On the global linear Zarankiewicz problem

(joint with Aris Papadopoulos)

Pantelis Eleftheriou

University of Leeds

Logic Seminar, Manchester, December 10, 2025

Theme

- recognise definable algebraic structure from combinatorial data

Theme

- recognise definable algebraic structure from combinatorial data
- linear vs non-linear structure $\mathcal{M} = \langle M, +, \ldots \rangle$

Theme

- recognise definable algebraic structure from combinatorial data
- linear vs non-linear structure $\mathcal{M} = \langle M, +, \dots \rangle$
- $f : X \subseteq M^m \to M^k$ is *linear* (affine) if for every $x, y, x+t, y+t \in X$,

$$f(x+t) - f(x) = f(y+t) - f(y).$$

Theme

- recognise definable algebraic structure from combinatorial data
- linear vs non-linear structure $\mathcal{M} = \langle M, +, \ldots \rangle$
- $f : X \subseteq M^m \to M^k$ is *linear* (affine) if for every $x, y, x + t, y + t \in X$,

$$f(x + t) - f(x) = f(y + t) - f(y).$$

- In $\langle \mathbb{R}, <, + \rangle$, $\langle \mathbb{Z}, <, + \rangle$ every definable function is piecewise linear.

Theme

- recognise definable algebraic structure from combinatorial data
- linear vs non-linear structure $\mathcal{M} = \langle M, +, \dots \rangle$
- $f : X \subseteq M^m \to M^k$ is *linear* (affine) if for every $x, y, x+t, y+t \in X$,

$$f(x+t) - f(x) = f(y+t) - f(y).$$

- In $\langle \mathbb{R}, <, + \rangle$, $\langle \mathbb{Z}, <, + \rangle$ every definable function is piecewise linear.
- In $\langle \mathbb{R}, <, +, \cdot_{\restriction (0,1)^2} \rangle$, $\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$, $\langle \mathbb{R}, <, +, 2^{\mathbb{Z}} \rangle$ not, but still some linearity may be present

Global Zarankiewicz Problem

Global Zarankiewicz Problem

- M set, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$

Global Zarankiewicz Problem

- M set, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$
- $E \subseteq M^{d_1} \times \cdots \times M^{d_r} = M^d$ relation

Global Zarankiewicz Problem

- M set, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$
- $E \subseteq M^{d_1} \times \cdots \times M^{d_r} = M^d$ relation
- B grid $= B_1 \times \cdots \times B_r$, where $B_i \subseteq M^{d_i}$

Global Zarankiewicz Problem

- M set, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$
- $E \subseteq M^{d_1} \times \cdots \times M^{d_r} = M^d$ relation
- B grid $= B_1 \times \cdots \times B_r$, where $B_i \subseteq M^{d_i}$
- If B finite, $n = n_B = \max_i |B_i|$

Global Zarankiewicz Problem

- M set, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$
- $E \subseteq M^{d_1} \times \cdots \times M^{d_r} = M^d$ relation
- B grid $= B_1 \times \cdots \times B_r$, where $B_i \subseteq M^{d_i}$
- If B finite, $n = n_B = \max_i |B_i|$, so $|E \cap B| \leq n^r$

Global Zarankiewicz Problem

- M set, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$
- $E \subseteq M^{d_1} \times \cdots \times M^{d_r} = M^d$ relation
- B grid $= B_1 \times \cdots \times B_r$, where $B_i \subseteq M^{d_i}$
- If B finite, $n = n_B = \max_i |B_i|$, so $|E \cap B| \leq n^r$
- Let $k \in \mathbb{N}$. Then E is *k-free* if it contains no k-grid B (that is, $B = B_1 \times \cdots \times B_r$ with each $|B_i| = k$).

Global Zarankiewicz Problem

- M set, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$
- $E \subseteq M^{d_1} \times \cdots \times M^{d_r} = M^d$ relation
- B grid $= B_1 \times \cdots \times B_r$, where $B_i \subseteq M^{d_i}$
- If B finite, $n = n_B = \max_i |B_i|$, so $|E \cap B| \leq n^r$
- Let $k \in \mathbb{N}$. Then E is *k-free* if it contains no k-grid B (that is, $B = B_1 \times \cdots \times B_r$ with each $|B_i| = k$).

Global Zarankiewicz (1951): E is k-free $\Rightarrow$ $|E \cap B| = O(n^?)$

E is k-free $\Rightarrow |E \cap B| = O(n^?)$

E is k-free $\Rightarrow |E \cap B| = O(n^{?})$

- Kővári–Sós–Turán (1954): $(r = 2)$, $O(n^{2-\frac{1}{k}})$

E is k-free $\Rightarrow$ $|E \cap B| = O(n^?)$

- Kővári–Sós–Turán (1954): $(r = 2)$, $O(n^{2-\frac{1}{k}})$, sharp $k = 2, 3$.

E is k-free $\Rightarrow$ $|E \cap B| = O(n^?)$

- Kővári–Sós–Turán (1954): $(r = 2)$, $O(n^{2-\frac{1}{k}})$, sharp $k = 2, 3$.
- Erdős (1964): $O(n^{r-\frac{1}{k^{r-1}}})$

E is k-free $\Rightarrow$ $|E \cap B| = O(n^?)$

- Kővári–Sós–Turán (1954): $(r = 2)$, $O(n^{2-\frac{1}{k}})$, sharp $k = 2, 3$.
- Erdős (1964): $O(n^{r-\frac{1}{k^{r-1}}})$
- Bohman-Keevash (2010): $(r = 2)$ lower bounds (complicated)

E is k-free $\Rightarrow$ $|E \cap B| = O(n^{?})$

- Kővári–Sós–Turán (1954): $(r = 2)$, $O(n^{2-\frac{1}{k}})$, sharp $k = 2, 3$.
- Erdős (1964): $O(n^{r-\frac{1}{k^{r-1}}})$
- Bohman-Keevash (2010): $(r = 2)$ lower bounds (complicated)
- Szemerédi-Trotter (1983): $E \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ point-line incidence, $O(n^{4/3})$, sharp

E is k-free $\Rightarrow |E \cap B| = O(n^?)$

- Kővári–Sós–Turán (1954): $(r = 2)$, $O(n^{2-\frac{1}{k}})$, sharp $k = 2, 3$.
- Erdős (1964): $O(n^{r-\frac{1}{k^{r-1}}})$
- Bohman-Keevash (2010): $(r = 2)$ lower bounds (complicated)
- Szemerédi-Trotter (1983): $E \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ point-line incidence, $O(n^{4/3})$, sharp
- Fox-Pach-Sheffer-Suk-Zahl (2017): $(r = 2)$ E semialgebraic, $O(n^{4/3})$ $(d_1 = d_2 = 2)$

E is k-free $\Rightarrow$ $|E \cap B| = O(n^?)$

- Kővári–Sós–Turán (1954): $(r = 2)$, $O(n^{2-\frac{1}{k}})$, sharp $k = 2, 3$.
- Erdős (1964): $O(n^{r-\frac{1}{k^{r-1}}})$
- Bohman-Keevash (2010): $(r = 2)$ lower bounds (complicated)
- Szemerédi-Trotter (1983): $E \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ point-line incidence, $O(n^{4/3})$, sharp
- Fox-Pach-Sheffer-Suk-Zahl (2017): $(r = 2)$ E semialgebraic, $O(n^{4/3})$ $(d_1 = d_2 = 2)$
- Do (2018): $(r > 2)$ E semialgebraic, complicated

