Graph Theory Notes

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CHAPTER 1

Introduction

This is an on going project. There will be changes to the material not yet covered, especially just before lectures. There may also be changes immediately after a lecture based on class questions, etc. Later corrections to previous material will be marked in red. For these reasons, you should continuously use the current version.

Use at your own risk. There are bound to be typos. Corrections are appreciated, and rewarded with small amounts of extra credit, especially when they indicate mathematical understanding. But you are also encouraged to ask questions in emails about things you do not understand. I will try to answer questions quickly, but if they are complicated, or there are many, I may need to wait until the next class.

These notes are meant to enhance, not replace, the lectures and class discussions. There are two parts to studying mathematical proofs. The first part is to understand the ideas and motivations behind the arguments. The second part is to learn the arguments. The lectures address the first part. These notes address the second part; they provide a concise record of the proofs, freeing students from the need to take careful notes during class. There are compromises. In striving for brevity, some details are suppressed.

1.1. Graphs

Formally a *graph* is an ordered pair $G = (V, E)$ where E is an irreflexive, symmetric, binary relation on a nonempty set V. Since E is symmetric there is no need to keep track of the order of pairs $(x, y) \in E$; since it is irreflexive there are no ordered singletons (x, x) in E. This leads to a more intuitive formulation. We take E to be a set of unordered pairs of elements from V. Elements of V are called *vertices*; elements of E are called *edges*. If $x, y \in V$ are vertices and $\{x, y\} \in E$ is an edge we usually (but not always) denote $\{x, y\}$ by the shorthand notation xy. So $xy = \{x, y\} = \{y, x\} = yx$. The vertices x and y are called *ends*, or *endpoints*, of the edge xy. The ends x and y of an edge xy are said to be *adjacent,* and the end x is said to be *incident* to the edge xy. We also say that x and y are *joined* or *linked* (not connected) by the edge xy. In this course, **every graph has finitely many vertices**, unless it is explicitly stated that it has infinitely many vertices. Graphs are illustrated by drawing dots for vertices and joining adjacent vertices by lines or curves. These illustrations play a huge role in understanding and developing the theory.

Our definition of a graph is what West calls a *simple graph.* Most of the time we will only be interested in simple graphs, and so we begin with the simplest definition. When necessary, we will introduce the more complicated notions of *directed graphs*, *multigraphs and hypergraphs,* but here is a quick hint. A directed graph $G = (V, E)$ is any binary relation (not necessarily irreflexive or symmetric) on V. In other words, E is any set of *ordered* pairs of V. A *multigraph* is obtained by letting E be a multiset; then two vertices can have more than one edge between them. Sometimes this is not enough, and we need to distinguish between edges with the same ends. Also, we may need edges of the form (v, v); these are called *loops*. Edges with distinct ends are called *links*. We will deal with these problems when they arise. A *hypergraph,* also called a *set system*, is obtained by letting E be a set of subsets of V, where the elements of E can have any size. If they all have size k then we get a k*-uniform hypergraph*, also called a k*-graph*. So ordinary graphs are 2-uniform hypergraphs, or 2-graphs.

The study of graph theory involves a huge number of parameters—see the front and back inside covers of West [**24**] or the last two pages of Diestel [**6**]. This can be quite daunting. My strategy is to introduce these parameters as they are needed. Please feel free to interrupt lectures to be reminded of their meanings. Most of the time my notation will agree with West, but it is also influenced by Diestel; I will try to emphasize differences. Next we introduce some very basic notation.

Given a graph G, $V(G)$ denotes the set of vertices of G and $E(G)$ denotes the set of edges of G. Set $|G| := |V(G)|$ and $\|G\| := |E(G)|$; this is from Diestel, and is not standard; instead West uses $v(G) := |V(G)|$ and $e(G) := |E(G)|$. Suppose $v \in V(G)$ is a vertex of G. Define

$$N_G(v) := \{w \in V(G) : vw \in E(G)\}; \qquad E_G(v) : = \{e \in E(G) : v \text{ is an end of } e\}.$$

The set $N_G(v)$ is called the (open) *neighborhood* of v, and its elements are called *neighbors* of v. So a vertex w is a neighbor of v if and only if it is adjacent to v. When there is no confusion with other graphs the subscript G is often dropped. The closed neighborhood of v is $N[v] := N(v) \cup \{v\}$—we dropped the subscript. West does not provide notation for $E_G(v)$. For simple graphs $|N(v)| = |E(v)|$. However for multigraphs this may not hold, since two vertices might be joined by several edges. Loops also cause a problem. The following simple definition is very useful.

