PHYS 331 – Numerical Techniques for the Sciences I

Homework 7: Fourier Transforms

Posted October 23, 2023.
Due November 3, 2023.

Problem 1: Simple cases and elementary properties of Fourier series [25pts + 5pts EC]

(a) Explain, using equations as well as full sentences, in what sense sines and cosines are orthogonal functions.

(b) Given the Fourier series of an arbitrary function $f(x)$, defined in a symmetric interval (say $[-L, L]$), show that the coefficients of the cosines vanish if the function is odd (i.e. if $f(-x) = -f(x)$), whereas the coefficients of the sines vanish if the function is even (i.e. if $f(-x) = f(x)$).

(c) Calculate analytically the Fourier series of

$$f(x) = \sin^3(\pi x), \tag{1}$$

defined in the interval $x \in [-1, 1]$.

Hint: Use trigonometric identities to write $f(x)$ as linear combinations of $\cos(n\pi x)$ and/or $\sin(m\pi x)$, where m and n are integers.

(d) Calculate analytically the Fourier series of the square wave function (note how it's different than the one in class)

$$f(x) = \begin{cases} -1 & \text{if } \; -\pi \leq x \leq -\pi/2 \\ 1 & \text{if } \; -\pi/2 \leq x \leq \pi/2 \\ -1 & \text{if } \; \pi/2 < x \leq \pi \end{cases} \tag{2}$$

defined in the interval $x \in [-\pi, \pi]$.

(e) Calculate analytically the Fourier series of the triangle wave function

$$f(x) = \begin{cases} \frac{\pi}{2} + x & \text{if } \; -\pi \leq x \leq 0 \\ \frac{\pi}{2} - x & \text{if } \; 0 < x \leq \pi \end{cases} \tag{3}$$

defined in the interval $x \in [-\pi, \pi]$.

(f) **Extra credit [5pts] :** Calculate analytically the Fourier series of $\cos^n(\pi x)$ for arbitrary integer $n > 0$ and $x \in [-1, 1]$. **Hints**: Use the complex exponential form of the Fourier series, rather than sines and cosines. Use the binomial theorem to express the n-th power of $\cos(\pi x) = (e^{i\pi x} + e^{-i\pi x})/2$.

Problem 2: Gibbs phenomenon [15pts]

(a) Rebuild the triangle wave you calculated above using $1, 5, 10, 50$, and 100 Fourier terms, and overlay the plots to show how the resulting function behaves as you add more terms. Plot the triangle wave as well.

(b) Rebuild the square wave you calculated above using $1, 5, 10, 50$, and 100 Fourier terms, and overlay the plots to show how the resulting function behaves as you add more terms. Plot the square wave as well.

(c) Describe your findings by comparing and contrasting the behavior of the above two cases.

Problem 3: "Slow" Discrete Fourier Transform [15pts]

(a) Implement the "straightforward" discrete Fourier Transform discussed in class. Specifically, write a function `sdft` that takes as arguments the array to be transformed (which takes the form of a vector, i.e. a one-dimensional array), and a parameter that controls the direction of the transform (forward or backward). (The output should also be a one-dimensional array.)

(b) Test your code with a function of your choosing, e.g. using the discrete form of one of the analytic cases you calculated above, for which you would take x to be `np.linspace(-np.pi,np.pi,`N`)`, for a fixed value of N even (take e.g. $N = 32, 64, 128$). Make sure that the inverse transform can reproduce the original function.

(c) Plot your chosen function before (one plot) and after (second plot) the transform. Note: the transformed function will have real and imaginary parts; plot them as two separate data sets.