E is k-free $\Rightarrow$ $|E \cap B| = O(n^?)$

- Kővári–Sós–Turán (1954): $(r = 2)$, $O(n^{2-\frac{1}{k}})$, sharp $k = 2, 3$.
- Erdős (1964): $O(n^{r-\frac{1}{k^{r-1}}})$
- Bohman-Keevash (2010): $(r = 2)$ lower bounds (complicated)
- Szemerédi-Trotter (1983): $E \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ point-line incidence, $O(n^{4/3})$, sharp
- Fox-Pach-Sheffer-Suk-Zahl (2017): $(r = 2)$ E semialgebraic, $O(n^{4/3})$ $(d_1 = d_2 = 2)$
- Do (2018): $(r > 2)$ E semialgebraic, complicated
- Basu-Raz (2017) / Chernikov-Galvin-Starchenko (2020): $(r = 2)$ E o-minimal, $O(n^{4/3})$ $(d_1 = d_2 = 2)$

E is k-free $\Rightarrow$ $|E \cap B| = O(n^?)$

- Kővári–Sós–Turán (1954): $(r = 2)$, $O(n^{2-\frac{1}{k}})$, sharp $k = 2, 3$.
- Erdős (1964): $O(n^{r-\frac{1}{k^{r-1}}})$
- Bohman-Keevash (2010): $(r = 2)$ lower bounds (complicated)
- Szemerédi-Trotter (1983): $E \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ point-line incidence, $O(n^{4/3})$, sharp
- Fox-Pach-Sheffer-Suk-Zahl (2017): $(r = 2)$ E semialgebraic, $O(n^{4/3})$ $(d_1 = d_2 = 2)$
- Do (2018): $(r > 2)$ E semialgebraic, complicated
- Basu-Raz (2017) / Chernikov-Galvin-Starchenko (2020): $(r = 2)$ E o-minimal, $O(n^{4/3})$ $(d_1 = d_2 = 2)$
- Tong (2025): $(r > 2)$ E o-minimal, complicated

E is k-free $\Rightarrow$ $|E \cap B| = O(n^{?})$

- Kővári–Sós–Turán (1954): $(r = 2)$, $O(n^{2-\frac{1}{k}})$, sharp $k = 2, 3$.
- Erdős (1964): $O(n^{r-\frac{1}{k^{r-1}}})$
- Bohman-Keevash (2010): $(r = 2)$ lower bounds (complicated)
- Szemerédi-Trotter (1983): $E \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ point-line incidence, $O(n^{4/3})$, sharp
- Fox-Pach-Sheffer-Suk-Zahl (2017): $(r = 2)$ E semialgebraic, $O(n^{4/3})$ $(d_1 = d_2 = 2)$
- Do (2018): $(r > 2)$ E semialgebraic, complicated
- Basu-Raz (2017) / Chernikov-Galvin-Starchenko (2020): $(r = 2)$ E o-minimal, $O(n^{4/3})$ $(d_1 = d_2 = 2)$
- Tong (2025): $(r > 2)$ E o-minimal, complicated
- Basit-Chernikov-Starchenko-Tao-Tran (2021): E semilinear, $O(n^{r-1})$

Global Linear Zarankiewicz Problem

- M, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$, $E \subseteq \prod_{i \in [r]} M^{d_i} = M^d$, B grid

Global Linear Zarankiewicz Problem

- M, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$, $E \subseteq \prod_{i \in [r]} M^{d_i} = M^d$, B grid
- $\mathcal{C}$ class of grids

Global Linear Zarankiewicz Problem

- M, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$, $E \subseteq \prod_{i \in [r]} M^{d_i} = M^d$, B grid
- $\mathcal{C}$ class of grids
- Let $k \in \mathbb{N}$. Say E is $\mathcal{C}$-k-*free* if it contains no k-grid $B \in \mathcal{C}$.

Global Linear Zarankiewicz Problem

- M, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$, $E \subseteq \prod_{i\in[r]} M^{d_i} = M^d$, B grid
- $\mathcal{C}$ class of grids
- Let $k \in \mathbb{N}$. Say E is *$\mathcal{C}$-k-free* if it contains no k-grid $B \in \mathcal{C}$.
- *E linear Z-bounds for $\mathcal{C}$* if $\exists \alpha \in \mathbb{R}^{>0}$, $\forall_{fin} B \in \mathcal{C}$, $|E \cap B| \leq \alpha n^{r-1}$.

Global Linear Zarankiewicz Problem

- M, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$, $E \subseteq \prod_{i\in[r]} M^{d_i} = M^d$, B grid
- $\mathcal{C}$ class of grids
- Let $k \in \mathbb{N}$. Say E is *$\mathcal{C}$-k-free* if it contains no k-grid $B \in \mathcal{C}$.
- *E linear Z-bounds for $\mathcal{C}$* if $\exists \alpha \in \mathbb{R}^{>0}$, $\forall_{fin} B \in \mathcal{C}$, $|E \cap B| \leq \alpha n^{r-1}$.

  $Zar(E, \mathcal{C})$: ($\exists k \in \mathbb{N}$, E $\mathcal{C}$-k-free) $\Rightarrow$ E linear Z-bounds for $\mathcal{C}$

Global Linear Zarankiewicz Problem

- M, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$, $E \subseteq \prod_{i \in [r]} M^{d_i} = M^d$, B grid
- $\mathcal{C}$ class of grids
- Let $k \in \mathbb{N}$. Say E is *$\mathcal{C}$-k-free* if it contains no k-grid $B \in \mathcal{C}$.
- *E linear Z-bounds for $\mathcal{C}$* if $\exists \alpha \in \mathbb{R}^{>0}$, $\forall_{fin} B \in \mathcal{C}$, $|E \cap B| \leq \alpha n^{r-1}$.

  $Zar(E, \mathcal{C})$: $(\exists k \in \mathbb{N}$, E $\mathcal{C}$-k-free$) \Rightarrow$ E linear Z-bounds for $\mathcal{C}$

  Write $Zar_\alpha(E, \mathcal{C})$ if bounds are witnessed by α.

Global Linear Zarankiewicz Problem

- M, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$, $E \subseteq \prod_{i\in[r]} M^{d_i} = M^d$, B grid
- $\mathcal{C}$ class of grids
- Let $k \in \mathbb{N}$. Say E is *$\mathcal{C}$-k-free* if it contains no k-grid $B \in \mathcal{C}$.
- *E linear Z-bounds for $\mathcal{C}$* if $\exists \alpha \in \mathbb{R}^{>0}$, $\forall_{fin} B \in \mathcal{C}$, $|E \cap B| \leq \alpha n^{r-1}$.

  $Zar(E, \mathcal{C})$: ($\exists k \in \mathbb{N}$, E $\mathcal{C}$-k-free) $\Rightarrow$ E linear Z-bounds for $\mathcal{C}$

  Write $Zar_\alpha(E, \mathcal{C})$ if bounds are witnessed by α.
- $\mathcal{E} = \{E_b\}_{b\in I}$ a family of relations E_b

Global Linear Zarankiewicz Problem

- M, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$, $E \subseteq \prod_{i \in [r]} M^{d_i} = M^d$, B grid
- $\mathcal{C}$ class of grids
- Let $k \in \mathbb{N}$. Say E is *$\mathcal{C}$-k-free* if it contains no k-grid $B \in \mathcal{C}$.
- *E linear Z-bounds for $\mathcal{C}$* if $\exists \alpha \in \mathbb{R}^{>0}$, $\forall_{fin} B \in \mathcal{C}$, $|E \cap B| \leq \alpha n^{r-1}$.

