zig-zag in H/A connecting h_i and h_j. Now, for any pair $i, j \in \mathcal{I}$ let $X_{i,j}$ be the pushout of (f_i, f_j) in $\mathcal{K}$ and Ξ_{ij} be the set of all zig-zags in $\mathcal{I}$ between i and j. For any $\xi \in \Xi_{ij}$ let X_ξ denote the colimit of the diagram $H(\xi)$ in $\mathcal{K}$ (this is obtained as a finite number of consecutive pushouts), and let $g_\xi\colon X_{i,j} \to X_\xi$ be the induced comparison; this gives a cone $(g_\xi\colon X_{i,j} \to X_\xi)_{\xi\in\Xi_{ij}}$ for any pair $i, j \in \mathcal{I}$. To conclude it is enough to note that to give an arrow $h\colon X_{i,j} \to A$ is the same as giving an arrow $X \to A$ which factors through f_i and f_j; moreover h factors through an arrow of the cone $(g_\xi)_{\xi\in\Xi_{ij}}$ if and only if the two factorizations of h are connected by a zig-zag in H/A. It follows at once that A is orthogonal with respect to f if and only if it is injective with respect to the cones $(f_i)_{i\in\mathcal{I}}$ and $(g_\xi)_{\xi\in\Xi_{ij}}$ for any $i, j \in \mathcal{I}$. □

As a consequence we recover:

Theorem 2.2.51. *[1, Theorem 4.17] Let $\mathcal{K}$ be locally presentable and $\mathcal{A}$ be a full subcategory of $\mathcal{K}$. Then $\mathcal{A}$ is accessible and accessibly embedded if and only if it is a cone-injectivity class in $\mathcal{K}$.*

Proof. Direct consequence of Theorem 2.2.32 and Proposition 2.2.50 above. □

2.3 Limits of accessible $\mathcal{V}$-categories

It is proved in [77, Theorem 5.1.6] that the 2-category **Acc**, of accessible categories, accessible functors, and natural transformations, admits all small bilimits and that the forgetful functor $U_0\colon \mathbf{Acc} \to \mathbf{CAT}$ preserves them. But more can be said: **Acc** has all flexible limits and U preserves them (as pointed out in [13, Remark 7.8]); then one can deduce from this that **Acc** has all pseudolimits and bilimits (see [61]).

In this section we are going to show that the corresponding result holds for accessible and conically accessible $\mathcal{V}$-categories as well. The corresponding result for locally presentable $\mathcal{V}$-categories was proved in Bird's thesis [14, Theorem 6.10].

2.3.1 The accessible case

Let $\mathcal{V}$-**Acc** be the 2-category of accessible $\mathcal{V}$-categories, accessible $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. When $\mathcal{V} = \mathbf{Set}$ we simply write **Acc** instead of **Set**-**Acc**.

Proposition 2.3.1. *Let $\mathcal{A}$ be an accessible $\mathcal{V}$-category and $\mathcal{C}$ be a small $\mathcal{V}$-category. Then $[\mathcal{C}, \mathcal{A}]$ is accessible.*

Proof. By Theorem 2.2.4 we can write $\mathcal{A} \simeq \mathrm{Mod}(\mathcal{S})$ for a sketch $\mathcal{S} = (\mathcal{B}, \mathbb{L}, \mathbb{C})$ and hence we can see $\mathcal{A}$ as a full subcategory of $[\mathcal{B}, \mathcal{V}]$. It follows at once that $[\mathcal{C}, \mathcal{A}]$ has a fully faithful embedding

$$J\colon [\mathcal{C}, \mathcal{A}] \hookrightarrow [\mathcal{C}, [\mathcal{B}, \mathcal{V}]] \simeq [\mathcal{C} \otimes \mathcal{B}, \mathcal{V}]$$

obtained by post-composition with the inclusion of $\mathcal{A}$ in $[\mathcal{B}, \mathcal{V}]$. Moreover $[\mathcal{C}, \mathcal{A}]$ can be described as the full subcategory of $[\mathcal{C} \otimes \mathcal{B}, \mathcal{V}]$ whose objects F are such that $F(c, -) \in \mathcal{A}$ for each $c \in \mathcal{C}$.

Now, for each $c \in \mathcal{C}$ and $\lambda\colon M \to \mathcal{B}(b, H-)$ in $\mathbb{L}$ consider the induced cylinder

$$\lambda_c\colon M \to (\mathcal{C} \otimes \mathcal{B})((c, b), (c, H-))$$

which acts constantly on c in the first component; let $\mathbb{L}_{\mathcal{C}}$ be the collection of all such cylinders. Similarly, for each $c \in \mathcal{C}$ and $\mu \in \mathbb{C}$ define the co-cylinder μ_c accordingly, so that we can a new family $\mathbb{C}_{\mathcal{C}}$.

It is easy to see that given $F\colon \mathcal{C} \otimes \mathcal{B} \to \mathcal{V}$, the functor $F(c, -)$ sends λ (respectively μ) to a (co)limiting cylinder if and only if F sends λ_c (respectively μ_c) to a (co)limiting cylinder. As a consequence $[\mathcal{C}, \mathcal{A}] \simeq \mathrm{Mod}(\mathcal{S}')$ for the sketch $\mathcal{S}' = (\mathcal{C} \otimes \mathcal{B}, \mathbb{L}_{\mathcal{C}}, \mathbb{C}_{\mathcal{C}})$, and thus it is accessible by Theorem 2.2.4. □

Remark 2.3.2. In the proof above as well as in the forthcoming ones we make use of the theory of sketches; it would be interesting instead to have a proof which does not rely on these but we do not have one. Moreover we do not know whether the proposition above is still true when accessibility is replaced by conical accessibility, nonetheless we are still able to show that conically accessible $\mathcal{V}$-categories are stable in $\mathcal{V}$-**CAT** under flexible colimits.

Corollary 2.3.3. *Let $\mathcal{A}$ be an accessible $\mathcal{V}$-category and $\mathcal{C}$ be an ordinary small category. Then the power $\mathcal{A}^{\mathcal{C}} := \mathcal{C} \pitchfork \mathcal{A}$ exists in $\mathcal{V}$-**Acc** and is preserved by the forgetful functor $U\colon \mathcal{V}\text{-}\mathbf{Acc} \to \mathcal{V}\text{-}\mathbf{CAT}$.*

Proof. Let $\mathcal{A}^{\mathcal{C}}$ be the power of $\mathcal{A}$ by $\mathcal{C}$ in $\mathcal{V}$-**CAT**. Since a $\mathcal{V}$-functor $\mathcal{B} \to \mathcal{A}^{\mathcal{C}}$ is accessible if and only if its transpose ordinary functor $\mathcal{C} \to \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{B},\mathcal{A})$ lands in $\mathcal{V}\text{-}\mathbf{Acc}(\mathcal{B},\mathcal{A})$, it is enough to show that $\mathcal{A}^{\mathcal{C}}$ is an accessible $\mathcal{V}$-category. This follows at once from the previous proposition since $\mathcal{A}^{\mathcal{C}} \cong [\mathcal{C}_{\mathcal{V}}, \mathcal{A}]$, where $\mathcal{C}_{\mathcal{V}}$ is the free $\mathcal{V}$-category on $\mathcal{C}$. □

The following will be needed to prove the main result of this section.

Lemma 2.3.4. *Let $F\colon \mathcal{A}_1 \to \mathcal{A}_2$ be an accessible $\mathcal{V}$-functor between accessible $\mathcal{V}$-categories; then there exist sketches $\mathcal{S}_1 = (\mathcal{B}_1, \mathbb{L}_1, \mathbb{C}_1)$ and $\mathcal{S}_2 = (\mathcal{B}_2, \mathbb{L}_2, \mathbb{C}_2)$ and a $\mathcal{V}$-functor $K\colon \mathcal{B}_2 \to \mathcal{B}_1$ for which:*

1. *$\mathcal{A}_1 \simeq \mathrm{Mod}(\mathcal{S}_1)$ and $\mathcal{A}_2 \simeq \mathrm{Mod}(\mathcal{S}_2)$:*
2. *The induced square*

$$\begin{tikzcd}
[\mathcal{B}_1,\mathcal{V}] \arrow[r, "-\circ K"] & [\mathcal{B}_2,\mathcal{V}] \\
\mathcal{A}_1 \arrow[u, hook] \arrow[r, "F"'] & \mathcal{A}_2 \arrow[u, hook]
\end{tikzcd}$$

commutes up to isomorphism.

Moreover $\mathcal{B}_2$ can be chosen to be $(\mathcal{A}_2)^{op}_{\alpha}$ for arbitrarily large cardinals α as in 2.1.20.

Proof. Given $F\colon \mathcal{A}_1 \to \mathcal{A}_2$ as above, let α be such that $\mathcal{A}_1, \mathcal{A}_2$, and F are α-accessible and $F((\mathcal{A}_1)_\alpha) \subseteq (\mathcal{A}_2)_\alpha$ (see 1.1.6 and 2.1.20). Then F is the left Kan extension of its restriction to $(\mathcal{A}_1)_\alpha$ and we can consider the commutative (up to isomorphism) diagram below

$$\begin{tikzcd}
[\mathcal{C},\mathcal{V}] \arrow[r, "\mathrm{Lan}_G"] & [\mathcal{B}_2,\mathcal{V}] \\
\mathcal{A}_1 \arrow[u, hook] \arrow[r, "F"'] & \mathcal{A}_2 \arrow[u, hook] \\
\mathcal{C}^{op} \arrow[u, hook] \arrow[r, "G^{op}"'] & \mathcal{B}_2^{op} \arrow[u, hook]
\end{tikzcd}$$

where $\mathcal{C} = (\mathcal{A}_1)^{op}_\alpha$, $\mathcal{B}_2 = (\mathcal{A}_2)^{op}_\alpha$, and G is the restriction of F^{op} to $\mathcal{C}$.

Now consider the $\mathcal{V}$-functor $L\colon \mathcal{B}_2 \to [\mathcal{C}^{op}, \mathcal{V}]$ sending B to $LB = \mathcal{B}_2(G-, B)$ (note that L^{op} is the virtual left adjoint to G^{op}). Let $\mathcal{B}_1$ be the full subcategory of $[\mathcal{C}^{op}, \mathcal{V}]$ spanned by the representables and the essential image of L; then we have a fully faithful inclusion $H\colon \mathcal{C} \hookrightarrow \mathcal{B}_1$ and a $\mathcal{V}$-functor $K\colon \mathcal{B}_2 \to \mathcal{B}_1$ induced by L.

The next step is to check that the triangle below commutes up to isomorphism.

$$\begin{tikzcd}
 & [\mathcal{B}_1,\mathcal{V}] \arrow[dr, "-\circ K"] & \\
[\mathcal{C},\mathcal{V}] \arrow[ur, hook, "\mathrm{Lan}_H"] \arrow[rr, "\mathrm{Lan}_G"'] & & [\mathcal{B}_2,\mathcal{V}]
\end{tikzcd}$$

Since all the $\mathcal{V}$-functors involved are cocontinuous it is enough to prove that the isomorphism holds for all representables in $[\mathcal{C},\mathcal{V}]$. Consider $C \in \mathcal{C}$: on one hand $\mathrm{Lan}_G(\mathcal{C}(C,-)) \cong \mathcal{B}_2(GC,-)$; on the other $\mathrm{Lan}_H(\mathcal{C}(C,-)) \cong \mathcal{B}_1(HC,-)$ which by precomposition with K gives

$$\mathcal{B}_1(HC,K-) \cong [\mathcal{C}^{op},\mathcal{V}](YC,L-) \cong L(-)(C) \cong \mathcal{B}_2(GC,-)$$

as desired.

Gluing this triangle with the diagram above we find the desired square in (2). To conclude it is then enough to note that $\mathcal{A}_2$ is identified with the full subcategory of $[\mathcal{B}_2,\mathcal{V}]$ spanned by the α-flat functors out of $\mathcal{B}_2$, and hence is the category of models of a sketch. The same holds for the inclusion of $\mathcal{A}_1$ in $[\mathcal{C},\mathcal{V}]$; moreover $[\mathcal{C},\mathcal{V}]$ itself is the category of models for a sketch on $\mathcal{B}_1$ (because the inclusion $H\colon \mathcal{C} \hookrightarrow \mathcal{B}_1$ is dense and therefore the essential image of Lan_H is given by a full subcategory of functors preserving specified colimits). It follows then that $\mathcal{A}_1$ is the category of models for a sketch in $\mathcal{B}_1$. $\square$

In the theorem below we say that a $\mathcal{V}$-functor is an isofibration if its underlying ordinary functor is one.

Theorem 2.3.5. *The 2-category $\mathcal{V}$-**Acc** has all flexible (and hence all pseudo- and bi-) limits, as well as all pullbacks along isofibrations, and the forgetful functors $U\colon \mathcal{V}\text{-}\mathbf{Acc} \to \mathcal{V}\text{-}\mathbf{CAT}$ and $(-)_0\colon \mathcal{V}\text{-}\mathbf{Acc} \to \mathbf{Acc}$ preserve them.*

Proof. We prove this using the fact that a 2-category has all flexible limits if and only if it has products, inserters, equifiers, and splitting of idempotent equivalences (see [13, Theorem 4.9]); moreover the latter comes for free in $\mathcal{V}$-**Acc** thanks to [13, Remark 7.6].

(a) *Pullbacks along isofibrations.* Let $F_1\colon \mathcal{A}_1 \to \mathcal{K}$ and $F_2\colon \mathcal{A}_2 \to \mathcal{K}$ be accessible $\mathcal{V}$-functors between accessible $\mathcal{V}$-categories, and assume that F_1 is an isofibration. Consider the pullback $\mathcal{A}_{12}$ of this pair in $\mathcal{V}$-**CAT**, with projections $P_1\colon \mathcal{A}_{12} \to \mathcal{A}_1$ and $P_2\colon \mathcal{A}_{12} \to \mathcal{A}_2$, and note that, since F_1 is an isofibration, this can be seen as a bipullback in $\mathcal{V}$-**CAT**. By Lemma 2.3.4 we can find small $\mathcal{V}$-categories $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{C}$ together with $\mathcal{V}$-functors $K_i\colon \mathcal{C} \to \mathcal{B}_1$ for which: $\mathcal{A}_i = \mathrm{Mod}(\mathcal{S}_i)$, $\mathcal{K} = \mathrm{Mod}(\mathcal{T})$ for some sketches $\mathcal{S}_i = (\mathcal{B}_i, \mathbb{L}_i, \mathbb{C}_i)$ and $\mathcal{T} = (\mathcal{B}, \mathbb{L}, \mathbb{C})$, and such that $-\circ K_i$ restricts to F_i (note that $\mathcal{C}$ can be chosen to be the same for both $\mathcal{A}_1$ and $\mathcal{A}_2$ thanks to the final assertion in the Lemma). Let $\mathcal{B}_{12}$ be the pushout of K_1 and K_2 in $\mathcal{V}$-**CAT**, with maps $J_i\colon \mathcal{B}_i \to \mathcal{B}_{12}$. Then $\mathcal{B}_{12}$ is sent to a pullback through $[-,\mathcal{V}]$ providing the commutative cube below

$$\begin{tikzcd}
\mathcal{A}_{12} \arrow[rr, "P_2"] \arrow[dd, "P_1"'] \arrow[dr, hook] \arrow[ddrr, phantom, "\lrcorner", very near start] & & \mathcal{A}_2 \arrow[dd] \arrow[dr, hook] & \\
& {[\mathcal{B}_{12},\mathcal{V}]} \arrow[rr, crossing over] \arrow[dd] \arrow[ddrr, phantom, "\lrcorner", very near start] & & {[\mathcal{B}_2,\mathcal{V}]} \arrow[dd, "-\circ K_2"] \\
\mathcal{A}_1 \arrow[rr] \arrow[dr, hook] & & \mathcal{K} \arrow[dr, hook] & \\
& {[\mathcal{B}_1,\mathcal{V}]} \arrow[rr, "-\circ K_1"'] & & {[\mathcal{C},\mathcal{V}]}
\end{tikzcd}$$

where the $\mathcal{V}$-functors not labelled are F_1, F_2, the precomposition functors $-\circ J_1, -\circ J_2$, and the inclusions. It follows that $\mathcal{A}_{12}$ can be identified with the full subcategory of $[\mathcal{B}_{12},\mathcal{V}]$ whose projections to $[\mathcal{B}_i,\mathcal{V}]$ land in $\mathcal{A}_i$.

Consider now the sketch $\mathcal{S}_{12} = (\mathcal{B}_{12}, \mathbb{L}_{12}, \mathbb{C}_{12})$ defined by

$$\mathbb{L}_{12} := \{J_i\eta \mid i = 1,2, \eta \in \mathbb{L}_i\}, \quad \mathbb{C}_{12} := \{J_i\mu \mid i = 1,2, \eta \in \mathbb{C}_i\}.$$

It follows at once that $F\colon \mathcal{B}_{12} \to \mathcal{V}$ is a model of $\mathcal{S}_{12}$ if and only if $F \circ J_i$ is a model of $\mathcal{S}_i$, for $i = 1, 2$; therefore $\mathcal{A} \simeq \mathrm{Mod}(\mathcal{S}_{12})$ is accessible.
To conclude, consider α such that $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{K}$ have, and F_1, F_2 preserve, all α-flat colimits; then it is a standard argument to check that $\mathcal{A}_{12}$ has α-flat colimits as well and P_1 and P_2 preserve them (this follows at once from the fact we are dealing with a bipullback, plus that the homs in $\mathcal{A}_{12}$ come as pullbacks of homs in $\mathcal{A}_1$,$\mathcal{A}_2$, and $\mathcal{K}$, and that α-flat colimits commute with them in $\mathcal{V}$). It is now routine to check that $\mathcal{A}_{12}$ is actually a pullback in $\mathcal{V}$-**Acc**.

(b) *Products.* Let $(\mathcal{A}_i)_{i\in I}$ be a small family of accessible $\mathcal{V}$-categories and denote by $\mathcal{A} = \prod_i \mathcal{A}_i$ the product in $\mathcal{V}$-**CAT**. For each $i \in I$ consider a sketch $\mathcal{S}_i = (\mathcal{B}_i, \mathbb{L}_i, \mathbb{C}_i)$ for which $\mathcal{A}_i \simeq \mathrm{Mod}(\mathcal{S}_i)$. Then we can see $\mathcal{A}$ as a full subcategory of

$$\prod_{i\in I}[\mathcal{B}_i, \mathcal{V}] \cong [\mathcal{B}, \mathcal{V}]$$

with $\mathcal{B} := \coprod_{i\in I} \mathcal{B}_i$ in $\mathcal{V}$-**CAT**. Now, for each $i \in I$, we can consider the coproduct inclusion $J_i\colon \mathcal{B}_i \to \mathcal{B}$ and define the set of cylinders

$$\mathbb{L} := \{J_i\eta \mid i \in I, \eta \in \mathbb{L}_i\}$$

and of cocylinders

$$\mathbb{C} := \{J_i\mu \mid i \in I, \eta \in \mathbb{C}_i\},$$

which identify a sketch $\mathcal{S} = (\mathcal{B}, \mathbb{L}, \mathbb{C})$. It follows then by construction that $F\colon \mathcal{B} \to \mathcal{V}$ lies in $\mathcal{A}$ if and only if its components F_i are models of $\mathcal{S}_i$ for each $i \in I$, if and only if F is a model of $\mathcal{S}$; thus $\mathcal{A} \simeq \mathrm{Mod}(\mathcal{S})$ is accessible. Moreover considering α such that each $\mathcal{A}_i$ is α-accessible, an easy calculation shows then that $\mathcal{A}$ has all α-flat colimits and these are preserved by the projections; it now easily follows that $\mathcal{B}$ is the product of $(\mathcal{A}_i)_{i\in I}$ in $\mathcal{V}$-**Acc**.

(c) *Inserters.* Let $F, G\colon \mathcal{A} \to \mathcal{K}$ be a parallel pair in $\mathcal{V}$-**Acc**; then their equifier $\mathcal{B}$ can be seen as the pullback

$$\begin{tikzcd}
\mathcal{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A} \arrow[d, "{(F,G)}"] \\
\mathcal{K}^{\mathbb{2}} \arrow[r, "\pi"'] & \mathcal{K} \times \mathcal{K}
\end{tikzcd}$$

where π is the projection induced by the inclusion $2 \to \mathbb{2}$. All the $\mathcal{V}$-categories involved are accessible (by the corollary above), as well as the $\mathcal{V}$-functors, and π is an isofibration. So the result follows from point (a).

(d) *Equifiers.* Let $\mu, \eta\colon F \Rightarrow G\colon \mathcal{A} \to \mathcal{K}$ be a parallel pair of 2-cells in $\mathcal{V}$-**Acc**. Arguing as above we can write their equifier $\mathcal{B}$ as the pullback

$$\begin{tikzcd}
\mathcal{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A} \arrow[d, "{(\bar\mu,\bar\eta)}"] \\
\mathcal{K}^{\mathbb{2}} \arrow[r, "\rho"'] & \mathcal{K}^{\mathcal{P}}
\end{tikzcd}$$

where $\mathcal{P}$ is the free living parallel pair and ρ is the diagonal induced by the projection $\mathcal{P} \to \mathbb{2}$. Again this exists in $\mathcal{V}$-**Acc** by point (a) since ρ is an isofibration and all the involved $\mathcal{V}$-categories and $\mathcal{V}$-functors are accessible.

The fact that $U\colon \mathcal{V}\text{-}\mathbf{Acc} \to \mathcal{V}\text{-}\mathbf{CAT}$ preserves the these limits is a direct consequence of the proof (since we took the limits in $\mathcal{V}\text{-}\mathbf{CAT}$ and then proved that they are still limits in $\mathcal{V}\text{-}\mathbf{Acc}$). Regarding the underlying functor $(-)_0\colon \mathcal{V}\text{-}\mathbf{Acc} \to \mathbf{Acc}$, this preserves all the limits in question since the forgetful functors $\mathcal{V}\text{-}\mathbf{CAT} \to \mathbf{CAT}$ and $\mathbf{Acc} \to \mathbf{CAT}$ do. $\square$

Remark 2.3.6. In the result above we show that, in addition to flexible limits, $\mathcal{V}\text{-}\mathbf{Acc}$ has pullbacks along isofibrations; this should not be too surprising in light of [20, Proposition A.1].

Similarly one obtains the same result for accessible $\mathcal{V}$-categories with limits of some class Ψ. Let $\mathcal{V}\text{-}\mathbf{Acc}_\Psi$ be the 2-category of the accessible $\mathcal{V}$-categories with Ψ-limits, Ψ-continuous and accessible $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations.

Corollary 2.3.7. *The 2-category $\mathcal{V}\text{-}\mathbf{Acc}_\Psi$ has all flexible (and hence all pseudo- and bi-) limits and the forgetful functor $\mathcal{V}\text{-}\mathbf{Acc}_\Psi \to \mathcal{V}\text{-}\mathbf{Acc}$ preserves them.*

Proof. This is a direct consequence of the results above since the following

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Acc}_\Psi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V}\text{-}\mathbf{CAT}_\Psi \arrow[d] \\
\mathcal{V}\text{-}\mathbf{Acc} \arrow[r] & \mathcal{V}\text{-}\mathbf{CAT}
\end{tikzcd}$$

is a pullback, where $\mathcal{V}\text{-}\mathbf{CAT}_\Psi$ is the 2-category of Ψ-complete $\mathcal{V}$-categories, Ψ-continuous $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations; this has all flexible limits and the forgetful functor $\mathcal{V}\text{-}\mathbf{CAT}_\Psi \to \mathcal{V}\text{-}\mathbf{CAT}$ preserves them (see [14]). $\square$

In particular, taking Ψ to be the class of all small limits, $\mathcal{V}\text{-}\mathbf{Acc}_\Psi$ can be identified with the 2-category $\mathcal{V}\text{-}\mathbf{Lp}$ of locally presentable $\mathcal{V}$-categories, accessible $\mathcal{V}$-functors with a left adjoint, and $\mathcal{V}$-natural transformations. Thus we recover part of [14, Theorem 6.10]:

Corollary 2.3.8. *The 2-category $\mathcal{V}\text{-}\mathbf{Lp}$ has all flexible (and hence all pseudo- and bi-) limits and the forgetful functor $\mathcal{V}\text{-}\mathbf{Lp} \to \mathcal{V}\text{-}\mathbf{CAT}$ preserves them.*

2.3.2 The conically accessible case

Analogous results hold for conical accessibility. Denote by $\mathcal{V}\text{-}\mathbf{cAcc}$ the 2-category of conically accessible $\mathcal{V}$-categories, conically accessible $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. Note that $\mathcal{V}\text{-}\mathbf{Acc}$ sits in $\mathcal{V}\text{-}\mathbf{cAcc}$ as a full subcategory by Corollary 2.1.20 and Proposition 2.1.23.

In the proof of the following theorem we rely on the facts that $\mathbf{Acc}$ has all flexible limits (which can be seen of course as a consequence of the previous theorem for $\mathcal{V} = \mathbf{Set}$) and that $(-)_0\colon \mathcal{V}\text{-}\mathbf{Acc} \to \mathbf{Acc}$ preserves them.

Theorem 2.3.9. *The 2-category $\mathcal{V}\text{-}\mathbf{cAcc}$ has all flexible (and hence all pseudo- and bi-) limits and the forgetful functors $U_c\colon \mathcal{V}\text{-}\mathbf{cAcc} \to \mathcal{V}\text{-}\mathbf{CAT}$, $(-)_0\colon \mathcal{V}\text{-}\mathbf{cAcc} \to \mathbf{Acc}$, and $J\colon \mathcal{V}\text{-}\mathbf{Acc} \hookrightarrow \mathcal{V}\text{-}\mathbf{cAcc}$ preserve them.*

Proof. (Powers) Let $\mathcal{A}$ be a conically accessible $\mathcal{V}$-category and $\mathcal{C}$ be an ordinary small category. Consider the power $\mathcal{A}^{\mathcal{C}} = [\mathcal{C}_{\mathcal{V}}, \mathcal{A}]$ in $\mathcal{V}\text{-}\mathbf{CAT}$. Let α be such that $\mathcal{A}$ has conical α-filtered colimits; then $\mathcal{A}^{\mathcal{C}}$ has them as well since they can be computed pointwise in $\mathcal{A}$.

Moreover, given $X \in \mathcal{A}^{\mathcal{C}}$, we can consider β such that each $X(c)$ is conically β-presentable in $\mathcal{A}$ and the end

$$\mathcal{A}^{\mathcal{C}}(X,-) \cong \int_{c\in\mathcal{C}} \mathcal{A}(X(c), \mathrm{ev}_c -)$$

is β-small, so that $\mathcal{A}^{\mathcal{C}}(X,-)$ preserves all β-filtered colimits. It follows then that each object of $\mathcal{A}^{\mathcal{C}}$ is conically presentable. Now $(\mathcal{A}^{\mathcal{C}})_0 \cong [\mathcal{C}, \mathcal{A}_0]$ is accessible since $\mathcal{A}_0$ is; thus the conical accessibility of $\mathcal{A}^{\mathcal{C}}$ follows from Proposition 2.1.17. The fact that $\mathcal{A}^{\mathcal{C}}$ is the desired power in $\mathcal{V}$-**cAcc** follows from the fact that a $\mathcal{V}$-functor $\mathcal{B} \to \mathcal{A}^{\mathcal{C}}$ is conically accessible if and only if its transpose ordinary functor $\mathcal{C} \to \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{B}, \mathcal{A})$ lands in $\mathcal{V}\text{-}\mathbf{cAcc}(\mathcal{B}, \mathcal{A})$.

(Pullbacks along isofibrations) Let $F\colon \mathcal{A} \to \mathcal{K}$ and $G\colon \mathcal{B} \to \mathcal{K}$ be conically accessible $\mathcal{V}$-functors between conically accessible $\mathcal{V}$-categories, and assume that F is an isofibration. Consider the pullback $\mathcal{C}$ of this in $\mathcal{V}$-**CAT**, with projections $P\colon \mathcal{C} \to \mathcal{A}$ and $Q\colon \mathcal{C} \to \mathcal{B}$, and note that, since F is an isofibration, this can be seen as a pseudo-pullback. Let α be such that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{K}$ have, and F,G preserve, all conical α-filtered colimits; then it is a standard argument to check that $\mathcal{C}$ has conical α-filtered colimits as well and P and Q preserve them. Similarly each object of $\mathcal{C}$ is conically presentable (here we use that each object of $\mathcal{A}$,$\mathcal{B}$, and $\mathcal{K}$ is). Moreover $\mathcal{C}_0$ is accessible being a pseudo-pullback of ordinary accessible categories. In conclusion $\mathcal{C}$ is conically accessible by Proposition 2.1.17, and it is now routine to check that it is actually a pseudo-pullback in $\mathcal{V}$-**Acc**.

The proof for products is very similar. Let $(\mathcal{A}_i)_{i\in I}$ be a small family of accessible $\mathcal{V}$-categories and denote by $\mathcal{A} = \prod_i \mathcal{A}_i$ their product in $\mathcal{V}$-**CAT**. Consider α such that each $\mathcal{A}_i$ has all conical α-filtered colimits, an easy calculation shows then that $\mathcal{A}$ has all conical α-filtered colimits as well and these are preserved by the projections. Moreover, given any $A = (A_i)_i \in \mathcal{A}$, consider β such that I is β-small and each A_i is conically β-presentable in $\mathcal{A}_i$; then A is conically β-presentable in $\mathcal{A}$. Finally $\mathcal{A}$ is conically accessible thanks to Proposition 2.1.17 since $\mathcal{A}_0 \cong \prod_i (\mathcal{A}_i)_0$ is an ordinary accessible category. It now easily follows that $\mathcal{A}$ is the product of $(\mathcal{A}_i)_{i\in I}$ in $\mathcal{V}$-**cAcc**.

Inserters and equifiers can be obtained from powers and pullbacks along isofibrations (see the proof of the previous theorem). Finally, splittings of idempotent equivalences come again for free thanks to [13, Remark 7.6]. $\square$

We can consider also the conical version of Corollary 2.3.7. Let $\mathcal{V}\text{-}\mathbf{cAcc}_\Psi$ be the 2-category of the conically accessible $\mathcal{V}$-categories with Ψ-limits, Ψ-continuous and conically accessible $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. The same argument then gives:

Corollary 2.3.10. *The 2-category $\mathcal{V}\text{-}\mathbf{cAcc}_\Psi$ has all flexible (and hence all pseudo- and bi-) limits and the forgetful functor $\mathcal{V}\text{-}\mathbf{cAcc}_\Psi \to \mathcal{V}\text{-}\mathbf{cAcc}$ preserves them.*

2.3.3 Sketches over a base

Definition 2.3.11. Let $\mathcal{S} = (\mathcal{B}, \mathbb{L}, \mathbb{C})$ be a sketch and $\mathcal{A}$ be a $\mathcal{V}$-category. A *model of a $\mathcal{S}$ in $\mathcal{A}$* is a $\mathcal{V}$-functor $F\colon \mathcal{B} \to \mathcal{A}$ which transforms each cylinder of $\mathbb{L}$ into a limit cylinder in $\mathcal{A}$, and each cocylinder of $\mathbb{C}$ into a colimit cocylinder in $\mathcal{A}$. We denote by $\mathrm{Mod}(\mathcal{S}, \mathcal{A})$ the full subcategory of $[\mathcal{B}, \mathcal{A}]$ spanned by the models of $\mathcal{S}$ in $\mathcal{A}$.

The accessibility of $\mathrm{Mod}(\mathcal{S}, \mathcal{A})$ when $\mathcal{A}$ is an accessible category depends on set theory in general (even in the ordinary case, see [1, Example A.19]), but when $\mathcal{A}$ has enough colimits something can be said, generalizing [1, Theorem 2.60].

Proposition 2.3.12. *Let $\mathcal{S} = (\mathcal{B}, \mathbb{L}, \mathbb{C})$ be a sketch and $\mathcal{A}$ be an accessible $\mathcal{V}$-category with M-colimits for any weight M appearing in $\mathbb{C}$. Then* $\mathrm{Mod}(\mathcal{S}, \mathcal{A})$ *is an accessible $\mathcal{V}$-category.*

Proof. Let $\mathcal{S}_{\mathbb{L}} = (\mathcal{B}, \mathbb{L})$ and $\mathcal{S}_{\mathbb{C}} = (\mathcal{B}, \mathbb{C})$ be the limit and colimit parts of $\mathcal{S}$. Then $\mathrm{Mod}(\mathcal{S}, \mathcal{A}) = \mathrm{Mod}(\mathcal{S}_{\mathbb{L}}, \mathcal{A}) \cap \mathrm{Mod}(\mathcal{S}_{\mathbb{C}}, \mathcal{A})$ and it is enough to prove that these two $\mathcal{V}$-categories are accessible.

Regarding the limit case, let $\mathcal{C} = \mathcal{A}_{\alpha}^{op}$, with inclusion $J \colon \mathcal{C} \to \mathcal{A}$, and consider the sketch $\mathcal{S}' = (\mathcal{C} \otimes \mathcal{B}, \mathbb{L}_{\mathcal{C}})$ defined as in the proof of Proposition 2.3.1 starting from $\mathcal{S}_{\mathbb{L}}$. Then $\mathrm{Mod}(\mathcal{S}_{\mathbb{L}}, \mathcal{A})$ can be seen as the intersection

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{S}_{\mathbb{L}}, \mathcal{A}) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mod}(\mathcal{S}') \arrow[d, hook] \\
{[\mathcal{B}, \mathcal{A}]} \arrow[r, hook, "K"'] & {[\mathcal{C} \otimes \mathcal{B}, \mathcal{A}]}
\end{tikzcd}$$

where K is the composite of the inclusion $[\mathcal{B}, \mathcal{A}(J,1)] \colon [\mathcal{B}, \mathcal{A}] \to [\mathcal{B}, [\mathcal{C}, \mathcal{A}]]$ with the isomorphism $[\mathcal{B}, [\mathcal{C}, \mathcal{A}]] \cong [\mathcal{C} \otimes \mathcal{B}, \mathcal{A}]$. Since $[\mathcal{B}, \mathcal{A}]$ and $\mathrm{Mod}(\mathcal{S}')$ are accessible and accessibly embedded in $[\mathcal{C} \otimes \mathcal{B}, \mathcal{A}]$ it follows that $\mathrm{Mod}(\mathcal{S}_{\mathbb{L}}, \mathcal{A})$ is accessible as well.

About the colimit case, for each cocylinder $(\eta : M \Rightarrow \mathcal{B}(H-, B)) \in \mathbb{C}$ consider the $\mathcal{V}$-functor $G_{\eta} \colon [\mathcal{B}, \mathcal{A}] \to \mathcal{A}^{\mathbb{2}}$ such that $G_{\eta}(F) \colon M * FH \to FB$ is the unique morphism induced by the cocylinder $F\eta$, and acts on hom-objects accordingly (to define G_{η} we are using that $\mathcal{A}$ has M-colimits). Now consider the full subcategory $\mathcal{A}^{\cong}$ of $\mathcal{A}^{\mathbb{2}}$ spanned by the isomorphisms of $\mathcal{A}$, this is accessible since it is equivalent to $\mathcal{A}$. Then we can see $\mathrm{Mod}(\mathcal{S}_{\mathbb{C}}, \mathcal{A})$ as the pullback below

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{S}_{\mathbb{C}}, \mathcal{A}) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_{\eta \in \mathbb{C}} \mathcal{A}^{\cong} \arrow[d, hook] \\
{[\mathcal{B}, \mathcal{A}]} \arrow[r, "(G_{\eta})_{\eta \in \mathbb{C}}"'] & \prod_{\eta \in \mathbb{C}} \mathcal{A}^{\mathbb{2}}
\end{tikzcd}$$

where the right vertical arrow is an isofibration. The three $\mathcal{V}$-categories involved in the limit are accessible by Theorem 2.3.5 and the $\mathcal{V}$-functors are easily seen to be accessible; thus $\mathrm{Mod}(\mathcal{S}_{\mathbb{C}}, \mathcal{A})$ is an accessible $\mathcal{V}$-category as well again by Theorem 2.3.5. □

Immediate consequences are:

Corollary 2.3.13. *For any accessible $\mathcal{V}$-category $\mathcal{A}$ and any limit sketch $\mathcal{S}$ the $\mathcal{V}$-category* $\mathrm{Mod}(\mathcal{S}, \mathcal{A})$ *is accessible.*

Corollary 2.3.14. *For any locally presentable $\mathcal{V}$-category $\mathcal{K}$ and any sketch $\mathcal{S}$ the $\mathcal{V}$-category* $\mathrm{Mod}(\mathcal{S}, \mathcal{K})$ *is accessible.*

As in the accessible case we can consider models of sketches over a conically accessible $\mathcal{V}$-category:

Proposition 2.3.15. *Let $\mathcal{S} = (\mathcal{B}, \mathbb{L}, \mathbb{C})$ be a sketch and $\mathcal{A}$ be a conically accessible $\mathcal{V}$-category with M-colimits for any weight M appearing in $\mathbb{C}$. Then* $\mathrm{Mod}(\mathcal{S}, \mathcal{A})$ *is a conically accessible $\mathcal{V}$-category.*

Proof. Same as that of Proposition 2.3.12. □

Corollary 2.3.16. *For any conically accessible $\mathcal{V}$-category $\mathcal{A}$ and any limit sketch $\mathcal{S}$ the $\mathcal{V}$-category* $\mathrm{Mod}(\mathcal{S}, \mathcal{A})$ *is conically accessible.*

CHAPTER

3

Flat vs. filtered colimits

The idea of flatness comes from homological algebra, but has since been incorporated into category theory in many contexts [15, 28, 29, 81], perhaps most importantly in the theory of accessible categories [1, 67, 77].

In Chapter 2 we have introduced the notions of accessible and conically accessible $\mathcal{V}$-category; the former is based on colimits weighed by flat $\mathcal{V}$-functors, while the latter on the more standard filtered colimits. The aim of this chapter will be to give an explicit description of flat-weighted colimits by using filtered colimits, for some specified bases of enrichment. We then use this to characterize accessible $\mathcal{V}$-categories in terms of the conically accessible ones in many cases.

This is a 40-year-old problem: in [56, Section 6.4], Kelly poses the question of whether, for any locally finitely presentable base, every flat presheaf is a filtered colimit of representables, and states his inability to prove this. As observed in [18] and above, this is actually false for the case $\mathcal{V} = \mathbf{Ab}$; but solving it in full generality is probably out of reach at this stage. The situation is analogous to the related hard problem of describing the absolute colimits over a given base: see [70, 79, 80, 95] for various instances of this.

Given any complete and cocomplete $\mathcal{V}$ and any small $\mathcal{V}$-category $\mathcal{C}$, we can form the underlying ordinary category $[\mathcal{C}^{op}, \mathcal{V}]_0$ of the presheaf $\mathcal{V}$-category: this consists of the $\mathcal{V}$-enriched presheaves and $\mathcal{V}$-enriched natural transformations. We can also form the presheaf category $[\mathcal{C}_0^{op}, \mathbf{Set}]$ on the underlying ordinary category. There is an adjunction

$$\begin{tikzcd} {[\mathcal{C}^{op}, \mathcal{V}]_0} \arrow[r, "\mathfrak{U}"'] \arrow[r, phantom, "\perp"] & {[\mathcal{C}_0^{op}, \mathbf{Set}]} \arrow[l, bend right, "\mathfrak{F}"'] \end{tikzcd}$$

between these, induced by the underlying functor $Y_0 \colon \mathcal{C}_0 \to [\mathcal{C}^{op}, \mathcal{V}]_0$ of the enriched Yoneda embedding: here $\mathfrak{U}$ sends $M \colon \mathcal{C}^{op} \to \mathcal{V}$ to $[\mathcal{C}^{op}, \mathcal{V}]_0(Y_0-, M) \cong \mathcal{V}_0(I, M_0-)$ and $\mathfrak{F}$ sends

N to the colimit $N * Y_0$ of Y_0 weighted by N (this can also be seen as the colimit of $\mathrm{El}(N) \xrightarrow{\pi} \mathcal{C}_0 \xrightarrow{Y_0} [\mathcal{C}^{op}, \mathcal{V}]_0$).

It is true in general that $\mathfrak{F}$ sends a flat (in the ordinary sense) presheaf on $\mathcal{C}_0$ to a flat (in the enriched sense) presheaf on $\mathcal{C}$, essentially because filtered colimits of flat presheaves are flat. It is not necessarily true that $\mathfrak{U}$ preserves flatness, but it is so in many examples, as we shall see. Our basic strategy will be to show in particular cases, sometimes under further assumptions on $\mathcal{C}$, that

(I) $\mathfrak{U}$ does preserve flatness;

(II) if M is a flat presheaf on $\mathcal{C}$, then the component $\epsilon_M \colon \mathfrak{F}\mathfrak{U}M \to M$ of the counit is invertible.

Given (II), any flat presheaf M on $\mathcal{C}$ is an $\mathfrak{F}\mathfrak{U}M$-weighted colimit of representables. Given (I), this colimit can be calculated as a filtered colimit. Combining these, it follows that the flat presheaves on $\mathcal{C}$ are the closure of the representables under filtered colimits.

In Section 3.1, we give conditions on $\mathcal{V}$ under which this is the case for *all* small $\mathcal{C}$, and deduce that for such $\mathcal{V}$, the existence and preservation of flat weighted colimits is equivalent to that of filtered colimits (Theorem 3.1.13). As a consequence the notions of α-accessibility and conical α-accessibility agree (Theorem 3.1.14). Examples of $\mathcal{V}$ satisfying these conditions include the cartesian closed categories **Set**, **Cat**, **SSet**, **Pos** (of partially ordered sets), and many others.

In Section 3.2, we give conditions on $\mathcal{V}$ under which (I) and (II) hold provided that $\mathcal{C}$ has certain $\mathcal{V}$-enriched absolute colimits (Proposition 3.2.14). It then follows easily (Theorem 3.2.19) that a $\mathcal{V}$-category is α-accessible if and only if it is conically α-accessible and has these absolute colimits. (An α-accessible category always has absolute colimits.) For the $\mathcal{V}$ which we study in Section 3.2, the absolute colimits in question are finite direct sums (biproducts) and copowers by dualizable objects. Examples of $\mathcal{V}$ satisfying these conditions include the monoidal categories **CMon** of commutative monoids, **Ab** of abelian groups, R-**Mod** of R-modules for a commutative ring R, and **GAb** of graded abelian groups.

In Section 3.3, we investigate the case where $\mathcal{V}$ is the cartesian closed category $\mathbf{Set}^G$ of G-sets, for a non-trivial finite group G, and show that in this case α-accessibility is strictly stronger than conical α-accessibility and the existence of absolute colimits (Corollary 3.3.4).

In Section 3.4, we investigate one further class of examples related to those in Section 3.2, and including for example $\mathcal{V} = \mathbf{DGAb}$. In this case (II) still holds when $\mathcal{C}$ has some finite direct sums and copowers by dualizable objects, while (I) does not seem to be true even with this further assumption. What is true is that, when $\mathcal{C}$ has those absolute colimits and M is flat, the ordinary functor $\mathfrak{U}M$ will be part of what we call a protofiltered diagram. Then we prove that flat colimits are generated by the absolute ones plus these protofiltered colimits (Theorem 3.4.16).

The content of this chapter has been published in
Advances in Mathematics [65].

3.1 When flat equals filtered

In this section we give sufficient conditions on the base $\mathcal{V}$ for α-flat colimits to reduce to the usual α-filtered ones. These conditions hold in many of the most important examples of locally presentable bases of enrichment, as Example 3.1.3 below illustrates. For most but not all of the bases which appear in Example 3.1.3, the canonical functor $\mathcal{V}_0(I, -): \mathcal{V}_0 \to$ **Set** is cocontinuous and strong monoidal, which in turn easily implies our conditions. One example for which this is not the case, but for which our sufficient conditions still hold, is the category of pointed sets, equipped as usual with the smash product.

Throughout this section we assume that $\mathcal{V}$ is locally α-presentable as a closed category and that the unit I satisfies the following conditions:

(a) $\mathcal{V}_0(I, -): \mathcal{V}_0 \to \mathbf{Set}$ is *weakly cocontinuous*: for any diagram $H: \mathcal{C} \to \mathcal{V}_0$ the comparison map $\operatorname{colim} \mathcal{V}_0(I, H-) \twoheadrightarrow \mathcal{V}_0(I, \operatorname{colim} H)$ is a surjection.

(b) $\mathcal{V}_0(I, -): \mathcal{V}_0 \to \mathbf{Set}$ is *weakly strong monoidal*: the comparison maps for the tensor product are surjections; in other words for each $X, Y \in \mathcal{V}_0$ the function $\mathcal{V}_0(I, X) \times \mathcal{V}_0(I, Y) \to \mathcal{V}_0(I, X \otimes Y)$, sending (x, y) to $x \otimes y$, is a surjection.

Remark 3.1.1. Note that condition (a) is equivalent to the fact that $\mathcal{V}_0(I, -)$ preserves cocones which are jointly a regular epimorphism; or even to the request that I is regular projective and $\mathcal{V}_0(I, -)$ weakly preserve coproducts. Therefore (a) is certainly satisfied whenever $\mathcal{V}_0(I, -)$ preserves coproducts and regular epimorphisms, and in particular when it is cocontinuous (most of our examples). Condition (b) comes for free when $\mathcal{V}_0(I, -)$ is strong monoidal (most of our examples), and in particular when $\mathcal{V}_0$ is endowed with the cartesian closed structure.

Remark 3.1.2. In condition (b) it might be natural to ask for the map $1_I: 1 \to \mathcal{V}_0(I, I)$ to be an epimorphism as well (and hence bijective), but it is not required for the results below. Nonetheless, this condition is satisfied for almost all our examples.

Examples 3.1.3. Here is a list of examples of such bases of enrichment. In the following group each base $\mathcal{V}$ is endowed with the cartesian structure and $\mathcal{V}_0(I, -)$ is cocontinuous:

1. $(\mathbf{Set}, \times, 1)$ for ordinary categories.
2. $(\mathbf{Cat}, \times, 1)$ for 2-categories.
3. $(\mathbf{SSet}, \times, 1)$ for simplicial categories.
4. $(\mathbb{2}, \times, 1)$ for posets.
5. The categories **Gpd** of groupoids, **Ord** of total orders, **Pos** of posets, and **rGra** of reflexive graphs with their cartesian closed structures.
6. Any presheaf category $([\mathcal{C}^{op}, \mathbf{Set}], \times, 1)$ for which $\mathcal{C}$ has a terminal object: the unit 1 is representable in $[\mathcal{C}^{op}, \mathbf{Set}]$ and hence homming out of it preserves all colimits.
7. $(2\text{-}\mathbf{Cat}_Q, \times, 1)$ the cartesian closed category of algebraically cofibrant 2-categories, see [27]. This base is locally presentable and the forgetful $V: 2\text{-}\mathbf{Cat}_Q \to 2\text{-}\mathbf{Cat}$ is a faithful left adjoint and is full on morphisms out of the terminal object, so that $2\text{-}\mathbf{Cat}_Q(1, -) \cong 2\text{-}\mathbf{Cat}(1, V-)$ is cocontinuous.

The following are examples of bases for which $\mathcal{V}_0(I, -)$ is cocontinuous and strong monoidal but the monoidal structure is not (necessarily) cartesian:

8. $(\mathcal{V}\text{-}\mathbf{Cat}, \otimes, \mathcal{I})$ with the tensor product inherited from $\mathcal{V}$, whenever $\mathcal{V}_0$ is locally presentable: $\mathcal{V}\text{-}\mathbf{Cat}$ is locally presentable by [58], and $\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{I}, -) \cong \mathrm{Ob}(-)$ is the functor that takes the underlying objects of a category; therefore it is cocontinuous and strong monoidal $(\mathrm{Ob}(\mathcal{A} \otimes \mathcal{B}) = \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{B}))$.

9. $(2\text{-}\mathbf{Cat}, \square, 1)$ with the "funny tensor product" [96, Section 2]: same reasons as above.
10. $(2\text{-}\mathbf{Cat}, \boxtimes, 1)$ with the pseudo Gray tensor product: same reasons as above (see [96, Section 6]).
11. $(\mathbf{Met}, \otimes, 1)$ of Lawvere metric spaces (see [3] and [72]).

In the next example the monoidal structure is not cartesian, the unit is not the terminal object, and $\mathcal{V}_0(I, -)$ is only weakly cocontinuous and weakly strong monoidal:

12. $(\mathbf{Set}_*, \wedge, I)$ the category of pointed sets endowed with the smash product. This is locally presentable being the co-slice $1/\mathbf{Set}$. Since the unit is given by the pointed set $I = (\{0, 1\}, 0)$, it follows that the functor $U := \mathbf{Set}_*(I, -) : \mathbf{Set}_* \to \mathbf{Set}$ is just the underlying set functor. Note now that the tensor product $(A, a) \wedge (B, b)$ is defined as a quotient of the set $A \times B$; thus U is weakly strong monoidal but not strong monoidal, nor does it preserve the unit. Moreover, it is easy to see that epimorphisms in $\mathbf{Set}_*$ are just surjections, and that the coproduct of a family $(A_i, a_i)_{i \in I}$ is the quotient of $\sum_{i \in I} A_i$ obtained identifying all he a_i's. As a consequence the functor U preserves all regular epimorphisms and weakly preserves all coproducts, but does not preserve coproducts in the usual sense.

Remark 3.1.4. The last example can be generalized as follows. Assume that $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ satisfies the conditions (a) and (b) above. Then $1/\mathcal{V}_0$ is still locally presentable and symmetric monoidal closed with monoidal structure given by the smash product $\wedge$ induced by the tensor product on $\mathcal{V}$ (see for example [40, Lemma 4.20]); moreover the forgetful functor $U : 1/\mathcal{V}_0 \to \mathcal{V}_0$ is monoidal and has a strong monoidal left adjoint F. Note now that, since the unit of $1/\mathcal{V}$ is FI, the functor $(1/\mathcal{V}_0)(FI, -) \cong \mathcal{V}_0(I, U-)$ weakly preserves the same colimits that both U and $\mathcal{V}_0(I, -)$ weakly preserves. Since the comparison map $UX \otimes UY \to U(X \wedge Y)$ is a regular epimorphism in $\mathcal{V}_0$ (by construction) and U is weakly cocontinuous, it follows that $(1/\mathcal{V}_0)(FI, -)$ is weakly cocontinuous and weakly strong monoidal as well. In conclusion $1/\mathcal{V} = (1/\mathcal{V}_0, \wedge, FI)$ still satisfies the conditions (a) and (b).

The following is an easy consequence of the two conditions above, but is also the foundation of the results of this section.

Lemma 3.1.5. *Let $M : \mathcal{C}^{op} \to \mathcal{V}$ and $H : \mathcal{C} \to \mathcal{V}$ be two $\mathcal{V}$-functors. Then for each arrow $x : I \to M * H$ there exist $C \in \mathcal{C}$, $y : I \to MC$, and $z : I \to HC$ for which the triangle*

$$\begin{tikzcd} I \arrow[r, dashed, "y \otimes z"] \arrow[dr, "x"'] & MC \otimes HC \arrow[d, "\rho_C"] \\ & M * H \end{tikzcd}$$

commutes; where the vertical map is taken from the colimiting cocone defining $M * H$.

Proof. The counit of a colimit $M * H$ determines a family

$$(\rho_C : MC \otimes HC \longrightarrow M * H)_{C \in \mathcal{C}}$$

which is jointly a regular epimorphism in $\mathcal{V}_0$. Since $\mathcal{V}_0(I, -)$ preserves such families it follows that x factors through ρ_C via a map $h : I \to MC \otimes HC$, for some $C \in \mathcal{C}$. Finally, by condition (2) on the unit, $h = y \otimes z$ for some $y : I \to MC$ and $z : I \to HC$. The claim then follows. $\square$

Remark 3.1.6. Given a pair $M\colon \mathcal{C}^{op} \to \mathcal{V}$ and $H\colon \mathcal{C} \to \mathcal{V}$, we can consider the weighted colimit $M * H$ in $\mathcal{V}$ and the ordinary weighted colimit $M_I * H_I$ in **Set**. Note that there is always a comparison map

$$c\colon M_I * H_I \longrightarrow \mathcal{V}_0(I, M * H).$$

It is easy to see that if conditions (1) and (2) hold then the map c is a surjection (use the Lemma above); thus in a certain sense the functor $\mathcal{V}_0(I,-)$ weakly preserves weighted colimits — this can be made precise using change of base for composition of profunctors. When $\mathcal{V}_0(I,-)$ is moreover cocontinuous and strong monoidal the comparison map c is actually an isomorphism: $\mathcal{V}_0(I, M*H) \cong M_I*H_I$ (this is a general fact about cocontinuous functors and strong monoidal change of base).

The following is a generalization of [57, Proposition 6.6] to our context; the proof is very similar to that and is based on the lemma above.

Corollary 3.1.7. *Let $M\colon \mathcal{C}^{op} \to \mathcal{V}$ be a weight for which $M * -\colon [\mathcal{C},\mathcal{V}] \to \mathcal{V}$ preserves α-small conical limits of representables and let $M_I := \mathcal{V}_0(I, M_0-)\colon \mathcal{C}_0^{op} \to$ **Set**; then the ordinary category $\mathrm{El}(M_I)$ is α-filtered, and so M_I is α-flat.*

Proof. Let $\mathcal{B}$ be an α-small category and $H\colon \mathcal{B} \to \mathrm{El}(M_I)$ be a functor; we need to prove that H has a cocone in $\mathrm{El}(M_I)$. Denote by $\pi\colon \mathrm{El}(M_I) \to \mathcal{C}_0$ the projection; then H induces a cone $(I \xrightarrow{Hb} M(\pi Hb))_{b\in\mathcal{B}}$, which is the same as an arrow

$$x\colon I \longrightarrow \lim_{b\in\mathcal{B}} M(\pi Hb).$$

Now, since $M * -$ preserves α-small limits of representables, we obtain the following isomorphisms

$$\lim_{b\in\mathcal{B}} M(\pi Hb) \cong \lim_{b\in\mathcal{B}} (M * \mathcal{C}(\pi Hb, -)) \cong M * (\lim_{b\in\mathcal{B}} \mathcal{C}(\pi Hb, -)).$$

Therefore, by the previous Lemma, there exist $C \in \mathcal{C}$, $y\colon I \to MC$, and $z\colon I \to \lim \mathcal{C}(\pi H-, C)$ which map down to x when taking the colimit. Note now that to give z is the same as to give a cone $\Delta I \to \mathcal{C}(\pi H-, C)$, which then corresponds to a cocone $(\eta_b\colon \pi Hb \to C)_{b\in\mathcal{B}}$ in $\mathcal{C}$. Finally the fact that $y \otimes z$ gets mapped down to x means that η is actually a cocone $(\eta_b\colon Hb \to (C, y))_{b\in\mathcal{B}}$ in $\mathrm{El}(M_I)$. ☐

This shows that in the current setting condition (I) from the Introduction holds. Next we turn to (II) — such an M is actually an α-filtered colimit of representables — but for that we need some work.

Definition 3.1.8. Let $h\colon Y \to X$ be a morphism in $\mathcal{V}$. Denote by $\mathbb{2}_Y$ the $\mathcal{V}$-category with two objects $*_1$ and $*_2$, and with hom-objects $\mathbb{2}_Y(*_i, *_i) = I$, $\mathbb{2}_Y(*_2, *_1) = 0$, and $\mathbb{2}_Y(*_1, *_2) = Y$. Let $N_h\colon \mathbb{2}_Y \to \mathcal{V}$ be the weight for which $N_h(*_1) = I$, $N_h(*_2) = X$ and determined on the hom-objects by $(N_h)_{*_1,*_2} = h$.
When h is the co-diagonal $\nabla\colon X + X \to X$ we write N_X for N_∇.

Note that, when X and Y are α-presentable, the weight N_h is α-small.

Example 3.1.9. When $\mathcal{V} =$ **Cat** and $h\colon \mathbb{2}_2 \to \mathbb{2}$ is the projection from the free category on a parallel pair to $\mathbb{2}$, then limits weighted by N_h correspond to equifiers (see [61]).

Consider the case of N_X. To give a diagram $H\colon \mathbb{2}_{X+X} \to \mathcal{K}$ is the same as to give two objects D_1, D_2 and arrows $g_1, g_2\colon X \to \mathcal{K}(D_1, D_2)$. In that case, if $\mathcal{K}$ has enough limits, $\{N_X, H\}$ can be seen as the equalizer:

$$\{N_X, H\} \rightarrowtail D_1 \underset{g_2^t}{\overset{g_1^t}{\rightrightarrows}} X \pitchfork D_2$$

where g_1^t and g_2^t are the transposes of g_1 and g_2. When $\mathcal{K} = \mathcal{V}$, to give an arrow $I \to \{N_X, H\}$ is then equivalent to giving $x\colon I \to D_1$ for which the diagram

$$\begin{tikzcd} & {[D_1, D_2] \otimes D_1} \arrow[dr, "\mathrm{ev}"] & \\ X \arrow[ur, "g_1 \otimes x"] \arrow[dr, "g_2 \otimes x"'] & & D_2 \\ & {[D_1, D_2] \otimes D_1} \arrow[ur, "\mathrm{ev}"'] & \end{tikzcd}$$

commutes. We are now ready to prove the following:

Proposition 3.1.10. *Let $M\colon \mathcal{C}^{op} \to \mathcal{V}$ be a weight for which $M * -\colon [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ preserves α-small limits of representables. Then*

$$M \cong \operatorname{colim}\left(\mathrm{El}(M_I)_{\mathcal{V}} \xrightarrow{\pi_{\mathcal{V}}} \mathcal{C} \xrightarrow{Y} [\mathcal{C}^{op}, \mathcal{V}]\right)$$

and so M is an α-filtered colimit of representables.

Proof. The category $\mathrm{El}(M_I)$ is α-filtered by Corollary 3.1.7; thus we only need to prove that the canonical map $\operatorname{colim}(Y \circ \pi_{\mathcal{V}}) \to M$ is invertible. Since colimits are computed pointwise it is enough to show that the canonical map $c\colon \operatorname{colim} \mathcal{C}(C, \pi_{\mathcal{V}} -) \to M(C)$ is invertible for any $C \in \mathcal{C}$.

Consider now a strong generator $\mathcal{G}$ of $\mathcal{V}$ made of α-presentable objects; then the morphism c above is invertible if and only if $\mathcal{V}_0(X, c)$ is invertible for any $X \in \mathcal{G}$. Since every object of $\mathcal{G}$ is α-presentable and $\mathrm{El}(M_I)$ is α-filtered, the functor $\mathcal{V}_0(X, -)$ preserves the colimit $\operatorname{colim} \mathcal{C}(C, \pi_{\mathcal{V}} -)$. Therefore it suffices to show that the canonical map

$$c_X\colon \operatorname{colim}\left(\mathrm{El}(M_I) \xrightarrow{\pi} \mathcal{C}_0^{op} \xrightarrow{\mathcal{V}_0(X, \mathcal{C}(C,-)_0)} \mathbf{Set}\right) \longrightarrow \mathcal{V}_0(X, M(C))$$

is an isomorphism for any $X \in \mathcal{G}$. Below we are going to consider the elements of the colimit on the left as equivalence classes defined in the standard way.
Consider $f\colon X \to MC$; then, since X-powers are α-small limits, we have

$$\mathcal{V}_0(X, M(C)) \cong \mathcal{V}_0(I, [X, M(C)]) \cong \mathcal{V}_0(I, M * (X \pitchfork \mathcal{C}(C, -)));$$

by Lemma 3.1.5 we find $D \in \mathcal{C}$, $x\colon I \to MD$, and $g\colon X \to \mathcal{C}(C, D)$ such that f coincides with

$$X \xrightarrow{g \otimes x} \mathcal{C}(C, D) \otimes MD \xrightarrow{M \otimes id} [MD, MC] \otimes MD \xrightarrow{ev} MC.$$

In other words $c_X[g, x] = f$, so that c_X is surjective. To prove the injectivity of c_X we need the α-small weight N_X introduced in Definition 3.1.8. Consider $g_i\colon X \to \mathcal{C}(C, D_i)$ and $x_i\colon I \to MD_i$, for $i = 1, 2$, such that $c_X[g_1, x_1] = c_X[g_2, x_2]$; we need to prove that $[g_1, x_1] = [g_1, x_1]$. First, since $\mathrm{El}(M_I)$ is filtered we can assume that $D = D_1 = D_2$ and

$x = x_1 = x_2$. Now $g_1, g_2 \colon X \to \mathcal{C}(C,D)$ determine a diagram $H \colon \mathbb{2}_{X+X} \to \mathcal{C}$, and x corresponds to an arrow $\bar{x} \colon I \to \{N_X, MH\}$ (see comments above). Since $M * -$ preserves α-small limits of representables we obtain

$$\{N_X, MH^{op}\} \cong \{N_X, M * YH\} \cong M * \{N_X, YH\};$$

then, using Lemma 3.1.5 again, we find that $\bar{x}$ factors through $y \colon I \to ME$ and $h \in \mathcal{C}_0(D,E)$, for some $E \in \mathcal{C}$. This means that $M(h)y = x$ and $\mathcal{C}(C,h) \circ g_1 = \mathcal{C}(C,h) \circ g_2$. Thus $[g_1, x_1] = [g_1, x_1]$ as desired and c_X is an isomorphism. $\square$

Remark 3.1.11. When $\mathcal{V}_0(I,-)$ is cocontinuous and strong monoidal, as in most of the examples, the proof becomes simpler. Consider any α-presentable X in $\mathcal{V}_0$; then

$$\begin{aligned}
\mathcal{V}_0(X, M(C)) &\cong \mathcal{V}_0(I, [X, M * \mathcal{C}(C,-)]) \\
&\cong \mathcal{V}_0(I, M * (X \pitchfork \mathcal{C}(C,-))) \\
&\cong M_I * (X \pitchfork \mathcal{C}(C,-)_I) \\
&\cong M_I * \mathcal{V}_0(X, \mathcal{C}(C,-)_0) \\
&\cong \operatorname{colim}\left(\mathrm{El}(M_I) \xrightarrow{\pi} \mathcal{C}_0 \xrightarrow{\mathcal{C}(C,-)_0} \mathcal{V}_0 \xrightarrow{\mathcal{V}_0(X,-)} \mathbf{Set}\right) \\
&\cong \mathcal{V}_0\left(X, \operatorname{colim}\left(\mathrm{El}(M_I) \xrightarrow{M} \mathcal{C}_0^{op} \xrightarrow{\mathcal{C}(C,-)_0} \mathcal{V}_0\right)\right).
\end{aligned}$$

Since the α-presentable objects form a strongly generating family the result follows.

Recall that we assume $\mathcal{V}$ to satisfy the conditions (a) and (b) from the beginning of this section; in this context we can characterize the α-flat $\mathcal{V}$-functors as follows:

Proposition 3.1.12. *Let $M \colon \mathcal{C}^{op} \to \mathcal{V}$ be a weight; the following are equivalent:*

1. *M is α-flat;*
2. *$M * - \colon [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ preserves α-small limits of representables;*
3. *M is a (conical) α-filtered colimit of representables.*

Proof. (1) $\Rightarrow$ (2) is trivial, (2) $\Rightarrow$ (3) is a direct consequence of the previous Proposition, while (3) $\Rightarrow$ (1) follows from the fact that representable functors are α-flat and these are closed under α-filtered colimits in $[\mathcal{C}, \mathcal{V}]$. $\square$

And as a consequence:

Theorem 3.1.13. *A $\mathcal{V}$-category $\mathcal{A}$ has α-flat colimits if and only if it has α-filtered colimits. A $\mathcal{V}$-functor from such an $\mathcal{A}$ preserves α-flat colimits if and only if it preserves α-filtered colimits.*

Proof. One direction is trivial. Conversely assume that $\mathcal{A}$ has all α-filtered colimits and let $M \colon \mathcal{C}^{op} \to \mathcal{V}$ be an α-flat weight. By Proposition 3.1.12 we can write $M \cong \operatorname{colim}(YK)$ where $Y \colon \mathcal{C} \to [\mathcal{C}^{op}, \mathcal{V}]$ is Yoneda and $K \colon \mathcal{D}_V \to \mathcal{C}$ in indexed on an α-filtered category $\mathcal{D}$. Therefore, given any $H \colon \mathcal{C} \to \mathcal{A}$, we obtain a chain of isomorphisms (either side existing if the other does):

$$\begin{aligned} M * H &\cong (\mathrm{colim} YK) * H \\ &\cong \mathrm{colim}(YK * H) \\ &\cong \mathrm{colim}(HK). \end{aligned}$$

Thus the $M * H$ exists since $\mathcal{A}$ has α-filtered colimits. For the same reason a functor $F\colon \mathcal{A} \to \mathcal{K}$ preserves $M * H$ if and only if it preserves $\mathrm{colim}(HK)$. □

Theorem 3.1.14. *A $\mathcal{V}$-category $\mathcal{A}$ is α-accessible if and only if it is conically α-accessible.*

Proof. By the theorem above an object A of $\mathcal{A}$ is α-presentable if and only if is conically α-presentable, so that $\mathcal{A}_\alpha = \mathcal{A}^c_\alpha$. Arguing as above, for any α-flat $M\colon \mathcal{C}^{op} \to \mathcal{V}$ and a diagram $H\colon \mathcal{C} \to \mathcal{A}_\alpha \subseteq \mathcal{A}$, the colimit $M * H$ can be replaced by an α-filtered one $\mathrm{colim}(HK)$, which still lands in $\mathcal{A}_\alpha$. Thus an object is an α-flat colimit of α-presentables if and only if it is an α-filtered colimit of (conically) α-presentables. The result then follows. □

We considered a notion of enriched sketch in Section 2.2.1; there we proved that being accessible is equivalent to being the $\mathcal{V}$-category of models of a sketch (Theorem 2.2.4). Putting this together with our result above we obtain:

Theorem 3.1.15. *Let $\mathcal{A}$ be a $\mathcal{V}$-category; the following are equivalent:*

1. *$\mathcal{A}$ is accessible;*
2. *$\mathcal{A}$ is conically accessible;*
3. *$\mathcal{A}$ is equivalent to the $\mathcal{V}$-category of models of a sketch.*

Remark 3.1.16. The paper [20] contains some powerful techniques for proving that a wide range of 2-categories of categories with structure are conically accessible as **Cat**-enriched categories; the structures in question should contain "no equations between objects", and the morphisms are functors which preserve the structure up to coherent isomorphism. One typical example is the 2-category of monoidal categories, strong monoidal functors, and monoidal natural transformations; another is the 2-category of regular categories, regular functors, and natural transformations. In [22], it was shown how to adapt these techniques to the simplicially enriched case, and in particular to show that most of the key examples of ∞-cosmoi studied in [89] are conically accessible as **SSet**-enriched categories. By the results proved here, these conically accessible **Cat**-enriched or **SSet**-enriched categories are in fact accessible; thus they are also sketchable, and the whole theory of enriched accessible categories applies. This in turn allows us, for example, to consider models of the corresponding enriched sketches in other (suitable) enriched categories than $\mathcal{V}$, and deduce the accessibility of the resulting enriched categories of models.