Definition 1. For a (multi)graph $G = (V, E)$ and pair $(v, e) \in V \times E$, set

$$\iota(v, e) := \begin{cases} 2 & \text{if } e \in E(v) \text{ and } e \text{ is a loop} \\ 1 & \text{if } e \in E(v) \text{ and } e \text{ is a link} \\ 0 & \text{if } e \notin E(v). \end{cases} \tag{1.1.1}$$

Thus $\iota(v, e)$ counts the number of times that v is an end of e. With this in mind, define the degree of a vertex v to be $d_G(v) := \sum_{e \in E} \iota(v, e)$. Now, if G is simple then $d_G(v) = |N_G(v)| = |E_G(v)|$, and if G is a loopless multigraph then $d_G(v) = |E_G(v)|$. A graph is k*-regular* if every vertex has degree k.

For $A, B \subseteq V$, the set of edges with one end in A and one end in B is denoted by $E_G(A, B)$; we often abbreviate $E_G(\{x\}, B)$ as $E_G(x, B)$. Set $\|x, B\|_G := |E_G(x, B)|$ and $\|A, B\|_G := \sum_{a \in A} |E_G(a, B)|$, that is, the number of edges in $E_G(A, B)$ counting those edges with both ends in $A \cap B$ twice.

For simplicity we will always choose notation so that if $x, y \in V(G)$ then $xy \notin V(G)$. Thus if $x, y \in V(G)$ then xy may be in $E(G)$, but it certainly is not in $V(G)$. With this convention we think of G as $V(G) \cup E(G)$. Thus if we know that $x, y \in V(G)$ then we can write $xy \in G$ instead of $xy \in E(G)$. However, this is a notational convention, not a formal definition; formally G is still the ordered pair (V, E).

We use the following set theoretic notation. The sets of natural numbers, integers and positive integers are denoted, respectively, by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Z}^+$. Note that $0 \in \mathbb{N}$. For real numbers a and b let $\{a, \ldots, b\}$ denote the set $\{i \in \mathbb{Z} : a \leq i \leq b\}$. Then $\{0, \ldots, -1\} = \emptyset$. For $n \in \mathbb{N}$ set $[n] := \{1, \ldots, n\}$; in particular $[0] = \emptyset$. For a set X and an element y, set $X + y := X \cup \{y\}$ and $X - y := X \smallsetminus \{y\}$. Finally, $\binom{X}{n}$ is the set of all n-element subsets of X, while $\binom{|X|}{n} = |\binom{X}{n}|$. An n-element (sub)set is called an *n-(sub)set.*

1.2. Proofs by Mathematical Induction

Most (not all) proofs in graph theory involve mathematical induction, or at least the Least Element Axiom. Here we quickly review this technique. Also see the discussion in West [**24**] on pages 19–20, and especially the *induction trap* on page 42.

Here we present mathematical induction in terms of the Least Element Axion (LEA). First we give a careful definition of "least element".

Definition 2. Let $B \subseteq \mathbb{N}$ be a set of natural numbers. A number $l \in \mathbb{N}$ is a least element of B if

(L1) $\{0, \ldots, l - 1\} \subseteq \mathbb{N} \smallsetminus B$, and
(L2) $l \in B$.

The following axiom (LEA) is fundamental.

Axiom (LEA). *Every nonempty set of natural numbers has a least element.*

Consider a set of "good" natural numbers S, and let $B = \mathbb{N} \smallsetminus S$ be the set of "bad" numbers. We would like to prove that every natural number is "good", i.e., $S = \mathbb{N}$. Here is a way to organize the argument.

