Tame Extensions of Generic Derivations on O-minimal Structures

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Summary

Definability of Types

Given a language $\mathcal{L}$, $n \in \mathbb{N}$, an $\mathcal{L}$-structure $\mathcal{M}$, and a subset $A \subseteq M$, an n-type $p(\overline{x}) \in S_n^{\mathcal{M}}(A)$ is said to be A**-definable** (or simply **definable** if $A = M$) if for any $\mathcal{L}(\varnothing)$-formula $\varphi(\overline{x}, \overline{y})$, there exists an $\mathcal{L}(A)$-formula $d\varphi(\overline{y})$ such that for all $\overline{b} \in A$, we have $\varphi(\overline{x}, \overline{b}) \in p(\overline{x})$ if and only if $\mathcal{M} \models d\varphi(\overline{b})$.

In a sense, $\overline{y}$ are the parameters, and whether the formula φ with parameters $\overline{b}$ is in the type p or not is completely determined by whether $\overline{b}$ realizes $d\varphi$ in $\mathcal{M}$. So in particular, $d\varphi$ depends on φ.

Fact

A theory T is stable if and only if for any $n \in \mathbb{N}$ and any $\mathcal{M} \models T$, every $p \in S_n^{\mathcal{M}}(M)$ is definable.

Let $\mathcal{R} \preceq \mathcal{S}$ be a pair of real closed fields, and pick $a \in S \backslash R$. Is $\mathrm{tp}(a/R)$ definable?

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Tame Extensions

From now on, T is a complete, model-complete o-minimal extension of the theory of real closed fields in a fixed language $\mathcal{L}$. Let $\mathcal{M} \models T$. An element $c \in M$, and subsets $A, B \subseteq M$, we denote $|A|$ as the set of all absolute values of elements in A, that is,

$$|A| = \{|a| : a \in A\},$$

and we write $c < A$ to mean that c is a lower bound of A, that is $c < a$ for every $a \in A$, and $A < B$ to mean that every a is a lower bound of B, that is $a < B$ for every $a \in A$. We denote $c > A$ and $A > B$ likewise (upper bound).

Let $\mathcal{N} \preceq \mathcal{M}$. We say that the pair $(\mathcal{M}, \mathcal{N})$ is **tame** if for every $a \in M$,

(i) $|a| > N$, in which case we say a is **infinite** with respect to N;

(ii) there exists some $b \in N$ such that b is **infinitesimal** to a (or $b - a$ is an **infinitesimal element**), that is $|b - a| < |N| \backslash \{0\}$; in the case that there exists such $b \in N$ that is infinitesimal to a, we say that a is N-**bounded**, and we call b the **standard part** of a, denoted by $\mathrm{st}(a) = b$.

Standard Part Map and Marker-Steinhorn Theorem

It is an easy exercise to see that the standard part of an element is necessarily unique and the function $\mathrm{st} : M \to N$ defined by

$$\mathrm{st}(a) = \begin{cases} b, & \text{if } b \in N \text{ and } b - a \text{ is an infinitesimal element} \\ 0, & \text{otherwise} \end{cases}$$

is definable in the pair structure $(\mathcal{M}, \mathcal{N})$. This function is called the **standard part map** induced on $\mathcal{M}$ by $\mathcal{N}$.

Theorem (Marker-Steinhorn, Theorem 2.1 in Marker and Steinhorn 1994)

Let $p(\overline{x}) \in S_n^{\mathcal{M}}(M)$. Then $p(\overline{x})$ is M-definable if and only if $\mathrm{dcl}_{\mathcal{L}}(M, \overline{a})$ is a tame extension of $\mathcal{M}$, where $\overline{a}$ is any n-tuple realizing $p(\overline{x})$.

It is well-known that o-minimal structures expanding groups have definable Skolem functions, and thus the $\mathcal{L}$-structure $\mathrm{dcl}_{\mathcal{L}}(M, \overline{a})$ is always an elementary extension of $\mathcal{M}$.

The Marker-Steinhorn Theorem is saying that $\overline{a}$ realizes a definable type if and only if each entry of the tuple $\overline{a}$ is either infinitely far away from M or infinitely close to some element of M.

