Enriched categories: theory and examples

Lecture notes from a postgraduate minicourse

University of Manchester

Giacomo Tendas

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Contents

Introduction

Enrichment is nowadays a standard tool in category theory; its range of applications is so vast that it reaches very different areas of mathematics, such as algebra [7, 14, 18], homotopy theory [4, 12, 17], computer science [3, 6, 16], and functional analysis [11, 15].

Even though additive and abelian categories were introduced earlier and can be understood as some (easy) examples of enrichment, it was only in the 60s, after the development of differentially graded categories, that people started to think about a general framework for dealing with categories whose homs have a much richer structure than that of a set.

The first to, independently, envisage the potentials of such a theory were Mac Lane [9] and Bénabou [1], as well as Linton[10] and Maranda [13]. However, Eilenberg and Kelly were the ones that actually developed a theory of enrichment in their monograph [5]. Afterwards, the theory started to get studied and many results from ordinary category theory were transferred into this richer setting, sometimes with effort and some other times very easily and elegantly.

Later, the theory evolved in new directions by introducing enrichment over bicategories [2, 19]; however that will not be the framework of this course, where we consider only enrichment over symmetric monoidal closed categories. A more complete account of all the results we discuss is given in Kelly's book [8].

CHAPTER 1

Week 1

1.1 Monoidal categories

In this section we introduce those categories which, equipped with additional structure, will be our bases of enrichment.

Given an ordinary category $\mathcal{C}$, we can describe the composition operation as a family of functions

$$\mathcal{C}(B,C) \times \mathcal{C}(A,B) \xrightarrow{-\circ-} \mathcal{C}(A,C)$$

for any $A, B, C \in \mathcal{C}$, satisfying certain axioms. This suggests that we could define a notion of category enriched over any given category with binary products. However, that would be very restrictive as, while we would capture many examples (2-dimensional, simplicial, etc) we would also miss many other important ones (for instance, additive and DG-categories).

To capture all these in the same framework we shall need a category endowed with a "tensor product":

Definition 1.1.1. A *monoidal category* $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$ is the data of a category $\mathcal{V}_0$ together with:

(1) an object $I \in \mathcal{V}_0$ called *unit*;

(2) a functor $-\otimes-\colon \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$, called *tensor product*;

(3) natural isomorphisms:

   (a) $\alpha_{A,B,C}\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ for any $A, B, C \in \mathcal{V}_0$;
   (b) $\lambda_A\colon I \otimes A \to A$ and $\rho_A\colon A \otimes I \to A$ for any $A \in \mathcal{V}_0$.

These are subject to the following coherence axioms:

(1) for any A, B, C and D in $\mathcal{V}_0$ the diagram below commutes;

$$\begin{tikzcd}
& A \otimes (B \otimes (C \otimes D)) \arrow[dl, "1 \otimes \alpha"'] \arrow[dr, "\alpha"] & \\
A \otimes ((B \otimes C) \otimes D) \arrow[d, "\alpha"'] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha"] \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha \otimes 1"'] & & ((A \otimes B) \otimes C) \otimes D
\end{tikzcd}$$

(2) for any A and B in $\mathcal{V}_0$ the diagram below commutes.

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\alpha"] \arrow[dr, "1 \otimes \lambda"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho \otimes 1"] \\
& A \otimes B &
\end{tikzcd}$$

Given a monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$, we usually write $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ and assume that α, λ and ρ are understood. Let us see some examples:

Example 1.1.2.

(1) If $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$ is a monoidal category, then

$$\mathcal{V}^{\text{op}} = (\mathcal{V}_0^{\text{op}}, \otimes^{\text{op}}, I, \alpha^{-1}, \lambda^{-1}, \rho^{-1})$$

is a monoidal category.

(2) If $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$ is a monoidal category, then

$$\mathcal{V}^{\text{rev}} = (\mathcal{V}_0, \hat{\otimes}, I, \hat{\alpha}, \hat{\lambda}, \hat{\rho})$$

is a monoidal category where:

(a) $A \mathbin{\hat{\otimes}} B := B \otimes A$;
(b) $\hat{\alpha}_{A,B,C} := \alpha^{-1}_{C,B,A}$;
(c) $\hat{\lambda}_A := \rho_A$ and $\hat{\rho}_A := \lambda_A$.

(3) Let $\mathcal{C}$ be any category with finite products; then $(\mathcal{C}, \times, 1)$ is a monoidal category where α, ρ and λ are induced by the universal property of the products; this structure is called the *cartesian* structure in $\mathcal{C}$. By (1) it follows that if $\mathcal{C}$ has finite coproducts, then $(\mathcal{C}, +, 0)$ is a monoidal category; this monoidal structure is called *cocartesian.*

(4) The category **Ab** of abelian groups and group homomorphisms, together with the tensor product $\otimes$ of abelian groups, and the unit $\mathbb{Z}$, forms a monoidal category. More generally, the category R-Mod of modules over a commutative ring R is monoidal with tensor product given by $\otimes_R$ and unit R.

(5) The category **DGAb** of differentially graded abelian groups (that is, chain complexes) is monoidal with the standard tensor product of chain complexes:

$$(A \otimes B)_n := \bigoplus_{i+j=n} A_i \otimes B_j$$

with differential generated by the rule

$$d^n_{A\otimes B}(a \otimes b) = d^i_A a \otimes b + (-1)^i a \otimes d^j_B b$$

where $a \in A_i$ and $b \in B_j$, with $i + j = n$. The unit I given by the chain complex with $\mathbb{Z}$ on degree 0 and (0) otherwise.

(6) Let $(G, \cdot, e)$ be a group (a monoid is enough) and G_0 be the discrete category on the underlying set of G. Then $\mathcal{G} := (G_0, \ \cdot\ , e)$ is a (strict) monoidal category where the tensor product of two objects is defined by multiplication in G.

To define what a category enriched over $\mathcal{V}$ we shall need only a monoidal category; however, to to certain constructions (like the opposite $\mathcal{V}$-category) we shall need the monoidal category $\mathcal{V}$ to also be symmetric:

Definition 1.1.3. A *symmetric monoidal category* $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is the data of a monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$ together with a natural isomorphism $\sigma_{A,B} \colon A \otimes B \to B \otimes A$ for any $A, B \in \mathcal{V}_0$, subject to the following coherences:

(1) for any $A, B \in \mathcal{V}_0$ we have $\sigma_{B,A} \circ \sigma_{A,B} = 1_{A\otimes B}$ and $\lambda_A \circ \sigma_{A,I} = \rho_A$;

(2) for any $A, B, C \in \mathcal{V}_0$ the diagram below commutes.

$$\begin{tikzcd}
 & (A \otimes B) \otimes C \arrow[dl, "\sigma \otimes 1"'] \arrow[dr, "\alpha"] & \\
(B \otimes A) \otimes C \arrow[d, "\alpha"'] & & A \otimes (B \otimes C) \arrow[d, "\sigma"] \\
B \otimes (A \otimes C) \arrow[dr, "1 \otimes \sigma"'] & & (B \otimes C) \otimes A \arrow[dl, "\alpha"] \\
 & B \otimes (C \otimes A) &
\end{tikzcd}$$

Again, we will write simply $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ for a symmetric monoidal category $\mathcal{V}$ where all the structure maps are understood. Often, we will say that a monoidal category $\mathcal{V}$ *is symmetric* to mean that $\mathcal{V}$ *comes equipped with a chosen symmetry* (note, a monoidal category might have more than one symmetry — see Exercise 1.3.4).

Example 1.1.4.

(1) If $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is symmetric monoidal, then

$$\mathcal{V}^{\mathrm{op}} = (\mathcal{V}_0^{\mathrm{op}}, \otimes^{\mathrm{op}}, I, \alpha^{-1}, \lambda^{-1}, \rho^{-1}, \sigma^{-1})$$

is symmetric monoidal.

(2) If $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is symmetric monoidal, then

$$\mathcal{V}^{\text{rev}} = (\mathcal{V}_0, \hat{\otimes}, I, \hat{\alpha}, \hat{\lambda}, \hat{\rho}, \hat{\sigma})$$

is symmetric monoidal where $\hat{\sigma}_{A,B} = \sigma_{B,A}$.

(3) Let $\mathcal{C}$ be any category with finite products; then $(\mathcal{C}, \times, 1)$ is symmetric monoidal where σ is induced by the universal property of the products. Similarly, if $\mathcal{C}$ has finite coproducts, then $(\mathcal{C}, +, 0)$ is symmetric monoidal.

(4) The monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$ is symmetric with symmetry

$$\sigma_{A,B} \colon A \otimes B \to B \otimes A$$

defined by sending a generator $a \otimes b \in A \otimes B$ to $\sigma_{A,B}(a \otimes b) := b \otimes a \in B \otimes A$. Similarly, the category R-Mod of modules over a commutative ring R is symmetric monoidal.

(5) The monoidal category **DGAb** of differentially graded abelian groups, with the standard tensor product, is symmetric. The symmetry is defined on the generators as

$$\sigma_{A,B}(a \otimes b) := (-1)^{ij} b \otimes a,$$

where $a \in A_i$ and $b \in B_j$.

(6) Let $(G, \cdot, e)$ be a group. Then $\mathcal{G} = (G_0, \cdot, e)$ is a symmetric monoidal if and only if the multiplication is G is commutative; that is, if and only if G is abelian.

Example 1.1.5. The category **Grp** of groups and group homomorphisms, with the monoidal closed structure given by the standard tensor product of groups is not symmetric.

The last concept, that will be needed to consider for instance enriched categories of functors, is that of symmetric closed monoidal category.

Definition 1.1.6. A *symmetric monoidal closed category* $\mathcal{V} = (\mathcal{V}_0, \otimes, [-,-])$ is the data of a symmetric monoidal category together with functor

$$[-,-] \colon \mathcal{V}_0^{\text{op}} \times \mathcal{V}_0 \to \mathcal{V}_0,$$

called *internal hom*, such that $(-) \otimes B \dashv [B, -]$ for any $B \in \mathcal{V}_0$. In other words, we have an isomorphism

$$\mathcal{V}_0(A \otimes B, C) \cong \mathcal{V}_0(A, [B, C])$$

natural in A, B, and C in $\mathcal{V}_0$.

Since the internal-hom functor $[-,-]$, when it exists, is uniquely determined (up to isomorphism) from the tensor product, we will still denote a symmetric monoidal closed category by just $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$.