Theorem 3 (Principle of Induction). *Suppose $S \subseteq \mathbb{N}$. Then $S = \mathbb{N}$, if*

$$(*) \qquad \forall n \in \mathbb{N}\ (\{0, \ldots, n - 1\} \subseteq S \to n \in S).$$

Proof. Arguing by contraposition, we suppose $S \neq \mathbb{N}$ and show that (*) fails. As $S \subsetneq \mathbb{N}$, $B := \mathbb{N} \smallsetminus S \neq \emptyset$. By LEA, B has a least element l. Thus by (L1), $\{0, \ldots, l-1\} \subseteq \mathbb{N} \smallsetminus B = S$. By (L2), $l \in B$, and so $l \notin S$. Thus (*) fails for S when $n = l$. □

By the Principle of Induction, when proving that a set $S \subseteq \mathbb{N}$ is equal to $\mathbb{N}$, it suffices to prove (*). For this we consider *any* natural number n. If $\{0, \ldots, n - 1\} \not\subseteq S$ then we are done, so we assume $\{0, \ldots, n - 1\} \subseteq S$. We use this "induction hypothesis" to prove $n \in S$.

When proving (*) the case $n = 0$ is special, since there is no natural number $k < n$, and so $\{0, \ldots, n - 1\} = \emptyset \subseteq S$; thus the induction hypothesis has given us no new information. More generally, the *base step* consists of the cases of the argument that do not rely the induction hypothesis. **Thus the case $n = 0$ is always part of the base step, but the base step may include more cases.** The *induction step* consists of those cases of the argument that do rely on the induction hypothesis. Here is an example. Notice that the statement of the theorem is carefully phrased because 0 and 1 do not have prime factors. This must be reflected in the definition of the set S.

Proposition 4. *Every natural number greater than 1 has a prime factor.*

Proof. Let $S = \{n \in \mathbb{N} : n \leq 1 \text{ or } n \text{ has a prime factor}\}$. It suffices to show (*). Consider any $n \in \mathbb{N}$ such that $\{0, \ldots, n-1\} \subseteq S$. We must show $n \in S$. If $n \leq 1$ then $n \in S$ by definition. So suppose $n \geq 2$. If n is prime then it is a prime factor of itself, and so it is in S. Otherwise, there exist integers a, b such that $1 < a, b < n$ and $ab = n$. Since $a < n$, we have $a \in S$. Since $1 < a$ this means that a has a prime factor p. Since p is a factor of a and a is a factor of n, p is a (prime) factor of n. $\square$

In the above argument, the cases $n \leq 1$ and n prime form the base step. Notice that in the induction step we never used that $n-1$ was in S; we only used that the factor a was in S, and it is easily seen that a is always less than $n-1$.

Here is an example from graph theory. We give it only to illustrate induction; after we develop some theory we will give a nicer proof. First we need some additional notation. For a graph G, let $\delta(G) = \min\{d(v) : v \in V(G)\}$. We say that a graph contains a cycle if it has a sequence $v_1, \ldots, v_s$ of distinct vertices with $s \geq 3$ such that $v_i v_{i+1} \in E$ for all $i \in [s-1]$ and $v_s v_1 \in E$; in this case we call $v_1 \ldots v_s v_1$ a cycle. The following (typical) statement is about all graphs, not all natural numbers, so it may be surprising that we can prove it by induction.

PROPOSITION 5. *Every graph $G = (V, E)$ with $\delta(G) \geq 2$ contains a cycle.*

PROOF. Let

$$S = \{n \in \mathbb{N} : \text{for all graphs } G \text{ with } |G| = n, \text{ if } \delta(G) \geq 2 \text{ then } G \text{ contains a cycle}\}.$$

Suppose (*) holds. Then by Theorem 3, $S = \mathbb{N}$, and as all graphs are finite, every graph G satisfies $|G| \in \mathbb{N} = S$; so if $\delta(G) \geq 2$ then G has a cycle. Thus it suffices to prove (*).