Theory of Tame Pairs

Let $\mathcal{L}_{\text{tame}} = \mathcal{L} \cup \{U, \text{st}\}$ be the language $\mathcal{L}$ expanded by a unary predicate symbol U and a unary function symbol st. We define T_{tame} to be the $\mathcal{L}_{\text{tame}}$-theory of proper tame elementary pairs of models of T. More precisely, $(\mathcal{M}, \mathcal{N}, \text{st}) \models T_{\text{tame}}$ if $\mathcal{M} \precnsim_{\text{tame}} \mathcal{N} \models T$, and $\text{st} : M \to N$ is the standard part function.

Theorem (Theorem 5.9 and Corollary 5.10 in Dries and Lewenberg 1995)

Suppose that T has quantifier elimination and is universally axiomatizable. Then T_{tame} has quantifier elimination. Without these assumptions on T, the theory T_{tame} is complete and model complete.

The condition T has quantifier elimination and is universally axiomatizable seems rather restrictive, but this is not the case if we study the geometry of definable sets in o-minimal structures expanding groups.

Expansion by Skolem Functions

For each $\mathcal{L}$-formula $\varphi(\overline{x}, y)$ such that

$$T \vdash \forall \overline{x} \exists ! y \varphi(\overline{x}, y),$$

Let f_φ be the new function symbol such that

$$\varphi(\overline{x}, f_\varphi(\overline{x})).$$

$\mathcal{L}^{df}$ is the language $\mathcal{L}$ expanded by all such f_φ and T^{df} is the theory expanding T by the corresponding axioms.

Since T has definable Skolem functions, this process does not generate new definable sets.

It is safe to assume that T has quantifier elimination and is universally axiomatizable for the purpose of this talk.

Stable Embedding

Proposition (van den Dries 2003a)

Let $(\mathcal{M}, \mathcal{N}, st) \models T_{tame}$ and $n \in \mathbb{N}$. If $X \subseteq M^n$ is definable in $(\mathcal{M}, \mathcal{N}, st)$, then $X \cap N^n$ is definable in $\mathcal{N}$.

The proposition implies the Marker-Steinhorn Theorem. To see that, we write the Marker-Steinhorn Theorem in an equivalent form as below.

Proposition (van den Dries 2003a)

Let $(\mathcal{M}, \mathcal{N}, st) \models T_{tame}$ and $n \in \mathbb{N}$. If $X \subseteq M^n$ is definable in $\mathcal{M}$, then $X \cap N^n$ is definable in $\mathcal{N}$.

If $\varphi(\overline{x}, \overline{y})$ is an $\mathcal{L}(\varnothing)$-formula, a φ-type $\mathrm{tp}_\varphi(\overline{a}/N)$ with $\overline{a} \in M^n$ is the same as an externally definable set $\{\overline{b} \in N^n : \varphi(\overline{a}, \overline{b})\}$ which is exactly the trace of an $\mathcal{L}(M)$-definable set.

G-Metric

Fix an ordered abelian group $\mathcal{G}$ and a set Z.

Definition

A function $d : Z \times Z \to G^{\geqslant 0}$ is called a ***G-metric*** (or ***G-valued distance function***) on Z if for all $x, y, z \in Z$,

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) + d(y, z) \geqslant d(x, z)$.

Given a G-metric d on Z, the value $d(x, y)$ is called the **distance between x and y in the G-metric d**, and for any positive $\varepsilon \in G$, the set

$$B_d(x, \varepsilon) := \{y \in Z \mid d(x, y) < \varepsilon\}$$

is called the **open ball centered at x of radius ε with respect to the G-metric d**. If d is clear from the context, we may omit it from the notation.

ε-Neighbourhood

Definition

Given a G-metric d on Z, the topology generated by the collection of all open balls $B_d(x, \varepsilon)$ for $x \in Z$ and positive $\varepsilon \in G$ is called the **G-metric topology on Z induced by d**. The triple (Z, G, d) is called a **G-metric space**. If d is clear from the context, we will simply say that Z is a G-metric space.

For $A \subseteq Z$ and positive $\varepsilon \in G$, the **ε-neighborhood** of A is defined as

$$U_d(A, \varepsilon) := \bigcup_{x \in A} B_d(x, \varepsilon).$$

If d is clear from the context, we may omit it from the notation.

Hausdorff Distance Function

Let $Z = M^n$ for some $n \in \mathbb{N}$ and d the Euclidean distance function. Let $\mathcal{K}(Z)$ denote the set of all nonempty $\mathcal{L}(M)$-definable closed and bounded subsets of Z. The **Hausdorff distance function** $D : \mathcal{K}(Z) \times \mathcal{K}(Z) \to M$ **induced by** d is defined by

$$D(A, B) = \max\{\inf\{\varepsilon \in M \mid B \subseteq U_d(A, \varepsilon)\}, \inf\{\varepsilon \in M \mid A \subseteq U_d(B, \varepsilon)\}\}.$$

Fact

$D(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}$. *In particular, the value* $D(A, B)$ *is* $\mathcal{L}(M)$*-definable for every pair of sets* $A, B \in \mathcal{K}(Z)$.