Corollary 3.1.17. *A $\mathcal{V}$-category is Cauchy complete if and only if idempotents split.*

Proof. If $\mathcal{C}$ is Cauchy complete as a $\mathcal{V}$-category then it certainly has splittings of idempotents. Conversely, if $\mathcal{C}$ has splittings of idempotents then it has all those conical colimits indexed on ordinary absolute diagrams. Let $M\colon \mathcal{B}^{op} \to \mathcal{V}$ be a Cauchy weight; this means that M is α-flat for each α. By Proposition 3.1.10 the ordinary category $\mathrm{El}(M_I)$ is then α-filtered for each α, and hence absolute in the ordinary sense. Arguing as above, M-weighted colimits in $\mathcal{C}$ can be reduced to conical colimits indexed on $\mathrm{El}(M_I)$. It follows then that $\mathcal{C}$ has M-weighted colimits and therefore is Cauchy complete. □

Remark 3.1.18. When $\mathcal{V}_0(I,-)$ is strong monoidal, the result above is given by [79, Proposition 3.2].

3.1.1 The cartesian closed case

We now give a more explicit characterization of α-flat $\mathcal{V}$-functors in the case where $\mathcal{V}_0$ is endowed with the cartesian monoidal structure and satisfies condition (a) from before (condition (b) is automatic).

In this case, for each $X \in \mathcal{V}_0$ the functor $\mathcal{V}_0(X,-)$ is (strong) monoidal; therefore induces a change of base 2-functor $\mathrm{W}_X : \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}$. Hence for every weight $M\colon \mathcal{C}^{op} \to \mathcal{V}$ we obtain an ordinary functor

$$M_X\colon \mathrm{W}_X\mathcal{C}^{op} \xrightarrow{\mathrm{W}_X M} \mathrm{W}_X\mathcal{V} \xrightarrow{\mathrm{W}_X\mathcal{V}(I,-)} \mathbf{Set}$$

generalizing an earlier notation M_I for $\mathcal{V}_0(I, M_0-)\colon \mathcal{C}_0^{op} \to \mathbf{Set}$.

Remark 3.1.19. For each $X \in \mathcal{V}$ and $M\colon \mathcal{C}^{op} \to \mathcal{V}$ we consider the category of elements $\mathrm{El}(M_X)$; this can be described explicitly as follows:

- an object is a pair $(C \in \mathcal{C}, x\colon X \to MC)$;
- a morphism $f\colon (C,x) \to (D,y)$ is an arrow $f\colon X \to \mathcal{C}(C,D)$ for which the triangle

$$\begin{tikzcd} X \arrow[r, "{(f,y)}"] \arrow[dr, "x"'] & \mathcal{C}(C,D) \times M(D) \arrow[d, "\mathrm{ev}_M"] \\ & M(C) \end{tikzcd}$$

  commutes (remember that M is contravariant);
- for each (C,x) the identity is given by $\mathrm{id}_{(C,x)}\colon X \xrightarrow{!} 1 \xrightarrow{1_C} \mathcal{C}(C,C)$;
- composition is as follows: given $f\colon (C,x) \to (D,y)$ and $g\colon (D,y) \to (E,z)$ the composite $g \circ f$ is

$$X \xrightarrow{(g,f)} \mathcal{C}(D,E) \times \mathcal{C}(C,D) \xrightarrow{M} \mathcal{C}(C,E)$$

  where M is the composition map in $\mathcal{C}$.

Note that $\mathrm{El}(M_X)$ is not in general $\mathrm{El}(\mathcal{V}_0(X, M_0-))$: they have same objects but a morphism in $\mathrm{El}(\mathcal{V}_0(X, M_0-))$ from (C,x) to (D,y) is a morphism $1 \to \mathcal{C}(C,D)$ such that the induced $X \to 1 \to \mathcal{C}(C,D)$ defines a morphism in $\mathrm{El}(M_X)$. Finally observe that for each X there is an induced functor

$$J_X\colon \mathrm{El}(M_1) \longrightarrow \mathrm{El}(M_X)$$

which acts by precomposition with the unique morphism $!\colon X \to 1$.

Proposition 3.1.20. *Let $\mathcal{V} = (\mathcal{V}_0, \times, 1)$ be as above and $\mathcal{G} \subseteq \mathcal{V}_\alpha$ a strong generator. A $\mathcal{V}$-functor $M\colon \mathcal{C}^{op} \to \mathcal{V}$ is α-flat if and only if:*

1. $\mathrm{El}(M_1)$ *is α-filtered;*

2. *for each $X \in \mathcal{G}$ the functor $J_X \colon \mathrm{El}(M_1) \longrightarrow \mathrm{El}(M_X)$ is final.*

Proof. By Propositions 3.1.10 and 3.1.12 above, M is α-flat if and only if $\mathrm{El}(M_1)$ is α-filtered and for any $C \in \mathcal{C}$ the following isomorphism holds

$$M(C) \cong \operatorname{colim}\left(\mathrm{El}(M_1) \xrightarrow{\pi} \mathcal{C}_0 \xrightarrow{\mathcal{C}(C,-)_0} \mathcal{V}_0\right).$$

Equivalently, since $\mathcal{G}$ is a strong generator, M is α-flat if and only if $\mathrm{El}(M_1)$ is α-filtered and for any $C \in \mathcal{C}$ and $X \in \mathcal{G}$ we have

$$\mathcal{V}_0(X, M(C)) \cong \operatorname{colim}\left(\mathrm{El}(M_1) \xrightarrow{\pi} \mathcal{C}_0 \xrightarrow{\mathcal{V}_0(X,\mathcal{C}(C,-)_0)} \mathbf{Set}\right).$$

Bearing in mind that $\mathrm{El}(M_1)$ is α-filtered, the isomorphism above holds if and only if for all $X \in \mathcal{G}$ and $C \in \mathcal{C}$:

- for any $x \colon X \to MC$ there exist $D \in \mathcal{C}$, $y \colon 1 \to MD$, and $g \colon X \to \mathcal{C}(C, D)$ such that x coincides with the composite

  $$X \xrightarrow{g \times x} \mathcal{C}(C, D) \times MD \xrightarrow{\mathrm{ev}_M} MC.$$

  In other words, such that g defines a morphism $(C, x) \to J_X(D, y)$ in $\mathrm{El}(M_X)$.

- for any (y_1, D_1, g_1) and (y_2, D_2, g_2) as above there exist $E \in \mathcal{C}$, $z \colon 1 \to ME$, and maps $h_i \colon D_i \to E$ such that $M_I(h_i)(z) = x_i$ and $\mathcal{C}(h_1, C) \circ g_1 = \mathcal{C}(h_2, C) \circ g_2$. In other words, such that we have a commutative square

  $$\begin{tikzcd} & J_X(D_1,y_1) \arrow[dr, "h_2"] & \\ (C,x) \arrow[ur, "g_1"] \arrow[dr, "g_2"'] & & J_X(E,z) \\ & J_X(D_2,y_2) \arrow[ur, "h_1"'] & \end{tikzcd}$$

  in $\mathrm{El}(M_X)$.

These conditions are easily seen to be equivalent to (2). $\square$

As a consequence all the categories $\mathrm{El}(M_X)$, for $X \in \mathcal{G}$, are also α-filtered (see Remark 1.3.3).

3.1.2 2-categories

We now further specialize to the case $\mathcal{V} = \mathbf{Cat}$ and give a characterization of α-flat 2-functors in terms of a certain double category.

Definition 3.1.21 ([47])**.** Let $M \colon \mathcal{C}^{op} \to \mathbf{Cat}$ be a 2-functor. The double category of elements $\mathbb{E}\mathrm{l}(M)$ of M is the one with:

- objects: pairs (C, x) with $C \in \mathcal{C}$ and $x \in M(C)$;
- horizontal arrows $f \colon (C, x) \to (D, y)$: morphisms $f \colon C \to D$ in $\mathcal{C}$ with $M(f)(y) = x$;

- vertical arrows $\xi\colon (C,x) \to (C,x')$: morphisms $\xi\colon x \to x'$ in $M(C)$.
- double cells:

$$\begin{tikzcd}
(C,x) \arrow[r, "f"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\Downarrow\alpha"] & (D,y) \arrow[d, "\mu"] \\
(C,x') \arrow[r, "g"'] & (D,y')
\end{tikzcd}$$

2-cells $\alpha\colon f \Rightarrow g$ in $\mathcal{C}$ for which $M(\alpha)(\mu) = \xi$.

Notation: Let $\mathbb{D}$ be a double category.

- We denote by $H(\mathbb{D})$ the *horizontal* category of $\mathbb{D}$; this has the same objects as $\mathbb{D}$ and the horizontal arrows as morphisms.
- We denote by $H_1(\mathbb{D})$ the category with vertical arrows as objects and double cells as morphisms, endowed with the horizontal composition of cells.

It follows that the horizontal category $H(\mathbb{El}(M))$ of $\mathbb{El}(M)$ corresponds to $\mathrm{El}(M_1)$, where $M_1 := \mathbf{Cat}_0(1, M_0-)\colon \mathcal{C}_0 \to \mathbf{Set}$ as usual. For any $\mathbb{D}$ there is a functor $1_H\colon : H(\mathbb{D}) \to H_1(\mathbb{D})$ which sends an object D to the vertical identity $D \to D$, and an arrow $f\colon D \to C$ to the identity 2-cell 1_f.

Definition 3.1.22. Let $\mathbb{D}$ be a double category; we say that the horizontal category $H(\mathbb{D})$ of $\mathbb{D}$ is *final* in $\mathbb{D}$ if the ordinary functor $1_H\colon H(\mathbb{D}) \to H_1(\mathbb{D})$ is final.

Remark 3.1.23. If $H(\mathbb{D})$ is final in $\mathbb{D}$, then double colimits indexed on $\mathbb{D}$ are the same as ordinary colimits on $H(\mathbb{D})$ in the following sense:

Let $\mathcal{C}$ be a 2-category and $\mathbb{K}\colon \mathbb{D} \to \mathcal{C}$ a double functor (where $\mathcal{C}$ is seen as a double category with identity vertical morphisms). Let $K\colon H(\mathbb{D}) \to \mathcal{C}$ be the induced horizontal functor; then the double colimit of $\mathbb{K}$ exists in $\mathcal{C}$ if and only if the conical colimit of K does so, and in that case they coincide.

Definition 3.1.24. We say that a double category $\mathbb{D}$ is *α-filtered* if the horizontal category $H(\mathbb{D})$ is α-filtered and final in $\mathbb{D}$. Equivalently, $\mathbb{D}$ is α-filtered if and only if:

1. $H(\mathbb{D})$ is α-filtered;
2. for any vertical morphism $\xi\colon C \to D$ there exists a square as below;

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "id"] \\
D \arrow[r, "g"'] & B
\end{tikzcd}$$

3. for any pair of double cells α and β

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\xi"', "\bullet" marking] \arrow[dr, phantom, "\Downarrow \alpha, \beta"] & B \arrow[d, "id", "\bullet"' marking] \\ D \arrow[r, "g"'] & B \end{tikzcd}$$

there exists a horizontal arrow $h\colon B \to A$ such that $\alpha h = \beta h$.

Proposition 3.1.25. *Let $\mathcal{C}$ be a 2-category and $M\colon \mathcal{C} \to \mathbf{Cat}$ a 2-functor; the following are equivalent:*

1. *M is α-flat;*
2. *the double category $\mathbb{E}\mathrm{l}(M)$ is α-filtered.*

Proof. Note that, by Corollary 3.1.20 applied to the strong generator $\{\mathbb{2}\}$, the 2-functor M is α-flat if and only if $\mathrm{El}(M_1)$ is α-filtered and the functor $J\colon \mathrm{El}(M_1) \longrightarrow \mathrm{El}(M_{\mathbb{2}})$, induced by precomposition with $!\colon \mathbb{2} \to 1$, is final. But $\mathrm{El}(M_1) = H(\mathbb{E}\mathrm{l}(M))$ is the underlying category spanned by the horizontal arrows of $\mathbb{E}\mathrm{l}(M)$, and $\mathrm{El}(M_{\mathbb{2}}) = H_1(\mathbb{E}\mathrm{l}(M))$ is the ordinary category with vertical arrows as objects and double cells as morphisms. Under this interpretation the functor J is the same as $1_H\colon H(\mathbb{E}\mathrm{l}(M)) \to H_1(\mathbb{E}\mathrm{l}(M))$; therefore the result follows by the definition of α-filtered double category. $\square$

3.2 When flat equals filtered plus absolute

In Section 3.2.1 we briefly recall the notion of dualizable object, then in Section 3.2.2 we introduce what we are calling the *locally dualizable categories*. Finally in Section 3.2.3 we show that if $\mathcal{V}$ is locally dualizable then α-flat colimits are generated by absolute colimits and the usual α-filtered ones. Key examples of locally dualizable categories can be found in Example 3.2.7 below.

3.2.1 Dualizable objects

Definition 3.2.1. We say that an object P of $\mathcal{V}$ is *dualizable* if there exist $P^* \in \mathcal{V}$ and morphisms $\eta_P\colon I \to P \otimes P^*$ and $\epsilon_P\colon P^* \otimes P \to I$, called unit and counit respectively, satisfying the triangle equalities. In that case P^* is unique up to isomorphism and is called the *dual* of P.

Note that the unit I is always dualizable and that if P is dualizable then P^* is too with $(P^*)^* \cong P$. When, as we assume here, $\mathcal{V}$ is symmetric monoidal closed, P is dualizable if and only if there exists $P^* \in \mathcal{V}$ such that $[P, -] \cong P^* \otimes -\colon \mathcal{V}_0 \to \mathcal{V}_0$.

The following is a well-known result about powers and copowers by dualizable objects:

Proposition 3.2.2. *Powers and copowers by dualizable objects are absolute; moreover for any $A \in \mathcal{A}$ and any dualizable P we have $P \pitchfork \mathcal{A} \cong P^* \cdot A$ (either side existing if the other does).*

Proof. Let P be dualizable in $\mathcal{V}$; then $P \otimes - \cong [P^*, -]$ is continuous and hence copowers by P are dualizable by [59, Theorem 6.22]. About the last statement, consider the following isomorphisms natural in $B \in \mathcal{A}$

$$\begin{aligned}\mathcal{A}(B, P \pitchfork A) &\cong [P, \mathcal{A}(B, A)] \\ &\cong P^* \otimes \mathcal{A}(B, A) \\ &\cong \mathcal{A}(B, P^* \cdot A)\end{aligned}$$

where the last isomorphism holds since copowers by P^* are absolute; therefore $P \pitchfork A \cong P^* \cdot A$. Finally, powers by P are absolute because they are the same as copowers by P^*. $\square$

3.2.2 Locally dualizable categories

We can now introduce the bases of enrichment considered in this section:

Definition 3.2.3. Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ be a cocomplete symmetric monoidal closed category; we say that $\mathcal{V}$ is *locally dualizable* if:

(a) $\mathcal{V}_0$ has finite direct sums;

(b) The unit I is regular projective and finitely presentable;

(c) $\mathcal{V}_0$ has a strong generator $\mathcal{G}$ made of dualizable objects;

(d) for every $A, B \in \mathcal{V}$ and every arrow $z \colon I \to A \otimes B$ there exists a dualizable object $P \in \mathcal{V}$ and maps $x \colon P \to A$ and $y \colon P^* \to B$ such that

$$\begin{tikzcd} I \arrow[r, "\eta_P"] \arrow[dr, "z"'] & P \otimes P^* \arrow[d, "x \otimes y"] \\ & A \otimes B \end{tikzcd}$$

commutes.

Notation: From now on we write simply $x \overset{P}{\otimes} y$ in place of the composite $(x \otimes y) \circ \eta_P$, for any dualizable object P, $x \colon P \to A$, and $y \colon P^* \to B$. Note in particular that if $P = I$ then $x \overset{I}{\otimes} y$ is just $x \otimes y$ up to the isomorphism $I \otimes I \cong I$.

By condition (a), we know in particular that $\mathcal{V}_0$ is the underlying category of a **CMon**-enriched category $\bar{\mathcal{V}}$; this notation will also be used later on.

For every $P \in \mathcal{G}$ we have $[P, -] \cong P^* \otimes -$, so that $[P, -]$ is cocontinuous and hence $\mathcal{V}_0(P, -) \cong \mathcal{V}_0(I, [P, -])$ is finitary and preserves regular epimorphisms. Therefore $\mathcal{G}$ is a strong generator made of finitely presentable regular projective objects and $\mathcal{V}_0$ is hence a finitary quasivariety [64, Definition 4.4], and in particular locally finitely presentable. Moreover, for any $P, Q \in \mathcal{G}$, the functor $\mathcal{V}_0(P \otimes Q, -) \cong \mathcal{V}_0(P, [Q, -])$ is finitary and preserves regular epimorphisms as well. This means that $\mathcal{V}$ is actually a symmetric monoidal finitary quasivariety in the sense of [64, Definition 4.11], and in particular $\mathcal{V}$ is locally finitely presentable as a closed category.

Proposition 3.2.4. *Assume that $\mathcal{V} = (\mathcal{V}_0, I, \otimes)$ satisfies the conditions $(a), (b)$ and (c) above. Then (d) holds if and only if:*

(d^*) *for every $A, B \in \mathcal{V}$, any arrow $z \colon I \to A \otimes B$ can be written as*

$$z = \sum_{i=1}^{n} (x_i \overset{P_i}{\otimes} y_i)$$

for some $P_i \in \mathcal{G}$, $x_i \colon P_i \to A$, and $y_i \colon P_i^* \to B$.

When writing the sum above we are seeing $\mathcal{V}_0$ as the **CMon**-*category $\bar{\mathcal{V}}$.*

Proof. Fix $A, B \in \mathcal{V}$ and an arrow $z \colon I \to A \otimes B$. If (d^*) holds then it is enough to consider $P := \bigoplus_{i=1}^n P_i$, $x := \sum_{i=1}^n x_i$, and $y := \sum_{i=1}^n y_i$. Then P is still dualizable, with dual $P^* \cong \bigoplus_{i=1}^n P_i^*$, and by construction $x \overset{P}{\otimes} y = z$.

Conversely, assume that (d) holds. Since $\mathcal{G}$ is a strong generator made of finitely presentable and projective objects, and P is finitely presentable and regular projective as well, we can find $P_1, \dots, P_n \in \mathcal{G}$ and a split epimorphism $q \colon Q := \bigoplus_{i=1}^n P_i \twoheadrightarrow P$, with section $s \colon P \rightarrowtail Q$ (see for example [64, Proposition 4.8.(4)]). It follows that the following triangle commutes

$$\begin{tikzcd} I \arrow[r, "\eta_Q"] \arrow[dr, "\eta_P"'] & Q \otimes Q^* \arrow[d, "q \otimes s^*"] \\ & P \otimes P^* \end{tikzcd}$$

in other words $\eta_P = q \overset{Q}{\otimes} s^*$. Therefore $z = x \overset{P}{\otimes} y = (x \circ q) \overset{Q}{\otimes} (y \circ s^*)$. Now it is enough to define x_i and y_i as the i-th components of $(x \circ q)$ and $(y \circ s^*)$ respectively. □

Remark 3.2.5. Assume that $\mathcal{V}$ satisfies conditions $(a), (b)$, and (c). Define the map

$$q_{A,B} := \textstyle\sum(x \overset{P}{\otimes} y) \colon \sum\limits_{(P, x\colon P \to A, y\colon P^* \to B)} I \longrightarrow A \otimes B.$$

It is then easy to see that $\mathcal{V}$ satisfies (d^*) if and only if $\mathcal{V}_0(I, q_{A,B})$ is a surjection. It is not true in general, not even for graded abelian groups, that the map $q_{A,B}$ is a regular epimorphism in $\mathcal{V}_0$. What is true in some cases, which include graded abelian groups, is that the following

$$\sum_{(P, Q, x\colon P \to A, y\colon P^* \otimes Q \to B)} Q \longrightarrow A \otimes B$$

is a regular epimorphism in $\mathcal{V}_0$; where the component (P, Q, x, y) is given by the composite

$$Q \cong I \otimes Q \xrightarrow{i_P} (P \otimes P^*) \otimes Q \cong P \otimes (P^* \otimes Q) \xrightarrow{x \otimes y} A \otimes B.$$

But condition (d) is all we need.

Proposition 3.2.6. *Let $\mathcal{V}$ be locally dualizable with strong generator $\mathcal{G}$, and let $\mathcal{C}$ be a small compact closed $\mathcal{V}$-category. Then $\mathcal{W}_0 := [\mathcal{C}, \mathcal{V}]_0$, endowed with Day's convolution as tensor products, is locally dualizable with strong generator*

$$\mathcal{G}' := \{P \cdot Yg \mid P \in \mathcal{G}, g \in \mathcal{C}\},$$

where $Yg = \mathcal{C}(g, -)$.

Proof. The category $\mathcal{W}$ is symmetric monoidal closed and cocomplete by construction. Observe that the unit of $\mathcal{W}_0$ is $Ye = \mathcal{C}(e, -)$, where e is the unit of $\mathcal{C}$; this is regular projective and finitely presentable because

$$\mathcal{W}_0(Ye, -) \cong \mathcal{V}_0(I, [\mathcal{C}, \mathcal{V}](Ye, -)) \cong \mathcal{V}_0(I, \mathrm{ev}_e -)$$

and the unit I of $\mathcal{V}$ is such. Moreover $\mathcal{W}$ has finite direct sums since $\mathcal{V}$ has them and the representables have duals $(Yg)^* = Y(g^*)$. For each $P \in \mathcal{G}$ and $A \in \mathcal{W}$ we have

$$[\mathcal{C}, \mathcal{V}]_0(P \cdot Yg, A) \cong \mathcal{V}_0(P, [\mathcal{C}, \mathcal{V}](Yg, A)) \cong \mathcal{V}_0(P, A(g)).$$

In particular an arrow $z\colon Ye \to A \otimes B$ in $\mathcal{W}$ corresponds to an arrow $\bar{z}\colon I \to (A \otimes B)(e)$ in $\mathcal{V}$. Now note that by definition

$$\begin{aligned}(A \otimes B)(e) &\cong \int^{g,h} \mathcal{C}(g \otimes h, e) \otimes A(g) \otimes B(h) \\ &\cong \int^{g,h} \mathcal{C}(h, g^*) \otimes A(g) \otimes B(h) \\ &\cong \int^{g} A(g) \otimes B(g^*)\end{aligned}$$

and that moreover we have a regular epimorphism in $\mathcal{V}$

$$\sum_g A(g) \otimes B(g^*) \twoheadrightarrow \int^g A(g) \otimes B(g^*).$$

Since I is finitely presentable and projective, $\bar{z}$ factors as a finite direct sum $\bar{z} = \sum_i \bar{y}_i$ for some $\bar{y}_i\colon I \to A(g_i) \otimes B(g_i^*)$; by hypothesis we have $\bar{y}_i = \bar{x}_i \overset{P_i}{\otimes} \bar{y}_i$ for some dualizable P_i in $\mathcal{V}$, $\bar{x}_i\colon P_i \to A(g_i)$ and $\bar{y}_i\colon P_i^* \to B(g_i^*)$. Let $x_i\colon P_i \cdot Yg_i \to A$ and $y_i\colon (P_i \cdot Yg_i)^* \cong (P_i^* \cdot Y(g_i^*)) \to B$ be the induced morphisms in $\mathcal{W}$. Then

$$z = \sum_i x_i \overset{P_i \cdot Yg_i}{\otimes} y_i,$$

and therefore $\mathcal{W}$ is locally dualizable by Proposition 3.2.4 with strong generator $\mathcal{G}' := \{P \cdot Yg \mid P \in \mathcal{G}, g \in \mathcal{C}\}$. □

As a consequence, for any compact closed **CMon**-category $\mathcal{C}$ the presheaf category $[\mathcal{C}^{op}, \mathbf{CMon}]$ is locally dualizable with strong generator given by the representables.

All the examples below can be constructed as above starting from the first.

Examples 3.2.7.

- The symmetric monoidal category **CMon** of commutative monoids is locally dualizable with strong generator $\mathcal{G} = \{\mathbb{N}\}$.
- The symmetric monoidal category **Ab** of abelian groups; more generally the symmetric monoidal closed category R-**Mod** of modules over a commutative ring R with $\mathcal{G} = \{R\}$.
- The symmetric monoidal category **G-Gr**(R-**Mod**) of **G**-graded R-modules, for an abelian group **G** and a commutative ring R, with $\mathcal{G} = \{S_g R\}_{g \in \mathbf{G}}$.

Proposition 3.2.8. *Let $\bar{\mathcal{V}}$ be a cocomplete symmetric monoidal closed* **CMon**-*category; then $\mathcal{V}$ is locally dualizable if and only if there exists a small compact closed $\mathcal{G}$ and a monoidal adjunction*

$$\begin{tikzcd} \bar{\mathcal{V}} \arrow[r, "U"'] \arrow[r, phantom, "\perp"] & {[\mathcal{G}^{op}, \mathbf{CMon}]} \arrow[l, bend right, "F"'] \end{tikzcd}$$

(where $[\mathcal{G}^{op}, \mathbf{CMon}]$ *has Day's convolution as monoidal structure) with* U *a conservative, filtered-colimit-preserving, and regular* $\mathbf{CMon}$*-functor for which the comparison maps*

$$UA \otimes UB \to U(A \otimes B)$$

are regular epimorphisms for any $A, B \in \mathcal{V}_0$*. The last requirement is also equivalent to*

$$UFX \otimes UFY \to UF(X \otimes Y)$$

being regular epimorphisms for any $X, Y \in [\mathcal{G}^{op}, \mathbf{CMon}]$*.*

Proof. Assume first that $\mathcal{V}$ is locally dualizable; as specified at the beginning we can assume $\mathcal{G}$ to contain the unit and be closed in $\mathcal{V}_0$ under tensor product and duals. For any $P \in \mathcal{G}$ the $\mathbf{CMon}$-functor $\bar{\mathcal{V}}(G, -)$ preserves finite direct sums, filtered colimits, regular epimorphisms, and so also all coproducts. We can then consider the induced $\mathbf{CMon}$-functor $U = \bar{\mathcal{V}}(\mathcal{G}, 1)\colon \bar{\mathcal{V}} \longrightarrow [\mathcal{G}^{op}, \mathbf{CMon}]$ which is then conservative, continuous, and preserves filtered colimits, direct sums and regular epimorphism. Moreover, since $\mathcal{V}_0$ is cocomplete, U has a left adjoint F which sends the representables to $\mathcal{G}$. This is a monoidal adjunction because F is strong monoidal: the restriction of F to $Y\mathcal{G}$ is strong monoidal by construction (being isomorphic to the identity) and hence, since $Y\mathcal{G}$ generates $[\mathcal{G}^{op}, \mathbf{CMon}]$ under colimits, the tensor product is cocontinuous in each variable, and F is itself cocontinuous, it follows that F is strong monoidal too. We are only left to show that the comparisons $UA \otimes UB \to U(A \otimes B)$, induced by the monoidal structure on U, are regular epimorphisms in $[\mathcal{G}^{op}, \mathbf{CMon}]$. This will be the case if and only if, for each $G \in \mathcal{G}$, the map $(UA \otimes UB)G \to U(A \otimes B)G$ is a surjection (of monoids); that is, if and only if the canonical map

$$\int^{P,H\in\mathcal{G}} \mathcal{G}(G, P \otimes H) \otimes \bar{\mathcal{V}}(P, A) \otimes \bar{\mathcal{V}}(H, B) \longrightarrow \bar{\mathcal{V}}(G, A \otimes B)$$

is surjective. But $\mathcal{G}(G, P \otimes H) \cong \mathcal{G}(P^* \otimes G, H)$, so the coend on the left hand side can be rewritten as

$$\int^{P\in\mathcal{G}} \bar{\mathcal{V}}(P, A) \otimes \bar{\mathcal{V}}(P^* \otimes G, B),$$

and this is covered by the coproduct of its components. As a consequence the morphism $(UA \otimes UB)G \to U(A \otimes B)G$ is a regular epimorphism if and only if the induced map

$$\sum_{P\in\mathcal{G}} \bar{\mathcal{V}}(P, A) \otimes \bar{\mathcal{V}}(P^* \otimes G, B) \longrightarrow \bar{\mathcal{V}}(G, A \otimes B)$$

is surjective. For that, given any $G \in \mathcal{G}$ and a map $f\colon G \to A \otimes B$, we can transpose f to obtain $f^t\colon I \to A \otimes (B \otimes G^*)$. By hypothesis this can be expressed as $f^t = \sum_{i=1}^n (x_i \overset{P_i}{\otimes} y_i^t)$ for some $P_i \in \mathcal{G}$, $x_i\colon P_i \to A$ and $y_i^t\colon P_i^* \to B \otimes G^*$. Transposing again we can then write

$$f = \sum_{i=1}^{n} (x_i \otimes y_i) \circ (\eta_{P_i} \otimes G)$$

where $y_i\colon P_i^* \otimes G \to B$ is the transpose of y_i^t, and $i_{P_i} \otimes G\colon G \to (P_i \otimes P_i^*) \otimes G \cong P_i \otimes (P_i^* \otimes G)$. This proves that the desired map is a surjection of monoids.

Assume conversely that we have such an adjunction. $\mathcal{V}_0$ has direct sums by hypothesis. Denote by $\mathcal{G}(-, J)$ the unit of $[\mathcal{G}^{op}, \mathbf{CMon}]$; then $I \cong F\mathcal{G}(-, J)$ and $\mathcal{V}_0(I, -) \cong$

$\mathbf{CMon}(\mathbb{N}, \mathrm{ev}_J U-)$ preserves filtered colimits and regular epimorphisms because U does. Moreover, since F is strong monoidal the image $F\mathcal{G}$ of the representables under F consists of dualizable objects; these form a strong generator of $\mathcal{V}_0$ since U is conservative and $\mathcal{G}$ is a strong generator of $[\mathcal{G}^{op}, \mathbf{CMon}]$. It remains to check property (d^*). Consider $z\colon I \to A \otimes B$, since $I \cong F\mathcal{G}(-, J)$ this corresponds to a map $z^t\colon \mathcal{G}(-, J) \to U(A \otimes B)$. By hypothesis the comparison $UA \otimes UB \to U(A \otimes B)$ is a regular epimorphism; thus z^t factors through it (since representables are projective), giving a map $z'\colon \mathcal{G}(-, J) \to UA \otimes UB$. Since $[\mathcal{G}^{op}, \mathbf{CMon}]$ is locally dualizable with strong generator given by the representables it follows that $z' = \sum_{i=1}^n (x_i^t \overset{P_i}{\otimes} y_i^t)$ for some $P_i \in \mathcal{G}$, $x_i^t\colon \mathcal{G}(-, P_i) \to UA$ and $y_i^t\colon \mathcal{G}(-, P_i)^* \to UB$. Transposing again through the adjunction we obtain maps $x_i\colon F\mathcal{G}(-, P_i) \to A$ and $y_i\colon F\mathcal{G}(-, P_i)^* \to B$ for which $z = \sum_{i=1}^n (x_i \overset{P_i}{\otimes} y_i)$. Therefore $\mathcal{V}_0$ is locally dualizable.

The last part of the statement is an easy consequence of the fact that whenever U is conservative the counit of the adjunction is pointwise a strong epimorphism and in **CMon** strong and regular epimorphisms coincide.

□

Remark 3.2.9. When $\mathcal{V}_0$ is exact this adjunction is monadic by Duskin's monadicity theorem [8, Theorem 9.1.3].

An immediate consequence is:

Proposition 3.2.10. *Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ be locally dualizable and $\bar{\mathcal{W}}$ be a symmetric monoidal closed* **CMon**-*category together with a monoidal* **CMon**-*adjunction*

$$\begin{tikzcd} \bar{\mathcal{W}} \arrow[r, "U"'] \arrow[r, phantom, bend left=20, "\perp"] & \bar{\mathcal{V}} \arrow[l, bend right=40, "F"'] \end{tikzcd}$$

with U a conservative, filtered colimit and regular epimorphism preserving $\mathcal{V}$-functor for which the comparison maps $UA \otimes UB \to U(A \otimes B)$ are regular epimorphisms. Then $\mathcal{W}$ is locally dualizable.

Remark 3.2.11. As a final observation before the next step take two $\mathcal{V}$-functors $M\colon \mathcal{C}^{op} \to \mathcal{V}$ and $H\colon \mathcal{C} \to \mathcal{V}$, where $\mathcal{C}$ has $\mathcal{G}$-copowers. Consider any $P \in \mathcal{G}$, $C \in \mathcal{C}$, and maps $x\colon P \to MC$ and $y\colon P^* \to HC$. Since $P \cong I \otimes P$ we can consider the transposes of x to get a map $I \to [P, MC]$ which, since M preserves such powers (being absolute), corresponds to a map $x'\colon I \to M(P \cdot C)$. Similarly, y corresponds to a morphism $I \to [P^*, HC] \cong P \otimes HC$ which, since H preserves such copowers, is the same as a map $y'\colon I \to H(P \cdot C)$. Then it is easy to see that the square below commutes.

$$\begin{tikzcd} & I \arrow[dl, "x \overset{P}{\otimes} y"'] \arrow[dr, "x' \otimes y'"] & \\ M(C) \otimes HC \arrow[dr, "q_C"'] & & M(P \cdot C) \otimes H(P \cdot C) \arrow[dl, "q_{P \cdot C}"] \\ & M * H & \end{tikzcd}$$

3.2.3 The characterization theorem

We are now ready to prove our results starting with an adapted version of Lemma 3.1.5. For the remainder of this section $\mathcal{V}$ is assumed to be locally dualizable.

Lemma 3.2.12. *Let $\mathcal{C}$ be a $\mathcal{V}$-category with finite direct sums and $\mathcal{G}$-copowers. Let $M\colon \mathcal{C}^{op} \to \mathcal{V}$ and $H\colon \mathcal{C} \to \mathcal{V}$ be two $\mathcal{V}$-functors. Then for each arrow $x\colon I \to M * H$ there exist $C \in \mathcal{C}$, $y\colon I \to MC$, and $z\colon I \to HC$ for which the triangle*

$$\begin{tikzcd} I \arrow[r, dashed, "y \otimes z"] \arrow[dr, "x"'] & MC \otimes HC \arrow[d, "\rho_C"] \\ & M * H \end{tikzcd}$$

commutes, where the vertical map is taken from the colimiting cocone defining $M * H$.

Proof. The weighted colimit $M * H$ can be seen as a coend; hence we have a regular epimorphism in $\mathcal{V}_0$

$$\sum_{C\in\mathcal{C}} MC \otimes HC \xrightarrow{\rho} \int^{C\in\mathcal{C}} MC \otimes HC \cong M * H.$$

Since $\bar{\mathcal{V}}(I, -)$ preserves regular epimorphisms and coproducts it follows that x factors through an element $h \in \sum_{C\in\mathcal{C}} \bar{\mathcal{V}}(I, MC \otimes HC)$ in **CMon**. Such an element is given by $h = \sum_{i=1}^n h_i$ with $h\colon I \to MD_i \otimes HD_i$ (by definition of coproduct in **CMon**). By the hypotheses on $\mathcal{V}$ we can write each h_i as a finite direct sum of elements of the form $y'_j \overset{P_j}{\otimes} z'_j$, for $y'_j\colon P_j \to M(D_i)$ and $z'_j\colon P_j^* \to H(D_i)$. In other words, relabelling the objects and the morphisms, we can write $h = \sum_{j=1}^n (y'_j \overset{P_j}{\otimes} z'_j)$ for some $P_j \in \mathcal{P}$, $y'_j\colon P_j \to M(D_j)$, and $z'_j\colon P_j^* \to H(D_j)$.
Remember now that powers by P_i and P_i^* are absolute, it then follows that each y'_j corresponds by transposition to $y_j\colon I \to M(P_j \cdot D_j)$, since M is contravariant, and each z'_j corresponds to $z_j\colon I \to H(P_j \cdot D_j)$. Therefore

$$x = \rho(h) = \rho(\sum_{j=1}^n y'_j \overset{P_j}{\otimes} z'_j) = \sum_{j=1}^n \rho(y'_j \overset{P_j}{\otimes} z'_j) = \sum_{j=1}^n \rho(y_j \otimes z_j)$$

as an element of $\bar{\mathcal{V}}(I, M * H)$, where the last equality holds thanks to Remark 3.2.11. Consider then the object of $\mathcal{C}$ given by

$$C := \bigoplus_{j=1}^n P_j \cdot D_j;$$

we can define $y := \sum_{j=1}^n y_j\colon I \to M(C)$ and $z := \sum_{j=1}^n z_j\colon I \to H(C)$ so that by construction we have

$$\rho(y \otimes z) = \sum_{j=1}^n \rho(y_j \otimes z_j) = x$$

as desired. $\square$

With the same hypotheses as in the Lemma above we can reduce flat colimits to filtered colimits, giving (under assumptions on $\mathcal{C}$) a proof of condition (I) from the introduction to the chapter.

Corollary 3.2.13. *Let $\mathcal{C}$ be a $\mathcal{V}$-category with finite direct sums and $\mathcal{G}$-copowers. Let $M\colon \mathcal{C}^{op} \to \mathcal{V}$ be a $\mathcal{V}$-functor for which $M * -\colon [\mathcal{C},\mathcal{V}] \to \mathcal{V}$ preserves α-small conical limits of representables, and define $M_I := \mathcal{V}_0(I, M_0-)\colon \mathcal{C}_0^{op} \longrightarrow$ **Set** as usual. Then the ordinary category $\mathrm{El}(M_I)$ is α-filtered, and so M_I is α-flat.*

Proof. Using the Lemma above the proof is exactly the same as that of Corollary 3.1.7. □

As a consequence we obtain the following proposition, showing that (II) also holds under these conditions.

Proposition 3.2.14. *Let $\mathcal{C}$ be a $\mathcal{V}$-category with finite direct sums and $\mathcal{G}$-copowers, and let $M\colon \mathcal{C}^{op} \to \mathcal{V}$ be a $\mathcal{V}$-functor. The following are equivalent:*

1. *M is α-flat;*
2. *$M * -\colon [\mathcal{C},\mathcal{V}] \to \mathcal{V}$ preserves α-small conical limits of representables;*
3. *$\mathrm{El}(M_I)$ is α-filtered;*
4. *M is an α-filtered colimit of representables.*

Moreover, in this case the canonical map:

$$\operatorname{colim}\left(\mathrm{El}(M_I)_{\mathcal{V}} \xrightarrow{\pi_{\mathcal{V}}} \mathcal{C} \xrightarrow{Y} [\mathcal{C}^{op},\mathcal{V}]\right) \longrightarrow M$$

is invertible.

Proof. (1) ⇒ (2) is trivial, (2) ⇒ (3) is true by the Corollary above, while (4) ⇒ (1) follows from the fact that representables $\mathcal{V}$-functors are α-flat and the α-flat $\mathcal{V}$-functors are closed under α-filtered colimits in $[\mathcal{C}^{op},\mathcal{V}]$.

It only remains to show (3) ⇒ (4). Since the category of elements is α-filtered, we only need to prove that the isomorphism in the statement holds. Colimits are computed pointwise in $[\mathcal{C}^{op},\mathcal{V}]$; thus it is enough to show that the canonical map

$$\operatorname{colim}\left(\mathrm{El}(M_I)_{\mathcal{V}} \xrightarrow{\pi_{\mathcal{V}}} \mathcal{C} \xrightarrow{\mathcal{C}(C,-)} \mathcal{V}\right) \longrightarrow M(C)$$

is invertible for any $C \in \mathcal{C}$. For that consider the strong generator $\mathcal{G}$; we know that $\mathcal{G}$-powers are absolute and that $\mathcal{G}$ consists of finitely presentable objects in $\mathcal{V}_0$. Moreover the ordinary functor $M_I = \mathcal{V}_0(I, M_0-)$ is the colimit of the corresponding ordinary diagram based on $\mathrm{El}(M_I)$. Thus for each $P \in \mathcal{G}$ we can write

$$\begin{aligned}
\mathcal{V}_0(P, M(C)) &\cong \mathcal{V}_0(I, [P, M(C)]) \\
&\cong \mathcal{V}_0(I, M(P\cdot C)) \\
&\cong M_I(P\cdot C) \\
&\cong \operatorname*{colim}_{(x,D)\in \mathrm{El}(M_I)} \mathcal{C}_0(P\cdot C, D) \\
&\cong \operatorname*{colim}_{(x,D)\in \mathrm{El}(M_I)} \mathcal{V}_0(I, \mathcal{C}(P\cdot C, D)) \\
&\cong \operatorname*{colim}_{(x,D)\in \mathrm{El}(M_I)} \mathcal{V}_0(I, [P, \mathcal{C}(C, D)]) \\
&\cong \operatorname*{colim}_{(x,D)\in \mathrm{El}(M_I)} \mathcal{V}_0(P, \mathcal{C}(C, D)) \\
&\cong \mathcal{V}_0(P, \operatorname*{colim}_{(x,D)\in \mathrm{El}(M_I)} \mathcal{C}(C, D)).
\end{aligned}$$

As a consequence, since the family $\mathcal{G}$ is strongly generating, it follows at once that $M(C) \cong \operatorname{colim} \mathcal{C}(C, \pi_{\mathcal{V}}-)$ as desired. $\square$

Remark 3.2.15. When $\alpha = \aleph_0$ it is enough to ask that $M * -$ preserve equalizers, since it already preserves finite products (being direct sums). Moreover, for $\mathcal{V} = \mathbf{Ab}$ and $\alpha = \aleph_0$ this is Theorem 3.2 of [81].

Therefore we obtain a characterization of α-flat $\mathcal{V}$-functors in general:

Proposition 3.2.16. *Let $M \colon \mathcal{C}^{op} \to \mathcal{V}$ be a $\mathcal{V}$-functor; the following are equivalent:*

1. *M is α-flat, or equivalently $M * - \colon [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ preserves α-small limits;*
2. *$M * - \colon [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ preserves α-small limits of representables;*
3. *M lies in the closure of the representables under $\mathcal{G}$-copowers, finite direct sums, and α-filtered colimits.*

Proof. (1) $\Rightarrow$ (2) is trivial, while (3) $\Rightarrow$ (1) follows from the fact that representables functors are α-flat and these are closed under α-filtered and absolute colimits in $[\mathcal{C}, \mathcal{V}]$.

(2) $\Rightarrow$ (3). Let $M \colon \mathcal{C}^{op} \to \mathcal{V}$ be as in (2) and $J \colon \mathcal{C} \hookrightarrow \mathcal{D}$ be the inclusion of $\mathcal{C}$ into its free completion under finite direct sums and $\mathcal{G}$-copowers. Consider the weight $M' := \operatorname{Lan}_{J^{op}} M$, this still has the property that $M' * - \colon [\mathcal{D}, \mathcal{V}] \to \mathcal{V}$ preserves α-small conical limits of representables: since finite direct sums and $\mathcal{G}$-copowers are α-small, the limit of any α-small diagram landing in $\mathcal{D}^{op}$ can be rewritten as the weighted limit of one landing in $\mathcal{C}^{op}$; thus $M' * -$ preserves α-small conical limits of representables by condition (2). Now the domain of $M' \colon \mathcal{D}^{op} \to \mathcal{V}$ satisfies the conditions of Proposition 3.2.14, therefore we can write $M' \cong \operatorname{colim}(YK)$ with $K \colon \mathcal{B}_{\mathcal{V}} \to \mathcal{D}$ an α-filtered diagram, and $Y \colon \mathcal{D} \to [\mathcal{D}^{op}, \mathcal{V}]$ the Yoneda embedding. Then one concludes since each element of $\mathcal{D}$ is a finite direct sum of $\mathcal{G}$-copowers of objects from $\mathcal{C}$. $\square$

Examples 3.2.17.

- For $\mathcal{V} = \mathbf{Ab}$ and $\alpha = \aleph_0$ we recover the characterization of flat additive functors from [81, Theorem 3.2].
- When $\mathcal{V} = \mathbf{GAb}$ we obtain that a $\mathcal{V}$-functor is flat if and only if it is a filtered colimit of suspensions of finite direct sums of representables.
- When $\mathcal{V} = R\text{-}\mathbf{Mod}$, $\alpha = \aleph_0$, and $\mathcal{C} = \mathcal{I}$ we recover Lazard's criterion [71]: an R-module is flat if and only if it is a filtered colimit of free modules.

This allows us to characterize α-flat colimits in terms of absolute and α-filtered ones.

Theorem 3.2.18. *A $\mathcal{V}$-category $\mathcal{A}$ has α-flat colimits if and only if it has finite direct sums, $\mathcal{G}$-copowers and α-filtered colimits. A $\mathcal{V}$-functor from such an $\mathcal{A}$ preserves α-flat colimits if and only if it preserves α-filtered colimits.*

Proof. Since α-filtered colimits and absolute colimits are α-flat, any $\mathcal{V}$-category $\mathcal{A}$ with α-flat colimits has all of them.

Vice versa, assume that $\mathcal{A}$ has the colimits above and consider an α-flat weight $M \colon \mathcal{C}^{op} \to \mathcal{V}$ together with a diagram $H \colon \mathcal{C} \to \mathcal{A}$ in $\mathcal{A}$. Let $J \colon \mathcal{C} \hookrightarrow \mathcal{D}$ be the inclusion of $\mathcal{C}$ into its free cocompletion under finite direct sums and $\mathcal{G}$-copowers. Since $\mathcal{A}$ has these colimits we can consider $H' := \operatorname{Lan}_J H$, while on the weighted side we take $M' := \operatorname{Lan}_{J^{op}} M$. By Lemma 1.3.2 the weight M' is still α-flat and, by construction, its domain satisfies the hypotheses of Proposition 3.2.14. Thus we can write $M' \cong \mathrm{colim} YF$ as an α-filtered colimit of representables; here $Y\colon \mathcal{D} \to [\mathcal{D}^{op}, \mathcal{V}]$ is Yoneda and $F\colon \mathcal{B}_{\mathcal{V}} \to \mathcal{D}$ is a functor with α-filtered domain. As a consequence $M * H$ exists if and only if $M' * H'$ exists, and so if and only if $\mathrm{colim}(H'F)$ exists (see the isomorphisms in the proof of 3.1.13) and they coincide. Thus the existence and preservation of the α-flat colimit $M * H$ is equivalent to the existence and preservation of the α-filtered colimit $\mathrm{colim} H'F$. □

Therefore:

Theorem 3.2.19. *A $\mathcal{V}$-category $\mathcal{A}$ is α-accessible if and only if it has finite direct sums and $\mathcal{G}$-copowers, and is conically α-accessible.*

Proof. By Theorem 3.2.18 above an object A of $\mathcal{A}$ is α-presentable if and only if is conically α-presentable, so that $\mathcal{A}_\alpha = \mathcal{A}^c_\alpha$. Arguing as above, for any α-flat $M\colon \mathcal{C}^{op} \to \mathcal{V}$ and diagram $H\colon \mathcal{C} \to \mathcal{A}_\alpha \subseteq \mathcal{A}$, the colimit $M * H$ can be replaced by an α-filtered one $\mathrm{colim}(H'F)$, where H' is the left Kan extension of H along the free cocompletion $\mathcal{D}$ of $\mathcal{C}$ under finite direct sums and $\mathcal{G}$-copowers, and $F\colon \mathcal{B}_{\mathcal{V}} \to \mathcal{D}$ has α-filtered domain. Then $H'F\colon \mathcal{B}_{\mathcal{V}} \to \mathcal{A}$ still lands in $\mathcal{A}_\alpha$ since this is closed in $\mathcal{A}$ under finite direct sums and $\mathcal{G}$-copowers. Thus an object of $\mathcal{A}$ is an α-flat colimit of α-presentables if and only if it is an α-filtered colimit of (conically) α-presentables. The result then follows. □

Remark 3.2.20. For $\mathcal{V} = \mathbf{Ab}$ and $\alpha = \aleph_0$ see Example 9.2 of [18].

Once more, using Theorem 3.2.19 and Theorem 2.2.4, we can compare conical accessible $\mathcal{V}$-categories and models of sketches as follows:

Theorem 3.2.21. *Let $\mathcal{A}$ be a $\mathcal{V}$-category; the following are equivalent:*

1. *$\mathcal{A}$ is accessible;*
2. *$\mathcal{A}$ has finite direct sums and $\mathcal{G}$-copowers, and is conically accessible;*
3. *$\mathcal{A}$ is equivalent to the $\mathcal{V}$-category of models of a sketch.*

As a direct consequence we characterize the Cauchy complete $\mathcal{V}$-categories.

Corollary 3.2.22. *Let $\mathcal{C}$ be a $\mathcal{V}$-category; the following are equivalent:*

1. *$\mathcal{C}$ is Cauchy complete;*
2. *$\mathcal{C}$ has finite direct sums, copowers (and hence powers) by dualizable objects, and splitting idempotents;*
3. *$\mathcal{C}$ has finite direct sums, $\mathcal{G}$-copowers, and splitting of idempotents.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial, and (3) $\Rightarrow$ (1) is a consequence of Theorem 3.2.18, using the fact that an absolute weight is one that is α-flat for any α.

□

For the next result consider the set $\langle \mathcal{G} \rangle$ given by the closure of $\mathcal{G} \cup \{I\}$ under tensor product.

Proposition 3.2.23. *Let $\mathcal{C}$ be any small $\mathcal{V}$-category. A weight $M\colon \mathcal{C}^{op} \to \mathcal{V}$ is absolute if and only if there exist objects $C_1, \dots, C_n \in \mathcal{C}$ and $P_1, \dots. P_n \in \langle \mathcal{G} \rangle$ such that M is a split subobject of*

$$\bigoplus_{i=1}^{n} P_i \cdot \mathcal{C}(-, C_i).$$

Proof. If the latter holds then M is an absolute colimit of absolute weights; thus it is absolute itself.

Assume now that M is absolute. Let $\mathcal{D}$ be the full subcategory of $[\mathcal{C}^{op},\mathcal{V}]$ spanned by the finite direct sums of $\langle\mathcal{G}\rangle$-copowers of representables; denote by $J\colon\mathcal{C}\hookrightarrow\mathcal{D}$ and note that $\mathcal{D}$ is closed in $[\mathcal{C}^{op},\mathcal{V}]$ under finite direct sums and $\mathcal{G}$-copowers. Consider now $M' := \mathrm{Lan}_{J^{op}}(M)$; then by Lemma 1.3.2 the weight M' is still absolute and, by construction, its domain satisfies the hypotheses of Proposition 3.2.14. It follows that M', seen in $[\mathcal{D}^{op},\mathcal{V}]_0$ is an ordinary absolute colimit of representables; therefore M' is a split subobject of $\mathcal{D}(-,D)$ for some $D\in\mathcal{D}$. As a consequence $M\cong M'\circ J^{op}$ is a split subobject of $\mathcal{D}(J-,D)$. Now, by construction of $\mathcal{D}$, the object D can be written as $\sum_{i=1}^n(P_i\cdot JC_i)$ for some $C_i\in\mathcal{C}$ and $P_i\in\langle\mathcal{G}\rangle$. Therefore $\mathcal{D}(J-,D)\cong\sum_{i=1}^n P_i\cdot\mathcal{C}(-,C_i)$ and the result follows. $\square$

Remark 3.2.24. If we consider $\mathcal{V}=\mathbf{GAb}$ we recover the results of section 6 from [80]. In fact we can consider $\mathcal{G}=\{S^nI\}_{n\in\mathbb{Z}}$ given by the suspensions of the unit. Then the proposition above is [80, Proposition 6.1] and Corollary 3.2.22 is [80, Proposition 6.2].

3.3 Flat does not equal filtered plus absolute in general

In this section we consider the base of enrichment to be the cartesian closed category $\mathcal{V}=\mathbf{Set}^G$ of G-sets, for a non-trivial finite group G. We will prove that in this case the flat $\mathcal{V}$-functors do not lie in the closure of representable functors under absolute and filtered colimits.

First of all note that we have an adjunction

$$\begin{tikzcd} \mathbf{Set}^G \arrow[r, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, bend right, "F"'] \end{tikzcd}$$

where $U=\mathbf{Set}^G(G,-)$ takes the underlying sets, $F=G\times -$ sends a set A to the G-set $G\times A$ with the free action, and we are denoting with $G\in\mathbf{Set}^G$ also the representable functor corresponding to the only object of the group G. Note that U is conservative, continuous, cocontinuous, strong monoidal, and strong closed.

The object G is finitely presentable and a strong generator for $\mathbf{Set}^G$; moreover the functors $\mathbf{Set}^G(1,-)$ and $\mathbf{Set}^G(G\times G,-)\cong\mathbf{Set}(G,U-)$ are finitary (since G is finite). Therefore $\mathbf{Set}^G$ is locally finitely presentable as a closed category.

Since U is strong monoidal it follows that there is an induced 2-functor:

$$U_*\colon\mathcal{V}\text{-}\mathbf{Cat}\longrightarrow\mathbf{Cat}$$

and a $\mathcal{V}$-functor $\hat{U}=(U_*\mathcal{V})(1,-)\colon U_*\mathcal{V}\to\mathbf{Set}$ which acts as U on objects. Now, for each $\mathcal{V}$-weight $M\colon\mathcal{C}^{op}\to\mathcal{V}$ we can define

$$M_U\colon U_*\mathcal{C}^{op}\xrightarrow{U_*M}U_*\mathcal{V}\xrightarrow{\hat{U}}\mathbf{Set}.$$

Since U is moreover cocontinuous it follows that for any $\mathcal{V}$-category $\mathcal{C}$, any $M\colon\mathcal{C}^{op}\to\mathcal{V}$ and $H\colon\mathcal{C}\to\mathcal{V}$, we have an isomorphism

$$U(M*H)\cong M_U*H_U$$

where the colimit on the right is an ordinary weighted colimit. See also Section 3.4.1 for related properties about change of base.

Note that to give a $\mathcal{V}$-category $\mathcal{C}$ is equivalently to give an ordinary category $\mathcal{C}$ whose homs are endowed with group actions $G \times \mathcal{C}(A, B) \to \mathcal{C}(A, B)$ for which the identities are fixed points and the composition maps are equivariant. It follows that $U_*\mathcal{C}$ is the same as $\mathcal{C}$ with the only difference being that the group actions on the homs are forgotten. The category $U_*\mathcal{C}$ should not be mistaken with the underlying category $\mathcal{C}_0$ of $\mathcal{C}$ which has homs $\mathcal{C}_0(A, B) = \mathrm{Fix}\ \mathcal{C}(A, B)$ given by the fixed points of the action on $\mathcal{C}(A, B)$.

Let us start with a result comparing enriched and ordinary flatness which can be seen as a consequence of [19, Theorem 18].

Proposition 3.3.1. *Let $\mathcal{C}$ be a $\mathcal{V}$-category and $M\colon \mathcal{C}^{op} \to \mathcal{V}$ be a $\mathcal{V}$-functor. Then M is α-flat if and only if M_U is α-flat.*

Proof. Assume first that M is α-flat; it is enough to prove that $\mathrm{El}(M_U)$ is α-filtered. Note that, since $U = \mathbf{Set}^G(G, -)$, the category $\mathrm{El}(M_U)$ can be described as in Remark 3.1.19 with G in place of X. Using that and the fact that $\mathbf{Set}^G(G, -)$ is cocontinuous and strong monoidal, one can easily adapt the proofs of Lemma 3.1.5 and Corollary 3.1.7 to show that $\mathrm{El}(M_U)$ is α-filtered (just replace with G all instances of I in the proofs).

Conversely assume that M_U is α-flat; we need to show that $M * -\colon [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ preserves all α-small conical limits and powers by G. Since U is continuous and conservative, $M * -$ preserves α-small conical limits if and only if $U(M * -)\colon [\mathcal{C}, \mathcal{V}]_0 \to \mathbf{Set}$ preserves them. But $U(M * -) \cong M_U * (-)_U$, where $(-)_U$ is continuous (since U is) and $M_U * -$ is α-continuous because M_U is α-flat. Thus we are left to prove that $M * -$ preserves powers by G. Let $H\colon \mathcal{C} \to \mathcal{V}$ be any $\mathcal{V}$-functor; then the comparison map $M * (G \pitchfork H) \to G \pitchfork (M * H)$ is invertible if and only if its image under U is so. Therefore

$$\begin{aligned} U(M * (G \pitchfork H)) &\cong M_U * (G \pitchfork H)_U \\ &\cong M_U * \mathbf{Set}(UG, H_U-) && (3.1) \\ &\cong \mathbf{Set}(UG, M_U * H_U) && (3.2) \\ &\cong \mathbf{Set}(UG, U(M * H)) \\ &\cong U(G \pitchfork (M * H)) && (3.3) \end{aligned}$$

where we used that $(G \pitchfork H)_U \cong \mathbf{Set}(UG, H_U-)$ for (3.1), that M_U is α-flat and G is finite for (3.2), and that U is strong closed for (3.3). It follows that M is α-flat. □

We are now ready to provide an explicit example of a $\mathcal{V}$-category for which flat presheaves on $\mathcal{C}$ are not in the closure of the representables under absolute and filtered colimit.

Define $\mathcal{C}$ as follows: the objects $\mathrm{Ob}(\mathcal{C}) = \mathbb{N}$ are natural numbers; for each $n, m \in \mathbb{N}$ we set $\mathcal{C}(n, m) = \emptyset$ if $n > m$, while $\mathcal{C}(n, m) = \{1_n\}$ consists only of the identity map (with trivial action) if $n = m$, and $\mathcal{C}(n, m) = G = F1$ (with action given by multiplication) if $n < m$. Composition $- \circ -\colon \mathcal{C}(m, l) \times \mathcal{C}(n, m) \to \mathcal{C}(n, l)$ is non-trivial only when $n < m < l$ and in that case is given by $g \circ h := g$, for any $g, h \in G$. It is now easy to see that the composition maps are equivariant and well defined, and that the identities are fixed points of the action; therefore we obtain a $\mathcal{V}$-category $\mathcal{C}$.

Proposition 3.3.2. *The $\mathcal{V}$-category $\mathcal{C}$ has absolute colimits and filtered colimits, and the Yoneda embedding $Y\colon \mathcal{C} \to [\mathcal{C}^{op}, \mathcal{V}]$ preserves them.*

Proof. To begin with note that the underlying category $\mathcal{C}_0$ of $\mathcal{C}$ is the discrete category $\mathbb{N}$; this is because $\mathcal{C}_0(n, n) = \{1_n\}$ and $\mathcal{C}_0(n, m) = \mathrm{Fix}\ \mathcal{C}(n, m) = \emptyset$ for any $n \neq m$ (here we are using the fact that the group G is non trivial). As a consequence the only filtered diagrams that exist in $\mathcal{C}$ are the constant ones, and these have as colimit the same object they pick; these colimits are clearly preserved by Y. (Thus in fact the filtered colimits in $\mathcal{C}$ are absolute.)

To conclude it is enough to show that every Cauchy $\mathcal{V}$-functor $M\colon \mathcal{C}^{op} \to \mathcal{V}$ is representable. Let then M be Cauchy; by Proposition 3.3.1 the ordinary functor $M_U\colon U_*\mathcal{C}^{op} \to$ **Set** is Cauchy as well (as usual use that a weight is Cauchy if and only if it is α-flat for every α). The category $U_*\mathcal{C}$ is Cauchy complete in the ordinary sense (since the only idempotents are the identities); therefore there exists $n \in U_*\mathcal{C}$ such that $M_U \cong (U_*\mathcal{C})(-, n)$. We wish to prove that actually $M \cong \mathcal{C}(-, n)$. For that, let $m \in \mathcal{C}$ be any other object and

$$M_{n,m}\colon \mathcal{C}(m, n) \longrightarrow [Mn, Mm]$$

be the action of M on morphisms in $\mathbf{Set}^G$. Since $UMn \cong M_U n \cong (U_*\mathcal{C})(n, n) = \{1_n\}$, it follows that $Mn = 1$ is the terminal object in $\mathbf{Set}^G$. As a consequence the maps $M_{n,m}$ are actually of the form

$$M_{n,m}\colon \mathcal{C}(m, n) \longrightarrow Mm$$

and define a $\mathcal{V}$-natural transformation $\mathcal{C}(-, n) \to M$. Since $M_U \cong (U_*\mathcal{C})(-, n)$ the maps $U(M_{n,m})$ are bijections of sets and hence, since U is conservative, the $\mathcal{V}$-natural transformation $M_{n,m}$ is an isomorphism. This proves that M is representable. □

Theorem 3.3.3. *The terminal object $\Delta 1$ in $[\mathcal{C}^{op}, \mathcal{V}]$ is flat but does not lie in the closure of the representables under absolute and filtered colimits.*

Proof. By the proposition above the closure of $\mathcal{C}$ in $[\mathcal{C}^{op}, \mathcal{V}]$ under absolute and filtered colimits is $\mathcal{C}$ itself; therefore it is enough to prove that $\Delta 1$ is flat but not representable.

By Proposition 3.3.1, the $\mathcal{V}$-functor $\Delta 1$ is flat if and only if the functor $\Delta 1\colon U_*\mathcal{C}^{op} \to$ **Set** is; and the latter is flat since its category of elements is equal to $U_*\mathcal{C}$, which is filtered. Finally, $\Delta 1\colon \mathcal{C}^{op} \to \mathcal{V}$ is not representable since $\mathcal{C}$ does not have a terminal object. □

Corollary 3.3.4. *The $\mathcal{V}$-category $\mathcal{C}$ is Cauchy complete and conically finitely accessible, but does not have all flat colimits. In particular $\mathcal{C}$ is not finitely accessible.*

Proof. Filtered colimits are trivial in $\mathcal{C}$; therefore every object is conically finitely presentable. Since $\mathcal{C}$ is small this is enough to imply that it is conically finitely accessible.

By the theorem above, the terminal $\mathcal{V}$-functor $\Delta 1\colon \mathcal{C}^{op} \to \mathcal{V}$ is flat; thus to conclude it is enough to show that the colimit $\Delta 1 * \mathrm{id}_{\mathcal{C}}$ does not exist in $\mathcal{C}$. In order to obtain a contradiction assume that $n \cong \Delta 1 * \mathrm{id}_{\mathcal{C}}$ exists; then

$$\mathcal{C}(n, n+1) \cong [\mathcal{C}^{op}, \mathcal{V}](\Delta 1, \mathcal{C}(-, n+1))$$

by the universal property of the colimit. But $\mathcal{C}(n, n+1)$ is not empty, while the underlying set of $[\mathcal{C}^{op}, \mathcal{V}](\Delta 1, \mathcal{C}(-, n+1))$ is empty since there are no maps $1 \to \mathcal{C}(n+2, n+1)$. Therefore $\Delta 1 * \mathrm{id}_{\mathcal{C}}$ does not exist. □

3.4 When flat equals protofiltered plus absolute

We begin this section by introducing the bases we are interested in, the main example to keep in mind being the symmetric monoidal closed category **DGAb** of chain complexes. In Section 3.4.2 we show that in this context the α-flat colimits are generated by the absolute colimits together with what we call *α-protofiltered colimits*. These include, but may not reduce to, the usual α-filtered colimits. In Section 3.4.3 we prove that α-protofiltered colimits are genuine α-flat colimits.

3.4.1 Setting

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ and $\mathcal{W} = (\mathcal{W}_0, \otimes, J)$ be symmetric monoidal closed complete and cocomplete categories for which $\mathcal{V}$ is locally dualizable with strong generator $\mathcal{G} \ni I$, and $\mathcal{W}$ has finite direct sums. Moreover we assume that there is a functor $U\colon \mathcal{W}_0 \to \mathcal{V}_0$ with adjoints $L \dashv U \dashv R$ such that:

(a) U is conservative, strong monoidal, and strong closed;

(b) $UL\mathcal{G}$ is still a strong generator of $\mathcal{V}$;

(c) for any $X \in \mathcal{G}$ the objects ULX is still dualizable.