Consider any $n \in \mathbb{N}$, and any graph $G = (V, E)$ with $|G| = n$ and $\delta(G) \geq 2$. (If there are no such G then n is trivially in S.) So $3 \leq |N[v]| \leq |G| = n$ for any vertex v. As $n - 1 \in \mathbb{N}$, we may assume $n - 1 \in S$. We aim to show that G contains a cycle.
Case 1: $\delta(G) \geq 3$. Let $v \in V$ and $G' = G - v$, i.e., form a graph G' by deleting the vertex v together with all edges incident to v. Then $|G'| = n - 1$, and $\delta(G') \geq 2$, since no vertex lost more than one neighbor. Since $n - 1 \in S$, there is a cycle contained in G', and this cycle is also contained in G.
Case 2: $\delta(G) = 2$. Pick $y \in V$ with $d(y) = 2$. Then $N(y) = \{x, z\}$ for some distinct vertices x and z. If $xz \in E$ then G contains the cycle $xyzx$. Otherwise, let G' be the graph formed by deleting the vertex y and the edges yx & yz, and adding the edge xz. Then $|G'| = n - 1$ and $\delta(G') \geq 2$, since the loss of y is compensated for by the fact that x and z are now new neighbors of each other. Since $n - 1 \in S$, there is a cycle C contained in G'. If x and z do not appear consecutively in C then C is a cycle contained in G. Else C has the form $C = xz \ldots v_s x$, so inserting y between x and z yields a cycle $xyz \ldots v_s x$ in G. $\square$

What were the base and induction steps of this proof? The proof only applies to finite graphs, why? Can you construct an infinite graph with $\delta(G) \geq 2$ and no cycle?

The proof of Proposition 5 illustrates an important technique. Suppose we want to prove that all graphs have a property P, and f is a function on graphs (such as $|G|$) whose values are natural numbers. We can *argue by induction on f* as follows. First set

$$S = \{n \in \mathbb{N} : \text{all graphs } G \text{ with } f(G) = n \text{ have property } P\}.$$

Then argue by induction that $S = \mathbb{N}$. Finally consider any graph G. As $f(G) \in \mathbb{N} = S$, the definition of S implies G has property P. Often the cleverness of a proof is in picking the *right* function f on which to do induction.

I have one last comment about a common mistake made by beginning students (see induction trap on page 42 of West [**24**]). When you argue by induction on f to prove a statement $\forall G(P(G) \to Q(G))$ about graphs G, the induction step starts with a big graph G that satisfies P; then G is converted into a *smaller* graph G', i.e., $f(G) > f(G')$, and you *prove* that G' also satisfies P; by the induction hypothesis G' satisfies Q; finally, you use that G' satisfies Q to *prove* that G satisfies Q. **You do not start with a small graph, and then make it bigger!**

The above process is illustrated in the proof of Proposition 5, where $f(G) = |G|$, P is the statement that $\delta(G) \geq 2$, and Q is the statement that G contains a cycle. Case 1 is fairly straightforward, but Case 2 is more interesting. We make a smaller graph by removing a special vertex—one whose degree is 2. The new graph might not satisfy P; in this case we change it so that it does. By the induction hypothesis, the changed graph satisfies Q—contains a cycle, but this cycle may not be a cycle in G; in this case we modify the cycle to make it a cycle in G.

1.3. Ramsey's Theorem for Graphs

Ramsey's Theorem is an important generalization of the Pigeonhole Principle. Here we only consider its simplest version applied to graphs; the general version, Theorem 7, is a statement about k-uniform hypergraphs. In the past, this simple version was presented as part of MAT 415, but it has been moved to MAT 416 because its presentation benefits from the language of graph theory.

Let $G = (V, E)$ be a graph, and suppose $X \subseteq V$. The set X is a *clique in* G if $\binom{X}{2} \subseteq E$, i.e., $xy \in E$ for all distinct vertices $x, y \in X$. It is an *independent set*, or *coclique*, in G if $xy \notin E$ for all vertices $x, y \in X$. A clique (coclique) X is a b-clique (b-coclique) if $|X| = b$. Let $\omega(G) := \max\{|X| : X \text{ is a clique in } G\}$, *and* $\alpha(G) := \max\{|X| : X \text{ is a coclique in } G\}$.

