

## Enriched purity: towards enriched model theory

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# Purity in logic

Let  $\mathbb{L}$  be a language with function and relation symbols. A **positive-primitive formula** is one of the form

$$\psi(x) := \exists y \, \varphi(x, y)$$

where  $\varphi$  is a conjunction of atomic formulas. E.g.  $\varphi(x, y) = (t(x, y) = s(x, y)) \wedge R(x, y)$ .

## Definition

A monomorphism of  $\mathbb{L}$ -structures  $f: M \rightarrow L$  is called **pure** if for any pp-formula  $\psi(x)$  and any  $a \in M^n$  we have

$$M \models \psi(a) \quad \text{iff} \quad L \models \psi(fa)$$

$$\begin{array}{ccc} \|\psi\|_M & \xrightarrow{\quad} & M^n \\ \downarrow & \lrcorner & \downarrow f^n \\ \|\psi\|_L & \xrightarrow{\quad} & L^n \end{array}$$

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## Theorem (classical)

The following are equivalent for a full subcategory  $\mathcal{H}$  of  $\mathbf{Str}(\mathbb{L})$ :

- $\mathcal{H} = \text{Mod}(\mathbb{T})$  for a regular  $\mathbb{L}$ -theory  $\mathbb{T}$ ;
- $\mathcal{H}$  is closed under products, filtered colimits, and **pure subobjects**.

# Injectivity classes

Note: to say that  $A$  satisfies  $\phi(x) = \exists y \psi(x, y)$  is the same as requiring that the composite

$$\psi_A = \{(a, b) \mid A \models \psi(a, b)\} \xrightarrow{i} A \times A \xrightarrow{\pi_1} A$$

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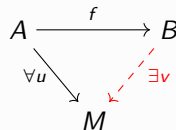
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The categorical analogue of regular theories are **injectivity classes**:

## Definition

An object  $M$  is injective w.r.t. a morphism  $f: A \rightarrow B$  in a category  $\mathcal{K}$  if

$$\forall u: A \rightarrow M, \exists v: B \rightarrow M \ (vf = u).$$



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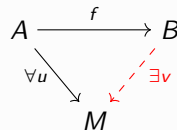
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Equivalently, if  $\mathcal{K}(f, M): \mathcal{K}(B, M) \rightarrow \mathcal{K}(A, M)$  is **surjective**.

An **injectivity class** in  $\mathcal{K}$  is a full subcategory of  $\mathcal{K}$  spanned by the objects injective with respect to a set  $\{f_i: A_i \rightarrow B_i\}_{i \in I}$ .

# Injectivity classes II

## Theorem (Rosický-Adámek-Borceux)

*TFAE for a full subcategory  $\mathcal{H}$  of a locally finitely presentable category  $\mathcal{K}$ :*

- $\mathcal{H}$  is a (finite) injectivity class in  $\mathcal{K}$ ;
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What is purity in this context?

## Definition

A morphism  $f: M \rightarrow L$  in  $\mathcal{K}$  is pure provided that in each commutative diagram on the right, where  $A$  and  $B$  are finitely presentable, there is a morphism  $t: B \rightarrow M$  such that  $tg = u$ .

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ u \downarrow & & \downarrow v \\ M & \xrightarrow{f} & L \end{array}$$

Note:  $\mathcal{H}$  is a (finite) injectivity class in some locally finitely presentable category  $\mathcal{K}$  if and only if  $\mathcal{M} \simeq \mathbf{Mod}(\mathbb{T})$  for some regular theory  $\mathbb{T}$  on a language  $\mathbb{L}$ .



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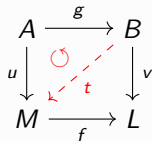
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# Enrichment

We fix:

- a symmetric monoidal closed category  $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$  which is locally presentable;
- a factorization system  $(\mathcal{E}, \mathcal{M})$  on  $\mathcal{V}$ ;

To keep in mind:

- ① **Set** and **Ab** with (epi, mono);
- ② **Met** with (dense, closed isometry);
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Then there is an enriched notion of  $\mathcal{E}$ -injectivity:

## Definition (Lack-Rosický)

An object  $M$  is  $\mathcal{E}$ -injective w.r.t. a morphism  $f: A \rightarrow B$  in a  $\mathcal{V}$ -category  $\mathcal{K}$  if the map

$$\mathcal{K}(f, M): \mathcal{K}(B, M) \longrightarrow \mathcal{K}(A, M)$$

lies in  $\mathcal{E}$ . An  $\mathcal{E}$ -injectivity class in  $\mathcal{K}$  is a full subcategory of  $\mathcal{K}$  spanned by the objects  $\mathcal{E}$ -injective with respect to a set  $\{f_i: A_i \rightarrow B_i\}_{i \in I}$ .