Remark 3.4.1. It is useful to note the following properties:

(i) U is conservative if and only if $L\mathcal{G}$ is a strong generator.

(ii) Given (a), similarly R is conservative if and only if $UL\mathcal{G}$ is a strong generator.

(iii) Given (a), an object $Y \in \mathcal{W}$ is dualizable if and only if UY is so, thus (c) is equivalent to the objects of $L\mathcal{G}$ being dualizable. And if $UL \cong ULI \otimes -$ then this is equivalent to ULI (or LI) being dualizable.

Examples 3.4.2.

- Let $\mathcal{W} = \mathbf{DGAb}$, $\mathcal{V} = \mathbf{GAb}$, and U be the forgetful functor. Then U has both adjoints and LI is the chain complex having (0) in all the degrees but $(LI)_0 = (LI)_{-1} = \mathbb{Z}$, with differential $d = \mathrm{id}$ between them. Moreover U is conservative, strong monoidal, and strong closed [39, Section 6]. Let now $\mathcal{G}$ be the strong generator of $\mathbf{GAb}$ consisting of the dualizable objects S_nI, for $n \in \mathbb{Z}$. Then $UL\mathcal{G} = \{S_nI \oplus S_{n-1}I\}_{n\in\mathbb{Z}}$ is still a strong generator of $\mathbf{GAb}$ made of dualizable objects. It follows that $\mathbf{DGAb}$ is an example of such base of enrichment.
- Let $\mathcal{V}$ be locally dualizable and H be a cocommutative Hopf algebra in $\mathcal{V}$ which is dualizable as an object of $\mathcal{V}$. We can consider $\mathcal{W}$ to be the symmetric monoidal closed category of H-modules with $U\colon \mathcal{W} \to \mathcal{V}$ the forgetful functor. Then U has both adjoints, satisfies $UL \cong H \otimes -$, and (a) holds. If $\mathcal{G}$ is a strong generator made of dualizable objects for $\mathcal{V}$, then the elements of $UL\mathcal{G}$ are of the form $H \otimes X$, for $X \in \mathcal{G}$. Then (c) holds by the remark above since H is dualizable, moreover $UL\mathcal{G}$ is still a strong generator since for every $X \in \mathcal{G}$ we have a split epimorphism $\epsilon \otimes 1\colon H \otimes X \to X$, where $\epsilon\colon H \to I$ is the counit of H. (The facts about Hopf algebras mentioned above can be found in Chapter 15 of [97]).

The following are a consequence of (a)-(c):

- $L \dashv U$ is an op-monoidal adjunction [55, Theorem 1.2] and $L(UY \otimes X) \cong A \otimes LX$;
- $U \dashv R$ is a monoidal adjunction [55, Theorem 1.2] and $R[UY, X] \cong [Y, RX]$;
- U is monadic by Beck's monadicity theorem [16, Theorem 4.4.4];
- $T = UL$ is an op-monoidal Hopf monad [24, Proposition 2.14].

In particular $LU \cong LI \otimes -$ and $RU \cong [LI, -]$.

Since U is strong monoidal it follows that there is an induced 2-functor:

$$U_* \colon \mathcal{W}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$$

and a $\mathcal{V}$-functor $\hat{U} = (U_*\mathcal{W})(I, -) \colon U_*\mathcal{W} \to \mathcal{V}$ which acts as U on objects. Now, for each $\mathcal{W}$-weight $M \colon \mathcal{C} \to \mathcal{W}$ we can consider the composite

$$M_U \colon U_*\mathcal{C} \xrightarrow{U_*M} U_*\mathcal{W} \xrightarrow{\hat{U}} \mathcal{V}$$

as a $\mathcal{V}$-weight. Given a $\mathcal{W}$-category $\mathcal{C}$ we denote by $S_\mathcal{C} \colon \mathcal{C}_0 \to (U_*\mathcal{C})_0$ the identity-on-objects functor which acts by applying U on morphisms: given a morphism $f \colon J \to \mathcal{C}(A, B)$ in $\mathcal{C}_0$ we define $S_\mathcal{C}(f) := Uf \colon I \cong UJ \to U\mathcal{C}(A, B) = (U_*\mathcal{C})(A, B)$.

The following lemmas are standard results about change of base along a monoidal functor which is continuous, cocontinuous, strong monoidal and strong closed.

Lemma 3.4.3. *Let $\mathcal{C}$ be a $\mathcal{W}$-category. Then:*

1. *for any ordinary $H \colon \mathcal{E} \to \mathcal{C}_0$, if the limit of H exists in $\mathcal{C}$ then it is also the limit of $S_\mathcal{C}H$ in $U_*\mathcal{C}$;*
2. *if the power $X \pitchfork A$ exists in $\mathcal{C}$ then it is also the power $UX \pitchfork A$ in $U_*\mathcal{C}$.*

The same property holds with the corresponding conical colimits and copowers.

Proof. (1) Assume that $\lim H$ exists in $\mathcal{C}$; we need to prove that it is also the limit of $(S_\mathcal{C}H)$, if seen as an object of $U_*\mathcal{C}$. Let C be any object of $U_*\mathcal{C}$; then

$$\begin{aligned}(U_*\mathcal{C})(C, \lim H) &\cong U(\lim \mathcal{C}(C, H-)_0)\\ &\cong \lim U \circ \mathcal{C}(C, H-)_0\\ &\cong \lim (U_*\mathcal{C})(C, S_\mathcal{C}H-)_0\end{aligned}$$

in $\mathcal{V}_0$, where we used that U is continuous and that $(U_*\mathcal{C})(C, S_\mathcal{C}-)_0 \cong U \circ \mathcal{C}(C, -)_0$. It follows that $\lim H$ is $\lim(S_\mathcal{C}H)$ in $U_*\mathcal{C}$.

(2) Assume that $X \pitchfork A$ exists in $\mathcal{C}$; then

$$\begin{aligned}(U_*\mathcal{C})(C, X \pitchfork A) &\cong U[X, \mathcal{C}(C, A)]\\ &\cong [UX, U\mathcal{C}(C, A)]\\ &\cong [UX, (U_*\mathcal{C})(C, A)]\end{aligned}$$

naturally in $C \in U_*\mathcal{C}$. It follows that $X \pitchfork A$, seen in $U_*\mathcal{C}$, is the power of A by UX.

The dual property involving colimits holds by the arguments above just replacing $\mathcal{C}$ with $\mathcal{C}^{op}$. $\square$

Lemma 3.4.4. *Let $\mathcal{C}$ be a $\mathcal{W}$-category. The ordinary functor*

$$(-)_U \colon [\mathcal{C}, \mathcal{W}]_0 \to [U_*\mathcal{C}, \mathcal{V}]_0$$

is continuous; moreover $\mathcal{C}(C, -)_U \cong (U_*\mathcal{C})(C, -)$.

Proof. To prove the first assertion note that $(-)_U$ can be written as the composite

$$[\mathcal{C}, \mathcal{W}]_0 \xrightarrow{S_{[\mathcal{C},\mathcal{W}]}} (U_*[\mathcal{C}, \mathcal{W}])_0 \xrightarrow{K} [U_*\mathcal{C}, U_*\mathcal{W}]_0 \xrightarrow{\hat{U}\circ -} [U_*\mathcal{C}, \mathcal{V}]_0$$

where $K = (U_*)_{\mathcal{C},\mathcal{W}}$ is the action of U_* on homs. Now, $S_{[\mathcal{C},\mathcal{W}]}$ is continuous by Lemma 3.4.3, and $\hat{U} \circ -$ preserves all limits that exist in $[U_*\mathcal{C}, U_*\mathcal{W}]_0$ since $\hat{U}$ does so (being representable). Finally, observe that K is fully faithful (since U is continuous and strong closed) and $(U_*[\mathcal{C},\mathcal{W}])_0$ contains the representables (since $K \circ U_*(Y_\mathcal{C}) \cong Y_{U_*\mathcal{C}}$); therefore K preserves all limits that exists in $(U_*[\mathcal{C},\mathcal{W}])_0$ (this is a general fact about full subcategories of presheaf categories which contain the representables). It follows at once that $(-)_U$ is continuous. Finally, that $\mathcal{C}(C,-)_U \cong (U_*\mathcal{C})(C,-)$ is an immediate consequence of fact that $K \circ U_*(Y_\mathcal{C}) \cong Y_{U_*\mathcal{C}}$. $\square$

Lemma 3.4.5. *Let $M\colon \mathcal{C}^{op} \to \mathcal{W}$ be a $\mathcal{W}$-weight and $H\colon \mathcal{C} \to \mathcal{W}$ be a $\mathcal{W}$-functor. Then*

$$U(M * H) \cong M_U * H_U.$$

Proof. The weighted colimit $M * H$ can be seen as a coend

$$\sum_{D,E\in\mathcal{C}} \mathcal{C}(D,E) \otimes MD \otimes HE \rightrightarrows \sum_{C\in\mathcal{C}} MC \otimes HC \twoheadrightarrow M * H.$$

Since U is cocontinuous and strong monoidal, the image of this under it leads to the coequalizer

$$\sum_{D,E\in U_*\mathcal{C}} (U_*\mathcal{C})(D,E) \otimes M_U D \otimes H_U E \rightrightarrows \sum_{C\in U_*\mathcal{C}} M_U C \otimes M_U C \twoheadrightarrow U(M * H),$$

where we used that $U\mathcal{C}(D,E) \cong (U_*\mathcal{C})(D,E)$ and that $UMC \cong M_U C$, similarly for H. Since $M_U * H_U$ can be seen as the coend above, it follows that $U(M * H) \cong M_U * H_U$. $\square$

Before moving on let us point out some properties of the base $\mathcal{W}$. First note that, even though $L\mathcal{G}$ is a strong generator of $\mathcal{W}$ made of dualizable objects, $\mathcal{W}$ may not be locally dualizable since the unit need not be projective.

Since the elements of $\mathcal{G}$ are finitely presentable and projective in $\mathcal{V}$, and U is cocontinuous, the elements of $L\mathcal{G}$ are finitely presentable and projective as well; it follows that $\mathcal{W}$ is a finitary quasivariety [64, Definition 4.4]. Moreover U sends a strong generator made of finitely presentable objects to one with the same property in $\mathcal{V}$. Finally note that for any $X, Y \in L\mathcal{G}$ the hom $\mathcal{W}_0(X \otimes Y, -) \cong \mathcal{W}_0(X, [Y,-]_0)$ preserves all colimits that $\mathcal{W}_0(X,-)$ preserves; therefore $\mathcal{W}$ is a symmetric monoidal closed finitary quasivariety and in particular locally finitely presentable as a closed category.

3.4.2 The characterization theorem

Fix $\mathcal{W}$, $\mathcal{V}$, and $U\colon \mathcal{W} \to \mathcal{V}$ as in the previous section.

Remark 3.4.6. Note that the α-small $\mathcal{W}$-weighted limits are generated by the α-small conical ones and powers by the strong generator $L\mathcal{G}$. Since the latter are dualizable objects, and powers by them are absolute, it follows that a $\mathcal{W}$-functor preserves α-small weighted limits of and only if it preserves α-small conical ones. Therefore a $\mathcal{W}$-weight M is α-flat if and only if $M * -$ preserves all α-small conical limits.

Given $M\colon \mathcal{C}^{op} \to \mathcal{W}$ we can consider two different categories of elements:

- $\mathrm{El}(M_J)$: this has objects pairs $(C \in \mathcal{C}, x\colon J \to M(C))$ and arrows $f\colon (C,x) \to (D,y)$ given by $f \in \mathcal{C}_0(C,D)$ with $Mf(y) = x$.

- $\mathrm{El}(M_{LI})$: objects are pairs $(C \in U_*\mathcal{C}, y\colon I \to M_U(C))$ and arrows $f\colon (C,x) \to (D,y)$ are given by $f \in (U_*\mathcal{C})_0(C,D)$ with $Mf(y) = x$.

Therefore we have an induced ordinary functor $S_M\colon \mathrm{El}(M_J) \longrightarrow \mathrm{El}(M_{LI})$ which makes the square below commute.

$$\begin{tikzcd} \mathrm{El}(M_J) \arrow[r, "S_M"] \arrow[d, "\pi"'] & \mathrm{El}(M_{LI}) \arrow[d, "\pi"] \\ \mathcal{C}_0 \arrow[r, "S_\mathcal{C}"'] & (U_*\mathcal{C})_0 \end{tikzcd}$$

Proposition 3.4.7. *Let $\mathcal{C}$ be a $\mathcal{W}$-category with copowers by LI and $M\colon \mathcal{C}^{op} \to \mathcal{W}$ be a $\mathcal{W}$-weight. Then:*

1. *the functor $S_\mathcal{C}\colon \mathcal{C}_0 \to (U_*\mathcal{C})_0$ has a left adjoint $T_\mathcal{C}$ given by $T_\mathcal{C}C\colon = LI \cdot C$;*
2. *the functor $S_M\colon \mathrm{El}(M_J) \to \mathrm{El}(M_{LI})$ has a left adjoint T_M which makes the square below commute.*

$$\begin{tikzcd} \mathrm{El}(M_J) \arrow[d, "\pi"'] & \mathrm{El}(M_{LI}) \arrow[l, "T_M"'] \arrow[d, "\pi"] \\ \mathcal{C}_0 & (U_*\mathcal{C})_0 \arrow[l, "T_\mathcal{C}"] \end{tikzcd}$$

Proof. (1). For any $C \in (U_*\mathcal{C})_0$ we obtain:

$$\begin{aligned} (U_*\mathcal{C})_0(C, S_\mathcal{C}-) &\cong \mathcal{V}_0(I, (U_*\mathcal{C})(C, S_\mathcal{C}-)_0) \\ &\cong \mathcal{V}_0(I, U \circ \mathcal{C}(C,-)_0) \\ &\cong \mathcal{W}_0(LI, \mathcal{C}(C,-)_0) \\ &\cong \mathcal{C}_0(LI \cdot C, -) \end{aligned}$$

where we used the fact that $(U_*\mathcal{C})(C, S_\mathcal{C}-)_0 \cong U \circ \mathcal{C}(C,-)_0$. Therefore $(U_*\mathcal{C})_0(C, S_\mathcal{C}-)$ is represented by the object $LI \cdot C$ of $\mathcal{C}_0$, and hence $LI \cdot (-)\colon (U_*\mathcal{C})_0 \to \mathcal{C}_0$ is a left adjoint to $S_\mathcal{C}$.

(2). By point (1) applied to $\mathcal{C} = \mathcal{W}^{op}$ the functor $S_\mathcal{W}\colon \mathcal{W}_0 \to (U_*\mathcal{W})_0$ has a right adjoint given pointwise by $[LI, -]$; notice that to give an arrow $X \to Y$ in $(U_*\mathcal{W})_0$ is the same as giving $UX \to UY$ in $\mathcal{V}_0$. Now, since M preserves powers by LI (these being absolute), the left adjoint $T_\mathcal{C}$ to $S_\mathcal{C}$ extends to a left adjoint of S_M as follows: $T(C,y) := (T_\mathcal{C}C, y^t)$ where $T_\mathcal{C}C = LI \cdot C$ as above and $y^t\colon J \to M(T_\mathcal{C}C) \cong [LI, MC]$ is the transpose of $y\colon I \cong UJ \to M_UC \cong UM(C)$ seen as a map $J \to M(C)$ in $(U_*\mathcal{W})_0$. A routine verification shows that the resulting T_M is left adjoint to S_M. $\square$

In the presence of such a nice change of base we can reduce $\mathcal{W}$-flatness to $\mathcal{V}$-flatness:

Proposition 3.4.8. *Let $\mathcal{C}$ be a $\mathcal{W}$-category with finite direct sums and copowers by $L\mathcal{G}$; let $M\colon \mathcal{C}^{op} \to \mathcal{W}$ be a $\mathcal{W}$-weight. The following are equivalent:*

1. *M is α-flat;*

2. *$M * -$ preserves conical α-small limits of representables;*

3. *M_U is α-flat as a $\mathcal{V}$-weight.*

Proof. (1) $\Rightarrow$ (2) is trivial. Let us consider (2) $\Rightarrow$ (3). To begin with, note that by Lemma 3.4.3 the $\mathcal{V}$-category $U_*\mathcal{C}$ has finite direct sums and copowers by $UL\mathcal{G}$, which is a strong generator of $\mathcal{V}$ made of dualizable objects. Thus, by Proposition 3.2.14, it is enough to prove that the category $\mathrm{El}(M_{LI}) = \mathrm{El}((M_U)_I)$ is α-filtered.

Consider therefore an α-small family of objects $(C_i, x_i)_{i\in I}$ in $\mathrm{El}(M_{LI})$; we need to find a cocone for that. Note that the family $(x_i)_i$ corresponds to a map $x\colon I \to \prod\limits_i UM(C_i)$ in $\mathcal{V}$. Moreover

$$\begin{aligned}
\prod_i UM(C_i) &\cong U(\prod_i M * \mathcal{C}(C_i, -)) \\
&\cong U(M * \prod_i \mathcal{C}(C_i, -)) \\
&\cong M_U * (\prod_i \mathcal{C}(C_i, -))_U \\
&\cong M_U * \prod_i U_*\mathcal{C}(C_i, -).
\end{aligned}$$

As a consequence, x corresponds to a map $x'\colon I \to M_U * \prod_i U_*\mathcal{C}(C_i, -)$. By Lemma 3.2.12 there exist then $D \in U_*\mathcal{C}$, $y\colon I \to M_U(D)$ and $f = (f_i)_i \in \prod_i (U_*\mathcal{C})_0(C_i, D)$ which map down to x'. In other words we obtained (D, y) and maps $f_i\colon (C_i, x, i) \to (D, y)$ in $\mathrm{El}(M_{LI})$, as desired.

Consider now an α-small family of parallel maps $\{f_i\colon (C, x) \to (D, y)\}_{i\in I}$ in $\mathrm{El}(M_{LI})$; we need to find an arrow coequalizing them. Since S_M has a left adjoint T_M we can consider the square below.

$$\begin{array}{ccc}
S_M T_M(C,x) & \xrightarrow{S_M T_M(f_i)} & S_M T_M(D,y) \\
\uparrow{\scriptstyle \eta_{(C,x)}} & & \uparrow{\scriptstyle \eta_{(D,y)}} \\
(C,x) & \xrightarrow[f_i]{} & (D,y)
\end{array}$$

It is then enough to find a map out of $S_M T(D, y)$ which coequalizes the $S_M T(f_i)$'s. Therefore, without loss of generality, we can assume $f_i = S_M(g_i)$ for some g_i in $\mathcal{C}_0$. Now we can argue as in the previous case: y defines an arrow $\bar{y}\colon I \to \mathrm{Eq}\ U(M(g_i))_i$ in $\mathcal{V}$ and we have

$$\begin{aligned}
\mathrm{Eq}\ U(M(g_i))_i &\cong U\left(\mathrm{Eq}(M * \mathcal{C}(g_i, -))_i\right) \\
&\cong U\left(M * \mathrm{Eq}\ \mathcal{C}(g_i, -)_i\right) \\
&\cong M_U * \mathrm{Eq}\ (U_*\mathcal{C})(f_i, -)_i
\end{aligned}$$

As a consequence, $\bar{y}$ corresponds to a map $y'\colon I \to M_U * \mathrm{Eq}(U_*\mathcal{C})(f_i, -)_i$. By Lemma 3.2.12 there exist then $E \in U_*\mathcal{C}$, $z\colon I \to M_U(E)$ and $f \in \mathrm{Eq}\ (U_*\mathcal{C})_0(f_i, E)$ which map down to x'. In other words we obtained an object (E, z) and a map $g\colon (D, y) \to (E, z)$ in $\mathrm{El}(M_{LI})$ coequalizing the f_i's. It follows that $\mathrm{El}(M_{LI})$ is α-filtered.

(3) $\Rightarrow$ (1). Assume now that M_U is α-flat; it is enough to prove that the $\mathcal{W}$-functor $M * -\colon [\mathcal{C}, \mathcal{W}] \to \mathcal{W}$ preserves all α-small conical limits (by Remark 3.4.6). Since U is continuous and conservative, that holds if and only if $U(M * -)\colon [\mathcal{C}, \mathcal{W}]_0 \to \mathcal{V}_0$ preserves all α-small conical limits. But $U(M * -) \cong M_U * (-)_U$, where $(-)_U$ is continuous (by

Lemma 3.4.4) and $M_U * -$ preserves α-small conical limits because M_U is α-flat. Thus the result follows. □

The following is a consequence of the results above and is what justifies the notion of *protofiltered* colimit introduced below.

Corollary 3.4.9. *Let $\mathcal{C}$ be a $\mathcal{W}$-category with finite direct sums and copowers by $L\mathcal{G}$, and let $M\colon \mathcal{C}^{op} \to \mathcal{W}$ be an α-flat $\mathcal{W}$-weight. Then:*

(i) $\mathrm{El}(M_{LI})$ *is α-filtered;*

(ii) $S_M\colon \mathrm{El}(M_J) \to \mathrm{El}(M_{LI})$ *is final.*

Proof. The first point follows directly by the proof of (1) ⇒ (3) above. The second point holds since S_M has a left adjoint by Proposition 3.4.7. □

Given a $\mathcal{W}$-category $\mathcal{A}$ consider the ordinary functor $S_{\mathcal{A}}\colon \mathcal{A}_0 \to (U_*\mathcal{A})_0$ introduced near the beginning of Section 3.4.1.

Definition 3.4.10. We say that an ordinary functor $S\colon \mathcal{E} \to \mathcal{F}$ is an *α-protofiltered index* if $\mathcal{F}$ is α-filtered and S is final.
An *S-indexed diagram* in a $\mathcal{W}$-category $\mathcal{A}$ is a pair of functors (H_1, H_2) making the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, dashed, "H_2"] & (U_*\mathcal{A})_0 \\
\mathcal{E} \arrow[u, "S"] \arrow[r, dashed, "H_1"'] & \mathcal{A}_0 \arrow[u, "S_{\mathcal{A}}"']
\end{tikzcd}$$

commute (strictly). We define its colimit, if it exists, as $\operatorname{colim}(H_1, H_2) := \operatorname{colim} H_1$ the conical colimit of H_1 in $\mathcal{A}$.

When $\operatorname{colim}(H_1, H_2)$ exists in $\mathcal{A}$ then $\operatorname{colim}(S_{\mathcal{A}}H_1)$ exists as a conical colimit in $U_*\mathcal{A}$ and

$$S_{\mathcal{A}}(\operatorname{colim} H_1) \cong \operatorname{colim}(S_{\mathcal{A}}H_1) \cong \operatorname{colim} H_2$$

where the first isomorphism holds because $S_{\mathcal{A}}$ preserves all conical colimits that exist in $\mathcal{A}$ (by Lemma 3.4.3), while the latter is true since S is final.

Remark 3.4.11. Note that if S is an α-protofiltered index the category $\mathcal{C}$ is in general not α-filtered (the protosplit coequalizer below gives a counterexample); see also 1.3.3.

Example 3.4.12.

- **Protosplit coequalizers**. The *protosplit index* is defined as the inclusion H of the free-living pair into the free-living split pair. Then protosplit coequalizers are just H-indexed colimits; these are α-filtered indexes for any α. For $\mathcal{W} = \mathbf{DGAb}$ protosplit coequalizers were first introduced in [80].

- **Categories of elements**. Let $M\colon \mathcal{C}^{op} \to \mathcal{W}$ be as in Proposition 3.4.8 above; then the functor $S_M\colon \mathrm{El}(M_J) \to \mathrm{El}(M_{LI})$ is an α-protofiltered index.

There is a way to express α-protofiltered colimits as honest α-flat weighted colimits: see Section 3.4.3.

The following is needed for the characterization of α-flat $\mathcal{W}$-functors.

Proposition 3.4.13. *Let $\mathcal{C}$ be a $\mathcal{W}$-category with finite direct sums and copowers by $L\mathcal{G}$, and let $M\colon \mathcal{C}^{op} \to \mathcal{W}$ be an α-flat $\mathcal{W}$-weight; then*

$$M \cong \operatorname{colim}\left(\mathrm{El}(M_J)_{\mathcal{W}} \xrightarrow{\pi_{\mathcal{W}}} \mathcal{C} \xrightarrow{Y} [\mathcal{C}^{op}, \mathcal{W}]\right).$$

Proof. As usual it is enough to prove that for any $C \in \mathcal{C}$ the comparison map between $M(C)$ and the colimit of $\mathcal{C}(C, \pi-)_0\colon \mathrm{El}(M_J) \to \mathcal{W}_0$ is an isomorphism. For that, note that we can consider the commutative square below.

$$\begin{tikzcd}
\mathrm{El}(M_{LI}) \arrow[r, "\pi"] & (U_*\mathcal{C})_0 \arrow[r, "(U_*\mathcal{C})(C,-)_0"] & \mathcal{V}_0 \\
\mathrm{El}(M_J) \arrow[u, "S_M"] \arrow[r, "\pi'"'] & \mathcal{C}_0 \arrow[u, "S_{\mathcal{C}}"'] \arrow[r, "\mathcal{C}(-,C)_0"'] & \mathcal{W}_0 \arrow[u, "U"']
\end{tikzcd}$$

Since S_M is final (Corollary 3.4.9) and U is conservative and cocontinuous, it is then enough to show that $M_U(C) = UM(C)$ is the colimit of $(U_*\mathcal{C})(C, \pi'-)_0$. But this is a consequence of Proposition 3.2.14 since M_U is α-flat by Proposition 3.4.8. □

Then, under the presence of some absolute colimits, we can express every α-flat colimit as an α-protofiltered colimit of representables:

Corollary 3.4.14. *Let $\mathcal{C}$ be a $\mathcal{W}$-category with finite direct sums and copowers by $L\mathcal{G}$, and let $M\colon \mathcal{C}^{op} \to \mathcal{W}$ be an α-flat $\mathcal{W}$-weight. Then M is an α-protofiltered colimit of representables; more precisely M is the α-protofiltered colimit of the diagram below.*

$$\begin{tikzcd}
\mathrm{El}(M_{LI}) \arrow[r, "\pi"] & (U_*\mathcal{C})_0 \arrow[r, "(U_*Y)_0"] & (U_*[\mathcal{C}^{op}, \mathcal{W}])_0 \\
\mathrm{El}(M_J) \arrow[u, "S_M"] \arrow[r, "\pi"'] & \mathcal{C}_0 \arrow[u, "S_{\mathcal{C}}"'] \arrow[r, "Y_0"'] & {[\mathcal{C}^{op}, \mathcal{W}]_0} \arrow[u, "R_{[\mathcal{C},\mathcal{W}]}"']
\end{tikzcd}$$

Proof. This is a direct consequence of the proposition above and Example 3.4.12. □

In conclusion:

Theorem 3.4.15. *If $M\colon \mathcal{C}^{op} \to \mathcal{W}$ is a $\mathcal{W}$-functor, the following are equivalent:*

1. *M is α-flat;*
2. *$M * -\colon [\mathcal{C}, \mathcal{W}] \to \mathcal{W}$ preserves α-small weighted limits of representables;*
3. *M lies in the closure of the representables under copowers by $L\mathcal{G}$, finite direct sums, and α-protofiltered colimits.*

Proof. Same as that of Proposition 3.2.16. □

Theorem 3.4.16. *A $\mathcal{W}$-category $\mathcal{A}$ has α-flat colimits if and only if it has finite direct sums, copowers by $L\mathcal{G}$, and α-protofiltered colimits. A $\mathcal{V}$-functor from such an $\mathcal{A}$ preserves α-flat colimits if and only if it preserves α-protofiltered colimits.*

Proof. Same as that of Theorem 3.2.18. □

Since in this case we do not know whether the flat colimits are generated by absolute and filtered colimits, we cannot compare accessible and conically accessible $\mathcal{V}$-categories (as we did in the previous sections). What we can say is the following:

Theorem 3.4.17. *Let $\mathcal{A}$ be a $\mathcal{W}$-category with α-flat colimits; then:*

1. *$A \in \mathcal{A}_\alpha$ if and only if $\mathcal{A}(A,-)$ preserves α-protofiltered colimits;*

2. *$\mathcal{A}$ is α-accessible if and only if $\mathcal{A}_\alpha$ is small and every object is an α-protofiltered colimit of α-presentable objects.*

Proof. (1) is a consequence of the Theorem above. To prove (2) we use Theorem 3.4.15 and argue as in the proof of Theorem 3.2.19. Given an α-flat $M\colon \mathcal{C}^{op} \to \mathcal{W}$ and a diagram $K\colon \mathcal{C} \to \mathcal{A}_\alpha \subseteq \mathcal{A}$, we can consider the free cocompletion $\mathcal{D}$ of $\mathcal{C}$ under finite direct sums and $L\mathcal{G}$-copowers, with inclusion $J\colon \mathcal{C} \to \mathcal{D}$. Let $M' := \mathrm{Lan}_{J^{op}} M$ and $K' := \mathrm{Lan}_J K$; then: $M * K \cong M' * K'$, the diagram K' still lands in $\mathcal{A}_\alpha$ (since this is closed in $\mathcal{A}$ under finite direct sums and $L\mathcal{G}$-copowers), the weight M' is still α-flat (Lemma 1.3.2), and the domain of M' satisfies the hypotheses of Corollary 3.4.14. Thus we can write $M' \cong \mathrm{colim}(Y_0 H_1, (U_*Y)_0 H_2)$ as an α-protofiltered colimit of representables, where (H_1, H_2) is an α-protofiltered diagram in $\mathcal{D}$. It follows that

$$M * K \cong \mathrm{colim}\ (K'_0 H_1, (U_*K')_0 H_2),$$

either side existing if the other does. Thus an object of $\mathcal{A}$ is an α-flat colimit of α-presentables if and only if it is an α-protofiltered colimit of α-presentables. The result then follows. $\square$

Remark 3.4.18. Let $\mathcal{A}$ be an α-accessible $\mathcal{W}$-category; then every object A of $\mathcal{A}$ can be written canonically as the α-protofiltered colimit of the diagram below.

$$\begin{tikzcd}
(U_*\mathcal{A}_\alpha)_0/A \arrow[r, "\pi"] & (U_*\mathcal{A})_0 \\
(\mathcal{A}_\alpha)_0/A \arrow[r, "\pi"'] \arrow[u, "s_A"] & \mathcal{A}_0 \arrow[u, "s_\mathcal{A}"']
\end{tikzcd}$$

Indeed, let $H\colon \mathcal{A}_\alpha \to \mathcal{A}$ be the inclusion; since $\mathcal{A}$ is α-accessible, every object A can be written as the α-flat colimit $\mathcal{A}(H-,A) * H$. Then the result follows thanks to Corollary 3.4.14 applied to $M = \mathcal{A}(H-,A)$ since $(\mathcal{A}_\alpha)_0/A = \mathrm{El}(\mathcal{A}(H-,A)_J)$ and $(U_*\mathcal{A}_\alpha)_0/A = \mathrm{El}(\mathcal{A}(H-,A)_{LI})$.

We say that an index $J\colon \mathcal{E} \to \mathcal{F}$ is *protoabsolute* if it is α-protofiltered for every α; equivalently if $\mathcal{F}$-colimits are Cauchy in the ordinary sense. The following result then generalizes [80, Theorem 7.2] to our setting:

Corollary 3.4.19. *Let $\mathcal{C}$ be a $\mathcal{W}$-category; the following are equivalent:*

1. *$\mathcal{C}$ is Cauchy complete;*

2. *$\mathcal{C}$ has finite direct sums, powers and copowers by dualizable objects, and protoabsolute colimits;*

3. *$\mathcal{C}$ has finite direct sums, copowers by $L\mathcal{G}$, and protoabsolute colimits.*

Proof. The proof is completely analogous to that of Corollary 3.2.22. $\square$

3.4.3 Protofiltered colimits as weighted colimits

Let $\mathcal{W}, \mathcal{V}$, and $U\colon \mathcal{W} \to \mathcal{V}$ be as in Section 3.4.1. Given an α-protofiltered index $S\colon \mathcal{E} \to \mathcal{F}$ we construct a $\mathcal{W}$-category $\mathcal{S}$ and a weight $\Delta\colon \mathcal{S}^{op} \to \mathcal{W}$ such that S-indexed colimits correspond to Δ-weighted colimits and Δ is α-flat.

Remark 3.4.20. The 2-category $\mathcal{W}$-**Cat** is locally finitely presentable as a 2-category. Indeed its underlying ordinary category $\mathcal{W}$-$\mathbf{Cat}_0$ is locally finitely presentable by [58, Theorem 4.5] and the finitely presentable objects are closed under copowers by $\mathbb{2}$ [58, Proposition 4.8]. It follows that every object which is finitely presentable in the ordinary sense is also finitely presentable in the 2-categorical sense. Therefore $\mathcal{W}$-**Cat** is locally finitely presentable by [57, 7.5].

Let us first define $\mathcal{S}$. Consider the 2-functor $S_*\colon \mathcal{W}\text{-}\mathbf{Cat} \to \mathbf{Cat}$ defined pointwise as the pullback

$$\begin{tikzcd} S_*\mathcal{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}(\mathcal{F}, (U_*\mathcal{A})_0) \arrow[d, "- \circ S"] \\ \mathbf{Cat}(\mathcal{E}, \mathcal{A}_0) \arrow[r, "S_{\mathcal{A}} \circ -"'] & \mathbf{Cat}(\mathcal{E}, (U_*\mathcal{A})_0) \end{tikzcd}$$

in **Cat** for any $\mathcal{A} \in \mathcal{W}\text{-}\mathbf{Cat}$, where $S_{\mathcal{A}}\colon \mathcal{A}_0 \to (U_*\mathcal{A})_0$ is the ordinary functor introduced in Section 3.4. Note that $U_*\colon \mathcal{W}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}$, $(-)_0\colon \mathcal{W}\text{-}\mathbf{Cat} \to \mathbf{Cat}$, and $(-)_0\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathbf{Cat}$ are continuous as 2-functors; indeed they preserves all conical limits as well as powers by $\mathbb{2}$ (since U is continuous and strong closed). Moreover they preserve α-filtered colimits for some α. Similarly $\mathbf{Cat}(\mathcal{E}, -)$ and $\mathbf{Cat}(\mathcal{F}, -)$ are continuous and preserve β-filtered colimits for some $\beta \geq \alpha$ (as is true for every object of a locally presentable category). It follows that S_* is a finite limit of 2-functors which are continuous and preserve β-filtered colimits. Since finite limits commute with β-filtered colimits in **Cat**, it follows that S_* is continuous and preserves β-filtered colimits as well. As a consequence, since $\mathcal{W}$-**Cat** and **Cat** are locally presentable 2-categories, it follows that S_* has a left adjoint [57, 7.9] and therefore S_* is a representable 2-functor. Now we can define $\mathcal{S}$ as the $\mathcal{W}$-category which represents S_*.

For any *small* $\mathcal{W}$-category $\mathcal{A}$ we have an isomorphism of categories $[\mathcal{S}, \mathcal{A}]_0 \cong S_*\mathcal{A}$; it is easy to see that this isomorphism actually holds for any (possibly large) $\mathcal{W}$-category $\mathcal{A}$. It follows in particular that to give a $\mathcal{W}$-functor $H\colon \mathcal{S} \to \mathcal{A}$ is the same as to give a S-indexed diagram (H_1, H_2) in $\mathcal{A}$. The same holds for $\mathcal{W}$-functors out of $\mathcal{S}^{op}$ just by moving the $(-)^{op}$ to the codomain.

Remark 3.4.21. Taking $\mathcal{A} = \mathcal{S}$, we note that the identity on $\mathcal{S}$ corresponds to functors $K_1\colon \mathcal{E} \to (\mathcal{S})_0$ and $K_2\colon \mathcal{F} \to (U_*\mathcal{S})_0$. These two functors induce the bijection just mentioned: given $H\colon \mathcal{S} \to \mathcal{A}$ then $H_1 = H_0 \circ K_1$ and $H_2 = (U_*H)_0 \circ K_2$.

In particular it follows that $H_1^t = H \circ K_1^t$ and $H_2^t = U_*H \circ K_2^t$, where $K_1^t\colon \mathcal{E}_{\mathcal{W}} \to \mathcal{S}$ and $K_2^t\colon \mathcal{F}_{\mathcal{V}} \to U_*\mathcal{S}$ are the transposes of K_1 and K_2 respectively. Therefore for any $A \in \mathcal{A}$ we obtain $\mathcal{A}(H-, A)_1^t \cong \mathcal{A}(H_1^t-, A)$, and for any $X \in \mathcal{V}$ we have $X \pitchfork H_1^t \cong (X \pitchfork H)_1^t$ whenever such powers exist in $\mathcal{A}$.

Next we define the weight $\Delta\colon \mathcal{S}^{op} \to \mathcal{W}$ as the one such that Δ^{op} corresponds under the isomorphism $\mathcal{W}\text{-}\mathbf{Cat}(\mathcal{S}, \mathcal{W}^{op}) \cong S_*\mathcal{W}^{op}$ to the S-indexed diagram

$$(\Delta J\colon \mathcal{E} \to \mathcal{W}_0^{op}, \Delta J\colon \mathcal{F} \to (U_*\mathcal{W})_0^{op}).$$

We can now prove that limits and colimits weighted by Δ are the same as S-indexed limits and colimits.

Proposition 3.4.22. *Let $H\colon \mathcal{S}^{op} \to \mathcal{W}$ be any $\mathcal{W}$-functor with induced S-index (H_1, H_2) in $\mathcal{W}$. Then*

$$[\mathcal{S}^{op}, \mathcal{W}](\Delta, H) \cong [\mathcal{E}^{op}_{\mathcal{W}}, \mathcal{W}](\Delta J^t, H_1^t)$$

where $\Delta J^t, H_1^t\colon \mathcal{E}^{op}_{\mathcal{W}} \to \mathcal{W}$ are the transposes of $\Delta J, H_1\colon \mathcal{E}^{op} \to \mathcal{W}_0$, and the isomorphism in induced by precomposition with K_1^t. In other words $\{\Delta, H\} \cong \lim H_1$.

Proof. The isomorphism above holds if and only if applying $\mathcal{W}_0(X, -)$ on each side we get a bijection of sets. But

$$\mathcal{W}_0(X, [\mathcal{S}^{op}, \mathcal{W}](\Delta, H)) \cong [\mathcal{S}^{op}, \mathcal{W}]_0(\Delta, X \pitchfork H),$$

similarly on the right-hand-side (using that $X \pitchfork H_1^t \cong (X \pitchfork H)_1^t$, see Remark 3.4.21). Therefore it is enough to prove that we have a bijection between $\mathcal{W}$-natural transformations $\eta\colon \Delta \Rightarrow H$ and $\eta_1\colon \Delta J \Rightarrow H_1$ for any H.

Since we have an isomorphism of categories $[\mathcal{S}, \mathcal{W}^{op}]_0 \cong S_*\mathcal{W}^{op}$, it follows that to give a $\mathcal{W}$-natural transformation $\eta\colon \Delta \Rightarrow H$ is the same as giving a pair of ordinary natural transformations $\eta_i\colon \Delta J \Rightarrow H_i$, for $i = 1, 2$, such that $S_{\mathcal{W}}\eta_1 = \eta_2 S$. Since S is final, η_2 is uniquely determined by $S_{\mathcal{W}}\eta_1$; therefore it is enough to give η_1 and hence we have the desired bijection. $\square$

An immediate consequence is the following:

Corollary 3.4.23. *Let $\mathcal{A}$ be any $\mathcal{W}$-category. For any $H\colon \mathcal{S} \to \mathcal{A}$ let (H_1, H_2) be the induced S-indexed diagram in $\mathcal{A}$; then*

$$\Delta * H \cong \operatorname{colim}(H_1, H_2)$$

either side existing if the other does.

Proof. It is enough to consider the following chain of isomorphisms for any $A \in \mathcal{A}$:

$$\begin{aligned}
\mathcal{A}(\Delta * H, A) &\cong [\mathcal{S}^{op}, \mathcal{W}](\Delta, \mathcal{A}(H-, A)) \\
&\cong [\mathcal{E}^{op}_{\mathcal{W}}, \mathcal{W}](\Delta J^t, \mathcal{A}(H-, A)_1^t) \\
&\cong [\mathcal{E}^{op}_{\mathcal{W}}, \mathcal{W}](\Delta J^t, \mathcal{A}(H_1^t-, A)) \\
&\cong \mathcal{A}(\operatorname{colim} H_1, A)
\end{aligned}$$

where the second isomorphism holds by the proposition above, while the third follows from the fact that $\mathcal{A}(H-, A)_1^t \cong \mathcal{A}(H_1^t-, A)$ from Remark 3.4.21. $\square$

The final step is to prove that the weight Δ is α-flat.

Proposition 3.4.24. *Let $S\colon \mathcal{E} \to \mathcal{F}$ be an α-protofiltered index; then $\Delta\colon \mathcal{S}^{op} \to \mathcal{W}$ is an α-flat $\mathcal{W}$-weight.*

Proof. First note that there is a commutative triangle as below,

$$\begin{tikzcd}
\mathcal{W}_0 \arrow[rr, "U"] \arrow[dr, "S_{\mathcal{W}}"'] & & \mathcal{V}_0 \\
& (U_*\mathcal{W})_0 \arrow[ur, "\hat{U}_0"'] &
\end{tikzcd}$$

where $\hat{U} = (U_*\mathcal{W})(I,-)\colon U_*\mathcal{W} \to \mathcal{V}$ was introduced in Section 3.4.1 and is such that $(-)_U = \hat{U} \circ U_*(-)$. Let $H\colon \mathcal{S} \to \mathcal{W}$ be a $\mathcal{W}$-functor; then by the corollary above we know that

$$U(\Delta * H) \cong U(\operatorname*{colim}_{\mathcal{E}} H_1) \cong \operatorname*{colim}_{\mathcal{E}}(\hat{U}_0 S_{\mathcal{W}} H_1) \cong \operatorname*{colim}_{\mathcal{E}}(\hat{U}_0 H_2 S) \cong \operatorname*{colim}_{\mathcal{F}}(\hat{U}_0 H_2)$$

and this can be rewritten as

$$U(\Delta * -) \cong \operatorname*{colim}_{\mathcal{F}}((-)_U \circ K_2^t)\colon [\mathcal{S},\mathcal{W}]_0 \to \mathcal{V}_0$$

where $K_2^t\colon \mathcal{F}_{\mathcal{V}} \to U_*\mathcal{E}_J$ is the transpose of the ordinary functor $K_2\colon \mathcal{F} \to (U_*\mathcal{S})_0$ introduced in Remark 3.4.21. Now, $(-)_U$ and $(- \circ K_2^t)$ are continuous and $\operatorname*{colim}_{\mathcal{F}}(-)$ preserves α-small ordinary limits ($\mathcal{F}$ is α-filtered); therefore $U(\Delta * -)$ preserves α-small limits as well. To conclude then note that, since U is continuous and conservative, $\Delta * -$ preserves all α-small conical limits, and this is enough to guarantee that Δ is α-flat by Remark 3.4.6. $\square$

CHAPTER 4

On continuity of accessible functors

Because of the variety of ways of characterizing locally presentable categories, different notions of morphisms between them have been considered in the literature. In this chapter we define a morphism between locally α-presentable categories to consist of a functor which is continuous and α-accessible; these, together with the natural transformations, identify a 2-category that we call $\mathbf{Lp}_\alpha$. Alternatively one could define a morphism between locally α-presentable categories to be a cocontinuous functor which preserves the α-presentable objects; this describes a 2-category biequivalent to $\mathbf{Lp}_\alpha^{op}$.

Our aim is then to give a characterization of the morphisms out of a locally α-presentable category $\mathcal{K} \in \mathbf{Lp}_\alpha$ in terms of those that preserve γ-small limits, for some determined γ. More specifically we prove that for any locally α-presentable category $\mathcal{K}$ there exists a regular cardinal γ such that an α-accessible $F\colon \mathcal{K} \to \mathcal{L}$, with $\mathcal{L}$ locally α-presentable, is continuous if and only if it preserves all γ-small limits (Theorem 4.1.6). The choice of γ depends entirely on the category $\mathcal{K}_\alpha$ (Remark 4.1.7).

In Section 4.1 we introduce the necessary background notions, prove the main result, and then give a few applications including a new adjoint functor theorem relative the α-presentable case. Then, in Section 4.2 we prove an enriched version of the main result based on the notion of locally presentable $\mathcal{V}$-category that we have already considered. We obtain again an adjoint functor theorem specialized to the α-accessible case (Theorem 4.2.4), and moreover we prove that a small $\mathcal{V}$-category is accessible if and only if it is Cauchy complete (Theorem 4.2.7).

The content of this chapter has been published in Applied Categorical Structures [101].

4.1 The Set-case

We start this section by recalling the notion of α-small functor:

Definition 4.1.1. [57, 4.1] A functor $M\colon \mathcal{C}^{op} \to \mathbf{Set}$ is called α-small if $\mathcal{C}$ is an (essentially) α-small category and M lands in $\mathbf{Set}_\alpha$.

Note that, assuming we are given an α-small category $\mathcal{C}$, to say that $M\colon \mathcal{C}^{op} \to \mathbf{Set}$ is α-small is the same as saying that it is α-presentable as an object of $[\mathcal{C}^{op}, \mathbf{Set}]$.

The result below first appeared (with a different choice of γ) in the proof of [77, Theorem 2.2.2].

Lemma 4.1.2. *Let $\mathcal{C}$ be such that $\mathcal{C}(B,C)$ is β-small for any B and C, and let $\gamma > \beta$. Then any γ-flat functor $M\colon \mathcal{C}^{op} \to \mathbf{Set}$ lands in $\mathbf{Set}_\beta$. If moreover $\mathcal{C}$ has less than γ objects, then M is γ-small.*

Proof. Assume that M does not land in $\mathbf{Set}_\beta$; then we can find $C \in \mathcal{C}$ and a family $(x_i \in M(C))_{i<\beta}$ of cardinality β where the x_i are all distinct. Since M is γ-flat, its category of elements $\mathrm{El}(M)$ is γ-filtered. Now, the x_i form a γ-small family of objects of $\mathrm{El}(M)$; thus there exists $y \in MD$ and morphisms $f_i\colon C \to D$ in $\mathcal{C}$ such that $M(f_i)(y) = x_i$ for any $i < \beta$. But by hypothesis the f_i cannot all be be distinct; contradicting the fact that all the x_i actually are. □

Lemma 4.1.3. *Every α-small and α-flat functor is Cauchy.*

Proof. Let $M\colon \mathcal{C}^{op} \to \mathbf{Set}$ be α-small and α-flat. Consider the free completion $\Phi^\dagger_\alpha\mathcal{C}$ of $\mathcal{C}$ under α-small limits, this comes together with the inclusion $J\colon \mathcal{C} \to \Phi^\dagger_\alpha\mathcal{C}$. Since M is α-small, $\mathrm{El}(M)$ is an essentially α-small category; thus we can consider the limit

$$X := \lim(\mathrm{El}(M) \xrightarrow{\pi} \mathcal{C} \xrightarrow{J} \Phi^\dagger_\alpha\mathcal{C})$$

in $\Phi^\dagger_\alpha\mathcal{C}$. Next we prove that $\mathrm{Lan}_J M$ is isomorphic to the representable $\Phi^\dagger_\alpha\mathcal{C}(X,-)$: since M is α-flat the Kan extension $\mathrm{Lan}_J(M)$ is α-flat too (by Lemma 1.3.2) and hence α-continuous; therefore it is enough to prove that $\mathrm{Lan}_J(M)$ and $\Phi^\dagger_\alpha\mathcal{C}(X,-)$ coincide when restricted to $\mathcal{C}$:

$$\begin{aligned}
\Phi^\dagger_\alpha\mathcal{C}(X, J-) &\cong \Phi^\dagger_\alpha\mathcal{C}(\lim_x(J\circ\pi x), J-) \\
&\cong \operatorname*{colim}_x \Phi^\dagger_\alpha\mathcal{C}(J\circ\pi x, J-) \qquad (4.1)\\
&\cong \operatorname*{colim}_x \mathcal{C}(\pi x, -) \\
&\cong M(-) \\
&\cong \mathrm{Lan}_J(M)(J-)
\end{aligned}$$

as required, where (4.1) holds since $\Phi^\dagger_\alpha\mathcal{C}(-, JC)$ is α-cocontinuous for any $C \in \mathcal{C}$ (property of the free completion). It follows that $\mathrm{Lan}_J M$ is representable, and hence a Cauchy weight (the left Kan extension of a representable functor along Yoneda is an evaluation map, and hence is continuous); by Lemma 1.3.2 then M is Cauchy as well. □

Remark 4.1.4. This Lemma is true more generally for a weakly sound class Φ: if M is such that $\mathrm{El}(M)$ is at the same time in Φ and Φ-filtered, then M is Cauchy.

As a consequence:

Corollary 4.1.5. *Let $\mathcal{C}$ be a small category and γ be a regular cardinal as in the last part of Lemma 4.1.2. Then every γ-flat functor $M\colon \mathcal{C}^{op} \to \mathbf{Set}$ is Cauchy.*

Proof. This is a direct consequence of Lemma 4.1.2 and 4.1.3. □

In short, it is enough to take $\gamma = \beta^+$, where β is such that $\mathcal{C}$ is β-small, but the description given in the Corollary above might provide a smaller cardinal. For example taking $\mathcal{K} = \mathbf{Set}$ and $\alpha = \aleph_0$, since $\mathbf{Set}_f$ is $\aleph_1$-small, then we can certainly consider $\gamma = \aleph_2$; however the hypotheses of Lemma 4.1.2 provide a better γ, in fact one can actually choose $\gamma = \aleph_1$ (since $\mathbf{Set}_f$ has countably many objects and the hom-sets are all finite).

We can now prove the main result of this section:

Theorem 4.1.6. *Let $\mathcal{K}$ be locally α-presentable. There exists a regular cardinal γ for which every α-accessible and γ-continuous $F\colon \mathcal{K} \to \mathcal{L}$, with $\mathcal{L}$ locally α-presentable, is in fact continuous.*

Proof. Let γ be the one given in Corollary 4.1.5 for $\mathcal{C} = \mathcal{K}_\alpha^{op}$, and denote by $J\colon \mathcal{K}_\alpha \hookrightarrow \mathcal{K}$ the inclusion.

Notice that a functor $F\colon \mathcal{K} \to \mathcal{L}$ as above is continuous if and only if $\mathcal{L}(A, F-)$ is such for any $A \in \mathcal{L}_\alpha$. Since $\mathcal{L}(A, F-)$ is still α-accessible and preserves all the limits that F preserves, we can assume without loss of generality $\mathcal{L} = \mathbf{Set}$. Therefore, we are given an α-accessible and γ-continuous $F\colon \mathcal{K} \to \mathbf{Set}$, and we need to prove that it is actually continuous. Since F is α accessible, it is the left Kan extension of its restriction $FJ\colon \mathcal{K}_\alpha \to \mathbf{Set}$; as a consequence the following triangles commute (up to isomorphism),

$$\begin{tikzcd}
{[\mathcal{K}_\alpha^{op}, \mathbf{Set}]} \arrow[dr, "\mathrm{Lan}_Y(FJ)"] & \\
\mathcal{K} \arrow[u, hook, "\mathcal{K}(J,1)"] \arrow[r, "F"] & \mathbf{Set} \\
\mathcal{K}_\alpha \arrow[u, hook, "J"] \arrow[ur, "FJ"'] &
\end{tikzcd}$$

where the vertical composite is the Yoneda embedding $Y\colon \mathcal{K}_\alpha \to [\mathcal{K}_\alpha^{op}, \mathbf{Set}]$, and $\mathcal{K}(J, 1)$ is continuous since it identifies $\mathcal{K}$ with the α-continuous functors out of $\mathcal{K}_\alpha^{op}$. Now, F is γ-continuous by hypothesis; thus $\mathrm{Lan}_Y(FJ)$ preserves γ-small limits of representables (these being γ-small limits in $\mathcal{K}$) and hence FJ is γ-flat by Proposition 1.2.3. By our choice of γ, it follows from Corollary 4.1.5 that FJ is Cauchy and thus $\mathrm{Lan}_Y(FJ)$ is continuous. Therefore F is continuous being the composite of two continuous functors. □

Remark 4.1.7. The optimal regular cardinal γ provided by our proofs is one for which:

- $\mathcal{K}_\alpha$ has less than γ objects (up to isomorphism);
- there exists β such that $\gamma > \beta > \#\mathcal{K}(X, Y)$ for each $X, Y \in \mathcal{K}_\alpha$.

By taking $\mathcal{K} = \mathbf{Set}$, $\alpha = \aleph_0$, and $\gamma = \aleph_1$ (thanks to the comments above the Theorem), it follows that a finitary functor $F\colon \mathbf{Set} \to \mathbf{Set}$ is continuous if and only if it preserves countable products and equalizers.

4.1.1 Some Applications

An(other) adjoint functor theorem

Freyd's *general adjoint functor theorem* says that if $\mathcal{K}$ is complete and $F\colon \mathcal{K} \to \mathcal{L}$ is continuous and satisfies the solution set condition, then it has a left adjoint. In the context of locally presentable categories this is implies that every continuous and accessible functor between locally presentable categories has a left adjoint [1, Theorem 1.66]. Our result is a specialization of this to the case of α-accessible functors:

Theorem 4.1.8. *Let $\mathcal{K}$ be locally α-presentable. There exists a regular cardinal γ such that for any α-accessible $U\colon \mathcal{K} \to \mathcal{L}$, with $\mathcal{L}$ locally α-presentable, the following are equivalent:*

1. *U has a left adjoint;*
2. *U is γ-continuous.*

Note that γ can be chosen again as in Remark 4.1.7.

Dualizable modules

Let $\mathcal{K} = R$-**Mod** be the monoidal category of R-modules for a commutative ring R, and $\alpha = \aleph_0$. Then we can use Theorem 4.1.6 to characterize the dualizable R-modules (that is, the dualizable objects of R-**Mod**).

First let us focus on the optimal choice of γ:

- if R if finite, then R-$\mathbf{Mod}_f$ has countably many objects and its hom-sets are all finite; so we can choose $\gamma = \aleph_1$;
- if $\alpha = \#R$ is infinite, then R-**Mod** has countably many objects but its hom-sets have cardinality α. Thus we can take $\gamma = \alpha^{++}$.

Let M be an R-module; then M is dualizable if and only if the functor $M \otimes -\colon R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ is continuous (it should actually be continuous as an enriched R-**Mod**-functor, but these are equivalent conditions, see also Section 4.2.1). Since every functor $M \otimes -$ is cocontinuous (and hence finitary), a consequence of Theorem 4.1.6 is:

Proposition 4.1.9. *An R-module M is dualizable if and only if it is flat and $M \otimes -$ preserves γ-small products.*

For a finite R this is saying that M needs to be flat and $M \otimes -$ needs to preserve countable products. Note however that the choice of γ is not optimal in general: for $R = \mathbb{Z}$ it is enough to take $\gamma = \aleph_1$ (easy to check), while by the results above we are given $\gamma = \aleph_2$.

Recognising the α-presentables

Let $\mathcal{K}$ be a locally α-presentable category for which $\mathcal{K}_\alpha$ is α-complete. Let $\widehat{\mathcal{K}} := \mathrm{Ind}_\alpha(\mathcal{K}_\alpha^{op})$ be the free cocompletion of $\mathcal{K}_\alpha^{op}$ under α-filtered colimits, so that $\widehat{\mathcal{K}} \simeq \alpha\text{-Cont}(\mathcal{K}_\alpha, \mathbf{Set})$; denote by $H\colon \mathcal{K}_\alpha \to \mathcal{K}$ and $J\colon \mathcal{K}_\alpha^{op} \to \widehat{\mathcal{K}}$ the inclusions. Then we can consider the following composite:

$$\mathcal{K} \xrightarrow{\mathcal{K}(H,1)} \alpha\text{-Cont}(\mathcal{K}_\alpha^{op}, \mathbf{Set}) \xrightarrow{\mathrm{Lan}_J} \alpha\text{-Filt}(\widehat{\mathcal{K}}, \mathbf{Set})$$

where $\mathcal{K}(H,1)$ is actually an equivalence, and α-Filt$(\widehat{\mathcal{K}},\mathbf{Set})$ is the full subcategory of $[\widehat{\mathcal{K}},\mathbf{Set}]$ spanned by the functors that preserve α-filtered colimits (which is equivalent to $[\mathcal{K}_\alpha^{op},\mathbf{Set}]$). Note that Lan_J is fully faithful and its essential image is given by those α-accessible functors which are also α-continuous.

Call the composite of these $G\colon \mathcal{K}\to\alpha$-Filt$(\widehat{\mathcal{K}},\mathbf{Set})$. It is easy to see that if $X\in\mathcal{K}_\alpha$ then $GX\cong\widehat{\mathcal{K}}(X,-)$ is representable and hence continuous; conversely if $F\colon\widehat{\mathcal{K}}\to\mathbf{Set}$ is continuous and preserves α-filtered colimits then $F\cong\widehat{\mathcal{K}}(X,-)$ for some $X\in\widehat{\mathcal{K}}_\alpha\simeq\mathcal{K}_\alpha^{op}$; thus $F\cong GX$. Let now γ be as in Remark 4.1.7, then:

Proposition 4.1.10. *An object X of $\mathcal{K}$ is α-presentable if and only if GX preserves γ-small products.*

Proof. Use the results above and Theorem 4.1.6, plus the fact that GX preserves γ-small limits if and only if it preserves γ-small products (since it already preserves finite limits). □

As an example, consider $\alpha=\aleph_0$ and $\mathcal{K}=\mathbf{Bool}$ the category of boolean algebras and morphisms between them. Now, since $\mathbf{Bool}_f\simeq\mathbf{Set}_f^{op}$, it follows that $\widehat{\mathbf{Bool}}=\mathbf{Set}$; moreover for any $B\in\mathbf{Bool}$ the functor $GB\colon\mathbf{Set}\to\mathbf{Set}$ defined above can be described as the one sending a set X to the set

$$GB(X)=\{f\colon X\to B \text{ of finite support }\mid \bigvee_i fi=1 \text{ and }\; fi\wedge fj=0\ \forall i\neq j\}.$$

(since this preserves filtered colimits and, with this definition of G, we have $GB(X)\cong\mathbf{Bool}(2^X,B)$ for any finite set X). Then by Remark 4.1.7 we can choose $\gamma=\aleph_1$, and hence the proposition above says that a boolean algebra B is finite if and only if the functor GB preserves countable products. The endofunctor GB is actually a monad on **Set** whose algebras are the B-sets (see [10] where $GB(X)$ is denoted by $X[B]^*$).

4.2 The enriched case

We now fix $\mathcal{V}=(\mathcal{V}_0,\otimes,I)$ to be a symmetric monoidal closed and locally presentable category, and consider α such that $\mathcal{V}$ is locally presentable as a closed category. Then we can immediately generalize Theorem 4.1.6 to the enriched setting:

Theorem 4.2.1. *Let $\mathcal{K}$ be a locally α-presentable $\mathcal{V}$-category. There exists a regular cardinal γ for which every α-accessible and γ-continuous $F\colon\mathcal{K}\to\mathcal{L}$, with $\mathcal{L}$ locally α-presentable, is in fact continuous.*

Proof. It is enough to consider γ as in Theorem 4.1.6 for $\mathcal{K}_0$, which is locally α-presentable as an ordinary category. Indeed, if F is γ-continuous in the weighted sense, then $F_0\colon\mathcal{K}_0\to\mathcal{L}_0$ is γ continuous as an ordinary functor, and hence continuous by Theorem 4.1.6. It follows then that F preserves all conical limits and powers by γ-presentable objects. This is enough to ensure that F is continuous since each object of $\mathcal{V}$ is a conical colimit of γ-presentable ones, thus each power in $\mathcal{K}$ is a conical limit of powers by γ-presentables, which are preserved by F. □

Recall that, in the enriched context, a weight $M\colon\mathcal{C}^{op}\to\mathcal{V}$ is called α-small if $\mathcal{C}$ has less than α objects (up to isomorphism), $\mathcal{C}(C,D)\in\mathcal{V}_\alpha$ for any $C,D\in\mathcal{C}$, and M lands in $\mathcal{V}_\alpha$.

Cauchy $\mathcal{V}$-functors have also been widely used and, like in the ordinary setting, can be characterized as those $\mathcal{V}$-functors whose left Kan extension along the Yoneda embedding is continuous, as well as those that are weights for absolute colimits (see for example [59, 95]).

We are now ready to deduce the enriched analogue of Corollary 4.1.5.

Corollary 4.2.2. *Let $\mathcal{C}$ be a small $\mathcal{V}$-category; then there exists γ such that every γ-flat $\mathcal{V}$-functor $M\colon \mathcal{C}^{op} \to \mathcal{V}$ is Cauchy.*

Proof. The weight M is γ-flat if and only if $\mathrm{Lan}_Y M\colon [\mathcal{C},\mathcal{V}] \to \mathcal{V}$ is γ-continuous, and is Cauchy if and only if $\mathrm{Lan}_Y M$ is continuous. Thus it is enough to take γ as in Theorem 4.2.1 for $\mathcal{K} = [\mathcal{C},\mathcal{V}]$, which is locally α-presentable, and $F = \mathrm{Lan}_Y M$ (which is cocontinuous, and hence α-accessible). □

Note also that Lemma 4.1.3 has an enriched version:

Lemma 4.2.3. *Every α-small and α-flat weight is Cauchy.*

Proof. Let $M\colon \mathcal{C}^{op} \to \mathcal{V}$ be α-small and α-flat. Consider the free completion $\Phi_\alpha\mathcal{C}$ of $\mathcal{C}$ under α-small weighted colimits, this comes together with the inclusion $J\colon \mathcal{C} \to \Phi_\alpha\mathcal{C}$. Since M is α-small we can consider the colimit $X := M * J$ in $\Phi_\alpha\mathcal{C}$. Then we prove that $\mathrm{Lan}_{J^{op}}M$ is isomorphic to the representable $\Phi_\alpha\mathcal{C}(-,X)$: since M is α-flat the Kan extension $\mathrm{Lan}_{J^{op}}M$ is α-flat too and hence α-continuous; therefore it is enough to prove that $\mathrm{Lan}_{J^{op}}M$ and $\Phi_\alpha\mathcal{C}(-,X)$ coincide when restricted to $\mathcal{C}$:

$$\begin{aligned}\Phi_\alpha\mathcal{C}(J-,X) &\cong \Phi_\alpha\mathcal{C}(J-,M*J)\\ &\cong M\square * \Phi_\alpha\mathcal{C}(J-,J\square)\\ &\cong M\square * \mathcal{C}(-,\square)\\ &\cong M(-).\end{aligned}$$

Since also $\mathrm{Lan}_{J^{op}}M$ restricts to M we are done. It follows that $\mathrm{Lan}_{J^{op}}M$ is a Cauchy weight (since every representable $\mathcal{V}$-functor is); by Lemma 1.3.2 then M is Cauchy too. □

4.2.1 Some Applications

An(other) adjoint functor theorem

As in the ordinary case we obtain an adjoint functor theorem specialized to the α-accessible $\mathcal{V}$-functors. This is again a consequence of Theorem 4.2.1 and of the fact that every continuous and accessible $\mathcal{V}$-functor between locally presentable $\mathcal{V}$-categories has a left adjoint.

Theorem 4.2.4. *Let $\mathcal{K}$ be a locally α-presentable $\mathcal{V}$-category. There exists a regular cardinal γ such that for any α-accessible $\mathcal{V}$-functor $U\colon \mathcal{K} \to \mathcal{L}$, with $\mathcal{L}$ locally α-presentable, the following are equivalent:*

1. *U has a left adjoint;*
2. *U is γ-continuous.*

Note that γ can be chosen again as in Remark 4.1.7 for $\mathcal{K}_0$.

Dualizable objects

Recall from Section 3.2.1 that an object $X \in \mathcal{V}$ is called dualizable if there exist $X^* \in \mathcal{V}$ and morphisms $\eta_X \colon I \to X \otimes X^*$ and $\epsilon_X \colon X^* \otimes X \to I$, called unit and counit respectively, satisfying the triangle equalities. Equivalently, since $\mathcal{V}$ is closed, X is dualizable if and only if there exists $X^* \in \mathcal{V}$ such that $X \otimes - \cong [X^*, -] \colon \mathcal{V}_0 \to \mathcal{V}_0$. By [59, Section 6], this is equivalent to $X \otimes -$ being continuous. Then a direct application of Theorem 4.2.1 to $F = M \otimes -$ gives:

Proposition 4.2.5. *There exists a regular cardinal γ such that an object $X \in \mathcal{V}$ is dualizable if and only if $X \otimes - \colon \mathcal{V} \to \mathcal{V}$ is γ-continuous.*

The following is an application of Lemma 4.2.3:

Proposition 4.2.6. *Let $\mathcal{V}$ be locally α-presentable as a closed category. An object $X \in \mathcal{V}$ is dualizable if and only if:*

1. *X is α-presentable;*

2. *X is α-flat, or equivalently $X \otimes -$ is α-continuous.*

Small accessible $\mathcal{V}$-categories

As an application of the main Theorem we can prove a generalization of [1, Proposition 2.6] which shows that a small ordinary category is accessible if and only if it has splittings of idempotents, or, equivalently, if it is Cauchy complete.

In the enriched context we have the notions of accessible and conically accessible $\mathcal{V}$-category. It is easy to see, using the ordinary characterization, that a small $\mathcal{V}$-category is conically accessible if and only if it has splittings of idempotents (that is, if the underlying category is Cauchy complete). As we see below, the analogue result for accessible $\mathcal{V}$-categories requires the enriched notion of Cauchy completeness.

