Enriched categories: theory and examples

Lecture notes from a postgraduate minicourse

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November 5, 2024

Contents

Introduction

Enrichment is nowadays a standard tool in category theory; its range of applications is so vast that it reaches very different areas of mathematics, such as algebra [7, 14, 18], homotopy theory [4, 12, 17], computer science [3, 6, 16], and functional analysis [11, 15].

Even though additive and abelian categories were introduced earlier and can be understood as some (easy) examples of enrichment, it was only in the 60s, after the development of differentially graded categories, that people started to think about a general framework for dealing with categories whose homs have a much richer structure than that of a set.

The first to, independently, envisage the potentials of such a theory were Mac Lane [9] and Bénabou [1], as well as Linton[10] and Maranda [13]. However, Eilenberg and Kelly were the ones that actually developed a theory of enrichment in their monograph [5]. Afterwards, the theory started to get studied and many results from ordinary category theory were transferred into this richer setting, sometimes with effort and some other times very easily and elegantly.

Later, the theory evolved in new directions by introducing enrichment over bicategories [2, 19]; however that will not be the framework of this course, where we consider only enrichment over symmetric monoidal closed categories. A more complete account of all the results we discuss is given in Kelly's book [8].

CHAPTER

1

Week 1

1.1 Monoidal categories

In this section we introduce those categories which, equipped with additional structure, will be our bases of enrichment.

Given an ordinary category $\mathcal{C}$, we can describe the composition operation as a family of functions

$$\mathcal{C}(B, C) \times \mathcal{C}(A, B) \xrightarrow{-\circ-} \mathcal{C}(A, C)$$

for any $A, B, C \in \mathcal{C}$, satisfying certain axioms. This suggests that we could define a notion of category enriched over any given category with binary products. However, that would be very restrictive as, while we would capture many examples (2-dimensional, simplicial, etc) we would also miss many other important ones (for instance, additive and DG-categories).

To capture all these in the same framework we shall need a category endowed with a "tensor product":

Definition 1.1.1. A *monoidal category* $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$ is the data of a category $\mathcal{V}_0$ together with:

(1) an object $I \in \mathcal{V}_0$ called *unit*;

(2) a functor $- \otimes -\colon \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$, called *tensor product*;

(3) natural isomorphisms:

  (a) $\alpha_{A,B,C}\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ for any $A, B, C \in \mathcal{V}_0$;
  (b) $\lambda_A\colon I \otimes A \to A$ and $\rho_A\colon A \otimes I \to A$ for any $A \in \mathcal{V}_0$.

These are subject to the following coherence axioms:

(1) for any A, B, C and D in $\mathcal{V}_0$ the diagram below commutes;

$$\begin{tikzcd}
 & A \otimes (B \otimes (C \otimes D)) \arrow[dl, "1 \otimes \alpha"'] \arrow[dr, "\alpha"] & \\
A \otimes ((B \otimes C) \otimes D) \arrow[d, "\alpha"'] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha"] \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha \otimes 1"'] & & ((A \otimes B) \otimes C) \otimes D
\end{tikzcd}$$

(2) for any A and B in $\mathcal{V}_0$ the diagram below commutes.

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\alpha"] \arrow[dr, "1 \otimes \lambda"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho \otimes 1"] \\
 & A \otimes B &
\end{tikzcd}$$

Given a monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$, we usually write $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ and assume that α, λ and ρ are understood. Let us see some examples:

Example 1.1.2.

(1) If $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$ is a monoidal category, then

$$\mathcal{V}^{\mathrm{op}} = (\mathcal{V}_0^{\mathrm{op}}, \otimes^{\mathrm{op}}, I, \alpha^{-1}, \lambda^{-1}, \rho^{-1})$$

is a monoidal category.