Remark 1.1.7. Fixed a symmetric monoidal closed category $\mathcal{V}$, we will often use the tensor-hom adjunction to "transpose" maps out of a tensor products to maps into an internal-hom. More precisely, to give a morphism

$$f \colon A \otimes B \longrightarrow C$$

is the same as to give a map
$$f^{\mathrm{t}}\colon A \to [B, C]$$
which is the transpose of f under $(-) \otimes B \dashv [B, -]$. Since $\mathcal{V}$ is also symmetric, the map f also corresponds to $f\sigma\colon B \otimes A \to C$ (up to isomorphism). Therefore to give f is also the same as to give
$${}^{\mathrm{t}}f\colon B \to [A, C].$$
We will use these natural bijections very often when constructing certain $\mathcal{V}$-categories. The notation with f, f^{t}, and ${}^{\mathrm{t}}f$ will not always be consistent with the one used above; for instance, if we start with $g\colon B \to [A, C]$ we will use the transpose to denote the other two maps.

The following examples will provide interesting instances of enrichment.

Example 1.1.8.

(1) The singleton $1 = \{*\}$, seen as a one-object discrete category, is (trivially) symmetric monoidal closed.

(2) The category **Set** of sets and functions with its cartesian structure is symmetric monoidal closed (also called, cartesian closed). The internal hom is simply given by
$$[A, B] := \mathbf{Set}(A, B).$$
Indeed, for any sets A, B, and C we have a natural isomorphism
$$\mathbf{Set}(A \times B, C) \cong \mathbf{Set}(A, \mathbf{Set}(B, C))$$
given by sending $f\colon A \times B \to C$ to $\hat{f}\colon A \to \mathbf{Set}(B, C)$ defined by $\hat{f}(a)(b) := f(a, b)$.

(3) Every (elementary) topos with its cartesian structure is symmetric monoidal closed by definition. The internal hom is usually denoted by $[A, B] := B^A$.

(4) The category **Cat** of categories is cartesian closed; the internal hom $[A, B]$ is the category of functors $A \to B$ and natural transformations between them.

(5) The monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$ is symmetric monoidal closed with internal hom $[A, B] := \mathbf{Ab}(A, B)$ endowed with the pointwise addition which makes it an abelian group. Similarly, the category R-Mod of modules over a commutative ring R is symmetric monoidal closed.

(6) The monoidal category **DGAb** of differentially graded abelian groups, with the standard tensor product, is symmetric monoidal closed. The internal hom is given by
$$[A, B]_n := \prod_{i \in \mathbb{Z}} \mathbf{Ab}(A_i, B_{i+n})$$
and with differential $d(f) = d_B \circ f - (-1)^n f \circ d_A$, where $f \in [A, B]_n$.

(7) Let $(G, \cdot, e)$ be an abelian group. Then $\mathcal{G} = (G_0, \ \cdot\ , e)$ is symmetric monoidal closed with internal hom
$$[g, h] := k \cdot h^{-1}$$
for any $g, h \in G$.

(8) Consider the arrow category $\mathbf{2} = \{0 \to 1\}$, then $(\mathbf{2}, \times, 1)$ is symmetric monoidal closed (it is cartesian closed as a full subcategory of **Set**).

(9) Consider the category $\mathbf{Set}_*$ of pointed sets: objects are pairs $(X \in \mathbf{Set}, x \in X)$, and morphisms $(X, y) \to (Y, y)$ are functions $f\colon X \to Y$ sending x to y. Between any two objects $A = (X, x)$ and $B = (Y, y)$ we can define a tensor product

$$A \wedge B := \left(\frac{X \times Y}{A \vee B},\ [A \vee B] \right)$$

where $A \vee B := (\{x\} \times Y) \cup (X \times \{y\}) \subseteq X \times Y$, and $[A \vee B]$ is the single point defined by the equivalence class of $A \vee B$ in the quotient. The unit is defined as $I := (\{0, 1\}, 0)$. It is easy to define the structure maps that make $(\mathbf{Set}_*, \wedge, I)$ into a symmetric monoidal category. This is in addition monoidal closed with internal hom defined as

$$[A, B] := (\mathbf{Set}_*(A, B), \Delta y)$$

where $B = (Y, y)$ and Δy is the constant function at $y \in Y$.

(10) Consider the poset $\overline{\mathbb{R}}_+ = ([0, \infty], \geq)$ as a category. Then $(\overline{\mathbb{R}}_+, +, 0)$ forms a symmetric monoidal closed category with internal hom

$$[a, b] = \begin{cases} b - a & \text{if } b \geq a \\ 0 & \text{otherwise} \end{cases}$$

for any $a, b \in \overline{\mathbb{R}}_+$.

(11) Consider the category $\mathbf{Met}$ whose objects are generalized metric spaces (we allow the distance to take value in $[0, \infty]$, everything else is as usual), and whose morphisms $f\colon (X, d_X) \to (Y, d_Y)$ are functions $f\colon X \to Y$ such that $d_Y(fx, fx') \leq d_X(x, x')$. On $\mathbf{Met}$ we have a symmetric monoidal closed structure with unit given by the singleton 1, tensor product

$$(X, d_X) \otimes (Y, d_Y) := (X \times Y,\ d_{X \otimes Y}((x, y), (x', y')) := d_X(x, x') + d_Y(y, y')\ ),$$

and internal hom

$$[(X, d_X), (Y, d_Y)] := (\mathbf{Met}((X, d_X), (Y, d_Y)),\ d(f, g) := \sup\{d_Y(fx, gy)\ |x \in X\})$$

for any $(X, d_X), (Y, d_Y) \in \mathbf{Met}$.

1.2 Enriched categories

From now on $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ is assumed to be a monoidal category.

Definition 1.2.1. A *$\mathcal{V}$-category* $\mathcal{C}$ is the data of:

(1) a class $\mathrm{Ob}(\mathcal{C})$ whose elements are called the *objects* of $\mathcal{C}$;

(2) for any $A, B \in \mathrm{Ob}(\mathcal{C})$ an object $\mathcal{C}(A, B) \in \mathcal{V}$ called the *hom-object* of morphisms from A to B;

(3) for any $A \in \mathrm{Ob}(\mathcal{C})$ an *identity map* $\mathrm{Id}_A\colon I \to \mathcal{C}(A, A)$ in $\mathcal{V}$;

(4) for any $A, B, C \in \mathrm{Ob}(\mathcal{C})$ a *composition map*

$$\circ_{A,B,C} \colon \mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,C)$$

in $\mathcal{V}$.

These are subject to the following axioms:

(1) *Identity*: for any $A, B \in \mathrm{Ob}(\mathcal{C})$ the diagram below commutes.

$$\begin{tikzcd} I \otimes \mathcal{C}(A,B) \arrow[dr, "\lambda"] \arrow[d, "\mathrm{Id}_B \otimes 1"'] & & \mathcal{C}(A,B) \otimes I \arrow[dl, "\rho"'] \arrow[d, "1 \otimes \mathrm{Id}_A"] \\ \mathcal{C}(B,B) \otimes \mathcal{C}(A,B) \arrow[r, "\circ_{A,B,B}"'] & \mathcal{C}(A,B) & \mathcal{C}(A,B) \otimes \mathcal{C}(A,A) \arrow[l, "\circ_{A,A,B}"] \end{tikzcd}$$

(2) *Associativity*: for any $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$ the following diagram

$$\begin{tikzcd} \mathcal{C}(C,D) \otimes (\mathcal{C}(B,C) \otimes \mathcal{C}(A,B)) \arrow[rr, "\alpha"] \arrow[d, "1 \otimes \circ_{A,B,C}"'] & & (\mathcal{C}(C,D) \otimes \mathcal{C}(B,C)) \otimes \mathcal{C}(A,B) \arrow[d, "\circ_{B,C,D} \otimes 1"] \\ \mathcal{C}(C,D) \otimes \mathcal{C}(A,C) \arrow[dr, "\circ_{A,C,D}"'] & & \mathcal{C}(B,D) \otimes \mathcal{C}(A,B) \arrow[dl, "\circ_{A,B,D}"] \\ & \mathcal{C}(A,D) & \end{tikzcd}$$

commutes.

The $\mathcal{V}$-category $\mathcal{C}$ is called *small* if the class of objects $\mathrm{Ob}(\mathcal{C})$ forms a small set.

From now on we will write $A \in \mathcal{C}$ to mean that $A \in \mathrm{Ob}(\mathcal{C})$. When it is clear from the context, we will simply write $\circ$ instead of $\circ_{A,B,C}$ for the composition map.

Given a $\mathcal{V}$-category $\mathcal{C}$ we can always construct an ordinary category $\mathcal{C}_0$ called the *underlying category of* $\mathcal{C}$. This has the same objects as $\mathcal{C}$ and hom-sets

$$\mathcal{C}_0(A,B) := \mathcal{V}_0(I, \mathcal{C}(A,B));$$

identities are the same as in $\mathcal{C}$ and the composition map is given by

$$\mathcal{C}_0(B,C) \times \mathcal{C}_0(A,B) \xrightarrow{\beta} \mathcal{V}_0(I, \mathcal{C}(B,C) \otimes \mathcal{C}(A,B)) \xrightarrow{\mathcal{V}_0(I, \circ_{A,B,C})} \mathcal{C}_0(A,C)$$

where $\beta(f,g) := (f \otimes g)\lambda_I^{-1}$. It is easy to see that the identity and associativity axioms still hold from $\mathcal{C}_0$, making it a category.

Notation 1.2.2. For an enriched category $\mathcal{C}$ we will always denote by $\mathcal{C}_0$ its underlying category as defined above. If we say that $f \colon A \to B$ is a morphism in $\mathcal{C}$, we actually mean that f is a morphism in $\mathcal{C}_0$, and therefore a map $f \colon I \to \mathcal{C}(A,B)$.
In particular, when we say that f is an isomorphism in $\mathcal{C}$, we mean that it is an isomorphism in $\mathcal{C}_0$.

We now turn to the definition of $\mathcal{V}$-functor:

Definition 1.2.3. Given $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$, a *$\mathcal{V}$-functor* $F\colon \mathcal{C} \to \mathcal{D}$ is the data of:

(1) a function $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$;

(2) for any $A, B \in \mathcal{C}$ a morphism

$$F_{A,B}\colon \mathcal{C}(A,B) \longrightarrow \mathcal{D}(FA,FB)$$

in $\mathcal{V}$.

