26.11.2025

Paradoxical sets

and the

Axiom of Choice

Logic Seminar - University of Manchester

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University of Leeds

Context

Axiom of Choice

Paradoxical sets

Axiom of Choice

Axiom of Choice

$\Longleftrightarrow$

Zorn's lemma

$\Longleftrightarrow$

Well-ordering principle

$\Downarrow$

Well-ordering on the reals

$WO(\mathbb{R})$

Paradoxical sets

- $P \subseteq \mathbb{R}^n \quad n = 1, 2, 3.$

P satisfies some counterintuitive property.

The existence of such a P involves Axiom of Choice

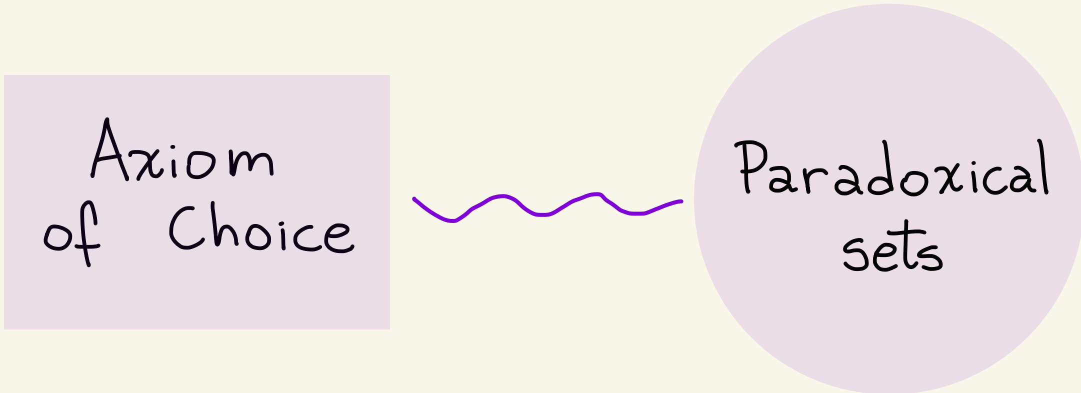
$\Downarrow$

$WO(\mathbb{R})$