Recall that a $\mathcal{V}$-category is called Cauchy complete if it has all colimits weighted by Cauchy $\mathcal{V}$-functors; this is equivalent to saying that every Cauchy $M \colon \mathcal{C}^{op} \to \mathcal{V}$ is representable. We are now ready to prove:

Theorem 4.2.7. *A small $\mathcal{V}$-category is accessible if and only if it is Cauchy complete.*

Proof. Every accessible $\mathcal{V}$-category is Cauchy complete since Cauchy weights are α-flat for every α. For the opposite direction consider any Cauchy complete and small $\mathcal{V}$-category $\mathcal{C}$. By Corollary 4.2.2 we can find γ such that every γ-flat $\mathcal{V}$-functor $M \colon \mathcal{C}^{op} \to \mathcal{V}$ is Cauchy. Since $\mathcal{C}$ is Cauchy complete, this means that every γ-flat weight out of $\mathcal{C}^{op}$ is representable; therefore $\mathcal{C} \simeq \gamma\text{-Flat}(\mathcal{C}^{op}, \mathcal{V})$ is accessible. □

CHAPTER

5

Accessible categories with limits of some class

In this chapter we characterize those accessible $\mathcal{V}$-categories that have limits of a specified class. We do this by introducing the notion of companion $\mathfrak{C}$ for a class of weights Ψ, as a collection of special types of colimit diagrams that are compatible with Ψ. We then characterize the accessible $\mathcal{V}$-categories with Ψ-limits as those accessibly embedded and $\mathfrak{C}$-*reflective* in a presheaf $\mathcal{V}$-category, and as the $\mathcal{V}$-categories of $\mathfrak{C}$-*models* of sketches. This allows us to recover the standard theorems for locally presentable, locally multipresentable, and locally polypresentable categories as instances of the same general framework. In addition, our theorem covers the case of any weakly sound class Ψ, and provides a new perspective on the case of weakly locally presentable categories.

We begin by introducing the general setting of the chapter (section 5.1). The main results are discussed in Section 5.2 where we introduce the notion of companions and prove the characterization theorems (5.2.16 and 5.2.17). In Section 5.2.2 we define the notion of $\mathfrak{C}$-model of a sketch whenever $\mathfrak{C}$ is a companion for a class Ψ, and give conditions on $\mathfrak{C}$ so that $\mathcal{V}$-categories of $\mathfrak{C}$-models characterize accessible $\mathcal{V}$-categories with Ψ-limits (Theorem 5.2.27).

Section 5.3 is entirely devoted to examples. We first discuss the case of a weakly sound class Ψ (Section 5.3.1); in this case the colimit types are actually classes of weights and, unlike in the general case, we are able to give a corresponding weakening of orthogonality. The main results of the section are Theorems 5.3.15 and 5.3.16. Next we consider the class of wide pullbacks (Section 5.3.2); here we recover the classical results on locally polypresentable categories and compare our results with those given in [50]. In Section 5.3.3 we consider a generalization of the weakly locally presentable categories to the enriched setting, while in Section 5.3.4 we discuss the case of accessible 2-categories with flexible limits.

Finally, in Section 5.4, we compare reflectivity with respect to a colimit type $\mathfrak{C}$ with the existing notions of weak reflectivity. We do this to obtain the characterization theorems of [1] for weakly locally presentable categories, and of [63] for accessible 2-categories with flexible limits, as instances of our theory.

5.1 The general setting

To begin, recall that by a *weight* we mean a $\mathcal{V}$-functor $M\colon \mathcal{C}^{op} \to \mathcal{V}$ with small domain. From now on we shall consider a class Ψ of weights representing the kind of limits that our accessible $\mathcal{V}$-categories will be assumed to have. In general, this class Ψ will not be locally small in the sense of [59], meaning that the free completion of a small category under Ψ-limits might not be small. Examples of such are the classes for connected limits, products, and wide pullbacks.

We denote (co)completions under Ψ-colimits and under Ψ-limits by $\Psi\mathcal{A}$ and $\Psi^\dagger\mathcal{A}$ respectively. When $\Psi = \mathcal{P}$ is the class of all weights, we recover the free (co)completions $\mathcal{P}\mathcal{A}$ and $\mathcal{P}^\dagger\mathcal{A}$ under small colimits and limits.

In this section we aim to capture a notion of Ψ-continuity, for a class Ψ as above, in the absence of Ψ-limits. Then we use this description to obtain a first instance of a characterization theorem for accessible $\mathcal{V}$-categories with Ψ-limits (Theorem 5.1.11).

To begin with, consider a *small* $\mathcal{V}$-category $\mathcal{A}$; we are interested in those $\mathcal{V}$-functors $M\colon \mathcal{A}^{op} \to \mathcal{V}$ for which $\mathrm{Lan}_Y M\colon [\mathcal{A},\mathcal{V}] \to \mathcal{V}$ preserves Ψ-limits of diagrams landing in $\mathcal{A}^{op}$, where $Y\colon \mathcal{A}^{op} \to [\mathcal{A},\mathcal{V}]$ is the Yoneda embedding. When $\mathcal{A}$ is Ψ-cocomplete, and so $\mathcal{A}^{op}$ is Ψ-complete, this is just requiring that M is a Ψ-continuous $\mathcal{V}$-functor.

Since $\mathrm{Lan}_Y M \cong M * -$, the condition above is saying that M-weighted colimits commute in $\mathcal{V}$ with Ψ-limits of representable $\mathcal{V}$-functors. That is, for any $N\colon \mathcal{D} \to \mathcal{V}$ in Ψ and $H\colon \mathcal{D} \to \mathcal{A}^{op}$ the canonical map defines an isomorphism

$$M * \{N, YH\} \cong \{N, M * YH\}. \tag{5.1}$$

This approach turns out still to be useful when the $\mathcal{V}$-category $\mathcal{A}$ is not assumed to be small, but $M\colon \mathcal{A}^{op} \to \mathcal{V}$ is a small $\mathcal{V}$-functor. In fact, in this case the collection of $\mathcal{V}$-functors $[\mathcal{A},\mathcal{V}]$ does not form a $\mathcal{V}$-category in general, but colimits of arbitrary $\mathcal{V}$-functors $\mathcal{A} \to \mathcal{V}$ weighted by M do exist (since M is small), and weighted limits of $\mathcal{V}$-functors are always defined pointwise in $\mathcal{V}$. Therefore, both sides of (5.1) still exist, as does the canonical comparison, and we can give the following definition:

Definition 5.1.1. Let Ψ be a class of weights, $\mathcal{A}$ be a $\mathcal{V}$-category, and $M\colon \mathcal{A}^{op} \to \mathcal{V}$ a small $\mathcal{V}$-functor. We say that M is Ψ*-precontinuous* if M-weighted colimits commute in $\mathcal{V}$ with Ψ-limits of representable $\mathcal{V}$-functors; in other words if $M * -$ preserves Ψ-limits of representables. Denote by $\Psi\text{-PCts}(\mathcal{A}^{op},\mathcal{V})$ the full subcategory of $[\mathcal{A}^{op},\mathcal{V}]$ spanned by the small Ψ-precontinuous $\mathcal{V}$-functors.

Note that we have the inclusions $\mathcal{A} \subseteq \Psi\text{-PCts}(\mathcal{A}^{op},\mathcal{V}) \subseteq \mathcal{P}\mathcal{A}$ so that $\Psi\text{-PCts}(\mathcal{A}^{op},\mathcal{V})$ is in particular a $\mathcal{V}$-category.

Example 5.1.2. If Ψ is a sound class of weights then Ψ-precontinuous and Ψ-flat $\mathcal{V}$-functors coincide; hence $\Psi\text{-PCts}(\mathcal{A}^{op},\mathcal{V}) = \Psi^+\mathcal{A}$, where Ψ^+ is the class of all the Ψ-flat weights.

Proposition 5.1.3. *Let $M\colon \mathcal{A}^{op} \to \mathcal{V}$ be Ψ-precontinuous; then M preserves any existing Ψ-limits. If $\mathcal{A}$ is Ψ-cocomplete, then a small M is Ψ-precontinuous if and only if it is Ψ-continuous.*

Proof. The Yoneda embedding $Y\colon \mathcal{A}^{op} \hookrightarrow [\mathcal{A},\mathcal{V}]$ preserves any existing Ψ-limits in $\mathcal{A}^{op}$ and $M * Y- \cong M$. Therefore if M is Ψ-precontinuous then it preserves any Ψ-limit that happens to exist. If $\mathcal{A}$ is Ψ-cocomplete then Ψ-limits of representables are still representables so Ψ-continuity implies Ψ-precontinuity. $\square$

Corollary 5.1.4. *The inclusion $V\colon \mathcal{A} \to \Psi\text{-PCts}(\mathcal{A}^{op},\mathcal{V})$ preserves any existing Ψ-colimits.*

Proof. This says that for any $M \in \Psi\text{-PCts}(\mathcal{A}^{op},\mathcal{V})$ the $\mathcal{V}$-functor $\Psi\text{-PCts}(\mathcal{A}^{op},\mathcal{V})(V-,M)$ preserves any existing Ψ-limits. But $\Psi\text{-PCts}(\mathcal{A}^{op},\mathcal{V})(V-,M) \cong M$, so that follows by the proposition above. $\square$

Corollary 5.1.5. *For any Ψ-cocomplete $\mathcal{V}$-category $\mathcal{A}$ we have an equality*

$$\Psi\text{-PCts}(\mathcal{A}^{op},\mathcal{V}) = \mathcal{P}\mathcal{A} \cap \Psi\text{-Cont}(\mathcal{A}^{op},\mathcal{V})$$

so that Ψ-precontinuous $\mathcal{V}$-functors out of $\mathcal{A}^{op}$ coincide with the small Ψ-continuous $\mathcal{V}$-functors. Moreover $V\colon \mathcal{A} \to \Psi\text{-PCts}(\mathcal{A}^{op},\mathcal{V})$ is Ψ-cocontinuous.

Proof. By Proposition 5.1.3, if $\mathcal{A}$ is Ψ-cocomplete then a small $\mathcal{V}$-functor $M\colon \mathcal{A}^{op} \to \mathcal{V}$ is Ψ-precontinuous if and only if it is Ψ-continuous. $\square$

Recall that the notion of *virtual cocompleteness* for a $\mathcal{V}$-category $\mathcal{A}$ was introduced in Definition 2.2.15; it says that the free completion $\mathcal{P}^\dagger\mathcal{A}$ has colimits of objects from $\mathcal{A}$.

Proposition 5.1.6. *The following are equivalent for a Ψ-complete $\mathcal{V}$-category $\mathcal{A}$:*

1. *$\mathcal{A}$ is virtually cocomplete;*
2. *$\mathcal{P}^\dagger\mathcal{A}$ is cocomplete;*
3. *$\Psi\text{-PCts}(\mathcal{A},\mathcal{V})^{op}$ is cocomplete;*
4. *$\Psi\text{-PCts}(\mathcal{A},\mathcal{V})^{op}$ has colimits of representables.*

Proof. (1) $\Leftrightarrow$ (2) is always true by the dual of [33, Theorem 3.8]. For the other implications note that, since $\mathcal{A}$ is Ψ-complete, $\Psi\text{-PCts}(\mathcal{A},\mathcal{V})^{op}$ consists of the small Ψ-continuous $\mathcal{V}$-functors out of $\mathcal{A}$, seen in $\mathcal{P}^\dagger\mathcal{A}$. It follows that (2) $\Rightarrow$ (3) since $\Psi\text{-PCts}(\mathcal{A},\mathcal{V})^{op}$ is then closed under colimits in $\mathcal{P}^\dagger\mathcal{A}$. That (3) $\Rightarrow$ (4) is trivial, while (4) $\Rightarrow$ (1) since the inclusion $\Psi\text{-PCts}(\mathcal{A},\mathcal{V})^{op} \hookrightarrow \mathcal{P}^\dagger\mathcal{A}$ preserves any existing limits. $\square$

A $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{K}$ has a left adjoint if each $\mathcal{K}(X,F-)\colon \mathcal{A} \to \mathcal{V}$ is representable; while it has a virtual left adjoint if each $\mathcal{K}(X,F-)$ is small. Similarly:

Definition 5.1.7. We say that a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{K}$ has a Ψ-virtual left adjoint if for each $X \in \mathcal{K}$ the $\mathcal{V}$-functor $\mathcal{K}(X,F-)$ is Ψ-precontinuous. If F is fully faithful we say that $\mathcal{A}$ is Ψ-virtually reflective in $\mathcal{K}$.

In other words, F has a Ψ-virtual left adjoint if and only if for each $X \in \mathcal{K}$ the functor $\mathcal{K}(X,F-)$ has a relative left V-adjoint, where $V\colon \mathcal{A} \hookrightarrow \Psi\text{-PCts}(\mathcal{A},\mathcal{V})^{op}$ is the inclusion. If $\Psi = \emptyset$ is the class with no weights, a $\emptyset$-precontinuous $\mathcal{V}$-functor is simply a small $\mathcal{V}$-functor; thus $\emptyset$-virtual left adjoints coincide with virtual left adjoints.

Proposition 5.1.8. *The following are equivalent for a Ψ-complete and virtually cocomplete $\mathcal{V}$-category $\mathcal{A}$, and a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$:*

1. *F is small and Ψ-continuous;*
2. *F has a Ψ-virtual left adjoint.*

Proof. A Ψ-virtual left adjoint $L\colon \mathcal{V} \to \Psi\text{-PCts}(\mathcal{A},\mathcal{V})^{op}$ exists if and only if the $\mathcal{V}$-functor $[X, F-]$ is small and Ψ-precontinuous, and in that case is given by $LX := [X, F-]$. Now, if F is small and Ψ-continuous it is in particular Ψ-precontinuous; thus $[X, F-]$ is Ψ-precontinuous too (since $[X, -]$ preserves all limits) and small (being the copower of F by X in $\mathcal{P}^\dagger\mathcal{A}$, which is cocomplete). Conversely, if F has a Ψ-virtual left adjoint then $F \cong [I, F-]$ is in $\Psi\text{-PCts}(\mathcal{A},\mathcal{V})^{op}$, and therefore it is small and Ψ-continuous. $\square$

In the accessible case we then obtain:

Corollary 5.1.9. *The following are equivalent for a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{K}$ between Ψ-complete accessible $\mathcal{V}$-categories:*

1. *F is accessible and Ψ-continuous;*
2. *F has a Ψ-virtual left adjoint.*

Proof. It follows from the proposition above plus the fact that every accessible $\mathcal{V}$-category is virtually cocomplete, and that, for $\mathcal{A}$ an accessible $\mathcal{V}$-category, $\mathcal{K}(X, F-)\colon \mathcal{A} \to \mathcal{V}$ is accessible if and only if it is small (Proposition 2.2.9). $\square$

Theorem 5.1.10. *Let $\mathcal{K}$ be an accessible $\mathcal{V}$-category with Ψ-limits and $\mathcal{A}$ a full subcategory of $\mathcal{K}$. The following are equivalent:*

1. *$\mathcal{A}$ is accessible, accessibly embedded, and closed under Ψ-limits;*
2. *$\mathcal{A}$ is accessibly embedded and Ψ-virtually reflective.*

Proof. (1) $\Rightarrow$ (2). This follows by Corollary 5.1.9 above.

(2) $\Rightarrow$ (1). Since $\mathcal{A}$ is Ψ-virtually reflective it is also virtually reflective; thus we are given a diagram as below

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook, "V"] \arrow[d, hook, "J"'] & \mathcal{B} \arrow[r, hook, "Z"] & \mathcal{P}^\dagger\mathcal{A} \arrow[d, hook, "\mathcal{P}^\dagger J"'] \arrow[d, phantom, shift left=2, "\vdash"] \\
\mathcal{K} \arrow[ur, "L"] \arrow[rr, hook, "Z'"] & & \mathcal{P}^\dagger\mathcal{K} \arrow[u, bend right, "L'"']
\end{tikzcd}$$

where $\mathcal{B} := \Psi\text{-PCts}(\mathcal{A},\mathcal{V})^{op}$, L is a relative left V-adjoint to J, and $ZL \cong L'Z'$. Being virtually reflective and accessibly embedded, $\mathcal{A}$ is also accessible and a virtual orthogonality class in $\mathcal{K}$ by Proposition 2.2.29. It only remains to prove that $\mathcal{A}$ is closed in $\mathcal{K}$ under Ψ-limits. By the proof of Proposition 2.2.29 the virtual orthogonality class defining $\mathcal{A}$ can be defined by the units $\eta_X\colon ZX \to (\mathcal{P}^\dagger J)ZLX$ of the virtual adjunction, for all $X \in \mathcal{K}_\alpha$ (and some α). This can equivalently be seen as the set

$$\mathcal{M} = \{\eta_X\colon ZX \to \{LX, Z'J\} \mid X \in \mathcal{K}_\alpha\}.$$

To conclude it is then enough to prove that any Ψ-limit in $\mathcal{K}$ of a diagram in $\mathcal{A}$ is still orthogonal to $\mathcal{M}$.

Consider then $Y=\{N,JS\}$ with $N\colon\mathcal{D}\to\mathcal{V}$ in Ψ and $S\colon\mathcal{D}\to\mathcal{A}=\mathcal{M}^{\perp}$, and let X be any object of $\mathcal{K}_\alpha$; then $LX=\mathcal{K}(X,J)\colon\mathcal{A}\to\mathcal{V}$ is Ψ-precontinuous. Consider then the following chain of isomorphisms

$$\begin{aligned}
\mathcal{K}(X,Y) &\cong \{N-,\mathcal{K}(X,JS-)\}\\
&\cong \{N-,\mathcal{B}(LX,VS-)\}\\
&\cong \{N-,LX\Box*\mathcal{B}(V\Box,VS-)\}\\
&\cong \{N-,LX\Box*\mathcal{A}(\Box,S-)\}\\
&\cong LX\Box*\{N-,\mathcal{A}(\Box,S-)\} \qquad (5.2)\\
&\cong LX\Box*\{N-,\mathcal{K}(J\Box,JS-)\}\\
&\cong LX\Box*\mathcal{K}(J\Box,Y)\\
&\cong \mathcal{P}^{\dagger}\mathcal{K}(\{LX,Z'J\},Z'Y)
\end{aligned}$$

where (5.2) is true by construction since LX is Ψ-precontinuous. This proves that Y is still orthogonal to $\mathcal{M}$, and therefore lies in $\mathcal{A}$. $\square$

Theorem 5.1.11. *The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:*

1. *$\mathcal{A}$ is accessible and Ψ-complete;*
2. *$\mathcal{A}$ is accessible and Ψ-$\mathrm{PCts}(\mathcal{A},\mathcal{V})^{op}$ is cocomplete;*
3. *$\mathcal{A}$ is accessible and Ψ-$\mathrm{PCts}(\mathcal{A},\mathcal{V})^{op}$ has colimits of representables;*
4. *$\mathcal{A}$ is accessibly embedded and Ψ-virtually reflective in $[\mathcal{C},\mathcal{V}]$ for some $\mathcal{C}$.*

In that case Ψ-$\mathrm{PCts}(\mathcal{A},\mathcal{V})$ consists of the small Ψ-continuous $\mathcal{V}$-functors.

Proof. (2) ⇒ (3) is trivial and (4) ⇒ (1) is a consequence of Theorem 5.1.10.

(1) ⇒ (2). $\mathcal{A}$ is accessible and therefore virtually cocomplete; thus it follows from Proposition 5.1.6 that Ψ-$\mathrm{PCts}(\mathcal{A},\mathcal{V})^{op}$ is cocomplete.

(3) ⇒ (4). Let α be such that $\mathcal{A}$ is α-accessible; then take $\mathcal{C}=\mathcal{A}_\alpha^{op}$ so that we have an accessible embedding $J\colon\mathcal{A}\hookrightarrow[\mathcal{C},\mathcal{V}]$. There is a $\mathcal{V}$-functor $G\colon\mathcal{C}^{op}\to\Psi$-$\mathrm{PCts}(\mathcal{A},\mathcal{V})^{op}$ which, up to isomorphism, is just the inclusion $\mathcal{A}_\alpha\subseteq\mathcal{A}\subseteq\Psi$-$\mathrm{PCts}(\mathcal{A},\mathcal{V})^{op}$. Since Ψ-$\mathrm{PCts}(\mathcal{A},\mathcal{V})^{op}$ has colimits of objects of $\mathcal{A}$ and G lands in $\mathcal{A}$ by construction, it has a essentially unique cocontinuous extension $L\colon[\mathcal{C},\mathcal{V}]\to\Psi$-$\mathrm{PCts}(\mathcal{A},\mathcal{V})^{op}$ which is a left V-adjoint to J, as desired.

$\square$

5.2 Main results

5.2.1 Colimit types and companions

The notion of Ψ-precontinuous $\mathcal{V}$-functor introduced in the previous section is not enough to capture the known characterization theorems for locally polypresentable and weakly locally presentable categories. To obtain them we need some explicit description of the Ψ-precontinuous functors. For this reason we now introduce the notions of *colimit type* and *companion*. This will also allow us to capture things like colimits of free groupoid actions which, as we saw in the introduction, arise in the characterization theorem of locally polypresentable categories.

Definition 5.2.1. A *colimit type* $\mathfrak{C}$ is the data of a full replete subcategory

$$\mathfrak{C}_M \hookrightarrow [\mathcal{C}, \mathcal{V}]$$

for any weight $M \colon \mathcal{C}^{op} \to \mathcal{V}$. This may equivalently be given by specifying the class

$$\mathfrak{C} = \{(M, H) \mid H \in \mathfrak{C}_M\}$$

of weighted diagrams.

Let us see some examples.

Example 5.2.2. If Φ is a class of weights there is a colimit type $\mathfrak{C}^\Phi$ with

$$\mathfrak{C}^\Phi_M = \begin{cases} [\mathcal{C}, \mathcal{V}] & \text{if } M \in [\mathcal{C}^{op}, \mathcal{V}] \text{ is in } \Phi, \\ \emptyset & \text{otherwise.} \end{cases}$$

Example 5.2.3. For $\mathcal{V} = \mathbf{Set}$, consider the colimit type $\mathfrak{F}$ defined by: $H \in \mathfrak{F}_M$, for $M \colon \mathcal{C}^{op} \to \mathbf{Set}$, if and only if $\mathcal{C}$ is a groupoid, $M = \Delta 1$, and $H \colon \mathcal{C} \to \mathbf{Set}$ is free, in the sense that

$$0 \longrightarrow HA \overset{Hf}{\underset{Hg}{\rightrightarrows}} HB$$

is an equalizer for any $f, g \colon A \to B$ with $f \neq g$.

Example 5.2.4. For $\mathcal{V} = \mathbf{Set}$, consider the colimit type $\mathfrak{R}$ defined by: $H \in \mathfrak{R}_M$, for $M \colon \mathcal{C}^{op} \to \mathbf{Set}$, if and only if $\mathcal{C} = \{x \rightrightarrows y\}$ is the free category on a pair of arrows, $M = \Delta 1$, and $H \colon \mathcal{C} \to \mathbf{Set}$ is a pseudo equivalence relation, in the sense that the pair of functions identified by H factors as

$$Hx \overset{e}{\twoheadrightarrow} Z \overset{h}{\underset{k}{\rightrightarrows}} Hy$$

an epimorphism e followed by a kernel pair (h, k). Such a factorization, when it exists, is unique since it will be given by the epi-mono factorization of the induced $Hx \to Hy \times Hy$. An equivalent definition is [29, Definition 6] which explains why these are called pseudo equivalence relations.

Given a class of weights Ψ and a colimit type $\mathfrak{C}$, we express the commutativity of Ψ-limits with colimits of diagrams indexed on $\mathfrak{C}$ as follows:

Definition 5.2.5. Let Ψ be a class of weights and $\mathfrak{C}$ be a colimit type; we say that $\mathfrak{C}$ *is compatible with* Ψ if, for any presheaf $M \colon \mathcal{C}^{op} \to \mathcal{V}$, either $\mathfrak{C}_M = \emptyset$ or $\mathfrak{C}_M \subseteq [\mathcal{C}, \mathcal{V}]$ is closed under Ψ-limits and the composite

$$\mathfrak{C}_M \hookrightarrow [\mathcal{C}, \mathcal{V}] \xrightarrow{M * -} \mathcal{V}$$

preserves them.

Remark 5.2.6. For our purposes it would be enough to require that $M * -$ preserve Ψ-limits of diagrams landing in $\mathfrak{C}_M$; however, since the condition above seems more natural and is satisfied by all the examples we have, we opted for that.

Example 5.2.7.

1. If $\mathfrak{C} = \mathfrak{C}^\Phi$, for a class of weights Φ, the compatibility condition says that Φ consists of (some or all) Ψ-flat weights (see Section 5.3.1).

2. The colimit type $\mathfrak{F}$ of free groupoid actions is compatible with wide pullbacks (see Section 5.3.2).

3. The colimit type $\mathfrak{R}$ of pseudo equivalence relations is compatible with small products (see Section 5.3.3).

With the following definition we introduce the $\mathcal{V}$-categories $\mathfrak{C}_1\mathcal{A}$ and $\mathfrak{C}_1^\dagger\mathcal{A}$ which can be interpreted as a generalization of free (co)completions to the context of our colimit types.

Definition 5.2.8. Let $\mathfrak{C}$ be a colimit type and $\mathcal{A}$ be a $\mathcal{V}$-category. We define $\mathfrak{C}_1\mathcal{A}$ to be the full subcategory of $\mathcal{P}\mathcal{A}$ consisting of:

1. the representables;

2. $M * YH$ for any $M\colon \mathcal{C}^{op} \to \mathcal{V}$ and $H\colon \mathcal{C} \to \mathcal{A}$ for which $\mathcal{A}(A, H-)\colon \mathcal{C} \to \mathcal{A}$ lies in $\mathfrak{C}_M$ for all $A \in \mathcal{A}$.

Dually, let $\mathfrak{C}_1^\dagger\mathcal{A} = \mathfrak{C}_1(\mathcal{A}^{op})^{op}$; this consists of certain $\mathcal{V}$-functors $F\colon \mathcal{A} \to \mathcal{V}$.

Proposition 5.2.9. *If $\mathfrak{C}$ is a colimit type that is compatible with Ψ then*

$$\mathfrak{C}_1\mathcal{A} \subseteq \Psi\text{-PCts}(\mathcal{A}^{op}, \mathcal{V}).$$

In particular every $F\colon \mathcal{A}^{op} \to \mathcal{V}$ in $\mathfrak{C}_1\mathcal{A}$ preserves any existing Ψ-limits and the inclusion $\mathcal{A} \hookrightarrow \mathfrak{C}_1\mathcal{A}$ preserves any existing Ψ-colimits.

Proof. Consider $X \in \mathfrak{C}_1\mathcal{A}$; if X is representable it is Ψ-precontinuous, so suppose that $X \cong M * YH$ for some $M\colon \mathcal{C}^{op} \to \mathcal{V}$ and $H\colon \mathcal{C} \to \mathcal{A}$ for which $\mathcal{A}(A, H-) \in \mathfrak{C}_M$ for all $A \in \mathcal{A}$. We need to show that X-weighted colimits commute with Ψ-limits of representables. For that, consider $N\colon \mathcal{D} \to \mathcal{V}$ in Ψ and $S\colon \mathcal{D} \to \mathcal{A}^{op}$; then:

$$\begin{aligned} X * \{N, YS\} &\cong (M * YH) * \{N, YS\} \\ &\cong M - *\{N\Box, \mathcal{A}(S\Box, H-)\} \\ &\cong \{N\Box, M - *\mathcal{A}(S\Box, H-)\} \qquad (5.3) \\ &\cong \{N, (M * YH) * YS)\} \\ &\cong \{N, X * YS)\} \end{aligned}$$

where (5.3) holds since $\mathcal{A}(SD, H-)$ lies in $\mathfrak{C}_M$ by hypothesis (for any $D \in \mathcal{D}$) and $M * -$ preserves Ψ-limits of diagrams landing in $\mathfrak{C}_M$. $\Box$

Corollary 5.2.10. *If $\mathfrak{C}$ is a colimit type compatible with Ψ and $\mathcal{A}$ is Ψ-cocomplete, then any $F \in \mathfrak{C}_1\mathcal{A}$ is Ψ-continuous and small.*

The following definition identifies when, for a given class of weights Ψ, a colimit type $\mathfrak{C}$ is rich enough to capture results in the spirit of Theorem 5.1.10 and Theorem 5.1.11.

Definition 5.2.11. We say that a colimit type $\mathfrak{C}$ is a *companion* for Ψ if:

(I) $\mathfrak{C}$ is compatible with Ψ;

(II) for any Ψ-complete and virtually cocomplete $\mathcal{A}$, each small Ψ-continuous $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$ lies in $\mathfrak{C}_1^\dagger\mathcal{A}$.

Assuming (I), condition (II) is equivalent to saying that for any Ψ-complete and virtually cocomplete $\mathcal{A}$ we have $\mathfrak{C}_1^\dagger\mathcal{A} = \Psi\text{-PCts}(\mathcal{A},\mathcal{V})^{op}$; that is, $\mathfrak{C}_1^\dagger\mathcal{A}$ consists of the small Ψ-continuous $\mathcal{V}$-functors $F\colon \mathcal{A} \to \mathcal{V}$.

Examples 5.2.12.

1. All the weakly sound classes from Section 5.3.1, where the classes of diagrams are actually classes of weights, give examples of companions.
2. $\mathcal{V} = \mathbf{Set}$, the colimit type $\mathfrak{F}$ given by the free groupoid actions is a companion for wide pullbacks diagrams — see Section 5.3.2.
3. $\mathcal{V} = \mathbf{Set}$, the colimit type $\mathfrak{R}$ given by the pseudo equivalence relations is a companion for small products — see Example 5.3.35 and Section 5.4.
4. More generally, we consider enriched colimit types similar to the class $\mathfrak{R}$ above when the class of weights is given by small products and powers by a dense generator — see Sections 5.3.3 and 5.4.
5. $\mathcal{V} = \mathbf{Cat}$, the colimit type $\mathfrak{P}$ of the pseudo-equivalence 2-relations is a companion for the class of flexible limits — see Sections 5.3.4 and 5.4.
6. $\mathcal{V} = \mathbf{Set}$, the colimit type $\mathfrak{S}^\lambda$ (including $\lambda = \infty$) of λ-sifted diagrams is a companion for the class of λ-small powers — see Section 5.3.5.

Note that, since $\mathfrak{C}_1^\dagger\mathcal{A}$ contains the representables, all existing colimits in $\mathfrak{C}_1^\dagger\mathcal{A}$ are computed pointwise, so that the inclusion $\mathfrak{C}_1^\dagger\mathcal{A} \hookrightarrow \mathcal{P}^\dagger\mathcal{A}$ always preserves any existing colimits.

Proposition 5.2.13. *Let $\mathfrak{C}$ be a companion for Ψ. The following are equivalent for a Ψ-complete $\mathcal{V}$-category $\mathcal{A}$:*

1. *$\mathcal{A}$ is virtually cocomplete;*
2. *$\mathcal{P}\mathcal{A}$ is cocomplete;*
3. *$\mathfrak{C}_1^\dagger\mathcal{A}$ is cocomplete;*
4. *$\mathfrak{C}_1^\dagger\mathcal{A}$ has colimits of representables.*

Proof. This follows from Proposition 5.1.6 since in this case $\mathfrak{C}_1^\dagger\mathcal{A} = \Psi\text{-PCts}(\mathcal{A},\mathcal{V})^{op}$. □

Once again we can generalize the notion of left adjoint and of virtual left adjoint to the case of a colimit type:

Definition 5.2.14. We say that a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{K}$ has a left $\mathfrak{C}$-adjoint if for each $X \in \mathcal{K}$ the $\mathcal{V}$-functor $\mathcal{K}(X, F-)$ lies in $\mathfrak{C}_1^\dagger\mathcal{A}$. If F is fully faithful we then say that $\mathcal{A}$ is $\mathfrak{C}$-reflective in $\mathcal{K}$.

In other words, F has a left $\mathfrak{C}$-adjoint if and only if it has a relative left V-adjoint, where $V\colon \mathcal{A} \hookrightarrow \mathfrak{C}_1^\dagger\mathcal{A}$ is the inclusion.

Proposition 5.2.15. *Let $\mathfrak{C}$ be a companion for Ψ. The following are equivalent for a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{K}$ between Ψ-complete accessible $\mathcal{V}$-categories:*

1. *F is accessible and Ψ-continuous;*

2. *F has a left $\mathfrak{C}$-adjoint.*

Proof. This is a consequence of Corollary 5.1.9 plus the fact that $\mathfrak{C}_1^\dagger\mathcal{A} = \Psi\text{-PCts}(\mathcal{A},\mathcal{V})^{op}$ under these assumptions. □

Theorem 5.2.16. *Let $\mathfrak{C}$ be a companion for Ψ, and $\mathcal{K}$ be an accessible $\mathcal{V}$-category with Ψ-limits. The following are equivalent for a full subcategory $\mathcal{A}$ of $\mathcal{K}$:*

1. *$\mathcal{A}$ is accessible, accessibly embedded, and closed under Ψ-limits;*

2. *$\mathcal{A}$ is accessibly embedded and $\mathfrak{C}$-reflective.*

Proof. (1) $\Rightarrow$ (2) by the proposition above, while (2) $\Rightarrow$ (1) is a consequence of Theorem 5.1.10 plus the fact that $\mathfrak{C}^\dagger\mathcal{A} \subseteq \Psi\text{-PCts}(\mathcal{A},\mathcal{V})^{op}$. □

Theorem 5.2.17. *Let $\mathfrak{C}$ be a companion for the class Ψ. The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:*

1. *$\mathcal{A}$ is accessible and Ψ-complete;*

2. *$\mathcal{A}$ is accessible and $\mathfrak{C}_1^\dagger\mathcal{A}$ is cocomplete;*

3. *$\mathcal{A}$ is accessible and $\mathfrak{C}_1^\dagger\mathcal{A}$ has colimits of representables;*

4. *$\mathcal{A}$ is accessibly embedded and $\mathfrak{C}$-reflective in $[\mathcal{C},\mathcal{V}]$ for some $\mathcal{C}$.*

Proof. Use Theorem 5.2.16 and apply the same proof of Theorem 5.1.11. □

Remark 5.2.18. In Section 5.3.1 we define a notion of Φ-orthogonality class for a class of weights Φ; this will not be possible in the case of a colimit type $\mathfrak{C}$. For example if we consider the type $\mathfrak{F}$ associated to wide pullbacks, then $\mathfrak{F}$-orthogonality in a locally presentable category would coincide with multiorthogonality since $\mathfrak{F}_1^\dagger\mathcal{K} = \text{Fam}^\dagger\mathcal{K}$ whenever $\mathcal{K}$ is a category of presheaves (if $\mathcal{K}$ has a terminal object, all free groupoid actions based in $\mathcal{K}$ must be indexed on the discrete groupoids). This would be in contrast with the existence of locally polypresentable categories which are not locally multipresentable.

5.2.2 Sketches

In this section we treat a notion of model of a sketch which differs from the usual one; to justify and better understand this notion it might be helpful to see models of sketches as *morphisms* in the category of sketches. This can be described as the category whose objects are sketches $\mathcal{S} = (\mathcal{B},\mathbb{L},\mathbb{C})$ and a morphisms $F\colon \mathcal{S}\to\mathcal{S}'$ are functors $F\colon\mathcal{B}\to\mathcal{B}'$ which send the classes of cylinders $\mathbb{L}$ and $\mathbb{C}$ to the classes $\mathbb{L}'$ and $\mathbb{C}'$ respectively.

Denote by $\mathcal{V}_\mathcal{P}$ the *large* sketch based on $\mathcal{V}$ itself and with the two specified classes given by all the limiting and colimiting cylinders in $\mathcal{V}$ (here we are allowing the base of our sketch to be large). Then, under this notation, a model of a sketch $\mathcal{S} = (\mathcal{B},\mathbb{L},\mathbb{C})$ is just a morphism of sketches $F\colon\mathcal{S}\to\mathcal{V}_\mathcal{P}$.

Consider now a colimit type $\mathfrak{C}$ and recall that, given a weight $M\colon\mathcal{D}^{op}\to\mathcal{V}$, we denote by $\mathfrak{C}_M\subseteq[\mathcal{D},\mathcal{V}]$ the full subcategory of those H for which $(M,H)\in\mathfrak{C}$. To introduce the notion of $\mathfrak{C}$-model consider instead of $\mathcal{V}_\mathcal{P}$ the sketch $\mathcal{V}_\mathfrak{C}$ given by $\mathcal{V}$ together with the class $\mathbb{L}$ of all limiting cylinders and the class $\mathbb{C}$ of all colimiting cylinders $\eta\colon M\Rightarrow\mathcal{V}(H-,X)$ for which $H\in\mathfrak{C}_M$. Then we define a $\mathfrak{C}$-model of a sketch $\mathcal{S} = (\mathcal{B},\mathbb{L},\mathbb{C})$ to be a morphism $F\colon\mathcal{S}\to\mathcal{V}_\mathfrak{C}$ in the category of sketches. More explicitly:

Definition 5.2.19. Let $\mathfrak{C}$ be a colimit type and $\mathcal{S} = (\mathcal{B}, \mathbb{L}, \mathbb{C})$ a sketch. A $\mathfrak{C}$-*model* of $\mathcal{S}$ is a $\mathcal{V}$-functor $F \colon \mathcal{B} \to \mathcal{V}$ satisfying the following conditions:

(i) for every γ in $\mathbb{L}$, its image $F\gamma$ is a limiting cylinder in $\mathcal{V}$;

(ii) for every η in $\mathbb{C}$, its image $F\eta$ is a colimiting cocylinder in $\mathcal{V}$;

(iii) for every $\eta \colon M \Rightarrow \mathcal{B}(H-, B)$ in $\mathbb{C}$, the functor FH lies in $\mathfrak{C}_M$.

Denote by $\mathrm{Mod}_{\mathfrak{C}}(\mathcal{S})$ the full subcategory of $[\mathcal{B}, \mathcal{V}]$ spanned by the $\mathfrak{C}$-models of $\mathcal{S}$ in $\mathcal{V}$.

In other words a $\mathfrak{C}$-model is a model of $\mathcal{S}$ (in the usual sense) which in addition satisfies condition (iii).

Remark 5.2.20. When $\mathfrak{C} = \mathfrak{C}^{\Phi}$ is the colimit type defined by a class of weights Φ, then $\mathcal{V}$-categories of $\mathfrak{C}^{\Phi}$-models of (general) sketches and $\mathcal{V}$-categories of models of limit/Φ-colimit sketches are the same. In fact, given a sketch $\mathcal{S} = (\mathcal{B}, \mathbb{L}, \mathbb{C})$, if $\mathbb{C}$ contains a weight which is not in Φ, then $\mathrm{Mod}_{\mathfrak{C}^{\Phi}}(\mathcal{S}) = \emptyset$; while, if all weights appearing in $\mathbb{C}$ lie in Φ, then $\mathrm{Mod}_{\mathfrak{C}^{\Phi}}(\mathcal{S}) = \mathrm{Mod}(\mathcal{S})$.

Accessible $\mathcal{V}$-categories with Ψ-limits can be seen as $\mathcal{V}$-categories of $\mathfrak{C}$-models:

Proposition 5.2.21. *Let $\mathfrak{C}$ be a companion for Ψ, and let $\mathcal{A}$ be accessible, accessibly embedded, and closed under Ψ-limits in $[\mathcal{C}, \mathcal{V}]$. Then there exist a fully faithful $J \colon \mathcal{C} \hookrightarrow \mathcal{B}$ and a sketch $\mathcal{S}$ on $\mathcal{B}$ such that* Ran_J *induces an equivalence*

$$\mathcal{A} \simeq \mathrm{Mod}_{\mathfrak{C}}(\mathcal{S}).$$

Proof. We argue as in the proof of Theorem 5.2.16. Fix $\mathcal{K} = [\mathcal{C}, \mathcal{V}]$ and an accessible embedding $J \colon \mathcal{A} \hookrightarrow \mathcal{K}$; by 5.2.16 we can consider the following diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook, "V"] \arrow[d, hook, "J"'] & \mathfrak{C}_1^{\dagger}\mathcal{A} \arrow[r, hook, "Z"] & \mathcal{P}^{\dagger}\mathcal{A} \arrow[d, hook, "\mathcal{P}^{\dagger}J"', shift right=2] \arrow[d, phantom, "\vdash"] \\
\mathcal{K} \arrow[ur, "L"'] \arrow[rr, hook, "Z'"] & & \mathcal{P}^{\dagger}\mathcal{K} \arrow[u, bend right, "L'"', shift right=2]
\end{tikzcd}$$

with L relative left V-adjoint to J and $ZL \cong L'Z'$. Moreover by Proposition 2.2.29 we can write $\mathcal{A}$ as a virtual orthogonality class in $\mathcal{K}$ with respect to

$$\mathcal{M} := \{\eta_X \colon ZX \to (\mathcal{P}^{\dagger}J)ZLX \mid X \in \mathcal{K}_{\alpha}\}.$$

Now, for each $X \in \mathcal{K}_{\alpha}$, we have an isomorphism $LX \cong \{M_X, VH_X\}$ with $\mathcal{A}(H_X-, A)$ in $\mathfrak{C}_{M_X}$ for any $A \in \mathcal{A}$. It follows that we can rewrite $\mathcal{M}$ as

$$\mathcal{M} = \{\eta_X \colon ZX \to \{M_X, Z'JH_X\} \mid X \in \mathcal{K}_{\alpha}\}.$$

Consider $\mathcal{B}^{op}$ to be the closure of $\mathcal{C}^{op}$ in $\mathcal{K}$ under β-small colimits, where β is such that for every $X \in \mathcal{K}_{\alpha}$, the functor JH_X lands in $\mathcal{B}^{op}$ (then $\mathcal{B}^{op}$ will coincide with $\mathcal{K}_{\beta}$). Now let $(H'_X)^{op}$ be the factorization of JH_X through the inclusion $\mathcal{B}^{op} \hookrightarrow \mathcal{K}$; then every η_X in $\mathcal{M}$ corresponds to a cocylinder $\gamma_X \colon M_X \to \mathcal{K}(X, JH_X-) \cong \mathcal{B}(H'_X-, X)$. It is proved in Proposition 2.2.30 that $\mathcal{A}$ is the equivalent to the $\mathcal{V}$-category of models of the sketch $\mathcal{S}$ on $\mathcal{B}$ with

- $\mathbb{L}$ consisting of all the β-small limiting cylinders in $\mathcal{B}$;

- $\mathbb{C}$ given by all the γ_X for $X \in \mathcal{K}_\alpha$;

The equivalence is $RJ \colon \mathcal{A} \to \mathrm{Mod}(\mathcal{S})$ where R is obtained by right Kan extending along the inclusion $\mathcal{C} \hookrightarrow \mathcal{B}$. It is now enough to prove that for this $\mathcal{S}$ we already have

$$\mathrm{Mod}(\mathcal{S}) = \mathrm{Mod}_{\mathfrak{C}}(\mathcal{S});$$

The inclusion $\mathrm{Mod}_{\mathfrak{C}}(\mathcal{S}) \subseteq \mathrm{Mod}(\mathcal{S})$ is obvious. For the other consider an object F of $\mathrm{Mod}(\mathcal{S})$; this is of the form RJA for some A in $\mathcal{A}$, thus for any η_X as above we are given the following commutative diagram

$$\begin{tikzcd}
 & \mathcal{A} \arrow[r, hook, "J"] & {[\mathcal{C}, \mathcal{V}]} \arrow[dd, hook, "R"] \\
\mathcal{D}^{op} \arrow[ur, "H_X"] \arrow[dr, "(H'_X)^{op}"'] & & \\
 & \mathcal{B}^{op} \arrow[uur, hook] \arrow[r, hook, "Y"'] & {[\mathcal{B}, \mathcal{V}]}
\end{tikzcd}$$

and obtain

$$RJA \circ H'_X \cong [\mathcal{B}, \mathcal{V}](Y(H'_X)^{op}-, RJA) \cong [\mathcal{C}, \mathcal{V}](JH_X-, JA) \cong \mathcal{A}(H_X-, A) \in \mathfrak{C}_{M_X}.$$

Thus $F \cong RJA \in \mathrm{Mod}_{\mathfrak{C}}(\mathcal{S})$.

□

And a consequence of the proof is the following:

Corollary 5.2.22. *Let $\mathfrak{C}$ be a companion for Ψ and $\mathcal{A}$ be an accessible $\mathcal{V}$-category with Ψ-limits. Then there exists a sketch $\mathcal{S}$ for which*

$$\mathcal{A} \simeq \mathrm{Mod}(\mathcal{S}) = \mathrm{Mod}_{\mathfrak{C}}(\mathcal{S}).$$

Remark 5.2.23. Note that, for arbitrary $\mathfrak{C}$ and $\mathcal{S}$, it is not necessarily true that $\mathrm{Mod}_{\mathfrak{C}}(\mathcal{S})$ is accessible since it may not be a $\mathcal{V}$-category of models in the usual sense., as we now see. For each M, the $\mathcal{V}$-category $\mathfrak{C}_M$ itself can be expressed as $\mathrm{Mod}_{\mathfrak{C}}(\mathcal{S})$ for a sketch $\mathcal{S}$: given $M \colon \mathcal{D}^{op} \to \mathcal{V}$, consider it as the colimit $M \cong M * Y$ in $[\mathcal{D}^{op}, \mathcal{V}]$ and denote by $\mathcal{B}$ the full subcategory of $[\mathcal{D}^{op}, \mathcal{V}]$ spanned by the representables and M; let $W \colon \mathcal{D} \hookrightarrow \mathcal{B}$ be the inclusion. If we consider the colimit cocylinder $\eta \colon M \to \mathcal{B}(W-, M)$, then

$$\mathfrak{C}_M \simeq \mathrm{Mod}_{\mathfrak{C}}(\mathcal{S})$$

where $\mathcal{S} = (\mathcal{B}, \mathbb{L} = \emptyset, \mathbb{C} = \{\eta\})$ and the equivalence is obtained left Kan extending along the inclusion W.

Thus, if we want any $\mathrm{Mod}_{\mathfrak{C}}(\mathcal{S})$ to be accessible for any sketch $\mathcal{S}$, we should at least ask $\mathfrak{C}_M$ to be accessible and accessibly embedded in its ambient $\mathcal{V}$-category, for each weight M. And this is all we need:

Definition 5.2.24. Let $\mathfrak{C}$ be a companion for Ψ; we say that $\mathfrak{C}$ is an *accessible companion* for Ψ if for each $M \colon \mathcal{D}^{op} \to \mathcal{V}$ the $\mathcal{V}$-category $\mathfrak{C}_M$ is accessible and accessibly embedded in $[\mathcal{D}, \mathcal{V}]$.

Recall that $\mathfrak{C}_M$, when non-empty, is assumed to be closed under Ψ-limits in $[\mathcal{C}, \mathcal{V}]$.

Example 5.2.25. The following are examples of accessible companions:

1. for every weakly sound class Ψ, the companion $\mathfrak{C}^{\Psi^+}$ given by the Ψ-flat weights: indeed $\mathfrak{C}^{\Psi^+}_M$ is either empty or the whole presheaf $\mathcal{V}$-category.

2. $\mathcal{V} = \mathbf{Set}$ and the companion $\mathfrak{F}$, of free groupoid diagrams, for the class of wide pullbacks: for each groupoid $\mathcal{G}$ consider the category $\mathcal{G}'$ obtained from $\mathcal{G}$ adding an initial object 0. On $\mathcal{G}'$ consider the sketch with limit diagrams in $\mathbb{L}$ the pairs $g, h \colon G \to H$, for every morphisms $g \neq h$ in $\mathcal{G}$, and with cone specification the unique arrow $0 \to G$. The only colimit diagram in $\mathbb{C}$ is the empty one with empty cocone given by the object 0. Let $J \colon \mathcal{G} \to \mathcal{G}'$ be the inclusion; then restriction along J induces an equivalence
$$\mathrm{Mod}(\mathcal{G}', \mathbb{L}, \mathbb{C}) \simeq \mathfrak{F}_{\mathcal{G}}.$$
On one side, if $F \colon \mathcal{G}' \to \mathbf{Set}$ is a model then FJ is clearly a free groupoid action; conversely, if $F \colon \mathcal{G} \to \mathbf{Set}$ is in $\mathfrak{F}_{\mathcal{G}}$ then $F' \colon \mathcal{G}' \to \mathbf{Set}$, defined extending F with $F'(0) = \emptyset$, is a model of the sketch (since F is a free groupoid action). It follows that each $\mathfrak{F}_{\mathcal{G}}$ is accessible and closed under filtered colimits in $[\mathcal{G}, \mathbf{Set}]$. Since the only colimit specification in the sketch is an initial object, $\mathfrak{F}_{\mathcal{G}}$ is closed under (all connected limits, and in particular) wide pullbacks.

3. $\mathcal{V} = \mathbf{Set}$ and the companion $\mathfrak{R}$, of pseudo-equivalence relations, for the class of products: let $\mathcal{C}$ be the category with two parallel non-identity arrows $f, g \colon X \to Y$, then $\mathfrak{R}_M$ is non empty only for $M = \Delta 1 \colon \mathcal{C}^{op} \to \mathbf{Set}$. Consider the category $\mathcal{C}'$ in $\mathcal{PC}$ spanned by: $\mathcal{C}$, the coequalizer q of (f, g), the kernel pair $h, k \colon Z \to Y$ of q, and the kernel pair (h', k') of the map $e \colon X \to Z$ induced by the kernel pair (h, k). Then define the sketch on $\mathcal{C}'$ with limit conditions $\mathbb{L}$ saying that (h, k) and (h', k') are the kernel pairs of q and e respectively; the only colimit conditions $\mathbb{C}$ are saying that q and e are the coequalizers of (h, k) and (h', k') respectively. It is then easy to see that
$$\mathfrak{R}_{\Delta 1} \simeq \mathrm{Mod}(\mathcal{C}', \mathbb{L}, \mathbb{C}),$$
and as a consequence it is accessible and closed under filtered colimits and products in $[\mathcal{C}, \mathbf{Set}]$.

4. See also examples from Section 5.3.3 and 5.3.4.

Proposition 5.2.26. *Let $\mathfrak{C}$ be an accessible companion for Ψ. For any sketch $\mathcal{S} = (\mathcal{B}, \mathbb{L}, \mathbb{C})$ the $\mathcal{V}$-category $\mathrm{Mod}_{\mathfrak{C}}(\mathcal{S})$ of $\mathfrak{C}$-models of $\mathcal{S}$ is accessible, accessibly embedded, and closed under Ψ-limits in $[\mathcal{B}, \mathcal{V}]$.*

Proof. First note that we can see $\mathrm{Mod}_{\mathfrak{C}}(\mathcal{S})$ as the intersection, in $[\mathcal{B}, \mathcal{V}]$, of all the $\mathrm{Mod}_{\mathfrak{C}}(\mathcal{S}_\eta)$, for all $\eta \in \mathbb{C}$ and $\mathcal{S}_\eta = (\mathcal{B}, \mathbb{L}, \{\eta\})$. Now, each $\mathrm{Mod}_{\mathfrak{C}}(\mathcal{S}_\eta)$ can be seen as the pullback

$$\begin{tikzcd}
\mathrm{Mod}_{\mathfrak{C}}(\mathcal{S}_\eta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{C}_M \arrow[d, hook] \\
\mathrm{Mod}(\mathcal{B}, \mathbb{L}) \arrow[r, "- \circ H"'] & {[\mathcal{C}, \mathcal{V}]}
\end{tikzcd}$$

with $\eta \colon M \Rightarrow \mathcal{B}(H-, B)$, $M \colon \mathcal{C}^{op} \to \mathcal{V}$, and $H \colon \mathcal{C} \to \mathcal{B}$. Since the inclusion of $\mathfrak{C}_M$ in $[\mathcal{C}, \mathcal{V}]$ is an isofibration, the $\mathcal{V}$-categories involved are accessible and Ψ-complete, and the $\mathcal{V}$-functors are accessible and Ψ-continuous, it follows by Corollary 2.3.7 that each $\mathrm{Mod}_{\mathfrak{C}}(\mathcal{S}_\eta)$

is accessible, accessibly embedded, and closed under Ψ-limits in $[\mathcal{B},\mathcal{V}]$. For the same reason $\mathrm{Mod}_{\mathfrak{C}}(\mathcal{S})$ is also an accessible and Ψ-complete $\mathcal{V}$-category. $\square$

Hence we can characterize accessible $\mathcal{V}$-categories with Ψ-limits as $\mathcal{V}$-categories of $\mathfrak{C}$-models:

Theorem 5.2.27. *Let Ψ be a class of weights, $\mathfrak{C}$ be an accessible companion for Ψ, and $\mathcal{A}$ be a $\mathcal{V}$-category; the following are equivalent:*

1. *$\mathcal{A}$ is accessible with Ψ-limits;*
2. *$\mathcal{A}$ is equivalent to the $\mathcal{V}$-category of $\mathfrak{C}$-models of a sketch.*

Proof. Put together Proposition 5.2.21 and 5.2.26. $\square$

$\mathfrak{C}$-sketches:

Instead of using the notion of $\mathfrak{C}$-model of a general sketch, we can also introduce a notion of $\mathfrak{C}$-sketch whose $\mathcal{V}$-categories of (standard) models characterize accessible $\mathcal{V}$-categories with Ψ-limits. In general this notion is more technical than that of $\mathfrak{C}$-model (as we see below); however in the case of pseudo-equivalence relations and free groupoid actions we recover the notions of limit/epi and galoisian sketches.

Let $\mathfrak{C}$ be an accessible companion for Ψ. By accessibility of $\mathfrak{C}$, for any $M\colon \mathcal{D}^{op}\to\mathcal{V}$ we can fix a fully faithful $W_M\colon \mathcal{D}\hookrightarrow\mathcal{D}_M$ and a sketch $\mathcal{S}_M=(\mathcal{D}_M,\mathbb{L}_M,\mathbb{C}_M)$ on $\mathcal{D}_M$ together with an equivalence

$$-\circ W_M\colon \mathrm{Mod}(\mathcal{S}_M)\longrightarrow \mathfrak{C}_M. \tag{5.4}$$

Using this we define the notion of $\mathfrak{C}$-sketch as follows:

Definition 5.2.28. A $\mathfrak{C}$-*sketch* $\mathcal{T}$ is determined by a sketch $\mathcal{S}=(\mathcal{B},\mathbb{L},\mathbb{C})$ for which every cocylinder in $\mathbb{C}$ is of the form

$$\eta\colon N\Rightarrow \mathcal{B}(H-,B)\colon \mathcal{D}_M^{op}\to\mathcal{V}$$

and such that $N\cong \mathrm{Lan}_{W_M^{op}}M$, for some weight M. Then $\mathcal{T}$, as a sketch, is given by $\mathcal{S}$ together with the additional classes of cylinders $H(\mathbb{L}_M)$ and of cocylinders $H(\mathbb{C}_M)$, for each $\eta\in\mathbb{C}$ and M as above. A model of $\mathcal{T}$ is then a model of

$$(\mathcal{B},\mathbb{L}\sqcup_{\eta\in\mathbb{C}}H(\mathbb{L}_M),\mathbb{C}\sqcup_{\eta\in\mathbb{C}}H(\mathbb{C}_M))$$

in the standard sense.

The following lemma will be important for the characterization theorem.

Lemma 5.2.29. *The following are equivalent for an accessible $\mathcal{V}$-category $\mathcal{A}$, a weight $M\colon\mathcal{D}^{op}\to\mathcal{V}$, and a diagram $H\colon\mathcal{D}^{op}\to\mathcal{A}$:*

1. *$\mathcal{A}(H-,A)\in\mathfrak{C}_M$ for any $A\in\mathcal{A}$;*
2. *H^{op} can be extended to a model $\widehat{H}^{op}\colon\mathcal{D}_M\to\mathcal{P}(\mathcal{A}^{op})$ of $\mathcal{S}_M$ in $\mathcal{P}(\mathcal{A}^{op})$.*

Proof. It is clear that if $\widehat{H}^{op}$ extends H^{op} and is a model of $\mathcal{S}_M$ then H lies representably in $\mathfrak{C}_M$. Conversely, if $H\colon\mathcal{D}^{op}\to\mathcal{A}$ lies representably in $\mathfrak{C}_M$, then the transpose $S\colon\mathcal{A}\to[\mathcal{D},\mathcal{V}]$ of $YH^{op}\colon\mathcal{D}\to\mathcal{P}(\mathcal{A}^{op})$ is accessible and lands in $\mathfrak{C}_M$. Then we can compose it with the given equivalence to obtain an accessible $\mathcal{V}$-functor $T\colon\mathcal{A}\to[\mathcal{D}_M,\mathcal{V}]$ that lands in $\mathrm{Mod}(\mathcal{S}_M)$. Transposing again we obtain a $\mathcal{V}$-functor $\widehat{H}^{op}\colon\mathcal{D}_M\to\mathcal{P}(\mathcal{A}^{op})$ which extends H^{op} and is a model of $\mathcal{S}_M$. $\square$

Then, accessible $\mathcal{V}$-categories with Ψ-limits can equivalently be described as $\mathcal{V}$-categories of models of a $\mathfrak{C}$-sketch:

Theorem 5.2.30. *Let Ψ be a class of weights, and $\mathfrak{C}$ be an accessible companion with a fixed sketch presentation as in (5.4). A $\mathcal{V}$-category $\mathcal{A}$ is accessible with Ψ-limits if and only if it is equivalent to the $\mathcal{V}$-category of models of a $\mathfrak{C}$-sketch.*

Proof. Let $\mathcal{T}$ be a $\mathfrak{C}$-sketch as in Definition 5.2.28; then a $\mathcal{V}$-functor $F\colon \mathcal{B} \to \mathcal{V}$ is a model of $\mathcal{T}$ if and only if it is a $\mathfrak{C}$-model of the sketch $\mathcal{S} = (\mathcal{B}, \mathbb{L}, \mathbb{C}')$, where $\mathbb{C}' = \{\eta \circ W_M^{op}\}_{\eta \in \mathbb{C}}$ (with η as in Definition 5.2.28). Thus $\mathrm{Mod}(\mathcal{T}) = \mathrm{Mod}_{\mathfrak{C}}(\mathcal{S})$ is accessible with Ψ-limits by Theorem 5.2.27.

Conversely, given an accessible $\mathcal{V}$-category $\mathcal{A}$ with Ψ-limits, we consider a fully faithful and accessible $J\colon \mathcal{A} \to \mathcal{K} = [\mathcal{C}, \mathcal{V}]$ and work in the setting of the proof of Proposition 5.2.21. The diagrams $H_X\colon \mathcal{D}^{op} \to \mathcal{A}$ considered there satisfy condition (1) of Lemma 5.2.29; thus they can be extended to some $\widehat{H}_X\colon \mathcal{D}_{M_X}^{op} \to \mathcal{P}^\dagger\mathcal{A}$ such that $\widehat{H}_X^{op}$ is a model of $\mathcal{S}_{M_X}$. Then, by possibly replacing $\mathcal{C}$ with a larger (but still small) $\mathcal{V}$-category, we can assume that every $\mathcal{P}^\dagger J \circ \widehat{H}_X\colon \mathcal{D}_{M_X}^{op} \to \mathcal{P}^\dagger\mathcal{K}$ lands in $\mathcal{K}$.

In addition, we take the $\mathcal{V}$-category $\mathcal{B}^{op} \subseteq \mathcal{K}$ (considered in the proof) to contain the images of all the $\widehat{H}_X$; call the resulting $\mathcal{V}$-functors $\widehat{H}'_X\colon \mathcal{D}_M \to \mathcal{B}$. Now consider the sketch $\mathcal{S} = (\mathcal{B}, \mathbb{L}, \mathbb{C})$ given in Proposition 5.2.21, and define the $\mathfrak{C}$-sketch $\mathcal{T}$ with same base $\mathcal{V}$-category $\mathcal{B}$, same set of cylinders $\mathbb{L}$, and cocylinders

$$\widehat{\gamma}_X\colon \mathrm{Lan}_{W_{M_X}^{op}} M_X \to \mathcal{B}(\widehat{H}'_X -, X)$$

induced by the γ_X specified in 5.2.21 for each given X. It is then easy to see that $\mathrm{Mod}_{\mathfrak{C}}(\mathcal{S}) = \mathrm{Mod}(\mathcal{T})$, and hence $\mathcal{A}$, being equivalent to $\mathrm{Mod}_{\mathfrak{C}}(\mathcal{T})$, is the $\mathcal{V}$-category of models of a $\mathfrak{C}$-sketch. $\square$

Example 5.2.31. In the case of the companion $\mathfrak{F}$ for wide pullbacks ($\mathcal{V} = \mathbf{Set}$), the remark above says that the sketches characterizing accessible categories with wide pullbacks have the following restrictions on the colimit cocones:

1. there is always an empty cocone, denote its vertex by 0;
2. any other cocone $H \to \Delta C$ is indexed on a groupoid $\mathcal{G}$ and comes together with a cone specification

$$\begin{tikzcd} & 0 \arrow[dl, dashed] \arrow[dr, dashed] & \\ HA \arrow[rr, shift left, "Hf"] \arrow[rr, shift right, "Hg"'] & & HB \end{tikzcd}$$

   for any parallel pair $f \neq g$ in $\mathcal{G}$.

By seeing every groupoid as (equivalent to) a coproduct of groups, these sketches can be easily recognized as the *galoisian* sketches of Ageron [6, Definition 4.14].

Example 5.2.32. We can apply the same arguments to the companion $\mathfrak{R}$ for products ($\mathcal{V} = \mathbf{Set}$). By the remark above, the only restriction for colimit cocones appearing in the sketches for accessible categories with products is that they need to be coequalizers of kernel pairs. Since we are working in $\mathbf{Set}$, this is equivalent to the specification of a set of maps which need to be sent to epimorphisms. That's exactly the data of a limit/epi sketch where all the cocones are as below.

$$\begin{tikzcd} B \arrow[r, "e"] \arrow[d, "e"'] & C \arrow[d, dashed, "1_C"] \\ C \arrow[r, dashed, "1_C"'] & C \end{tikzcd}$$

Thus we recover the characterization of [2, Theorem 4.13]. The same argument can be applied in the more general setting of Section 5.3.3.

5.2.3 The Set-case

For this section we take $\mathcal{V}$ to be **Set** and give a further characterization of companions. To do that we need to make the following assumption:

Assumption 5.2.33. All the components $\mathfrak{C}_M$, of a colimit type $\mathfrak{C}$, are assumed to be accessible, accessibly embedded, and such that the sketches defining them involve only colimits and connected limits.

All the classes of colimits in Example 5.2.25 satisfy this condition.

The following lemma is a straightforward extension of the well-known special case where $\mathcal{A}$ is small.

Lemma 5.2.34. *Let $F\colon \mathcal{A}^{op} \to \mathbf{Set}$ be a small functor and $\mathrm{El}(F)$ be its category of elements, so that we have a projection $q\colon \mathrm{El}(F) \to \mathcal{A}$. Then*

$$\mathcal{P}(\mathrm{El}(F)) \simeq \mathcal{P}\mathcal{A}/F$$

and, under this equivalence, $\mathcal{P}q$ is the projection $Q\colon \mathcal{P}\mathcal{A}/F \to \mathcal{P}\mathcal{A}$.

Proof. For the proof it is easier to see $\mathrm{El}(F)$ as the slice category $\mathcal{A}/F$, which comes together a fully faithful inclusion $J\colon \mathcal{A}/F \to [\mathcal{A}^{op}, \mathbf{Set}]/F$. It is well-known that we have an equivalence at the level of presheaves $E\colon [\mathcal{A}^{op}, \mathbf{Set}]/F \to [(\mathcal{A}/F)^{op}, \mathbf{Set}]$ given by $Ep \cong [\mathcal{A}^{op}, \mathbf{Set}]/F(J-, p)$ for any $p\colon G \to F$. To conclude it is then enough to show that if $p\colon G \to F$ has small domain then Ep is small. Indeed $\mathcal{P}\mathcal{A}/F$ would then be a full subcategory of $\mathcal{P}(\mathcal{A}/F)$ which contains the representables and is closed under colimits; thus it is equivalent to $\mathcal{P}(\mathcal{A}/F)$. For any $p\colon G \to F$ with G small in $[\mathcal{A}^{op}, \mathbf{Set}]$, consider a small category $\mathcal{C}$ and $H\colon \mathcal{C} \to \mathcal{A}$ such that $\operatorname{colim} YH \cong G$ (here Y is the Yoneda embedding). Then we can express $Ep\colon (\mathcal{A}/F)^{op} \to \mathbf{Set}$ as the colimit

$$Ep \cong \operatorname{colim}\left(\mathcal{C}/(GH) \xrightarrow{p^*} \mathcal{C}/FH \xrightarrow{H^*} \mathcal{A}/F \longrightarrow [(\mathcal{A}/F)^{op}, \mathbf{Set}]\right)$$

where p^* and H^* are the obvious maps induced by p and H. The fact that Ep is indeed isomorphic to the colimit above can be checked pointwise using that both the colimit evaluated at (A, x) and $Ep(A, x)$ coincide with $p_A^{-1}(x) \subseteq GA$. The fact that $\mathcal{P}q \simeq Q$ is a direct consequence of how E is defined.