  $Zar(E, \mathcal{C})$: ($\exists k \in \mathbb{N}$, E $\mathcal{C}$-k-free) $\Rightarrow$ E linear Z-bounds for $\mathcal{C}$

  Write $Zar_\alpha(E, \mathcal{C})$ if bounds are witnessed by α.
- $\mathcal{E} = \{E_b\}_{b \in I}$ a family of relations E_b

  $Zar(\mathcal{E}, \mathcal{C})$: $\exists \alpha \in \mathbb{R}^{>0}$, $\forall b \in I$, $Zar_\alpha(E_b, \mathcal{C})$

Global Linear Zarankiewicz Problem

- M, $d_1, \ldots, d_r \in \mathbb{N}^{>0}$, $E \subseteq \prod_{i\in[r]} M^{d_i} = M^d$, B grid
- $\mathcal{C}$ class of grids
- Let $k \in \mathbb{N}$. Say E is *$\mathcal{C}$-k-free* if it contains no k-grid $B \in \mathcal{C}$.
- *E linear Z-bounds for $\mathcal{C}$* if $\exists \alpha \in \mathbb{R}^{>0}$, $\forall_{fin} B \in \mathcal{C}$, $|E \cap B| \leq \alpha n^{r-1}$.

  $Zar(E, \mathcal{C})$: ($\exists k \in \mathbb{N}$, E $\mathcal{C}$-k-free) $\Rightarrow$ E linear Z-bounds for $\mathcal{C}$

  Write $Zar_\alpha(E, \mathcal{C})$ if bounds are witnessed by α.
- $\mathcal{E} = \{E_b\}_{b\in I}$ a family of relations E_b

  $Zar(\mathcal{E}, \mathcal{C})$: $\exists \alpha \in \mathbb{R}^{>0}$, $\forall b \in I$, $Zar_\alpha(E_b, \mathcal{C})$

  Omit $\mathcal{C}$ if it is the class of all grids.

Linear and Semibounded Zarankiewicz

Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be o-minimal.

Linear and Semibounded Zarankiewicz

Let $\mathcal{M} = \langle M, <, +, \dots \rangle$ be o-minimal.

Theorem 0 (Basit-Chernikov-Starchenko-Tao-Tran, 2021)

The following are equivalent:

Linear and Semibounded Zarankiewicz

Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be o-minimal.

Theorem 0 (Basit-Chernikov-Starchenko-Tao-Tran, 2021)

The following are equivalent:

a. $\mathcal{M}$ *linear*

Linear and Semibounded Zarankiewicz

Let $\mathcal{M} = \langle M, <, +, \dots \rangle$ be o-minimal.

Theorem 0 (Basit-Chernikov-Starchenko-Tao-Tran, 2021)

The following are equivalent:

a. $\mathcal{M}$ *linear* ($\Leftrightarrow$ *definable functions are piecewise linear*
$\Leftrightarrow$ *no infinite definable field*)

Linear and Semibounded Zarankiewicz

Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be o-minimal.

Theorem 0 (Basit-Chernikov-Starchenko-Tao-Tran, 2021)

The following are equivalent:

a. $\mathcal{M}$ *linear* ($\Leftrightarrow$ *definable functions are piecewise linear*
$\Leftrightarrow$ *no infinite definable field*)

b. *for every definable* $\mathcal{E}$, *Zar*$(\mathcal{E})$ *holds.*

Linear and Semibounded Zarankiewicz

Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be o-minimal.

Theorem 0 (Basit-Chernikov-Starchenko-Tao-Tran, 2021)

The following are equivalent:

a. $\mathcal{M}$ *linear* ($\Leftrightarrow$ *definable functions are piecewise linear*
$\Leftrightarrow$ *no infinite definable field*)

b. *for every definable* $\mathcal{E}$, $Zar(\mathcal{E})$ *holds.*

Examples: $\langle \mathbb{R}, <, + \rangle$, $\mathbb{R}_{vec} = \langle \mathbb{R}, <, +, \{x \to rx\}_{r \in \mathbb{R}} \rangle$.

Linear and Semibounded Zarankiewicz

Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be o-minimal.

Theorem 0 (Basit-Chernikov-Starchenko-Tao-Tran, 2021)

The following are equivalent:

a. $\mathcal{M}$ *linear* ($\Leftrightarrow$ *definable functions are piecewise linear*
$\Leftrightarrow$ *no infinite definable field*)

b. *for every definable* $\mathcal{E}$, $Zar(\mathcal{E})$ *holds.*

Examples: $\langle \mathbb{R}, <, + \rangle$, $\mathbb{R}_{vec} = \langle \mathbb{R}, <, +, \{x \to rx\}_{r \in \mathbb{R}} \rangle$.

Theorem 1

The following are equivalent:

Linear and Semibounded Zarankiewicz

Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be o-minimal.

Theorem 0 (Basit-Chernikov-Starchenko-Tao-Tran, 2021)

The following are equivalent:

a. $\mathcal{M}$ *linear* ($\Leftrightarrow$ *definable functions are piecewise linear*
$\Leftrightarrow$ *no infinite definable field*)

b. *for every definable* $\mathcal{E}$, $Zar(\mathcal{E})$ *holds.*

Examples: $\langle \mathbb{R}, <, + \rangle$, $\mathbb{R}_{vec} = \langle \mathbb{R}, <, +, \{x \to rx\}_{r \in \mathbb{R}} \rangle$.

Theorem 1

The following are equivalent:

a. $\mathcal{M}$ *semibounded*

Linear and Semibounded Zarankiewicz

Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be o-minimal.

Theorem 0 (Basit-Chernikov-Starchenko-Tao-Tran, 2021)

The following are equivalent:

a. $\mathcal{M}$ *linear* ($\Leftrightarrow$ *definable functions are piecewise linear*
$\Leftrightarrow$ *no infinite definable field*)

b. *for every definable* $\mathcal{E}$, $Zar(\mathcal{E})$ *holds.*

Examples: $\langle \mathbb{R}, <, + \rangle$, $\mathbb{R}_{vec} = \langle \mathbb{R}, <, +, \{x \to rx\}_{r \in \mathbb{R}} \rangle$.

Theorem 1

The following are equivalent:

a. $\mathcal{M}$ *semibounded* ($\Leftrightarrow$ *definable functions are eventually linear*
$\Leftrightarrow$ *no definable field on* M)

Linear and Semibounded Zarankiewicz

Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be o-minimal.

Theorem 0 (Basit-Chernikov-Starchenko-Tao-Tran, 2021)

The following are equivalent:

a. *$\mathcal{M}$ linear ($\Leftrightarrow$ definable functions are piecewise linear*
$\Leftrightarrow$ no infinite definable field)

b. *for every definable $\mathcal{E}$, $Zar(\mathcal{E})$ holds.*

Examples: $\langle \mathbb{R}, <, + \rangle$, $\mathbb{R}_{vec} = \langle \mathbb{R}, <, +, \{x \to rx\}_{r \in \mathbb{R}} \rangle$.

Theorem 1

The following are equivalent:

a. *$\mathcal{M}$ semibounded ($\Leftrightarrow$ definable functions are eventually linear*
$\Leftrightarrow$ no definable field on M)

b. *for every definable $\mathcal{E}$, there is $m \in M^{>0}$, such that for $\mathcal{C}_m$ the class of all m-distant grids, $Zar(\mathcal{E}, \mathcal{C}_m)$ holds.*

Linear and Semibounded Zarankiewicz

Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be o-minimal.

Theorem 0 (Basit-Chernikov-Starchenko-Tao-Tran, 2021)

The following are equivalent:

a. $\mathcal{M}$ *linear* ($\Leftrightarrow$ *definable functions are piecewise linear*
$\Leftrightarrow$ *no infinite definable field*)

b. *for every definable* $\mathcal{E}$, $Zar(\mathcal{E})$ *holds.*

Examples: $\langle \mathbb{R}, <, + \rangle$, $\mathbb{R}_{vec} = \langle \mathbb{R}, <, +, \{x \to rx\}_{r \in \mathbb{R}} \rangle$.