(2) If $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$ is a monoidal category, then

$$\mathcal{V}^{\mathrm{rev}} = (\mathcal{V}_0, \hat{\otimes}, I, \hat{\alpha}, \hat{\lambda}, \hat{\rho})$$

is a monoidal category where:

(a) $A \mathbin{\hat{\otimes}} B := B \otimes A$;
(b) $\hat{\alpha}_{A,B,C} := \alpha^{-1}_{C,B,A}$;
(c) $\hat{\lambda}_A := \rho_A$ and $\hat{\rho}_A := \lambda_A$.

(3) Let $\mathcal{C}$ be any category with finite products; then $(\mathcal{C}, \times, 1)$ is a monoidal category where α, ρ and λ are induced by the universal property of the products; this structure is called the *cartesian* structure in $\mathcal{C}$. By (1) it follows that if $\mathcal{C}$ has finite coproducts, then $(\mathcal{C}, +, 0)$ is a monoidal category; this monoidal structure is called *cocartesian.*

(4) The category **Ab** of abelian groups and group homomorphisms, together with the tensor product $\otimes$ of abelian groups, and the unit $\mathbb{Z}$, forms a monoidal category. More generally, the category R-Mod of modules over a commutative ring R is monoidal with tensor product given by $\otimes_R$ and unit R.

(5) The category **DGAb** of differentially graded abelian groups (that is, chain complexes) is monoidal with the standard tensor product of chain complexes:

$$(A \otimes B)_n := \bigoplus_{i+j=n} A_i \otimes B_j$$

with differential generated by the rule

$$d^n_{A\otimes B}(a \otimes b) = d^i_A a \otimes b + (-1)^i a \otimes d^j_B b$$

where $a \in A_i$ and $b \in B_j$, with $i + j = n$. The unit I given by the chain complex with $\mathbb{Z}$ on degree 0 and (0) otherwise.

(6) Let $(G, \cdot, e)$ be a group (a monoid is enough) and G_0 be the discrete category on the underlying set of G. Then $\mathcal{G} := (G_0, \ \cdot\ , e)$ is a (strict) monoidal category where the tensor product of two objects is defined by multiplication in G.

To define what a category enriched over $\mathcal{V}$ we shall need only a monoidal category; however, to to certain constructions (like the opposite $\mathcal{V}$-category) we shall need the monoidal category $\mathcal{V}$ to also be symmetric:

Definition 1.1.3. A *symmetric monoidal category* $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is the data of a monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho)$ together with a natural isomorphism $\sigma_{A,B} \colon A \otimes B \to B \otimes A$ for any $A, B \in \mathcal{V}_0$, subject to the following coherences:

(1) for any $A, B \in \mathcal{V}_0$ we have $\sigma_{B,A} \circ \sigma_{A,B} = 1_{A\otimes B}$ and $\lambda_A \circ \sigma_{A,I} = \rho_A$;

(2) for any $A, B, C \in \mathcal{V}_0$ the diagram below commutes.

$$\begin{tikzcd}
 & (A \otimes B) \otimes C \arrow[dl, "\sigma \otimes 1"'] \arrow[dr, "\alpha"] & \\
(B \otimes A) \otimes C \arrow[d, "\alpha"'] & & A \otimes (B \otimes C) \arrow[d, "\sigma"] \\
B \otimes (A \otimes C) \arrow[dr, "1 \otimes \sigma"'] & & (B \otimes C) \otimes A \arrow[dl, "\alpha"] \\
 & B \otimes (C \otimes A) &
\end{tikzcd}$$

Again, we will write simply $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ for a symmetric monoidal category $\mathcal{V}$ where all the structure maps are understood. Often, we will say that a monoidal category $\mathcal{V}$ *is symmetric* to mean that $\mathcal{V}$ *comes equipped with a chosen symmetry* (note, a monoidal category might have more than one symmetry — see Exercise **??**).

Example 1.1.4.

(1) If $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is symmetric monoidal, then

$$\mathcal{V}^{\mathrm{op}} = (\mathcal{V}_0^{\mathrm{op}}, \otimes^{\mathrm{op}}, I, \alpha^{-1}, \lambda^{-1}, \rho^{-1}, \sigma^{-1})$$

is symmetric monoidal.