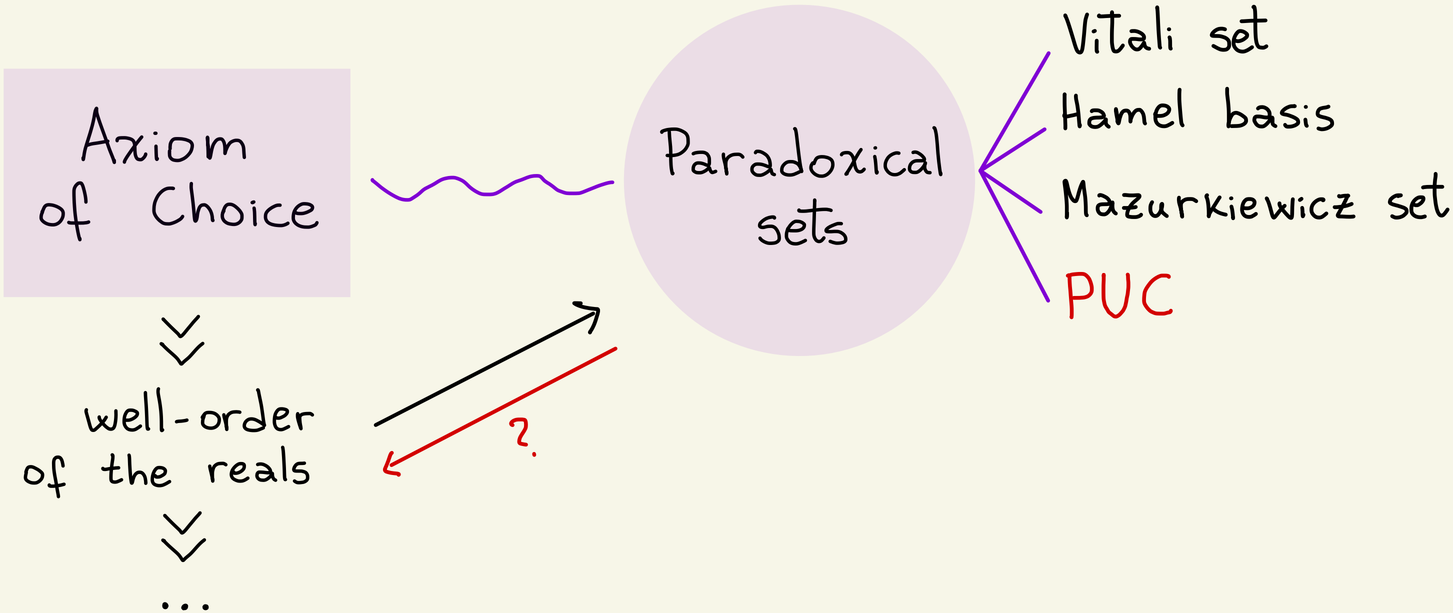
$\rightsquigarrow$

transfinite induction
on the reals

Context

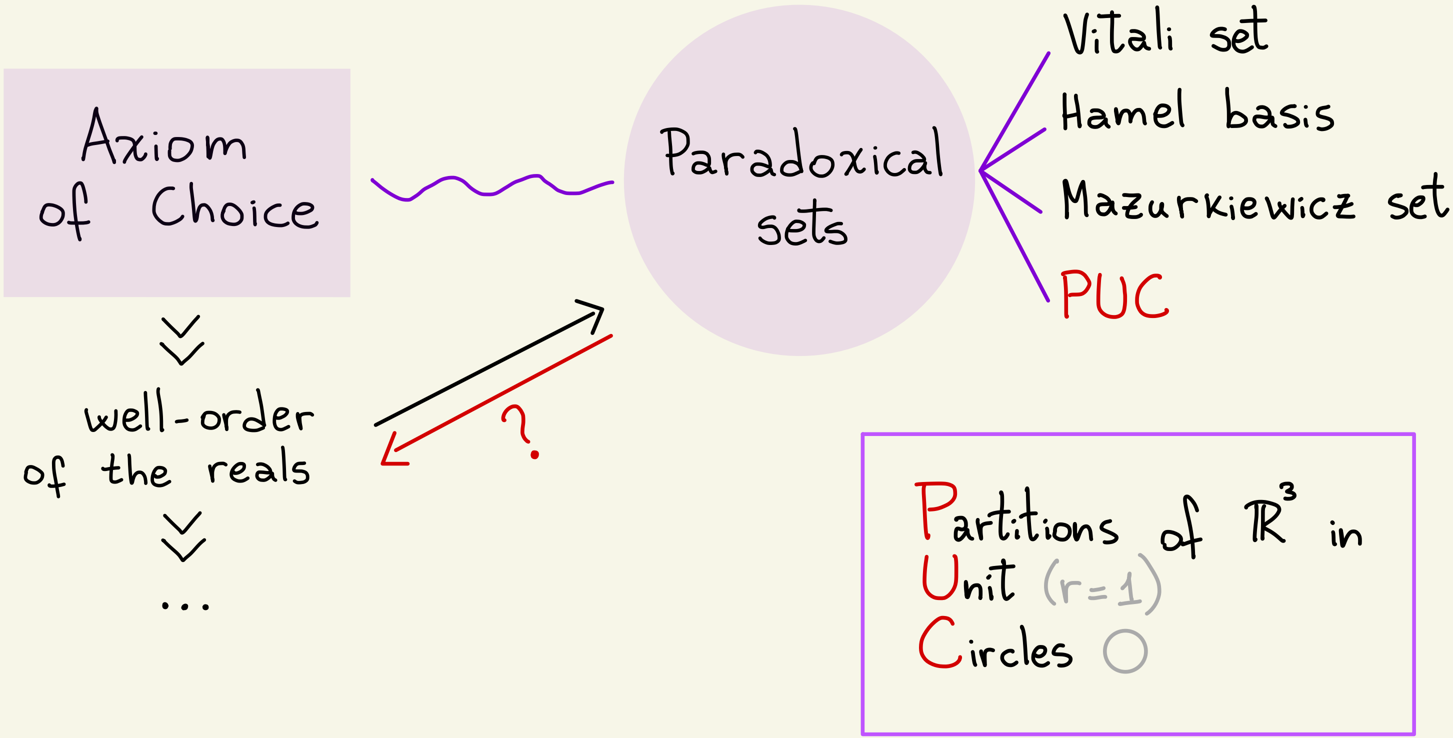


Context



Axiom of Choice
well-order of the reals
...
Paradoxical sets
?
Vitali set
Hamel basis
Mazurkiewicz set
PUC

Context


Axiom of Choice
Paradoxical sets
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PUC
well-order of the reals
...
?
Partitions of $\mathbb{R}^3$ in
Unit (r = 1)
Circles


TOMMY

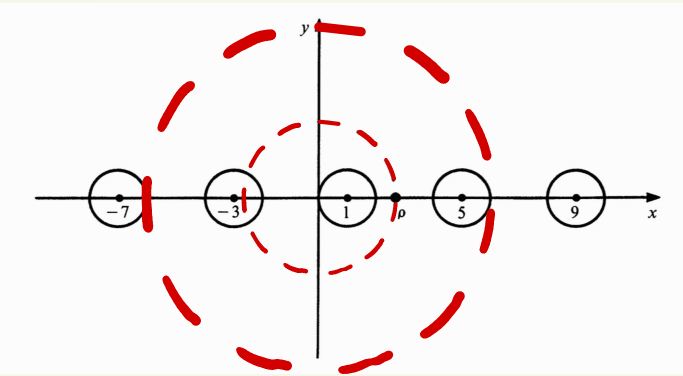
Partitions in circles

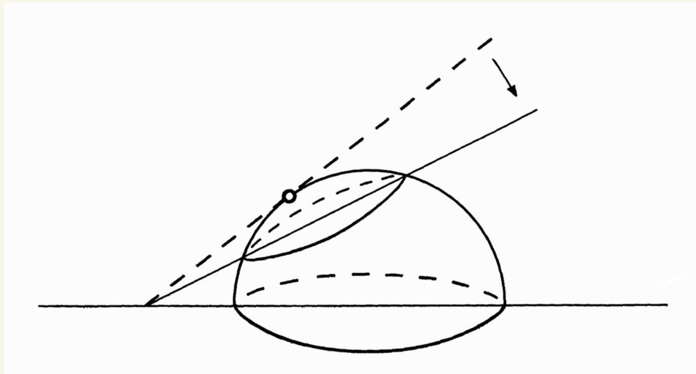
Theorem (ZF) (Szulkin)
$\mathbb{R}^3$ can be partitioned in circles.