Theorem 1

The following are equivalent:

a. $\mathcal{M}$ *semibounded* ($\Leftrightarrow$ *definable functions are eventually linear*
$\Leftrightarrow$ *no definable field on* M)

b. *for every definable* $\mathcal{E}$, *there is* $m \in M^{>0}$, *such that for* $\mathcal{C}_m$ *the class of all* m*-distant grids,* $Zar(\mathcal{E}, \mathcal{C}_m)$ *holds.*

B is *m-distant* if for all i and $x, y \in B_i$, $|x - y| > m$.

Linear and Semibounded Zarankiewicz

Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be o-minimal.

Theorem 0 (Basit-Chernikov-Starchenko-Tao-Tran, 2021)

The following are equivalent:

a. *$\mathcal{M}$ linear ($\Leftrightarrow$ definable functions are piecewise linear*
$\Leftrightarrow$ no infinite definable field)

b. *for every definable $\mathcal{E}$, $Zar(\mathcal{E})$ holds.*

Examples: $\langle \mathbb{R}, <, + \rangle$, $\mathbb{R}_{vec} = \langle \mathbb{R}, <, +, \{x \to rx\}_{r \in \mathbb{R}} \rangle$.

Theorem 1

The following are equivalent:

a. *$\mathcal{M}$ semibounded ($\Leftrightarrow$ definable functions are eventually linear*
$\Leftrightarrow$ no definable field on M)

b. *for every definable $\mathcal{E}$, there is $m \in M^{>0}$, such that for $\mathcal{C}_m$ the class of all m-distant grids, $Zar(\mathcal{E}, \mathcal{C}_m)$ holds.*

B is *m-distant* if for all i and $x, y \in B_i$, $|x - y| > m$.

Examples: $\langle \mathbb{R}, <, +, \cdot_{\restriction (0,1)^2} \rangle$, $\langle \mathbb{R}_{vec}, \cdot_{\restriction (0,1)^2} \rangle$.

Theorems

0. Linear Zarankiewicz
1. Semibounded Zarankiewicz

Theorems

0. Linear Zarankiewicz
1. Semibounded Zarankiewicz
2. Let $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$. Then for every definable E, $Zar(E)$.

Theorems

0. Linear Zarankiewicz
1. Semibounded Zarankiewicz
2. Let $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$. Then for every definable E, $Zar(E)$.
   - For $\mathcal{E} = \{E_t\}_{t \in \mathbb{Z}}$ with $E_t = [0, t]$, $Zar(\mathcal{E})$ does not hold.

Theorems

0. Linear Zarankiewicz
1. Semibounded Zarankiewicz
2. Let $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$. Then for every definable E, $Zar(E)$.
   - For $\mathcal{E} = \{E_t\}_{t \in \mathbb{Z}}$ with $E_t = [0, t]$, $Zar(\mathcal{E})$ does not hold.
   - Proof by reduction to subgroups of powers of groups

Theorems

0. Linear Zarankiewicz
1. Semibounded Zarankiewicz
2. Let $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$. Then for every definable E, $Zar(E)$.
   - For $\mathcal{E} = \{E_t\}_{t \in \mathbb{Z}}$ with $E_t = [0, t]$, $Zar(\mathcal{E})$ does not hold.
   - Proof by reduction to subgroups of powers of groups
3. Let $\mathcal{M} = \langle \mathbb{R}, <, +, \mathbb{Z} \rangle$. Then for every definable E, $Zar(E)$.

Theorems

0. Linear Zarankiewicz
1. Semibounded Zarankiewicz
2. Let $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$. Then for every definable E, $Zar(E)$.
    - For $\mathcal{E} = \{E_t\}_{t \in \mathbb{Z}}$ with $E_t = [0, t]$, $Zar(\mathcal{E})$ does not hold.
    - Proof by reduction to subgroups of powers of groups
3. Let $\mathcal{M} = \langle \mathbb{R}, <, +, \mathbb{Z} \rangle$. Then for every definable E, $Zar(E)$.
4. Abstract Zarankiewicz for saturated $\mathcal{M}$, operator cl, grid class $\mathcal{C}$.

Theorems

0. Linear Zarankiewicz
1. Semibounded Zarankiewicz
2. Let $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$. Then for every definable E, $Zar(E)$.
   - For $\mathcal{E} = \{E_t\}_{t \in \mathbb{Z}}$ with $E_t = [0, t]$, $Zar(\mathcal{E})$ does not hold.
   - Proof by reduction to subgroups of powers of groups
3. Let $\mathcal{M} = \langle \mathbb{R}, <, +, \mathbb{Z} \rangle$. Then for every definable E, $Zar(E)$.
4. Abstract Zarankiewicz for saturated $\mathcal{M}$, operator cl, grid class $\mathcal{C}$.

Applications:

a. Let $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$ saturated, $\mathcal{E}$ definable family, $\mathcal{C}_{\mathbb{Z}}$ class of all $\mathbb{Z}$-distant grids. Then $Zar(\mathcal{E}, \mathcal{C}_{\mathbb{Z}})$.

Theorems

0. Linear Zarankiewicz
1. Semibounded Zarankiewicz
2. Let $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$. Then for every definable E, $Zar(E)$.
   - For $\mathcal{E} = \{E_t\}_{t \in \mathbb{Z}}$ with $E_t = [0, t]$, $Zar(\mathcal{E})$ does not hold.
   - Proof by reduction to subgroups of powers of groups
3. Let $\mathcal{M} = \langle \mathbb{R}, <, +, \mathbb{Z} \rangle$. Then for every definable E, $Zar(E)$.
4. Abstract Zarankiewicz for saturated $\mathcal{M}$, operator cl, grid class $\mathcal{C}$.

Applications:

a. Let $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$ saturated, $\mathcal{E}$ definable family, $\mathcal{C}_{\mathbb{Z}}$ class of all $\mathbb{Z}$-distant grids. Then $Zar(\mathcal{E}, \mathcal{C}_{\mathbb{Z}})$.
   - B is $\mathbb{Z}$-*distant* if for every i and $x, y \in B_i$, $x - y \notin \mathbb{Z}^{d_i}$.

Theorems

0. Linear Zarankiewicz
1. Semibounded Zarankiewicz
2. Let $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$. Then for every definable E, $Zar(E)$.
   - For $\mathcal{E} = \{E_t\}_{t \in \mathbb{Z}}$ with $E_t = [0, t]$, $Zar(\mathcal{E})$ does not hold.
   - Proof by reduction to subgroups of powers of groups
3. Let $\mathcal{M} = \langle \mathbb{R}, <, +, \mathbb{Z} \rangle$. Then for every definable E, $Zar(E)$.
4. Abstract Zarankiewicz for saturated $\mathcal{M}$, operator cl, grid class $\mathcal{C}$.

Applications:

a. Let $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$ saturated, $\mathcal{E}$ definable family, $\mathcal{C}_{\mathbb{Z}}$ class of all $\mathbb{Z}$-distant grids. Then $Zar(\mathcal{E}, \mathcal{C}_{\mathbb{Z}})$.
   - B is *$\mathbb{Z}$-distant* if for every i and $x, y \in B_i$, $x - y \notin \mathbb{Z}^{d_i}$.

b. Zarankiewicz for stable 1-based theories with *nfcp*.