□

The following then provides an easy way to recognize companions:

Proposition 5.2.35. *Let $\mathfrak{C}$ be a colimit type as in Assumption 5.2.33 and Ψ a class of weights. Then $\mathfrak{C}$ is a companion for Ψ if and only if:*

(i) $\mathfrak{C}$ is compatible with Ψ;

(ii) for any Ψ-complete and virtually cocomplete $\mathcal{B}$, the category $\mathfrak{C}_1^\dagger\mathcal{B}$ has an initial object.

Proof. If $\mathfrak{C}$ is a companion for Ψ then (i) holds by definition. For (ii), note that by property (II) defining a companion, and the fact the constant functor $\Delta 1\colon \mathcal{B}^{op} \to \mathbf{Set}$ is Ψ-continuous, it follows that $\Delta 1 \in \mathfrak{C}_1^\dagger\mathcal{B}$ which then has an initial object.

Conversely assume that (i) and (ii) hold. We only need to prove that for any Ψ-complete and virtually cocomplete $\mathcal{A}$, each small Ψ-continuous functor $F\colon \mathcal{A} \to \mathcal{V}$ lies in $\mathfrak{C}_1(\mathcal{A}^{op})$. Given F as above, the category $\mathcal{B} := \mathrm{El}(F)^{op}$ is Ψ-complete and virtually cocomplete (since by Lemma 5.2.34 $\mathcal{P}(\mathcal{B}^{op}) \simeq \mathcal{P}(\mathcal{A}^{op})/F$ and $\mathcal{A}$ is virtually cocomplete). Thus, by (ii), the category $\mathfrak{C}_1(\mathcal{B}^{op})^{op} \simeq \mathfrak{C}_1^\dagger\mathcal{B}$ has an initial object, which must coincide with the constant functor $\Delta 1\colon \mathcal{B} \to \mathbf{Set}$.
To conclude it is then enough to prove that the projection $q\colon \mathcal{B} \to \mathcal{A}$ induces a functor $\mathfrak{C}_1(q^{op})\colon \mathfrak{C}_1(\mathcal{B}^{op}) \to \mathfrak{C}_1(\mathcal{A}^{op})$ which is the restriction of $\mathcal{P}(q^{op})$. This will suffice since then $\mathfrak{C}_1(q^{op})(\Delta 1)$ coincides with $\Delta 1 * Yq \cong F$ which then lies in $\mathfrak{C}_1(\mathcal{A}^{op})$. Now, if $X : \mathcal{B} \to \mathbf{Set}$ is in $\mathfrak{C}_1(\mathcal{B}^{op})$, then we can write it as $X \cong M * YH$ with $\mathcal{B}(H-, B) \in \mathfrak{C}_M$ for any $B \in \mathcal{B}$. Note that the diagram $H\colon \mathcal{C} \to \mathcal{B}$ above is representably in $\mathcal{B}$ if and only if $YH\colon \mathcal{B} \to \mathcal{P}(\mathcal{B}^{op})$ is a model of the sketch corresponding to $\mathfrak{C}_M$. Moreover, by our assumption on $\mathfrak{C}$, the sketch defining $\mathfrak{C}_M$ can be assumed to be such that its limit specifications are all connected. By Lemma 5.2.34 the induced functor $\mathcal{P}(q^{op})$ is, up to equivalence, the projection $Q\colon \mathcal{P}(\mathcal{A}^{op})/F \to \mathcal{P}(\mathcal{A}^{op})$ and thus it preserves all colimits and all connected limits. It follows that $(\mathcal{P}q^{op})YH^{op}$ is a model in $\mathcal{P}(\mathcal{A}^{op})$ of the sketch corresponding to $\mathfrak{C}_M$; since $(\mathcal{P}q^{op})YH^{op} \cong Y \circ (qH)^{op}$ this means that qH lies representably in $\mathfrak{C}_M$. Therefore $\mathcal{P}q^{op}(X) \cong M * Y(qH)$ lies in $\mathfrak{C}_1(\mathcal{A}^{op})$. □

5.3 Examples

5.3.1 The weakly sound case

In this section we focus on the case where our companion $\mathfrak{C}$, for a class of weights Ψ, is determined by a class of weights Φ; that is $\mathfrak{C} = \mathfrak{C}^\Phi$. For simplicity we denote the colimit type simply by Φ instead of $\mathfrak{C}^\Phi$, but note that whenever we say that Φ is a companion we mean that the corresponding $\mathfrak{C}^\Phi$ is one.

We denote with $\Phi_1^\dagger\mathcal{A} = (\mathfrak{C}^\Phi)_1^\dagger\mathcal{A}$ the full subcategory of $\mathcal{P}^\dagger\mathcal{A}$ spanned by the representables and Φ-limits of those. This is not in general the free completion of $\mathcal{A}$ under Φ-limits (which is denoted by $\Phi^\dagger\mathcal{A}$). When it is so for all $\mathcal{A}$, we say that Φ is a *pre-saturated* class of weights (Section A.1), which for simplicity we shall henceforth assume for any class Φ considered in this section.

In this context then Definition 5.2.11 translates into the following: let Ψ and Φ be classes of weights; then Φ is a companion for Ψ if and only if:

(I) Φ-colimits commute in $\mathcal{V}$ with Ψ-limits;

(II) for any Ψ-complete and virtually cocomplete $\mathcal{A}$, each small Ψ-continuous $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$ lies in $\Phi^\dagger\mathcal{A}$.

Recall that, unlike in the case of the Ψ-accessible $\mathcal{V}$-categories of Section 2.1.1, we do not assume our class Ψ to be small; in other words, completions under Ψ-limits of small $\mathcal{V}$-categories might be large.

Remark 5.3.1. If Ψ is a *small* class of weights and Φ is a class of weights compatible with Ψ, then it is enough to check the companion property (II) above only for small $\mathcal{V}$-categories. Indeed, suppose that the condition holds for any small $\mathcal{V}$-category, let $\mathcal{A}$ be any (possibly large) Ψ-complete $\mathcal{V}$-category, and suppose that $F\colon \mathcal{A} \to \mathcal{V}$ is small and Ψ-continuous. Then $F = \mathrm{Lan}_J(FJ)$, with $J\colon \mathcal{C} \hookrightarrow \mathcal{A}$ small; since Ψ is small we can suppose that $\mathcal{C}$ is closed in $\mathcal{A}$ under Ψ-limits, and hence that FJ is Ψ-continuous. By hypothesis then FJ is a Φ-colimit of representables, but $\mathrm{Lan}_J(-)$ preserves colimits as well as the representables, so F is a Φ-colimit of representables too.

We begin by extending the definition of flat $\mathcal{V}$-functors and soundness to a possibly large class of weights Ψ:

Definition 5.3.2. Let Ψ be a class of weights. We say that a small presheaf $M\colon \mathcal{A}^{op} \to \mathcal{V}$ is Ψ-*flat* if M-colimits commute in $\mathcal{V}$ with Ψ-limits. We call Ψ-flat weight a Ψ-flat presheaf with small domain. Denote by Ψ^+ the class given by the Ψ-flat weights.

As noted at the beginning of Section 5.1, when $M\colon \mathcal{A}^{op} \to \mathcal{V}$ is a small $\mathcal{V}$-functor, M-weighted colimits exist in any cocomplete $\mathcal{V}$-category making the notion above well defined. Recall moreover that when we talk about a Ψ-flat weight $M \in \Psi^+$, we assume M to have a small domain (and hence it is just any presheaf).

Definition 5.3.3. A class of weights Ψ is called *weakly sound* if every Ψ-continuous and small $\mathcal{V}$-functor $M\colon \mathcal{A} \to \mathcal{V}$, from a virtually cocomplete and Ψ-complete $\mathcal{A}$, is Ψ-flat.

When Ψ is locally small, thanks to the same argument of Remark 5.3.1, we recover the usual notion of weakly sound class of Section 1.3 that only involves small Ψ-complete $\mathcal{V}$-categories.

The relationship between Ψ being weakly sound and it having a companion identified by a class of weights is explained by the following proposition.

Proposition 5.3.4. *If a class Φ of weights is a companion for Ψ then Ψ is a weakly sound class and $\Phi \subseteq \Psi^+$. Conversely, if Ψ is a weakly sound class then Ψ^+ is a companion for Ψ.*

Proof. For the first assertion consider $M\colon \mathcal{A} \to \mathcal{V}$ to be small and Ψ-continuous (with $\mathcal{A}$ virtually cocomplete and Ψ-complete); then M is a Φ-colimit of representables since Φ is a companion for Ψ. As a consequence $M * -$ is a Φ-colimit of evaluation functors, which are continuous. Then, since Φ-colimits commute with Ψ-limits in $\mathcal{V}$, it follows at once that $M * -$ preserves Ψ-limits, and hence that Ψ is weakly sound.

For the second part we already know, by definition, that Ψ^+-colimits commute in $\mathcal{V}$ with Ψ-limits; therefore consider again $M\colon \mathcal{A} \to \mathcal{V}$ small and Ψ-continuous, we need to prove that it is a Ψ^+-colimit of representables. By assumption, $M \cong \mathrm{Lan}_J(MJ)$ for some small full subcategory $J\colon \mathcal{C} \to \mathcal{A}$. Therefore, since M is Ψ-flat (being Ψ-continuous), MJ is Ψ-flat as well by Lemma 1.3.2 and hence $M \cong MJ * ZJ$ (where Z is the inclusion of $\mathcal{A}^{op}$ in $\mathcal{P}(\mathcal{A}^{op})$) is a Ψ^+-colimit of representables. This shows that Ψ^+ is a companion for Ψ. $\square$

As a consequence, given a companion Φ for Ψ and any $\mathcal{V}$-category $\mathcal{A}$ the following inclusions hold:

$$\Phi\mathcal{A} \subseteq \Psi^+\mathcal{A};$$

and that becomes an equality whenever $\mathcal{A}^{op}$ is Ψ-complete and virtually cocomplete. If in addition every Ψ-flat $\mathcal{V}$-functor is a Φ-colimit of representables, then the equality

$$\Phi\mathcal{A} = \Psi^+\mathcal{A}$$

always holds.

Remark 5.3.5. It is not true in general, given a companion Φ for Ψ, that all Ψ-flat $\mathcal{V}$-functors are Φ-colimits of representables. For example, when $\mathcal{V} = \mathbf{Ab}$, the filtered colimits form a companion for the class of finite limits, but flat **Ab**-functors are not just filtered colimits of representables. In fact they are filtered colimits of finite sums of representables (see [81, Theorem 3.2] or our Section 3.2).

Example 5.3.6. In the following examples Ψ is a weakly sound class; we spell out when the companion can be taken to consist of some, but not all, the Ψ-flat weights. See Example 1.3.6 for explanations of why most of these classes are sound.

1. $\Psi = \mathcal{P}$ is the class of all small weights. Then the class $\Phi = \emptyset$ is a companion for $\mathcal{P}$ since $\emptyset$-colimits (trivially) commute in $\mathcal{V}$ with all limits, and any continuous $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$, from a complete and virtually cocomplete $\mathcal{A}$, has a left adjoint by Lemma 5.3.8 below. A $\mathcal{P}$-flat weight is a Cauchy (or absolute) weight. An accessible $\mathcal{V}$-category with all limits is a locally presentable $\mathcal{V}$-category.

2. $\Psi = \emptyset$. Then $\Psi^+ = \mathcal{P}$ is the class of all weights. When $\mathcal{V} = \mathbf{Set}$, the class Φ of all small categories is a companion for Ψ. This choice of Ψ will classify accessible $\mathcal{V}$-categories (with no limits specified).

3. $\mathcal{V}$ locally α-presentable as a closed category, Ψ is the class of α-small weights. Then the class Φ of α-filtered categories is a companion for Ψ; in general not every α-flat $\mathcal{V}$-functor is a filtered colimit of representables.

4. $\mathcal{V}$ symmetric monoidal closed finitary quasivariety, Ψ is the class for finite products and finitely presentable projective powers. Then the class Φ of sifted categories is a companion for Ψ; we do not know whether Ψ-flat $\mathcal{V}$-functors are always sifted colimits of representables.

5. $\mathcal{V}$ cartesian closed, Ψ is the class of finite discrete diagrams. Then M is Ψ-flat if and only if $\mathrm{Lan}_\Delta M \cong M \times M$.

6. $\mathcal{V} = \mathbf{Set}$, Ψ is the class of connected categories. Then the class Φ of discrete categories is a companion for Ψ; every Ψ-flat weight is a split subobject of coproducts of representables. The fact that connected limits commute in **Set** with coproducts is standard. Moreover, if $F\colon \mathcal{A} \to \mathbf{Set}$ is small and preserves connected limits, then the connected components of the category of elements of F are (α-filtered for any α and hence) absolute; this makes F a coproduct of representable functors. An accessible category with connected limits is called a locally multipresentable category.

7. $\mathcal{V} = \mathbf{Set}$, $\Psi = \{\emptyset\}$. Then Ψ^+ is generated by the class of connected categories [4].

8. $\mathcal{V} = \mathbf{Set}$, Ψ consists of the finite connected categories. Then Ψ^+ is generated by coproducts of filtered categories [4].

9. $\mathcal{V} = \mathbf{Set}$, Ψ is the class of finite non empty categories. Then Ψ^+ is generated by the filtered categories plus the empty category.

10. $\mathcal{V} = \mathbf{Set}$, Ψ is the class of finite discrete non empty categories. Then Ψ^+ is generated by the sifted categories plus the empty category.

11. $\mathcal{V} = \mathbf{Cat}$, Ψ is the class generated by connected conical limits and powers by connected categories. Then the class Φ of discrete categories is a companion for Ψ; every Ψ-flat weight is a split subobject of coproducts of representables. The proof is based on that of (6) using that every small 2-functor $F\colon \mathcal{A} \to \mathbf{Cat}$ can be written as the coproduct of a conical connected colimit of connected copowers of representable 2-functors.

12. $\mathcal{V} = ([\mathcal{C}^{op}, \mathbf{Set}], \otimes, I)$ with the representables closed under the monoidal structure, Ψ is the class defined by powers by representables. Then the class Φ of all conical weights is a companion for Ψ (see Example 1.3.6(10)).

In this context, companions of locally small classes of weights can be characterized as follows. When $\Phi = \Psi^+$ part of this was proved in [59, Theorem 8.11].

Proposition 5.3.7. *Let Φ and Ψ be a pair of classes of weights with Ψ locally small. The following are equivalent:*

1. *Φ is a companion for Ψ.*
2. *Φ-colimits commute in $\mathcal{V}$ with Ψ-limits, and for any $\mathcal{V}$-category $\mathcal{A}$ every object of $\mathcal{P}\mathcal{A}$ is a Φ-colimit of objects from $\Psi\mathcal{A}$.*
3. *$W := \mathrm{Lan}_K Z\colon \Phi \circ \Psi(-) \to \mathcal{P}(-)$ is an equivalence of endofunctors on $\mathcal{V}$-$\mathbf{CAT}$, where $Z\colon \Psi(-) \to \mathcal{P}(-)$ and $K\colon \Psi(-) \to \Phi \circ \Psi(-)$ are the inclusions.*
4. *For each Ψ-cocomplete $\mathcal{A}$ the inclusion $V\colon \mathcal{A} \to \Phi\mathcal{A}$ is Ψ-cocontinuous, and freely adding Φ-colimits induces a 2-functor*
$$\Phi(-)\colon \Psi\text{-}\mathbf{Coct}\text{-}\mathbf{CAT} \longrightarrow \mathbf{Coct}\text{-}\mathbf{CAT}$$
from the 2-category of Ψ-cocomplete $\mathcal{V}$-categories to that of cocomplete $\mathcal{V}$-categories.

Proof. (1) $\Rightarrow$ (2). The first part is already in the definition of companion. For the latter assume first that $\mathcal{A}$ is small, so that $\Psi\mathcal{A}$ is small as well, and consider $F\colon \mathcal{A}^{op} \to \mathcal{V}$ (an object of $\mathcal{P}\mathcal{A}$). Let $W\colon \mathcal{A} \to \Psi\mathcal{A} = \Psi^\dagger(\mathcal{A}^{op})^{op}$ be the inclusion; then $\mathrm{Ran}_{W^{op}}F$ is Ψ-continuous (and small, since $\Psi\mathcal{A}$ is). The fact that Φ is a companion for Ψ implies that $\mathrm{Ran}_{W^{op}}F$ is a Φ-colimit of representables: there exist M in $\Phi\mathcal{C}$ and $H\colon \mathcal{C} \to \Psi\mathcal{A}$ such that $\mathrm{Ran}_{W^{op}}F \cong M * YH$, where Y is the Yoneda embedding. Since pre-composition with W is cocontinuous, it follows that $F \cong (M * YH)W \cong M * ZH$, where $Z\colon \Psi\mathcal{A} \to \mathcal{P}\mathcal{A}$ is the inclusion, as desired.

Consider now the case of a general $\mathcal{A}$; let $F\colon \mathcal{A}^{op} \to \mathcal{V}$ be small, then $F \cong \mathrm{Lan}_J(FJ)$ for some small $J\colon \mathcal{B}^{op} \to \mathcal{A}^{op}$. By the argument above we know that $FJ \cong M * Z_{\mathcal{B}}H$ is a Φ-colimit of objects from $\Psi\mathcal{B}$; thus
$$F \cong \mathrm{Lan}_J(M * Z_{\mathcal{B}}H) \cong M * \mathrm{Lan}_J Z_{\mathcal{B}}H$$
but $\mathrm{Lan}_J Z_{\mathcal{B}}$ takes values in $\Psi\mathcal{A}$, thus F is a Φ-colimit of objects from $\Psi\mathcal{A}$.

(2) $\Rightarrow$ (3). Let $W\colon \Phi(\Psi\mathcal{A}) \to \mathcal{P}\mathcal{A}$ be as in (3). The fact that every object of $\mathcal{P}\mathcal{A}$ is a Φ-colimit of objects from $\Psi\mathcal{A}$ is equivalent to the request of W being essentially surjective. Similarly, we will now show that commutativity of Φ-colimits with Ψ-limits in $\mathcal{V}$ is equivalent to W being fully faithful; this will be enough to show (3).

Since Φ-colimits commute in $\mathcal{V}$ with Ψ-limits, the Ψ-continuous $M\colon \Psi(\mathcal{A})^{op} \to \mathcal{V}$ are closed under Φ-colimits. Of course the representables are also Ψ-continuous; thus any $N\colon \Psi(\mathcal{A})^{op} \to \mathcal{V}$ in $\Phi(\Psi\mathcal{A})$ is Ψ-continuous, and so the canonical $N \to \mathrm{Ran}_J(NJ)$ is invertible, where $J\colon \mathcal{A}^{op} \to (\Psi\mathcal{A})^{op}$ is the inclusion. Note moreover that W acts by precomposition with J; indeed, by the arguments above, this defines a Φ-cocontinuous $\mathcal{V}$-functor $\Phi(\Psi\mathcal{A}) \to \mathcal{P}\mathcal{A}$ which coincides with Z when restricted to $\Psi\mathcal{A}$. Now for any $M, N\colon \Psi(\mathcal{A})^{op} \to \mathcal{V}$ in $\Phi(\Psi\mathcal{A})$ we have

$$\begin{aligned}\Phi(\Psi\mathcal{A})(M,N) &\cong [\Psi(\mathcal{A})^{op},\mathcal{V}](M,N)\\ &\cong [\Psi(\mathcal{A})^{op},\mathcal{V}](M,\mathrm{Ran}_J(NJ))\\ &\cong [\mathcal{A}^{op},\mathcal{V}](MJ,NJ)\end{aligned}$$

giving the fully faithfulness of W.

(3) $\Rightarrow$ (4). Let $\mathcal{A}$ be Ψ-complete and consider the induced square

$$\begin{tikzcd}
\Phi\mathcal{A} \arrow[rr, hook, "\Phi J"'] \arrow[rr, phantom, "\perp"] & & \Phi\Psi\mathcal{A} \arrow[ll, bend right, "\Phi L"'] \\
\mathcal{A} \arrow[u, hook, "V"] \arrow[rr, hook, "J"'] \arrow[rr, phantom, "\perp"] & & \Psi\mathcal{A} \arrow[u, hook, "V'"'] \arrow[ll, bend right, "L"']
\end{tikzcd}$$

where L exists since $\mathcal{A}$ is Ψ-cocomplete. Note that V' is Ψ-cocontinuous because it coincides, up to equivalence, with the inclusion $\Psi\mathcal{A} \hookrightarrow \mathcal{P}\mathcal{A}$. Thus it is easy to see that V must be Ψ-cocontinuous as well.

For the second statement, consider the endofunctors

$$\mathrm{Id}(-), \Psi(-)\colon \Psi\text{-}\mathbf{Coct}\text{-}\mathbf{CAT} \to \Psi\text{-}\mathbf{Coct}\text{-}\mathbf{CAT}$$

given respectively by the identity and by freely adding Ψ-colimits. Since we are restricted to Ψ-cocomplete categories, the inclusion $\mathrm{Id}(-) \hookrightarrow \Psi(-)$ has a left adjoint $\Psi(-) \to \mathrm{Id}(-)$; by applying $\Phi(-)$ this induces an adjunction

$$\begin{tikzcd}
\Phi(-) \arrow[r, hook] \arrow[r, phantom, "\perp"] & \Phi\circ\Psi(-) \arrow[l, bend right]
\end{tikzcd}$$

but $\Phi \circ \Psi(-) \simeq \mathcal{P}(-)$. Thus $\Phi\mathcal{A}$ is cocomplete whenever $\mathcal{A}$ is Ψ-cocomplete, and ΦF is cocontinuous whenever F is Ψ-cocontinuous.

(4) $\Rightarrow$ (1). Let us prove first that Φ-colimits commute in $\mathcal{V}$ with Ψ-limits. Let $M\colon \mathcal{C}^{op} \to \mathcal{V}$ be a weight in Φ, we need to prove that $M * -\colon [\mathcal{C},\mathcal{V}] \to \mathcal{V}$ is Ψ-continuous. For this, consider $\mathcal{A} = [\mathcal{C},\mathcal{V}]^{op}$; by the assumptions $V\colon \mathcal{A} \to \Phi\mathcal{A}$ is Ψ-cocontinuous and hence for each $X \in \Phi\mathcal{A}$ the functor $\Phi\mathcal{A}(V-,X)\colon \mathcal{A}^{op} \to \mathcal{V}$ is Ψ-continuous. Take now $X := M*VY$, where $Y\colon \mathcal{C} \to [\mathcal{C},\mathcal{V}]^{op} = \mathcal{A}$ is the Yoneda embedding, then

$$\Phi\mathcal{A}(V-,X) \cong M * \mathcal{A}(-,Y) \cong M * [\mathcal{C},\mathcal{V}](Y,-) \cong M * -$$

is Ψ-continuous, as required.

It remains to prove property (II) from the definition of companion; by Remark 5.3.1 we can reduce it to the case when $\mathcal{A}$ is small and Ψ-complete. Let $\mathcal{A}$ be small and Ψ-complete, and $F\colon \mathcal{A} \to \mathcal{V}$ be Ψ-continuous; then by (4) the $\mathcal{V}$-category $\Phi^\dagger\mathcal{A} = \Phi(\mathcal{A}^{op})^{op}$ is complete and $\Phi^\dagger F$ is continuous, moreover F has a virtual left adjoint (since $\mathcal{A}$ is small). By Lemma

5.3.8 below it follows that $\Phi^\dagger F$ has a left adjoint. Equivalently $\mathrm{Ran}_V F\colon \Phi^\dagger\mathcal{A}\to\mathcal{V}$ has a left adjoint L; hence $F\cong L(I)\in\Phi^\dagger\mathcal{A}$ is a Φ-colimit of representables. □

Lemma 5.3.8. *Let $F\colon\mathcal{A}\to\mathcal{B}$ be such that $\Phi^\dagger\mathcal{A}$ is complete and $\Phi^\dagger F\colon\Phi^\dagger\mathcal{A}\to\Phi^\dagger\mathcal{B}$ is continuous; then F has a virtual left adjoint if and only if $\Phi^\dagger F$ has a left adjoint.*

Proof. Recall that F has a virtual left adjoint if and only if $\mathcal{B}(B,F-)$ is small for any $B\in\mathcal{B}$; while $\Phi^\dagger F$ has a left adjoint if and only if $\mathcal{B}(B,F-)$ lies in $\Phi^\dagger\mathcal{A}$ for any $B\in\mathcal{B}$. Thus, if $\Phi^\dagger F$ has a left adjoint then F has a virtual left adjoint.

Conversely, assume that $\mathcal{B}(B,F-)$ is small for any $B\in\mathcal{B}$ and let $V\colon\mathcal{A}\hookrightarrow\Phi^\dagger\mathcal{A}$ and $W\colon\mathcal{B}\hookrightarrow\Phi^\dagger\mathcal{B}$ be the inclusions. To conclude it is enough to show that $\Phi^\dagger\mathcal{B}(WB,\Phi^\dagger F-)$ is representable for any $B\in\mathcal{B}$; this would in fact define a relative left adjoint of F with respect to V and hence say that $\mathcal{B}(B,F-)$ lies in $\Phi^\dagger\mathcal{A}$.

Since $\Phi^\dagger F$ is continuous, the $\mathcal{V}$-functor $\Phi^\dagger\mathcal{B}(WB,\Phi^\dagger F-)$ is continuous too, call this G. By [56, Theorem 4.80] it is enough to show that $\{G,1_{\Phi^\dagger\mathcal{A}}\}$ exists in $\Phi^\dagger\mathcal{A}$ and is preserved by G. Since $G\colon\Phi^\dagger\mathcal{A}\to\mathcal{V}$ is continuous, $G\cong\mathrm{Ran}_V(GV)$, and $GV=\mathcal{B}(B,F-)$ which is small. Thus $\{GV,V\}$ exists in $\Phi^\dagger\mathcal{A}$ and

$$\{GV,V\}\cong\{G,\mathrm{Ran}_V V\}\cong\{G,1_{\Phi^\dagger\mathcal{A}}\}.$$

Likewise

$$G\{G,1_{\Phi^\dagger\mathcal{A}}\}\cong G\{GV,V\}\cong\{GV,GV\}\cong\{G,\mathrm{Ran}_V(GV)\}\cong\{G,G\}.$$

□

Corollary 5.3.9. *Let Ψ be locally small and Φ be a companion for Ψ. For each Ψ-cocomplete $\mathcal{A}$ the $\mathcal{V}$-category $\Phi\mathcal{A}$ is the free cocompletion of $\mathcal{A}$ relative to Ψ-colimits. In other words the following is a bi-adjunction*

$$\begin{tikzcd}\mathbf{Coct\text{-}CAT} \arrow[r, "U"', ""{name=U}] & \Psi\text{-}\mathbf{Coct\text{-}CAT} \arrow[l, bend right, "\Phi(-)"', ""{name=P}] \arrow[from=P, to=U, phantom, "\perp"]\end{tikzcd}$$

where U is the forgetful functor.

It is now time to introduce the ingredients of the characterization theorem for accessible $\mathcal{V}$-categories with Ψ-limits. We begin by introducing Φ-orthogonality classes which generalize the usual notion of orthogonality and that of virtual orthogonality class of Chapter 2.

Definition 5.3.10. Let Φ be a companion for Ψ, and $\mathcal{K}$ be a $\mathcal{V}$-category with inclusion $Z\colon\mathcal{K}\hookrightarrow\Phi^\dagger\mathcal{K}$. Let $f\colon ZX\to P$ be a morphism in $\Phi^\dagger\mathcal{K}$ with representable domain. We say that an object A of $\mathcal{K}$ is *orthogonal with respect to f* if

$$\Phi^\dagger\mathcal{K}(f,ZA)\colon\Phi^\dagger\mathcal{K}(P,ZA)\longrightarrow\Phi^\dagger\mathcal{K}(ZX,ZA)$$

is an isomorphism in $\mathcal{V}$.

Given an object P in $\Phi^\dagger\mathcal{K}$, we can write it as a Φ-limit of representables $P\cong\{M,ZH\}$. Thus to give $f\colon ZX\to P$ is the same as giving a cylinder $\bar{f}\colon M\to\mathcal{K}(X,H-)$; moreover $\Phi^\dagger\mathcal{K}(ZX,ZA)\cong\mathcal{K}(X,A)$ and $\Phi^\dagger\mathcal{K}(P,ZA)\cong M*\mathcal{K}(H-,A)$. As a consequence, an object A of $\mathcal{K}$ is orthogonal with respect to $f\colon ZX\to P$ if and only if the map

$$M*\mathcal{K}(H-,A)\to\mathcal{K}(X,A)$$

induced by $\bar{f}\colon M \to \mathcal{K}(X, H-)$ is an isomorphism.

Definition 5.3.11. Let Φ be a companion for Ψ. Given a $\mathcal{V}$-category $\mathcal{K}$ and a small collection $\mathcal{M}$ of morphisms in $\Phi^\dagger\mathcal{K}$ of the form $f\colon ZX \to P$, we denote by $\mathcal{M}^\perp$ the full subcategory of $\mathcal{K}$ spanned by the objects which are orthogonal with respect to each $f \in \mathcal{M}$. We call Φ-*orthogonality class* any full subcategory of $\mathcal{K}$ which arises in this way.

Equivalently, a Φ-orthogonality class in $\mathcal{K}$ is a virtual orthogonality class of $\mathcal{K}$ for which the morphisms that define it lie in $\Phi^\dagger\mathcal{K} \subseteq \mathcal{P}^\dagger\mathcal{K}$.

Examples 5.3.12. Let $\mathcal{K}$ be a $\mathcal{V}$-category;

- if $\Psi = \mathcal{P}$ and $\Phi = \emptyset$, then a Φ-orthogonality class in $\mathcal{K}$ is an orthogonality class in the usual sense.
- if $\Psi = \emptyset$ and $\Phi = \mathcal{P}$, then a Φ-orthogonality class in $\mathcal{K}$ is a virtual orthogonality class in the sense of Section 2.2.4.
- if $\mathcal{V} = \mathbf{Set}$, Ψ is the class of connected categories, and Φ that of the discrete ones, then a Φ-orthogonality class in $\mathcal{K}$ is a multiorthogonality class in the sense of [35].

The content of Definition 5.2.14 in this setting specializes to the following:

Definition 5.3.13. Let Φ be a companion for Ψ. We say that a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{K}$ has a *left* Φ-*adjoint* if for each $X \in \mathcal{K}$ the $\mathcal{V}$-functor $\mathcal{K}(X, F-)$ is a Φ-colimit of representables. If F is fully faithful we say that $\mathcal{A}$ is Φ-*reflective* in $\mathcal{K}$.

Equivalently, F has a left Φ-adjoint if and only if $\mathcal{K}(X, F-)$ lies in $\Phi^\dagger\mathcal{A}$, seen as a full subcategory of $\mathcal{P}^\dagger\mathcal{A}$. In other words, $F\colon \mathcal{A} \to \mathcal{K}$ has a Φ-left adjoint if and only if F has a relative left adjoint with respect to the inclusion $V\colon \mathcal{A} \hookrightarrow \Phi^\dagger\mathcal{A}$, if and only if $\Phi^\dagger F\colon \Phi^\dagger\mathcal{A} \to \Phi^\dagger\mathcal{K}$ has a left adjoint.

Examples 5.3.14. Let $F\colon \mathcal{A} \to \mathcal{V}$ be a $\mathcal{V}$-functor.

- if $\Psi = \mathcal{P}$ and $\Phi = \emptyset$, then a left Φ-adjoint for F is a left adjoint.
- if $\Psi = \emptyset$ and $\Phi = \mathcal{P}$, then a left Φ-adjoint for F is a virtual left adjoint.
- if $\mathcal{V} = \mathbf{Set}$, Ψ is the class of connected categories, and Φ that of the discrete ones, then a left Φ-adjoint is a left multiadjoint.

Theorem 5.3.15. *Let Φ and Ψ be classes of weights, Φ be a companion for Ψ, and $\mathcal{A}$ be a full subcategory of some $[\mathcal{C}, \mathcal{V}]$. The following are equivalent:*

1. *$\mathcal{A}$ is accessible, accessibly embedded, and closed under Ψ-limits in $[\mathcal{C}, \mathcal{V}]$;*
2. *$\mathcal{A}$ is accessibly embedded and Φ-reflective in $[\mathcal{C}, \mathcal{V}]$;*
3. *$\mathcal{A}$ is a Φ-orthogonality class in $[\mathcal{C}, \mathcal{V}]$.*

Proof. (1) $\Leftrightarrow$ (2) is a consequence of Theorem 5.2.16.
(2) $\Rightarrow$ (3). Let $\mathcal{K} = [\mathcal{C}, \mathcal{V}]$ and denote by L' the left $V_\mathcal{A}$-adjoint to J; in particular $Z_\mathcal{A}L'$ is a left adjoint to J relative to $Z_\mathcal{A}V_\mathcal{A}$, which means that $\mathcal{A}$ is virtually reflective. Therefore we have:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook, "V_{\mathcal{A}}"] \arrow[d, hook, "J"'] & \Phi^\dagger\mathcal{A} \arrow[r, hook, "Z_{\mathcal{A}}"] \arrow[d, hook, "\Phi^\dagger J"] & \mathcal{P}^\dagger\mathcal{A} \arrow[d, hook, "R"', shift right] \\
\mathcal{K} \arrow[r, hook, "V_{\mathcal{K}}"'] \arrow[ur, "L'"] & \Phi^\dagger\mathcal{K} \arrow[r, hook, "Z_{\mathcal{K}}"'] & \mathcal{P}^\dagger\mathcal{K} \arrow[u, bend right, "L"', shift right] \arrow[u, phantom, "\vdash"]
\end{tikzcd}$$

where $Z_{\mathcal{A}}L' \cong LZ_{\mathcal{K}}V_{\mathcal{K}}$ since they are both left $(Z_{\mathcal{A}}V_{\mathcal{A}})$-adjoints to J.
By the virtual case we know that $\mathcal{A}$ is the virtual orthogonality class defined by

$$\mathcal{M} := \{\eta_X \colon Z_{\mathcal{K}}V_{\mathcal{K}}X \to RLZ_{\mathcal{K}}V_{\mathcal{K}}X |\ X \in \mathcal{K}_\alpha\}.$$

for some α, where η is the unit of $L \dashv R$. We shall show that this is actually a Φ-orthogonality class. For each $X \in \mathcal{K}_\alpha$ we have

$$RLZ_{\mathcal{K}}V_{\mathcal{K}}X \cong RZ_{\mathcal{A}}L'X \cong Z_{\mathcal{K}}(\Phi^\dagger J)L'X;$$

therefore $\mathcal{M}$ is contained in $\Phi^\dagger\mathcal{K}$ and coincides (up to isomorphism) with

$$\mathcal{M}' := \{\eta_X \colon V_{\mathcal{K}}X \to (\Phi^\dagger J)L'X |\ X \in \mathcal{K}_\alpha\},$$

which exhibits $\mathcal{A}$ as a Φ-orthogonality class.

(3) $\Rightarrow$ (1). $\mathcal{A}$ is accessible by Theorem 2.2.32 because it is in particular a virtual orthogonality class. Let $\mathcal{K} = [\mathcal{C}, \mathcal{V}]$ and $\mathcal{M}$ be the set of arrows in $\Phi^\dagger\mathcal{K}$ defining $\mathcal{A}$; then $\mathcal{A}$ can be seen as the pullback

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook] \arrow[d, hook] & \mathcal{K} \arrow[d, hook, "V_{\mathcal{K}}"] \\
\mathcal{M}^\perp \arrow[r, hook] & \Phi^\dagger\mathcal{K}
\end{tikzcd}$$

where $\mathcal{M}^\perp$ is the an orthogonality class of $\Phi^\dagger\mathcal{K}$ in the usual sense. Now note that $\mathcal{M}^\perp$ is closed under all small limits that exist in $\Phi^\dagger\mathcal{K}$, and $V_{\mathcal{K}}$ preserves Ψ-limits since Φ is a companion for Ψ; it follows then that $\mathcal{A}$ is closed in $\mathcal{K}$ under Ψ-limits as well.

□

And for a general $\mathcal{V}$-category $\mathcal{A}$ we obtain the following.

Theorem 5.3.16. *Let Φ and Ψ be classes of weights for which Φ is a companion for Ψ. The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:*

1. *$\mathcal{A}$ is accessible with Ψ-limits;*
2. *$\mathcal{A}$ is accessible and $\Phi^\dagger\mathcal{A}$ is cocomplete;*
3. *$\mathcal{A}$ is accessible and $\Phi^\dagger\mathcal{A}$ has colimits of representables;*
4. *$\mathcal{A}$ is accessibly embedded and Φ-reflective in some $[\mathcal{C}, \mathcal{V}]$;*
5. *$\mathcal{A}$ is a Φ-orthogonality class in some $[\mathcal{C}, \mathcal{V}]$;*
6. *$\mathcal{A}$ is the category of models of a limit/Φ-colimit sketch.*

Proof. Note that Φ is an accessible companion for Ψ since every $\mathfrak{C}^\Phi$ is either empty or the whole presheaf $\mathcal{V}$-category. Thus (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are a consequence of Theorem 5.2.17 and (4) $\Rightarrow$ (5) $\Rightarrow$ (1) follow from Theorem 5.3.15 above. Finally, (1) $\Leftrightarrow$ (6) follows from Theorem 5.2.27 and Remark 5.2.20. $\square$

Remark 5.3.17. Given a class of weights Φ denote by Φ^- the class of weights whose indexed limits commute in $\mathcal{V}$ with Φ-colimits. It has been shown in [59, Example 5.8] that even for a (weakly) sound class Ψ we do not have the equality $\Psi^{+-} = \Psi^*$, where Ψ^* is the saturation of Ψ; the fact that makes this fail is that Ψ^* need not contain the absolute weights; that's exactly what one needs to obtain an equality, as we now show.

Let Ψ be a locally small weakly sound class. Then Ψ^+ is both a companion for Ψ and for Ψ^{+-}; therefore by Theorem 5.3.16 above it follows that an accessible $\mathcal{V}$-category $\mathcal{A}$ is Ψ-complete if and only if it is Ψ^{+-}-complete. When $\mathcal{A}$ is small, by Theorem 4.2.7, this is saying that a small $\mathcal{V}$-category is Cauchy complete and Ψ-complete if and only if it is Ψ^{+-}-complete (remember that Ψ^{+-} already contains the absolute weights). Similarly, a $\mathcal{V}$-functor from such a $\mathcal{V}$-category into $\mathcal{V}$ is Ψ-continuous if and only if it is Ψ^{+-}-continuous (since they both correspond to Ψ^+-colimits of representables). Thus if we denote by $\mathcal{Q}$ the class of absolute weights we obtain the equality

$$\Psi^{+-} = (\Psi \cup \mathcal{Q})^*$$

expressing Ψ^{+-} as the saturation of Ψ together with $\mathcal{Q}$.

5.3.2 Wide Pullbacks

In this section we let $\mathcal{V} = \mathbf{Set}$ and let Ψ consist of the weights for wide pullbacks. The colimit type $\mathfrak{F}$ we consider is the one given by the groupoid indexed diagrams in **Set** which induce a free action in the sense of Example 5.2.3. Equivalently, a groupoid indexed functor $G\colon \mathcal{G} \to \mathbf{Set}$ is in $\mathfrak{F}$ if and only if, writing $\mathcal{G}$ as a coproduct of groups $(\mathcal{G}_i)_i$, for every non identity $g \in \mathcal{G}_i$ the function $G(g)$ has no fixed points.

Recall the notion of polylimit in a category:

Definition 5.3.18. [68, Definition 0.12] Let $H\colon \mathcal{C} \to \mathcal{A}$ be a diagram in a category $\mathcal{A}$. A *polylimit* of H is given by a family of cones $(c_i\colon \Delta A_i \to H)_{i\in I}$, with $A_i \in \mathcal{A}$, with the following property: for any cone $c\colon \Delta C \to H$ there exists a unique $i \in I$ and a map $f\colon C \to A_i$ in $\mathcal{A}$ such that $c = c_i \circ \Delta f$, moreover f is unique up to unique automorphism of A_i.

Proposition 5.3.19. *Let $\mathcal{A}$ be a category, $V\colon \mathcal{A} \hookrightarrow \mathfrak{F}_1\mathcal{A}$ be the inclusion, and $H\colon \mathcal{C} \to \mathcal{A}$ be a diagram. Then H has a polylimit in $\mathcal{A}$ if and only if VH has a limit in $\mathfrak{F}_1\mathcal{A}$. In particular then $\mathcal{A}$ has (α-small) polylimits if and only if $\mathfrak{F}_1\mathcal{A}$ has (α-small) limits of representables.*

Proof. (This is an adaptation of [50, Proposition 3.4].) Assume first that H has a polylimit $(\Delta A_i \to H)_{i\in I}$ in $\mathcal{A}$; then consider the groupoid indexed diagram $G\colon \sum_i \mathrm{Aut}(A_i) \to \mathcal{A}$ given simply by the inclusions of the A_i's together with their automorphisms. The polylimit property implies that $\mathcal{A}(A, G-)$ lies in $\mathfrak{F}$ for any $A \in \mathcal{A}$: if $f\colon A \to A_i$ and $g \in \mathrm{Aut}(A_i)$ satisfy $gf = f$ then, since f corresponds to a cone $\Delta A \to H$, such a g must be unique, but the identity also satisfies the equality; thus $g = 1_{A_i}$.

As a consequence the colimit X of VG is an object of $\mathfrak{F}_1\mathcal{A}$. Moreover the maps $(c_i\colon \Delta A_i \to H)_{i\in I}$ define a cocone out of VG in $\mathfrak{F}_1\mathcal{A}$ which in turn induces a map

$c\colon \Delta X \to VH$. It is now easy to see that, by the polylimit properties of the A_i's, the map c exhibits X as the limit of VH in $\mathfrak{F}_1\mathcal{A}$.

Conversely, let X be the limit of VH in $\mathfrak{F}_1\mathcal{A}$; then we can write X as the colimit of VG where $G\colon \sum_i \mathcal{G}_i \to \mathcal{A}$ is representably in $\mathfrak{F}$ and each $\mathcal{G}_i$ is a group. Let A_i be the image in $\mathcal{A}$ of each group component $\mathcal{G}_i$; then the limiting cone of X induces a family $(\Delta A_i \to H)_{i\in I}$ of cones over H. We prove that these exhibit $(A_i)_i$ as the polylimit of $\mathcal{A}$. To give a cone for $A \in \mathcal{A}$ over H is the same as giving an arrow $VA \to X$; note now that $\mathfrak{F}_1\mathcal{A}(VA, -)$ preserves the colimit of VG defining X, and therefore $\mathfrak{F}_1\mathcal{A}(VA, X) \cong \operatorname{colim}\mathcal{A}(A, G-)$ in **Set**. It follows that giving an arrow $VA \to X$ is the same as giving a map $A \to A_i$, for a unique i, determined up to composition with some $G(g)\colon A_i \to A_i$; finally this g is also unique because $\mathcal{A}(A, G-)$ is a free groupoid action by hypothesis. It follows that the family $(A_i)_i$ is the polylimit of H in $\mathcal{A}$. $\square$

Recall the following lemma of Lamarche:

Lemma 5.3.20 (Lemma 0.13 of [68])**.** *Let $\mathcal{B}$ be a category with wide pushouts. Then $\mathcal{B}$ has a polyterminal object if and only if it has a weakly terminal family.*

Thanks to this and Proposition 5.2.35 we can easily prove the following:

Proposition 5.3.21. *The class $\mathfrak{F}$ is an accessible companion for the class of wide pullbacks.*

Proof. Accessibility of $\mathfrak{F}$ is given by Example 5.2.25, which also shows that the limit specifications in the sketches defining $\mathfrak{F}$ are all connected. Thus we can use Proposition 5.2.35 to show that $\mathfrak{F}$ is a companion for the class of wide pullbacks. That $\mathfrak{F}$ is compatible with wide pullbacks is given by [50, Proposition 1.4] — see Section 5.3.2 for a comparison of our work with that in [50]. Thanks to Proposition 5.3.19, to show property (ii) of Proposition 5.2.35, we need to prove that every virtually cocomplete category $\mathcal{B}$ with wide pullbacks has a polyinitial object. This follows at once by the dual of Lemma 5.3.20 since every virtually cocomplete category $\mathcal{B}$ has a weakly initial family. Indeed, by virtual cocompleteness, the functor $\Delta 1\colon \mathcal{B} \to \mathbf{Set}$ is small; thus it is the left Kan extension of its restriction to a small full subcategory $\mathcal{C}$ of $\mathcal{B}$. The elements of $\mathcal{C}$ then form a weakly initial family in $\mathcal{B}$. $\square$

Definition 5.3.22. We say that a functor $F\colon \mathcal{A} \to \mathcal{B}$ has a *left polyadjoint* if it has a left $\mathfrak{F}$-adjoint; if F is fully faithful we say that $\mathcal{A}$ is *polyreflective* in $\mathcal{B}$.

Traditionally, one says that $F\colon \mathcal{A} \to \mathcal{B}$ has a left polyadjoint if, for any $B \in \mathcal{B}$, the category $B/F = \mathrm{El}(\mathcal{B}(B, F-))^{op}$ has a polyinitial object [68, Page 35]. By Proposition 5.3.19 and the arguments of Section 5.2.3, this is equivalent to saying that $\mathcal{B}(B, F-)$ lies in $\mathfrak{F}_1^\dagger(\mathcal{A})$. Thus our definition coincides with the classical notion of left polyadjoint.

Remark 5.3.23. In [99, Section 1.2] Taylor shows that $F\colon \mathcal{A} \to \mathcal{B}$ has a left polyadjoint if and only if the induced functor $F/A\colon \mathcal{A}/A \to \mathcal{B}/FA$ has a left adjoint for any $A \in \mathcal{A}$.

Now, since $\mathfrak{F}$ is a companion for the class of wide pullbacks and thanks to Theorem 5.2.16 we obtain:

Theorem 5.3.24. *Let $\mathcal{K}$ be an accessible category with wide pullbacks and $\mathcal{A}$ a full subcategory of $\mathcal{K}$. The following are equivalent:*

1. *$\mathcal{A}$ is accessible, accessibly embedded, and closed under wide pullbacks in $\mathcal{K}$;*

2. *$\mathcal{A}$ is accessibly embedded and polyreflective in $\mathcal{K}$.*

Regarding sketches, note that, by the presentation of $\mathfrak{F}$ as in Example 5.2.31, the categories of $\mathfrak{F}$-models of sketches are the same as the categories of models of galoisian sketches [6]. Thus, thanks to Theorem 5.2.17 and the dual of Proposition 5.3.19, we obtain the characterization below.

Theorem 5.3.25. *Let $\mathcal{A}$ be a category; the following are equivalent:*

1. *$\mathcal{A}$ is accessible with wide pullbacks;*
2. *$\mathcal{A}$ is accessible and $\mathfrak{F}_1^\dagger\mathcal{A}$ is cocomplete;*
3. *$\mathcal{A}$ is accessible and polycocomplete;*
4. *$\mathcal{A}$ is accessibly embedded and polyreflective in $[\mathcal{C}, \mathbf{Set}]$ for some $\mathcal{C}$;*
5. *$\mathcal{A}$ is the category of models of a galoisian sketch.*

The equivalence of (1) and (3) was first given by Lamarche in [68, Theorem 0.20], then Ageron further added condition (5) in [6, Theorem 4.19].

Quasi-coproducts

The notion appearing below has been used in [50]:

Definition 5.3.26. A groupoid indexed diagram $H\colon \mathcal{G} \to \mathcal{B}$ is called *quasi-discrete* if for each non-initial $B \in \mathcal{B}$ the functor $\mathcal{B}(B, H-)$ is a free action in **Set**; that is, if $\mathcal{B}(B, H-) \in \mathfrak{F}$ for any non-initial B. A *quasi-coproduct* is the colimit of such a diagram.

Given a category $\mathcal{B}$ with quasi-coproducts, Hu and Tholen define $\mathcal{B}_q$ to be the full subcategory of $\mathcal{B}$ consisting of those objects B for which $\mathcal{B}(B, -)$ preserves quasi-coproducts; they call $\mathcal{B}$ *quasi-based* if each object is a quasi-coproduct of objects from $\mathcal{B}_q$.

The following shows that our notion of diagram representably in $\mathfrak{F}$ is comparable with that of quasi-discrete diagram.

Proposition 5.3.27. *Let $\mathcal{A}$ be a category, $V\colon \mathcal{A} \hookrightarrow \mathfrak{F}_1\mathcal{A}$ be the inclusion, and $H\colon \mathcal{G} \to \mathcal{A}$ be a groupoid indexed diagram. Then $\mathcal{A}(A, H-) \in \mathfrak{F}$ for any A in $\mathcal{A}$ if and only if VH is quasi-discrete in $\mathfrak{F}_1\mathcal{A}$.*

Similarly, let $\mathcal{B}$ be a quasi-based category and $H\colon \mathcal{G} \to \mathcal{B}_q$ be a groupoid indexed diagram. Then H is quasi-discrete in $\mathcal{B}$ if and only if $\mathcal{B}_q(B, H-) \in \mathfrak{F}$ for any B in $\mathcal{B}_q$.

Proof. Assume that $\mathcal{A}(A, H-) \in \mathfrak{F}$ for any A and that there exists a non-initial object $X \in \mathfrak{F}_1\mathcal{A}$ such that $\mathfrak{F}_1\mathcal{A}(X, VH-)$ is not free. Since X is not initial and is a colimit of of elements from $\mathcal{A}$, there exists a map $VA \to X$ for some A in $\mathcal{A}$. It then follows that $\mathcal{A}(A, H-) \cong \mathfrak{F}_1\mathcal{A}(VA, VH-)$ is not free as well, leading to a contradiction. Conversely, assume that VH is quasi-discrete in $\mathfrak{F}_1\mathcal{A}$. Since the terminal object of $\mathfrak{F}_1\mathcal{A}$ is computed as in $[\mathcal{A}^{op}, \mathbf{Set}]$ it cannot lie in $\mathcal{A}$; thus $\mathcal{A}(A, H-)$ is in $\mathfrak{F}$ for any $A \in \mathcal{A}$ by definition.

The same proof applies to the second statement since the initial object 0 of $\mathcal{B}$ does not lie in $\mathcal{B}_q$; indeed $\mathcal{B}(0, -)$ does not preserve coproducts, and hence cannot preserve quasi-coproducts. □

By the proposition above, it follows that a category $\mathcal{B}$ with quasi-coproducts is quasi-based if and only if $\mathcal{B} \simeq \mathfrak{F}_1\mathcal{B}_q$. Thus, the content of [50, Proposition 3.4] coincides with that of our Proposition 5.3.19.

The difference between our approach and that of Hu and Tholen is simply that, while in [50] they are interested in recognising those categories that arise as free cocompletions under colimits of free groupoid actions, we want to construct such free cocompletions starting from a given category. In fact, when freely adding colimits of free groupoid actions to a category $\mathcal{A}$, one needs to consider those diagrams $H : \mathcal{G} \to \mathcal{A}$ that lie representably in $\mathfrak{F}$; however, when determining if a category $\mathcal{B}$ is a free completion under colimits of free groupoid actions the right notion to consider is that of quasi-discrete diagram.

5.3.3 Products and powers by a dense generator

Let $\mathcal{V}$ be symmetric monoidal closed and locally presentable as usual, and $\mathcal{G}$ be a (possibly large) dense generator of $\mathcal{V}_0$ containing the unit and closed under tensor product.

Definition 5.3.28. Let $\mathscr{E} \subseteq \mathcal{V}^2$ be the class of maps e for which $G \pitchfork e$ is a regular epimorphism for any $G \in \mathcal{G}$. In particular every map in $\mathscr{E}$ is a regular epimorphism.

Note: from now on we assume one of the following conditions:

(I) the unit I is regular projective;

(II) if $f \circ g$ is a regular epimorphism in $\mathcal{V}$ then so is f, and $\mathscr{E}$ is closed under products in $\mathcal{V}^2$.

Any base of enrichment listed in Example 3.1.3 satisfies (I), while any locally dualizable base of Section 3.2 satisfies (II).

Lemma 5.3.29. *If condition (I) holds then: $e \in \mathscr{E}$ if and only if $\mathcal{V}_0(P, e)$ is surjective for any $P \in \mathcal{G}$. In particular $\mathscr{E}$ is closed under products and under composition.*

Proof. If $e \in \mathscr{E}$ and $P \in \mathcal{G}$ then $\mathcal{V}_0(P, e) \cong \mathcal{V}_0(I, P \pitchfork e)$ is surjective since I is regular projective in $\mathcal{V}_0$ and $P \pitchfork e$ is a regular epimorphism. Conversely, assume that $\mathcal{V}_0(P, e)$ is surjective for any $P \in \mathcal{G}$. Let H be the inclusion of $\mathcal{G}$ in $\mathcal{V}_0$; by hypothesis we have a fully faithful $J = \mathcal{V}_0(H, 1) : \mathcal{V}_0 \hookrightarrow \mathcal{P}\mathcal{G}$ which has a left adjoint L since $\mathcal{V}_0$ is cocomplete. Then our hypothesis is saying that Je is a regular epimorphism in $\mathcal{P}\mathcal{G}$: the kernel pair of Je exists in $\mathcal{P}\mathcal{G}$ since it is the image through J of the kernel pair of e in $\mathcal{V}_0$; hence Je, being by definition a pointwise surjection, is the coequalizer of its kernel pair. Thus $e \cong LJe$ is a regular epimorphism in $\mathcal{V}_0$ by cocontinuity of L. Now, given $G \in \mathcal{G}$, the morphism $G \pitchfork e$ still satisfies that $\mathcal{V}_0(P, G \pitchfork e)$ is surjective for any $P \in \mathcal{G}$ (since $\mathcal{G}$ is closed under tensor product); thus $G \pitchfork e$ is a regular epimorphism by the previous argument. It follows that $e \in \mathscr{E}$. $\square$

Definition 5.3.30. We say that a pair $f, g : X \to Y$ in $\mathcal{V}$ is a *$\mathcal{G}$-pseudo equivalence relation* if it factors as a map $e : X \twoheadrightarrow Z$ in $\mathscr{E}$ followed by a kernel pair $h, k : Z \to Y$ whose coequalizer lies in $\mathscr{E}$. Denote by $\mathfrak{C}$ the colimit type generated by the $\mathcal{G}$-pseudo equivalence relations: $\mathfrak{C}_M$ is non-empty only for $M = \Delta I : \mathcal{C}^{op} \to \mathcal{V}$, where $\mathcal{C}$ is the free $\mathcal{V}$-category on a pair of arrows, and in that case $\mathfrak{C}_{\Delta I}$ is the full subcategory of $[\mathcal{C}, \mathcal{V}]$ spanned by the $\mathcal{G}$-pseudo equivalence relations in $\mathcal{V}$.

Proposition 5.3.31. *The class $\mathfrak{C}$ is a companion for the class of products and $\mathcal{G}$-powers. If $\mathcal{G}$ is small then $\mathfrak{C}$ is an accessible companion.*

Proof. That $\mathfrak{C}$ is compatible with products and $\mathcal{G}$-powers in $\mathcal{V}$ is a consequence of the fact that the maps in $\mathscr{E}$ are stable under them, and kernel pairs commute with any limit.

Assume now that $\mathcal{A}$ is virtually cocomplete with products and $\mathcal{G}$-powers, and consider $F\colon \mathcal{A} \to \mathcal{V}$ to be a small functor which preserves these limits. Let $Y\colon \mathcal{A}^{op} \to \mathcal{P}(\mathcal{A}^{op})$ be the Yoneda embedding; then, by smallness of F and since $\mathcal{G}$ is a dense generator, there is a regular epimorphism

$$q\colon \sum_i (P_i \cdot YA_i) \twoheadrightarrow F$$

with $P_i \in \mathcal{G}$ for any i. Consider now $A = \prod_i (P_i \pitchfork A_i)$ in $\mathcal{A}$ and the comparison $\sum_i (P_i \cdot YA_i) \to YA$. Since F preserves products and $\mathcal{G}$-powers, q factorizes through the comparison via a map

$$e\colon YA \longrightarrow F.$$

We wish to prove that e lies pointwise in $\mathscr{E}$. The proof will depend on which condition, (I) or (II), holds in $\mathcal{V}$. Assume that (I) holds; since both YA and F preserve $\mathcal{G}$-powers, we have an isomorphism $G \pitchfork e_B \cong e_{G\pitchfork B}$ for any $G \in \mathcal{G}$ and $B \in \mathcal{A}$. Therefore

$$\mathcal{V}_0(G, e_B) \cong \mathcal{V}_0(I, G \pitchfork e_B) \cong \mathcal{V}_0(I, e_{G\pitchfork B})$$

is surjective since $\mathcal{V}_0(I, q)$ was thanks to condition (I); thus $e_B \in \mathscr{E}$ for any B. On the other hand, if (II) holds then e_B is a regular epimorphism for any $\mathcal{B}$ in $\mathcal{A}$ (since q_B was), and thus also $G \pitchfork e_B \cong e_{G\pitchfork B}$ is. As a consequence $e_B \in \mathscr{E}$.

Now, since $\mathcal{A}$ is virtually cocomplete, $\mathcal{P}(\mathcal{A}^{op})$ is complete, and the kernel pair K of e is still small and preserves the same limits as F; hence by the same arguments we can find a map $e'\colon YA' \to K$ which lies pointwise in $\mathscr{E}$. It follows that F can be expressed as the coequalizer of a $\mathcal{G}$-pseudo equivalence relation between representables; in other words it lies in $\mathfrak{C}_1(\mathcal{A}^{op})$. This proves that $\mathfrak{C}$ is a companion for products and $\mathcal{G}$-powers.

Assume now that $\mathcal{G}$ is small. We define a sketch for $\mathfrak{C}_{\Delta I}$ in the same spirit of Example 5.2.25(3). Consider the $\mathcal{V}$-category $\mathcal{C}'$ in $\mathcal{PC}$ spanned by: $\mathcal{C}$, the coequalizer q of (f, g), the kernel pair $h, k\colon Z \to Y$ of q, the kernel pair (h', k') of the map $e\colon X \to Z$ induced by the kernel pair (h, k), and $\mathcal{G}$-powers of all these. Then define the sketch on $\mathcal{C}'$ with limit conditions $\mathbb{L}$ specifying that (h, k) and (h', k') as the kernel pairs of q and e respectively. The only colimit conditions $\mathbb{C}$ specify, for any $G \in \mathcal{G}$, the maps $G \pitchfork q$ and $G \pitchfork e$ as the coequalizers of $(G \pitchfork h, G \pitchfork k)$ and $(G \pitchfork h', G \pitchfork k')$ respectively. Since these conditions force q and e to be identified with maps in $\mathscr{E}$, it is then easy to see that

$$\mathfrak{C}_{\Delta I} \simeq \mathrm{Mod}(\mathcal{C}', \mathbb{L}, \mathbb{C}).$$

As a consequence it is accessible and closed under filtered colimits, products and $\mathcal{G}$-powers in $[\mathcal{C}, \mathcal{V}]$. $\square$

We can now apply Theorem 5.2.16 to obtain:

Theorem 5.3.32. *Let $\mathcal{K}$ be an accessible $\mathcal{V}$-category with products and $\mathcal{G}$-powers, and let $\mathcal{A}$ be a full subcategory of $\mathcal{K}$. The following are equivalent:*

1. *$\mathcal{A}$ is accessible, accessibly embedded, and closed under products and $\mathcal{G}$-powers;*
2. *$\mathcal{A}$ is accessibly embedded and $\mathfrak{C}$-reflective.*

Regarding sketches, note that, by the presentation of $\mathfrak{C}$ as in Proposition 5.3.31 and Section 5.2.2, the $\mathcal{V}$-categories of $\mathfrak{C}$-models of sketches are the same as the $\mathcal{V}$-categories of models of limit sketches where in addition some maps are specified to lie in $\mathscr{E}$; these kind of sketches are commonly called limit/$\mathscr{E}$ sketches. Thus, Theorems 5.2.17 and 5.2.27 in this case become:

Theorem 5.3.33. *The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:*

1. *$\mathcal{A}$ is accessible with products and $\mathcal{G}$-powers;*
2. *$\mathcal{A}$ is accessible and $\mathfrak{C}_1^\dagger\mathcal{A}$ is cocomplete;*
3. *$\mathcal{A}$ is accessible and $\mathfrak{C}_1^\dagger\mathcal{A}$ has colimits of representables;*
4. *$\mathcal{A}$ is accessibly embedded and $\mathfrak{C}$-reflective in $[\mathcal{C},\mathcal{V}]$ for some $\mathcal{C}$.*

If $\mathcal{G}$ is small, then they are further equivalent to:

5. *$\mathcal{A}$ is the $\mathcal{V}$-category of models of a limit/$\mathscr{E}$ sketch.*

For some of the examples outlined below, this theorem relates to the results of [63], although at this point, it does not capture them completely. We deal with this in Section 5.4.

Example 5.3.34 (*Products and small powers).* Assume that $\mathcal{V}$ satisfies (I) and consider $\mathcal{G} = \mathcal{V}_0$ as the dense generator. Then, using Lemma 5.3.29, it is easy to see that the class $\mathscr{E}$ consists exactly of the split epimorphisms in $\mathcal{V}$.

Example 5.3.35 (*Products and projective powers).* Let $\mathcal{V}$ be a symmetric monoidal quasivariety as in [64]; in this section we consider $\mathcal{G}$ to consist of the enriched finitely presentable and regular projective objects of $\mathcal{V}$. Then $\mathcal{V}$ satisfies condition (II) being regular. The class $\mathscr{E}$ is simply given by the regular epimorphisms in $\mathcal{V}$. The corresponding colimit type $\mathfrak{R}$, like in the ordinary case, is the one formed by the pseudo-equivalence relations in $\mathcal{V}$: we say that a pair $f, g\colon X \to Y$ in $\mathcal{V}$ is a *pseudo-equivalence relation* if it factors as a regular epimorphism $e\colon X \twoheadrightarrow Z$ followed by a kernel pair $h, k\colon Z \to Y$.
Note that a Cauchy complete $\mathcal{V}$-category has products and $\mathcal{G}$-powers if and only if it has products and powers by projective objects. This is because every projective object of $\mathcal{V}$ is a split subobject of coproducts of elements of $\mathcal{G}$ [64, Proposition 4.8]. Therefore Theorem 5.3.33 provides a characterization of the accessible $\mathcal{V}$-categories with products and projective powers.
In the ordinary case, what we obtain is not exactly the traditional characterization of [1, Chapter 4] which uses weak colimits and weak left adjoints. See Section 5.4 for the relation between left $\mathfrak{R}$-adjoints and weak left adjoints, and between colimits in $\mathfrak{R}_1^\dagger\mathcal{A}$ and weak colimits.

Example 5.3.36 (*Products and powers by $\mathbb{2}$).* Let $\mathcal{V} = \mathbf{Cat}$, which satisfies condition (I), and consider $\mathcal{G} = \{\mathbb{2}^n\}_{n\in\mathbb{N}}$ together with the induced class $\mathscr{E}$.
By Lemma 5.3.29, $f\colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{Cat}$ lies in $\mathscr{E}$ if and only if it is *surjective on cubes*; that is, if $\mathbb{2}^n \pitchfork f$ is surjective on objects for any $n \in \mathbb{N}$. The colimit type $\mathfrak{H}$ induced by $\mathscr{E}$ can be described as the one formed by the pairs $f, g\colon \mathcal{C} \to \mathcal{D}$ in $\mathbf{Cat}$ which factor as a surjective on cubes functor $e\colon \mathcal{C} \to \mathscr{E}$ followed by a kernel pair $h, k\colon \mathscr{E} \to \mathcal{D}$ whose coequalizer is surjective on cubes.
Given the ordinal $\mathbb{n} := \{0 \to 1 \cdots \to n-1\}$, one could try to consider those morphisms e such that $\mathbb{n} \pitchfork e$ is a regular epimorphism for each $n \geq 0$. Since $\mathbb{n}$ is a split subobject of $\mathbb{2}^{n-1}$, it follows that every cube epimorphism also satisfies this property. However

the converse does not hold: consider the inclusion of the non-commutative square into the commutative square, this satisfies the condition on ordinals in **Cat** but is not surjective on cubes. Note that, by [83, Proposition 6.2] and the observation just made, every surjective on cubes functor in **Cat** is an effective descent morphism.

Example 5.3.37 (*Products and finite powers)***.** Let $\mathcal{V}$ be locally finitely presentable as a closed category with a regular projective unit, so that condition (I) is satisfied. Consider $\mathcal{G} = \mathcal{V}_f$, then the maps in $\mathscr{E}$ are usually called *pure epimorphisms*: these are the morphisms e for which $\mathcal{V}_0(A, e)$ is surjective for any $A \in \mathcal{V}_f$.

5.3.4 Flexible Limits

For this section we let $\mathcal{V} = \mathbf{Cat}$ and consider the class of weights **Flex** for flexible limits; these are generated by products, inserters, equifiers, and splittings of idempotents. See [61] as a reference for 2-limits.

Remark 5.3.38. Note that, when dealing with accessible categories, it is equivalent to consider **Flex** or the class **PIE** of PIE-limits. This is because every accessible 2-category is Cauchy complete, and flexible limits are generated by PIE-limits and splittings of idempotents.

Before describing a companion for **Flex** let us recall some facts about retract equivalences, coisoidentifiers, and a notion of kernel in the 2-categorical context.