These are subject to the following axioms:

(1) for any $A \in \mathcal{C}$ the triangle below commutes;

$$\begin{tikzcd} & I \arrow[dl, "\mathrm{Id}^{\mathcal{C}}_A"'] \arrow[dr, "\mathrm{Id}^{\mathcal{D}}_{FA}"] & \\ \mathcal{C}(A,A) \arrow[rr, "F_{A,A}"'] & & \mathcal{D}(FA,FA) \end{tikzcd}$$

(2) for any $A, B, C \in \mathcal{C}$ the following square

$$\begin{tikzcd} \mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \arrow[r, "\circ^{\mathcal{C}}"] \arrow[d, "F_{B,C} \otimes F_{A,B}"'] & \mathcal{C}(A,C) \arrow[d, "F_{A,C}"] \\ \mathcal{D}(FB,FC) \otimes \mathcal{D}(FA,FC) \arrow[r, "\circ^{\mathcal{D}}"'] & \mathcal{D}(FA,FC) \end{tikzcd}$$

commutes.

Given a $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ between enriched categories, we can define an ordinary functor $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$, called *underlying functor of* F, between the corresponding underlying categories. The functor F_0 acts on objects as F and the action on the hom-sets is defined as

$$(F_0)_{A,B} := \mathcal{V}_0(I, F_{A,B})\colon \mathcal{C}_0(A,B) \longrightarrow \mathcal{D}_0(FA,FB).$$

It is easy to see that F_0 preserves identities and composition (since F does) and is therefore a functor.

For any pair of $\mathcal{V}$-functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ we define the *composite $\mathcal{V}$-functor* $GF\colon \mathcal{C} \to \mathcal{E}$ by setting $GF(A) := G(F(A))$ and

$$(GF)_{A,B}\colon \mathcal{C}(A,B) \xrightarrow{F_{A,B}} \mathcal{D}(FA,FB) \xrightarrow{G_{FA,FB}} \mathcal{E}(GFA,GFB)$$

for any $A, B \in \mathcal{C}$. The fact that GF satisfies the axioms of a $\mathcal{V}$-functor follows by stacking together the commutativity conditions of F and G.

For any $\mathcal{V}$-category $\mathcal{C}$ we have an identity $\mathcal{V}$-functor $1_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ defined by $1_{\mathcal{C}}(A) = A$ and $(1_{\mathcal{C}})_{A,B} = 1_{\mathcal{C}(A,B)}$ in $\mathcal{V}$.

Notation 1.2.4. For an enriched functor F we will always denote by F_0 its underlying functor as defined above. Small $\mathcal{V}$-categories, $\mathcal{V}$-functors, identity $\mathcal{V}$-functors, and composition of $\mathcal{V}$-functors, form a category $\mathcal{V}$-**Cat**. The underlying category/functor construction defines a functor

$$(-)_0 \colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}\,.$$

When the $\mathcal{V}$-categories are not assumed to be small, we call the induced category $\mathcal{V}$-**CAT**.

One-object $\mathcal{V}$-categories are important on their own:

Example 1.2.5. Let $\mathcal{M}$ be a one object $\mathcal{V}$-category; then $M := \mathcal{M}(*,*)$ (where $*$ is the unique object of $\mathcal{M}$) is endowed with a monoid structure induced by the composition map

$$\circ \colon M \otimes M \to M$$

and the identity $\mathrm{Id}_* \colon I \to M$ (the unitality and associativity axioms follow from those of a $\mathcal{V}$-category). Conversely, any monoid (M, μ, η) in $\mathcal{V}$ induces a unique $\mathcal{V}$ category $\mathcal{M}$ with a single object $*$, hom-object $\mathcal{M}(*,*) := M$, composition given by μ, and identity given by η. Under this interpretation, a $\mathcal{V}$-functor between single-object $\mathcal{V}$-categories is just a monoid morphism.

We now turn to some more concrete examples:

Example 1.2.6.

(1) Let $\mathcal{V} = (\mathbf{Set}, \times, 1)$. Then a $\mathcal{V}$-category is just an ordinary category and a $\mathcal{V}$-functor is just a functor; therefore $\mathcal{V}$-$\mathbf{Cat} \cong \mathbf{Cat}$.

(2) Let $\mathcal{V} = 1$ be the discrete category with one object (with the trivial monoidal structure). Then a $\mathcal{V}$-category is completely determined by its objects (every hom-objects is the same and composition and identities are forced to be the identities of 1). It follows that $\mathcal{V}$-$\mathbf{Cat} \cong \mathbf{Set}$.

(3) Let $\mathcal{V} = (\mathbf{Cat}, \times, 1)$ be the cartesian closed category of small categories. Then a $\mathcal{V}$-category is commonly called a *2-category*. We will study these in more detail later.

(4) Let $\mathcal{V} = (\mathbf{Ab}, \otimes, \mathbb{Z})$ the the symmetric monoidal closed category of abelian groups. These $\mathcal{V}$-categories are commonly called *pre-additive* categories. To give a pre-additive category $\mathcal{C}$ is the same as giving an ordinary category whose hom-sets $\mathcal{C}(A, B)$ come with an abelian group structure, and such that composition $- \circ -$ is linear in both variables.

(5) Let $\mathcal{V} = (\mathbf{DGAb}, \times, I)$ be the symmetric monoidal closed category of chain complexes. Then a $\mathcal{V}$-category is commonly called a *DG-category*. We will study these in more detail later.

(6) Let $\mathcal{V} = (\mathbf{2}, \times, 1)$, where $\mathbf{2} = \{0 \to 1\}$ is the arrow category. We will show that to give a $\mathcal{V}$-category is the same as to give a preorder. Given a $\mathcal{V}$-category $\mathcal{C}$, for any pair of objects A, B, we have

$$\mathcal{C}(A, B) \in \{0, 1\}$$

and, the fact that we have maps $\mathrm{Id}_A \colon 1 \to \mathcal{C}(A, A)$, says that $\mathcal{C}(A, A) = 1$ for each A (and that Id_A is the identity map in $\mathbf{2}$). The composition map

$$\circ_{A,B,C} \colon \mathcal{C}(B, C) \times \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$$

is uniquely determined when at least one between $\mathcal{C}(B,C)$ and $\mathcal{C}(A,B)$ is 0; when they are both 1 one necessarily has that also $\mathcal{C}(A,C) = 1$ (and hence the composition map is the identity). Finally, the identity and commutativity axioms do not add any information (every diagram commutes in **2**). Thus, given $\mathcal{C}$, consider the set (or class) $\Sigma := \mathrm{Ob}(C)$, and define the relation: $A \leq B$ if and only if $\mathcal{C}(A,B) = 1$. Because of the arguments above it follows that $A \leq A$ for any $A \in \Sigma$, and that whenever $A \leq B \leq C$ then also $A \leq C$ (composition law). Therefore $(\Sigma, \leq)$ forms a preorder.

Conversely, if $(\Sigma, \leq)$ is a preorder we can define a $\mathcal{V}$-category $\mathcal{C}$ with $\mathrm{Ob}(C) := \Sigma$ and

$$\mathcal{C}(A,B) = \begin{cases} 1 & \text{if } A \leq B \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that this is a well defined $\mathcal{V}$-category. Similarly, one shows that to give a $\mathcal{V}$-functor, under this translation, is the same as giving an order-preserving function. It follows that $\mathcal{V}$-**Cat** $\cong$ **Prd** is isomorphic to the category of preorders.

It is straightforward to see that a $\mathcal{V}$-category $\mathcal{C}$ is skeletal (that is, if $A \to B$ is an isomorphisms then $A = B$) if and only if the corresponding preorder $(\Sigma, \leq)$ satisfies the antisymmetry law ($A \leq B \leq A$ implies $A = B$), if and only if $(\Sigma, \leq)$ is a poset. As a consequence, the full subcategory $\mathcal{V}$-$\mathbf{Cat}_{\mathrm{sk}}$ of $\mathcal{V}$-**Cat** spanned by the skeletal $\mathcal{V}$-categories is isomorphic to **Pos**, the category of posets and order preserving maps.

(7) Let $(\overline{\mathbb{R}}_+, +, 0)$ be the monoidal category where $\overline{\mathbb{R}}_+ = ([0,\infty], \geq)$. Then a small $\overline{\mathbb{R}}_+$-category is a *Lawvere metric space*: to give a small $\overline{\mathbb{R}}_+$-category we need

(a) a set X (the objects);
(b) for any $x, y \in X$ a real number $d(x,y) \in [0,\infty]$ (the hom-objects);
(c) (identity maps) for any $x \in X$ we have $0 \geq d(x,x)$;
(d) (composition maps) for any $x, y, z \in X$ we have

$$d(y,z) + d(x,y) \geq d(x,z).$$

There are no axioms to satisfy since all diagrams commute in a poset. It follows that a $\overline{\mathbb{R}}_+$-category is determined by a set X together with a metric $d\colon X \times X \to [0,\infty]$ which satisfies the triangle inequality and for which $d(x,x) = 0$ for any $x \in X$ (note that d may not be symmetric).

In such a $\overline{\mathbb{R}}_+$-category, two points might be distinct and still have distance equal to 0. This can be avoided (arguing as in the previous example) by considering skeletal $\overline{\mathbb{R}}_+$-categories.

Similarly, it is easy to see that a $\mathcal{V}$-functor between Lawvere metric spaces (X, d_X) and (Y, d_Y) is given by a function $f\colon X \to Y$ such that for any $x, y \in X$ we have

$$d_Y(fx, fy) \leq d_X(x,y);$$

indeed, f itself is the assignment on objects, while the condition above is expressed by the structure maps $f_{x,y}$. It follows that $\mathcal{V}$-functors can be identified with non-increasing maps between Lawvere metric spaces.

1.3 Exercises

Exercise 1.3.1. Prove that the category **Ab** of abelian groups is not cartesian closed.

Exercise 1.3.2. Let **Rng** be the category of rings with unit and ring homomorphisms. Show that there is no symmetric monoidal closed structure on **Rng** with unit $\mathbb{Z}$.

Exercise 1.3.3. Show that the category **Met**, of generalized metric spaces and non-increasing maps, is not cartesian closed.