Theorem (ZFC) (Conway-Croft / Kharazishvili)
$\mathbb{R}^3$ can be partitioned in unit circles.

Partitions in circles

Theorem (ZF) (Szulkin)
$\mathbb{R}^3$ can be partitioned in circles.



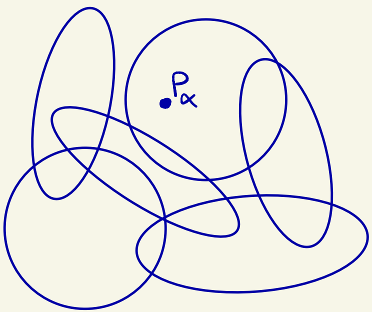


Partitions in unit circles

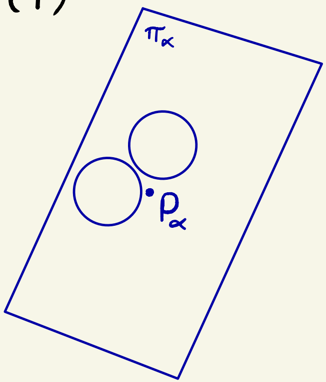
Theorem (ZFC) (Conway-Croft / Kharazishvili)
$\mathbb{R}^3$ can be partitioned in unit circles.

Let $\mathbb{R}^3 = \{p_\alpha\}_{\alpha < \mathfrak{c}}$.

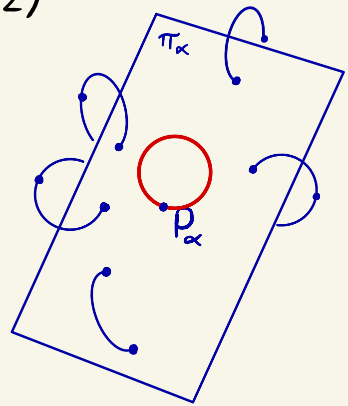
(0)



(1)



(2)



Partitions in unit circles

Observation: The proof shows that any partial PUC of cardinality $<\mathfrak{c}$ can be extended to a (complete) PUC.

Question: Can we always extend a partial PUC to a (complete) PUC?

– Sometimes there is not enough "space" to extend a partial PUC.

Context

Axiom of Choice

Paradoxical sets

- Vitali set
- Hamel basis
- Mazurkiewicz set
- PUC

$\Downarrow$

well-order of the reals

?

$\Downarrow$

...

Partitions of $\mathbb{R}^3$ in
Unit (r = 1)
Circles

How does constructing models look like?

Set theory

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$$(M,\ \mathbb{P},\ g \subseteq \mathbb{P}) \longrightarrow \boxed{\text{Forcing}} \rightsquigarrow M[g]$$

Set theory

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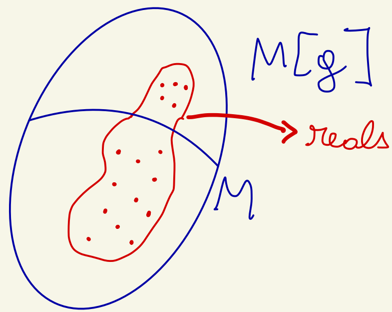


The model(s)

1. Cohen-Halpern-Lévy model:

$$\boxed{H := L(A)}$$

where g is $\mathbb{C}(\omega)$-generic over L, and
$A = \{c_n : n < \omega\}$ is the set of Cohen reals added by g.

2.

$$\boxed{W = L(\mathbb{R}, \mathcal{P})^{L[\tilde{g}, h]}}$$

where $\tilde{g}$ is $\mathbb{C}(\omega_1)$-generic over L,
h is $\mathbb{P}$-generic over $L[\tilde{g}]$, and
$\mathcal{P} = \bigcup h$ is the PUC added by h.

Recent results

Models of $ZF + \neg C + \exists P$

- First Cohen model $L(A)$:

  $ZF + \neg AC_\omega + \ldots$ + Hamel basis [BSWY]

  + Mazurkiewicz set [BS]

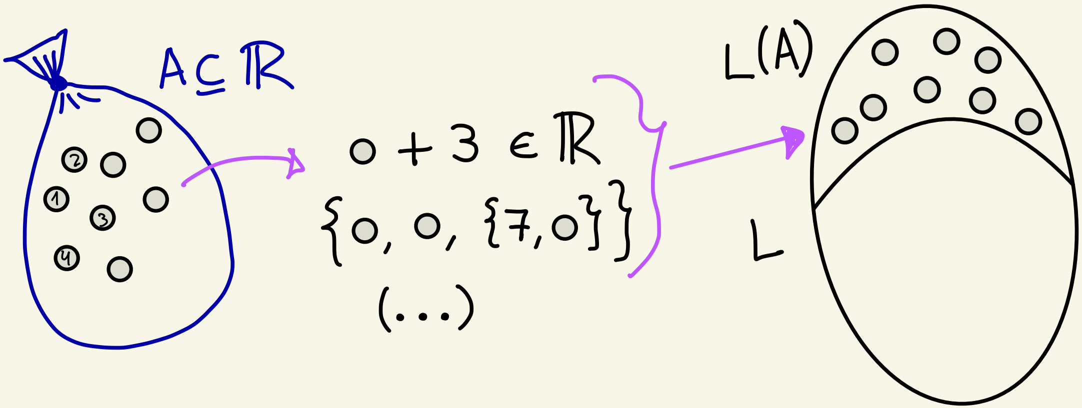
  + Partition $\mathbb{R}^3$ in unit circles [F]

Cohen-Halpern-Lévy model

Cohen-Halpern-Lévy model:

$$\boxed{H := L(A)}$$

where g is $\mathbb{C}(\omega)$-generic over L, and A is the set of Cohen reals added by g.