(2) If $\mathcal{V} = (\mathcal{V}_0, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is symmetric monoidal, then

$$\mathcal{V}^{\mathrm{rev}} = (\mathcal{V}_0, \hat{\otimes}, I, \hat{\alpha}, \hat{\lambda}, \hat{\rho}, \hat{\sigma})$$

is symmetric monoidal where $\hat{\sigma}_{A,B} = \sigma_{B,A}$.

(3) Let $\mathcal{C}$ be any category with finite products; then $(\mathcal{C}, \times, 1)$ is symmetric monoidal where σ is induced by the universal property of the products. Similarly, if $\mathcal{C}$ has finite coproducts, then $(\mathcal{C}, +, 0)$ is symmetric monoidal.

(4) The monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$ is symmetric with symmetry

$$\sigma_{A,B} \colon A \otimes B \to B \otimes A$$

defined by sending a generator $a \otimes b \in A \otimes B$ to $\sigma_{A,B}(a \otimes b) := b \otimes a \in B \otimes A$. Similarly, the category R-Mod of modules over a commutative ring R is symmetric monoidal.

(5) The monoidal category **DGAb** of differentially graded abelian groups, with the standard tensor product, is symmetric. The symmetry is defined on the generators as

$$\sigma_{A,B}(a \otimes b) := (-1)^{ij} b \otimes a,$$

where $a \in A_i$ and $b \in B_j$.

(6) Let $(G, \cdot, e)$ be a group. Then $\mathcal{G} = (G_0, \ \cdot\ , e)$ is a symmetric monoidal if and only if the multiplication is G is commutative; that is, if and only if G is abelian.

Example 1.1.5. The category **Grp** of groups and group homomorphisms, with the monoidal closed structure given by the standard tensor product of groups is not symmetric.

The last concept, that will be needed to consider for instance enriched categories of functors, is that of symmetric closed monoidal category.

Definition 1.1.6. A *symmetric monoidal closed category* $\mathcal{V} = (\mathcal{V}_0, \otimes, [-,-])$ is the data of a symmetric monoidal category together with functor

$$[-,-] \colon \mathcal{V}_0^{\mathrm{op}} \times \mathcal{V}_0 \to \mathcal{V}_0,$$

called *internal hom*, such that $(-) \otimes B \dashv [B,-]$ for any $B \in \mathcal{V}_0$. In other words, we have an isomorphism

$$\mathcal{V}_0(A \otimes B, C) \cong \mathcal{V}_0(A, [B, C])$$

natural in A, B, and C in $\mathcal{V}_0$.

Since the internal-hom functor $[-,-]$, when it exists, is uniquely determined (up to isomorphism) from the tensor product, we will still denote a symmetric monoidal closed category by just $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$.

Remark 1.1.7. Fixed a symmetric monoidal closed category $\mathcal{V}$, we will often use the tensor-hom adjunction to "transpose" maps out of a tensor products to maps into an internal-hom. More precisely, to give a morphism

$$f \colon A \otimes B \longrightarrow C$$

is the same as to give a map

$$f^{\mathrm{t}} : A \to [B, C]$$

which is the transpose of f under $(-) \otimes B \dashv [B, -]$. Since $\mathcal{V}$ is also symmetric, the map f also corresponds to $f\sigma : B \otimes A \to C$ (up to isomorphism). Therefore to give f is also the same as to give

$${}^{\mathrm{t}}f : B \to [A, C].$$

We will use these natural bijections very often when constructing certain $\mathcal{V}$-categories. The notation with f, f^{t}, and ${}^{\mathrm{t}}f$ will not always be consistent with the one used above; for instance, if we start with $g : B \to [A, C]$ we will use the transpose to denote the other two maps.

The following examples will provide interesting instances of enrichment.

Example 1.1.8.

(1) The singleton $1 = \{*\}$, seen as a one-object discrete category, is (trivially) symmetric monoidal closed.

(2) The category **Set** of sets and functions with its cartesian structure is symmetric monoidal closed (also called, cartesian closed). The internal hom is simply given by $[A, B] := \mathbf{Set}(A, B)$.