Definable Family and Fibers

Let $\mathcal{M} \models T$ be $\aleph_1$-saturated such that $\mathbb{R} \prec M$. Fix $m \in \mathbb{N}$. Let $A^* \subseteq \mathcal{M}^{m+k}$ be $\mathcal{L}(\varnothing)$-definable. Let $\Pi_m : \mathcal{M}^{m+k} \to \mathcal{M}^m$ be the projection onto the first m coordinates. Set $(A')^* := \Pi_m(A^*)$. For each $a \in (A')^*$, denote

$$A_a^* := \{x \mid (a, x) \in A^*\}, \quad F(A^*) := \{A_a^* \mid a \in (A')^*\}.$$

Since $\mathbb{R}$ is Dedekind complete, we can define $\mathrm{st} : M^k \to \mathbb{R}^k \cup \{\infty\}$ by

$$\mathrm{st}(s) := \begin{cases} r & \text{if there exists } r \in \mathbb{R}^k \text{ such that } d(r, s) \text{ is infinitesimal,} \\ \infty & \text{otherwise.} \end{cases}$$

Let $A' := (A')^* \cap \mathbb{R}^m$ and $A := A^* \cap \mathbb{R}^{m+k}$. We also assume that for each $a \in A'$, the set A_a is closed and bounded with respect to the metric d.

Hausdorff Limits and Standard Parts

The next lemma explains how the Hausdorff limit of a sequence from a definable family is related to the standard part of an externally definable set.

Lemma

$X \in cl(F(A))$ if and only if there exists $a \in (A')^*$ such that $X = st(A^*_a)$.

A consequence of the Marker-Steinhorn Theorem is

Theorem

Any Hausdorff limit of a sequence from a definable family in $\mathbb{R}$ is definable in $\mathbb{R}$.

A consequence of the Stable Embedding Property is

Theorem

$cl(F(A))$ is definable in $\mathbb{R}$.

T-derivation and its Model Completion

Let $\mathcal{L}^\delta = \mathcal{L} \cup \{\delta\}$ where δ is a unary function symbol. Define T^δ to be the theory expanding T by an additional axiom schema saying that δ is **compatible** with every $\mathcal{L}(\varnothing)$-definable $\mathcal{C}^1$ function $f : U \to M$ with $U \subseteq M^n$ open, in the following sense:

$$\delta f(\overline{u}) = \mathbf{J}_f(\overline{u})\delta\overline{u},$$

for each $\overline{u} \in U$, where $\mathbf{J}_f(\overline{u})$ is the Jacobian matrix of f at $\overline{u}$ (the Jacobian is computed with respect to the standard derivative on real closed fields; see Chapter 7 of Dries 2003b for details on differentiations on o-minimal structures extending fields).

It is not hard to see that a T-derivation is indeed a derivation (see Lemma 2.2 in Fornasiero and Kaplan 2021). Fornasiero and Kaplan showed that T^δ has a model completion T^δ_g, which is T^δ with extra axioms of genericity.

Jet-Space

A very important notion associated with structures expanding ordered differential fields is the n-jet-space.

Definition

Let $(\mathcal{M}, \delta)$ be an $\mathcal{L}^{\delta}$-structure. For a k-tuple $(n_1, \ldots, n_k) \in \mathbb{N}^k$ and $A \subseteq M^k$, the $(n_1, \ldots, n_k)$***-jet-space*** of A is the set

$$\mathrm{Jet}^{\delta}_{(n_1,\ldots,n_k)}(A) := \{(x_1, \delta x_1, \ldots, \delta^{n_1} x_1, \ldots, x_k, \delta x_k, \ldots, \delta^{n_k} x_k) \mid (x_1, \ldots, x_k) \in A\}.$$

If $k = 1$, then we simply write $\mathrm{Jet}^{\delta}_{n_1}(A)$ instead, and if δ is clear from the context, we will drop the superscript and write $\mathrm{Jet}_{(n_1,\ldots,n_k)}(A)$ instead.