Cohen-Halpern-Lévy model

1. Cohen-Halpern-Lévy model:

$$\boxed{H := L(A)}$$

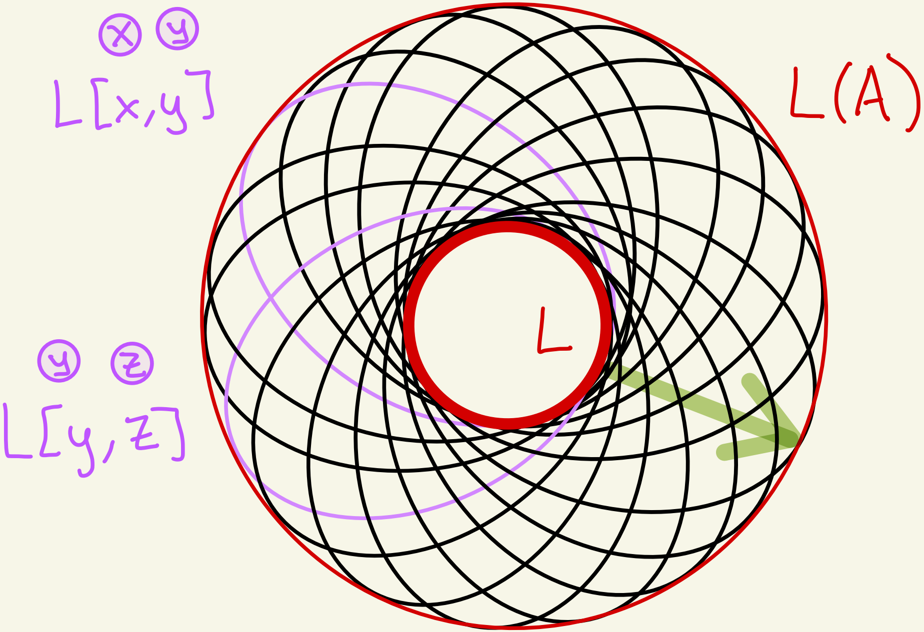
Facts about H

(i) There is no well-ordering of the reals.

(ii) There is no countable subset of $A \subseteq \mathbb{R}$

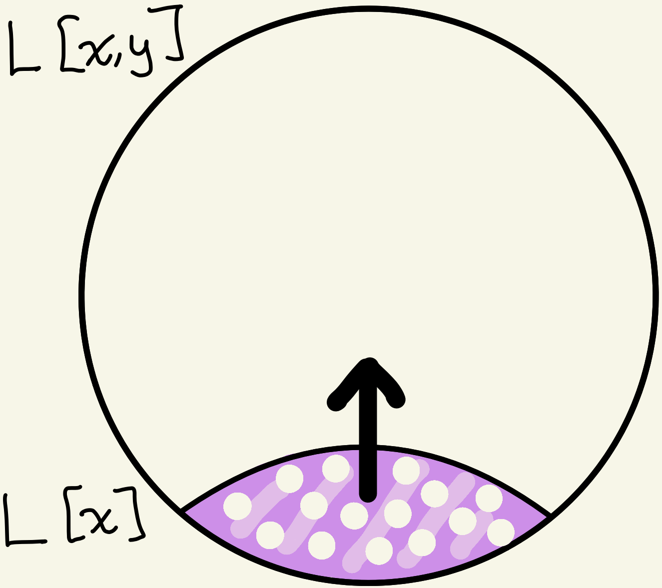
(iii) $$\mathbb{R} \cap H = \bigcup_{a \in [A]^{<\omega}} (\mathbb{R} \cap L[a])$$

Construction of a PUC in H

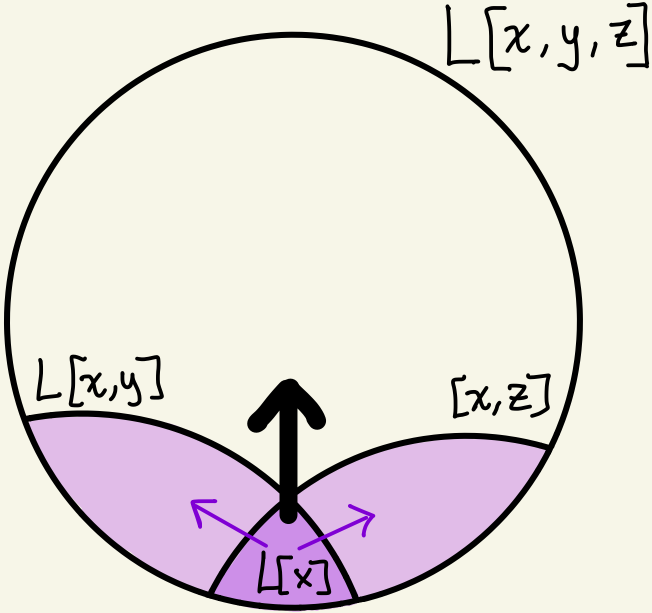


Strategy: use AC in each inner model to construct PUC's, and be careful to glue them well.