Exercise 1.3.4. Prove that the monoidal category **GAb** of graded abelian groups (with unit and tensor product defined as for **DGAb** forgetting about the differential) has exactly two different symmetries given by:

$$\sigma_{A,B}(a \otimes b) := b \otimes a \quad \text{and} \quad \tau_{A,B}(a \otimes b) := (-1)^{ij} b \otimes a$$

where $a \in A_i$ and $b \in B_j$.

Exercise 1.3.5. Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ be symmetric monoidal closed and consider the functor

$$(-)^* := [-, I] \colon \mathcal{V}_0^{\mathrm{op}} \longrightarrow \mathcal{V}_0.$$

Exhibit a natural transformation $(-) \Rightarrow (-)^{**}$ and show that it is not in general an isomorphism.

Exercise 1.3.6. Assume that $\mathcal{V}$ is symmetric monoidal closed. Rewrite the definition of $\mathcal{V}$-category and $\mathcal{V}$-functor in an equivalent form that only uses the unit I and the internal hom $[-,-]$, but not the tensor product $\otimes$.

Exercise 1.3.7. Let $\mathcal{V}$-**Mon** be the full subcategory of $\mathcal{V}$-**Cat** spanned by the 1-object $\mathcal{V}$-categories; we know this to be isomorphic to the category of monoids in $\mathcal{V}$. Show that:

(1) for $\mathcal{V} = \mathbf{Ab}$ we have **Ab**-**Mon** $\cong$ **Rng**;

(2) given any $R \in \mathbf{Rng}$ denote by $\bar{R}$ the corresponding **Ab**-category, then to give an **Ab**-functor $M \colon \bar{R} \to \mathbf{Ab}$ is the same as to give an R-module;

(3) for any $R \in \mathbf{Rng}$, to give an **Ab**-functor $\bar{R} \to \mathbf{Ab}$ is the same as to give a left R-module.

Exercise 1.3.8. Consider the functor $U \colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathbf{Set}$ which sends a $\mathcal{V}$-category $\mathcal{C}$ to its set of objects. Give conditions on $\mathcal{V}$ so that U has a right adjoint, and compute the adjoint explicitly.

Exercise 1.3.9. Let $(G, \cdot, e)$ be a group. A *G-torsor* is the data of a set T together with a group action $a \colon G \times T \to T$ satisfying the following condition: for any $s, t \in T$ there is a unique $g \in G$ for which $a(g, s) = t$.
Consider the monoidal category $\mathcal{G} = (G_0, \cdot, e)$ induced from G as in Example 1.1.2. Show that every G-torsor can be interpreted as a $\mathcal{G}$-category. What condition do you need to impose on a $\mathcal{G}$-category so that it is induced by a G-torsor?

Exercise 1.3.10. A category $\mathcal{C}$ with both a terminal and a initial object that are isomorphic, is called *pointed*. Show that:

(1) $\mathbf{Set}_*$ is pointed;

(2) every pointed category is naturally enriched over $\mathbf{Set}_*$;

(3) if $\mathcal{D}$ is a $\mathbf{Set}_*$-category, then: $\mathcal{D}_0$ has an initial and a terminal object if and only if it is pointed.

Exercise 1.3.11. Denote by **CMon** the symmetric monoidal closed category of *commutative monoids*; the tensor product is defined by the property that monoid maps $A \otimes B \to C$ are in natural bijection with bilinear mas $A \times B \to C$ (as in **Ab**).
We say that a pointed category $\mathcal{C}$ has finite *direct sums* if for any $A_1, A_2 \in \mathcal{C}$ there is an object $A_1 \oplus A_2$ together with maps : $A_i \xrightarrow{\iota_i} A_1 \oplus A_2 \xrightarrow{\pi_i} A_i$, for $i = 1, 2$, such that $(A_1 \oplus A_2, \iota_1, \iota_2)$ is a coproduct, $(A_1 \oplus A_2, \pi_1, \pi_2)$ is a product, and the composite $\pi_j \circ \iota_i$ is 1_{A_i} for $i = j$ and 0 otherwise. Show that:

(1) **CMon** has finite direct sums;

(2) every category with finite direct sums is naturally enriched over **CMon**;

(3) if $\mathcal{D}$ is a **CMon**-category, then: $\mathcal{D}_0$ has a finite products if and only if it has finite coproducts if and only if it has finite direct sums.

CHAPTER

2

Week 2

2.1 Several constructions

When $\mathcal{V}$ is symmetric monoidal closed we can endow $\mathcal{V}_0$ with a structure of $\mathcal{V}$-category:

Proposition 2.1.1. *Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ be symmetric monoidal closed. Then there is a $\mathcal{V}$-category, still denoted by $\mathcal{V}$, whose objects are the same as $\mathcal{V}_0$ and for which*

$$\mathcal{V}(A, B) := [A, B].$$

Moreover, the underlying category of $\mathcal{V}$ is $\mathcal{V}_0$ (so there is no clash in the notation).

Proof. We need to define the identity and composition maps of $\mathcal{V}$. The fact that the underlying category of $\mathcal{V}$ is $\mathcal{V}_0$ will follow from the isomorphism $\mathcal{V}_0(I, [A, B]) \cong \mathcal{V}_0(A, B)$.

For any $A \in \mathcal{V}$ we define the identity on A as the map

$$\lambda_A^t \colon I \to [A, A]$$

which is transpose to $\lambda_A \colon I \otimes A \to A$ under the internal-hom adjunction. Now, given any pair of objects $A, B \in \mathcal{V}$ we define

$$\mathrm{ev}_{A,B} \colon [A, B] \otimes A \to B$$

as the transpose of $1_{[A,B]} \colon [A, B] \to [A, B]$ under the same adjunction. It follows that, given $A, B, C \in \mathcal{V}$ we can define the composition map $\circ_{A,B,C}$ as the transpose of

$$[B, C] \otimes [A, C] \otimes A \xrightarrow{1_{[B,C]} \otimes \mathrm{ev}_{A,B}} [B, C] \otimes B \xrightarrow{\mathrm{ev}_{B,C}} C.$$

The fact that these satisfy the identity and associativity axioms follows (arguing by transposition) from the fact that $\otimes$ and I satisfy the coherence axioms of a monoidal category. □

When $\mathcal{V}$ is symmetric we can talk about opposite $\mathcal{V}$-categories and $\mathcal{V}$-functors:

Definition 2.1.2. Assume that $\mathcal{V}$ is symmetric. For any $\mathcal{V}$-category $\mathcal{C}$ we define a $\mathcal{V}$-category $\mathcal{C}^{\mathrm{op}}$ with:

(1) $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$;

(2) for any $A, B \in \mathrm{Ob}(\mathcal{C})$ we set $\mathcal{C}^{\mathrm{op}}(A,B) := \mathcal{C}(B,A) \in \mathcal{V}$;

(3) for any $A \in \mathrm{Ob}(\mathcal{C})$ an the identity map $I \to \mathcal{C}^{\mathrm{op}}(A,A) = \mathcal{C}(A,A)$ is the same as in $\mathcal{C}$;

(4) for any $A, B, C \in \mathrm{Ob}(\mathcal{C})$ the composition map $\circ^{\mathcal{C}^{\mathrm{op}}}_{A,B,C}$ is defined as

$$\mathcal{C}^{\mathrm{op}}(B,C) \otimes \mathcal{C}^{\mathrm{op}}(A,B) \xrightarrow{\sigma} \mathcal{C}(B,A) \otimes \mathcal{C}(C,B) \xrightarrow{\circ_{C,B,A}} \mathcal{C}(C,A) = \mathcal{C}^{\mathrm{op}}(A,C)$$

in $\mathcal{V}$.

If $F\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{V}$-functor, then there is an induced $\mathcal{V}$-functor

$$F^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{op}}$$

defined by:

(1) $F^{\mathrm{op}}(A) := F(A)$ for any $A \in \mathcal{C}^{\mathrm{op}}$;

(2) $F^{\mathrm{op}}_{A,B} := F_{B,A}$ for any $A, B \in \mathcal{C}^{\mathrm{op}}$.

It follows that taking opposites defines a functor

$$(-)^{\mathrm{op}}\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$$

moreover, since $\sigma \circ \sigma = 1$, this is an involution: $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$. It is also easy to check that $(\mathcal{C}^{\mathrm{op}})_0 = (\mathcal{C}_0)^{\mathrm{op}}$.

Example 2.1.3. Let $\mathcal{V} = (\mathbf{2}, \times, 1)$, so that a $\mathcal{V}$-category is a preorder $(\Sigma, \leq)$. Then

$$(\Sigma, \leq)^{\mathrm{op}} = (\Sigma, \geq)$$

is the preorder obtained from Σ by reflecting the relationship.

Example 2.1.4. Let $\mathcal{V} = (\overline{\mathbb{R}}_+, +, 0)$, so that a $\mathcal{V}$-category is a Lawvere metric space $X = (M, d)$. Under this identification, $X^{\mathrm{op}} := (M, d^{\mathrm{op}})$ is the Lawvere metric space with underlying set M and distance

$$d^{\mathrm{op}}(x,y) := d(y,x).$$

It follows that a Lawvere metric X space satisfies the symmetry condition (these are usually called generalized metric spaces) if and only if $X = X^{\mathrm{op}}$.

Now that we have defined opposite $\mathcal{V}$-categories and seen that $\mathcal{V}$ itself has a structure of $\mathcal{V}$-category, we can construct representable $\mathcal{V}$-functors.

Definition 2.1.5. Let $\mathcal{C}$ be a small $\mathcal{V}$-category and $C \in \mathcal{C}$; we denote by

$$\mathcal{C}(-, C) \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{V}$$

the $\mathcal{V}$-functor defined as follows:

(1) an object $A \in \mathcal{C}$ is sent to $\mathcal{C}(-, C)(A) := \mathcal{C}(A, C) \in \mathcal{V}$;

(2) for any $A, B \in \mathcal{C}$ the map

$$\mathcal{C}(-, C)_{A,B} \colon \mathcal{C}(B, A) \longrightarrow [\mathcal{C}(A, C), \mathcal{C}(B, C)]$$

is defined as the transpose of the composition morphism $\circ_{B,A,C}$.

That this preserves composition and identities follows from the fact that composition in $\mathcal{C}$ is associative and unital.