A graph H is a *subgraph* of G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. It is an *induced subgraph* of G if $H \subseteq G$ and $E(H) = \{xy \in E(G) : x \in V(H) \text{ and } y \in V(H)\}$. For $X \subseteq V$, $G[X]$ is the induced subgraph of G that has vertex set X. If $H \subseteq G$ we may write $G[H]$ for $G[V(H)]$. The *complement* of G is the graph, $\overline{G} := (V(G), \overline{E}(G))$, where $\overline{E}(G) := \binom{V(G)}{2} \smallsetminus E(G)$.

THEOREM 6 (Ramsey's Theorem [**18**]). *For all graphs $G = (V, E)$ and $a, b \in \mathbb{Z}^+$, if $|G| \geq 2^{a+b-2}$ then $\omega(G) \geq a$ or $\alpha(G) \geq b$.*

PROOF. Argue by induction on $n := a + b$. (That is, let S be the set of natural numbers n such that for all positive integers a and b if $n = a + b$ and G is a graph with $|G| \geq 2^{n-2}$, then $\omega(G) \geq a$ or $\alpha(G) \geq b$. Show that for all $n \in \mathbb{N}$ if $\{0, \ldots, n-1\} \subseteq S$ then $n \in S$.) Consider any $n \in \mathbb{N}$, $a, b \in \mathbb{Z}^+$, and graph G with $n = a + b$ and $|G| \geq 2^{n-2}$.
Base step: $\min\{a, b\} = 1$. Let $v \in V$. As $\{v\}$ is both a clique and an independent set, both $\omega(G) \geq 1$ and $\alpha(G) \geq 1$. So we are done regardless of whether $a = 1$ or $b = 1$.
Induction Step: $\min\{a, b\} \geq 2$ (so $a - 1, b - 1 \in \mathbb{Z}^+$). (We assume the induction hypothesis: $n - 1 \in S$) Let $v \in V$. Then

$$1 + d_G(v) + d_{\overline{G}}(v) = |G| \geq 2^{n-2} = 2^{n-3} + 2^{n-3}.$$

By the pigeonhole principle, either $d_G(v) \geq 2^{n-3}$ or $d_{\overline{G}}(v) \geq 2^{n-3}$.

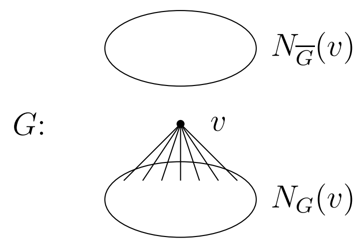


FIGURE 1.3.1. Ramsey's Theorem

Case 1: $d_G(v) \geq 2^{n-3}$. Set $H := G[N_G(v)]$. Then $|H| = d_G(v) \geq 2^{a-1+b-2}$. By the induction hypothesis, $\omega(H) \geq a-1$ or $\alpha(H) \geq b$. In the first case, H has an $(a-1)$-clique X, and $X+v$ is an a-clique in G. Else, H has a b-coclique Y, and Y is a b-coclique in G.
Case 2: $d_{\overline{G}}(v) \geq 2^{n-3}$. Set $H := G[N_{\overline{G}}(v)]$. Then $|H| = d_{\overline{G}}(v) \geq 2^{a+b-1-2}$. By the induction hypothesis, $\omega(H) \geq a$ or $\alpha(H) \geq b-1$. In the first case, H has an a-clique X, and X is an a-clique in G. Else H has a $(b-1)$-coclique Y, and $Y+v$ is a b-coclique in G. □

For $a, b \in \mathbb{Z}^+$, define $\mathrm{Ram}(a,b)$ to be the least positive integer n such that every graph G with $|G| \geq n$ satisfies $\omega(G) \geq a$ or $\alpha(G) \geq b$. By Theorem 6, $\mathrm{Ram}(a,b)$ exists and satisfies $\mathrm{Ram}(a,b) \leq 2^{a+b-2}$. Note that to prove $\mathrm{Ram}(a,b) = n$ you need to show:

(1) for every graph G, if $|G| = n$ then G has either an a-clique or a b-coclique; and
(2) there is a graph G with $|G| = n-1$ that has no a-clique and no b-coclique (or $n = 1$).