Theorems

0. Linear Zarankiewicz
1. Semibounded Zarankiewicz
2. Let $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$. Then for every definable E, $Zar(E)$.
   - For $\mathcal{E} = \{E_t\}_{t \in \mathbb{Z}}$ with $E_t = [0, t]$, $Zar(\mathcal{E})$ does not hold.
   - Proof by reduction to subgroups of powers of groups
3. Let $\mathcal{M} = \langle \mathbb{R}, <, +, \mathbb{Z} \rangle$. Then for every definable E, $Zar(E)$.
4. Abstract Zarankiewicz for saturated $\mathcal{M}$, operator cl, grid class $\mathcal{C}$.

Applications:

a. Let $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$ saturated, $\mathcal{E}$ definable family, $\mathcal{C}_{\mathbb{Z}}$ class of all $\mathbb{Z}$-distant grids. Then $Zar(\mathcal{E}, \mathcal{C}_{\mathbb{Z}})$.
   - B is *$\mathbb{Z}$-distant* if for every i and $x, y \in B_i$, $x - y \notin \mathbb{Z}^{d_i}$.

b. Zarankiewicz for stable 1-based theories with *nfcp*.

c. Zarankiewicz for locally modular regular types p in a stable theory (eg. the generic type of the solution set of the Heat differential equation), and $\mathcal{C}$ the class of all cl_p-independent grids.

Zarankiewicz for subgroups of powers of groups

Zarankiewicz for subgroups of powers of groups

- For $j \in [r]$, let $p_j : \prod_{i \in [r]} M^{d_i} \to \prod_{i \neq j} M^{d_i}$ omit j-th coordinates

Zarankiewicz for subgroups of powers of groups

- For $j \in [r]$, let $p_j : \prod_{i \in [r]} M^{d_i} \to \prod_{i \neq j} M^{d_i}$ omit j-th coordinates
- If some $p_{j \restriction E}$ is N-to-1, then E linear Z-bounds, witnessed by N.

Zarankiewicz for subgroups of powers of groups

- For $j \in [r]$, let $p_j : \prod_{i \in [r]} M^{d_i} \to \prod_{i \neq j} M^{d_i}$ omit j-th coordinates
- If some $p_{j \restriction E}$ is N-to-1, then E linear Z-bounds, witnessed by N. Indeed: $|E \cap B| \leq N |p_j(E) \cap p_j(B)| \leq N n^{r-1}$

Zarankiewicz for subgroups of powers of groups

- For $j \in [r]$, let $p_j : \prod_{i \in [r]} M^{d_i} \to \prod_{i \neq j} M^{d_i}$ omit j-th coordinates
- If some $p_{j \restriction E}$ is N-to-1, then E linear Z-bounds, witnessed by N. Indeed: $|E \cap B| \leq N|p_j(E) \cap p_j(B)| \leq Nn^{r-1}$
- Now let $\langle M, + \rangle$ be a group.

Zarankiewicz for subgroups of powers of groups

- For $j \in [r]$, let $p_j : \prod_{i \in [r]} M^{d_i} \to \prod_{i \neq j} M^{d_i}$ omit j-th coordinates
- If some $p_{j\restriction E}$ is N-to-1, then E linear Z-bounds, witnessed by N.
  Indeed: $|E \cap B| \leq N|p_j(E) \cap p_j(B)| \leq Nn^{r-1}$
- Now let $\langle M, + \rangle$ be a group.
- Let $V \leqslant \langle M^d, + \rangle = \prod_{i \in [r]} \langle M^{d_i}, + \rangle$

Zarankiewicz for subgroups of powers of groups

- For $j \in [r]$, let $p_j : \prod_{i\in[r]} M^{d_i} \to \prod_{i\neq j} M^{d_i}$ omit j-th coordinates
- If some $p_{j\restriction E}$ is N-to-1, then E linear Z-bounds, witnessed by N.
  Indeed: $|E \cap B| \leq N|p_j(E) \cap p_j(B)| \leq Nn^{r-1}$
- Now let $\langle M, +\rangle$ be a group.
- Let $V \leqslant \langle M^d, +\rangle = \prod_{i\in[r]}\langle M^{d_i}, +\rangle$

\- $p_{j\restriction V}$ is N-to-1 (some $N \in \mathbb{N}$) $\Leftrightarrow$ $\ker p_{j\restriction V}$ finite

Zarankiewicz for subgroups of powers of groups

- For $j \in [r]$, let $p_j : \prod_{i\in[r]} M^{d_i} \to \prod_{i\neq j} M^{d_i}$ omit j-th coordinates
- If some $p_{j\restriction E}$ is N-to-1, then E linear Z-bounds, witnessed by N. Indeed: $|E \cap B| \leq N|p_j(E) \cap p_j(B)| \leq Nn^{r-1}$
- Now let $\langle M, +\rangle$ be a group.
- Let $V \leqslant \langle M^d, +\rangle = \prod_{i\in[r]}\langle M^{d_i}, +\rangle$
  - $p_{j\restriction V}$ is N-to-1 (some $N \in \mathbb{N}$) $\Leftrightarrow$ $\ker p_{j\restriction V}$ finite
  - Let $U_j = \pi_j(\ker p_{j\restriction V})$.

Zarankiewicz for subgroups of powers of groups

- For $j \in [r]$, let $p_j : \prod_{i \in [r]} M^{d_i} \to \prod_{i \neq j} M^{d_i}$ omit j-th coordinates
- If some $p_{j\restriction E}$ is N-to-1, then E linear Z-bounds, witnessed by N.
  Indeed: $|E \cap B| \leq N|p_j(E) \cap p_j(B)| \leq Nn^{r-1}$
- Now let $\langle M, + \rangle$ be a group.
- Let $V \leqslant \langle M^d, + \rangle = \prod_{i \in [r]} \langle M^{d_i}, + \rangle$
  - $p_{j\restriction V}$ is N-to-1 (some $N \in \mathbb{N}$) $\Leftrightarrow$ $\ker p_{j\restriction V}$ finite
  - Let $U_j = \pi_j(\ker p_{j\restriction V})$. Then $U_1 \times \cdots \times U_r \subseteq V$

Zarankiewicz for subgroups of powers of groups

- For $j \in [r]$, let $p_j : \prod_{i \in [r]} M^{d_i} \to \prod_{i \neq j} M^{d_i}$ omit j-th coordinates
- If some $p_{j \restriction E}$ is N-to-1, then E linear Z-bounds, witnessed by N. Indeed: $|E \cap B| \leq N|p_j(E) \cap p_j(B)| \leq Nn^{r-1}$
- Now let $\langle M, + \rangle$ be a group.
- Let $V \leqslant \langle M^d, + \rangle = \prod_{i \in [r]} \langle M^{d_i}, + \rangle$
  - $p_{j \restriction V}$ is N-to-1 (some $N \in \mathbb{N}$) $\Leftrightarrow$ $\ker p_{j \restriction V}$ finite
  - Let $U_j = \pi_j(\ker p_{j \restriction V})$. Then $U_1 \times \cdots \times U_r \subseteq V$
  - Therefore $Zar(V)$ holds.

Zarankiewicz for subgroups of powers of groups

- For $j \in [r]$, let $p_j : \prod_{i\in[r]} M^{d_i} \to \prod_{i\neq j} M^{d_i}$ omit j-th coordinates
- If some $p_{j\restriction E}$ is N-to-1, then E linear Z-bounds, witnessed by N. Indeed: $|E \cap B| \leq N|p_j(E) \cap p_j(B)| \leq Nn^{r-1}$
- Now let $\langle M, +\rangle$ be a group.
- Let $V \leqslant \langle M^d, +\rangle = \prod_{i\in[r]}\langle M^{d_i}, +\rangle$
  - $p_{j\restriction V}$ is N-to-1 (some $N \in \mathbb{N}$) $\Leftrightarrow$ $\ker p_{j\restriction V}$ finite
  - Let $U_j = \pi_j(\ker p_{j\restriction V})$. Then $U_1 \times \cdots \times U_r \subseteq V$
  - Therefore $Zar(V)$ holds.
- For $C \subseteq M^d$, $Sh(C) :=$ subgroup of $\langle M^d, +\rangle$ generated by C.