Observe that

$$\mathrm{Jet}_{(n_1,\ldots,n_k)}(M^k) = \mathrm{Jet}_{n_1}(M) \times \cdots \times \mathrm{Jet}_{n_k}(M).$$

For $k, m \in \mathbb{N}$, let $\Pi_m : M^{m+k} \to M^m$ be the projection map to the first m coordinates.

T_g^δ

Definition

A T-derivation δ on $\mathcal{M}$ is said to be **generic** if for every $n \in \mathbb{N}$ and every $\mathcal{L}(M)$-definable set $A \subseteq M^{n+1}$, if

$$\dim_{\mathcal{L}}(\Pi_n(A)) = n,$$

then there exists $a \in M$ such that $\mathrm{Jet}_n(a) \in A$. Let T_g^δ be the $\mathcal{L}^\delta$-theory extending T^δ by the axiom schema which asserts that δ is generic.

Theorem (Theorem 4.8 in Fornasiero and Kaplan 2021)

T_g^δ is the model completion of T^δ, and if T has quantifier elimination and is universally axiomatizable, then T_g^δ has quantifier elimination.

Lemma (Lemma 4.11 in Fornasiero and Kaplan 2021)

For every $\mathcal{L}^\delta$-formula φ (possibly with parameters), there exist some $n \in \mathbb{N}$ and some $\mathcal{L}$-formula $\tilde{\varphi}$ such that

$$T_g^\delta \vdash \forall \overline{x}\,[\varphi(\overline{x}) \leftrightarrow \tilde{\varphi}(Jet_n(\overline{x}))].$$

Tame Pairs of T_g^δ

Set $\mathcal{L}_{\text{tame}}^\delta := \mathcal{L}_{\text{tame}} \cup \{\delta\}$. We define T_{tame}^δ to be the theory such that $(\mathcal{M}, \mathcal{N}, \text{st}, \delta) \models T_{\text{tame}}^\delta$ if $(\mathcal{N}, \delta|_N), (\mathcal{M}, \delta) \models T^\delta$, and $(\mathcal{M}, \mathcal{N}, \text{st}) \models T_{\text{tame}}$. Let $T_{g,\text{tame}}^\delta$ be the theory such that $(\mathcal{M}, \mathcal{N}, \text{st}, \delta) \models T_{\text{tame}}^\delta$ if $(\mathcal{N}, \delta|_{\mathcal{N}}), (\mathcal{M}, \delta) \models T_g^\delta$ and $(\mathcal{M}, \mathcal{N}, \text{st}) \models T_{\text{tame}}$.

Theorem

Suppose that T has quantifier elimination and is universally axiomatizable. Then, the theory $T_{g,tame}^\delta$ has quantifier elimination.

Proposition

Let $(\mathcal{M}, \mathcal{N}, st, \delta) \models T_{g,tame}^\delta$ and $n \in \mathbb{N}$. If $X \subseteq M^n$ is definable in $(\mathcal{M}, \mathcal{N}, st, \delta)$, then $X \cap N^n$ is definable in $(\mathcal{N}, \delta|_N)$.

Proposition

Suppose that $A = dcl_{(\mathcal{M},\delta)}(A)$ and $\overline{a} \in M$. Then $tp^{(\mathcal{M},\delta)}(\overline{a}/A)$ is A-definable if and only if $dcl_{(\mathcal{M},\delta)}(A, \overline{a})$ is a tame extension of A as $\mathcal{L}$-structures.

δ-topology

Fact (Fornasiero and Kaplan 2021)

Let $A \subseteq M^n$ be an $\mathcal{L}^\delta(M)$-definable set. Then the Euclidean closure $cl(A)$ of A is $\mathcal{L}(M)$-definable.

Another problem with the Euclidean topology is that δ is highly discontinuous with respect to it.

Let $I = (-1, 1)$. Then $\delta^{-1}(I) = \{x \in M : -1 < \delta x < 1\}$. The set $\delta^{-1}(I) \subseteq M$ is a dense and codense subset.

Definition

Let $A \subseteq M^n$ be defined by the $\mathcal{L}^\delta(M)$-formula $\varphi(\overline{x})$. Suppose that there exists an $\mathcal{L}(M)$-formula $\tilde{\varphi}(\overline{y})$ such that

$$T_g^\delta \vdash \forall \overline{x}\,[\varphi(\overline{x}) \leftrightarrow \tilde{\varphi}(\mathrm{Jet}_n(\overline{x}))],$$

and $\tilde{\varphi}$ defines a Euclidean open set. Then A is called **δ-open**. The topology generated by all δ-open sets is called the **δ-topology** on M^n.