HW 1. (B) Prove that for $a, b \in \mathbb{Z}^+$ with $a, b \geq 2$:

(1) $\mathrm{Ram}(a,b) = \mathrm{Ram}(b,a)$.
(2) $\mathrm{Ram}(a,1) = 1$.
(3) $\mathrm{Ram}(a,2) = a$.
(4) $\mathrm{Ram}(3,3) = 6$.
(5) $\mathrm{Ram}(a,b) \leq \mathrm{Ram}(a-1,b) + \mathrm{Ram}(a,b-1)$.

HW 2. Prove:

(1) (+) $\mathrm{Ram}(3,4) = 9$ (tricky, see Proposition 11).
(2) (B) $\mathrm{Ram}(4,4) \leq 18$ (use HW 1(5) and (1) even if you did not do them).
(3) (+) $\mathrm{Ram}(4,4) \geq 18$.

It is known that $\mathrm{Ram}(4,5) = 25$ and $43 \leq \mathrm{Ram}(5,5) \leq 48$, and it is conjectured that 43 is the right answer. Proving it would make for a truly exceptional doctoral thesis; in fact any improvement on these bounds would be an outstanding thesis. Almost certainly, any proof would rely on heavy use of a computer.

Ramsey's theorem can be easily extended as follows: For all $c, a_1, \ldots, a_c \in \mathbb{Z}^+$, there exists $n \in \mathbb{Z}^+$ such that for all sets V with $|V| \geq n$ and for all partitions $\{E_1, \ldots, E_c\}$ of $\binom{V}{2}$, either $\omega(V, E_1) \geq a_1$ or $\ldots$ or $\omega(V, E_c) \geq a_c$. To see that this extends Theorem 6, notice that for $c = 2$ we can think of $E_1 = E$, $G = (V, E_1)$, $E_2 = \overline{E} := \binom{V}{2} \smallsetminus E$, and $\overline{G} = (V, E_2)$.

Ramsey's real theorem is for all r-uniform hypergraphs. For an r-uniform hypergraph G, let $\omega(G)$ be the size of the largest subset $X \subseteq V(G)$ such that $\binom{X}{r} \subseteq E(G)$.

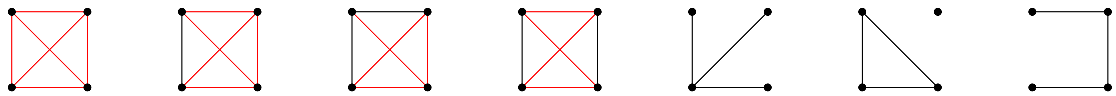


Figure 1.4.1. Isomorphism types of graphs on four vertices; those with 4, 5, 6 edges are shown in red. Numbers indicate the number of graphs in each type.

Theorem 7 (General Ramsey Theorem for Graphs[**18**]). *For all $r, c, a_1, \dots, a_c \in \mathbb{Z}^+$ there exists $n \in \mathbb{Z}^+$ such that for all sets V with $|V| \geq n$ and all partitions $\{E_1, \dots, E_c\}$ of $\binom{V}{r}$, either $\omega(V, E_1) \geq a_1$ or $\dots$ or $\omega(V, E_c) \geq a_c$.*

Theorem 7 can be proved in the style of Theorem 6, but requires a "double induction" on r and then $\sum_{i=1}^{c} a_i$. Strong doctoral students (with effort and maybe some hints) should work this out from scratch. Our discussion of Ramsey's Theorem is just the tip of the iceberg. Whole books have been written on the subject. See Graham, Rothchild & Spencer [**10**] for a friendly introduction.