Zarankiewicz for subgroups of powers of groups

- For $j \in [r]$, let $p_j : \prod_{i\in[r]} M^{d_i} \to \prod_{i\neq j} M^{d_i}$ omit j-th coordinates
- If some $p_{j\restriction E}$ is N-to-1, then E linear Z-bounds, witnessed by N. Indeed: $|E \cap B| \leq N|p_j(E) \cap p_j(B)| \leq Nn^{r-1}$
- Now let $\langle M, +\rangle$ be a group.
- Let $V \leqslant \langle M^d, +\rangle = \prod_{i\in[r]}\langle M^{d_i}, +\rangle$
  - $p_{j\restriction V}$ is N-to-1 (some $N \in \mathbb{N}$) $\Leftrightarrow$ $\ker p_{j\restriction V}$ finite
  - Let $U_j = \pi_j(\ker p_{j\restriction V})$. Then $U_1 \times \cdots \times U_r \subseteq V$
  - Therefore $Zar(V)$ holds.
- For $C \subseteq M^d$, $Sh(C) :=$ subgroup of $\langle M^d, +\rangle$ generated by C.

General strategy for $Zar(E)$: Show that E is a finite union of sets C, each satisfying, after translating to 0:

$$(*)\quad C \text{ is } k\text{-free} \Rightarrow Sh(C)\ k\text{-free}$$

(*) for linear and Presburger cells C containing 0

(*) for linear and Presburger cells C containing 0

- C k-free ($k \in \mathbb{N}$) $\Rightarrow$ $Sh(C)$ 2-free

(*) for linear and Presburger cells C containing 0

- C k-free ($k \in \mathbb{N}$) $\Rightarrow$ $Sh(C)$ 2-free

Linear:

(*) for linear and Presburger cells C containing 0

- C k-free ($k \in \mathbb{N}$) $\Rightarrow$ $Sh(C)$ 2-free

Linear:

- C is open in $Sh(C)$

(*) for linear and Presburger cells C containing 0

- C k-free ($k \in \mathbb{N}$) $\Rightarrow$ $Sh(C)$ 2-free

Linear:

- C is open in $Sh(C)$
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$

(*) for linear and Presburger cells C containing 0

- C k-free ($k \in \mathbb{N}$) $\Rightarrow$ $Sh(C)$ 2-free

Linear:

- C is open in $Sh(C)$
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$,
  with each $U_i \subseteq M^{d_i}$ definable infinite and close to 0

(*) for linear and Presburger cells C containing 0

- C k-free ($k \in \mathbb{N}$) $\Rightarrow$ $Sh(C)$ 2-free

Linear:

- C is open in $Sh(C)$
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$,
  with each $U_i \subseteq M^{d_i}$ definable infinite and close to 0
- $C \cap (U_1 \times \cdots \times U_r)$ contains a k-grid, for any $k \in \mathbb{N}$.

(*) for linear and Presburger cells C containing 0

- C k-free ($k \in \mathbb{N}$) $\Rightarrow$ $Sh(C)$ 2-free

Linear:

- C is open in $Sh(C)$
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$,
  with each $U_i \subseteq M^{d_i}$ definable infinite and close to 0
- $C \cap (U_1 \times \cdots \times U_r)$ contains a k-grid, for any $k \in \mathbb{N}$.

Presburger:

(*) for linear and Presburger cells C containing 0

- C k-free ($k \in \mathbb{N}$) $\Rightarrow$ $Sh(C)$ 2-free

Linear:

- C is open in $Sh(C)$
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$,
  with each $U_i \subseteq M^{d_i}$ definable infinite and close to 0
- $C \cap (U_1 \times \cdots \times U_r)$ contains a k-grid, for any $k \in \mathbb{N}$.

Presburger:

- Let $N \in \mathbb{N}$. Say $x \in C$ is *N-internal* if $B_N(x) \cap Sh(C) \subseteq C$.

(*) for linear and Presburger cells C containing 0

- C k-free ($k \in \mathbb{N}$) $\Rightarrow$ $Sh(C)$ 2-free

Linear:

- C is open in $Sh(C)$
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$, with each $U_i \subseteq M^{d_i}$ definable infinite and close to 0
- $C \cap (U_1 \times \cdots \times U_r)$ contains a k-grid, for any $k \in \mathbb{N}$.

Presburger:

- Let $N \in \mathbb{N}$. Say $x \in C$ is *N-internal* if $B_N(x) \cap Sh(C) \subseteq C$.

- For every $N \in \mathbb{N}$, C contains an N-internal point a.

(*) for linear and Presburger cells C containing 0

- C k-free ($k \in \mathbb{N}$) $\Rightarrow$ $Sh(C)$ 2-free

Linear:

- C is open in $Sh(C)$
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$,
  with each $U_i \subseteq M^{d_i}$ definable infinite and close to 0
- $C \cap (U_1 \times \cdots \times U_r)$ contains a k-grid, for any $k \in \mathbb{N}$.

Presburger:

- Let $N \in \mathbb{N}$. Say $x \in C$ is *N-internal* if $B_N(x) \cap Sh(C) \subseteq C$.
- For every $N \in \mathbb{N}$, C contains an N-internal point a.
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$

(*) for linear and Presburger cells C containing 0

- C k-free ($k \in \mathbb{N}$) $\Rightarrow$ $Sh(C)$ 2-free

Linear:

- C is open in $Sh(C)$
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$, with each $U_i \subseteq M^{d_i}$ definable infinite and close to 0
- $C \cap (U_1 \times \cdots \times U_r)$ contains a k-grid, for any $k \in \mathbb{N}$.

Presburger:

- Let $N \in \mathbb{N}$. Say $x \in C$ is *N-internal* if $B_N(x) \cap Sh(C) \subseteq C$.

- For every $N \in \mathbb{N}$, C contains an N-internal point a.
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$, with each $U_i \cap \mathbb{Z}^{d_i}$ infinite

(*) for linear and Presburger cells C containing 0

- C k-free ($k \in \mathbb{N}$) $\Rightarrow$ $Sh(C)$ 2-free

Linear:

- C is open in $Sh(C)$
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$, with each $U_i \subseteq M^{d_i}$ definable infinite and close to 0
- $C \cap (U_1 \times \cdots \times U_r)$ contains a k-grid, for any $k \in \mathbb{N}$.

Presburger:

- Let $N \in \mathbb{N}$. Say $x \in C$ is *N-internal* if $B_N(x) \cap Sh(C) \subseteq C$.
- For every $N \in \mathbb{N}$, C contains an N-internal point a.
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$, with each $U_i \cap \mathbb{Z}^{d_i}$ infinite
- $Sh(C) \cap \mathbb{Z}^d$ contains a k-grid $B \subseteq B_N(0)$ (for any $k \in \mathbb{N}$).

(*) for linear and Presburger cells C containing 0

- C k-free ($k \in \mathbb{N}$) $\Rightarrow$ $Sh(C)$ 2-free

Linear:

- C is open in $Sh(C)$
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$, with each $U_i \subseteq M^{d_i}$ definable infinite and close to 0
- $C \cap (U_1 \times \cdots \times U_r)$ contains a k-grid, for any $k \in \mathbb{N}$.

Presburger:

- Let $N \in \mathbb{N}$. Say $x \in C$ is *N-internal* if $B_N(x) \cap Sh(C) \subseteq C$.
- For every $N \in \mathbb{N}$, C contains an N-internal point a.
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$, with each $U_i \cap \mathbb{Z}^{d_i}$ infinite
- $Sh(C) \cap \mathbb{Z}^d$ contains a k-grid $B \subseteq B_N(0)$ (for any $k \in \mathbb{N}$). Let $a \in C$ N-internal.