A retract equivalence in a 2-category $\mathcal{K}$ is a morphism $q\colon D \to E$ for which there exist a section $s\colon E \to D$, so that $qs = 1_E$, and an invertible 2-cell $\sigma\colon sq \cong 1_D$. In **Cat** retract equivalences are precisely those equivalences that are moreover surjective on objects.

We shall present retract equivalences in **Cat** as part of a kernel-quotient system ([21], see also Section 5.4). The *kernel* of this system is given by what we call an isokernel cell.

Definition 5.3.39. Let $q\colon D \to E$ be a morphism in a 2-category $\mathcal{K}$; the *isokernel cell* of q is the universal invertible 2-cell

$$\begin{tikzcd} C \arrow[r, bend left, "\pi_1"] \arrow[r, bend right, "\pi_2"'] \arrow[r, phantom, "\Downarrow\phi"] & D \end{tikzcd}$$

such that $q\phi = id$.

In **Cat**, the category C is given by the full subcategory of D^2 whose objects are the isomorphisms $f\colon x \to y$ of D for which $q(f) = id$. Then π_1 and π_2 are the domain and codomain projections, and $\phi_f = f\colon \pi_1(f) \to \pi_2(f)$.

The *quotient* of the system is given by coisoidentifiers:

Definition 5.3.40. Let $\phi\colon \pi_1 \Rightarrow \pi_2\colon C \to D$ be an invertible 2-cell in a 2-category $\mathcal{K}$. The *coisoidentifier* $q\colon D \to E$ of ϕ is the universal morphism out of D satisfying the equality $q\phi = \mathrm{id}$.

See Section 5.4 for a description in terms of weighted colimits.

The next proposition shows that retract equivalences are the (split) colimit of their isokernel cells.

Proposition 5.3.41. *Every retract equivalence in a 2-category $\mathcal{K}$ is the coisoidentifier of its isokernel cell.*

Proof. Let $q\colon D \to E$ be a retract equivalence in $\mathcal{K}$; then we can take a section $s\colon E \to D$ and an invertible 2-cell $\sigma\colon sq \cong 1_D$. Given the isokernel cell $\phi\colon \pi_1 \Rightarrow \pi_2\colon \mathcal{C} \to \mathcal{D}$ of q, by the universal property of the limit applied to σ there exists $v\colon \mathcal{D} \to \mathcal{C}$ with $\pi_1 v = sq$, $\pi_2 v = 1_D$, and $\phi v = \sigma$. It now follows easily that q is the (split) coisoidentifier of ϕ: given any map $h\colon D \to F$ such that $h\phi = \mathrm{id}$, then $(hs)q = h\pi_1 v = h\pi_2 v = h$ so that h factors through q. The factorization is unique since q is an epimorphism. □

In general, it is not true that the coisoidentifier of any isokernel cell is a retract equivalence. For instance, in **Cat**, if $\phi\colon \pi_1 \Rightarrow \pi_2\colon C \to D$ is the isokernel cell of some map $r\colon D \to F$, and D has a non-identity isomorphism $f\colon x \to x$ such that $r(f) = id$, then the colimit $q\colon D \to E$ of ϕ will send f to the identity map. Thus q is not faithful and in particular not a retract equivalence.

We can avoid this problem by introducing the notion of acyclic isokernel cell:

Definition 5.3.42. An isokernel cell $\phi\colon \pi_1 \Rightarrow \pi_2\colon C \to D$ in **Cat** is called *acyclic* if $\phi_c = id$ whenever $\pi_1 c = \pi_2 c$.

Remark 5.3.43. Let $k\colon B \to C$ be the equalizer of π_1 and π_2, then ϕ is acyclic if and only if $\phi k = id$. Equivalently, ϕ is acyclic if and only if the equalizer k of π_1 and π_2 coincides with the identifier of ϕ.

Proposition 5.3.44. *An isokernel cell $\phi\colon \pi_1 \Rightarrow \pi_2\colon C \to D$ in* **Cat** *is acyclic if and only if its coisoidentifier is a retract equivalence. In this case, ϕ is also the isokernel cell of its coisoidentifier.*

Proof. Assume first that the coisoidentifier $q\colon D \to E$ of ϕ is a retract equivalence. By the property of the colimit, every morphism in $\mathcal{D}$ of the form $\phi_c\colon d \to d$ is sent to the identity morphism by q. But q is an equivalence; thus $\phi_c = 1_d$ and ϕ is acyclic.

Conversely let us assume that ϕ, as above, is an acyclic isokernel cell in **Cat**. We shall give an explicit construction of its coisoidentifier.

Consider the following equivalence relation on the objects of D: two objects d and e of $\mathcal{D}$ are related if and only if there exists c in $\mathcal{C}$ such that ϕ_c connects them:

$$d \xrightarrow{\phi_c} e$$

in other words: if $\pi_1 c = d$ and $\pi_2 c = e$. This is actually an equivalence relation on the objects of D since ϕ is an isokernel cell (the inverse or composition of any maps of the form ϕ_c is still of the form $\phi_{c'}$ for some $c' \in C$). Now, for each equivalence class of objects choose a representative. Let E be the full subcategory of D consisting of the chosen representatives, and $s\colon E \to D$ be the inclusion; clearly this is essentially surjective on objects and so an equivalence.

For each $d \in D$, let $qd \in E$ be the chosen representative of the equivalence class of d, and let $\sigma_d\colon sqd \to d$ be the unique isomorphism of the form ϕ_c for some unique $c \in C$ (the uniqueness of σ_d follows from the fact that ϕ is acyclic: given any other σ'_d the composite $\sigma_d^{-1}\sigma'_d$ must be the identity map). Then q defines a functor $q\colon D \to E$ in such a way that $qs = 1_E$ and the σ_d define a natural isomorphism $\sigma\colon sq \cong 1_D$. It is now easy to see that ϕ is also the isokernel cell of q and thus, by the universal property of the limit, there is a unique $v\colon D \to C$ with $\pi_1 v = 1_D$, $\pi_2 v = sq$, and $\phi v = \sigma$. Now q is a (split) coisoidentifier of ϕ. □

We will now construct a companion $\mathfrak{P}$ for **Flex** by considering retract equivalences and acyclic isokernel cells in **Cat**.

Definition 5.3.45. Let $\mathfrak{P}$ be the colimit type given by: $\mathfrak{P}_M$ is non empty only when $M = \Delta 1 \colon \mathcal{W}^{op} \to \mathbf{Cat}$, where $\mathcal{W} = \{\cdot \cong \cdot\}$ is the 2-category freely generated by an invertible 2-cell. In that case $\mathfrak{P}_M$ consists of the (invertible) 2-cells

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \psi"] & Y \end{tikzcd}$$

in **Cat** which factor as a retract equivalence $e \colon X \to Z$ followed by an acyclic isokernel cell $\phi \colon \pi_1 \Rightarrow \pi_2 \colon Z \to Y$ (so that $f = \pi_1 e$, $g = \pi_2 e$, and $\psi = \phi e$).

Remark 5.3.46. If ψ in $\mathfrak{P}_M$ as above, since e is in particular surjective on objects the coisoidentifier of ψ and ϕ coincide and is a retract equivalence by Proposition 5.3.44 above. Moreover ϕ is then the isokernel cell of such a coisoidentifier, and e is the map induced by the universal property of the limit.

In the next proposition we will use [23, Theorem 6.2], which states that if $\mathcal{A}$ is a 2-category with flexible limits and $F \colon \mathcal{A} \to \mathbf{Cat}$ is a 2-functor which preserves them and (whose underlying functor) satisfies the solution-set condition, then there exists $A \in \mathcal{A}$ and a pointwise retract equivalence $q \colon \mathcal{A}(A, -) \to F$.

Proposition 5.3.47. *The colimit type $\mathfrak{P}$ is an accessible companion for* **Flex**.

Proof. Let us first show that $\mathfrak{P}$ is compatible with **Flex**; for that we need to prove that the inclusion $J \colon \mathfrak{P}_M \hookrightarrow [\mathcal{W}, \mathbf{Cat}]$ (with M and $\mathcal{W}$ as in the definition above) and the composite of J with the colimit 2-functor $[\mathcal{W}, \mathbf{Cat}] \to \mathbf{Cat}$ preserve flexible limits.

Consider the 2-category $\mathcal{Z}$ generated by the following data below.

$$\begin{tikzcd} w \arrow[r, "e"] & x \arrow[r, bend left, "\pi_1"] \arrow[r, bend right, "\pi_2"'] \arrow[r, phantom, "\cong \phi"] & y \arrow[r, "q"] & z \end{tikzcd}$$

with $q\phi = 1$. There is a continuous and cocontinuous 2-functor $T \colon [\mathcal{Z}, \mathbf{Cat}] \to [\mathcal{W}, \mathbf{Cat}]$ which acts by sending a diagram (e, ϕ, q) to the invertible 2-cell ϕe.

Let now $\mathcal{P}$ be the full subcategory of $[\mathcal{Z}, \mathbf{Cat}]$ consisting of those diagrams for which e and q are retract equivalences and ϕ is the isokernel cell of q. We will now see that T restricts to an equivalence $T' \colon \mathcal{P} \to \mathfrak{P}_M$. Note first that if (e, ϕ, q) is in $\mathcal{P}$ then ϕe lies in $\mathfrak{P}_M$ since ϕ is acyclic by Proposition 5.3.44. Moreover, consider the 2-functor $S \colon \mathfrak{P}_M \to \mathcal{P}$ defined by sending a 2-cell ψ to the triple (e, ϕ, q) where q is the coisoidentifier of ψ, ϕ is the isokernel cell of q, and e is the map induced by ψ into the domain of ϕ; this is well defined by Remark 5.3.46. It is easy to see that S is an inverse for T', and hence T' is an equivalence of 2-categories.

Now, to prove that the inclusion $J \colon \mathfrak{P}_M \hookrightarrow [\mathcal{W}, \mathbf{Cat}]$ and the restriction of the colimit 2-functor $[\mathcal{W}, \mathbf{Cat}] \to \mathbf{Cat}$ to $\mathfrak{P}_M$ preserve flexible limits, it is enough to show that the inclusion $\mathcal{P} \hookrightarrow [\mathcal{Z}, \mathbf{Cat}]$ and the the 2-functor $[\mathcal{Z}, \mathbf{Cat}] \to \mathbf{Cat}$, evaluating at z, preserve them. The latter is continuous and cocontinuous (being an evaluation 2-functor) and the former preserves flexible limits since retract equivalences are stable in **Cat** under them (see for example [63, Section 9]) and isokernel cells are stable under any limits. Thus it follows that $\mathfrak{P}$ is compatible with flexible limits in **Cat**. Moreover the same arguments also show that $\mathfrak{P}_M$ is closed in $[\mathcal{W}, \mathbf{Cat}]$ under filtered colimits.

Let us now prove the companion property (II) from Definition 5.2.11. Consider a virtually cocomplete 2-category $\mathcal{A}$ with flexible limits, and a small flexible-limit preserving $F\colon \mathcal{A} \to \mathbf{Cat}$. By virtual cocompleteness of $\mathcal{A}$, the $\mathcal{V}$-category $\mathcal{P}(\mathcal{A}^{op})$ is complete; thus for any $X \in \mathbf{Cat}$ the 2-functor $[X, F-]$ is still small and flexible-limit preserving. Therefore there exists a regular epimorphism

$$\sum_i (P_i \cdot \mathcal{A}(A_i, -)) \twoheadrightarrow [X, F-]$$

which, since $[X, F-]$ preserves products and powers, factors through the comparison as a map

$$\mathcal{A}(\prod_i P_i \cdot A_i, -) \twoheadrightarrow [X, F-]$$

which is, in particular, pointwise surjective on objects. The corresponding morphism $\eta\colon X \to F(\prod_i P_i \cdot A_i)$ has the property that any $x\colon X \to FA$ factorizes through η via some $\prod_i P_i \cdot A_i \to A$; thus F satisfies the solution-set condition.

By [23, Theorem 6.2] there exists then a pointwise retract equivalence $q\colon \mathcal{A}(A, -) \to F$. Form now the isokernel cell $\phi\colon \pi_1 \Rightarrow \pi_2\colon G \to \mathcal{A}(A, -)$ of q. By completeness of $\mathcal{P}(\mathcal{A}^{op})$, the 2-functor G is still small and, since it preserves flexible limits, we can obtain again a pointwise retract equivalence $p\colon \mathcal{A}(B, -) \to G$. It follows that $f := \pi_1 p$, $g := \pi_2 p$, and $\psi := \phi p$ define a 2-cell that lies pointwise in $\mathfrak{P}_M$ and which has coisoidentifier equal to F. This shows that $\mathfrak{P}$ is a companion for **Flex**.

To conclude, we are only left to prove that $\mathfrak{P}_M$ is accessible and accessibly embedded in $[\mathcal{W}, \mathbf{Cat}]$. By the arguments above, it is enough to show that the 2-category $\mathcal{P}$ is accessible and accessibly embedded in $[\mathcal{Z}, \mathbf{Cat}]$. Let $\mathcal{Z}'$ be the full subcategory of $\mathcal{P}^{\dagger}\mathcal{Z}$ spanned by the representables, the isokernel cell $\phi'\colon \pi_1' \Rightarrow \pi_2'$ of the map $e\colon w \to x$, and the equalizers l and l' of (π_1, π_2) and (π_1', π_2') respectively.

Then define the sketch on $\mathcal{Z}'$ with limit conditions $\mathbb{L}$ specifying ϕ and ϕ' as the isokernel cells of q and e respectively, l as both the equalizer of (π_1, π_2) and the identifier of ϕ, and similarly l' as both the equalizer of (π_1', π_2') and the identifier of ϕ'. The colimit cocones $\mathbb{C}$ specify simply the maps q and e as the coisoidentifiers of ϕ and ϕ' respectively. These conditions say exactly that ϕ and ϕ' are acyclic isokernel cells (Remark 5.3.43), and that q and e are retract equivalences (by Proposition 5.3.44); thus it is then easy to see that

$$\mathfrak{P}_M \simeq \mathcal{P} \simeq \mathrm{Mod}(\mathcal{W}', \mathbb{L}, \mathbb{C})$$

is therefore accessible.

□

As we did in the previous section, by the presentation of $\mathfrak{P}_M$ given above and Section 5.2.2, the 2-categories of $\mathfrak{P}$-models of sketches are the same as the 2-categories of models of limit sketches where in addition some maps are specified to be retract equivalences. If we let $\mathscr{E}$ be the class of retract equivalences in **Cat**, we recover the notion of limit/$\mathscr{E}$ sketch. Thus, Theorem 5.2.17 in this case becomes:

Theorem 5.3.48. *Let $\mathcal{A}$ be a 2-category; the following are equivalent:*

1. *$\mathcal{A}$ is accessible with flexible limits;*
2. *$\mathcal{A}$ is accessible and $\mathfrak{P}_1^{\dagger}\mathcal{A}$ is cocomplete;*
3. *$\mathcal{A}$ is accessible and $\mathfrak{P}_1^{\dagger}\mathcal{A}$ has colimits of representables;*

4. *$\mathcal{A}$ is accessibly embedded and $\mathfrak{P}$-reflective in $[\mathcal{C}, \mathcal{V}]$ for some $\mathcal{C}$;*
5. *$\mathcal{A}$ is the $\mathcal{V}$-category of models of a limit/$\mathscr{E}$ sketch.*

This gives a characterisation of accessible 2-categories with flexible limits similar to that of [63, Theorem 9.4] and [23, Section 9.3]; for the relation between these two characterization see Section 5.4. Notice that in [63] and [23] they consider the conical version of accessibility while we deal with the flat one; however the two notions coincide by Theorem 3.1.14.

5.3.5 Powers

In this section we denote by Λ the class of weights for powers by λ-small sets.

Definition 5.3.49. [5, Section 4] A functor $H \colon \mathcal{D} \to \mathbf{Set}$, with small domain $\mathcal{D}$, is called λ-*sifted* if the following conditions hold:

1. given less than λ elements $x_i \in Hd_i$ there exists an object $d \in \mathcal{D}$ such that each (d_i, x_i) lies in the same component of $\mathrm{El}(H)$ as some element of Hd.
2. given a set I of cardinality less that λ, and families $(d, x_i)_{i \in I}, (d', x'_i)_{i \in I}$ in $\mathrm{El}(H)$ such that for each i the pair $(d, x_i), (d', x'_i)$ lies in one connected component of $\mathrm{El}(H)$, there exists a zig-zag Z in $\mathcal{D}$ connecting d and d' such that each of the pair above can be connected by a zig-zag in $\mathrm{El}(H)$ whose underlying zig-zag is Z.

Denote by $\mathfrak{S}^\lambda$ the colimit type given by the λ-sifted functors, with weight $\Delta 1$. We allow λ to be ∞, meaning that we have no restriction on the cardinality of I, and define $\mathfrak{S}^\infty$ accordingly.

Proposition 5.3.50. *[5, Section 4] The colimit of a functor $H \colon \mathcal{C} \to \mathbf{Set}$ commutes with λ-small powers if and only if H is λ-sifted. In particular $\mathfrak{S}^\lambda$ is compatible with Λ.*

This allows us to prove that $\mathfrak{S}^\lambda$ is a companion for Λ, and to establish a relationship between being λ-sifted and being Λ-precontinuous.

Proposition 5.3.51. *The colimit type $\mathfrak{S}^\lambda$ is a companion for Λ. Moreover, for any $\mathcal{A}$ we have an equality*

$$\mathfrak{S}^\lambda_1 \mathcal{A} = \Lambda\text{-PCts}(\mathcal{A}^{op}, \mathbf{Set}).$$

Proof. We already know that $\mathfrak{S}^\lambda$ is compatible with Λ, then to conclude it is enough to prove the equality above.

Compatibility with Λ gives the inclusion $\mathfrak{S}^\lambda_1 \mathcal{A} \subseteq \Lambda\text{-PCts}(\mathcal{A}^{op}, \mathbf{Set})$. Consider now a Λ-precontinuous functor $F \colon \mathcal{A}^{op} \to \mathbf{Set}$, then since F is small we can find $H \colon \mathcal{C} \to \mathcal{A}$, with $\mathcal{C}$ small, such that $F \cong \operatorname{colim} YH$. By Λ-precontinuity of F, for each $X \in \mathbf{Set}_\lambda$ and $A \in \mathcal{A}$ we have

$$\operatorname{colim}(X \pitchfork \mathcal{A}(A, H-)) \cong X \pitchfork (\operatorname{colim} \mathcal{A}(A, H-)).$$

Therefore, by the proposition above, $\mathcal{A}(A, H-)$ is λ-sifted for any $A \in \mathcal{A}$; in other words $\mathcal{A}(A, H-) \in \mathfrak{S}^\lambda$ for all $A \in \mathcal{A}$. Thus $F \cong \operatorname{colim} YH$ lies in $\mathfrak{S}^\lambda_1 \mathcal{A}$. □

Remark 5.3.52. This can be done more generally for any $\mathcal{V}$ and any class of objects $\mathcal{G}$ in $\mathcal{V}$. Consider the class of weights $\pitchfork_\mathcal{G}$ for powers by elements of $\mathcal{G}$, and let $\mathfrak{C}_\mathcal{G}$ be the colimit type given by the pairs (M, H) for which

$$M * (G \pitchfork H) \cong G \pitchfork (M * H)$$

for any $G \in \mathcal{G}$.

5.4 Weak reflections

In this section we plan to capture the standard characterization theorems of [1] for accessible categories with products in terms of weak reflection and weak cocompleteness. We also obtain the results of [63] involving accessible 2-categories with flexible limits. To do this we make use of the notion of *kernel-quotient system* developed in [21, Section 2].

Definition 5.4.1. Let us fix an object $X \in \mathcal{V}$ together with a map $x\colon X \to I$; we define a $\mathcal{V}$-category $\mathbb{F}$ with three objects $2, 1, 0$ and homs $\mathbb{F}(2,2) = \mathbb{F}(1,1) = \mathbb{F}(0,0) = I$, $\mathbb{F}(2,1) = X$, $\mathbb{F}(2,0) = \mathbb{F}(1,0) = I$, and $\mathbb{F}(0,1) = \mathbb{F}(0,2) = \mathbb{F}(1,2) = 0$; the only non-trivial composition map is $x\colon \mathbb{F}(1,0) \otimes \mathbb{F}(2,1) \to \mathbb{F}(2,0)$. Let now $\mathbb{K}$ be the full subcategory of $\mathbb{F}$ with objects 2 and 1; we depict $\mathbb{F}$ and $\mathbb{K}$ as below.

$$\begin{tikzcd} 2 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\mathbb{K}"] & 1 \arrow[r, two heads] & 0 \end{tikzcd}$$

We denote the inclusions by $k\colon \mathbb{K} \to \mathbb{F}$ and $h\colon \mathbb{2} \to \mathbb{F}$, and consider the adjunction below (as in [21])

$$\begin{tikzcd} {[\mathbb{2}, \mathcal{V}]} \arrow[r, "K"'] \arrow[r, phantom, "\perp"] & {[\mathbb{K}, \mathcal{V}]} \arrow[l, bend right, "Q"'] \end{tikzcd}$$

where $K = k^* \circ \mathrm{Ran}_h$ and $Q = h^* \circ \mathrm{Lan}_k$. Given a map f in $\mathcal{V}$ we call Kf the $\mathbb{F}$-kernel of f, and given a diagram H on $\mathbb{K}$ we call the map QH the $\mathbb{F}$-quotient of H.

Remark 5.4.2. Note that, for any $H\colon \mathbb{K} \to \mathcal{V}$ and $f\colon A \to H2$ in $\mathcal{V}$, pre-composition with f induces a diagram $H^f\colon \mathbb{K} \to \mathcal{V}$ with $(H^f)1 = H1, (H^f)0 = H0$, and $(H^f)2 = A$.

Lemma 5.4.3. *Every $\mathbb{F}$-quotient is an epimorphism in $\mathcal{V}$. Moreover, for any $H\colon \mathbb{K} \to \mathcal{V}$ and any epimorphism $e\colon A \to H2$ in $\mathcal{V}$, we have $QH \cong Q(H^e)$.*

Proof. Consider an $\mathbb{F}$-quotient map $e = QH\colon H1 \to P$ and any pair $f, g\colon P \to B$ such that $fe = ge$. Then fe is a cocone for H and factors through QH by f and g; by the universal property of the colimit then $f = g$. For the last part of the statement note that, since e is an epimorphism, giving a cocone for H is equivalent to giving a cocone for H^e. Therefore $QH \cong Q(H^e)$. □

Assumption 5.4.4. Let $\mathscr{E}$ be a collection of maps in $\mathcal{V}$. From now on we assume that $\mathbb{F}$ and $\mathscr{E}$ satisfy the following proprieties:

1. Every map in $\mathscr{E}$ is the $\mathbb{F}$-quotient of its $\mathbb{F}$-kernel.
2. $\mathscr{E}$ is closed under composition.

If $\mathscr{E}$ consists of all the $\mathbb{F}$-quotient maps, then (1) is saying that $\mathbb{F}$-quotient maps are effective in the sense of [21].

Examples 5.4.5. The following examples satisfy the conditions above:

1. Let $\mathcal{V}$ be either a regular category of have a regular projective unit. Let $\mathcal{G}$ be a dense generator of $\mathcal{V}_0$, and $\mathscr{E}$ as in Definition 5.3.28 with the properties assumed in Section 5.3.3. Then we can consider the kernel-quotient system for kernel pairs and coequalizers ($X = I + I$ and x is the co-diagonal).

2. Let $\mathcal{V} = \mathbf{Cat}$; consider $\mathbb{F}$ generated by $X = \{\cdot \cong \cdot\}$ the free-living isomorphism and $x\colon X \to 1$ the unique map. Then $\mathbb{F}$-quotients are coisoidentifiers and $\mathbb{F}$-kernels are isokernel cells. We take $\mathscr{E}$ to consist of the retract equivalences (which are $\mathbb{F}$-quotients, but not all $\mathbb{F}$-quotients are retract equivalences).

The data of a kernel-quotient system and a class of maps $\mathscr{E}$ induces a colimit type $\mathfrak{C}$:

Definition 5.4.6. Given $\mathbb{F}$ and $\mathscr{E}$ as above we can define a colimit type $\mathfrak{C}$ as follows: $\mathfrak{C}_M$ is non-empty only for $M = \mathbb{F}(k-,0)\colon \mathbb{K}^{op} \to \mathcal{V}$ and in that case $\mathfrak{C}_M$ is the full subcategory of $[\mathbb{K},\mathcal{V}]$ spanned by the diagrams of the form $(Kq)\circ e$ for any compatible $e, q \in \mathscr{E}$.

Example 5.4.7. In the case of Example 5.4.5(1) the colimit type induced is that of $\mathcal{G}$-pseudo equivalence relations (Section 5.3.3); in the case of (2) we obtain the colimit type of pseudo equivalence 2-relations (Section 5.3.4).

Let $Y\colon \mathcal{A}^{op} \to [\mathcal{A},\mathcal{V}]$ be the Yoneda embedding.

Definition 5.4.8. Say that a $\mathcal{V}$-functor $P\colon \mathcal{A}^{op} \to \mathcal{V}$ is *$\mathscr{E}$-weakly representable* if there exists a map $f\colon YA \to P$ which is pointwise in $\mathscr{E}$. Denote by $\mathrm{W}_{\mathscr{E}}(\mathcal{A})$ the full subcategory of $[\mathcal{A}^{op},\mathcal{V}]$ spanned by the $\mathscr{E}$-weakly representables.

Note that by construction we then have $\mathfrak{C}_1(\mathcal{A}) \subseteq \mathrm{W}_{\mathscr{E}}(\mathcal{A})$.

Definition 5.4.9. We say that a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ is *$\mathscr{E}$-weakly reflective* if $\mathcal{B}(B, F-)$ is $\mathscr{E}$-weakly representable for any $B \in \mathcal{B}$.

In the following proposition we assume $\mathfrak{C}_1(\mathcal{K}^{op})$ to have $\mathbb{F}$-kernels of representables; that is true in particular when it has all limits of representables and hence whenever $\mathcal{K}$ is cocomplete.

Proposition 5.4.10. *The following are equivalent for a fully faithful inclusion $J\colon \mathcal{A} \hookrightarrow \mathcal{K}$ and a $\mathcal{V}$-category $\mathcal{K}$ for which $\mathfrak{C}_1(\mathcal{K}^{op})$ has $\mathbb{F}$-kernels of representables:*

1. *$\mathcal{A}$ is $\mathscr{E}$-weakly reflective in $\mathcal{K}$;*
2. *$\mathcal{A}$ is $\mathfrak{C}$-reflective in $\mathcal{K}$.*

Proof. Note that (1) says that each $\mathcal{K}(K, J-)$ is in $\mathrm{W}_{\mathscr{E}}(\mathcal{A}^{op})$, while (2) says that it is in $\mathfrak{C}_1(\mathcal{A}^{op})$. Then (2) $\Rightarrow$ (1) is trivial since $\mathfrak{C}_1(\mathcal{A}^{op}) \subseteq \mathrm{W}_{\mathscr{E}}(\mathcal{A}^{op})$.

For (1) $\Rightarrow$ (2) assume that $\mathcal{A}$ is $\mathscr{E}$-weakly reflective in $\mathcal{K}$; we need to prove that $\mathcal{K}(K, J-)$ actually lies in $\mathfrak{C}_1(\mathcal{A}^{op})$. By hypothesis $\mathcal{K}(K, J-)$ is $\mathscr{E}$-weakly representable, so there exists $A \in \mathcal{A}$ together with a map pointwise in $\mathscr{E}$

$$q\colon \mathcal{A}(A,-) \twoheadrightarrow \mathcal{K}(K, J-).$$

Such a q determines a map $K \to JA$ in $\mathcal{K}$, and this in turn induces a morphism

$$q'\colon \mathcal{K}(JA,-) \longrightarrow \mathcal{K}(K,-)$$

which, when restricted to $\mathcal{A}$, gives back q. Now, by the hypothesis on $\mathfrak{C}_1(\mathcal{K}^{op})$, the $\mathbb{F}$-kernel Kq' of q', with domain S, lies $\mathfrak{C}_1(\mathcal{K}^{op})$. In particular we obtain a diagram

$$\begin{tikzcd} \mathcal{K}(Q,-) \arrow[r, two heads, "s"] & S \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "Kq'"] & \mathcal{K}(JA,-) \arrow[r, "q'"] & \mathcal{K}(K,-) \end{tikzcd}$$

where s is a map pointwise in $\mathscr{E}$, and hence an epimorphism by Lemma 5.4.3. Now we can restrict this diagram to $\mathcal{A}$ by pre-composing with J and, since pre-composition in continuous, $(Kq')J \cong Kq$ is the $\mathbb{F}$-kernel of q. Moreover q is the $\mathbb{F}$-quotient of Kq by our initial assumptions on $\mathscr{E}$. Note also that sJ is still pointwise in $\mathscr{E}$ and hence an epimorphism.

By hypothesis $\mathcal{K}(Q, J-)$ is $\mathscr{E}$-weakly representable, so there exists $r\colon \mathcal{A}(B,-) \twoheadrightarrow \mathcal{K}(Q, J-)$ which lies pointwise in $\mathscr{E}$. The map $e = sJ \circ r$ is still pointwise in $\mathscr{E}$ by condition (2); thus we have a presentation as below

$$\begin{tikzcd} \mathcal{A}(B,-) \arrow[r, two heads, "e"] & SJ \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "Kq"] & \mathcal{A}(A,-) \arrow[r, two heads, "q"] & \mathcal{K}(K, J-) \end{tikzcd}$$

showing that $\mathcal{K}(X, J-)$ can be written as an $\mathbb{F}$-quotient of representables (by condition (1) and Lemma 5.4.3). Moreover the diagram is constructed so that it lies pointwise in $\mathfrak{C}$, witnessing that $\mathcal{K}(X, J-)$ lies in $\mathfrak{C}_1(\mathcal{A}^{op})$. □

Corollary 5.4.11. *The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:*

1. $\mathrm{W}_{\mathscr{E}}(\mathcal{A})$ *has limits of representables;*
2. $\mathfrak{C}_1(\mathcal{A})$ *has limits of representables.*

Proof. (2) ⇒ (1) is trivial. To show that (1) ⇒ (2) let us prove the dual statement, that is: if $\mathrm{W}^{\dagger}_{\mathscr{E}}(\mathcal{A})$ has colimits of representables then so does $\mathfrak{C}^{\dagger}_1\mathcal{A}$. Let $\mathcal{K} = \mathcal{P}\mathcal{A}$ and $J\colon \mathcal{A} \hookrightarrow \mathcal{K}$ be the inclusion; then $\mathcal{K}$ is cocomplete and hence satisfies the hypothesis of Proposition 5.4.10. Moreover J has an $\mathscr{E}$-weak left adjoint: given $X \in \mathcal{K}$ we can write it as a colimit $X \cong M * JH$ of objects from $\mathcal{A}$, then $\mathcal{K}(X, J-) \cong \{M, YH\}$ is a limit of representables in $[\mathcal{A}, \mathcal{V}]$. In other words $\mathcal{K}(X, J-)$ is a colimit of representables when seen in the opposite $\mathcal{V}$-category, by our assumption then $\mathcal{K}(X, J-)$ lies in $\mathrm{W}^{\dagger}_{\mathscr{E}}(\mathcal{A})$, as desired.

It follows by the proposition above that J is $\mathfrak{C}$-reflective, and thus $\mathfrak{C}^{\dagger}_1\mathcal{A}$ has colimits of representables: compute the colimits in $\mathcal{K}$ and then transport them into $\mathfrak{C}^{\dagger}_1\mathcal{A}$ through the relative left adjoint. □

Remark 5.4.12. Assume that the unit I and the object X (defining $\mathbb{F}$) are α-presentable; then $\mathbb{F}$-kernels are α-small limits. Thus, if we replace $\mathcal{P}\mathcal{A}$ in the proof above with the free cocompletion under α-small colimits, we can prove that $\mathrm{W}_{\mathscr{E}}(\mathcal{A})$ has α-small limits of representables if and only if $\mathfrak{C}_1\mathcal{A}$ has them.

Now we can apply Proposition 5.4.10 and its corollary in the context of Example 5.4.5(1), where $\mathscr{E}$ is the class of those regular epimorphisms that are stable under $\mathcal{G}$-powers. Then Theorem 5.3.33 becomes:

Theorem 5.4.13. *Let $\mathcal{A}$ be a $\mathcal{V}$-category; the following are equivalent:*

1. *$\mathcal{A}$ is accessible with products and $\mathcal{G}$-powers;*
2. *$\mathcal{A}$ is accessible and $\mathrm{W}^{\dagger}_{\mathscr{E}}(\mathcal{A})$ has colimits of representables;*
3. *$\mathcal{A}$ is accessibly embedded and $\mathscr{E}$-weakly reflective in $[\mathcal{C}, \mathcal{V}]$ for some $\mathcal{C}$;*
4. *$\mathcal{A}$ is the $\mathcal{V}$-category of models of a limit/$\mathscr{E}$ sketch.*

In this way we recover the characterization of ordinary accessible categories with products given in [2, Chapter 4]; we also obtain an enriched version of it in the context of categories enriched over finitary quasivarieties.

Similarly, we can apply Proposition 5.4.10 and its corollary in the context of Example 5.4.5(2), where $\mathcal{V} = \mathbf{Cat}$ and $\mathscr{E}$ is the class of retract equivalences. Then Theorem 5.3.48 becomes:

Theorem 5.4.14. *Let $\mathcal{A}$ be a 2-category; the following are equivalent:*

1. *$\mathcal{A}$ is accessible with flexible limits;*
2. *$\mathcal{A}$ is accessible and $\mathrm{W}^{\dagger}_{\mathscr{E}}(\mathcal{A})$ has colimits of representables;*
3. *$\mathcal{A}$ is accessibly embedded and $\mathscr{E}$-weakly reflective in $[\mathcal{C}, \mathcal{V}]$ for some $\mathcal{C}$;*
4. *$\mathcal{A}$ is the $\mathcal{V}$-category of models of a limit/$\mathscr{E}$ sketch.*

As a consequence we obtain part of [63, Theorem 9.4] characterizing accessible 2-categories with flexible limits in terms of weak cocompleteness.

CHAPTER

6

Dualities for accessible categories with limits

Gabriel and Ulmer showed in [44] that a locally finitely presentable category $\mathcal{K}$ can be described, starting from its finitely cocomplete subcategory $\mathcal{K}_f$ of finitely presentable objects, as the category $\mathrm{Lex}(\mathcal{K}_f^{op}, \mathbf{Set})$ of the finite-limit preserving functors from $\mathcal{K}_f^{op}$ into $\mathbf{Set}$. Therefore we obtain a duality between the 2-category of locally finitely presentable categories and that of the small and finitely complete ones; this was generalized to the enriched context in [57] and [18].

When completeness is dropped, and we deal just with the existence of some limits (as in Chapter 5), some dualities have been considered in the literature. In the absence of limits, Makkai and Pare gave a duality between the 2-category determined by the finitely accessible categories and the 2-category of presheaf categories, lex left-adjoint functors between them, and natural transformations. Similarly, Diers gave a duality in [35] between locally finitely multipresentable categories (finitely accessible with connected limits) and finitely complete categories which are the free cocompletions of a small category under coproducts. In the case of accessible categories with products, Hu gave a duality [49] between the 2-category of the weakly locally finitely presentable categories and that of exact categories with enough projectives.

We recover all these dualities as part of the framework of companions. In particular we shall show in Section 6.2 that, if $\mathfrak{C}$ is a companion for Ψ and satisfies some additional properties, then the 2-category of finitely accessible $\mathcal{V}$-categories with Ψ-limits is biequivalent to the opposite of the 2-category whose objects are "lex $\mathfrak{C}$-cocompletions" of small $\mathcal{V}$-categories. This applies for instance to the case of any weakly sound class Ψ (see Section 6.3.1 where some examples are spelled out explicitly) and in the context of enriched weakly locally presentable categories (Section 6.3.3).

6.1 More on colimit types

Given a colimit type $\mathfrak{C}$, in Section 5.2.1 we introduced the $\mathcal{V}$-category $\mathfrak{C}_1\mathcal{A}$ as that obtained from $\mathcal{A}$ by freely adding colimits of type $\mathfrak{C}$ of elements of $\mathcal{A}$. Here we generalize that by defining a $\mathcal{V}$-category $\mathfrak{C}\mathcal{A}$ as a "free cocompletion" of $\mathcal{A}$ under $\mathfrak{C}$-colimits, so that in particular one has the inclusion $\mathfrak{C}_1\mathcal{A} \subseteq \mathfrak{C}\mathcal{A}$. Then we adapt the results of Section 5.2.1 to the setting where $\mathfrak{C}_1\mathcal{A}$ is replaced by $\mathfrak{C}\mathcal{A}$. This construction will be needed to prove the duality theorem.

Definition 6.1.1. Let $\mathfrak{C}$ be a class of diagrams and $\mathcal{A}$ be a $\mathcal{V}$-category. We say that a weight $M\colon \mathcal{C}^{op} \to \mathcal{V}$ together with $H\colon \mathcal{C} \to \mathcal{P}\mathcal{A}$ is *a colimit diagram of type* $\mathfrak{C}$ if it is representably in $\mathfrak{C}$ when restricted to $\mathcal{A}$:

$$\mathrm{ev}_A \circ H \in \mathfrak{C}_M$$

for any $A \in \mathcal{A}$, where $\mathrm{ev}_A\colon \mathcal{P}\mathcal{A} \to \mathcal{V}$ is the evaluation at $A \in \mathcal{A}$. Dually, a pair $N\colon \mathcal{C} \to \mathcal{V}$ together with $H\colon \mathcal{C} \to \mathcal{P}^\dagger\mathcal{A}$ is *a limit diagram of type* $\mathfrak{C}$ if (M, H^{op}) is a colimit diagram of type $\mathfrak{C}$.

Equivalently, M and H as above define a colimit diagram of type $\mathfrak{C}$ if $\mathcal{P}\mathcal{A}(YA, H-) \in \mathfrak{C}_M$ for any $A \in \mathcal{A}$. When H lands in $\mathcal{A}$, since $\mathcal{P}\mathcal{A}(YA, H-) \cong \mathcal{A}(A, H-)$, we recover the notion of diagram used to define the elements of $\mathfrak{C}_1\mathcal{A}$.

Definition 6.1.2. Given a $\mathcal{V}$-category $\mathcal{A}$ we define $\mathfrak{C}\mathcal{A}$ to be the smallest full subcategory of $\mathcal{P}\mathcal{A}$ which contains the representables and is closed under colimits of type $\mathfrak{C}$. Dually, let $\mathfrak{C}^\dagger\mathcal{A} = \mathfrak{C}(\mathcal{A}^{op})^{op}$.

Remark 6.1.3. The $\mathcal{V}$-category $\mathfrak{C}\mathcal{A}$ can be described as the intersection of all those full subcategories $\mathcal{B}$ of $\mathcal{P}\mathcal{A}$ which contain the representables and are closed under colimits of type $\mathfrak{C}$. Alternatively we can define $\mathfrak{C}\mathcal{A}$ by transfinite recursion on the full subcategories $\mathfrak{C}_\gamma\mathcal{A}$ of $\mathcal{P}\mathcal{A}$ defined as follows: let $\mathfrak{C}_0\mathcal{A} = \mathcal{A}$, then $\mathfrak{C}_{\gamma+1}\mathcal{A}$ consists of $\mathfrak{C}_\gamma\mathcal{A}$ together with all the colimits in $\mathcal{P}\mathcal{A}$ of diagrams in $\mathfrak{C}_\gamma\mathcal{A}$ which are of type $\mathfrak{C}$. Take unions at the limit steps. Then $\mathfrak{C}\mathcal{A} = \mathfrak{C}_\lambda\mathcal{A}$ for an opportune inaccessible cardinal λ.

Note that, by construction, the $\mathcal{V}$-category $\mathfrak{C}_1\mathcal{A}$ that appears above is exactly what we have been using in Section 5.2.1.

Proposition 6.1.4. *Let Ψ be a class of weights and $\mathfrak{C}$ a colimit type compatible with Ψ. Then the following inclusion holds for any $\mathcal{V}$-category $\mathcal{A}$*

$$\mathfrak{C}\mathcal{A} \subseteq \Psi\text{-}\mathrm{PCts}[\mathcal{A}^{op}, \mathcal{V}]$$

as full subcategories of $\mathcal{P}\mathcal{A}$.

Proof. Since $\Psi\text{-}\mathrm{PCts}[\mathcal{A}^{op}, \mathcal{V}]$ contains the representables, it is enough to show that it is closed in $\mathcal{P}\mathcal{A}$ under colimits of type $\mathfrak{C}$. Let $M\colon \mathcal{C}^{op} \to \mathcal{V}$ be a weight and $H\colon \mathcal{C} \to \mathcal{P}\mathcal{A}$ be a diagram of type $\mathfrak{C}$ which lands in $\Psi\text{-}\mathrm{PCts}[\mathcal{A}^{op}, \mathcal{V}]$. We need to prove that $M * H$ is Ψ-precontinuous; in other words we need to show that $(M * H) * -\colon [\mathcal{A}, \mathcal{V}] \to \mathcal{V}$ preserves Ψ-limits of representables. Note that $(M * H) * -$ can be written as the composite of $H * -\colon [\mathcal{A}, \mathcal{V}] \to [\mathcal{C}, \mathcal{V}]$ (given pointwise by $(H * F)(C) \cong HC * F$) and of $M * -\colon [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$.

Now, since for any $A \in \mathcal{A}$ the $\mathcal{V}$-functor HA is Ψ-precontinuous, it follows that $H * -$ preserves Ψ-limits of diagrams landing in $\mathcal{A}^{op}$; denote the data of such a diagram by $N: \mathcal{D} \to \mathcal{V} \in \Psi$ and $S: \mathcal{D} \to \mathcal{A}^{op}$. Then, for any $D \in \mathcal{D}$, the image of $YSD := \mathcal{A}(SD, -)$ through $H * -$ is $H * YSD \cong YSD * H(-) \cong \mathrm{ev}_{SD} \circ H$ and lies in $\mathfrak{C}_M$ by hypothesis (because H is of type $\mathfrak{C}$). Since $M * -$ preserves Ψ-limits of diagrams landing in $\mathfrak{C}_M$ it follows then that $(M * H) * -$ preserves Ψ-limits of representables. $\square$

Corollary 6.1.5. *If $\mathfrak{C}$ is a colimit type compatible with Ψ and $\mathcal{A}^{op}$ is Ψ-complete, then any $F \in \mathfrak{C}\mathcal{A}$ is Ψ-continuous and small.*

Let us consider now the case where $\mathfrak{C}$ is a companion for Ψ; from now on it will be more convenient to use $\mathfrak{C}^\dagger\mathcal{A}$ instead of $\mathfrak{C}\mathcal{A}$. Condition (II) of the notion of companion then implies in particular that in some cases the one-step completion is already closed under colimits of type $\mathfrak{C}$:

Corollary 6.1.6. *Let $\mathfrak{C}$ be a companion for Ψ; then for every Ψ-complete and virtually cocomplete $\mathcal{A}$ we have $\mathfrak{C}^\dagger_1\mathcal{A} = \mathfrak{C}^\dagger\mathcal{A}$.*

Proof. The $\mathcal{V}$-categories $\mathfrak{C}^\dagger_1\mathcal{A}$ and $\mathfrak{C}^\dagger\mathcal{A}$ are both contained in $\Psi\text{-PCts}[\mathcal{A}, \mathcal{V}]^{op}$. Moreover, since $\mathfrak{C}$ is a companion for Ψ, we also have the equality $\mathfrak{C}^\dagger_1\mathcal{A} = \Psi\text{-PCts}[\mathcal{A}, \mathcal{V}]^{op}$. $\square$

Then we can adapt the content of Proposition 5.2.13 to this framework:

Proposition 6.1.7. *Let $\mathfrak{C}$ be a companion for Ψ and let $\mathcal{A}$ be Ψ-complete; the following are equivalent:*

1. *$\mathcal{A}$ is virtually cocomplete;*
2. *$\mathfrak{C}^\dagger\mathcal{A}$ is cocomplete;*
3. *$\mathfrak{C}^\dagger\mathcal{A}$ has colimits of representables.*

Proof. The same proof as in Proposition 5.2.13 applies thanks to Corollary 6.1.6 above. $\square$

Finally one can also obtain a version of Theorem 5.2.16:

Theorem 6.1.8. *Let $\mathfrak{C}$ be a companion for Ψ and $\mathcal{K}$ be an accessible $\mathcal{V}$-category with Ψ-limits. The following are equivalent for a full subcategory $J: \mathcal{A} \hookrightarrow \mathcal{K}$:*

1. *$\mathcal{A}$ is accessible, accessibly embedded, and closed under Ψ-limits;*
2. *$\mathcal{A}$ is accessibly embedded and J has a relative left adjoint with respect to the inclusion $V: \mathcal{A} \hookrightarrow \mathfrak{C}^\dagger\mathcal{A}$.*

Proof. The implication (1) $\Rightarrow$ (2) is a consequence of the same implication of Theorem 5.2.16 since $\mathfrak{C}^\dagger_1\mathcal{A} \subseteq \mathfrak{C}^\dagger\mathcal{A}$, while (2) $\Rightarrow$ (1) is a consequence of Theorem 5.1.10 plus the fact that $\mathfrak{C}^\dagger\mathcal{A} \subseteq \Psi\text{-PCts}[\mathcal{A}, \mathcal{V}]^{op}$. $\square$

Theorem 6.1.9. *Let $\mathfrak{C}$ be a companion for Ψ and $\mathcal{A}$ be a category; the following are equivalent:*

1. *$\mathcal{A}$ is accessible and Ψ-complete;*
2. *$\mathcal{A}$ is accessible and $\mathfrak{C}^\dagger\mathcal{A}$ is cocomplete;*
3. *$\mathcal{A}$ is accessible and $\mathfrak{C}^\dagger\mathcal{A}$ has colimits of representables;*
4. *$\mathcal{A}$ is accessibly embedded in $[\mathcal{C}, \mathcal{V}]$ for some $\mathcal{C}$, and the inclusion has a relative left adjoint with respect to $V: \mathcal{A} \hookrightarrow \mathfrak{C}^\dagger\mathcal{A}$.*

Proof. Use the theorem above and apply the same proof as in Theorem 5.1.11. $\square$

6.2 The duality

Let $\mathfrak{C}$ be a companion for Ψ, and α be a fixed regular cardinal; in this section we assume two further conditions:

(a) given any $M\colon \mathcal{C}^{op} \to \mathcal{V}$, the category $\mathfrak{C}_M$ is closed in $[\mathcal{C}, \mathcal{V}]$ under α-flat colimits.

(b) for any α-accessible $\mathcal{V}$-category $\mathcal{A}$ with Ψ-limits, every $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$ in $\mathcal{P}(\mathcal{A}^{op})$ preserving Ψ-limits and α-flat colimits is a colimit of type $\mathfrak{C}$ of elements from $\mathcal{A}_\alpha$.

Condition (b) can be rephrased as: there exist $M\colon \mathcal{C}^{op} \to \mathcal{V}$ and $H\colon \mathcal{C}^{op} \to \mathcal{A}$ landing in $\mathcal{A}_\alpha$ such that $\mathcal{A}(H-, A) \in \mathfrak{C}_M$ for any $A \in \mathcal{A}$, and $F \cong M * YH$. Moreover, condition (a) is automatically true (for some α) if $\mathfrak{C}$ is an accessible companion for Ψ.

Examples 6.2.1. The companions listed below satisfy (a) and (b).

1. Ψ a weakly sound class in $\mathcal{V}$ and $\mathfrak{C} = \mathfrak{C}^{\Psi^+}$, see section 6.3.1.

2. $\mathcal{V} = \mathbf{Set}$, Ψ the class for wide pullbacks, and $\mathfrak{F}$ given by the free groupoid actions by [50, Proposition 4.2]. See also Section 6.3.2.

3. $\mathcal{V} = \mathbf{Set}$, Ψ the class for products, and $\mathfrak{R}$ the pseudo equivalence relations. For any α-accessible $\mathcal{A}$ with products and any $F\colon \mathcal{A} \to \mathcal{V}$ preserving products and α-filtered colimits, we can cover F with a representable $q\colon YA \twoheadrightarrow F$ and we can write A as an α-filtered colimit of α-presentables; then since F preserves this colimit q factors through a map $e\colon YB \twoheadrightarrow F$, with $B \in \mathcal{A}_\alpha$, which is still a regular epimorphism. Now argue as usual by taking the kernel pair of e and repeating the argument. This exhibits F as a colimit of type $\mathfrak{R}$ of elements of $\mathcal{A}_\alpha$.

4. The argument of (3) generalizes to the enriched setting of Section 5.3.3. For a detailed treatment see Sections 6.3.3 and 6.3.4.

Given an α-accessible $\mathcal{V}$-category $\mathcal{A}$ and the inclusion $J\colon \mathcal{A}_\alpha \hookrightarrow \mathcal{A}$, condition (a) allows us to restrict $\mathcal{P}^\dagger J$ to a $\mathcal{V}$-functor $\mathfrak{C}_1^\dagger J$ as shown below.

$$\begin{tikzcd}
\mathfrak{C}_1^\dagger \mathcal{A}_\alpha \arrow[r, hook, "\mathfrak{C}_1^\dagger J"] & \mathfrak{C}_1^\dagger \mathcal{A} \\
\mathcal{A}_\alpha \arrow[u, hook] \arrow[r, hook, "J"'] & \mathcal{A} \arrow[u, hook]
\end{tikzcd}$$

This follows from the fact that given any pair (M, H) in $\mathcal{A}_\alpha$ for which $\mathcal{A}_\alpha(H-, A) \in \mathfrak{C}_M$ for any $A \in \mathcal{A}_\alpha$, then (M, JH) still satisfies the same property with respect to any $A \in \mathcal{A}$: given $A \in \mathcal{A}$ we can write it as an α-flat colimit $A \cong N * JK$ of elements from $\mathcal{A}_\alpha$; therefore $\mathcal{A}(JH-, A) \cong N\Box * \mathcal{A}_\alpha(H-, K\Box)$ is an α-flat colimit of objects from $\mathfrak{C}_M$ and hence is itself in $\mathfrak{C}_M$ by (a).

Condition (b) then says that for any α-accessible $\mathcal{A}$ with Ψ-limits we obtain

$$\mathfrak{C}_1^\dagger \mathcal{A}_\alpha \simeq \alpha\text{-}\mathrm{Acc}_\Psi(\mathcal{A}, \mathcal{V})^{op} \tag{6.1}$$

where $\alpha\text{-}\mathrm{Acc}_\Psi(\mathcal{A}, \mathcal{V})$ is the full subcategory of $[\mathcal{A}, \mathcal{V}]$ spanned by those $\mathcal{V}$-functors which preserve Ψ-limits and α-flat colimits, and the equivalence is given by left Kan extending along the inclusion $J\colon \mathcal{A}_\alpha \hookrightarrow \mathcal{A}$. Indeed, the observation above says that if $F\colon \mathcal{A}_\alpha \to \mathcal{V}$ is in $\mathfrak{C}_1^\dagger\mathcal{A}_\alpha$ then $\mathrm{Lan}_J F$ is in $\mathfrak{C}_1^\dagger\mathcal{A}$ and hence is Ψ-continuous, it moreover preserves α-flat colimits because $\mathcal{A}$ is α-accessible. The converse is just a rephrasing of condition (b).

We now introduce the notion of $\mathfrak{C}$-cocontinuous and $\mathfrak{C}$-continuous $\mathcal{V}$-functors between $\mathcal{V}$-categories of the form $\mathfrak{C}\mathcal{B}$.

Definition 6.2.2. Let $\mathcal{B}$ and $\mathcal{B}'$ be $\mathcal{V}$-categories. A $\mathcal{V}$-functor $F\colon \mathfrak{C}\mathcal{B} \to \mathfrak{C}\mathcal{B}'$ is called $\mathfrak{C}$-*cocontinuous* if it preserves diagrams of type $\mathfrak{C}$ as well as their colimits. We define $\mathfrak{C}$-*continuous* functors accordingly.

In other words, $F\colon \mathfrak{C}\mathcal{B} \to \mathfrak{C}\mathcal{B}'$ is $\mathfrak{C}$-cocontinuous if for any weight $M\colon \mathcal{C}^{op} \to \mathcal{V}$ and diagram $H\colon \mathcal{C} \to \mathfrak{C}\mathcal{B}$ of type $\mathfrak{C}$ (with respect to $\mathcal{B}$), then M and FH also form a diagram of type $\mathfrak{C}$ (with respect to $\mathcal{B}'$) and F preserves the colimit $M * H$. Note that $\mathfrak{C}$-cocontinuous $\mathcal{V}$-functors are then the left Kan extension of their restriction to $\mathcal{B}$, but not everything which arises in this way is $\mathfrak{C}$-cocontinuous: they will preserve the colimits but may not preserve the diagrams of type $\mathfrak{C}$.

Proposition 6.2.3. *Let $\mathcal{A}$ be an α-accessible $\mathcal{V}$-category with Ψ-limits; then $\mathfrak{C}_1^\dagger\mathcal{A} = \mathfrak{C}^\dagger\mathcal{A}$ and $\mathfrak{C}_1^\dagger\mathcal{A}_\alpha = \mathfrak{C}^\dagger\mathcal{A}_\alpha$. Moreover $\mathfrak{C}^\dagger\mathcal{A}_\alpha$ is α-cocomplete and $\mathcal{P}^\dagger J$ restricts to*

$$\mathfrak{C}^\dagger J\colon \mathfrak{C}^\dagger\mathcal{A}_\alpha \hookrightarrow \mathfrak{C}^\dagger\mathcal{A}$$

which is α-cocontinuous and $\mathfrak{C}$-continuous.

Proof. We know that $\mathfrak{C}_1^\dagger\mathcal{A} = \Psi\text{-Cont}(\mathcal{A},\mathcal{V})^{op} \cap \mathcal{P}^\dagger\mathcal{A}$, this is closed in $\mathcal{P}^\dagger\mathcal{A}$ under limits of type $\mathfrak{C}$ since the diagrams involved are pointwise in $\mathfrak{C}$ and colimits of type $\mathfrak{C}$ commute in $\mathcal{V}$ with Ψ-limits; thus $\mathfrak{C}_1^\dagger(\mathcal{A}) = \mathfrak{C}^\dagger(\mathcal{A})$. Consider now the inclusions

$$\begin{tikzcd}
\mathcal{P}^\dagger\mathcal{A}_\alpha \arrow[r, hook, "\mathcal{P}^\dagger J"] & \mathcal{P}^\dagger\mathcal{A} \\
\mathfrak{C}^\dagger\mathcal{A}_\alpha \arrow[u, hook, "W'"] \arrow[r, dashed, "\mathfrak{C}^\dagger J"] & \mathfrak{C}^\dagger\mathcal{A} \arrow[u, hook, "W"'] \\
\mathcal{A}_\alpha \arrow[u, hook, "V'"] \arrow[r, hook, "J"'] & \mathcal{A} \arrow[u, hook, "V"']
\end{tikzcd}$$

to prove that $\mathcal{P}^\dagger J$ restricts to a $\mathfrak{C}$-continuous $\mathfrak{C}^\dagger J\colon \mathfrak{C}^\dagger\mathcal{A}_\alpha \hookrightarrow \mathfrak{C}^\dagger\mathcal{A}$ it is enough to show that given a limit diagram $(M\colon \mathcal{C} \to \mathcal{V}, H\colon \mathcal{C} \to \mathfrak{C}^\dagger\mathcal{A}_\alpha)$ of type $\mathfrak{C}$, then $(M, (\mathcal{P}^\dagger J)W'H)$ lands in $\mathfrak{C}^\dagger\mathcal{A}$ and is of type $\mathfrak{C}$ as well. This is done by induction on the construction of $\mathfrak{C}^\dagger\mathcal{A}$. We already know that $\mathfrak{C}_1^\dagger J$ is well defined and $\mathfrak{C}$-continuous. Assume now that also $\mathfrak{C}_\gamma^\dagger J$ is such, and that $H\colon \mathcal{C} \to \mathfrak{C}^\dagger\mathcal{A}_\alpha$ as above lands in $\mathfrak{C}_\gamma^\dagger\mathcal{A}_\alpha$, then by inductive hypothesis $\mathcal{P}^\dagger J \circ W'H$ lands in $\mathfrak{C}_\gamma^\dagger\mathcal{A}$ and therefore in $\mathfrak{C}^\dagger\mathcal{A}$. Then we only need to prove that $(M, (\mathcal{P}^\dagger J)W'H)$ is of type $\mathfrak{C}$ with respect to $\mathcal{A}$.

Note first that for any $X \in \mathcal{P}^\dagger\mathcal{A}_\alpha$ the $\mathcal{V}$-functor

$$\mathcal{P}^\dagger\mathcal{A}((\mathcal{P}^\dagger J)X, WV-) \cong (\mathcal{P}^\dagger J)X \cong \mathrm{Lan}_J X$$

preserves α-flat colimits since $\mathcal{A}$ is α-accessible and J is the inclusion of the α-presentable objects in $\mathcal{A}$. Now, given $A \in \mathcal{A}$, we can write it as an α-flat colimit $A \cong N * JK$ of

α-presentable objects; thus

$$\begin{aligned} \mathcal{P}^\dagger\mathcal{A}((\mathcal{P}^\dagger J)W'H-, WVA) &\cong N\Box * \mathcal{P}^\dagger\mathcal{A}((\mathcal{P}^\dagger J)W'H-, WVJK\Box) \\ &\cong N\Box * \mathcal{P}^\dagger\mathcal{A}_\alpha(W'H-, W'V'K\Box) \\ &\cong N\Box * \mathfrak{C}^\dagger\mathcal{A}_\alpha(H-, V'K\Box) \end{aligned}$$

where the first isomorphism holds since, by the argument above, $\mathcal{P}^\dagger\mathcal{A}((\mathcal{P}^\dagger J)W'H-, WV\Box)$ preserves α-flat colimits in the second variable. Thus $\mathcal{P}^\dagger\mathcal{A}((\mathcal{P}^\dagger J)W'H-, WVA)$ is an α-flat colimit of elements of $\mathfrak{C}_M$, and hence is in $\mathfrak{C}_M$. This proves that $(M, \mathcal{P}^\dagger J \circ W'H)$ is of type $\mathfrak{C}$; thus $\mathcal{P}^\dagger J$ restricts to $\mathfrak{C}^\dagger_{\gamma+1}\mathcal{A}$. The limit steps are easy, and in the end we obtain ans induced $\mathfrak{C}$-continuous $\mathfrak{C}^\dagger J \colon \mathfrak{C}^\dagger\mathcal{A}_\alpha \hookrightarrow \mathfrak{C}^\dagger\mathcal{A}_\alpha$.

To conclude, consider now $\mathfrak{C}^\dagger_1\mathcal{A}_\alpha \simeq \alpha\text{-Acc}_\Psi(\mathcal{A}, \mathcal{V})^{op}$ which is easily seen to be closed in $\mathfrak{C}^\dagger\mathcal{A} \simeq \Psi\text{-Cont}(\mathcal{A}, \mathcal{V})^{op} \cap \mathcal{P}^\dagger\mathcal{A}$ under α-small colimits. Thus $\mathfrak{C}^\dagger_1\mathcal{A}_\alpha = \mathfrak{C}^\dagger\mathcal{A}_\alpha$ is α-cocomplete and $\mathfrak{C}^\dagger J$ is α-cocontinuous. $\Box$

Next we introduce the notion of $(\mathfrak{C}, \alpha)$-regular $\mathcal{V}$-functor:

Definition 6.2.4. Let $\mathcal{B}$ be a $\mathcal{V}$-category for which $\mathfrak{C}\mathcal{B}$ is α-complete; we say that a $\mathcal{V}$-functor $F \colon \mathfrak{C}\mathcal{B} \to \mathcal{V}$ is $(\mathfrak{C}, \alpha)$-*regular* if it is α-continuous and preserves colimits of diagrams in $\mathfrak{C}\mathcal{B}$ which are of type $\mathfrak{C}$. Denote by $\mathfrak{C}\text{-Reg}_\alpha(\mathfrak{C}\mathcal{B}, \mathcal{V})$ the full subcategory of $[\mathfrak{C}\mathcal{B}, \mathcal{V}]$ spanned by those functors.

In general $\mathfrak{C}\mathcal{B}$ is large even for a small $\mathcal{V}$-category $\mathcal{B}$, so the fact that $\mathfrak{C}\text{-Reg}_\alpha(\mathfrak{C}\mathcal{B}, \mathcal{V})$ is actually a $\mathcal{V}$-category is guaranteed by the following:

Proposition 6.2.5. *Let $\mathcal{B}$ be a small Cauchy complete $\mathcal{V}$-category for which $\mathfrak{C}\mathcal{B}$ is α-complete; then*

$$\mathfrak{C}\text{-Reg}_\alpha(\mathfrak{C}\mathcal{B}, \mathcal{V}) \simeq \alpha\text{-Flat}(\mathcal{B}, \mathcal{V})$$

is α-accessible with Ψ-limits, and the equivalence is induced by restricting along the inclusion. Moreover for any $(\mathfrak{C}, \alpha)$-regular $G \colon \mathfrak{C}\mathcal{B} \to \mathcal{V}$ and any diagram (M, H) of type $\mathfrak{C}$ landing in $\mathfrak{C}\mathcal{B}$, we have $G \circ H \in \mathfrak{C}_M$.

Proof. Let $J \colon \mathcal{B} \hookrightarrow \mathfrak{C}\mathcal{B}$ be the inclusion; the fact that $\mathfrak{C}\text{-Reg}_\alpha(\mathfrak{C}\mathcal{B}, \mathcal{V}) \simeq \alpha\text{-Flat}(\mathcal{B}, \mathcal{V})$ is routine: if $F \colon \mathcal{B} \to \mathcal{V}$ is α-flat then $\text{Lan}_J F$ is α-flat as well (Lemma 1.3.2) and hence α-continuous, moreover it preserves the colimits which in $\mathfrak{C}\mathcal{B}$ are computed pointwise; thus $\text{Lan}_J F$ is $(\mathfrak{C}, \alpha)$-regular. Conversely, any $(\mathfrak{C}, \alpha)$-regular $\mathcal{V}$-functor $G \colon \mathfrak{C}\mathcal{B} \to \mathcal{V}$ is the left Kan extension of its restriction to $\mathcal{B}$ (since it preserves colimits of type $\mathfrak{C}$ diagrams that generate $\mathfrak{C}\mathcal{B}$), and that is α-flat (again by Lemma 1.3.2). It follows that $\mathcal{A} := \mathfrak{C}\text{-Reg}_\alpha(\mathfrak{C}\mathcal{B}, \mathcal{V})$ is α-accessible with α-presentable objects $\mathcal{A}_\alpha \simeq \mathcal{B}^{op}$.

Before proving that $\mathcal{A}$ has Ψ-limits let us show that the last part of the statement holds. Let (M, H) be a diagram of type $\mathfrak{C}$ in $\mathfrak{C}\mathcal{B}$ and consider an α-presentable G in $\mathcal{A}$; then

$$G \cong \text{Lan}_J \mathcal{B}(B, -) \cong \mathfrak{C}\mathcal{B}(JB, -)$$

for some $B \in \mathcal{B}$, and $G \circ H \cong \mathfrak{C}\mathcal{B}(JB, H-)$ which is in $\mathfrak{C}_M$ by definition. In general, any $G \in \mathcal{A}$ is an α-flat colimit of elements of $\mathcal{A}_\alpha$; thus $G \circ H \in \mathfrak{C}_M$ since by condition (a) the full subcategory $\mathfrak{C}_M$ is closed under α-flat colimits in its presheaf category.

To conclude we need to show that $\mathcal{A}$ has Ψ-limits. Let $N \colon \mathcal{B} \to \mathcal{V}$ be in Ψ and $S \colon \mathcal{B} \to \mathcal{A}$ be a diagram in $\mathcal{A}$, and consider the limit $G := \{N, S\}$ in $[\mathfrak{C}\mathcal{B}, \mathcal{V}]$. Then G is still α-continuous, we need to show that it preserves colimits of type $\mathfrak{C}$. Let (M, H) be such a diagram; then

$$
\begin{aligned}
M * GH &\cong M\Box * \{N-, S(-) \circ H\Box\} \\
&\cong \{N-, M\Box * (S(-) \circ H\Box)\} \qquad (6.2)\\
&\cong \{N-, S(-)(M * H)\} \qquad (6.3)\\
&\cong G(M * H)
\end{aligned}
$$

where in (6.2) we can make the Ψ-limit and the colimit commute since, by the arguments above, $S(X) \circ H$ is an element of $\mathfrak{C}_M$ for any X in the domain of S. The isomorphism (6.3) holds since precomposition by H is cocontinuous. Thus $G \in \mathcal{A}$ which has therefore Ψ-limits. $\Box$

Now we are ready to define the 2-categories involved in the duality theorem.

Definition 6.2.6. Let α-$\mathbf{Acc}_\Psi$ be the 2-category of α-accessible $\mathcal{V}$-categories with Ψ-limits, Ψ-continuous $\mathcal{V}$-functors which preserves α-flat colimits, and $\mathcal{V}$-natural transformations.

Note that an α-flat colimit preserving $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ between α-accessible $\mathcal{V}$-categories is Ψ-continuous if and only if it has a $\mathfrak{C}$-left adjoint by Proposition 5.2.15.