Obstacles to glue the PUCs

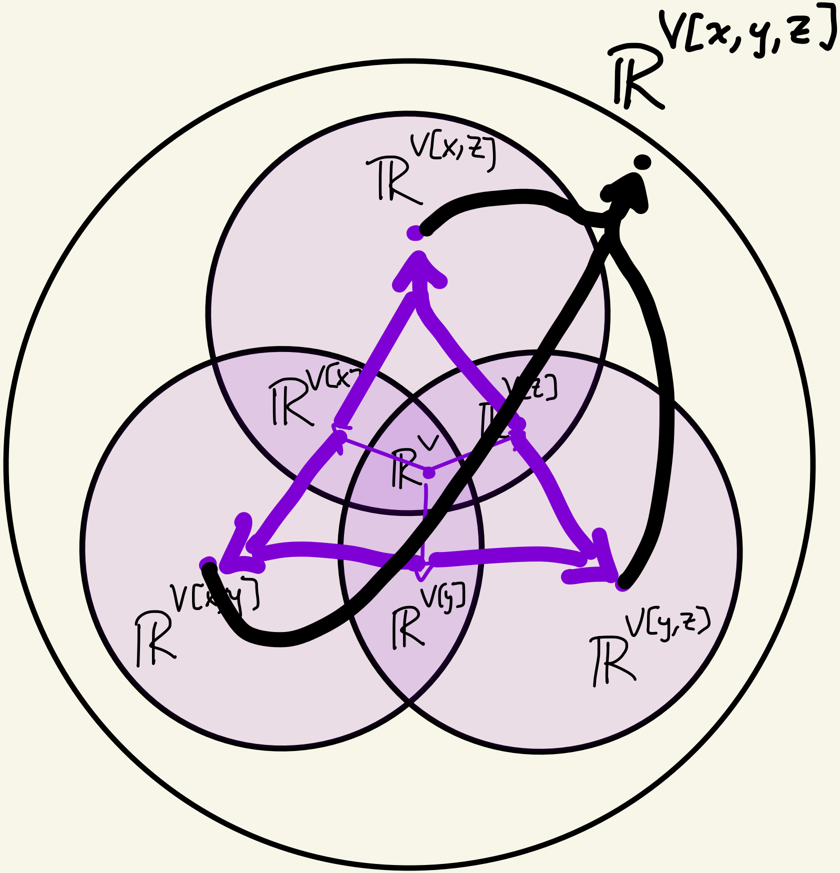


Extendability



2-Amalgamation*
(<ω - Amalgamation*)