Applying the definition to $\mathcal{C}^{\mathrm{op}}$ (since $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$) we also obtain the covariant representable $\mathcal{V}$-functors

$$\mathcal{C}(C, -) \colon \mathcal{C} \longrightarrow \mathcal{V}$$

for any $C \in \mathcal{C}$.

Example 2.1.6. Consider $\mathcal{V} = (\mathbf{2}, \times, 1)$, so that a $\mathcal{V}$-category is just a preorder Σ. For any $x \in \Sigma$, the representable $\mathcal{V}$-functor

$$\Sigma(-, x) \colon \Sigma^{\mathrm{op}} \to \mathbf{2}$$

sends y to 1 if $y \leq x$, and to 0 otherwise. So, it can be identified with the lower set $I_x \subseteq \Sigma$ defined by all elements of Σ that are smaller than x.

Example 2.1.7. Consider $\mathcal{V} = (\overline{\mathbb{R}}_+, +, 0)$, for which $\mathcal{V}$-categories are Lawvere metric spaces. Given a Lawvere metric space X and $x \in X$, the representable

$$X(-, x) \colon X^{\mathrm{op}} \to \overline{\mathbb{R}}$$

is simply the distance function $d_X(-, x)$.

Notation 2.1.8. Given a morphism $f \colon A \to B$ in a $\mathcal{V}$-category $\mathcal{C}$, for any $C \in \mathcal{C}$ we have induced morphisms

$$\mathcal{C}(C, f) \colon \mathcal{C}(C, A) \longrightarrow \mathcal{C}(C, B)$$

and

$$\mathcal{C}(f, C) \colon \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C)$$

in $\mathcal{V}$. The former is defined as the composite

$$\mathcal{C}(C, A) \xrightarrow{\lambda^{-1}} I \otimes \mathcal{C}(C, A) \xrightarrow{f \otimes 1} \mathcal{C}(A, B) \otimes \mathcal{C}(C, A) \xrightarrow{\circ} \mathcal{C}(A, C),$$

the latter is defined similarly using ρ^{-1}.

2.2 $\mathcal{V}$-categories of $\mathcal{V}$-functors

For the following definition we assume $\mathcal{V}$ to be a monoidal category.

Definition 2.2.1. Given $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$, and $\mathcal{V}$-functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a $\mathcal{V}$-*natural transformation* $\eta\colon F \Rightarrow G$ is given by a family

$$\{\eta_A\colon FA \to GA\}_{A\in\mathcal{C}}$$

of morphisms in $\mathcal{D}$, subject to the commutativity of the square

$$\begin{tikzcd} \mathcal{C}(A,B) \arrow[r, "F_{A,B}"] \arrow[d, "G_{A,B}"'] & \mathcal{D}(FA,FB) \arrow[d, "\mathcal{D}(FA,\eta_B)"] \\ \mathcal{D}(GA,GB) \arrow[r, "\mathcal{D}(\eta_A,GB)"'] & \mathcal{D}(FA,GB) \end{tikzcd}$$

for any $A, B \in \mathcal{C}$.

Any $\mathcal{V}$-natural transformation $\eta\colon \mathcal{F} \Rightarrow G$ also defines an ordinary natural transformation

$$\eta\colon F_0 \Rightarrow G_0$$

between the underlying functors. Indeed, pre-composing the square above with maps $I \to \mathcal{C}(A,B)$, which are the morphisms of $\mathcal{C}_0$, one obtains the usual naturality square.

Remark 2.2.2. By adding $\mathcal{V}$-natural transformations as 2-cells, we can see $\mathcal{V}$-**Cat** as a 2-category. Most of the results involving $\mathcal{V}$-**Cat** can then be extended to the 2-dimensional setting. However, for the purposes of this course we will keep considering $\mathcal{V}$-**Cat** as an ordinary category.

Example 2.2.3. Let $\mathcal{V} = (\mathbf{2}, \times, 1)$. Then, to give two $\mathcal{V}$-functors is the same as giving two order preserving maps $f, g\colon \Sigma \to \Theta$ between preorders. To give a natural transformation $\eta\colon f \Rightarrow g$ is the same as specifying that for any $x \in \Sigma$ we have

$$fx \leq gx,$$

since we have a (necessarily unique) map $1 \to \Sigma(fx, gx)$ if and only if $fx \leq gx$ (the naturality square becomes trivial in this case). It follows that, given f, g as above, we can have at most one $\mathcal{V}$-natural transformation from f to g, denote $f \leq g$, and this exists if and only if $fx \leq gx$ for any $x \in \Sigma$.

Example 2.2.4. Let $\mathcal{V} = (\overline{\mathbb{R}}_+, +, 0)$. Then, to give two $\mathcal{V}$-functors is the same as giving two non-increasing maps $f, g\colon X \to Y$ between Lawvere metric spaces. To give a natural transformation $\eta\colon f \Rightarrow g$ is the same as saying that for any $x \in X$ we have

$$d_Y(fx, gx) = 0$$

since the only morphism out of the unit 0 in $\overline{\mathbb{R}}_+$ is the identity. It follows that, given f, g as above, we can have at most one $\mathcal{V}$-natural transformation from f to g and this exists if and only if $d_Y(fx, gx) = 0$ for any $x \in \Sigma$.

From now on we assume $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ to be symmetric monoidal closed and complete; this is needed define a $\mathcal{V}$-category structure on $\mathcal{V}$-$\mathbf{Cat}(\mathcal{B}, \mathcal{C})$.

Definition 2.2.5. Given a small $\mathcal{V}$-category $\mathcal{B}$ and a $\mathcal{V}$-category $\mathcal{C}$ we define a new $\mathcal{V}$-category $[\mathcal{B}, \mathcal{C}]$ for which:

(1) The class of objects $\mathrm{Ob}([\mathcal{B}, \mathcal{C}])$ is given by all $\mathcal{V}$-functors $\mathcal{B} \to \mathcal{C}$.

(2) For any $F, G \in [\mathcal{B}, \mathcal{C}]$ the hom-object is defined as the equalizer below

$$\begin{tikzcd}
{[\mathcal{B},\mathcal{C}](F,G)} \arrow[d, tail, "e_{F,G}"'] \\
\prod_{B\in\mathcal{B}} \mathcal{C}(FB,GB) \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] \\
\prod_{C,D\in\mathcal{B}} [\mathcal{B}(C,D), \mathcal{C}(FC,GD)]
\end{tikzcd}$$

in $\mathcal{V}$, where the components of u and v at $C, D \in \mathcal{B}$ are given respectively by the transposes of the maps below.

$$\begin{tikzcd}
\mathcal{B}(C,D) \otimes (\prod_{B\in\mathcal{B}} \mathcal{C}(FB,GB)) \arrow[d, "G_{C,D}\otimes\pi_C"'] & (\prod_{B\in\mathcal{B}} \mathcal{C}(FB,GB)) \otimes \mathcal{B}(C,D) \arrow[d, "\pi_D\otimes F_{C,D}"] \\
\mathcal{C}(GC,GD) \otimes \mathcal{C}(FC,GC) \arrow[d, "\circ"'] & \mathcal{C}(FD,GD) \otimes \mathcal{C}(FC,FD) \arrow[d, "\circ"] \\
\mathcal{C}(FC,GD) & \mathcal{C}(FC,GD)
\end{tikzcd}$$

(3) For any $F \in [\mathcal{B}, \mathcal{C}]$ the identity map $\mathrm{Id}_F \colon I \to [\mathcal{B}, \mathcal{C}](F, F)$ is induced by the universal property of the equalizer since the diagonal map below

$$\begin{tikzcd}
I \arrow[r, dashed, "\mathrm{Id}_F"] \arrow[dr, "(\mathrm{Id}_{FB})_{B\in\mathcal{B}}"'] & {[\mathcal{B},\mathcal{C}](F,F)} \arrow[d, tail, "e_{F,F}"] \\
& \prod_{B\in\mathcal{B}} \mathcal{C}(FB,FB)
\end{tikzcd}$$

equalizes u and v.

(4) for any $F, G, H \in [\mathcal{B}, \mathcal{C}]$ the composition map

$$\circ \colon [\mathcal{B},\mathcal{C}](G,H) \otimes [\mathcal{B},\mathcal{C}](F,G) \longrightarrow [\mathcal{B},\mathcal{C}](F,H)$$

is induced by the universal property of the equalizer since the bottom left composite below

$$\begin{tikzcd}
{[\mathcal{B},\mathcal{C}](G,H) \otimes [\mathcal{B},\mathcal{C}](F,G)} \arrow[r, dashed, "\circ"] \arrow[d, "(e^B_{G,H} \otimes e^B_{F,G})_{B\in\mathcal{B}}"'] & {[\mathcal{B},\mathcal{C}](F,H)} \arrow[d, tail, "e_{F,H}"] \\
\prod_{B\in\mathcal{B}}(\mathcal{C}(GB,HB) \otimes \mathcal{C}(FB,GB)) \arrow[r, "\prod\limits_{B\in\mathcal{B}} \circ^{\mathcal{C}}"'] & \prod_{B\in\mathcal{B}} \mathcal{C}(FB,HB)
\end{tikzcd}$$

equalizes the u and v involved in the definition of $[\mathcal{B},\mathcal{C}](F,H)$.

The fact that composition is associative and unital can be checked componentwise using that composition in $\mathcal{C}$ is associative and unital.

Remark 2.2.6. It is easy to see that to give a map $\alpha\colon I \to [\mathcal{B},\mathcal{C}](F,G)$ is the same as to give a $\mathcal{V}$-natural transformation from F to G. It follows then that

$$[\mathcal{B},\mathcal{C}]_0$$

is the category if $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations between them.

For a small $\mathcal{C}$, we call $[\mathcal{C}^{\mathrm{op}},\mathcal{V}]$ the $\mathcal{V}$-category of *(enriched) presheaves* on $\mathcal{C}$.

Example 2.2.7. Consider $\mathcal{V} = (\mathbf{2},\times,1)$, so that a $\mathcal{V}$-category is just a preorder. It is easy to see that, given two preorders Σ and Ω the $\mathcal{V}$-category of presheaves

$$[\Sigma,\Omega]$$

is the preorder whose elements are order preserving functions $f\colon \Sigma \to \Omega$, and where $f \le g$ if and only if $f(x) \le g(x)$ in Ω for any $x \in \Sigma$.