(*) for linear and Presburger cells C containing 0

- C k-free ($k \in \mathbb{N}$) $\Rightarrow$ $Sh(C)$ 2-free

Linear:

- C is open in $Sh(C)$
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$, with each $U_i \subseteq M^{d_i}$ definable infinite and close to 0
- $C \cap (U_1 \times \cdots \times U_r)$ contains a k-grid, for any $k \in \mathbb{N}$.

Presburger:

- Let $N \in \mathbb{N}$. Say $x \in C$ is *N-internal* if $B_N(x) \cap Sh(C) \subseteq C$.
- For every $N \in \mathbb{N}$, C contains an N-internal point a.
- $Sh(C)$ not 2-free $\Rightarrow$ $\exists U_1 \times \cdots \times U_r \subseteq Sh(C)$, with each $U_i \cap \mathbb{Z}^{d_i}$ infinite
- $Sh(C) \cap \mathbb{Z}^d$ contains a k-grid $B \subseteq B_N(0)$ (for any $k \in \mathbb{N}$). Let $a \in C$ N-internal. Then

$$a + B \subseteq B_N(a) \cap Sh(C) \subseteq C.$$

Abstract Zarankiewicz in saturated setting

Abstract Zarankiewicz in saturated setting

Theorem (Inspired by BCSTT, 2021)

Let $\mathcal{M}$ saturated, $\mathrm{cl} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$*, and $\mathcal{C}$ a class of grids that contains all* cl*-independent grids (each B_i is* cl*-independent).*

Abstract Zarankiewicz in saturated setting

Theorem (Inspired by BCSTT, 2021)

Let $\mathcal{M}$ saturated, $\mathrm{cl} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$, and $\mathcal{C}$ a class of grids that contains all cl-independent grids (each B_i is cl-independent). Assume:

(DEF) *Let $(a, b) \in M^{k+m}$ and $A \subseteq M$. If $a \in \mathrm{cl}(Ab)$, then there is an A-definable set $X \subseteq M^{k+m}$ with $(a, b) \in X$, and for every $(a', b') \in X$, $a' \in \mathrm{cl}(Ab')$.*

Abstract Zarankiewicz in saturated setting

Theorem (Inspired by BCSTT, 2021)

Let $\mathcal{M}$ saturated, $\mathrm{cl} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$, and $\mathcal{C}$ a class of grids that contains all cl*-independent grids (each B_i is* cl*-independent). Assume:*

(DEF) *Let $(a, b) \in M^{k+m}$ and $A \subseteq M$. If $a \in \mathrm{cl}(Ab)$, then there is an A-definable set $X \subseteq M^{k+m}$ with $(a, b) \in X$, and for every $(a', b') \in X$, $a' \in \mathrm{cl}(Ab')$.*

(TIGHT) *Let E definable, $\mathcal{C}$-k-free (some $k \in \mathbb{N}$). Then for every $a = (a_1, \ldots, a_r) \in E$, the set $\{a_1, \ldots, a_r\}$ is* acl*-dependent.*

Abstract Zarankiewicz in saturated setting

Theorem (Inspired by BCSTT, 2021)

Let $\mathcal{M}$ saturated, $\mathrm{cl} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$, and $\mathcal{C}$ a class of grids that contains all cl*-independent grids (each B_i is* cl*-independent). Assume:*

(DEF) *Let $(a, b) \in M^{k+m}$ and $A \subseteq M$. If $a \in \mathrm{cl}(Ab)$, then there is an A-definable set $X \subseteq M^{k+m}$ with $(a, b) \in X$, and for every $(a', b') \in X$, $a' \in \mathrm{cl}(Ab')$.*

(TIGHT) *Let E definable, $\mathcal{C}$-k-free (some $k \in \mathbb{N}$). Then for every $a = (a_1, \ldots, a_r) \in E$, the set $\{a_1, \ldots, a_r\}$ is* acl*-dependent.*

(UB) *Let $\mathcal{X} = \{X_b\}_{b \in I}$ definable family of sets in M^{d_i}. Then $\exists N \in \mathbb{N}$ such that $\forall b \in I$ with X_b finite, and $Y \in \mathcal{C}$, we have $|X_b \cap \pi_i(Y)| \leq N$.*

Abstract Zarankiewicz in saturated setting

Theorem (Inspired by BCSTT, 2021)

Let $\mathcal{M}$ saturated, $\mathrm{cl} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$, *and* $\mathcal{C}$ *a class of grids that contains all* cl*-independent grids (each* B_i *is* cl*-independent). Assume:*

(DEF) *Let* $(a, b) \in M^{k+m}$ *and* $A \subseteq M$. *If* $a \in \mathrm{cl}(Ab)$, *then there is an* A*-definable set* $X \subseteq M^{k+m}$ *with* $(a, b) \in X$, *and for every* $(a', b') \in X$, $a' \in \mathrm{cl}(Ab')$.

(TIGHT) *Let* E *definable,* $\mathcal{C}$*-*k*-free (some* $k \in \mathbb{N}$*). Then for every* $a = (a_1, \dots, a_r) \in E$, *the set* $\{a_1, \dots, a_r\}$ *is* acl*-dependent.*

(UB) *Let* $\mathcal{X} = \{X_b\}_{b \in I}$ *definable family of sets in* M^{d_i}. *Then* $\exists N \in \mathbb{N}$ *such that* $\forall b \in I$ *with* X_b *finite, and* $Y \in \mathcal{C}$, *we have* $|X_b \cap \pi_i(Y)| \leq N$.

Then for every definable family $\mathcal{E}$, $Zar(\mathcal{E}, \mathcal{C})$.

a. $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$. Let $\mathrm{cl} = \mathrm{acl}$ and $\mathcal{C}$ all $\mathbb{Z}$-distant grids.

Abstract Zarankiewicz in saturated setting

Theorem (Inspired by BCSTT, 2021)

Let $\mathcal{M}$ saturated, $\mathrm{cl} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$, and $\mathcal{C}$ a class of grids that contains all cl*-independent grids (each B_i is* cl*-independent). Assume:*

(DEF) *Let $(a, b) \in M^{k+m}$ and $A \subseteq M$. If $a \in \mathrm{cl}(Ab)$, then there is an A-definable set $X \subseteq M^{k+m}$ with $(a, b) \in X$, and for every $(a', b') \in X$, $a' \in \mathrm{cl}(Ab')$.*

(TIGHT) *Let E definable, $\mathcal{C}$-k-free (some $k \in \mathbb{N}$). Then for every $a = (a_1, \ldots, a_r) \in E$, the set $\{a_1, \ldots, a_r\}$ is* acl*-dependent.*

(UB) *Let $\mathcal{X} = \{X_b\}_{b \in I}$ definable family of sets in M^{d_i}. Then $\exists N \in \mathbb{N}$ such that $\forall b \in I$ with X_b finite, and $Y \in \mathcal{C}$, we have $|X_b \cap \pi_i(Y)| \leq N$.*

Then for every definable family $\mathcal{E}$, $Zar(\mathcal{E}, \mathcal{C})$.

a. $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$. Let $\mathrm{cl} = \mathrm{acl}$ and $\mathcal{C}$ all $\mathbb{Z}$-distant grids.
b. $Th(\mathcal{M})$ stable, 1-based, nfcp. Let $\mathrm{cl} = \mathrm{acl}$, $\mathcal{C}$ all grids.