A corresponding notion of purity was missing.

# Enriched purity

The result below is based on some assumptions on  $\mathcal{E}$  and a class of objects  $\mathcal{G} \subseteq \mathcal{V}$  such that powers by  $\mathcal{G}$  satisfy a stability condition with respect to  $\mathcal{E}$ . Then we can prove:

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What is an  $\mathcal{E}$ -pure morphism? Definition by examples:

**Met:**

$$d(fu, vg) \leq \varepsilon$$

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ u \downarrow & & \downarrow v \\ K & \xrightarrow{f} & L \end{array}$$

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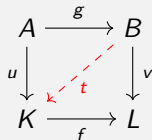
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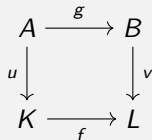
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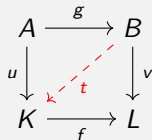
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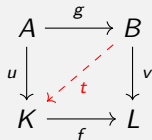
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**DGab:**

$$du = dv = 0$$

$$dt \neq 0 \text{ generally}$$



# A logical interpretation

- The canonical language  $\mathbb{L}$  on a locally finitely presentable  $\mathcal{V}$ -category  $\mathcal{K}$  has **sorts and function symbols** given by the objects and morphisms of  $\mathcal{K}_f^{\text{op}}$ .

Every object  $M$  of  $\mathcal{K}$  defines an  $\mathbb{L}$ -structure in  $\mathcal{V}$ :

- a sort  $A$  is assigned to  $M_A := \mathcal{K}(A, M)$ ;
- a function symbol  $f : (A, B)$  is assigned to  $M_f := \mathcal{K}(f, M) : M_A \rightarrow M_B$ .

## pp-formulas:

$$\psi(x) \equiv \exists y \phi(x, y)$$

where  $\phi(x, y)$  is a conjunction of

$$(f(x) = g(y))$$

for some  $f : B \rightarrow A$ ,  $g : B \rightarrow C$ .

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e.g.: 
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$$\begin{array}{ccccc} \|\phi\|_M & \xrightarrow{\quad} & M_A \times M_C & \xrightarrow{p_1} & M_A \\ & \searrow \varepsilon & & \nearrow \mathcal{M} & \\ & & \|\psi\|_M & & \end{array}$$

# A logical interpretation II

## Definition

Let  $f: M \rightarrow L$  be a morphism in a locally finitely-presentable  $\mathcal{V}$ -category  $\mathcal{K}$ . We say that  $f$  is **elementary with respect to a pp-formula  $\psi$**  if the square below is a pullback.

$$\begin{array}{ccc} \|\psi\|_M & \xrightarrow{\quad} & M_A \\ \downarrow & \lrcorner & \downarrow f_A \\ \|\psi\|_L & \xrightarrow{\quad} & L_A \end{array}$$

## Theorem

Let  $\mathcal{K}$  be a locally finitely presentable  $\mathcal{V}$ -category and  $f: M \rightarrow L$  be a morphism in it. Then  $f$  is  **$\mathcal{E}$ -pure** if and only if it is **elementary with respect to any pp-formula** in the canonical language associated to  $\mathcal{K}$ .

Can we talk about languages, structures, terms, and formulas in general?

# Towards enriched model theory

Introduce enriched languages, structures, and terms:

## Definition

- A (single-sorted) **enriched (operational) language**  $\mathbb{L}$  is the data of a set of operation symbols  $f: (X, Y)$  whose arities  $X$  and  $Y$  are objects of  $\mathcal{V}_f$ .

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- The class of  **$\mathbb{L}$ -terms** is defined recursively as follows:
  - ① Every morphism  $f: Y \rightarrow X$  of  $\mathcal{V}_f$  is an  $(X, Y)$ -ary term;
  - ② Every operation symbol  $f: (X, Y)$  of  $\mathbb{L}$  is an  $(X, Y)$ -ary term;
  - ③ If  $t$  is a  $(X, Y)$ -ary term and  $Z$  is an arity, then  $t^Z$  is a  $(Z \otimes X, Z \otimes Y)$ -ary term;
  - ④ If  $t$  and  $s$  are  $(X, Y)$ -ary and  $(Y, W)$ -ary terms; then  $s \circ t$  is a  $(X, W)$ -ary term.

Then one can define **equational theories**: to appear soon.

What about **regular theories**???

**Thank You**