PHYS 331 – Numerical Techniques for the Sciences I

Homework 3: Complex numbers, Newton-Raphson in 2D, Fixed Point Iteration

Posted Tuesday September 12, 2023.
Due Friday September 29, 2023.

Problem 1 – Complex numbers: basics (15 points)
Given a complex number $z = x + iy = re^{i\phi}$, where i is the imaginary unit, i.e. $i^2 = -1$ and $x = \mathrm{Re}[z]$ and $y = \mathrm{Im}[z]$ are real numbers (respectively the real and imaginary parts of z), show that

(a)

$$\mathrm{Re}[z^2] = x^2 - y^2, \qquad \mathrm{Im}[z^2] = 2xy \tag{1}$$

(b)

$$z^* z = x^2 + y^2, \tag{2}$$

where $z^* = x - iy$ is the *complex conjugate* of z.

(c) Using the previous result, show that

$$\mathrm{Re}\left[\frac{1}{z}\right] = \frac{x}{x^2+y^2}, \qquad \mathrm{Im}\left[\frac{1}{z}\right] = \frac{-y}{x^2+y^2}. \tag{3}$$

(d)

$$e^z = e^x(\cos y + i \sin y) \tag{4}$$

(e)

$$\sinh(z) = \sinh(x)\cos(y) + i\cosh(x)\sin(y) \tag{5}$$

(f)

$$z^n = r^n(\cos n\phi + i \sin n\phi) \tag{6}$$

(g)

$$\ln(z) = \ln r + i(\phi + 2\pi n), \tag{7}$$

where n is any integer (i.e. show that the right-hand side works as the logarithm of z for any integer n).

Problem 2 – Root-Finding in Two Dimensions (10 points):
Use the "generating" complex function for a fractal,

$$f(z) = z^3 - 1 \tag{8}$$

to test a basic two-dimensional Newton-Raphson method using the procedure described below.

(a) Rewrite Eq. 8 as a set of two equations of two variables, namely

$$\begin{aligned} g_1(x,y) &= \mathrm{Re}[f(z)] \qquad (9) \\ g_2(x,y) &= \mathrm{Im}[f(z)], \qquad (10) \end{aligned}$$

where $z = x + iy$.

(b) Write down the Jacobian for your two equations, and its inverse.

(c) Write a simple two-dimensional Newton-Raphson method as a function named `rf_newton2d` in the notebook `problem2.ipynb` using the inverse Jacobian you calculated in part (b). Test the following initial guesses for your root-finder,

$$\begin{aligned} x_1 &= (1.01, 0.01) \qquad (11) \\ x_2 &= (-0.51, 0.866) \qquad (12) \\ x_3 &= (-0.51, -0.866). \qquad (13) \end{aligned}$$

To which root do they converge and in how many iterations for a tolerance of 10^{-3}? How do your results change for a tolerance of 10^{-6}?

Repeat the above test but with

$$\begin{aligned} x_4 &= (-0.1, -0.866) \qquad (14) \\ x_5 &= (0.2, -0.866) \qquad (15) \\ x_6 &= (0.51, 0.866). \qquad (16) \end{aligned}$$

To which root do they converge and in how many iterations for a tolerance of 10^{-3}? How do your results change for a tolerance of 10^{-6}?

Problem 3 – Fixed-Point Iteration (10pts):
Here, we explore the fixed-point iteration method to find the roots of the function

$$f(x) = x^4 - 20x^2 + x. \tag{17}$$

(a) How many roots do you expect? Justify. Can you simplify the problem before even using a computational method?

(b) For the fixed-point iteration, we need to bring the function $f(x) = 0$ into the form $g(x) = x$. Find three such functions $g_1(x)$, $g_2(x)$, $g_3(x)$ to re-write the problem. *Hint: Use the simplified form of the problem you derived above instead of the full form of $f(x)$.*

(c) Calculate the derivative $g'(x)$ for all three functions g_1, g_2, g_3, and determine the region where the condition $|g'(x)| < 1$ is met.

(d) Plot (with paper and pencil) the three functions and try to find a root graphically, starting with $x = 3$. *Hint: Like we saw in class, also draw $y = x$ on your diagram. Start at $x = 3$, evaluate $g(3)$, draw a horizontal line to $y = x$, take the resulting x and evaluate $g(x)$. Draw a horizontal line to $y = x$ and repeat.*