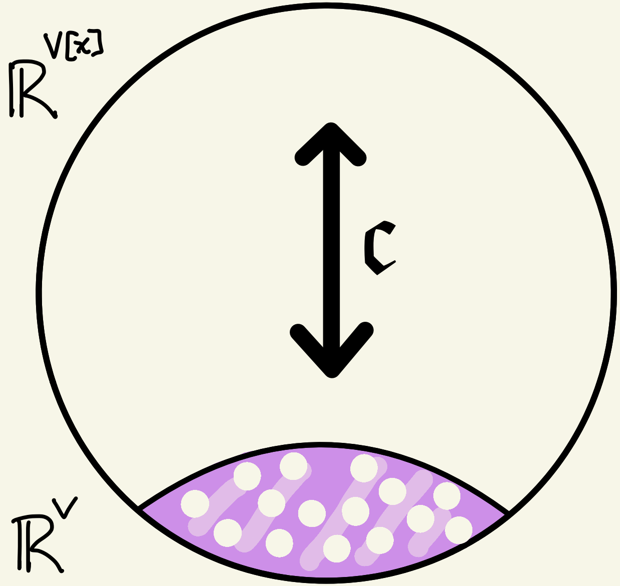
Obstacles to glue the PUCs



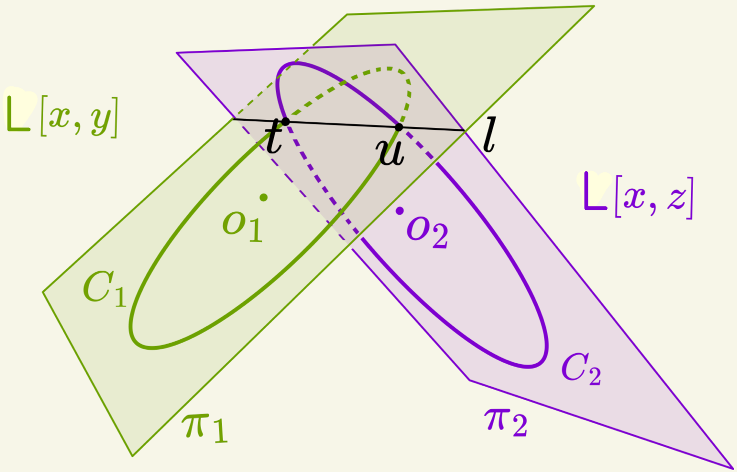
3 - Amalgamation*

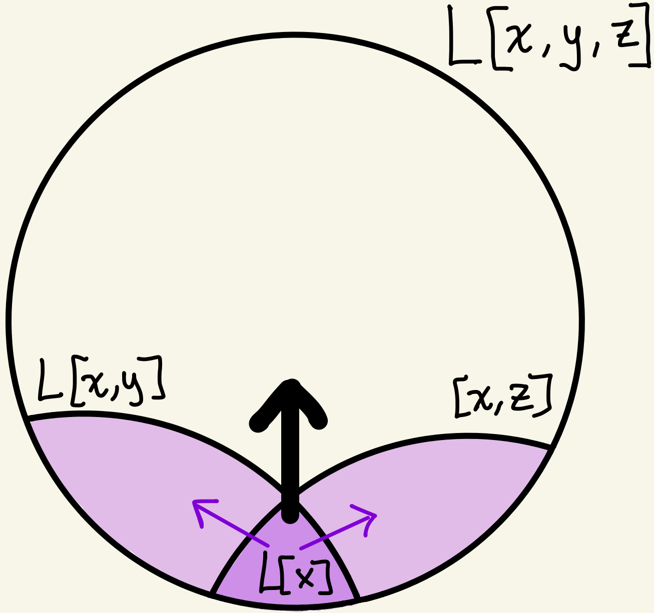
Making space

Fact: Let $V \models ZFC$ and $V[x]$ be a generic extension obtained by adding one Cohen real. Then the transcendence degree of $\mathbb{R}^{V[x]}$ over $\mathbb{R}^V$ is $\mathfrak{c}$.



Construction of a PUC in H



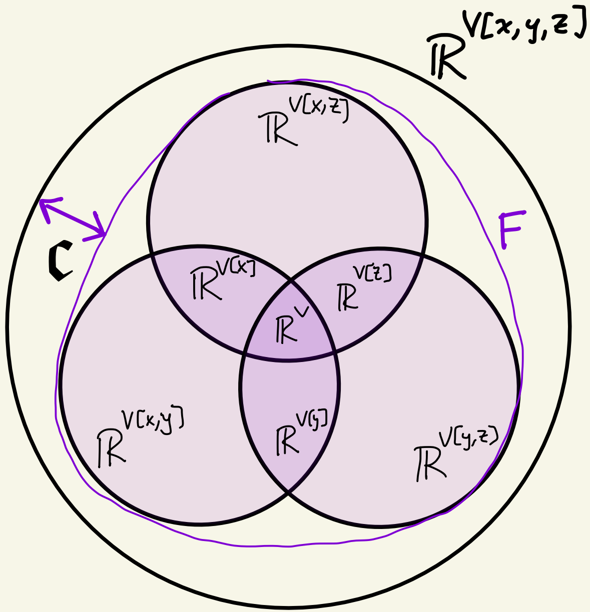


2- Amalgamation*

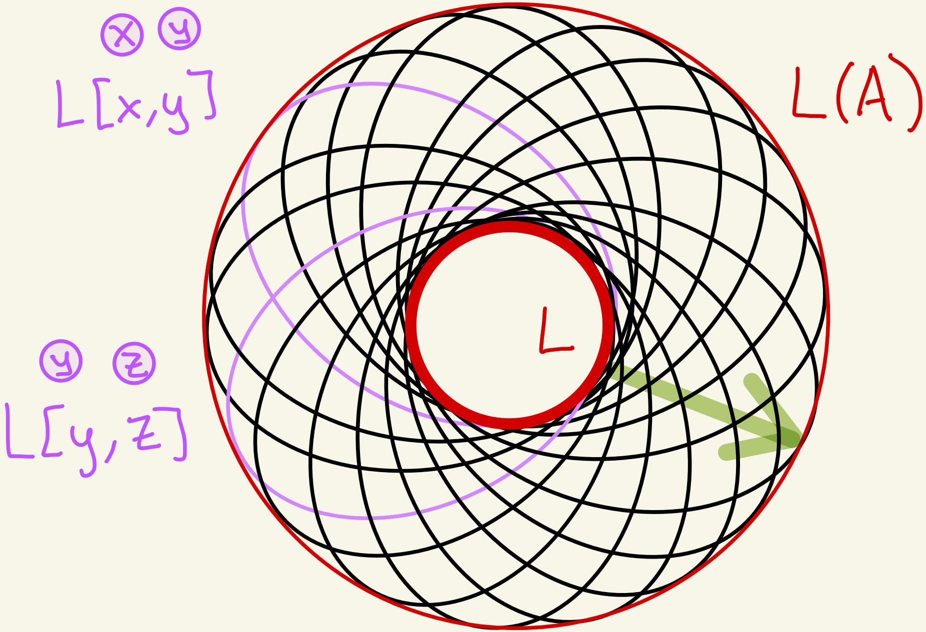
Algebraic theorem (any n)

Theorem (F., Schindler; 2025) Let V be a model of ZFC and let S be a finite set of mutually generic Cohen reals over V. Consider F the minimum field containing $\bigcup_{\substack{T \subseteq S \\ |T| = |S|-1}} \mathbb{R}^{V[T]}$.

Then the transcendence degree of $\mathbb{R}^{V[S]}$ over F is $\mathfrak{c}$.