Abstract Zarankiewicz in saturated setting

Theorem (Inspired by BCSTT, 2021)

Let $\mathcal{M}$ saturated, $\mathrm{cl} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$, and $\mathcal{C}$ a class of grids that contains all cl*-independent grids (each B_i is* cl*-independent). Assume:*

(DEF) *Let $(a, b) \in M^{k+m}$ and $A \subseteq M$. If $a \in \mathrm{cl}(Ab)$, then there is an A-definable set $X \subseteq M^{k+m}$ with $(a, b) \in X$, and for every $(a', b') \in X$, $a' \in \mathrm{cl}(Ab')$.*

(TIGHT) *Let E definable, $\mathcal{C}$-k-free (some $k \in \mathbb{N}$). Then for every $a = (a_1, \ldots, a_r) \in E$, the set $\{a_1, \ldots, a_r\}$ is* acl*-dependent.*

(UB) *Let $\mathcal{X} = \{X_b\}_{b \in I}$ definable family of sets in M^{d_i}. Then $\exists N \in \mathbb{N}$ such that $\forall b \in I$ with X_b finite, and $Y \in \mathcal{C}$, we have $|X_b \cap \pi_i(Y)| \leq N$.*

Then for every definable family $\mathcal{E}$, $Zar(\mathcal{E}, \mathcal{C})$.

a. $\mathcal{M} \models Th(\langle \mathbb{Z}, <, + \rangle)$. Let $\mathrm{cl} = \mathrm{acl}$ and $\mathcal{C}$ all $\mathbb{Z}$-distant grids.
b. $Th(\mathcal{M})$ stable, 1-based, nfcp. Let $\mathrm{cl} = \mathrm{acl}$, $\mathcal{C}$ all grids.
c. $\mathcal{M} = p(\mathcal{U})$, p locally modular regular type. Let $\mathrm{cl}_p(B) = \{b \in M : b \not\perp\!\!\!\perp B\}$, $\mathcal{C}$ all cl_p-independent grids

$\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$ and open questions

$\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$ and open questions

- $\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$: Special case $E = S + J + D$, where S Presburger cell, $J + D$ linear cell, J 'purely unbounded', D bounded.

$\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$ and open questions

- $\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$: Special case $E = S + J + D$, where S Presburger cell, $J + D$ linear cell, J 'purely unbounded', D bounded.

  Adjust (*): $E = S + J + D$ k-free $\Rightarrow$ $Sh(S) + Sh(J) + D$ k-free

  $\Rightarrow \exists j \exists N \in \mathbb{N}, p_{j \restriction E}$ N-to-1

$\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$ and open questions

- $\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$: Special case $E = S + J + D$, where S Presburger cell, $J + D$ linear cell, J 'purely unbounded', D bounded.
  Adjust (*): $E = S + J + D$ k-free $\Rightarrow$ $Sh(S) + Sh(J) + D$ k-free
  $\Rightarrow \exists j \exists N \in \mathbb{N}, p_{j \restriction E}$ N-to-1
- Zarankievicz for OAGs, $\langle \mathbb{R}, <, +, 2^{\mathbb{Z}} \rangle$, $\langle \mathbb{Z}, <, +, 2^{\mathbb{N}} \rangle$, 'linear type', no definable field.

$\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$ and open questions

- $\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$: Special case $E = S + J + D$, where S Presburger cell, $J + D$ linear cell, J 'purely unbounded', D bounded.

  Adjust (*): $E = S + J + D$ k-free $\Rightarrow$ $Sh(S) + Sh(J) + D$ k-free
  $\Rightarrow \exists j \exists N \in \mathbb{N}, p_{j \restriction E}$ N-to-1
- Zarankievicz for OAGs, $\langle \mathbb{R}, <, +, 2^{\mathbb{Z}} \rangle$, $\langle \mathbb{Z}, <, +, 2^{\mathbb{N}} \rangle$, 'linear type', no definable field.
- Let $\mathcal{M} = \langle \mathbb{R}, <, +, \ldots \rangle$ o-minimal. For what $\mathcal{C}$, TFAE:
  a. the exponential exp is not definable in M.
  b. for every binary definable $E \subseteq M^{d_1} \times M^{d_2}$, $Zar(E, \mathcal{C})$ holds.

$\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$ and open questions

- $\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$: Special case $E = S + J + D$, where S Presburger cell, $J + D$ linear cell, J 'purely unbounded', D bounded.

  Adjust (*): $E = S + J + D$ k-free $\Rightarrow$ $Sh(S) + Sh(J) + D$ k-free

  $$\Rightarrow \exists j \exists N \in \mathbb{N}, p_{j \restriction E} \text{ } N\text{-to-1}$$

- Zarankievicz for OAGs, $\langle \mathbb{R}, <, +, 2^{\mathbb{Z}} \rangle$, $\langle \mathbb{Z}, <, +, 2^{\mathbb{N}} \rangle$, 'linear type', no definable field.
- Let $\mathcal{M} = \langle \mathbb{R}, <, +, \ldots \rangle$ o-minimal. For what $\mathcal{C}$, TFAE:
  a. the exponential exp is not definable in M.
  b. for every binary definable $E \subseteq M^{d_1} \times M^{d_2}$, $Zar(E, \mathcal{C})$ holds.
- Elekes-Szabó: recognise group structure from combinatorial data.

$\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$ and open questions

- $\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$: Special case $E = S + J + D$, where S Presburger cell, $J + D$ linear cell, J 'purely unbounded', D bounded.
  Adjust (*): $E = S + J + D$ k-free $\Rightarrow$ $Sh(S) + Sh(J) + D$ k-free
  $\Rightarrow$ $\exists j \exists N \in \mathbb{N}, p_{j \restriction E}$ N-to-1
- Zarankievicz for OAGs, $\langle \mathbb{R}, <, +, 2^{\mathbb{Z}} \rangle$, $\langle \mathbb{Z}, <, +, 2^{\mathbb{N}} \rangle$, 'linear type', no definable field.
- Let $\mathcal{M} = \langle \mathbb{R}, <, +, \ldots \rangle$ o-minimal. For what $\mathcal{C}$, TFAE:
  a. the exponential exp is not definable in M.
  b. for every binary definable $E \subseteq M^{d_1} \times M^{d_2}$, $Zar(E, \mathcal{C})$ holds.
- Elekes-Szabó: recognise group structure from combinatorial data. Let $\mathcal{M}$ reduct of an o-minimal $\langle M, <, +, \ldots \rangle$. Recover $+$.

$\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$ and open questions

- $\langle \mathbb{R}, <, +, \mathbb{Z} \rangle$: Special case $E = S + J + D$, where S Presburger cell, $J + D$ linear cell, J 'purely unbounded', D bounded.
  Adjust (*): $E = S + J + D$ k-free $\Rightarrow$ $Sh(S) + Sh(J) + D$ k-free
  $\Rightarrow \exists j \exists N \in \mathbb{N}, p_{j\restriction E}$ N-to-1
- Zarankievicz for OAGs, $\langle \mathbb{R}, <, +, 2^{\mathbb{Z}} \rangle$, $\langle \mathbb{Z}, <, +, 2^{\mathbb{N}} \rangle$, 'linear type', no definable field.
- Let $\mathcal{M} = \langle \mathbb{R}, <, +, \ldots \rangle$ o-minimal. For what $\mathcal{C}$, TFAE:
  a. the exponential exp is not definable in M.
  b. for every binary definable $E \subseteq M^{d_1} \times M^{d_2}$, $Zar(E, \mathcal{C})$ holds.
- Elekes-Szabó: recognise group structure from combinatorial data. Let $\mathcal{M}$ reduct of an o-minimal $\langle M, <, +, \ldots \rangle$. Recover $+$.

Thank you!