Approximation of δ-topology

Let $\mathcal{T}_\infty$ be the δ-topology. It is not $\mathcal{L}^\delta$-definable in the sense that there does not exists an $\mathcal{L}^\delta(\varnothing)$-formula $\varphi(\overline{x}, \overline{y})$ such that $\mathcal{T}_\infty$ is generated by the set defined by $\varphi(\overline{x}, \overline{a})$ where $\overline{a}$ ranges through $M^{|\overline{y}|}$.

To make preimage of Euclidean open sets open under δ, we need the M-metric topology generated by the M-metric

$$d_{1,k}(\overline{x}, \overline{y}) = \sum_{j=1}^{k} \sqrt{(x_j - y_j)^2 + (\delta x_j - \delta y_j)^2}.$$

Let this topology be $\mathcal{T}_1$. Let $I_1 = \delta^{-1}(I)$ and then $\delta^{-1}(I_1) = \{x \in M : -1 < \delta^2 x < 1\}$ not in $\mathcal{T}_1$. So we need

$$d_{2,k}(\overline{x}, \overline{y}) = \sum_{j=1}^{k} \sqrt{(x_j - y_j)^2 + (\delta x_j - \delta y_j)^2 + (\delta^2 x_j - \delta^2 y_j)^2}.$$

Call it $\mathcal{T}_2$.

$d_{n,k}$-metric

Define $\mathcal{T}_n$ the metric topology induced by the M-metric

$$d_{n,k}(\overline{x},\overline{y}) = \sum_{i=0}^{n} \sqrt{\sum_{j=1}^{k} (\delta^i x_j - \delta^i y_j)^2}.$$

It is not hard to see that $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$ and $\mathcal{T}_\infty = \bigcup_{i=0}^{\infty} \mathcal{T}_i$, where $\mathcal{T}_0$ is the Euclidean topology. Note that for all $n < \infty$, the topology $\mathcal{T}_n$ is $\mathcal{L}^\delta$-definable.

Let $(\mathcal{M}, \delta) \models T_g^\delta$ be $\aleph_1$-saturated such that $\mathbb{R} \subseteq M$ and $\mathrm{dcl}_{(\mathcal{M},\delta)}(\mathbb{R}) = \mathbb{R}$. In particular, $(\mathbb{R}, \delta|_{\mathbb{R}}) \models T^\delta$. Fix $n \in \mathbb{N}$. Let $A^* \subseteq \mathcal{M}^{m+k}$ be $\mathcal{L}^\delta(\varnothing)$-definable. Recall that $\Pi_m : \mathcal{M}^{m+k} \to \mathcal{M}^m$ denotes the projection onto the first m coordinates. Set $(A')^* := \Pi_m(A^*)$. For each $a \in (A')^*$, denote

$$A_a^* := \{x \mid (a,x) \in A^*\}, \quad F(A^*) := \{A_a^* \mid a \in (A')^*\}.$$

Since $\mathbb{R}$ is Dedekind complete, we can define $\mathrm{st}_{n,k} : M^k \to \mathbb{R}^k \cup \{\infty\}$ by

$$\mathrm{st}_{n,k}(s) := \begin{cases} r & \text{if there exists } r \in \mathbb{R}^k \text{ such that } d_{n,k}(r,s) \text{ is infinitesimal,} \\ \infty & \text{otherwise.} \end{cases}$$

Definability of Hausdorff limits

Let $A' := (A')^* \cap \mathbb{R}^m$ and $A := A^* \cap \mathbb{R}^{m+k}$. We also assume that for each $a \in A'$, the set A_a is closed and bounded with respect to the metric $d_{n,k}$. The next lemma explains how the Hausdorff limit of a sequence from a definable family is related to the standard part of an externally definable set.

Lemma

If $X \in cl_n(F(A))$, then there exists $a \in (A')^*$ such that $X = st_{n,k}(A_a^*)$. The converse holds if $(\mathbb{R}, \delta|_{\mathbb{R}})$ is an elementary substructure of $(\mathcal{M}, \delta)$.

Theorem

Any Hausdorff limit of a sequence from a definable family in $(\mathbb{R}, \delta|_{\mathbb{R}})$ is definable in $(\mathbb{R}, \delta|_{\mathbb{R}})$.

Theorem

$cl_n(F(A))$ is definable in $(\mathbb{R}, \delta|_{\mathbb{R}})$.

Reference

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