Construction of a PUC in H



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Recipe I

Dish: $L(\mathbb{R}, \bigcup h)^{V[g,h]} \models ZF + DC + \neg WO(\mathbb{R}) + \Psi(\mathcal{P})$

Ingredients:

- $V \models ZFC$.
- $\mathbb{Q} = \mathbb{C}(\omega_1)$, g $\mathbb{Q}$-generic filter over V.
- $\mathbb{P} \in V[g]$, h $\mathbb{P}$-generic filter over $V[g]$, $\mathcal{P} := \bigcup h$.
  - $\hookrightarrow$ $\mathbb{P}$ adds a real partition according to Ψ.

Preparation:

- Prove $\mathbb{P}$ satisfies Extendability and 2-Amalgamation.

Recipe III

<u>Dish</u>: $L(\mathbb{R}, \bigcup h)^{V[g,h]} \models ZF + DC + \cancel{\neg WO(\mathbb{R})} + \Psi(\mathcal{P})$

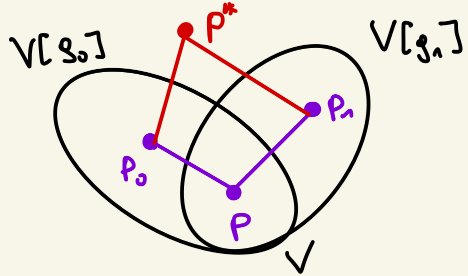
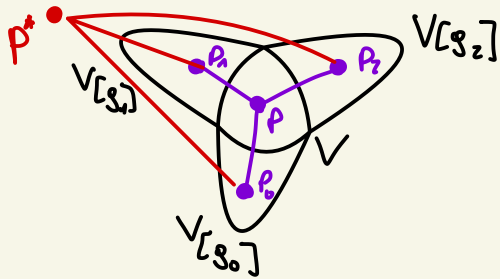
$\neg UI(\omega)$

<u>Ingredients</u>:

- $V \models ZFC$.
- $\mathbb{Q} = \mathbb{C}(\omega_1)$, g $\mathbb{Q}$-generic filter over V.
- $\mathbb{P} \in V[g]$, h $\mathbb{P}$-generic filter over $V[g]$, $\mathcal{P} := \bigcup h$.
  - ↳ $\mathbb{P}$ adds a real partition according to Ψ.

<u>Preparation</u>:

- Prove $\mathbb{P}$ satisfies Extendability and $(n, n-1)$-Amalgamation.

Source	Luke's preprint	My thesis	Our work	
Object	SEA orders	Hamel basis	AP orders (fair social welfare order)	
Criteria				
	GST-placid $\forall g_0, g_1$ generics $/V$ $\forall p\ \exists \bar{p}$ virtual condition $\leq p$ $\forall p_0, p_1 \leq \bar{p}\quad \exists p^* \leq p_0, p_1$.	(2,1)-Amalgamation $\forall g_0, g_1$ $\mathbb{C}$-gen. $/V$ $\exists$ densely many p $\forall p_0, p_1 \leq p,\ \exists p^* \leq p_0, p_1$.	GST-(3,2)-balanced Similar - g_0, g_1, g_2 are generics $/V$, pairwise mutually generic	(3,2)-Amalgamation Similar - g_0, g_1, g_2 are $\mathbb{C}$-generics $/V$, pairwise $\mathbb{C}$-mutually generic.
Model	$M[g]$ ↓ Solovay's ↘ $\mathbb{P}$-generic	$L(\mathbb{R}, U_g)^{L[h,g]}$ ↙ $\mathbb{P}$-gen $/L[h]$ ↓ $\mathbb{C}(\omega_1)$-gen $/L$	$M[g]$	$L(\mathbb{R}, U_g)^{L[h,g]}$
Result(*)	$$\text{Model} \models ZF + DC + \text{object} + \neg Uf(\omega).$$			

(*) For the models GST-style we assume the existence of an inaccessible.

Recent results

Models of ZF + $\neg$C + $\exists$P

* Models of ZF + DC + $\neg$WO($\mathbb{R}$) +
  - Hamel basis [SY]
  - Hamel basis + ... [B'CSWY]
  - Mazurkiewicz set [BS]
  - Partition of $\mathbb{R}^3$ in unit circles [F]

Coming back

Axiom of Choice $\sim$ Paradoxical sets

Paradoxical sets:
- Vitali set
- Hamel basis
- Mazurkiewicz set
- PUC

Axiom of Choice

$\Downarrow$

well-order of the reals $\Rightarrow$ Paradoxical sets

well-order of the reals $\not\Leftarrow$ Paradoxical sets

$\Downarrow$

...

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Partitions in circles

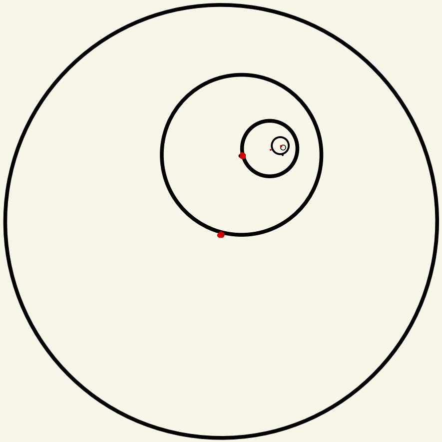
PUC := partition of $\mathbb{R}^3$ in unit circles

Question: (i) Why $\mathbb{R}^3$?