Example 2.2.8. Consider $\mathcal{V} = (\overline{\mathbb{R}}_+,+,0)$, for which $\mathcal{V}$-categories are Lawvere metric spaces. Given Lawvere metric spaces X and Y, then

$$[X,Y] = (\mathbf{Met}(X,Y), d_{[X,Y]}),$$

where $\mathbf{Met}(X,Y)$ is the set of non-increasing functions from X to Y and

$$d_{[X,Y]}(f,g) := \sup\{d_Y(fx,gy) \mid x \in X\}.$$

2.3 Exercises

Exercise 2.3.1. Assume that in $\mathcal{V}$ coproducts exist, and that the tensor product functor preserves these in both variables (the latter happens in particular whenever $\mathcal{V}$ is symmetric monoidal closed). Show then that $(-)_0\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}$ has a left adjoint

$$(-)_{\mathcal{V}}\colon \mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}\,.$$

Hint: given an ordinary category $\mathcal{B}$, use the coproducts of the unit to define the hom-objects of $\mathcal{B}_{\mathcal{V}}$.

Proof. Given an ordinary category $\mathcal{B}$, we define the $\mathcal{V}$-category $\mathcal{B}_{\mathcal{V}}$ as follows:

(1) $\mathcal{B}_{\mathcal{V}}$ has the same objects as $\mathcal{B}$;

(2) for any $A, B \in \mathcal{B}$ the hom-object is

$$\mathcal{B}_{\mathcal{V}}(A,B) := \sum_{\mathcal{B}(A,B)} I.$$

Note now that, since $I \otimes -$ and $- \otimes I$ preserve coproducts, for any sets X, Y we have a canonical isomorphism

$$\sum_{X\times Y} I \cong \sum_X \sum_Y (I \otimes I) \cong \sum_X I \otimes \sum_Y I.$$

Under this isomorphism, to define the composition maps in $\mathcal{B}_{\mathcal{V}}$ it is enough to give morphisms

$$\circ_{A,B.C} \colon \sum_{\mathcal{B}(B,C)\times\mathcal{B}(A,B)} I \longrightarrow \sum_{\mathcal{B}(A,C)} I;$$

then, given $(f, g) \in \mathcal{B}(B,C) \times \mathcal{B}(A,B)$ and using the universal property of the coproduct, we define the (f, g)-component of $\circ_{A,B.C}$ as the coproduct inclusion $\iota_{f\circ g} \colon I \to \mathcal{B}_{\mathcal{V}}(A,C)$, where $f \circ g$ the composite in $\mathcal{B}$. Similarly, for any $A \in \mathcal{B}$ we define the identity on A as the coproduct inclusion $\iota_{1_A} \colon I \to \mathcal{B}_{\mathcal{V}}(A,A)$, where 1_A is the identity on A in $\mathcal{B}$. It is easy to see that $\mathcal{B}_{\mathcal{V}}$ is a well defined $\mathcal{V}$-category.

We now need to prove that for any $\mathcal{V}$-category $\mathcal{C}$ and ordinary category $\mathcal{B}$ we have a bijection

$$\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{B}_{\mathcal{V}}, \mathcal{C}) \cong \mathbf{Cat}(\mathcal{B}, \mathcal{C}_0)$$

natural both in $\mathcal{C}$ and $\mathcal{B}$. Given a $\mathcal{V}$-functor $F \colon \mathcal{B}_{\mathcal{V}} \to \mathcal{C}$ we can define an ordinary functor $G \colon \mathcal{B} \to \mathcal{C}_0$ which acts as F on objects and sends $f \in \mathcal{B}(A,B)$ to

$$Gf \colon I \xrightarrow{\iota_f} \mathcal{B}_{\mathcal{V}}(A,B) \xrightarrow{F_{A,B}} \mathcal{C}(A,B).$$

Conversely, given an ordinary functor $G \colon \mathcal{B} \to \mathcal{C}_0$ we define a $\mathcal{V}$-functor $F \colon \mathcal{B}_{\mathcal{V}} \to \mathcal{C}$ which acts as G on objects and whose action on hom-objects is the arrow

$$\mathcal{B}_{\mathcal{V}}(A,B) = \sum_{\mathcal{B}(A,B)} I \longrightarrow \mathcal{C}(A,B)$$

whose component at $f \in \mathcal{B}(A,B)$ is given by $I \xrightarrow{\iota_{Gf}} \mathcal{C}(A,B)$. It is easy to see that these are well defined (as a $\mathcal{V}$-functor and functor respectively) and that are one the inverse of the other. $\square$

Exercise 2.3.2. Assume that $\mathcal{V}$ is symmetric monoidal closed. Show that the tensor product and internal hom induce $\mathcal{V}$-functors

$$X \otimes -, [X, -] \colon \mathcal{V} \longrightarrow \mathcal{V}$$

for any $X \in \mathcal{V}$.

Exercise 2.3.3. Assume that I is a generator of $\mathcal{V}_0$; that is, $\mathcal{V}_0(I, -) \colon \mathcal{V}_0 \to \mathbf{Set}$ is faithful. Show that to give a $\mathcal{V}$-natural transformation $F \Rightarrow G$ between $\mathcal{V}$-functors is the same as giving a natural transformation $F_0 \Rightarrow G_0$ between the underlying functors.

Exercise 2.3.4. Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ and $\mathcal{W} = (\mathcal{W}_0, \boxtimes, J)$ be two monoidal categories and

$F\colon \mathcal{V} \to \mathcal{W}$ a monoidal functor between them: this comes equipped with morphisms

$$J \xrightarrow{\epsilon} F(I) \qquad \text{and} \qquad F(A) \boxtimes F(B) \xrightarrow{\mu_{A,B}} F(A \otimes B),$$

in $\mathcal{W}$ for any $A, B \in \mathcal{V}$, satisfying certain associativity and unitality axioms. Show that F induces a functor

$$F^*\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{W}\text{-}\mathbf{Cat}\,.$$

Exercise 2.3.5. Assume that $\mathcal{V}$ is symmetric monoidal closed and complete. Define a symmetric monoidal closed structure $\mathcal{V}$-**Cat** whose interna-hom coincides with the $\mathcal{V}$-category $[\mathcal{C}, \mathcal{D}]$ of Definition 2.2.5

Exercise 2.3.6. Given $\mathcal{V}$-functors $F, G\colon \mathcal{B} \to \mathcal{C}$ and an object $X \in \mathcal{V}$; prove that there is a bijection (natural in F, G, and X) between:

(1) morphisms $X \to [\mathcal{B}, \mathcal{C}](F, G)$ in $\mathcal{V}$;

(2) $\mathcal{V}$-natural transformations $F \Rightarrow [X, G-]$;

(3) $\mathcal{V}$-natural transformations $X \otimes F(-) \Rightarrow G$.

In other words

$$\mathcal{V}_0(X, [\mathcal{B}, \mathcal{C}](F, G)) \cong [\mathcal{B}, \mathcal{C}]_0(F, [X, G-]) \cong [\mathcal{B}, \mathcal{C}]_0(X \otimes F(-), G).$$

CHAPTER 3

Week 3

3.1 Yoneda

We now wish to prove the enriched Yoneda lemma, stating that for any $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ and $C \in \mathcal{C}$ we have an isomorphism

$$FC \cong [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](\mathcal{C}(-,C), F)$$

in $\mathcal{V}$. To do so we fist need to construct the morphism $\eta_{C,F}\colon FC \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](\mathcal{C}(-,C), F)$, between these two objects, that will be proven to be invertible. This is induced by the universal property of the equalizer defining $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}](\mathcal{C}(-,C), F)$, and applied to the diagonal map

$$\begin{tikzcd}
FC \arrow[r, dashed, "\eta_{C,F}"] \arrow[dr, "(F^{\mathrm{t}}_{C,A})_{A\in\mathcal{C}}"'] & {[\mathcal{C}^{\mathrm{op}}, \mathcal{V}](\mathcal{C}(-,C), F)} \arrow[d, tail] \\
& \prod_{A\in\mathcal{C}} [\mathcal{C}(A,C), FA]
\end{tikzcd}$$

which equalizes the u and v involved in the definition.

Lemma 3.1.1. *For any small $\mathcal{V}$-category $\mathcal{C}$, $\mathcal{V}$-functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$, and object $C \in \mathcal{C}$ the map*

$$\eta_{C,F}\colon FC \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](\mathcal{C}(-,C), F)$$

is an isomorphism in $\mathcal{V}$.

Proof. By definition of $\eta_{C,F}$, it is enough to show that the diagram below

$$FC \xrightarrow{w} \prod_{A\in\mathcal{C}}[\mathcal{C}(A,C),FA] \overset{u}{\underset{v}{\rightrightarrows}} \prod_{B,B'\in\mathcal{C}}[\mathcal{C}(B',B),[\mathcal{C}(B,C),FB']]$$

is an equalizer, where $w := (F^{\mathrm{t}}_{C,A})_{A\in\mathcal{C}}$. Note that for any $X \in \mathcal{V}$, by unwinding the definition of u and v, to give a map $\alpha\colon X \to \prod_{A\in\mathcal{C}}[\mathcal{C}(A,C),FA]$ equalizing u and v is the same as giving a family of arrows

$$(\alpha_A\colon X \to [\mathcal{C}(A,C),FA])_{A\in\mathcal{C}}$$

for which the following square commutes

$$\begin{tikzcd}
X\otimes\mathcal{C}(A,C)\otimes\mathcal{C}(B,A) \arrow[r, "\alpha^{\mathrm{t}}_A\otimes 1"] \arrow[d, "1\otimes\circ_{B,A,C}"'] & FA\otimes\mathcal{C}(B,A) \arrow[d, "{}^{\mathrm{t}}F_{A,B}"] \\
X\otimes\mathcal{C}(B,C) \arrow[r, "\alpha^{\mathrm{t}}_B"'] & FB
\end{tikzcd}$$

for any $A,B\in\mathcal{C}$, where $(-)^{\mathrm{t}}$ and ${}^{\mathrm{t}}(-)$ denote (different, but uniquely determined) transpositions under the tensor-hom adjunction.