1.4. Graph Isomorphism and the Reconstruction Conjecture

In order to study graph theory we need to know when two graphs are essentially the same.

Definition 8. Two graphs G and H are *isomorphic* if there exists a bijection

$$f : V(G) \to V(H) \text{ such that } xy \in E(G) \text{ iff } f(x)f(y) \in E(H) \text{ for all } x, y \in V(G).$$

In this case we say that f is an isomorphism from G to H and write $G \cong H$. The isomorphism relation is an equivalence relation on the class of graphs. The equivalence classes of this relation are called *isomorphism types.* In graph theory we generally do not differentiate between two isomorphic graphs. We say that H is a *copy* of G to mean that $G \cong H$.

For example, let $V = \{a, b, c, d\}$ and $\mathcal{G} = \{G : G \text{ is a graph with } V(G) = V\}$. Then $G \in \mathcal{G}$ if and only if $V(G) = V$ and $E(G) \subseteq \binom{V}{2}$. Thus $|\mathcal{G}| = 2^6 = 64$. Then $\mathcal{G}$ has 1 isomorphism type whose graphs have 0 edges, and it contains 1 graph; 1 isomorphism type whose graphs have 1 edge, and it contains 6 graphs; 2 isomorphism types whose graphs have 2 edges, one with 3 graphs and one 12 graphs. See Figure 1.4.1. It follows (why) that $\mathcal{G}$ has 1 isomorphism type whose graphs have 6 edges, and it contains 1 graph; 1 isomorphism type whose graphs have 5 edges, and it contains 6 graphs; 2 isomorphism types whose graphs have 4 edges, one with 3 graphs and one 12 graphs. Finally $\mathcal{G}$ has 3 isomorphism types whose graphs have 3 edges, one with 12 graphs and two with 4 graphs.

HW 3. (B) Let $f : V(G) \to V(H)$ be an isomorphism between two graphs G and H. Prove carefully that $d_G(v) = d_H(f(v))$ for all $v \in V(G)$. [Hint: Fix v and construct a bijection between $N_G(v)$ and $N_H(f(v))$.]

HW 4. (*) Here we *do* distinguish between isomorphic graphs. Let $V = \{v, w, x, y, z\}$ be a set of five vertices, and $\mathcal{G} = \{G : G \text{ is a graph with } V(G) = V \text{ and } \|G\| = 4\}$. Determine $|\mathcal{G}|$, the number of isomorphism types of $\mathcal{G}$ and the size of each isomorphism type.

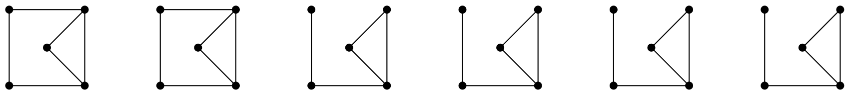


FIGURE 1.4.2. The vertex deleted subgraphs of a graph G. What is $|G|$? What is $\|G\|$? What is the isomorphism type of G?

If x is a vertex of a graph G then $G-x$ is the induced subgraph $G[V(G)-x]$. The graph $G-x$ is called a *vertex deleted* subgraph of G.

DEFINITION 9. A *complete set of vertex deleted subgraphs* of a graph $G=(V,E)$ is a set $\mathcal{G}$ such that there exists a bijection $\psi : V \to \mathcal{G}$ with $\psi(x) \cong G-x$ for all $x \in V$. Notice that for distinct vertices x and y it may be that $G-x \cong G-y$. In this case $\psi(x)$ and $\psi(y)$ are *distinct copies* of $G-x$. See Figure 1.4.2. *A* complete set of vertex deleted subgraphs $\mathcal{G}$ of G is also called a *deck* of G, and the elements of $\mathcal{G}$ are called cards. Two graphs are *hypomorphic* if they have the same deck.