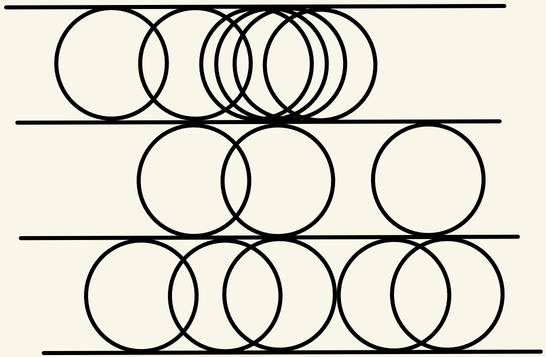
(ii) Why partitions? (1-covering)

(iii) Why circles? Why unit circles?

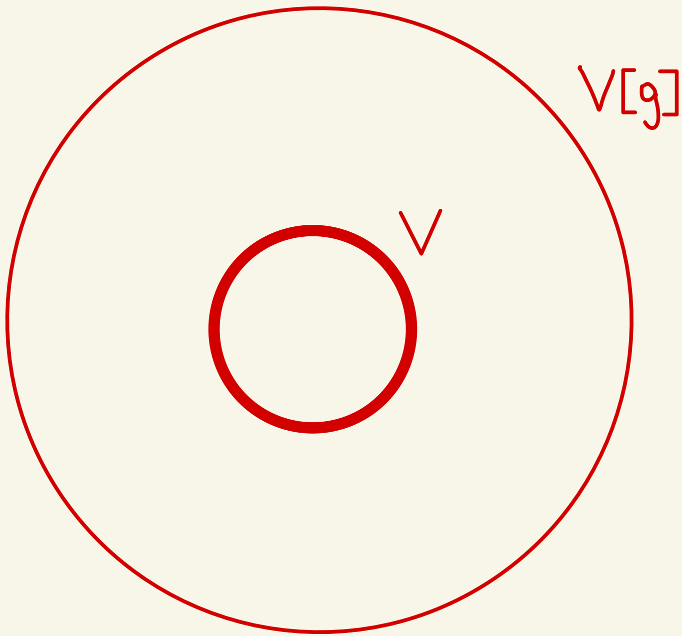
(i)



(ii)



Recipe I



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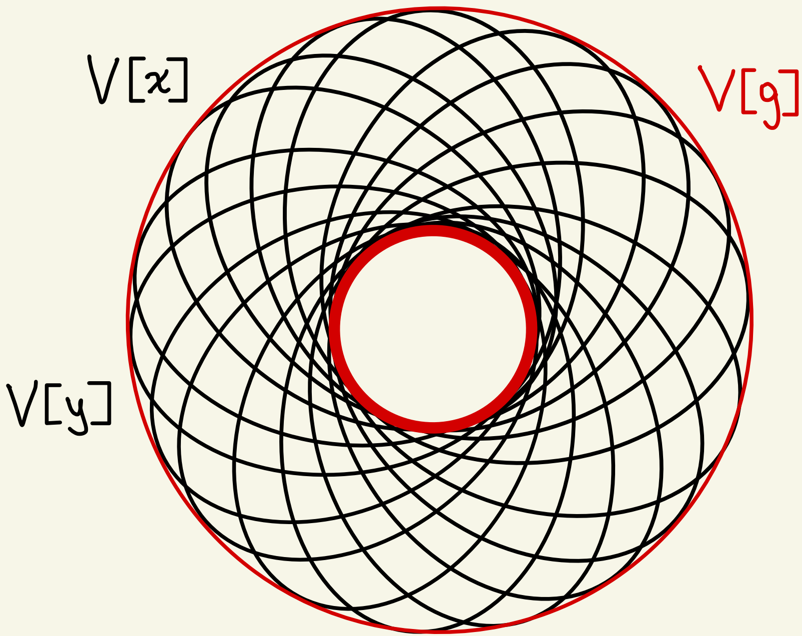

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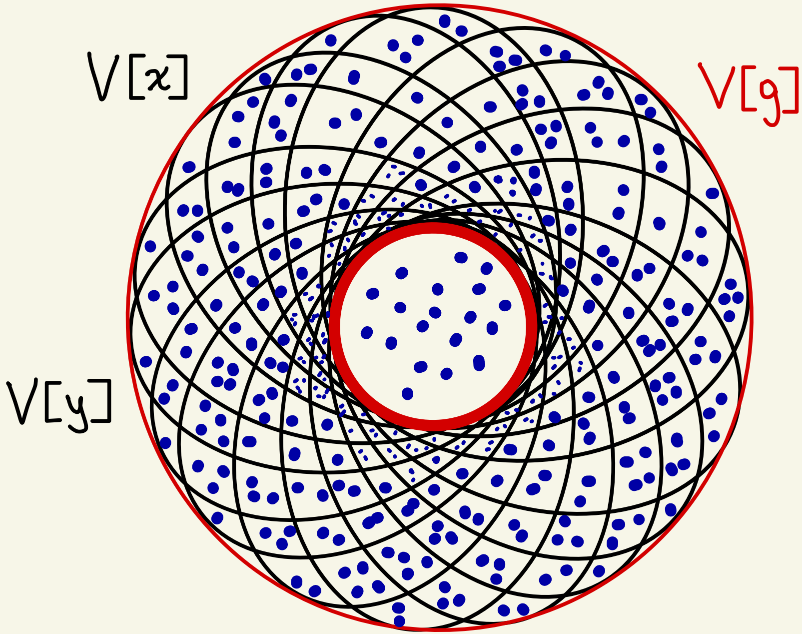

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