On the other hand consider:

Definition 6.2.7. Let $\mathfrak{C}$-$\mathbf{Ex}_\alpha$ be the 2-category with objects pairs $(\mathcal{B}, \mathfrak{C}\mathcal{B})$ where $\mathcal{B}$ is small and Cauchy complete and $\mathfrak{C}\mathcal{B}$ is α-complete; morphisms are the α-continuous and $\mathfrak{C}$-cocontinuous $\mathcal{V}$-functors $\mathfrak{C}\mathcal{B} \to \mathfrak{C}\mathcal{B}'$, and 2-cells are $\mathcal{V}$-natural transformations between them.

Remark 6.2.8. Note that we could not have denoted the 2-category introduced above by $\mathfrak{C}$-$\mathbf{Reg}_\alpha$ since $(\mathfrak{C}, \alpha)$-regular $\mathcal{V}$-functors are not the same as α-continuous and $\mathfrak{C}$-cocontinuous $\mathcal{V}$-functors. The notation we use is inspired to that of Section 6.3.3 where exact categories play a central role.

Since $\mathfrak{C}\mathcal{B}$ is determined by $\mathcal{B}$, we will often just take $\mathcal{B}$ to represent an object of $\mathfrak{C}$-$\mathbf{Ex}_\alpha$. Note that in general $\mathcal{B}$ cannot be described starting from $\mathfrak{C}\mathcal{B}$ alone, so we cannot take just $\mathfrak{C}\mathcal{B}$ to represent an object of $\mathfrak{C}$-$\mathbf{Ex}_\alpha$.

Between these 2-categories we consider the 2-functor

$$\mathfrak{C}\text{-Reg}_\alpha(-, \mathcal{V})\colon \mathfrak{C}\text{-}\mathbf{Ex}_\alpha^{op} \longrightarrow \alpha\text{-}\mathbf{Acc}_\Psi$$

which sends the pair $(\mathcal{B}, \mathfrak{C}\mathcal{B})$ to $\mathfrak{C}\text{-Reg}_\alpha(\mathfrak{C}\mathcal{B}, \mathcal{V})$ and acts by precomposition on morphisms; this is well defined by Proposition 6.2.5.

In the other direction consider the 2-functor

$$((-)_\alpha^{op}, \alpha\text{-Acc}_\Psi(-, \mathcal{V}))\colon \alpha\text{-}\mathbf{Acc}_\Psi \longrightarrow \mathfrak{C}\text{-}\mathbf{Ex}_\alpha^{op}.$$

This sends an α-accessible $\mathcal{V}$-category $\mathcal{A}$ with Ψ-limits to the small Cauchy complete $\mathcal{V}$-category $\mathcal{A}_\alpha^{op}$ together with its completion

$$\mathfrak{C}(\mathcal{A}_\alpha^{op}) \simeq \alpha\text{-Acc}_\Psi(\mathcal{A}, \mathcal{V})$$

which is α-complete by Proposition 6.2.3. The action on morphisms is given by precomposition: for any $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{K}$ which is Ψ-continuous and α-flat-colimit preserving, since $\mathfrak{C}(\mathcal{A}_\alpha^{op}) \simeq \alpha\text{-Acc}_\Psi(\mathcal{A},\mathcal{V})$ (same for $\mathcal{K}$), precomposition with F induces a $\mathcal{V}$-functor

$$F_\alpha^*\colon \mathfrak{C}(\mathcal{K}_\alpha^{op}) \to \mathfrak{C}(\mathcal{A}_\alpha^{op}).$$

Then the 2-functor sends F to $F_\alpha^* \simeq \alpha\text{-Acc}_\Psi(F,\mathcal{V})$. This is α-continuous and $\mathfrak{C}$-cocontinuous since in the square below

$$\begin{tikzcd}
\mathfrak{C}(\mathcal{K}^{op}) \arrow[r, "F^*"] & \mathfrak{C}(\mathcal{A}^{op}) \\
\mathfrak{C}(\mathcal{K}_\alpha^{op}) \arrow[u, hook] \arrow[r, "F_\alpha^*"'] & \mathfrak{C}(\mathcal{A}_\alpha^{op}) \arrow[u, hook]
\end{tikzcd}$$

the two vertical legs are α-continuous and $\mathfrak{C}$-cocontinuous (by 6.2.3) and the top arrow is continuous (limits are computed pointwise and precomposition with F preserves them) and $\mathfrak{C}$-cocontinuous as well: for this it is again enough to prove that F^* sends colimit diagrams of type $\mathfrak{C}$ for $\mathcal{K}$ to ones of type $\mathfrak{C}$ for $\mathcal{A}$. Let (M,H) be of type $\mathfrak{C}$ in $\mathfrak{C}(\mathcal{K}^{op})$ then for each $A \in \mathcal{A}$ the composite

$$\text{ev}_A \circ F^*H = \text{ev}_{FA} \circ H$$

is in $\mathfrak{C}_M$ by assumption; thus (M, F^*H) is of type $\mathfrak{C}$. Therefore the 2-functor is well defined.

Remark 6.2.9. It follows from the considerations above that, given a Ψ-continuous and α-flat-colimit preserving $F\colon \mathcal{A} \to \mathcal{K}$, the opposite of the $\mathcal{V}$-functor $\alpha\text{-Acc}_\Psi(F,\mathcal{V})$ is, up to equivalence, the restriction of the virtual left adjoint $L\colon \mathcal{P}^\dagger\mathcal{K} \to \mathcal{P}^\dagger\mathcal{A}$ of F to the free completion under limits of type $\mathfrak{C}$.

Finally we can prove the following duality theorem which, as we will see in the next sections, captures the known dualities for locally presentable, multipresentable, polypresentable, and weakly locally presentable categories as instances of the same theory.

Theorem 6.2.10. *The 2-functors*

$$((-)_\alpha^{op}, \alpha\text{-Acc}_\Psi(-,\mathcal{V}))\colon \alpha\text{-}\mathbf{Acc}_\Psi \rightleftarrows \mathfrak{C}\text{-}\mathbf{Ex}_\alpha^{op} : \mathfrak{C}\text{-Reg}_\alpha(-,\mathcal{V})$$

form a biequivalence of 2-categories.

Proof. On one hand, given $(\mathcal{B}, \mathfrak{C}\mathcal{B}) \in \mathfrak{C}\text{-}\mathbf{Ex}_\alpha^{op}$ we obtain

$$\mathfrak{C}\text{-Reg}_\alpha(\mathfrak{C}\mathcal{B},\mathcal{V})_\alpha \simeq \alpha\text{-Flat}(\mathcal{B},\mathcal{V})_\alpha \simeq \mathcal{B}^{op}$$

by Proposition 6.2.5 and the Cauchy completeness of $\mathcal{B}$; moreover

$$\alpha\text{-Acc}_\Psi(\mathfrak{C}\text{-Reg}_\alpha(\mathfrak{C}\mathcal{B},\mathcal{V}),\mathcal{V}) \simeq \mathfrak{C}(\mathfrak{C}\text{-Reg}_\alpha(\mathfrak{C}\mathcal{B},\mathcal{V})_\alpha^{op}) \simeq \mathfrak{C}(\mathcal{B}^{op})$$

where the first equivalence holds by 6.1 and the last by the equivalence above. Conversely, given an α-accessible $\mathcal{V}$-category $\mathcal{A}$ with Ψ-limits, then

$$\mathcal{A} \simeq \alpha\text{-Flat}(\mathcal{A}_\alpha^{op},\mathcal{V}) \simeq \mathfrak{C}\text{-Reg}_\alpha(\mathfrak{C}(\mathcal{A}_\alpha^{op}),\mathcal{V}) \simeq \mathfrak{C}\text{-Reg}_\alpha(\alpha\text{-Acc}_\Psi(\mathcal{A},\mathcal{V}),\mathcal{V})$$

where the second equivalence holds by applying Proposition 6.2.5 to $\mathcal{B} = \mathcal{A}_\alpha^{op}$ (we can do that since $\mathfrak{C}(\mathcal{A}_\alpha^{op})$ is α-complete by the dual of Proposition 6.2.3), while the third holds by 6.1. The action on morphisms and 2-cells follows easily from this. $\square$

Thanks to this the objects of $\mathfrak{C}$-$\mathbf{Ex}_\alpha$ can be more easily described.

Corollary 6.2.11. *If $\mathcal{B}$ is in $\mathfrak{C}$-$\mathbf{Ex}_\alpha$ then $\mathfrak{C}\mathcal{B} = \mathfrak{C}_1\mathcal{B}$ is a one-step closure.*

Proof. Follows from the Theorem above and Proposition 6.2.3. □

Moreover, for $\mathcal{V}$-categories of the form $\mathfrak{C}\mathcal{B}$, having α-small limits of elements of $\mathcal{B}$ is enough to guarantee α-completeness:

Corollary 6.2.12. *The following are equivalent for any small and Cauchy complete $\mathcal{V}$-category $\mathcal{B}$:*

1. *$\mathfrak{C}\mathcal{B}$ is α-complete;*
2. *$\mathfrak{C}\mathcal{B}$ has α-small limits of objects in $\mathcal{B}$.*

Proof. (1) $\Rightarrow$ (2) is trivial. For the converse assume that $\mathcal{B}$ is small, Cauchy complete, and that $\mathfrak{C}\mathcal{B}$ has α-small limits of representables. Then the α-accessible $\mathcal{V}$-category $\mathcal{A} := \alpha\text{-Flat}(\mathcal{B},\mathcal{V})$ is equivalent to the full subcategory of $[\mathfrak{C}\mathcal{B},\mathcal{V}]$ spanned by those $\mathcal{V}$-functors which preserve colimits of type $\mathfrak{C}$ and α-small limits of diagrams in $\mathcal{B}$ (here we are using the fact that the class of α-small weights is sound, not just weakly sound, by Proposition 1.3.5). Thus $\mathcal{A}$ has Ψ-limits, since such $\mathcal{V}$-functors are stable under them. It follows from the duality theorem that $\mathfrak{C}\mathcal{B} \simeq \mathfrak{C}(\mathcal{A}_\alpha^{op})$ is α-complete. □

6.2.1 A nicer setting

Suppose that in addition each $\mathfrak{C}_M$ is accessible (and thus accessibly embedded in $[\mathcal{D},\mathcal{V}]$) as in Section 5.2.2. Then for any $M\colon \mathcal{D}^{op} \to \mathcal{V}$ we can fix a fully faithful $W_M\colon \mathcal{D} \hookrightarrow \mathcal{D}_M$ and a sketch $\mathcal{S}_M = (\mathcal{D}_M, \mathbb{L}_M, \mathbb{C}_M)$ on $\mathcal{D}_M$ together with an equivalence

$$- \circ W_M \colon \mathrm{Mod}(\mathcal{S}_M) \longrightarrow \mathfrak{C}_M.$$

What happens in the main examples treated below is that, whenever $\mathcal{B}$ lies in $\mathfrak{C}$-$\mathbf{Ex}_\alpha$ the $\mathcal{V}$-category $\mathfrak{C}\mathcal{B}$ is "rich enough" to ensure the existence of an extension $\hat{H}\colon \mathcal{D}_M \to \mathfrak{C}\mathcal{B}$ of $H\colon \mathcal{D} \to \mathfrak{C}\mathcal{B}$ for which a pair (M,H) in $\mathfrak{C}\mathcal{B}$ is of type $\mathfrak{C}$ if and only if $\hat{H}$ lies in $\mathrm{Mod}(\mathcal{S}_M,\mathfrak{C}\mathcal{B})$. This is now a condition on $\mathfrak{C}\mathcal{B}$, which is independent from $\mathcal{B}$.

We still do not know how to appropriately formalize the statement outlined above; in particular the properties identifying the $\mathcal{V}$-category $\mathfrak{C}\mathcal{B}$ as "rich enough" are not yet well determined. However, assuming that such property holds, we can define $\mathfrak{C}$-cocompleteness and $\mathfrak{C}$-cocontinuity in any $\mathcal{V}$-category $\mathcal{E}$ using the approach above: say that (M,H) is a diagram of type $\mathfrak{C}$ in $\mathcal{E}$ if H has an extension $\hat{H}$ that lies in $\mathrm{Mod}(\mathcal{S}_M,\mathcal{E})$, a $\mathfrak{C}$-cocontinuous functor is then one that preserves diagrams of type $\mathfrak{C}$ and their colimits.

Moreover, for any such $\mathcal{E}$ we can define the full subcategory $\mathcal{E}_\mathfrak{C}$ spanned by the $\mathfrak{C}$-presentable objects of $\mathcal{E}$: those $E \in \mathcal{E}$ for which $\mathcal{E}(E,-)$ preserves colimits of diagrams of type $\mathfrak{C}$. It is then easy to check that whenever $\mathcal{B}$ is in $\mathfrak{C}$-$\mathbf{Ex}_\alpha$ we have $(\mathfrak{C}\mathcal{B})_\mathfrak{C} \simeq \mathcal{B}$.

Following the notation of [50], we call $\mathfrak{C}$-*based* any $\mathfrak{C}$-cocomplete $\mathcal{V}$-category $\mathcal{E}$ with a small $\mathcal{E}_\mathfrak{C}$ and such that $\mathcal{E} \simeq \mathfrak{C}(\mathcal{E}_\mathfrak{C})$.

Under this setting $\mathfrak{C}$-$\mathbf{Ex}_\alpha$ can be described as the 2-category of $\mathfrak{C}$-based and α-complete $\mathcal{V}$-categories, $\mathfrak{C}$-cocontinuous and α-continuous $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. The duality theorem then becomes simpler to state: it says that homming into $\mathcal{V}$ on both sides

$$\alpha\text{-Acc}_\Psi(-,\mathcal{V})\colon \alpha\text{-}\mathbf{Acc}_\Psi \underset{\longleftarrow}{\overset{\longrightarrow}{}} \mathfrak{C}\text{-}\mathbf{Ex}_\alpha^{op} : \mathfrak{C}\text{-Reg}_\alpha(-,\mathcal{V}).$$

induces a biequivalence of 2-categories.

6.3 Examples

6.3.1 The weakly sound case

We now restrict to the setting of Section 5.3.1 by considering a weakly sound class of weights Ψ. Then by Proposition 5.3.4, the class Ψ^+ of the Ψ-flat weights is an accessible companion for Ψ and $\Psi\text{-Flat}(\mathcal{C},\mathcal{V}) = \Psi^+\mathcal{C}$ for any $\mathcal{C}$.

It follows that Ψ^+ satisfies the hypotheses (a) and (b) of Section 6.2 for any α. Indeed, condition (a) is trivial since Ψ^+ is an actual class of weights. For condition (b), let $\mathcal{A}$ be α-accessible with Ψ-limits and $F\colon \mathcal{A} \to \mathcal{V}$ be Ψ-continuous and α-flat colimit preserving. Let $H\colon \mathcal{A}_\alpha \hookrightarrow \mathcal{A}$ be the inclusion; then $F \cong \mathrm{Lan}_H FH$ and $FH\colon \mathcal{A}_\alpha \to \mathcal{V}$ is Ψ-flat by Proposition 1.3.2 (since F is). By our assumption then FH is a Ψ^+-colimit of representables; therefore taking its left Kan extension along H we can write F as a Ψ^+-colimit of representables from $\mathcal{A}_\alpha$, as desired.

In this case we can directly apply the results of Section 6.2.1. The notion of Ψ^+-cocompleteness and Ψ^+-cocontinuity are the usual ones, so that for a Ψ^+-cocomplete $\mathcal{E}$ the category $\mathcal{E}_{\Psi^+}$ is just the full subcategory of the Ψ^+-presentable objects. A Ψ^+-based $\mathcal{V}$-category $\mathcal{E}$ is then one which is the free cocompletion of a small $\mathcal{V}$-category under Ψ^+-colimits; assuming Cauchy completeness, that small $\mathcal{V}$-category coincides up to equivalence with $\mathcal{E}_{\Psi^+}$ by [59, Proposition 7.5].

It follows that $\Psi^+\text{-}\mathbf{Ex}_\alpha$ can be described as the 2-category of Ψ^+-based and α-complete $\mathcal{V}$-categories, Ψ^+-cocontinuous and α-continuous $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. Thus the duality can be rewritten as:

Theorem 6.3.1. *The 2-functors*

$$\alpha\text{-Acc}_\Psi(-,\mathcal{V})\colon \alpha\text{-}\mathbf{Acc}_\Psi \rightleftarrows \Psi^+\text{-}\mathbf{Ex}_\alpha^{op} : \Psi^+\text{-Reg}_\alpha(-,\mathcal{V})$$

form a biequivalence of 2-categories.

Each of the weakly sound classes from Example 5.3.6 then provides a duality theorem. Let us see some in particular.

When $\Psi = \mathcal{P}$ and $\Psi^+ = \mathcal{Q}$ we recover the Gabriel-Ulmer duality for locally α-presentable categories which in the enriched context was first proved by Kelly:

Theorem 6.3.2 ([57])**.** *The 2-functors*

$$\mathrm{L}\alpha\mathrm{p}(-,\mathcal{V})\colon \mathbf{L}\alpha\mathbf{p} \rightleftarrows \mathbf{Lex}_\alpha^{op} : \mathrm{Lex}_\alpha(-,\mathcal{V}).$$

form a biequivalence of 2-categories.

When $\mathcal{V} = \mathbf{Set}$, Ψ is the class for connected limits, and $\Psi^+ = \mathbf{Fam}$ is the class generated by coproducts, we recover the Diers duality for locally α-multipresentable categories. Below the 2-category $\mathbf{Fam\text{-}Lex}_\alpha$ has α-complete coproduct cocompletions of small categories as objects, α-continuous and coproduct-preserving functors as morphisms, and natural transformations as 2-cells.

Theorem 6.3.3 ([35])**.** *The 2-functors*

$$\alpha\text{-Lmp}(-,\mathbf{Set})\colon \alpha\text{-}\mathbf{Lmp} \rightleftarrows \mathbf{Fam\text{-}Lex}_\alpha^{op} : \mathrm{Fam\text{-}Lex}_\alpha(-,\mathbf{Set}).$$

form a biequivalence of 2-categories.

A 2-categorical version of this duality can be obtained by taking $\mathcal{V} = \mathbf{Cat}$ and Ψ to be the weakly sound class consisting of the connected 2-limits described in Example 5.3.6(12).

When $\mathcal{V}$ is general, $\Psi = \emptyset$, and $\Psi^+ = \mathcal{P}$ we obtain a duality for α-accessible categories. Below the 2-category $\mathcal{P}\text{-}\mathbf{Lex}_\alpha$ has presheaf $\mathcal{V}$-categories as objects, α-continuous and cocontinuous $\mathcal{V}$-functors as morphisms, and $\mathcal{V}$-natural transformations as 2-cells.

Theorem 6.3.4. *The 2-functors*

$$\alpha\text{-Acc}(-,\mathcal{V})\colon \alpha\text{-}\mathbf{Acc} \rightleftarrows \mathcal{P}\text{-}\mathbf{Lex}_\alpha^{op} : \mathcal{P}\text{-Lex}_\alpha(-,\mathcal{V}).$$

form a biequivalence of 2-categories.

For $\alpha = \aleph_0$ and $\mathcal{V} = \mathbf{Set}$ the duality first appeared as [77, Proposition 4.2.1], moreover this is part of the Scott adjunction between accessible categories with filtered colimits and Grothendieck topoi [34]; in Section A.4 we construct an enriched version of it.

When Ψ is a locally small class of weights, then Ψ^+-cocompletions of small $\mathcal{V}$-categories are just free cocompletions of small $\mathcal{V}$-categories under Ψ-flat colimits, and these (by Proposition 2.1.5) coincide with what we have called Ψ-accessible $\mathcal{V}$-categories in Section 2.1.1. It follows that $\Psi^+\text{-}\mathbf{Ex}_\alpha = \Psi\text{-}\mathbf{Acc}_\alpha$ is the same as the 2-category of Ψ-accessible $\mathcal{V}$-categories with α-small limits, α-continuous and Ψ^+-cocontinuous $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. Thus the duality becomes:

Theorem 6.3.5. *Let Ψ be a small and weakly sound class of weights; then the 2-functors*

$$\alpha\text{-Acc}_\Psi(-,\mathcal{V})\colon \alpha\text{-}\mathbf{Acc}_\Psi \rightleftarrows \Psi\text{-}\mathbf{Acc}_\alpha^{op} : \Psi\text{-Acc}_\alpha(-,\mathcal{V}).$$

form a biequivalence of 2-categories.

In particular, when Ψ is the class of α-small weights, a Ψ-accessible $\mathcal{V}$-category is just an α-accessible $\mathcal{V}$-category. Thus $\Psi\text{-}\mathbf{Acc}_\alpha = \alpha\text{-}\mathbf{Acc}_\alpha$ is the 2-category of α-complete and α-accessible $\mathcal{V}$-categories, α-continuous and α-flat-colimit preserving $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. Therefore we obtain the following:

Theorem 6.3.6. *There is a biequivalence of 2-categories*

$$\alpha\text{-}\mathbf{Acc}_\alpha \simeq (\alpha\text{-}\mathbf{Acc}_\alpha)^{op}$$

induced by the 2-functor $\alpha\text{-Acc}_\alpha(-,\mathcal{V})$.

Let $\Sigma = \alpha\text{-Acc}_\alpha(-,\mathcal{V})$ be the 2-functor involved in the duality; then in particular Σ is a bi-involution: $\Sigma^2 \simeq 1$. In addition, we can give a more direct way to describe the action of Σ on objects and morphisms as follows.

Given an α-accessible $\mathcal{V}$-category $\mathcal{A}$, left Kan extending along the inclusion induces an equivalence $[\mathcal{A}_\alpha,\mathcal{V}] \simeq \alpha\text{-Acc}(\mathcal{A},\mathcal{V})$; if $\mathcal{A}$ is moreover α-complete then, by Lemma 1.3.2, the equivalence restrict to $\alpha\text{-Flat}(\mathcal{A}_\alpha,\mathcal{V}) \simeq \alpha\text{-Acc}_\alpha(\mathcal{A},\mathcal{V})$. Thus

$$\Sigma\mathcal{A} \simeq \alpha\text{-Flat}(\mathcal{A}_\alpha,\mathcal{V}),$$

or equivalently: $\Sigma(\alpha\text{-Flat}(\mathcal{C}^{op},\mathcal{V})) \simeq \alpha\text{-Flat}(\mathcal{C},\mathcal{V})$. Similarly, given a morphism $F\colon \mathcal{A} \to \mathcal{B}$ in $\alpha\text{-}\mathbf{Acc}_\alpha$ with inclusions $J\colon \mathcal{A}_\alpha \to \mathcal{A}$ and $H\colon \mathcal{B}_\alpha \to \mathcal{B}$, a few calculations show that the 2-functor Σ acts as follows:

$$\begin{aligned} \alpha\text{-Flat}(\mathcal{B}_\alpha,\mathcal{V}) &\xrightarrow{\Sigma F} \alpha\text{-Flat}(\mathcal{A}_\alpha,\mathcal{V}) \\ X &\longmapsto (\mathrm{Lan}_H X)FJ. \end{aligned}$$

The resulting $\mathcal{V}$-functor $(\mathrm{Lan}_H X)FJ$ is still α-flat thanks to Lemma 1.3.2, since X is α-flat and F is α-continuous and α-flat-colimit preserving.

Remark 6.3.7. The duality can be further generalized to the setting of a locally small weakly sound class Ψ in place of the class of α-small weights; as it usually happens in these cases the proofs given above generalize simply by replacing α-small limits with Ψ-limits. The theorem then will say that the 2-category Ψ-$\mathbf{Acc}_\Psi$ of Ψ-accessible categories with Ψ-limits, Ψ-continuous and Ψ-flat-colimit preserving $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations, is dual to itself.

6.3.2 Wide pullbacks

Here we consider the case when Ψ consists of the conical weights for wide pullbacks and $\mathfrak{C} = \mathfrak{F}$ consists of the free groupoid actions in **Set** as in Section 5.3.2. For simplicity let us restrict to $\alpha = \aleph_0$.

We now wish to describe the 2-category $\mathfrak{F}$-**Ex**. An object there is a Cauchy complete category $\mathcal{B}$ with a lex (finitely complete) $\mathfrak{F}\mathcal{B}$. By 6.2.12 and 5.3.19 we know that this is equivalent to $\mathcal{B}$ being Cauchy complete and finitely polycomplete. Following the ideas of Section 6.2.1, we define:

Definition 6.3.8. Let $\mathcal{E}$ be a category with an initial object. A diagram $H\colon \mathcal{G} \to \mathcal{E}$, indexed on a groupoid $\mathcal{G}$, is called a *free action* if for each $g \neq h$ in $\mathcal{G}$ the equalizer of (Hg, Hh) is the initial object of $\mathcal{E}$.

It is easy to see that a diagram in $\mathfrak{F}\mathcal{B}$ is of type $\mathfrak{F}$ if and only if it is a free action in $\mathfrak{F}\mathcal{B}$ (this is a consequence of the fact that each diagram of type $\mathfrak{F}$ is pointwise in $\mathfrak{F}$, and that equalizers and the terminal object are computed pointwise in $\mathfrak{F}\mathcal{C}$).

Let $\mathcal{E}$ be a category with colimits of free actions; we denote by $\mathcal{E}_\mathfrak{F}$ the full subcategory of $\mathcal{E}$ spanned by those objects E for which $\mathcal{E}(E,-)$ preserves colimits of free actions (just the colimits not the diagrams).

Definition 6.3.9. Let $\mathcal{E}$ be a category with colimits of free actions; we say that $\mathcal{E}$ is $\mathfrak{F}$-*based* if the category $\mathcal{E}_\mathfrak{F}$ is small and every object of $\mathcal{E}$ is the colimit of a free action of objects from $\mathcal{E}_\mathfrak{F}$.

In other words, a category $\mathcal{E}$ with colimits of free actions is $\mathfrak{F}$-based if and only if $\mathcal{E}_\mathfrak{F}$ is small and $\mathcal{E} \simeq \mathfrak{F}_1(\mathcal{E}_\mathfrak{F})$. When $\mathcal{E}$ has finite limits, the last requirement is equivalent to $\mathcal{E} \simeq \mathfrak{F}(\mathcal{E}_\mathfrak{F})$ by Corollary 6.2.11.

Remark 6.3.10. By the results of Section 5.3.2 comparing our notions with those of [50], an $\mathfrak{F}$-based category is just a quasi-based category in the sense of Hu and Tholen.

Thus, if $\mathcal{B}$ is an object of $\mathfrak{F}$-**Ex**, then $\mathfrak{F}\mathcal{B}$ is $\mathfrak{F}$-based and moreover $(\mathfrak{F}\mathcal{B})_\mathfrak{F} \simeq \mathcal{B}$; indeed for every $B \in \mathcal{B}$ the hom functor $\mathfrak{F}\mathcal{B}(B,-)$ preserves colimits of free actions by construction, and if $B \in (\mathfrak{F}\mathcal{B})_\mathfrak{F}$ then, being also a colimit of a free action of representables, it is a split subobject of representables. Hence $B \in \mathcal{B}$ because $\mathcal{B}$ is Cauchy complete.

Conversely, if $\mathcal{E}$ is lex and $\mathfrak{F}$-based then $(\mathcal{E}_\mathfrak{F}, \mathcal{E})$ is an object of $\mathfrak{F}$-**Ex**. Moreover any lex functor $F\colon \mathcal{E} \to \mathcal{E}'$, between $\mathfrak{F}$-based lex categories, is $\mathfrak{F}$-cocontinuous (when seen as a functor $\mathfrak{F}(\mathcal{E}_\mathfrak{F}) \to \mathfrak{F}(\mathcal{E}'_\mathfrak{F})$) if and only if it preserves colimits of free actions. Indeed, if F is

$\mathfrak{F}$-cocontinuous then it preserves colimits of free actions by definition (since a diagram in $\mathfrak{F}(\mathcal{E}_{\mathfrak{F}})$ is of type $\mathfrak{F}$ if and only if it is a free action). Conversely, if F preserves colimits of free actions, then in particular it preserves the initial object of $\mathcal{E}$, and thus, since it it is lex, it also preserves diagrams of type $\mathfrak{F}$.

The same argument shows that a functor $F\colon \mathcal{E} \simeq \mathfrak{F}(\mathcal{E}_{\mathfrak{F}}) \to \mathcal{V}$ is $\mathfrak{F}$-regular if and only if it is lex and preserves colimits of free actions. In particular the notions of lex $\mathfrak{F}$-cocontinuous and $\mathfrak{F}$-regular functors coincide.

It follows that $\mathfrak{F}$-**Ex** can be described as the 2-category of $\mathfrak{F}$-based lex categories, free-action-colimit preserving lex functors, and natural transformations. We denote by **Lfpp** the 2-category of locally finitely polypresentable categories, wide-pullback-preserving and finitary functors, and natural transformations. Then the duality theorem becomes:

Theorem 6.3.11 ([50])**.** *The 2-functors*

$$\mathrm{Lfpp}(-,\mathbf{Set})\colon \mathbf{Lfpp} \rightleftarrows \mathfrak{F}\text{-}\mathbf{Ex}^{op} : \mathfrak{F}\text{-Reg}(-,\mathbf{Set})$$

form a biequivalence of 2-categories.

Let us now give an equivalent description of colimits of free actions, that will allow us to recognize their existence more easily.

Definition 6.3.12. Let $\mathcal{E}$ be a category with an initial object. A groupoid diagram $H\colon \mathcal{G} \to \mathcal{E}$ in $\mathcal{E}$ is called a *weakly-free action* if for each $g, h \in \mathcal{G}$ for which $Hg \neq Hh$ the equalizer of (Hg, Hh) is the initial object of $\mathcal{E}$.

An easy consequence of the definition is then:

Proposition 6.3.13. *Let $H\colon \mathcal{G} \to \mathcal{E}$ be a groupoid indexed diagram in $\mathcal{E}$, and let $H = H' \circ F$ be its (b.o and full, faithful)-factorisation, with $H'\colon \mathcal{G}' \to \mathcal{E}$ ($\mathcal{G}'$ is still a groupoid). Then H is a weakly-free action if and only if H' is a free action. Moreover* $\mathrm{colim}H = \mathrm{colim}H'$ *whenever one of them exists.*

Now, since every groupoid is equivalent to the sum of some groups, for a category $\mathcal{E}$ to have colimits of free groupoid actions is the same as having small coproducts and colimits of free *group* actions. Since every group $\mathcal{G}$ is covered by an opportune free group $*_{i\in I}\mathbb{Z}$, it follows by the previous proposition that $\mathcal{E}$ has colimits of free group actions if and only if it has colimits of weakly-free actions by free groups $*_{i\in I}\mathbb{Z}$. Finally, note that the colimit of a weakly free action $*_{i\in I}\mathbb{Z} \to \mathcal{E}$ can also be seen as the cointersection (wide pushout) of the colimits of each i-component separately, which are weakly-free actions by $\mathbb{Z}$. In conclusion:

Proposition 6.3.14. *A category $\mathcal{E}$ has colimits of free actions if and only if it has:*

1. *coproducts;*
2. *colimits of weakly-free actions by $\mathbb{Z}$;*
3. *small co-intersections of quotients as in (2).*

A functor F preserves colimits of free actions if and only if it preserves the colimits above.

As a final remark note that a weakly-free action by $\mathbb{Z}$ in $\mathcal{E}$ is the data of an automorphism $f\colon E \to E$ in $\mathcal{E}$ for which every non identity f^n is fixed-point-free, meaning that the equalizer of f^n and id_E is the initial object of $\mathcal{E}$.

6.3.3 Products and projective powers

Let $\mathcal{V}$ be a symmetric monoidal finitary variety as in [64] for which every finitely accessible category is conically finitely accessible. For simplicity we restrict ourselves to the case $\alpha = \aleph_0$.

Example 6.3.15. All the symmetric monoidal finitary varieties of Example 3.1.3, as well as any locally dualizable base of Section 3.2, satisfy the conditions above.

We now consider the case when Ψ is the class of weights for products and projective powers as in Example 5.3.35. The colimit type $\mathfrak{R}$ consists of the pseudo-equivalence relations in $\mathcal{V}$; this is an accessible companion for Ψ by Proposition 5.3.31. Moreover $\mathfrak{R}$ satisfies the conditions (a) and (b) of Section 6.2 for the same reasons explained in Example 6.2.1(3), since by our assumptions on $\mathcal{V}$ we can use filtered colimits rather than flat colimits.

Remark 6.3.16. We deal with the more general case of products and powers by a dense generator in the section below; this is because the already existing notions of regularity and exactness of [64] will play an important role in this case.

Recall that a $\mathcal{V}$-category $\mathcal{B}$ is called regular if it has all finite weighted limits, coequalizers of kernel pairs, and regular epimorphisms are stable under pullbacks and powers by finite projective objects [64, Definition 5.1]. In addition, $\mathcal{B}$ is called exact if it is regular and its underlying ordinary category is exact (that is, all equivalence relations are kernel pairs).

Remark 6.3.17. Note that, when $\mathcal{V} = \mathbf{Set}$, given a small Cauchy complete category $\mathcal{B}$, by Corollary 6.2.12 and 5.4.12, it follows that $\mathfrak{R}\mathcal{B}$ is lex if and only if $\mathcal{B}$ has weak finite limits.

It is easy to see, for $\mathcal{B} \in \mathfrak{R}$-$\mathbf{Ex}$, that every kernel pair in $\mathfrak{R}\mathcal{B}$ is a diagram of type $\mathfrak{R}$; therefore $\mathfrak{R}\mathcal{B}$ is closed in $[\mathcal{B}^{op}, \mathcal{V}]$ under finite limits and coequalizers of kernel pairs. Since $[\mathcal{B}^{op}, \mathcal{V}]$ is regular then $\mathfrak{R}\mathcal{B}$ is a regular $\mathcal{V}$-category as well. Moreover, every equivalence relation (h, k) in $\mathfrak{R}\mathcal{B}$ is a kernel pair in $[\mathcal{B}^{op}, \mathcal{V}]$ (since $\mathcal{V}$ is a finitary variety, the presheaf $\mathcal{V}$-category is exact), and hence a diagram of type $\mathfrak{R}$ in $\mathfrak{R}\mathcal{B}$; thus the coequalizer of (h, k) exists in $\mathfrak{R}\mathcal{B}$ and the pair (h, k) is its kernel pair. This shows that every equivalence relation in $\mathcal{B}$ is effective; therefore $\mathfrak{R}\mathcal{B}$ is also an exact $\mathcal{V}$-category.

Remark 6.3.18. When $\mathcal{V} = \mathbf{Set}$ and $\mathcal{B}$ is a category with weak finite limits, then $\mathfrak{R}\mathcal{B}$ coincides with the free exact completion of $\mathcal{B}$ as a weakly lex category described in [29]. Indeed, such an exact completion $\mathcal{E}$ is described as the full subcategory of $[\mathcal{B}^{op}, \mathbf{Set}]$ spanned by the coequalizers of pseudo-equivalence relations from $\mathcal{B}$ in the sense of [29, Definition 6], and these are the same as pairs of type $\mathfrak{R}$ (see before Definition 6 and Theorem 26 of [29]). The universal property of this free completion says that, for each exact category $\mathcal{D}$, precomposition with the inclusion induces an equivalence

$$\mathrm{Reg}(\mathcal{E}, \mathcal{D}) \simeq \mathrm{Lco}(\mathcal{B}, \mathcal{D}),$$

where those on the right are the left-covering functors as in [29].

Consider now a pair of arrows (f, g) in $\mathfrak{R}\mathcal{B}$ and the induced regular factorization given by a regular epimorphism e followed by a monic pair (h, k). It is easy to see that (f, g) is of type $\mathfrak{R}$ if and only if (h, k) is a kernel pair. It follows at once that a $\mathcal{V}$-functor $F \colon \mathfrak{R}\mathcal{B} \to \mathcal{V}$ is $\mathfrak{R}$-regular if and only if it is regular in the usual sense, and that a lex functor $G \colon \mathfrak{R}\mathcal{B} \to \mathfrak{R}\mathcal{B}'$ is $\mathfrak{R}$-cocontinuous if and only if it preserves coequalizers of kernel pairs, if and only if it is a regular $\mathcal{V}$-functor.

Now, given an exact category $\mathcal{E}$ denote by $\mathcal{E}_P$ the full subcategory spanned by its (regular) projective objects: those $E \in \mathcal{E}$ for which $\mathcal{E}(E,-)$ preserves regular epimorphisms.

Then for any $\mathcal{B}$ in $\mathfrak{R}$-**Ex**, we obtain $(\mathfrak{R}\mathcal{B})_P \simeq \mathcal{B}$. Indeed if $B \in \mathcal{B}$ then $\mathfrak{R}\mathcal{B}(B,-)$ preserves regular epimorphisms since they are computed pointwise in $\mathfrak{R}\mathcal{B} \subseteq [\mathcal{B}^{op}, \mathcal{V}]$ and B is a representable object of $\mathfrak{R}\mathcal{B}$. Conversely, given any projective $B \in \mathfrak{R}\mathcal{B}$, since B is also a regular quotient of some $B' \in \mathcal{B}$, it is in particular split subobject of B'. But $\mathcal{B}$ is Cauchy complete by hypothesis, thus $B \in \mathcal{B}$ as desired.

We can therefore consider the following definition. For $\mathcal{V} = \mathbf{Set}$ this has been studied for instance in [29].

Definition 6.3.19. We say that a regular $\mathcal{V}$-category $\mathcal{E}$ has enough projectives if every object of $\mathcal{E}$ is a regular quotient of an object from $\mathcal{E}_P$.

Proposition 6.3.20. *A lex $\mathcal{V}$-category $\mathcal{E}$ is exact with enough projectives if and only if $\mathcal{E} \simeq \mathfrak{R}(\mathcal{E}_P)$.*

Proof. If $\mathcal{E} \simeq \mathfrak{R}(\mathcal{E}_P)$, it follows by the arguments above that $\mathcal{E}$ has enough projectives and is an exact $\mathcal{V}$-category.

Conversely, if $\mathcal{E}$ is exact and has enough projectives then every object of $\mathcal{E}$ is the coequalizer of a pseudo equivalence relation in $\mathcal{E}_P$ (arguing as usual). This shows that $\mathcal{E}_P$ is dense in $\mathcal{E}$, so that there is a regular embedding $J\colon \mathcal{E} \hookrightarrow [\mathcal{E}_P^{op}, \mathcal{V}]$, and that such inclusion J factors through $\mathfrak{R}_1(\mathcal{E}_P) \subseteq [\mathcal{E}_P^{op}, \mathcal{V}]$. Therefore $\mathcal{E} \subseteq \mathfrak{R}_1(\mathcal{E}_P)$. Consider now $X \in \mathfrak{R}_1(\mathcal{E}_P)$, then we can find a pair (f,g) in $\mathcal{E}_P$ whose image factorization in $[\mathcal{E}_P^{op}, \mathcal{V}]$ is a regular epimorphism e followed by a kernel pair (h,k). Since $\mathcal{E}$ is regularly embedded in $[\mathcal{E}_P^{op}, \mathcal{V}]$ the maps e, h, and k lie in $\mathcal{E}$ and e is still a regular epimorphism. Moreover, by exactness of $\mathcal{E}$ the pair (h,k) is still a kernel pair in $\mathcal{E}$. It follows that the coequalizer of (f,g) exists in $\mathcal{E}$ and coincides with X. This show $\mathcal{E} \simeq \mathfrak{R}_1(\mathcal{E}_P)$; since $\mathcal{E}$ is closed in $[\mathcal{E}_P^{op}, \mathcal{V}]$ under coequalizers of pseudo equivalence relations it follows that actually $\mathcal{E} \simeq \mathfrak{R}(\mathcal{E}_P)$. □

Remark 6.3.21. Note that a regular $\mathcal{V}$-category $\mathcal{E}$ with enough projectives is not in general equivalent to $\mathfrak{R}(\mathcal{E}_P)$, but we only have the inclusion $\mathcal{E} \subseteq \mathfrak{R}(\mathcal{E}_P)$.

If $\mathcal{E}$ is exact with enough projectives then $\mathcal{E}_P$ is Cauchy complete, and thus an object of $\mathfrak{R}$-**Ex**. It follows that the 2-category $\mathfrak{R}$-**Ex** can be described as the 2-category **pEx** of small exact $\mathcal{V}$-categories with enough projectives, regular $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations.

Definition 6.3.22. We say that a $\mathcal{V}$-category $\mathcal{A}$ is *weakly locally finitely presentable* if it is finitely accessible with products and projective powers.

Remark 6.3.23. By Theorem 5.2.17, a $\mathcal{V}$-category $\mathcal{A}$ is weakly locally finitely presentable if and only if it is finitely accessible and $\mathfrak{R}_1^\dagger\mathcal{A}$ has colimits of representables. Now, by Corollary 5.4.11, we know that $\mathfrak{R}_1^\dagger\mathcal{A}$ has colimits of representables if and only if $W_{\mathscr{E}}^\dagger\mathcal{A}$ has them; which is as saying that $\mathcal{A}$ is weakly cocomplete in the enriched sense. Thus, in the ordinary case, we recover the standard characterization of [1].

We denote by **wLfp** the 2-category of weakly locally finitely presentable $\mathcal{V}$-categories, finitary $\mathcal{V}$-functors that preserve products and projective powers (these are called definable in [64]), and $\mathcal{V}$-natural transformations. In conclusion the duality of Theorem 6.2.10 can be expressed as follows:

Theorem 6.3.24. *The 2-functors*

$$\mathrm{wLfp}(-,\mathcal{V})\colon \mathbf{wLfp} \rightleftarrows \mathbf{pEx}^{op} : \mathrm{Reg}(-,\mathcal{V})$$

form a biequivalence of 2-categories.

When $\mathcal{V} = \mathbf{Set}$ this appeared first as [49, Theorem 5.11]. Moreover, the duality above is a restriction of that between definable and exact $\mathcal{V}$-categories which was considered in [88] for $\mathcal{V} = \mathbf{Ab}$, in [60] for $\mathcal{V} = \mathbf{Set}$, and in [64] for a general $\mathcal{V}$ as in this section. In particular this says that a definable category $\mathcal{D}$ is finitely accessible if and only if $\mathrm{Def}(\mathcal{D},\mathcal{V})$ has enough projectives. See also Section A.3 for more about definable categories and their relationship with the colimit type $\mathfrak{R}$.

6.3.4 Products and powers by a dense generator

Here we work in the context of Section 5.3.3 and generalize the results of the previous section to that more general context.

Let us fix a dense generator $\mathcal{G} \subseteq (\mathcal{V}_0)_f$ which contains the unit and is closed under tensor products. Assume moreover that $\mathcal{V}$ satisfies at least one of conditions (I) and (II) of Section 5.3.3 and that every finitely accessible $\mathcal{V}$-category is conically finitely accessible.

Example 6.3.25. Any base of enrichment listed in Example 3.1.3 satisfies (I) and the condition on accessibility. Similarly, any locally dualizable base of Section 3.2 satisfies (II) and the accessibility condition.

Then we can consider the collection $\mathscr{E}$ of all the regular epimorphisms in $\mathcal{V}$ that are stable under $\mathcal{G}$-powers, and the corresponding colimit type $\mathfrak{C}$ (Definition 5.3.30). It follows that $\mathfrak{C}$ is an accessible companion for the class Ψ of products and $\mathcal{G}$-powers by Proposition 5.3.31; moreover it satisfies the conditions (a) and (b) of Section 6.2 by the same arguments given in Example 6.2.1.

Arguing as in the previous sections we can consider the following definition:

Definition 6.3.26. Let $\mathcal{E}$ be a $\mathcal{V}$-category with $\mathcal{G}$-powers; we say that a map $e\colon X \to Y$ in $\mathcal{E}$ is a *$\mathcal{G}$-epimorphism* if $G \pitchfork e$ is a regular epimorphism for each $G \in \mathcal{G}$. Denote by $\mathcal{E}_{\mathcal{G}}$ the full subcategory of $\mathcal{E}$ spanned by the objects $E \in \mathcal{E}$ for which $\mathcal{E}(E,-)$ preserves $\mathcal{G}$-epimorphism.

The $\mathcal{G}$-epimorphisms of $\mathcal{V}$ are simply the elements of $\mathscr{E}$.

Definition 6.3.27. We say that a $\mathcal{V}$-category $\mathcal{E}$ is *$\mathcal{G}$-regular* if there exists a fully faithful $J\colon \mathcal{E} \to [\mathcal{B},\mathcal{V}]$ closed under limits and coequalizers of kernel pairs which are $\mathcal{G}$-epimorphisms. A $\mathcal{V}$-functor between $\mathcal{G}$-regular $\mathcal{V}$-categories is *$\mathcal{G}$-regular* if it preserves finite limits and $\mathcal{G}$-epimorphisms.

Remark 6.3.28. This corresponds to a notion of regularity in the sense of [45] with respect to a suitable class of lex weights.

Definition 6.3.29. A $\mathcal{G}$-regular category $\mathcal{E}$ is called *$\mathcal{G}$-exact with enough projectives* if for any $X \in \mathcal{E}$ there exist a $\mathcal{G}$-epimorphism $e\colon A \to X$ with $A \in \mathcal{E}_P$, and the induced $\mathcal{V}$-functor $\mathcal{E} \to [\mathcal{E}_P^{op},\mathcal{V}]$ reflects kernel pairs of $\mathcal{G}$-epimorphisms.

Arguing as in the previous section, it is easy to see that a lex $\mathcal{V}$-category $\mathcal{E}$ is $\mathcal{G}$-exact with enough $\mathcal{G}$-projectives if and only if $\mathcal{E} \simeq \mathfrak{C}(\mathcal{E}_{\mathcal{G}})$. Moreover, if $\mathcal{B}$ is Cauchy complete and $\mathfrak{C}\mathcal{B}$ is lex, then $\mathfrak{C}\mathcal{B}$ is $\mathcal{G}$-exact with enough $\mathcal{G}$-projectives and $(\mathfrak{C}\mathcal{B})_{\mathcal{G}} \simeq \mathcal{B}$. Similarly, in the presence of finite limits, a $\mathcal{V}$-functor is lex $\mathfrak{C}$-cocontinuous (Definition 6.2.2) if and only if it is $\mathfrak{C}$-regular (Definition 6.2.4) if and only if it is $\mathcal{G}$-regular in the sense defined above.

It follows that $\mathfrak{C}$-**Ex** can be described as the 2-category $\mathcal{G}$-**pEx** of small $\mathcal{G}$-exact $\mathcal{V}$-categories with enough $\mathcal{G}$-projectives, $\mathcal{G}$-regular $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations.

Finally, say that a $\mathcal{V}$-category $\mathcal{A}$ is *$\mathcal{G}$-weakly locally finitely presentable* if it is finitely accessible with products and $\mathcal{G}$-powers. By Theorem 5.3.33 and Corollary 5.4.11 this is the same as being finitely accessible and $\mathscr{E}$-weakly cocomplete. Denote by $\mathcal{G}$-**wLfp** the corresponding 2-category of $\mathcal{G}$-weakly locally finitely presentable categories, finitary $\mathcal{V}$-functors which preserve products and $\mathcal{G}$-powers, and $\mathcal{V}$-natural transformations. Then the duality of Theorem 6.2.10 becomes:

Theorem 6.3.30. *The 2-functors*

$$\mathcal{G}\text{-wLfp}(-,\mathcal{V})\colon \mathcal{G}\text{-}\mathbf{wLfp} \rightleftarrows \mathcal{G}\text{-}\mathbf{pEx}^{op} : \mathcal{G}\text{-Reg}(-,\mathcal{V})$$

form a biequivalence of 2-categories.

Example 6.3.31. Let $\mathcal{V} = \mathbf{Cat}$ and $\mathscr{E}$ be the class of the surjective on cube functors in **Cat** as in Example 5.3.36; this is determined by the dense generator $\mathcal{H} = \{\mathbb{2}^n\}_{n\in\mathbb{N}}$. Then the theorem above gives a duality between the 2-category of finitely accessible 2-categories with products and powers by $\mathbb{2}$, and the 2-category of small $\mathcal{H}$-exact categories with enough $\mathcal{H}$-projectives.

Example 6.3.32. Let $\mathcal{V}$ be locally finitely presentable and such that $\mathcal{V}_0(I,-)$ is weakly cocontinuous and weakly strong monoidal as in Section 3.1; then $\mathcal{V}$ satisfies the conditions required for this section and we can consider the class $\mathscr{E}$ consisting of the pure epimorphisms in $\mathcal{V}$; this is induced by the dense generator $\mathcal{G} = \mathcal{V}_f$ (as in Example 5.3.37). Then the theorem above provides a duality between the 2-category of the finitely accessible $\mathcal{V}$-categories with products and finite powers, and the 2-category of small $\mathcal{V}_f$-exact $\mathcal{V}$-categories with enough $\mathcal{V}_f$-projectives.

6.3.5 Flexible limits

Let $\mathcal{V} = \mathbf{Cat}$ and consider Ψ to be the class of all flexible weights together with its companion $\mathfrak{P}$ formed by the pseudo equivalence 2-relations (Definition 5.3.45).

Unfortunately, we do not know whether Ψ and $\mathfrak{P}$ satisfy condition (b) of Section 6.2, so we cannot deduce a duality theorem in this context. However there are corresponding notions of regularity and exactness for 2-categories which are strictly related to accessible 2-categories with flexible limits.

Definition 6.3.33. We say that a 2-category $\mathcal{B}$ is *regular* if it has finite limits, coisoidentifiers of acyclic isokernel cells, and admits a fully faithful $J\colon \mathcal{B} \hookrightarrow [\mathcal{C}, \mathbf{Cat}]$ (for some small $\mathcal{B}$) which preserves finite limits and said coisoidentifiers. A *regular* 2-functor between regular 2-categories is one that preserves finite limits and coisoidentifiers of acyclic isokernel cells.

Remark 6.3.34. This corresponds to a notion of regularity in the sense of [45], but (apparently) not in the sense of [21]. Moreover it would be interesting to have an intrinsic definition of regular 2-category, and not one relying on an embedding into a 2-category of presheaves.

Definition 6.3.35. We call an object P of a regular 2-category $\mathcal{B}$ *2-projective* if $\mathcal{C}(P,-)$ preserves coisoidentifiers of acyclic isokernel cells; denote by $\mathcal{B}_P$ the full subcategory of $\mathcal{B}$ spanned by the 2-projectives. We say that a regular $\mathcal{B}$ is *exact with enough 2-projectives* if every object B admits a map $q\colon P \to B$ with $P \in \mathcal{B}_P$ and q a coisoidentifier of an acyclic isokernel cell, and the induced $\mathcal{E} \to [\mathcal{E}_P^{op}, \mathbf{Cat}]$ reflects acyclic isokernel cells.

Proposition 6.3.36. *A lex 2-category $\mathcal{B}$ is exact with enough 2-projectives if and only if $\mathcal{E} \simeq \mathfrak{P}(\mathcal{E}_P)$.*

Proof. Argue as in the proof Proposition 6.3.20 by replacing coequalizers with coisoidentifiers and kernel pairs with acyclic isokernel cells. □

Proposition 6.3.37. *Let $\mathcal{E}$ be a small regular 2-category; then*

1. *the 2-category* $\mathrm{Reg}(\mathcal{E}, \mathbf{Cat})$ *is accessible with flexible limits;*
2. *if $\mathcal{E}$ is exact with enough 2-projectives then* $\mathrm{Reg}(\mathcal{E}, \mathbf{Cat})$ *is finitely accessible and has flexible limits.*

Conversely, if $\mathcal{A}$ is accessible with flexible limits then the 2-category $\mathrm{FlatFlex}(\mathcal{A}, \mathbf{Cat})$*, of 2-functors preserving flat colimits and flexible limits, is exact.*

Proof. Given a small regular 2-category $\mathcal{E}$, since coisoidentifiers of acyclic isokernel cells in $\mathbf{Cat}$ are just the retract equivalences, the 2-category $\mathrm{Reg}(\mathcal{E}, \mathbf{Cat})$ is clearly sketchable by limit/$\mathscr{E}$ sketch in the sense of Theorem 5.3.48, and hence is accessible with flexible limits.

If $\mathcal{E}$ is moreover exact with enough projectives, then arguing as in the proof of Proposition 6.2.5 and using that $\mathcal{E} \simeq \mathfrak{P}(\mathcal{E}_P)$ by the proposition above, we obtain an equivalence

$$\mathrm{Reg}(\mathcal{E}, \mathbf{Cat}) \simeq \alpha\text{-}\mathrm{Flat}(\mathcal{E}_P, \mathbf{Cat})$$

induced by precomposition with the inclusion $\mathcal{E}_P \hookrightarrow \mathcal{E}$. Thus $\mathrm{Reg}(\mathcal{E}, \mathbf{Cat})$ is finitely accessible.

Finally, if $\mathcal{A}$ is accessible with flexible limits then $\mathrm{FlatFlex}(\mathcal{A}, \mathbf{Cat})$ is closed in $[\mathcal{A}, \mathbf{Cat}]$ under finite limits and coisoidentifiers of equivalence 2-relations; thus it is exact since $[\mathcal{A}, \mathbf{Cat}]$ is.

□

APPENDIX

A

Additional results

A.1 On saturation and pre-saturation

In this section we study those classes of indexing categories (or weights, in the enriched setting) for which free cocompletions under colimits of that shape arise as one-step closures.

Let us consider first the unenriched and conical case. Given a class Λ of indexing categories, we denote by $\Lambda\mathcal{C}$ the free completion of a category $\mathcal{C}$ under Λ-colimits; this can be defined as the closure of $\mathcal{C}$ in $\mathcal{P}\mathcal{C}$ under Λ-colimits.

Definition A.1.1 ([7])**.** Let Λ be a class of small categories. The *saturation* Λ^* of Λ is the class of all small categories $\mathcal{B}$ for which every Λ-cocomplete category is $\mathcal{B}$-cocomplete and every Λ-cocontinuous functor is $\mathcal{B}$-cocontinuous. Given this, we say that:

1. Λ is *pre-saturated* if for any category $\mathcal{C}$ every object of $\Lambda\mathcal{C}$ is a Λ-colimit of objects from $\mathcal{C}$;
2. Λ is *saturated* if $\Lambda = \Lambda^*$.

By construction we always have the inclusion $\Lambda \subseteq \Lambda^*$.

Remark A.1.2. It has been shown in [7] that for $\mathcal{B}$ to be in Λ^* it suffices that Λ-cocomplete implies $\mathcal{B}$-cocomplete: preservation of $\mathcal{B}$-colimits by Λ-continuous functors is actually a consequence of that.

Example A.1.3. It is well known that the class **Fin** of finite categories is pre-saturated. However, it is not saturated. The saturation of **Fin** is the class of all small categories which have a final finite subcategory; these are called *L-finite* in Section 3 of [82].

Example A.1.4. The class **FD** of finite discrete categories is pre-saturated but not saturated. Its saturation is the class of all categories with finite connected components and local terminal objects.

Example A.1.5. The class α-**DPos** of small α-directed posets is pre-saturated but not saturated. Its saturation is the class α-**DFilt** of the small α-filtered categories.

The fact that the saturations in the examples above can be described in that way, can be seen as a consequence of Proposition A.1.11.

In [7] it is proved that every saturated class of weights is pre-saturated; our main goal is to prove the corresponding statement for classes of categories, for which there seems to be more work to do. In fact, we first need an intermediate step.

Definition A.1.6. Let Λ be a class of categories. We say that Λ is a *stable* class if given any small category $\mathcal{C}$, any $\mathcal{D} \in \Lambda$, and $H\colon \mathcal{D} \to [\mathcal{C}^{op}, \mathbf{Set}]$ such that all the Hd are a Λ-colimit of representables; then there exists $\mathcal{H} \in \Lambda$ together with

$$\begin{tikzcd}
\mathcal{H} \arrow[r, dashed, "\pi"] \arrow[d, dashed, "K"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{D} \arrow[d, "H"] \\
\mathcal{C} \arrow[r, hook, "Y"'] & {[\mathcal{C}^{op}, \mathbf{Set}]}
\end{tikzcd}$$

for which $\mathrm{Lan}_\pi(YK) \cong H$.

To give the natural transformation as in the square above, since $\mathrm{Lan}_\pi(YK) \cong H$, it is enough to give a suitable functor $\mathcal{H} \to Y/H$.

Proposition A.1.7. *Let Λ be a stable class of categories; then Λ is pre-saturated.*

Proof. Let $\mathcal{C}$ be any category and let $\Lambda_1\mathcal{C}$ be the full subcategory of $[\mathcal{C}^{op}, \mathbf{Set}]$ spanned by the Λ-colimits of representables. It is enough to prove that $\Lambda_1\mathcal{C}$ is closed in $[\mathcal{C}^{op}, \mathbf{Set}]$ under Λ-colimits; then we would have $\Lambda_1\mathcal{C} = \Lambda\mathcal{C}$ showing that Λ is pre-saturated.

Let $\mathcal{D} \in \Lambda$ and $H\colon \mathcal{D} \to [\mathcal{C}^{op}, \mathbf{Set}]$ be a functor landing in $\Lambda_1\mathcal{C}$. By construction H satisfies the hypothesis given in the stability condition; since Λ is stable we can then consider $\mathcal{H}$, $K\colon \mathcal{H} \to \mathcal{C}$, and $\pi\colon \mathcal{H} \to \mathcal{D}$ as in the definition above. Now, the colimit of H in $[\mathcal{C}^{op}, \mathbf{Set}]$ can be denoted a the functor $\operatorname{colim} H\colon 1 \to [\mathcal{C}^{op}, \mathbf{Set}]$ out of the terminal category. The fact that this the colimit of H says exactly that $\operatorname{colim} H \cong \mathrm{Lan}_! H$, where $!\colon \mathcal{D} \to 1$ is the unique functor. Then

$$\operatorname{colim} H \cong \mathrm{Lan}_! H \cong \mathrm{Lan}_! \mathrm{Lan}_\pi(YK) \cong \mathrm{Lan}_!(YH) \cong \operatorname{colim}(YH).$$

But the latter is a Λ-colimit of representable; therefore $\operatorname{colim} H \in \Lambda_1\mathcal{C}$ as desired. $\square$

Proposition A.1.8. *Any saturated class is stable, and hence pre-saturated.*

Proof. Let Λ be saturated and $H\colon \mathcal{D} \to [\mathcal{C}^{op}, \mathbf{Set}]$ be as in the definition of stability. For each $d \in \mathcal{D}$ we choose $\mathcal{E}_d \in \Lambda$ such that $Hd \cong \operatorname{colim}(YL_d)$ for some $L_d\colon \mathcal{E}_d \to \mathcal{C}$; denote by $\eta^d_e\colon YL_d(e) \to Hd$ the colimiting cocone. Consider now $L := (L_d)_d\colon \sum_d \mathcal{E}_d \to \mathcal{C}$ and the functor $S\colon \sum_d \mathcal{E}_d \to \mathcal{D}$ sending each object and morphism of $\mathcal{E}_d$ to d and 1_d respectively; then the colimiting cocones define a natural transformation $\eta\colon YL \Rightarrow HS$. Let Y/H be the comma object of Y and H; then there is an induced $T\colon \sum_d \mathcal{E}_d \to Y/H$ making the following diagram commute

$$\begin{tikzcd}
\sum_d \mathcal{E}_d \arrow[dr, "T"] \arrow[drr, bend left, "S"] \arrow[ddr, bend right, "L"'] & & \\
& Y/H \arrow[r, "P_1"] \arrow[d, "P_2"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{D} \arrow[d, "H"] \\
& \mathcal{C} \arrow[r, hook, "Y"'] & {[\mathcal{C}^{op}, \mathbf{Set}]}
\end{tikzcd}$$

T acts on objects by sending $(d \in \mathcal{D}, e \in \mathcal{E}_d)$ to $T(d,e) = (d \in \mathcal{D}, L_d(e) \in \mathcal{C}, \eta^d_e)$. Then we define $\mathcal{H}$ to be given by the (bijective on objects, fully faithful) factorization of T; this is the full subcategory of Y/H spanned by $T(d,e)$ for any $d \in \mathcal{D}$ and $e \in \mathcal{E}_d$. Denote by $J\colon \mathcal{H} \to Y/H$ the inclusion; then we define $K := P_2 J$ and $\pi = P_1 J$; the natural transformation is the restriction of that associated to the co-comma object.

The next step is to prove that $\mathcal{H}$, K, and π have the required properties. For any $d \in \mathcal{D}$ consider the slice π/d together with its projection $Q_d\colon \pi/d \to \mathcal{H}$. Since this will be central for the rest of the proof let us spell out explicitly the objects and morphisms of π/d:

- objects are triples $(d_1, e_1, f_1) = (d_1 \in \mathcal{D}, e_1 \in \mathcal{E}_{d_1}, f_1\colon d_1 \to d)$;
- morphisms $(d_1, e_1, f_1) \to (d_2, e_2, f_2)$ are pairs (h, k), with $h\colon d_1 \to d_2$ in $\mathcal{D}$ and $k\colon L_{d_1}(e_1) \to L_{d_2}(e_2)$ in $\mathcal{C}$, for which the diagrams

$$\begin{tikzcd}
YL_{d_1}(e_1) \arrow[r, "Y(k)"] \arrow[d, "\eta^{d_1}_{e_1}"'] & YL_{d_2}(e_2) \arrow[d, "\eta^{d_2}_{e_2}"] \\
Hd_1 \arrow[r, "H(h)"'] & Hd_2
\end{tikzcd}
\qquad
\begin{tikzcd}
d_1 \arrow[rr, "h"] \arrow[dr, "f_1"'] & & d_2 \arrow[dl, "f_2"] \\
& d &
\end{tikzcd}$$

  commute in $[\mathcal{C}^{op}, \mathbf{Set}]$ and $\mathcal{D}$ respectively.

It follows that there is an induced functor $F_d\colon \mathcal{E}_d \to \pi/d$ which sends an object $e \in \mathcal{E}_d$ to $F_d(e) := (d, e, 1_d)$ and a morphism $g\colon e \to e'$ to $F_d(g) := (1_d, L_d(g))$. As a consequence the following diagram commutes

$$\begin{tikzcd}
\pi/d \arrow[r, "Q_d"] & \mathcal{H} \arrow[d, "K"] \\
\mathcal{E}_d \arrow[u, "F_d"] \arrow[r, "L_d"'] & \mathcal{C}
\end{tikzcd}$$

Now we prove that F_d is final for any $d \in \mathcal{D}$. Let (d_1, e_1, f_1) be an object of π/d; then we can consider the composite $H(f_1) \circ \eta^{d_1}_{e_1}\colon YL_{d_1}(e_1) \to Hd$ in $[\mathcal{C}^{op}, \mathbf{Set}]$. But $Hd \cong \operatorname{colim}(YL_d)$ and since $YL_{d_1}(e_1)$ is representable we have

$$[\mathcal{C}^{op}, \mathbf{Set}](YL_{d_1}(e_1), Hd) \cong \operatorname{colim}\ \mathcal{C}(L_{d_1}(e_1), L_d-)$$

it follows that $H(f_1) \circ \eta^{d_1}_{e_1}$ factors as $\eta^d_e \circ Y(k)$ for some $e \in \mathcal{E}_d$ and $k\colon L_{d_1}(e_1) \to L_d(e)$ in $\mathcal{C}$. As a consequence we obtain a map $(f_1, k)\colon (d_1, e_1, f_1) \to F_d(e)$. Given any other map $(f_1, k')\colon (d_1, e_1, f_1) \to F_d(e')$ in π/d (the first component is always forced to be f_1) we have that $\eta^d_e \circ Y(k) = \eta^d_{e'} \circ Y(k')$; thus k and k' induce the same map in the set

$\operatorname{colim}\mathcal{C}(L_{d_1}(e_1), L_d-)$, and hence there exists a zig-zag in $\mathcal{E}_d$ connecting e and e' whose image in $L_{d_1}(e_1)/L_d$ completes to a zig-zag between k and k'. The same zig-zag connects (f_1, k) and (f_1, k') in π/d. Thus F_d is final.

Thanks to that we can now prove that $\mathrm{Lan}_\pi(YK) \cong H$: the natural transformation $YK \Rightarrow H\pi$ induces a map $\gamma\colon \mathrm{Lan}_\pi(YK) \Rightarrow H$; to conclude it is enough to prove that its components are isomorphisms. Consider $d \in \mathcal{D}$, then γ_d is given by the composites of the following isomorphisms:

$$\begin{aligned}
\mathrm{Lan}_\pi(YK)(d) &\cong \operatorname{colim}(\pi/d \xrightarrow{Q_d} \mathcal{H} \xrightarrow{K} \mathcal{C} \xrightarrow{Y} [\mathcal{C}^{op}, \mathbf{Set}]) \\
&\cong \operatorname{colim}(\mathcal{E}_d \xrightarrow{F_d} \pi/d \xrightarrow{Q_d} \mathcal{H} \xrightarrow{K} \mathcal{C} \xrightarrow{Y} [\mathcal{C}^{op}, \mathbf{Set}]) \\
&\cong \operatorname{colim}(\mathcal{E}_d \xrightarrow{L_d} \mathcal{C} \xrightarrow{Y} [\mathcal{C}^{op}, \mathbf{Set}]) \\
&\cong Hd
\end{aligned}$$

as desired.