Now, for any such $\alpha = (\alpha_A\colon X \to [\mathcal{C}(A,C),FA])_{A\in\mathcal{C}}$, we can define the composite

$$\hat{\alpha}\colon X \xrightarrow{\rho^{-1}} X\otimes I \xrightarrow{1\otimes 1_C} X\otimes\mathcal{C}(C,C) \xrightarrow{\alpha^{\mathrm{t}}_C} FC.$$

We first show that $w\hat{\alpha}=\alpha$. This holds if and only if $F^{\mathrm{t}}_{C,A}\hat{\alpha}=\alpha_A$ for any $A\in\mathcal{C}$, if and only if the following triangle

$$\begin{tikzcd}
X\otimes\mathcal{C}(A,C) \arrow[r, "\hat{\alpha}\otimes 1"] \arrow[dr, "\alpha^{\mathrm{t}}_A"'] & FC\otimes\mathcal{C}(A,C) \arrow[d, "{}^{\mathrm{t}}F_{C,A}"] \\
& FA
\end{tikzcd}$$

commutes for any $A\in\mathcal{C}$. But this can be decomposed into the diagram below

$$\begin{tikzcd}
X\otimes\mathcal{C}(A,C) \arrow[r, "(1\otimes 1_C)\rho^{-1}\otimes 1"] \arrow[dr, "1"'] & X\otimes\mathcal{C}(C,C)\otimes\mathcal{C}(A,C) \arrow[r, "\alpha^{\mathrm{t}}_C\otimes 1"] \arrow[d, "1\otimes\circ_{A,C,C}"] & FC\otimes\mathcal{C}(A,C) \arrow[d, "{}^{\mathrm{t}}F_{C,A}"] \\
& X\otimes\mathcal{C}(A,C) \arrow[r, "\alpha^{\mathrm{t}}_A"'] & FA
\end{tikzcd}$$

where the top composite is still $\hat{\alpha}\otimes 1$, the left triangle commutes by unitality of composition, and the square on the right commutes since α equalizes u and v. Thus, $w\hat{\alpha}=\alpha$; it remains to show that $\hat{\alpha}$ is the unique with this property.

Given any other $x\colon X\to FC$ such that $wx=\alpha$, then looking at the component of w at C, we obtain $F^{\mathrm{t}}_{C,C}x=\alpha_C$. Consider then the following diagram

$$\begin{tikzcd}
X \arrow[rr, "x"] \arrow[d, "(1\otimes 1_C)\rho^{-1}"'] & & FC \arrow[d, "1\otimes 1_C"'] \arrow[dd, bend left, "1"] \\
X\otimes \mathcal{C}(C,C) \arrow[rr, "x\otimes 1"] \arrow[drr, "\alpha^{\mathrm{t}}_C"'] & & FC\otimes \mathcal{C}(C,C) \arrow[d, "{}^{\mathrm{t}}F_{C,C}"'] \\
& & FC
\end{tikzcd}$$

where the top part commutes by unitality, and the bottom triangle commutes by transposing the equality $F^{\mathrm{t}}_{C,C}x = \alpha_C$. This says exactly that $x = \hat{\alpha}$, and we are done. $\square$

Definition 3.1.2. We say that a $\mathcal{V}$-functor $F\colon \mathcal{C}\to\mathcal{D}$ is *fully faithful* if for any A, B in $\mathcal{C}$ the structure map

$$F_{A,B}\colon \mathcal{C}(A,B) \longrightarrow \mathcal{D}(FA,FB)$$

is an isomorphism in $\mathcal{V}$. If moreover, for any $D\in\mathcal{D}$ there exists $C\in\mathcal{C}$ and an isomorphism

$$FC \xrightarrow{\cong} D$$

in $\mathcal{D}$, we say that F is an *equivalence of $\mathcal{V}$-categories*.

Let's make this more concrete with our favourite examples:

Example 3.1.3. Given $\mathcal{V} = (\mathbf{2},\times,1)$, for which $\mathcal{V}$-category are preorder. Then, an order preserving map $f\colon(\Sigma,\le)\to(\Theta,\le)$ is fully faithful if and only if it is order reflecting:

$$\text{if } f(x)\le f(y) \text{ then } x\le y$$

for any $x, y\in\Sigma$. Moreover, f is an equivalence if in addition for any $y\in\Theta$ there exists $x\in\Sigma$ such that $f(x)\le y\le f(x)$.

Example 3.1.4. Given $\mathcal{V} = (\overline{\mathbb{R}}_+, +, 0)$, so that $\mathcal{V}$-categories are Lawvere metric spaces. Then, a non-increasing map $f\colon (X,d_X)\to(Y,d_Y)$ is fully faithful if and only if it is an isometry:

$$d_Y(fx,fy) = d_X(x,y)$$

for any $x, y\in X$. In addition, the map f if an equivalence if for every $y\in Y$ there exists $x\in X$ such that $d_Y(fx,y)=0$ and $d_Y(y,fx)=0$.

Let us now construct the Yoneda $\mathcal{V}$-functor (which we shall prove is fully faithful):

Definition 3.1.5. Given a small $\mathcal{V}$-category $\mathcal{C}$ we define a $\mathcal{V}$-functor

$$\mathcal{Y}\colon \mathcal{C}\longrightarrow [\mathcal{C}^{\mathrm{op}},\mathcal{V}]$$

as follows:

(1) for any $C\in\mathcal{C}$ we set $\mathcal{Y}C := \mathcal{C}(-,C)$;

(2) For any $B, C\in\mathcal{C}$ the map

$$\mathcal{Y}_{B,C}\colon \mathcal{C}(B,C)\longrightarrow [\mathcal{C}^{\mathrm{op}},\mathcal{V}](\mathcal{C}(-,B),\mathcal{C}(-,C))$$

is $\eta_{B,\mathcal{C}(-,C)}$ from the Yoneda isomorphism above. This is induced by the universal property of the equalizer since the diagonal map below

$$\begin{tikzcd}
\mathcal{C}(B,C) \arrow[r, dashed, "\mathcal{Y}_{B,C}"] \arrow[dr, "(\mathcal{C}(A,-)_{B,C})_{A\in\mathcal{C}}"'] & {[\mathcal{C}^{\mathrm{op}},\mathcal{V}](\mathcal{C}(-,B),\mathcal{C}(-,C))} \arrow[d, tail] \\
& \prod_{A\in\mathcal{C}}[\mathcal{C}(A,B),\mathcal{C}(A,C)]
\end{tikzcd}$$

equalizes the u and v involved in the definition of $[\mathcal{C}^{\mathrm{op}},\mathcal{V}](\mathcal{C}(-,B),\mathcal{C}(-,C))$.

The fact that this is a well-defined $\mathcal{V}$-functor follows from the fact that composition in $\mathcal{C}$ is associative and unital (checking everything componentwise).

Theorem 3.1.6. *For any small $\mathcal{V}$-category $\mathcal{C}$ the $\mathcal{V}$-functor*

$$\mathcal{Y}\colon \mathcal{C} \longrightarrow [\mathcal{C}^{\mathrm{op}},\mathcal{V}]$$

is fully faithful.

Proof. This follows immediately from Lemma 3.1.1 since the structure maps of $\mathcal{Y}$ are all of the form $\eta_{B,\mathcal{C}(-,C)}$, and this are isomorphisms. □

Exercise 3.1.7. Consider $\mathcal{V} = 1$, where we have seen that $\mathcal{V}$-**Cat** $\cong$ **Set**; then for any set X the presheaves

$$[X,1] \cong 1$$

are all trivial, since there is only one map into $\mathcal{V}$. Then the Yoneda lemma says that the (unique) function

$$X \longrightarrow 1$$

is fully faithful. This, at first, might seem very counter-intuitive.

Note that, under the isomorphism $\mathcal{V}$-**Cat** $\cong$ **Set**, the underlying category functor $(-)_0\colon$ **Set** $\to$ **Cat** sends a set X to the *indiscrete* category with X-many objects and where all homs are singletons (so every object is uniquely isomorphic to any other object). Then the Yoneda embedding is just saying that every set X, seen as an indiscrete category, is equivalent to the terminal category.

Example 3.1.8. Consider $\mathcal{V} = (\mathbf{2},\times,1)$, where $\mathcal{V}$-categories are preorders. Given a preorder Σ, order preserving maps $f\colon \Sigma^{\mathrm{op}} \to \mathbf{2}$ are in bijection with (possibly empty) *lower sets* in Σ; that is, subsets $I \subseteq \Sigma$ such that for any $x \in I$ and $y \in \Sigma$, if $y \leq x$ then $y \in I$. The bijection is given by defining $x \in I$ if and only if $fx = 1$.

For any $x \in \Sigma$ we have a principal lower set I_x defined by all the elements lower than x; this corresponds to the representable $\Sigma(-,x)\colon \Sigma^{\mathrm{op}} \to \mathbf{2}$. Denote by $\mathcal{L}(\Sigma)$ the poset of lower sets in Σ ordered by inclusion; then

$$[\Sigma^{\mathrm{op}},\mathbf{2}] \cong \mathcal{L}(\Sigma)$$

and the Yoneda embedding says that the map

$$\Sigma \longrightarrow \mathcal{L}(\Sigma) : x \mapsto I_x$$

is order preserving and reflecting.

3.2 Weighted limits for ordinary categories

For this section we go back to ordinary category theory and take a different perspective on the notion of limit.

Given a category $\mathcal{C}$ and a functor $H\colon \mathcal{D} \to \mathcal{C}$, out of a small $\mathcal{D}$, we say that the limit of H in $\mathcal{C}$ exists if there is an object $\lim H \in \mathcal{C}$ together with a natural transformation (that is, a cone) $\eta\colon \Delta(\lim H) \Rightarrow H$ inducing a bijection

$$\mathcal{C}(A, \lim H) \cong [\mathcal{D}, \mathcal{C}](\Delta(A), H)$$

for any $A \in \mathcal{C}$; here $\Delta(A)\colon \mathcal{D} \to \mathcal{C}$ is the functor taking constant value at A. This expresses the usual universal property of a limit in terms of bijections between hom-sets: every cone $\gamma\colon \Delta(A) \Rightarrow H$ (an element of the right-hand-side) factors uniquely as

$$\gamma = \eta \circ \Delta(f)$$

for a unique $f\colon A \to \lim H$ (an element of the left-hand-side).

Remark 3.2.1. Generalizing this notion to the enriched context, already presents some difficulties: if $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$-categories, the constant assignment $\Delta(A)\colon \mathcal{D} \dashrightarrow \mathcal{C}$ doesn't define a $\mathcal{V}$-functor in general (unless the unit of $\mathcal{V}$ is the terminal object). This problem doesn't arise when we take $\mathcal{D} = \mathcal{B}_{\mathcal{V}}$, the free $\mathcal{V}$-category on an ordinary category $\mathcal{B}$.

