Problem 1

a) Two vectors are orthogonal if their dot product equals zero. However, functions, like sine and cosine generate vectors with ∞ entries. So, the dot product or inner product of 2 functions becomes multiplying them together and integrating them.

$$\langle f, g \rangle = \frac{1}{L}\int_{-L}^{L} f(x)g(x)\,dx$$

$\sin\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{m\pi x}{L}\right)$ form an orthogonal basis provided $m \neq n$. This also works for cosine.

even x odd = odd   even x even = even   odd x odd = even

b) $\boxed{\bullet \text{ for } A_n\text{'s}}$; $\frac{1}{L}\int F(x)\cos\left(\frac{n\pi x}{L}\right)dx$

so, when $F(t)$ is odd and multiplied with cosine, an even function, the result is an odd function, call it $h(x)$

$$\int_{-L}^{L} h(x)\,dx = \int_{-L}^{0} h(x)\,dx + \int_{0}^{L} h(x)\,dx$$

since $h(x)$ is odd, $\int_{-L}^{0} h(x)dx =$

$$-\int_{-L}^{0} h(-x)\,dx = -\int_{0}^{L} h(x)\,dx$$

Thus $\int_{-L}^{L} F(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0$

$\boxed{\bullet \text{ for } B_n\text{'s}}$ its the same story except since sine is odd, $F(x)$ needs to be even for the resulting $h(x)$ to be odd.

c) $f(x) = \sin^3(\pi x) = \dfrac{3\sin(\pi x) - \sin(3\pi x)}{4}$

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

$L = 1$

$$A_0 = \frac{1}{2L} \cdot \int_{-L}^{L} f(x)\,dx$$
$$A_n = \frac{1}{L} \cdot \int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$
$$B_n = \frac{1}{L} \cdot \int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

• $\sin^3(\pi x)$ is odd, so cos coeff. $A_n = 0$.

$$B_n = 1 \cdot \int_{-1}^{1} \frac{3\sin(\pi x) - \sin(3\pi x)}{4} \cdot \sin(n\pi x)\,dx$$

$$f(x) = \left(\frac{3}{4}\sin(\pi x) - \frac{1}{4}\sin(3\pi x)\right) \frac{\sin(n\pi x)}{L}$$

$b_1 = \frac{3}{4}$ $\quad b_3 = -\frac{1}{4}$ $\quad b_k = 0$ if $k \neq 1, 3$

$a_k = 0 \quad \forall k$

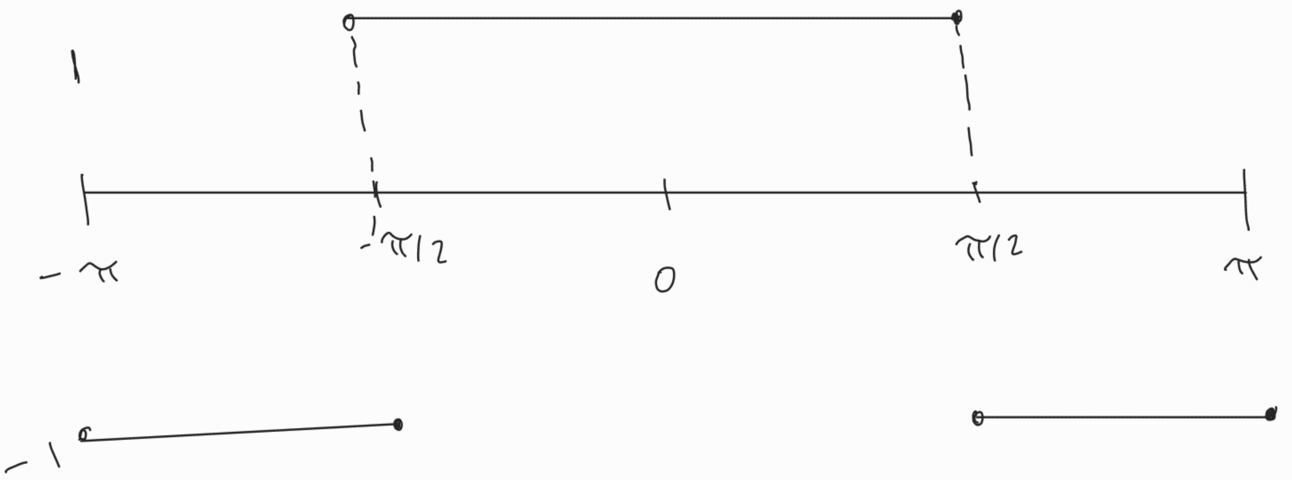
d) $f(x) = \begin{cases} -1 & \text{if } -\pi \le x \le -\pi/2 \\ 1 & \text{if } -\pi/2 \le x \le \pi/2 \\ -1 & \text{if } \pi/2 < x \le \pi \end{cases}$