Note now that, since Λ is saturated, each slice π/d lies in Λ: existence and preservation of colimits indexed by π/d can be reduced to that of colimits indexed on $\mathcal{E}_d$, which is in Λ by definition. We are only left to prove that $\mathcal{H}$ itself is in Λ; for that it is enough to show that every Λ-cocomplete category is also $\mathcal{H}$-cocomplete (again since Λ is saturated). Let $\mathcal{A}$ be Λ-cocomplete and $F\colon \mathcal{H} \to \mathcal{A}$ be any functor; then $\mathrm{Lan}_\pi F\colon \mathcal{D} \to \mathcal{A}$ exists (this is defined pointwise as a Λ-colimit in $\mathcal{A}$ since each π/d is in Λ) and $\operatorname{colim}(\mathrm{Lan}_\pi F) \cong \operatorname{colim} F$. It follows that the colimit of F exists in $\mathcal{A}$ and hence $\mathcal{H} \in \Lambda$. □

Corollary A.1.9. *Let Λ be a saturated class of small categories and let α be an infinite regular cardinal. Then the class Λ_α spanned by the α-small categories of Λ is stable, and hence pre-saturated.*

Proof. Let $\mathcal{C}$ be any small category and $H\colon \mathcal{D} \to [\mathcal{C}^{op}, \mathbf{Set}]$ be as in Definition A.1.6 for Λ_α; our aim is to argue as in the proof of Proposition A.1.8. Consider $\mathcal{E}_d \in \Lambda_\alpha$ together with functors $L_d\colon \mathcal{E}_d \to \mathcal{C}$ for which $Hd \cong \operatorname{colim}(YL_d)$ with colimit cocone $\eta^d_e\colon YL_d(e) \to Hd$. Since $\mathcal{D}$ and each $\mathcal{E}_d$ are α-small we can find an α-small subcategory $J\colon \mathcal{C}_1 \to \mathcal{C}$ (not necessarily full) which contains the image of each $\mathcal{E}_d$ and such that every morphism $H(f\colon d \to d')$ is the colimit of maps in $\mathcal{C}_1$ between $L_d(\mathcal{E}_d)$ and $L_{d'}(\mathcal{E}_{d'})$. In particular it follows that each L_d lands in $\mathcal{C}_1$ and hence that $Hd \cong \operatorname{colim}(YJL_d)$. Now we proceed as in the proof of A.1.8 above with the only difference that instead of the comma Y/H we consider the comma YJ/H and define $\mathcal{H}$ as a full subcategory of it, this comes with its projections $K\colon \mathcal{H} \to \mathcal{C}_1$ and $\pi\colon \mathcal{H} \to \mathcal{D}$. Notice now that, since $\mathcal{D}$ and $\mathcal{C}_1$ are α-small, the category YJ/H has α-small hom-sets (but in general has more than α objects); as a consequence $\mathcal{H}$ is α-small (since $\mathcal{D}$ and all the $\{\mathcal{E}_d\}_{d\in\mathcal{D}}$ are) and moreover it still satisfies all the properties shown in the proof above; in particular $\mathcal{H} \in \Lambda$. If follows by definition then that $\mathcal{H} \in \Lambda_\alpha$ and satisfies $\mathrm{Lan}_\pi(Y(JK)) \cong H$ as desired. □

Remark A.1.10. As a consequence, since the class of all small categories is clearly saturated it follows that the classes of all the α-small categories, for a given α, are pre-saturated (as we already knew). Similarly, for given cardinals α and β, the class of the β-small α-filtered categories is pre-saturated (as proved also in [30, Corollary 4.11]).

Proposition A.1.11. *Let Λ be a class of small categories. The following are equivalent*

1. *Λ is pre-saturated*

2. *$\mathcal{B} \in \Lambda^*$ if and only if there exists a final functor $K\colon \mathcal{A} \to \mathcal{B}$ for some $\mathcal{A} \in \Lambda$.*

Proof. Assume that (1) holds. If $K\colon \mathcal{A} \to \mathcal{B}$ is final and $\mathcal{A} \in \Lambda$, then the existence and preservation of $\mathcal{B}$-colimits is a consequence of the existence and preservation of $\mathcal{A}$-colimits. Thus $\mathcal{B} \in \Lambda^*$ by definition.
Conversely assume that $\mathcal{B}$ is in Λ^* and consider the free cocompletion $\Lambda\mathcal{B}$ as a full subcategory of $[\mathcal{B}^{op}, \mathbf{Set}]$, which is closed under Λ-colimits. By definition of Λ^* the category $\Lambda\mathcal{B}$ has Λ^*-colimits and these are computed as in $[\mathcal{B}^{op}, \mathbf{Set}]$. Consider now the terminal functor $\Delta 1\colon \mathcal{B}^{op} \to \mathbf{Set}$; then $\Delta 1 \cong \Delta 1 * Y = \operatorname{colim}(Y) \in \Lambda\mathcal{B}$ and hence, since Λ is pre-saturated, $\Delta 1$ can be expressed a Λ-colimit of representables. There exists then $\mathcal{A} \in \Lambda$ and $H\colon \mathcal{A} \to \mathcal{B}$ such that $\Delta 1 \cong \operatorname{colim}(YH) \cong \Delta 1 * YH$. But then $\Delta 1 \cong \operatorname{Lan}_{H^{op}}\Delta 1 * Y \cong \operatorname{Lan}_{H^{op}}\Delta 1$, which means that H is a final functor.

Assume now that (2) holds. Since Λ^* is saturated (by construction) then it is also pre-saturated by Proposition A.1.8; hence for any $\mathcal{C}$ the free cocompletion $\Lambda^*\mathcal{C}$ is a one-step closure. But $\Lambda\mathcal{C} = \Lambda^*\mathcal{C}$; thus for any $X \in \Lambda\mathcal{C}$ we can find $\mathcal{B}$ in Λ^* and $F\colon \mathcal{B} \to \mathcal{C}$ such that $X \cong \operatorname{colim}\ JF$, where $J\colon \mathcal{C} \to \Lambda\mathcal{C}$ is the inclusion. Let $K\colon \mathcal{A} \to \mathcal{B}$ be as in condition (2); then $X \cong \operatorname{colim}(JFK)$ is a Λ-colimit of elements of $\mathcal{C}$. Thus $\Lambda\mathcal{C}$ is a one-step closure and Λ is pre-saturated. $\square$

Hence we can characterize the saturated classes as follows:

Corollary A.1.12. *Let Λ be a class of small categories. The following are equivalent:*

1. *Λ is saturated;*

2. *Λ is stable and such that: $\mathcal{A} \in \Lambda$ and $H\colon \mathcal{A} \to \mathcal{B}$ final implies $\mathcal{B} \in \Lambda$;*

3. *Λ is pre-saturated and such that: $\mathcal{A} \in \Lambda$ and $H\colon \mathcal{A} \to \mathcal{B}$ final implies $\mathcal{B} \in \Lambda$.*

Proof. (1) $\Rightarrow$ (2) is given by Proposition A.1.8 and the definition of saturated class, (2) $\Rightarrow$ (3) follows by Proposition A.1.7, and (3) $\Rightarrow$ (1) follows from Proposition A.1.11 above since it implies that $\Lambda = \Lambda^*$. $\square$

We now move to the enriched context and prove the corresponding versions of Proposition A.1.11 and Corollary A.1.12 above. Let us fix as base of enrichment a symmetric monoidal closed, complete, and cocomplete category $(\mathcal{V}_0, \otimes, I)$.

Definition A.1.13 ([7])**.** Let Φ be a class of weights. We define its saturation Φ^* as the class of all weights N for which any Φ-cocomplete $\mathcal{V}$-category is N-cocomplete and any Φ-cocontinuous $\mathcal{V}$-functor is N-cocontinuous. Given this, we say that:

1. Φ is *pre-saturated* if for any category $\mathcal{A}$ the free cocompletion $\Phi\mathcal{A}$ is a one-step closure;

2. Φ is *saturated* if $\Phi = \Phi^*$.

Any class of categories Λ corresponds to a class of weights Φ_Λ defined by all the conical weights $\Delta I\colon \mathcal{C}_{\mathcal{V}}^{op} \to \mathcal{V}$ with $\mathcal{C} \in \Lambda$. It is easy to see that if Λ is pre-saturated in ordinary sense then Φ_Λ is also pre-saturated in the enriched sense; however the same does not hold for saturation. For instance, the class **Filt** of small filtered categories is saturated when seen as a class of categories, but the corresponding class of conical weights is not saturated: by Proposition A.1.14 below its saturation is the class of those weights $N\colon \mathcal{D}^{op} \to \mathcal{V}$ which arise as the left Kan extension of $\Delta I\colon \mathcal{C}_{\mathcal{V}}^{op} \to \mathcal{V}$, for a filtered $\mathcal{C}$, along some $H\colon \mathcal{C}_{\mathcal{V}} \to \mathcal{D}$.

Proposition A.1.14. *Let Φ be a class of weights. The following are equivalent*

1. *Φ is pre-saturated*
2. *$N \in \Phi^*$ only if $N \cong \mathrm{Lan}_H M$ for some H and $M \in \Phi$.*

Proof. Assume first that Φ is pre-saturated. If $N \cong \mathrm{Lan}_H M$ for some H and $M \in \Phi$, then the existence and preservation of N-colimits is a consequence of the existence and preservation of M-colimits. Since Φ^* is saturated then $N \in \Phi^*$.
Conversely assume that $N \colon \mathcal{C}^{op} \to \mathcal{V}$ is in Φ^* and consider the free cocompletion $\Phi\mathcal{C}$ as a full subcategory of $[\mathcal{C}^{op}, \mathcal{V}]$ closed under Φ-colimits. By definition of Φ^* the $\mathcal{V}$-category $\Phi\mathcal{C}$ has Φ^*-colimits and these are computed as in $[\mathcal{C}^{op}, \mathcal{V}]$. Therefore $N \cong N * Y \in \Phi\mathcal{C}$ and hence, since Φ is pre-saturated, N is a Φ-colimit of representables. There exists then $M \colon \mathcal{D}^{op} \to \mathcal{V}$ in Φ and $H^{op} \colon \mathcal{D} \to \mathcal{C}$ such that $N \cong M * YH^{op}$. But then

$$N \cong M * YH^{op} \cong \mathrm{Lan}_H M * Y \cong \mathrm{Lan}_H M$$

as requested.

Assume now that (2) holds. Since Φ^* is saturated (by construction) then it is also pre-saturated and hence for any $\mathcal{A}$ the free cocompletion $\Phi^*\mathcal{A}$ is a one-step closure. But $\Phi\mathcal{A} = \Phi^*\mathcal{A}$; thus for any $X \in \Phi\mathcal{A}$ we can find $N \colon \mathcal{C}^{op} \to \mathcal{V}$ in Ψ^* and $F \colon \mathcal{C} \to \mathcal{A}$ such that $X \cong N * JF$, where $J \colon \mathcal{A} \to \Phi\mathcal{A}$ is the inclusion (when $\mathcal{A}$ is small we can take $N = X$ and $F = 1_{\mathcal{A}}$). Let M and H be as in condition (2); then $X \cong N * JF \cong M * JFH^{op}$ is a Φ-limit of elements of $\mathcal{A}$. Thus $\Phi\mathcal{A}$ is a one-step closure and Φ is pre-saturated. $\square$

Every saturated class of weights is pre-saturated by [7]. The converse does not hold in general but we can give a necessary and sufficient condition for a pre-saturated class to be saturated:

Corollary A.1.15. *A class Φ is saturated if and only if it is pre-saturated and closed under left Kan extensions.*

Proof. If Φ is saturated then it is pre-saturated by [7] and closed under left Kan extension by definition (since colimits weighted by $\mathrm{Lan}_N M$ are the same as colimits weighted by M). Vice versa, if Φ is pre-saturated and closed under left Kan extension then $\Phi = \Phi^*$ by Proposition A.1.14, and hence Φ is saturated. $\square$

A.2 Enriched pettiness

The aim of this section is to generalize to the context of enriched category theory the notion of petty functor introduced by Freyd [41], as well as the solution-set condition. With that we wish then to extend the content of Section 2.2.6 comparing cone and virtual reflectivity to this more general context.

We assume $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ to be any symmetric monoidal closed, complete, and cocomplete category which is closed under regular subobjects in some fixed universe enlargement $\mathcal{V}'$ as in [56, Section 3.11]. This is true for most of the bases of enrichment considered in practice; for instance whenever $\mathcal{V}$ consists of the "small" objects of a suitable larger $\mathcal{V}'$, then the inclusion $\mathcal{V} \hookrightarrow \mathcal{V}'$ will satisfy the desired property. Standard examples are $\mathcal{V} =$ **Set**, **Ab**, **Cat**, **SSet**, **Gra**, and **CGTop** (compactly generated topological spaces), where $\mathcal{V}'$ is the corresponding category of "large objects".

The first thing we require is an enriched notion of pettiness:

Definition A.2.1. We say that $F\colon \mathcal{A}^{op} \to \mathcal{V}$ is *petty* if there exist $A_i \in \mathcal{A}$ and $X_i \in \mathcal{V}$, for i in a small set I, together with a regular epimorphism

$$\sum_{i\in I} X_i \cdot \mathcal{A}(-, A_i) \twoheadrightarrow F.$$

Let $\mathrm{Pt}(\mathcal{A})$ be the full subcategory of $[\mathcal{A}^{op}, \mathcal{V}]$ spanned by the petty $\mathcal{V}$-functors.

Note that $\mathrm{Pt}(\mathcal{A})$ is a $\mathcal{V}$-category even though $[\mathcal{A}^{op}, \mathcal{V}]$ may not be one; indeed for any $F, G \in \mathrm{Pt}(\mathcal{A})$, we can cover F by coproducts of copowers of representables as above, and hence obtain a regular monomorphism

$$\mathrm{Pt}(\mathcal{A})(F, G) \overset{q}{\rightarrowtail} \prod_{i\in I} [X_i, GA_i].$$

which is an object of $\mathcal{V}$ by our initial assumption.

Remark A.2.2. When $\mathcal{V} = \mathbf{Set}$ we recover Freyd's notion of pettiness since every copower of a representable functor by a set X coincides with the coproduct of the same representable indexed on X.

Moreover the free cocompletion $\mathcal{P}\mathcal{A}$ is contained in $\mathrm{Pt}(\mathcal{A})$ since any small $\mathcal{V}$-functor $F\colon \mathcal{A}^{op} \to \mathcal{V}$ is weighted colimit of representables, and thus can be written as a coequalizer

$$\sum_{h\in H} Y_j \cdot \mathcal{A}(-, B_h,) \rightrightarrows \sum_{i\in I} X_i \cdot \mathcal{A}(-, A_i) \overset{q}{\twoheadrightarrow} F.$$

The map q shows that F is then a petty $\mathcal{V}$-functor.

Definition A.2.3. We say that a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ satisfies the *enriched solution-set condition* if $\mathcal{B}(B, F-)$ is petty for any $B \in \mathcal{B}$.

The following then generalizes Proposition 2.2.44.

Proposition A.2.4. *The following are equivalent for a fully faithful $J\colon \mathcal{A} \hookrightarrow \mathcal{K}$ and virtually cocomplete $\mathcal{K}$:*

1. *J satisfies the enriched solution-set condition;*
2. *J has a virtual left adjoint.*

Proof. (2) $\Rightarrow$ (1) is trivial since each small $\mathcal{V}$-functor is petty. For (1) $\Rightarrow$ (2) assume that J satisfies the enriched solution-set condition; we need to prove that $\mathcal{K}(K, J-)$ is actually small for any $K \in \mathcal{K}$. By hypothesis $\mathcal{K}(K, J-)$ is petty, so there exist $A_i \in \mathcal{A}$ and $X_i \in \mathcal{V}$, for $i \in I$, together with a regular epimorphism

$$q\colon \sum_{i\in I} X_i \cdot \mathcal{A}(A_i, -) \twoheadrightarrow \mathcal{K}(K, J-).$$

Such an arrow determines a family of maps $(X_i \to \mathcal{K}(K, JA_i))_{i\in I}$, and this in turn induces a morphism

$$q'\colon \sum_{i\in I} X_i \cdot \mathcal{K}(JA_i, -) \to \mathcal{K}(K, -)$$

which, when restricted to $\mathcal{A}$, gives back q. The domain of q' is still a small $\mathcal{V}$-functor and hence, since $\mathcal{K}$ is virtually cocomplete, the kernel pair P of q' will then be a (small and hence) petty $\mathcal{V}$-functor. In particular we obtain

$$\sum_{j\in J} Y_j\cdot\mathcal{K}(K_j,-)\xrightarrow{s'}\!\!\!\!\rightarrow P \overset{u'}{\underset{v'}{\rightrightarrows}} \sum_{i\in I} X_i\cdot\mathcal{K}(JA_i,-)\xrightarrow{q'}\!\!\!\!\rightarrow \mathcal{K}(K,-)$$

where s is a (regular) epimorphism, and q' is the coequalizer of (u',v'), and therefore also of $(u's',v's')$. Now we can restrict this diagram to $\mathcal{A}$ by pre-composing with J and, since pre-composition in continuous and cocontinuous, the following is still a coequalizer

$$\sum_{j\in J} Y_j\cdot\mathcal{K}(K_j,J-) \overset{u}{\underset{v}{\rightrightarrows}} \sum_{i\in I} X_i\cdot\mathcal{A}(A_i,-)\xrightarrow{q}\!\!\!\!\rightarrow \mathcal{K}(K,J-)$$

where $u=u's'J$ and $v=v's'J$. By hypothesis each $\mathcal{K}(K_j,J-)$ is petty, so it is covered through a (regular) epimorphism by coproducts of copowers of representables; thus, since epimorphisms are stable under coproducts and copowers, also $\sum_{j\in J} Y_j\cdot\mathcal{K}(K_j,J-)$ is covered through an epimorphism by coproducts of copowers of representables. Since the coequalizer of a pair does not change is the pair is pre-composed with an epimorphism, we can obtain $\mathcal{K}(X,J-)$ as a coequalizer

$$\sum_{h\in H} Y_j\cdot\mathcal{A}(B_h,-) \overset{u}{\underset{v}{\rightrightarrows}} \sum_{i\in I} X_i\cdot\mathcal{A}(A_i,-)\xrightarrow{q}\!\!\!\!\rightarrow \mathcal{K}(K,J-)$$

witnessing that $\mathcal{K}(X,J-)$ is small. □

Finally, the corollary below is a generalization of Proposition 2.2.42.

Corollary A.2.5. *The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:*

1. $\mathrm{Pt}(\mathcal{A})$ *has limits of representables;*
2. $\mathcal{P}\mathcal{A}$ *has limits of representables;*
3. $\mathcal{P}\mathcal{A}$ *is complete.*

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are trivial and (2) $\Rightarrow$ (3) is given by [33, Theorem 3.8]. To show that (1) $\Rightarrow$ (2) let us prove that dual statement, that is: if $\mathrm{Pt}^\dagger(\mathcal{A})$ has colimits of representables then so does $\mathcal{P}^\dagger\mathcal{A}$. Let $\mathcal{K}=\mathcal{P}\mathcal{A}$ and $J\colon\mathcal{A}\hookrightarrow\mathcal{K}$ be the inclusion; then $\mathcal{K}$ is cocomplete and hence virtually cocomplete. Moreover J satisfies the enriched solution-set condition: given $X\in\mathcal{K}$ we can write it as a colimit $X\cong M*JH$ of objects from $\mathcal{A}$, then $\mathcal{K}(X,J-)\cong\{M,YH\}$ is a limit of representables in $[\mathcal{A},\mathcal{V}]$. In other words $\mathcal{K}(X,J-)$ is a colimit of representables when seen in the opposite category, by our assumption then $\mathcal{K}(X,J-)$ lies in $\mathrm{Pt}^\dagger(\mathcal{A})$, as desired. It follows by the proposition above that J has a virtual left adjoint, and thus $\mathcal{P}^\dagger\mathcal{A}$ has colimits of representables. □

A.3 Definable categories

Definable categories were first introduced in the additive context by Prest [86] and then studied also in the ordinary setting by setting by Kuber and Rosický [60]; a general enriched approach has been given in [64].

One of the problems of the theory, even in the ordinary and additive context, is to give a satisfactory intrinsic notion of definability; in fact definable categories are generally introduced as particular subcategories $\mathcal{D}$ of some locally finitely presentable category. In this section, given an accessible category $\mathcal{D}$ with products, we use the colimit type $\mathfrak{R}$ of Example 5.3.35 to identify a specific locally finitely presentable category $\mathcal{L}$, uniquely determined by $\mathcal{D}$, where it is possible to establish whether $\mathcal{D}$ is a definable category or not.

For simplicity we consider just the case of $\mathcal{V} = \mathbf{Set}$ and $\mathcal{V} = \mathbf{Ab}$; however everything can be extended to the context of [64]. In particular when talking about $\mathcal{V}$-categories and $\mathcal{V}$-functors we mean ordinary categories and ordinary functors, or preadditive categories and preadditive functors.

Definable $\mathcal{V}$-categories can be described in multiple ways: as those of the form $\mathrm{Reg}(\mathcal{C}, \mathcal{V})$, for some small regular $\mathcal{V}$-category $\mathcal{C}$; as finite injectivity classes of some locally finitely presentable $\mathcal{V}$-category; and as full subcategories of a locally finitely presentable $\mathcal{V}$-category closed under products, filtered colimits, and pure subobjects.

Let $\mathcal{D}$ be an accessible $\mathcal{V}$-category with products; consider the free completion $\mathfrak{R}^\dagger\mathcal{D}$ of $\mathcal{D}$ under equalizers of pseudo equivalence relations (see Example 5.3.35 and Section 6.3.3 where the dual notion is considered). Since $\mathfrak{R}$ is a companion for the class of products and $\mathcal{D}$ has them, it follows that $\mathfrak{R}^\dagger\mathcal{D}$ is cocomplete and can be identified with the full subcategory of $[\mathcal{D}, \mathcal{V}]^{op}$ spanned by the accessible $\mathcal{V}$-functors out of $\mathcal{D}$ which also preserve products.

Recall that, when $\mathcal{D}$ also has filtered colimits, we denote by $\mathrm{Def}(\mathcal{D}, \mathcal{V})$ the full subcategory of $[\mathcal{D}, \mathcal{V}]$ whose objects are the $\mathcal{V}$-functors that preserves products and filtered colimits.

Proposition A.3.1. *Let $\mathcal{D}$ be an accessible $\mathcal{V}$-category with products and filtered colimits; then*

$$(\mathfrak{R}^\dagger\mathcal{D})_f \simeq \mathrm{Def}(\mathcal{D}, \mathcal{V})^{op}.$$

Moreover $(\mathfrak{R}^\dagger\mathcal{D})_f$ is a small $\mathcal{V}$-category.

Proof. The proof is inspired by that of [100, Lemma 4.2.6]. Denote by $J\colon \mathcal{D} \to \mathfrak{R}^\dagger\mathcal{D}$ the inclusion, then the embedding of $\mathfrak{R}^\dagger\mathcal{D}$ in $[\mathcal{D}, \mathcal{V}]^{op}$ sends an object X to $\mathfrak{R}^\dagger\mathcal{D}(X, J-)$. Since J preserves products and filtered colimits it follows at once that $(\mathfrak{R}^\dagger\mathcal{D})_f \subseteq \mathrm{Def}(\mathcal{D}, \mathcal{V})^{op}$.

Let us now take a definable $\mathcal{V}$-functor $F\colon \mathcal{D} \to \mathcal{V}$, by the observations above this can be seen as an object F of $\mathfrak{R}^\dagger\mathcal{D}$ and $F \cong \mathfrak{R}^\dagger\mathcal{D}(F, J-)$; thus homming out of F in $\mathfrak{R}^\dagger\mathcal{D}$ preserves filtered colimits of objects from $\mathcal{D}$. We need to prove that it actually preserves all of them; by [1, Corollary 1.7] it is enough to show that $\mathfrak{R}^\dagger\mathcal{D}(F, -)$ preserves colimits of smooth chains: diagrams $(L_\beta)_{\beta<\alpha}$ indexed by an ordinal α and for which $L_\lambda = \mathrm{colim}_{\beta<\lambda} L_\beta$ for every limit $\lambda < \alpha$.

Before moving on, note that $\mathfrak{R}^\dagger\mathcal{D}$ is a coregular category and coregularly embedded in $[\mathcal{D}, \mathcal{V}]^{op}$; thus a regular monomorphism $m\colon L \to S$ in $\mathfrak{R}^\dagger\mathcal{D}$ is just a regular epimorphism $\mathfrak{R}^\dagger\mathcal{D}(S, J-) \to \mathfrak{R}^\dagger\mathcal{D}(T, J-)$ in $[\mathcal{D}, \mathcal{V}]$. As a consequence $m\colon L \to S$ is a regular monomorphism in $\mathfrak{R}^\dagger\mathcal{D}$ if and only if every object of $\mathcal{D}$ is injective with respect to it.

Consider hence a smooth chain $(L_\beta)_{\beta<\alpha}$ in $\mathfrak{R}^\dagger\mathcal{D}$ with connecting maps $d_{\beta,\gamma}\colon L_\beta \to L_\gamma$; for each $\beta < \alpha$ we define by transfinite induction a presentation expressing L_β as an equalizer of a pseudo-equivalence relation from $\mathcal{D}$:

$$L_\beta \overset{s_\beta}{\rightarrowtail} JS_\beta \underset{v_\beta}{\overset{u_\beta}{\rightrightarrows}} M_\beta \overset{t_\beta}{\rightarrowtail} JT_\beta;$$

here s_β will be a regular monomorphism in $\mathfrak{R}^\dagger\mathcal{D}$ into $\mathcal{D}$ (this exists by properties of $\mathfrak{R}^\dagger\mathcal{D}$), the pair (u_β, v_β) will be the cokernel pair of s_β, and t_β will again be a regular monomorphism into $\mathcal{D}$. We also define smooth chains of such presentations compatibly with $(L_\beta)_{\beta<\alpha}$.

If $\beta = 0$ any such presentation for L_0 is fine. Suppose now that everything is defined at level $\beta < \alpha$, then we define a presentation for $L_{\beta+1}$ and the connecting maps

$$\begin{tikzcd}
L_\beta \arrow[r, tail, "s_\beta"] \arrow[d, "d_{\beta,\beta+1}"'] & JS_\beta \arrow[r, shift left, "u_\beta"] \arrow[r, shift right, "v_\beta"'] \arrow[d, dashed, "e_{\beta,\beta+1}"'] & M_\beta \arrow[r, tail, "t_\beta"] \arrow[d, dashed, "f_{\beta,\beta+1}"'] & JT_\beta \arrow[d, dashed, "g_{\beta,\beta+1}"'] \\
L_{\beta+1} \arrow[r, tail, dashed, "s_{\beta+1}"'] & JS_{\beta+1} \arrow[r, shift left, dashed, "u_{\beta+1}"] \arrow[r, shift right, dashed, "v_{\beta+1}"'] & M_{\beta+1} \arrow[r, tail, dashed, "t_{\beta+1}"'] & JT_{\beta+1}
\end{tikzcd}$$

as follows: take the pushout $\tilde{S}_{\beta+1}$ of s_β and $d_{\beta,\beta+1}$ and call the two induced maps $\tilde{s}_{\beta+1}\colon L_{\beta+1} \rightarrowtail \tilde{S}_{\beta+1}$ and $\tilde{e}_{\beta,\beta+1}\colon S_\beta \to \tilde{S}_{\beta+1}$, where $\tilde{s}_{\beta+1}$ is a regular monomorphism because $\mathfrak{R}^\dagger\mathcal{D}$ is coregular. Consider now a regular monomorphism $r_{\beta+1}\colon \tilde{S}_{\beta+1} \rightarrowtail JS_{\beta+1}$ into $\mathcal{D}$; it is then enough to consider $s_{\beta+1} := r_{\beta+1} \circ \tilde{s}_{\beta+1}$, which is still a regular monomorphism, and $e_{\beta,\beta+1} := r_{\beta+1} \circ \tilde{e}_{\beta,\beta+1}$. We define $(u_{\beta+1}, v_{\beta+1})$ as the cokernel pair of $s_{\beta+1}$, while $f_{\beta,\beta+1}$ is induced by the universal property of u_β and v_β. Finally define $t_{\beta+1}$ and $g_{\beta,\beta+1}$ as in the first step. This gives a presentation for $L_{\beta+1}$ which is compatible with the chain already defined.

If $\lambda < \alpha$ is a limit ordinal, we take as presentation associated to L_λ the one obtained as the colimit of the presentations defined so far, in other words we consider $x_\lambda := \operatorname{colim}_{\beta<\lambda}(x_\beta)$ for $x = s, u, v, t$. It is easy to check that s_λ and t_λ are still regular monomorphisms, indeed every object of $\mathcal{D}$ is injective with respect to them: given an arrow $f\colon L_\lambda \to T \in \mathcal{D}$, define by induction compatible arrows $\tilde{f}_\beta\colon S_\beta \to T$, for $\beta < \lambda$, such that $f \circ d_{\beta,\lambda} = \tilde{f}_\beta \circ s_\beta$; then the colimit of the $\tilde{f}_\beta$ induces a factorization of f through s_λ. As a consequence, since in addition cokernel pairs commute with filtered colimits, the one just defined is a presentation for L_λ. Moreover, by construction, the colimit cocones induce maps $e_{\beta,\lambda}$, $f_{\beta,\lambda}$ and $g_{\beta,\lambda}$ which are compatible with the chains defined so far. We can then take the colimit of these chains:

$$\operatorname{colim}_{\beta<\alpha}(L_\beta) \overset{s}{\rightarrowtail} J\operatorname{colim}_{\beta<\alpha}(S_\beta) \overset{u}{\underset{v}{\rightrightarrows}} \operatorname{colim}_{\beta<\alpha}(M_\beta) \overset{t}{\rightarrowtail} J\operatorname{colim}_{\beta<\alpha}(T_\beta)$$

By the previous arguments this is a presentation for $\operatorname{colim}_{\beta<\alpha}(L_\beta)$. Now consider the functor $\mathfrak{R}^\dagger\mathcal{D}(F,-)$, this preserves filtered colimits from $\mathcal{D}$ as well as the equalizers in such presentations. It follows that:

$$\begin{aligned}
\mathfrak{R}^\dagger\mathcal{D}(F, \operatorname{colim}_{\beta<\alpha}L_\beta) &\cong \operatorname{eq}(\mathfrak{R}^\dagger\mathcal{D}(F, t\circ u), \mathfrak{R}^\dagger\mathcal{D}(F, t\circ v)) \\
&\cong \operatorname{eq}(\operatorname{colim}_{\beta<\alpha}\mathfrak{R}^\dagger\mathcal{D}(F, t_\beta\circ u_\beta), \operatorname{colim}_{\beta<\alpha}\mathfrak{R}^\dagger\mathcal{D}(F, t_\beta\circ v_\beta)) \\
&\cong \operatorname{colim}_{\beta<\alpha}(\operatorname{eq}(\mathfrak{R}^\dagger\mathcal{D}(F, t_\beta\circ u_\beta), \mathfrak{R}^\dagger\mathcal{D}(F, t_\beta\circ v_\beta))) \\
&\cong \operatorname{colim}_{\beta<\alpha}\mathfrak{R}^\dagger\mathcal{D}(F, L_\beta)
\end{aligned}$$

as desired. Therefore $F \in (\mathfrak{R}^\dagger\mathcal{D})_f$.

Finally the fact that $\operatorname{Def}(\mathcal{D},\mathcal{V})$, and hence $(\mathfrak{R}^\dagger\mathcal{D})_f$, is small can be deduced as follows. Let α be a regular cardinal for which $\mathcal{D}$ is α-accessible; then since $\mathcal{D}$ also has products it follows that it is an α-definable category, and thus the $\mathcal{V}$-category α-$\operatorname{Def}(\mathcal{D},\mathcal{V})$ spanned by the α-definable functors is small (thanks to the infinitary version of [64, Theorem 9.7]). In conclusion $\operatorname{Def}(\mathcal{D},\mathcal{V})$ is small because it is contained in α-$\operatorname{Def}(\mathcal{D},\mathcal{V})$. $\square$

The point of the following result is that it allows us to test the definability of a $\mathcal{V}$-category $\mathcal{D}$ on a particular locally finitely presentable $\mathcal{V}$-category $\mathcal{L}$ that is completely determined by $\mathcal{D}$ itself (unlike in the definition).

Corollary A.3.2. *Let $\mathcal{D}$ be an accessible $\mathcal{V}$-category with products and filtered colimits; let $\mathcal{L}$ be the closure of $(\mathfrak{R}^\dagger\mathcal{D})_f$ in $\mathfrak{R}^\dagger\mathcal{D}$ under filtered colimits (this is a locally finitely presentable $\mathcal{V}$-category). Then $\mathcal{D}$ is a definable $\mathcal{V}$-category if and only if it is a definable subcategory of $\mathcal{L}$.*

Proof. If $\mathcal{D}$ is a definable subcategory of $\mathcal{L}$ then it is by definition a definable $\mathcal{V}$-category. Conversely, if $\mathcal{D}$ is definable then $\mathcal{D} \simeq \mathrm{Reg}(\mathcal{C},\mathcal{V})$, with $\mathcal{C} = \mathrm{Def}(\mathcal{C},\mathcal{V})$, and is a definable subcategory of $\mathrm{Lex}(\mathcal{C},\mathcal{V})$. By the previous Proposition $(\mathfrak{R}^\dagger\mathcal{D})_f \simeq \mathrm{Def}(\mathcal{D},\mathcal{V})^{op} = \mathcal{C}^{op}$; thus $\mathcal{L} \simeq \mathrm{Lex}(\mathcal{C},\mathcal{V})$ and the thesis follows. □

We can now give our interpretation of Makkai's notion of pp-object into a more general setting: [76]:

Definition A.3.3. Let $\mathcal{D}$ be an accessible $\mathcal{V}$-category with products and filtered colimits. An object $X \in \mathcal{D}$ is called a *pp-object* if $\mathcal{D}(X,-)$ is the limit in $[\mathcal{D},\mathcal{V}]$ of a co-directed diagram whose limit cone consists of regular epimorphisms $\mathcal{D}(X,-) \twoheadrightarrow F$ with codomain is a definable functor.

When $\mathcal{V} = \mathbf{Set}$ and $\mathcal{D} = \mathrm{Reg}(\mathcal{C},\mathbf{Set})$, for an exact category $\mathcal{C}$, Makkai defines the pp-objects of $\mathcal{D}$ in [76, Definition 4.2] and gives a more categorical description in [76, Definition 4.2']. It is easy to see that the latter translates into the following: $X \in \mathcal{D}$ is a pp-object if and only if $X \in \mathrm{Lex}(\mathcal{C},\mathbf{Set})$ is a filtered colimit of representable functors whose colimit cocone consists of weak reflections into $\mathcal{D}$. Using this, the embedding of $\mathcal{D}$ and $\mathrm{Lex}(\mathcal{C},\mathbf{Set})$ in $[\mathcal{D},\mathbf{Set}]^{op}$, that $\mathcal{C} \simeq \mathrm{Def}(\mathcal{D},\mathbf{Set})$ (see [64]), and that weak reflections into $\mathcal{D}$ are just regular epimorphisms in $[\mathcal{D},\mathcal{V}]$, we recover Makkai's notion of pp-object from ours.

Both for $\mathcal{V} = \mathbf{Set}$ and $\mathcal{V} = \mathbf{Ab}$ the pp-objects of a definable category generate it under filtered colimits; this is proved, using model theory, in [76, Proposition 4.4] for $\mathcal{V} = \mathbf{Set}$ and in [87, Theorem 4.9] for $\mathcal{V} = \mathbf{Ab}$.

Consider now a generalized notion of purity:

Definition A.3.4. Let $\mathcal{K}$ be a cocomplete $\mathcal{V}$-category, we say that a morphism $f\colon X \to Y$ in $\mathcal{K}$ is *pure* if it lies in the closure $\mathcal{L}$ of $\mathcal{K}_f$ under filtered colimits and is a pure morphism in $\mathcal{L}$ in the usual sense [1, Definition 2.27].

Then we can conclude the section with the following result:

Theorem A.3.5. *Let $\mathcal{V} = \mathbf{Set}$ or $\mathcal{V} = \mathbf{Ab}$. A $\mathcal{V}$-category $\mathcal{D}$ is definable if and only if:*

1. *$\mathcal{D}$ is accessible with products and filtered colimits;*
2. *the pp-objects generate $\mathcal{D}$ under filtered colimits;*
3. *$\mathcal{D}$ is closed in $\mathfrak{R}^\dagger\mathcal{D}$ under pure subobjects.*

Proof. Let $\mathcal{L}$ be the closure of $(\mathfrak{R}^\dagger\mathcal{D})_f$ in $\mathfrak{R}^\dagger\mathcal{D}$ under filtered colimits. If $\mathcal{D}$ is definable then it is a definable subcategory of $\mathcal{L}$ (see the proof of the corollary above) and is accessible. Thus (1) and (3) hold, point (2) holds by the result of Makkai and Prest mentioned above. Conversely, assume that $\mathcal{D}$ satisfies the three conditions. Since the pp-objects generate

$\mathcal{D}$ under filtered colimits, and these are contained in $\mathcal{L}$, the whole category $\mathcal{D}$ is a full subcategory of $\mathcal{L}$. By construction $\mathcal{D}$ is closed in $\mathcal{L}$ under products and filtered colimits. Thus, if moreover $\mathcal{D}$ is closed in $\mathcal{L}$ under pure subobjects, it is a finite injectivity class in $\mathcal{L}$ [92, Theorem 2.2] and hence a definable subcategory of it. $\square$

A.4 An adjunction for $\mathcal{V}$-topoi

We have seen, as part of the duality in Theorem 6.3.4, that if $\mathcal{A}$ is a finitely accessible category then $\aleph_0$-Acc$(\mathcal{A}, \mathbf{Set})$ is equivalent to the presheaf category $[\mathcal{A}_f, \mathbf{Set}]$. Conversely, given any category of presheaves $[\mathcal{C}, \mathbf{Set}]$ the category $\mathcal{P}$-Lex$([\mathcal{C}, \mathbf{Set}], \mathbf{Set}) \cong$ Flat$(\mathcal{C}^{op}, \mathbf{Set})$ is finitely accessible. This is actually part of a wider adjunction between the 2-category of accessible categories with filtered colimits and that of Grothendieck topoi (see [34]). We shall prove in this section that such an adjunction can be extended to the enriched setting by considering accessible $\mathcal{V}$-categories with flat (or just filtered) colimits and the notion of $\mathcal{V}$-topos introduced below.

We assume for simplicity that $\mathcal{V}$ is locally finitely presentable as a closed category; nonetheless, everything can be carried out as usual in the infinitary case.

Definition A.4.1. We say that a $\mathcal{V}$-category $\mathcal{E}$ is a *$\mathcal{V}$-topos* if it is locally presentable and a left exact localization of a presheaf $\mathcal{V}$-category $[\mathcal{C}^{op}, \mathcal{V}]$. In other words, $\mathcal{E}$ is a $\mathcal{V}$-topos if there exists a fully faithful and accessible $J\colon \mathcal{E} \to [\mathcal{C}^{op}, \mathcal{V}]$ which has a lex left adjoint.

By [45, Proposition 2.6] $\mathcal{C}$ can be chosen to be a full subcategory of $\mathcal{E}$. Morphisms between them are cocontinuous lex $\mathcal{V}$-functors. Denote by $\mathcal{V}$-**Top** the 2-category of $\mathcal{V}$-topoi, morphisms between them, and $\mathcal{V}$-natural transformations.

Remark A.4.2. The one above is not the standard definition of $\mathcal{V}$-topos: local presentability is usually omitted. We need to add it for the proofs below to work. When $\mathcal{V}_0$ is a Grothendieck topos then the additional condition is automatic (if $\mathcal{E}$ is $\mathcal{V}$-topos then $\mathcal{E}_0$ is a Grothendieck topos and hence locally presentable. This is enough to guarantee that $\mathcal{E}$ is also locally presentable as a $\mathcal{V}$-category.) We do not know whether it is true in general that every left exact localization of a presheaf $\mathcal{V}$-category is locally presentable.

Denote by $\mathbf{cAcc}^{\aleph_0}$ the 2-category of conically accessible $\mathcal{V}$-categories with filtered colimits, filtered-colimit-preserving $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. Similarly we denote by $\mathbf{Acc}^{\aleph_0}$ the 2-category of accessible $\mathcal{V}$-categories with flat colimits, flat-colimit-preserving $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations; then we have a faithful inclusion $Z\colon \mathbf{Acc}^{\aleph_0} \hookrightarrow \mathbf{cAcc}^{\aleph_0}$.

Then the following can be seen as an extension of the Scott adjunction [34] to the enriched context.

Proposition A.4.3. *The following*

$$\mathrm{cAcc}(-,\mathcal{V})^{\aleph_0} \colon \mathbf{cAcc}^{\aleph_0} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathcal{V}\text{-}\mathbf{Top}^{op} \colon \mathcal{V}\text{-Top}(-,\mathcal{V})$$

defines a 2-adjunction.

Proof. The only non-trivial part is to show that the 2-functors are well defined, the universal property that defines the adjunction is in fact an easy consequence of the commutativity of filtered colimits with finite limits in $\mathcal{V}$. Let $\mathcal{E}$ be a $\mathcal{V}$-topos; then $\mathcal{V}$-$\mathbf{Top}(\mathcal{E}, \mathcal{V})$ is closed in $[\mathcal{E}, \mathcal{V}]$ under filtered colimits, so that we only need to prove that it is accessible. By hypothesis $\mathcal{E}$ is locally presentable and we can take $H\colon \mathcal{C} \hookrightarrow \mathcal{E}$ small such that

$J := \mathcal{E}(H,1)\colon \mathcal{E} \to \mathcal{PC} = [\mathcal{C}^{op},\mathcal{V}]$ has a lex left adjoint L. Consider now α such that $\mathcal{E}$ is locally α-presentable; we will prove that the square below is a bipullback.

$$\begin{tikzcd} \mathcal{V}\text{-}\mathbf{Top}(\mathcal{E},\mathcal{V}) \arrow[r, "-\circ L"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V}\text{-}\mathbf{Top}(\mathcal{PC},\mathcal{V}) \arrow[d, hook] \\ \alpha\text{-}\mathbf{Acc}(\mathcal{E},\mathcal{V}) \arrow[r, "-\circ L"'] & \alpha\text{-}\mathbf{Acc}(\mathcal{PC},\mathcal{V}) \end{tikzcd}$$

Now, we have the equivalence $\mathcal{V}\text{-}\mathbf{Top}(\mathcal{PC},\mathcal{V}) \simeq \mathrm{Flat}(\mathcal{C},\mathcal{V})$, while $\alpha\text{-}\mathbf{Acc}(\mathcal{E},\mathcal{V}) \simeq [\mathcal{E}_\alpha,\mathcal{V}]$, and $\alpha\text{-}\mathbf{Acc}(\mathcal{PC},\mathcal{V}) \simeq [(\mathcal{PC})_\alpha,\mathcal{V}]$ since $\mathcal{E}$ and $\mathcal{PC}$ are locally α-presentable (note that for a locally α-presentable category there is no difference between $\alpha\text{-}\mathbf{Acc}(\mathcal{E},\mathcal{V})$ and $\alpha\text{-}\mathbf{cAcc}(\mathcal{E},\mathcal{V})$). Now, to prove that the square is a bipullback it is enough to notice that a $\mathcal{V}$-functor $F\colon \mathcal{E} \to \mathcal{V}$ is lex-cocontinuous if and only if FL is. One direction is clear (since L is lex-cocontinuous), for the other assume that FL is lex-cocontinuous, then $F \cong FLJ$ is lex because J preserves all limits, moreover for any diagram $D\colon \mathcal{D} \to \mathcal{E}$ and weight $M\colon \mathcal{D}^{op} \to \mathcal{V}$ we obtain:

$$\begin{aligned} F(M * D) &\cong F(M * LJD) \\ &\cong FL(M * JD) \\ &\cong M * FLJD \\ &\cong M * FD \end{aligned}$$

so that F is cocontinuous. It follows that $\mathcal{V}\text{-}\mathbf{Top}(\mathcal{E},\mathcal{V})$ can be seen as a bipullback of accessible $\mathcal{V}$-categories along accessible $\mathcal{V}$-functors; thus it is itself accessible and in particular conically accessible.

Let now $\mathcal{A}$ be a conically accessible $\mathcal{V}$-category with filtered colimits and consider α such that $\mathcal{A}$ is conically α-accessible. Denote by $J\colon \mathbf{cAcc}^{\aleph_0}(\mathcal{A},\mathcal{V}) \hookrightarrow \alpha\text{-}\mathbf{cAcc}(\mathcal{A},\mathcal{V}) \simeq [\mathcal{A}_\alpha,\mathcal{V}]$ the inclusion (this preserves all colimits and all finite limits). Now define $\mathcal{B} := \mathrm{Ind}(\mathcal{A}_\alpha)$, so that $\mathcal{B}$ is conically finitely accessible; then we have induced $\mathcal{V}$-functors $S\colon \mathcal{A} \to \mathcal{B}$, which preserves α-filtered colimits and extends the inclusion of $\mathcal{A}_\alpha$ in $\mathcal{B}$ to $\mathcal{A}$, and $T\colon \mathcal{B} \to \mathcal{A}$, which preserves filtered colimits and extends the inclusion of $\mathcal{A}_\alpha$ in $\mathcal{A}$ to $\mathcal{B}$. By construction they satisfy $TS \cong \mathrm{id}_\mathcal{A}$. Arguing as in the chain of isomorphisms above it follows that a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$ preserves filtered colimits if and only if FT preserves them. As a consequence we can see $\mathbf{cAcc}(\mathcal{A},\mathcal{V})^{\aleph_0}$ as the bipullback below.

$$\begin{tikzcd} \mathbf{cAcc}^{\aleph_0}(\mathcal{A},\mathcal{V}) \arrow[r, "-\circ T"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{cAcc}^{\aleph_0}(\mathcal{B},\mathcal{V}) \arrow[d, hook] \\ \alpha\text{-}\mathbf{cAcc}(\mathcal{A},\mathcal{V}) \arrow[r, "-\circ T"'] & \alpha\text{-}\mathbf{cAcc}(\mathcal{B},\mathcal{V}) \end{tikzcd}$$

Since $\mathbf{cAcc}^{\aleph_0}(\mathcal{B},\mathcal{V}) \simeq [\mathcal{A}_\alpha,\mathcal{V}] \simeq \alpha\text{-}\mathbf{cAcc}(\mathcal{A},\mathcal{V})$ and $\alpha\text{-}\mathbf{cAcc}(\mathcal{B},\mathcal{V}) \simeq [\mathcal{B}_\alpha,\mathcal{V}]$ are locally presentable and the functors between them are accessible, it follows that $\mathcal{E} := \mathbf{Acc}(A,\mathcal{V})^{\aleph_0}$ is accessible as well. Moreover since $\mathcal{E}$ is closed in $[\mathcal{A},\mathcal{V}]$ under colimits, it is also locally presentable. Now let $H\colon \mathcal{C} \hookrightarrow \mathcal{E}$ be a small dense full subcategory of $\mathcal{E}$ closed under finite weighted limits (a small dense subcategory exists by local presentability, then take the closure of this in $\mathcal{E}$ under finite limits). The fully faithful functor $\mathcal{E}(H,1)\colon \mathcal{E} \hookrightarrow [\mathcal{C}^{op},\mathcal{V}]$ has a left adjoint given by $\mathrm{Lan}_Y H\colon [\mathcal{C}^{op},\mathcal{V}] \to \mathcal{E}$; to conclude that $\mathcal{E}$ is a $\mathcal{V}$-topos then it is enough to prove that $\mathrm{Lan}_Y H$ is lex. Consider the composite $J(\mathrm{Lan}_Y H)$, since this is cocontinuous and it restricts to JH it follows that $J(\mathrm{Lan}_Y H) \cong \mathrm{Lan}_Y(JH)$. But $\mathrm{Lan}_Y(JH)$ is lex by [45, Proposition 2.4(4)] (since $[\mathcal{A}_\alpha, \mathcal{V}]$ is a $\mathcal{V}$-topos); thus also $\mathrm{Lan}_Y H$ is lex, because J is fully faithful and preserves finite limits. As a consequence $\mathcal{E}$ is a left exact localization of $[\mathcal{C}^{op}, \mathcal{V}]$ and hence a $\mathcal{V}$-topos. □

The proof shows that the adjunction above restricts to the full image of the inclusion $Z\colon \mathbf{Acc}^{\aleph_0} \hookrightarrow \mathbf{cAcc}^{\aleph_0}$, so that on the left-hand-side one can have the 2-category of accessible $\mathcal{V}$-categories with filtered colimits, filtered-colimit-preserving $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations.

We can also show that the adjunction still holds if we consider $\mathbf{Acc}^{\aleph_0}$ instead. Note that this is not a restriction of the adjunction above, since for a general accessible $\mathcal{V}$-category $\mathcal{A}$ with filtered colimits we only have an inclusion $\mathbf{Acc}(\mathcal{A}, \mathcal{V})^{\aleph_0} \subseteq \mathbf{cAcc}(\mathcal{A}, \mathcal{V})^{\aleph_0}$.

Proposition A.4.4. *The following*

$$\mathrm{Acc}(-, \mathcal{V})^{\aleph_0} \colon \mathbf{Acc}^{\aleph_0} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathcal{V}\text{-}\mathbf{Top}^{op} : \mathcal{V}\text{-}\mathbf{Top}(-, \mathcal{V})$$

defines a 2-adjunction.

Proof. Again it is enough to show that the 2-functors involved are well defined. Given a $\mathcal{V}$-topos $\mathcal{E}$, the proof above shows that $\mathcal{V}\text{-}\mathbf{Top}(\mathcal{E}, \mathcal{V})$ is an accessible $\mathcal{V}$-category moreover it has flat colimits since this commute in $\mathcal{V}$ with colimits and finite weighted limits. For the other direction one can argue as above, just replace (α-)filtered colimits by (α-)flat colimits everywhere. □

Remark A.4.5. This can be further generalized in two different directions by considering companions and weakly sound classes:

1. Given a class of weights Ψ and a companion $\mathfrak{C}$ for Ψ, replace $\mathcal{V}$-**Top** with the 2-category whose objects are the accessible and left exact localizations of finitely complete $\mathcal{V}$-categories of the form $\mathfrak{C}\mathcal{B}$, for some small $\mathcal{B}$. Morphisms between them are lex and $\mathfrak{C}$-cocontinuous $\mathcal{V}$-functors, and 2-cells as 2-natural transformations. On the other hand, instead of $\mathbf{cAcc}^{\aleph_0}$, consider the 2-category of conically accessible categories with Ψ-limits and filtered colimits, Ψ-continuous and finitary $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. Then we recover the adjunction above by taking $\Psi = \emptyset$ and $\mathfrak{C} = \mathcal{P}$.

2. Given a locally small weakly sound class of weights Φ. On one hand, we could replace the $\mathcal{V}$-topoi above with Φ-*topoi*: accessible and reflective subcategories of a presheaf $\mathcal{V}$-category with a Φ-continuous left adjoint; morphisms between them are cocontinuous and Φ-continuous $\mathcal{V}$-functors. On the other hand one considers accessible (or conically accessible) $\mathcal{V}$-categories with Φ-flat colimits and $\mathcal{V}$-functors which preserves them. Then we recover the adjunction above by taking Φ to be the class of finite weights.

Bibliography

[1] J. Adámek and J. Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[2] J. Adámek and J. Rosický. On sifted colimits and generalized varieties. *Theory and Applications of Categories*, 8(3):33–53, 2001.

[3] J. Adámek and J. Rosickỳ. Approximate injectivity and smallness in metric-enriched categories. *Journal of Pure and Applied Algebra*, 226(6):106974, 2022.

[4] J. Adámek, F. Borceux, S. Lack, and J. Rosický. A classification of accessible categories. *Journal of Pure and Applied Algebra*, 175(1):7 – 30, 2002. Special Volume celebrating the 70th birthday of Professor Max Kelly.

[5] J. Adámek, V. Koubek, and J. Velebil. A duality between infinitary varieties and algebraic theories. *Commentationes Mathematicae Universitatis Carolinae*, 41(3):529–541, 2000.

[6] P. Ageron. The logic of structures. *Journal of Pure and Applied Algebra*, 79(1):15–34, 1992.

[7] M. Albert and G. Kelly. The closure of a class of colimits. *Journal of Pure and Applied Algebra*, 51(1):1–17, 1988.

[8] M. Barr and C. Wells. *Toposes, Triples, and Theories.* Springer-Verlag, 2000.

[9] T. Beke. Sheafifiable homotopy model categories. *Mathematical Proceedings of the Cambridge Philosophical Society*, 129(3):447–475, 2000.

[10] G. M. Bergman. Actions of boolean rings on sets. *Algebra Universalis*, 28:153–187, 1991.

[11] R. Betti, A. Carboni, R. Street, and R. Walters. Variation through enrichment. *Journal of Pure and Applied Algebra*, 29(2):109–127, 1983.

[12] R. Betti and S. Kasangian. Tree automata and enriched category theory. 1985.

[13] G. Bird, G. Kelly, A. Power, and R. Street. Flexible limits for 2-categories. *Journal of Pure and Applied Algebra*, 61(1):1 – 27, 1989.

[14] G. J. Bird. *Limits in 2-categories of locally-presented categories.* PhD thesis, University of Sydney, 1984; published as a Sydney Category Seminar Report.

[15] A. Blass and A. Ščedrov. Classifying topoi and finite forcing. *Journal of Pure and Applied Algebra*, 28(2):111–140, 1983.

[16] F. Borceux. *Handbook of Categorical Algebra: Volume 2, Categories and Structures.* Cambridge University Press, 1994.

[17] F. Borceux and C. Quinteiro. Enriched accessible categories. *Bulletin of the Australian Mathematical Society*, 54(3):489–501, 1996.

[18] F. Borceux, C. Quinteiro, and J. Rosický. A theory of enriched sketches. *Theory and Applications of Categories*, 4:47–72, 1998.

[19] F. Borceux and J. Rosický. On filtered weighted colimits of presheaves. *Cahiers de topologie et géométrie différentielle catégoriques*, 49(4):243–266, 2008.

[20] J. Bourke. Accessible aspects of 2-category theory. *Journal of Pure and Applied Algebra*, 225(3):106519, 2021.

[21] J. Bourke and R. Garner. Two-dimensional regularity and exactness. *Journal of Pure and Applied Algebra*, 218(7):1346–1371, 2014.

[22] J. Bourke and S. Lack. Accessible ∞-cosmoi. *arXiv preprint arXiv:2111.00147*, 2021.

[23] J. Bourke, S. Lack, and L. Vokřínek. Adjoint functor theorems for homotopically enriched categories. *arXiv preprint arXiv:2006.07843*, 2020.

[24] A. Bruguières, S. Lack, and A. Virelizier. Hopf monads on monoidal categories. *Advances in Mathematics*, 227(2):745–800, 2011.

[25] M. Bunge and J. Funk. On a bicomma object condition for KZ-doctrines. *Journal of Pure and Applied Algebra*, 143(1-3):69–105, 1999.

[26] J. Bénabou. Catégories avec multiplication. *C.R. Acad. Sci. Paris*, 256:1887 – 1890, 1963.

[27] A. Campbell. The model category of algebraically cofibrant 2-categories. *see https://conferences.inf.ed.ac.uk/ct2019/slides/13.pdf*, In preparation.

[28] O. Caramello. Yoneda representations of flat functors and classifying toposes. *Theory and Applications of Categories*, 26(21):538–553, 2012.

[29] A. Carboni and E. M. Vitale. Regular and exact completions. *Journal of Pure and Applied Algebra*, 125(1-3):79–116, 1998.

[30] R. Chen. On sifted colimits in the presence of pullbacks. *Theory and Applications of Categories*, 37(35):1176–1193, 2021.

[31] J.-M. Cordier and T. Porter. Homotopy coherent category theory. *Transactions of the American Mathematical Society*, 349(1):1–54, 1997.

[32] B. Day. Density presentations of functors. *Bulletin of the Australian Mathematical Society*, 16:427–448, 1977.

[33] B. J. Day and S. Lack. Limits of small functors. *Journal of Pure and Applied Algebra*, 210(3):651 – 663, 2007.

[34] I. Di Liberti. General facts on the Scott adjunction. *Applied Categorical Structures*, 30:569–591, 2022.

[35] Y. Diers. Catégories localement multiprésentables. *Archiv der Mathematik*, 34:344–356, 1980.

[36] D. Dugger. Combinatorial model categories have presentations. *Advances in Mathematics*, 164(1):177–201, 2001.

[37] D. Dugger. Universal homotopy theories. *Advances in Mathematics*, 164(1):144–176, 2001.

[38] C. Ehresmann. Esquisses et types des structures algébriques. *Bul. Inst. Politeh. Iași, N. Ser.*, 14(1-2):1–14, 1968.

[39] S. Eilenberg and G. M. Kelly. Closed categories. In *Proceedings of the Conference on Categorical Algebra*, pages 421–562. Springer, 1966.

[40] A. D. Elmendorf and M. A. Mandell. Permutative categories, multicategories and algebraic k–theory. *Algebraic & Geometric Topology*, 9(4):2391–2441, 2009.

[41] P. Freyd. Several new concepts: Lucid and concordant functors, pre-limits, pre-completeness, the continuous and concordant completions of categories. *Hilton P.J. (eds) Category Theory, Homology Theory and Their Applications III. Lecture Notes in Mathematics*, 99, 1969.

[42] P. Freyd. Aspects of topoi. *Bulletin of the Australian Mathematical Society*, 7:1–76, 1972.

[43] P. J. Freyd. *Abelian categories*, volume 1964. Harper & Row New York, 1964.

[44] P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien.* Lecture Notes in Mathematics, Vol. 221. Springer-Verlag, Berlin-New York, 1971.

[45] R. Garner and S. Lack. Lex colimits. *Journal of Pure and Applied Algebra*, 216(6):1372–1396, 2012.

[46] M. Gerner. *Le lien entre la logique et la géométrie via les esquisses.* Ph.D. thesis, Denis Diderot University - Paris 7, 1994.

[47] M. Grandis and R. Paré. Limits in double categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 40(3):162–220, 1999.

[48] R. Guitart and C. Lair. Existence de diagrammes localement libres. *Diagrammes*, 6:GL1–GL13, 1981.

[49] H. Hu. *Dualities for accessible categories.* Ph.D. thesis, Mc Gill University, 1992.

[50] H. Hu and W. Tholen. Quasi-coproducts and accessible categories with wide pullbacks. *Applied Categorical Structures*, 4:387–402, 1996.

[51] M. Hyland and J. Power. Discrete Lawvere theories and computational effects. *Theoretical Computer Science*, 366(1-2):144–162, 2006.

[52] S. Johnson. Small Cauchy completions. *Journal of Pure and Applied Algebra*, 62(1):35–45, 1989.

[53] B. Keller. Deriving DG categories. In *Annales scientifiques de l'Ecole normale supérieure*, volume 27, pages 63–102, 1994.

[54] G. Kelly and S. Lack. Finite-product-preserving functors, Kan extensions, and strongly-finitary 2-monads. *Applied Categorical Structures*, 1(1):85–94, 1993.

[55] G. M. Kelly. Doctrinal adjunction. In G. M. Kelly, editor, *Category Seminar*, pages 257–280, Berlin, Heidelberg, 1974. Springer Berlin Heidelberg.

[56] G. M. Kelly. *Basic Concepts of Enriched Category Theory.* Cambridge University Press, 1982.

[57] G. M. Kelly. Structures defined by finite limits in the enriched context. I. *Cahiers de Topologie et Géométrie Différentielle*, 23(1):3–42, 1982. Third Colloquium on Categories, Part VI (Amiens, 1980).

[58] G. M. Kelly and S. Lack. V-Cat is locally presentable or locally bounded if V is so. *Theory and Applications of Categories*, 8:555–575, 2001.

[59] G. M. Kelly and V. Schmitt. Notes on enriched categories with colimits of some class. *Theory and Applications of Categories*, 14(17):399–423, 2005.

[60] A. Kuber and J. Rosický. Definable categories. *Journal of Pure and Applied Algebra*, 222(5):1006–1025, 2018.

[61] S. Lack. A 2-categories companion. In *Towards higher categories*, pages 105–191. Springer, 2010.

[62] S. Lack and J. Rosický. Notions of Lawvere theory. *Applied Categorical Structures*, 19:363–391, 2011.

[63] S. Lack and J. Rosický. Enriched weakness. *Journal of Pure and Applied Algebra*, 216:1807–1822, 2012.

[64] S. Lack and G. Tendas. Enriched regular theories. *Journal of Pure and Applied Algebra*, 224(6):106268, 2020.

[65] S. Lack and G. Tendas. Flat vs. filtered colimits in the enriched context. *Advances in Mathematics*, 404:108381, 2022.

[66] S. Lack and G. Tendas. Virtual concepts in the theory of accessible categories. *Journal of Pure and Applied Algebra*, 227(2):107196, 2023.

[67] C. Lair. Catégories modelables et catégories esquissables. *Diagrammes*, 6:L1–L20, 1981.

[68] F. Lamarche. *Modelling polymorphism with categories.* Ph.D. thesis, Mc Gill University, 1989.

[69] S. M. Lane. Categorical algebra. *Bulletin of the American Mathematical Society*, 71(1):40 – 106, 1965.

[70] F. W. Lawvere. Metric spaces, generalized logic, and closed categories. *Rendiconti del seminario matematico e fisico di Milano*, 43(1):135–166, 1973.

[71] D. Lazard. Sur les modules plats. *Comptes rendus de l'Académie des Sciences, Paris*, 258:6313–6316, 1964.

[72] M. Lieberman and J. Rosický. Metric abstract elementary classes as accessible categories. *The Journal of Symbolic Logic*, 82(3):1022–1040, 2017.

[73] F. Linton. Autonomous categories and duality of functors. *Journal of Algebra*, 2(3):315–349, 1965.

[74] F. E. Linton. An outline of functorial semantics. In *Seminar on triples and categorical homology theory*, pages 7–52. Springer, 1969.

[75] J. Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 2009.

[76] M. Makkai. A theorem on Barr-exact categories, with an infinitary generalization. *Annals of Pure and Applied Logic 47*, pages 225–268, 1990.

[77] M. Makkai and R. Paré. *Accessible Categories: The Foundations of Categorical Model Theory*, volume 104. Springer, 1989.

[78] A. Neeman. *Triangulated categories.* Number 148. Princeton University Press, 2001.

[79] B. Nikolić. Cauchy completeness and causal spaces. *arXiv preprint arXiv:1712.00560*, 2017.

[80] B. Nikolić, R. Street, and G. Tendas. Cauchy completeness for DG-categories. *Theory and Applications of Categories*, 37(28):940–963, 2021.

[81] U. Oberst and H. Rohrl. Flat and coherent functors. *Journal of Algebra*, 14(1):91 – 105, 1970.

[82] R. Paré. Simply connected limits. *Canadian Journal of Mathematics*, 42(4):731–746, 1990.

[83] M. C. Pedicchio and W. Tholen. *Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2003.

[84] J. W. Pelletier. Dual functors and the Radon-Nikodym property in the category of Banach spaces. *Journal of the Australian Mathematical Society*, 27(4):479–494, 1979.

[85] A. J. Power. Enriched Lawvere theories. *Theory and Applications of Categories*, 6(7):83–93, 1999.

[86] M. Prest. *Definable Additive Categories: Purity and Model Theory.* Memoirs of the American Mathematical Society, 2011.

[87] M. Prest. Strictly atomic modules in definable categories. *arXiv preprint arXiv:2109.13750*, 2021.

[88] M. Prest and R. Rajani. Structure sheaves of definable additive categories. *Journal of Pure and Applied Algebra*, 214:1370–1383, 2010.

[89] E. Riehl and D. Verity. *Elements of ∞-Category Theory.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2022.

[90] J. Rosický. *Representace konkrétních kategorií.* Doctoral thesis, University of Brno, 1983.

[91] J. Rosický. Cartesian closed exact completions. *Journal of Pure and Applied Algebra*, 142:261–270, 1999.

[92] J. Rosický, J. Adámek, and F. Borceux. More on injectivity in locally presentable categories. *Theory and Applications of Categories*, 10(7):148–161, 2002.

[93] J. Rosický and W. Tholen. Accessibility and the solution set condition. *Journal of Pure and Applied Algebra*, 98:189–208, 1995.

[94] J.-P. Serre. Géométrie algébrique et géométrie analytique. *Annales de l'Institut Fourier*, 6:1–42, 1956.

[95] R. Street. Absolute colimits in enriched categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 24(4):377–379, 1983.

[96] R. Street. Categorical structures. *Handbook of algebra*, 1:529–577, 1996.

[97] R. Street. *Quantum Groups: A Path to Current Algebra.* Australian Mathematical Society Lecture Series. Cambridge University Press, 2007.

[98] P. Taylor. Locally finitely poly-presentable categories. *preprint, https://paultaylor.eu/stable/locfpp.pdf*, 1990.

[99] P. Taylor. The trace factorisation of stable functors. *Preprint, Imperial College*, 1990.

[100] G. Tendas. Enriched regular theories. *MRes Thesis, Macquarie University, http://hdl.handle.net/1959.14/1270426*, 2019.

[101] G. Tendas. On continuity of accessible functors. *Applied Categorical Structures*, 30:930–947, 2022.

[102] W. Tholen. Pro-categories and multiadjoint functors. *Canadian Journal of Mathematics*, 36(1):144–155, 1984.

[103] R. F. Walters. Sheaves and Cauchy-complete categories. *Cahiers de topologie et géométrie différentielle catégoriques*, 22(3):283–286, 1981.