Using this notion of limit, one says that a category $\mathcal{C}$ has *$\mathcal{D}$-limits*, or *limits of shape $\mathcal{D}$*, if $\lim H$ exists for any $H\colon \mathcal{D} \to \mathcal{C}$. If $\mathbb{D}$ is a collection od small categories we say that $\mathcal{C}$ has *$\mathbb{D}$-limits* if it has $\mathcal{D}$-limits for any $\mathcal{D} \in \mathbf{D}$.

Example 3.2.2. If $\mathbb{D}$ is the collection of all small discrete categories (those with only identities morphisms), then a category $\mathcal{C}$ has $\mathbb{D}$-limits if and it has all (small) products.

Using class of categories $\mathbb{D}$ to characterize those categories that have limits of a given shape, turns out to be very useful as a formal way of expressing existence and preservation of such limits (especially in the construction of free limit completions). However, there are certain important type of limits whose existence in a category $\mathcal{C}$ cannot be expressed as the property of having $\mathbb{D}$-limits for some class of categories $\mathbb{D}$; this is because the "type" of limit may not be determined just from its shape.

A straightforward example is given by powers:

Definition 3.2.3. We say that a category $\mathcal{C}$ has *(small) powers* if for any $C \in \mathcal{C}$ and $S \in \mathbf{Set}$ the product

$$C^S := \prod_{s \in S} C,$$

of C with itself $\#(S)$-times, exists in $\mathcal{C}$.

Then every category with products certainly has powers, but the converse is not true. More importantly: there exists no class of categories $\mathbb{D}$ for which a category has $\mathbb{D}$-limits if and only if it has powers (see Exercise 3.3.3).

Example 3.2.4. The same problem seems to appear if we consider categories with kernel pairs. Recall that the kernel pair of a morphism $f\colon A \to B$ in $\mathcal{C}$ is defined to be the pullback of f along itself.

This problem can be solved with the use of weighted limits. To introduce this notion, let's go back to the definition of limit: given a category $\mathcal{C}$ and a functor $H\colon \mathcal{D} \to \mathcal{C}$, to give a cone $\Delta A \to H$ is the same as giving a cone

$$\Delta(1) \to \mathcal{C}(A, H-)$$

where $\mathcal{C}(A, H-)\colon \mathcal{D} \to \mathbf{Set}$ is the representable $\mathcal{C}(A, -)$ restricted along H, and $\Delta(1)\colon \mathcal{D} \to \mathbf{Set}$ is constant at 1.

It follows that the limit of H in $\mathcal{C}$ exists if and only if there is an object $\lim H \in \mathcal{C}$ together with a a cone $\eta\colon \Delta(1) \Rightarrow \mathcal{C}(\lim H, H-)$ inducing a bijection

$$\mathcal{C}(A, \lim H) \cong [\mathcal{D}, \mathbf{Set}](\Delta(1), \mathcal{C}(A, H-))$$

for any $A \in \mathcal{C}$.

Now it is enough to argue that, if we want to obtain a more general notion of limit, one possibility would be to replace $\Delta(1)\colon \mathcal{D} \to \mathbf{Set}$ with any functor $\mathcal{W}\colon \mathcal{D} \to \mathbf{Set}$. This is exactly how weighted limits are defined:

Definition 3.2.5. Let $\mathcal{D}$ be a small category and $W\colon \mathcal{D} \to \mathbf{Set}$ be a functor, from now on called *weight*. Given a category $\mathcal{C}$ and $H\colon \mathcal{D} \to \mathcal{C}$ we say that the *limit of H weighted by W* exists in $\mathcal{C}$ if there is an object $\lim^W(H) \in \mathcal{C}$ together with a natural transformation $\eta\colon W \Rightarrow \mathcal{C}(\lim^W(H), H-)$ inducing a bijection

$$\mathcal{C}(A, \lim^W(H)) \cong [\mathcal{D}, \mathbf{Set}](W, \mathcal{C}(A, H-))$$

for any $A \in \mathcal{C}$. This says that for any $\gamma\colon W \Rightarrow \mathcal{C}(A, H-)$ there exists a unique $f\colon A \to \lim^W(H)$ in $\mathcal{C}$ such that $\gamma = \mathcal{C}(f, H-) \circ \eta$.

As for classes of indexing categories, we can consider classes of indexing weights.

Definition 3.2.6. Let Φ be a collection of weights of the form $W\colon \mathcal{D} \to \mathbf{Set}$. We say that a category $\mathcal{C}$ has Φ-weighted limits if $\lim^W(H)$ exists in $\mathcal{C}$ for any $W \in \Phi$ and $H\colon \mathcal{D} \to \mathcal{C}$.

The arguments at the beginning of this section basically show the following proposition:

Proposition 3.2.7. *Given a small category $\mathcal{D}$ denote by $\Delta(1)\colon \mathcal{D} \to \mathbf{Set}$ the constant functor at $1 \in \mathbf{Set}$. Then, for any $\mathcal{C}$ and $H\colon \mathcal{D} \to \mathcal{C}$ we have*

$$\lim H \cong \lim^{\Delta(1)}(H) \in \mathcal{C},$$

either side existing if the other does.
In particular, given a class $\mathbb{D}$ of small categories, if we can consider the class of weights

$$\Phi_{\mathbb{D}} := \{\Delta(1)\colon \mathcal{D} \to \mathbf{Set} \mid \mathcal{D} \in \mathbb{D}\},$$

then a category has $\mathbb{D}$-limits if and only if it has $\Phi_{\mathbb{D}}$-weighted limits.

As promised, we will now exhibit weights whose limits describe powers.

Example 3.2.8. Given a set $S \in \mathbf{Set}$, consider the weight out of the terminal category

$$\hat{S}\colon \{*\} \longrightarrow \mathbf{Set}$$

that picks out S. Then, for any category $\mathcal{C}$, to give a diagram $H\colon \{*\} \to \mathcal{C}$ is the same as specifying an object $C \in \mathcal{C}$. Under these assumption, the universal property of $\lim^{\hat{S}} H$ says that

$$\begin{aligned}
\mathcal{C}(A, \lim^{\hat{S}}(H)) &\cong [\mathcal{D}, \mathbf{Set}](W, \mathcal{C}(A, H-)) \\
&\cong \mathbf{Set}(S, \mathcal{C}(A, C)) \\
&\cong \prod_{s \in S} \mathcal{C}(A, C).
\end{aligned}$$

This is exactly say that $\lim^{\hat{S}}(H)$ is the product of C with itself $\#(S)$-times; that is, $\lim^{\hat{S}}(H) \cong C^S$.
Thus, if we consider the class of weights

$$\Phi_{pow} := \{\hat{S}\colon \{*\} \to \mathbf{Set} \ \mid S \in \mathbf{Set}\},$$

then a category has Φ_{pow}-weighted limits if and only if it has powers.

Example 3.2.9. We will see in Exercise 3.3.4 that having kernel pairs can be expressed by having limits of a certain class of weights.

The example above provides a class of weights whose weighted limits cannot be parametrized by a class of categories; for this reason weighted limits are more expressive than the usual (category indexed) limits. However, if we not focus on the shapes, every weighted limit can be rewritten as one of the usual kind, in the sense made precise below. This is the reason why weighted colimits in ordinary categories are not commonly used.

Theorem 3.2.10. *Let $W\colon \mathcal{D} \to \mathbf{Set}$ be a weight, $\mathcal{C}$ a category, and $H\colon \mathcal{D} \to \mathcal{C}$ a functor. Then the weighted limit $\lim^W(H)$ exists if and only if the limit*

$$\lim \left(\mathrm{El}(W) \xrightarrow{\pi} \mathcal{D} \xrightarrow{H} \mathcal{C} \right)$$

exists in $\mathcal{C}$, and in that case they coincide. Here $\mathrm{El}(W)$ is the category of elements of W.

Proof. See [8, 3.33]. □

3.3 Exercises

Exercise 3.3.1. Given $\mathcal{V} = (\mathbf{2}, \times, 1)$, so that $\mathcal{V}$-categories coincide with preorders. Find an order preserving map $f\colon (\Sigma, \leq) \to (\Theta, \leq)$ between preorders, which is an equivalence of $\mathcal{V}$-categories but is not injective, nor surjective. What conditions on $(\Sigma, \leq)$ and $(\Theta, \leq)$ can we impose so that an equivalence between them is just an isomorphism of preorders?

Exercise 3.3.2. Given $\mathcal{V} = (\overline{\mathbb{R}}_+, +, 0)$, so that $\mathcal{V}$-categories are the Lawvere metric spaces. Find an non-increasing map $f\colon (X, d_X) \to (Y, d_Y)$ between Lawvere metric spaces, which is an equivalence of $\mathcal{V}$-categories but is not injective, nor surjective. What conditions on (X, d_X) and (Y, d_Y) can we impose so that an equivalence between them is just an isomorphism in the usual metric-sense?

Exercise 3.3.3. In the context of ordinary category theory. Show that there exists no class of categories $\mathbb{D}$ for which a category has $\mathbb{D}$-limits if and only if it has powers.

Exercise 3.3.4. In the context of ordinary category theory. Find a weight $K\colon \mathcal{D} \to \mathbf{Set}$ for which K-weighted limits are the same as kernel pairs. Is there a class of categories $\mathbb{D}$ for which a category has $\mathbb{D}$-limits if and only if it has kernel pairs? (Genuine question.)

Exercise 3.3.5. Let $\mathcal{V} = \mathbf{Set}_*$ be the monoidal category of pointed sets, so that a $\mathcal{V}$-category is just a category with 0-morphisms, preserved by composition. Given a $\mathcal{V}$-category $\mathcal{C}$ and a morphism $f\colon A \to B$, the *kernel* of f is the data of an object $K \in \mathcal{C}$ and a morphism $k\colon K \to A$ such that:

(1) $f \circ k = 0$ and

(2) for any $h\colon C \to A$ such that $f \circ h = 0$ then there exists a unique $l\colon C \to K$ for which $h = k \circ l$.

Find a weight $W\colon \mathcal{D} \to \mathbf{Set}_*$ for which limits weighted by W are kernels.

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