(3) Every (elementary) topos with its cartesian structure is symmetric monoidal closed by definition. The internal hom is usually denoted by $[A, B] := B^A$.

(4) The category **Cat** of categories is cartesian closed; the internal hom $[A, B]$ is the category of functors $A \to B$ and natural transformations between them.

(5) The monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$ is symmetric monoidal closed with internal hom $[A, B] := \mathbf{Ab}(A, B)$ endowed with the pointwise addition which makes it an abelian group. Similarly, the category R-Mod of modules over a commutative ring R is symmetric monoidal closed.

(6) The monoidal category **DGAb** of differentially graded abelian groups, with the standard tensor product, is symmetric monoidal closed. The internal hom is given by

$$[A, B]_n := \prod_{i \in \mathbb{Z}} \mathbf{Ab}(A_i, B_{i+n})$$

and with differential $d(f) = d_B \circ f - (-1)^n f \circ d_A$, where $f \in [A, B]_n$.

(7) Let $(G, \cdot, e)$ be an abelian group. Then $\mathcal{G} = (G_0, \ \cdot\ , e)$ is symmetric monoidal closed with internal hom

$$[g, h] := k \cdot h^{-1}$$

for any $g, h \in G$.

(8) Consider the arrow category $\mathbf{2} = \{0 \to 1\}$, then $(\mathbf{2}, \times, 1)$ is symmetric monoidal closed (it is cartesian closed as a full subcategory of **Set**).

(9) Consider the category $\mathbf{Set}_*$ of pointed sets: objects are pairs $(X \in \mathbf{Set}, x \in X)$, and morphisms $(X, y) \to (Y, y)$ are functions $f : X \to Y$ sending x to y. Between any two objects $A = (X, x)$ and $B = (Y, y)$ we can define a tensor product

$$A \wedge B := \left(\frac{X \times Y}{A \vee B}, \ [A \vee B] \right)$$

where $A \vee B := (\{x\} \times Y) \cup (X \times \{y\}) \subseteq X \times Y$, and $[A \vee B]$ is the single point defined by the equivalence class of $A \vee B$ in the quotient. The unit is defined as $I := (\{0,1\}, 0)$. It is easy to define the structure maps that make $(\mathbf{Set}_*, \wedge, I)$ into a symmetric monoidal category. This is in addition monoidal closed with internal hom defined as

$$[A, B] := (\mathbf{Set}_*(A, B), \Delta y)$$

where $B = (Y, y)$ and Δy is the constant function at $y \in Y$.

(10) Consider the poset $\overline{\mathbb{R}}_+ = ([0, \infty], \geq)$ as a category. Then $(\overline{\mathbb{R}}_+, +, 0)$ forms a symmetric monoidal closed category with internal hom

$$[a, b] = \begin{cases} b - a & \text{if } b \geq a \\ 0 & \text{otherwise} \end{cases}$$

for any $a, b \in \overline{\mathbb{R}}_+$.

(11) Consider the category $\mathbf{Met}$ whose objects are generalized metric spaces (we allow the distance to take value in $[0, \infty]$, everything else is as usual), and whose morphisms $f\colon (X, d_X) \to (Y, d_Y)$ are functions $f\colon X \to Y$ such that $d_Y(fx, fx') \leq d_X(x, x')$. On $\mathbf{Met}$ we have a symmetric monoidal closed structure with unit given by the singleton 1, tensor product

$$(X, d_X) \otimes (Y, d_Y) := (X \times Y,\ d_{X \otimes Y}((x, y), (x', y')) := d_X(x, x') + d_Y(y, y')\ ),$$

and internal hom

$$[(X, d_X), (Y, d_Y)] := (\mathbf{Met}((X, d_X), (Y, d_Y)),\ d(f, g) := \sup\{d_Y(fx, gy)\ |x \in X\})$$

for any $(X, d_X), (Y, d_Y) \in \mathbf{Met}$.

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