Non-univalent families

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Introduction

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(I will try to make it clearer.)

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There is a large part of the history of Topology behind that problem, from Riemann to Grothendieck.

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and how Grassmannians are approximations to groupoids of bundles.

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It is the universal family of subsets in the sense that any subset $U \subseteq X$ is classified uniquely by a characteristic function $\chi_U : X \to \{0,1\}$.

More precisely: in the sense there exists a bijection $Sub(X) \simeq \{0,1\}^X$, which is natural in X (= isomorphism of functors $Set^{op} \to Set$).

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Another way to say this is that the natural operations that exist on families provide an algebraic structure on the universe.

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But we will see that it is, in fact, *not* univalent.

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It is subtle, but essentially because the symmetries of the sets do not make the pullback square unique (unless the sets are empty or singletons, which was the previous example of subsets).

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In fact, there cannot exist a universal family of covering spaces in covering

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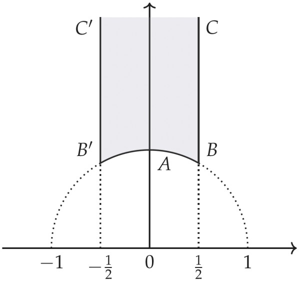
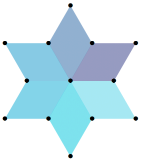
Topologically, this quotient is a sphere minus one point, and pinched at two points (A and B corresponding to the square and hexagonal lattices that have extra symmetries). [2]





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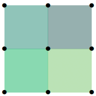

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This true for $X = 1$ by design, but this is false for a general X.

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One could expect that isomorphism classes of such bundles are in bijection with continuous maps $X \to \mathbb{H}/SL_2(\mathbb{Z})$.

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There cannot be a bijection $\{\text{iso. cl. tori bundle on } S^1\} \simeq (\mathbb{H}/SL_2(\mathbb{Z}))^{S^1}$

Universal families

But what is, formally, a universal family?

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\begin{aligned}
Set^{op} \text{ (or } Top^{op}) &\longrightarrow Set \\
I &\longmapsto Fam(I) \\
I \to J &\longmapsto \text{reindexing} : Fam(J) \to Fam(I) \,.
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The object U is called the universe for the structure.

Finally a definition

If (U, U') is a family that repeat some member, then

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A family (U, U') is uni-versal if it is univalent and strongly versal

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As it happens, all notions that can be indexed by sets, can also be indexed by groupoids.

When the indexing objects are topological spaces, this was the motivation to define stacks (which are, loosely, spaces with a groupoid of points).

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1. cartesian products, coproducts, exponentials are functors $Fin^2 \to Fin$
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These natural constructions are the very reason Eilenberg and Mac Lane invented categories and functors.

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All this is nice.

But now that we know what univalent families are, I want to forget about them and talk about non-univalent families.

It is just a rhetorical trick. We will see that the most interesting non-univalent families are in fact univalent but for an other structure.

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How does one do that?

By adding some extra structure which does not admit symmetries.

Get rid of symmetries: two strategies

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$$
\begin{tikzcd}
 & \mathrm{A} \arrow[dl, two heads] \arrow[dr, two heads] & \\
E \arrow[rr, dashed] & & F
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[rr, dashed] \arrow[dr, tail] & & F \arrow[dl, tail] \\
 & \Omega &
\end{tikzcd}
$$

There is at most one function from E to F compatible with A or Ω.

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The "true" moduli space is the quotient by this action $\mathbb{H} \to \mathbb{H}/Sl_2(\mathbb{Z})$.

$$\begin{tikzcd} & \mathbb{H} \arrow[d] \\ X \arrow[ur, dashed, "\text{bundle + surjection } \mathbb{C} \twoheadrightarrow -"] \arrow[r, "\text{tori bundle}"'] & \mathbb{H}/Sl_2(\mathbb{Z}) \end{tikzcd}$$

Finite sets

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It is the classifier for bundles with fibers $\underline{n}$ embedded in a trivial bundle with fiber $\mathbb{R}^\infty$

$$\begin{tikzcd} X' \arrow[rr, hook] \arrow[dr, "\text{cover degree } n"'] & & \mathbb{R}^\infty \times X \arrow[dl] \\ & X\,. & \end{tikzcd}$$

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The "true" classifier of covering of degree n is the groupoid (stack) BS_n.

$$\begin{tikzcd} Gr(\underline{n}, \infty)' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (BS_n)' \subset Fin' \arrow[d] \\ Gr(\underline{n}, \infty) \arrow[r] & BS_n \subset Fin \end{tikzcd}
\qquad
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The "true" bundle classifier is the stack $BAut(K)$.

$$\begin{tikzcd} Gr(K,\infty)' \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & BAut(K)' = \underline{K}/Aut(K) \arrow[d] \\ Gr(K,\infty) \arrow[r] & BAut(K) \end{tikzcd}$$

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There is a uniquely defined cartesian square

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\begin{tikzcd}
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I \arrow[ur, dashed, "\text{fam. + total order}"] \arrow[r, "\text{fam. of sets}"'] & Fin
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The operations of finite products and coproduct on finite sets can be lifted to $\mathbb{N}$ by the usual products and sums.

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Any ordinal has an underlying set $\mathbb{O} \to Set$

$$\begin{tikzcd} \mathbb{O}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Set' \arrow[d] \\ \mathbb{O} \arrow[r] & Set \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{O} \arrow[d] \\ I \arrow[ur, dashed, "\text{fam. + good order on fibers}"] \arrow[r, "\text{fam. of sets}"'] & Set \end{tikzcd}$$

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This is the purpose of ZF.

Zermelo–Fraenkel

The problem of building an injection $E \to \Omega$ has a distinguished solution.

[3] $\mathbb{ZF}$ This is a fixed point of the powerset functor $E \mapsto Sub_{<\kappa}(E)$. It needs a bound on sizes κ to exist (and should be denoted $\mathbb{ZF}_\kappa$), cf. Joyal–Moerdijk.

Zermelo–Fraenkel

The problem of building an injection $E \to \Omega$ has a distinguished solution.

There exists a (large enough) set $\mathbb{ZF}$ such that

$$\mathbb{ZF} \simeq Sub(\mathbb{ZF}) = \coprod_E E \rightarrowtail \mathbb{ZF}$$

(where the coproduct is taken over small enough iso. cl. of sets).

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Every such forest has an underlying set E_0. This provides a diagram

$$\begin{tikzcd} \mathbb{ZF}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & Set' \arrow[d] \\ \mathbb{ZF} \arrow[r, "E_0"] & Set \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{ZF} \arrow[d] \\ I \arrow[r, "\text{fam. of sets}"'] \arrow[ur, dashed, "\text{fam. + forest structure}"] & Set \end{tikzcd}$$

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$\mathbb{ZF}$ is morally an extension of $\mathbb{O}$ that will allow the definition of powersets and infinite products.

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But we lost the natural associativity/unitality of these operations.

How to get back to univalent families

We have seen examples of groupoid-like universes covered by set-like universes $\mathbb{U} \twoheadrightarrow U$

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We have seen that such constructions can be lifted to the set-like universes $\mathbb{U}$. But a construction on $\mathbb{U}$ need not come from (descent to) U (need not be natural wrt to the isomorphism of families).

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1. The cumulative hierarchy $\mathbb{ZF}$,
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3. The Grassmannian $Gr(K, \infty)$,
4. The family $\mathbb{N}' \to \mathbb{N}$,
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They are the proof that "non-univalent" families should be considered along with univalent ones.

Thank you!