$x \in [-\pi, \pi]$

$\cos(k\pi) = (-1)^k$

$\sin\left(\frac{k\pi}{2}\right) = (-1)^{k+1}$

$$A_n = \frac{1}{L} \cdot \int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$
$$B_n = \frac{1}{L} \cdot \int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$



$$a_0 = \frac{1}{2\pi}\left(\int_{-\pi}^{-\pi/2} (-1)\,dx + \int_{-\pi/2}^{\pi/2} dx + \int_{\pi/2}^{\pi} (-1)\,dx = 0\right.$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$$

$\sin\left(\frac{k\pi}{2}\right) = 0$ if k is even

$$= \frac{1}{\pi}\left[-\int_{-\pi}^{-\pi/2}\cos(kx)\,dx + \int_{-\pi/2}^{\pi/2}\cos(kx)\,dx - \int_{\pi/2}^{\pi}\cos(kx)\,dx\right]$$

$$-\frac{1}{k}\sin(kx)\Big|_{-\pi}^{-\pi/2} + \frac{1}{k}\sin(kx)\Big|_{-\pi/2}^{\pi/2} - \frac{1}{k}\sin(kx)\Big|_{\pi/2}^{\pi}$$

$$a_n = \frac{4\sin\left(\frac{\pi k}{2}\right)}{\pi k}$$

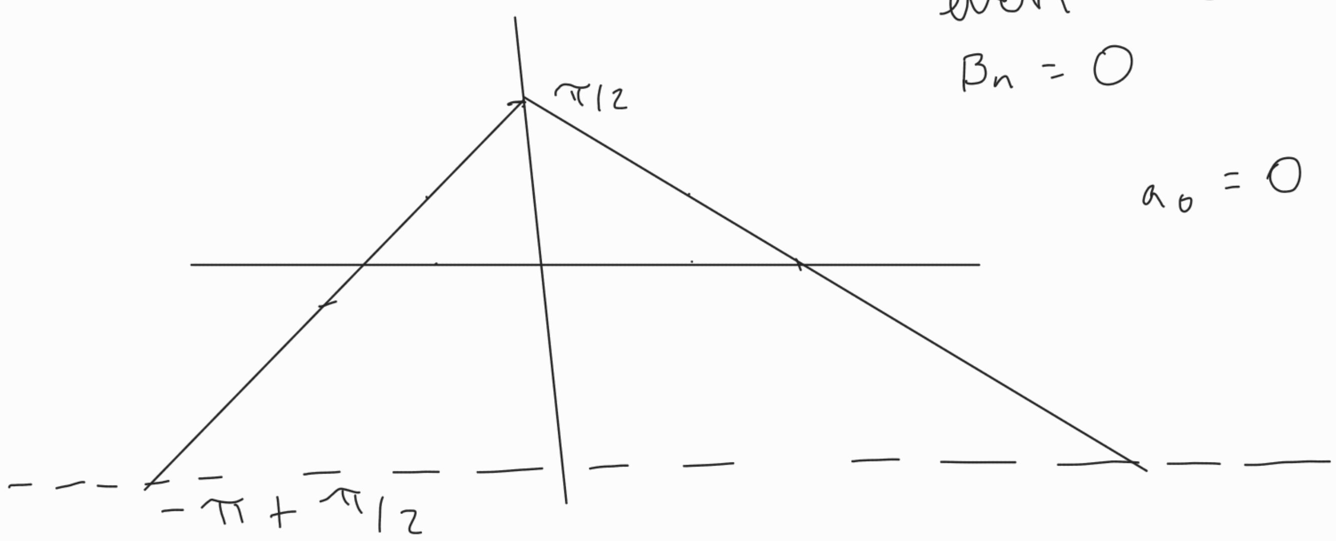
$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(-1)\sin(nx)\,dx + \int_{-\pi/2}^{\pi/2}\sin(nx)\,dx + \int_{\pi/2}^{\pi}(-1)\sin(nx)\,dx\right]$$

$$= 0$$

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}$$

$$f(x) = \sum_{n=1}^{\infty}\frac{4\sin\left(\frac{\pi n}{2}\right)}{\pi n}\cos\left(\frac{n\pi x}{L}\right)$$

e)



even so
$B_n = 0$

$a_0 = 0$

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(\pi/2 + x)\,dx + \int_{0}^{\pi}(\pi/2 - x)\,dx\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi/2 + x)\cos(nx)\,dx + \int_{0}^{\pi}(\pi/2 - x)\cos(nx)\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi/2 + x)\sin(nx)\,dx + \int_{0}^{\pi}\left(\frac{\pi}{2} - x\right)\sin(nx)\,dx\right]$$

$$a_0 = 0$$

$$a_n = \frac{2 - 2(-1)^n}{\pi n^2}$$

$$b_n = 0$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2-2(-1)^n}{\pi n^2} \cos\left(\frac{n\pi x}{L}\right)$$

Forward: $F_k = \sum_{j=0}^{n-1} f_j e^{-i 2\pi kj/n}$ $F_k \rightarrow F[k]$

$f_j = f[j]$

Backward: $f_j = \sum_{k=0}^{n-1} F_k e^{i 2\pi kj/n}$ length n

Δ wave: $A_k = \frac{-2}{\pi R^2} \left((-1)^k - 1 \right)$

F = j + np.zeroes(N)

f =