Notice that G has infinitely many decks, but each deck $\mathcal{G}$ of G satisfies $|\mathcal{G}| = |G|$. We cannot determine $V(G)$ from its deck $\mathcal{G}$, but can we determine the isomorphism type of G from its deck? Let $V=\{x,y\}$, $G=(V,\{xy\})$ and $H=(V,\emptyset)$. Both have the same deck $\{(\{x\},\emptyset),(\{y\},\emptyset)\}$, but G is not isomorphic to H. The following famous conjecture asserts that we can determine the isomorphism type of G from it deck for all G with $|G| \geq 3$.

CONJECTURE 10 (Reconstruction Conjecture 1.3.12). *Any two hypomorphic graphs with at least three vertices are isomorphic.*

Now we take a little detour before proving Proposition 12. An edge of a multigraph is called a link if it has two distinct ends and a loop if both ends are the same. The *degree* $d(v)$ of a vertex of a multigraph is the number of links incident to v plus twice the number of loops incident to v. In a multigraph there may be cycles of length 1—one loop—and cycles of length 2—two links between the same two vertices. This definition is designed so that each edge is counted twice when the degrees of the vertices are summed.

PROPOSITION 11 (Handshaking 1.3.3.). *Every (multi-)graph $G := (V,E)$ satisfies*

$$\sum_{v\in V} d(v) = 2\,\|G\|\,.$$

In particular, G has an even number of vertices with odd degree.

PROOF. Since $d(v) = \sum_{e\in E} \iota(v,e)$ for all $v \in V$ and $\sum_{v\in V} \iota(v,e) = 2$ for all $e \in E$,

$$\sum_{v\in V} d(v) = \sum_{v\in V}\sum_{e\in E} \iota(v,e) = \sum_{e\in E}\sum_{v\in V} \iota(v,e) = \sum_{e\in E} 2 = 2\,\|G\|\,. \qquad \square$$

The *degree sequence* of a graph $G=(V,E)$ is a nondecreasing sequence of integers $d_1,\dots,d_{|G|}$, where $V=\{v_1,\dots,v_{|G|}\}$ and $d_i = d(v_i)$ for all $i \in [|G|]$. For example the degree sequence for the graph J in Figure 1.4.2 is $1,2,2,2,3$. The next Proposition shows that we can determine the number of edges of a graph and its degree sequence from its deck.

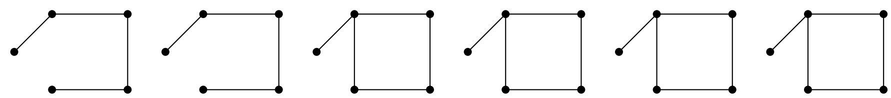
$\mathcal{G} =$ 

FIGURE 1.4.3. HW 6

PROPOSITION 12 (1.3.11). *For all simple graphs $G = (V, E)$ with $|G| \geq 3$ and $w \in V$,*

$$\|G\| = \frac{\sum_{v \in V} \|G - v\|}{|G| - 2} \qquad \text{and} \qquad d_G(w) = \|G\| - \|G - w\|\,.$$

PROOF. Any edge $e \in E$ satisfies $e \in E(G - v)$ if and only if $\iota(v, e) = 0$. Thus

$$\sum_{v \in V} \|G - v\| = \sum_{v \in V} \sum_{e \in E} (1 - \iota(v, e)) = \sum_{e \in E} \sum_{v \in V} (1 - \iota(v, e)) = \sum_{e \in E} (|G| - 2) = \|G\| \, (|G| - 2).$$

So the first equality holds. The second follows from $E = E(G - w) \cup E(w)$. □

EXAMPLE 13. Let $\mathcal{G}$ be a deck for some unknown graph, and suppose $\mathcal{G}$ consists of two distinct copies of H and four distinct copies of J as shown in Figure 1.4.2. Find (with proof) a graph Q that has deck $\mathcal{G}$, and show that if G is any other graph with deck $\mathcal{G}$ then $G \cong Q$.

SOLUTION. Let Q be the graph shown in Figure 1.4.4, and check that $\mathcal{G}$ is a deck of Q. Now consider any graph $G = (V, E)$ with deck $\mathcal{G}$. Then $|G| = |\mathcal{G}| = 6$. Using Proposition 12, we have:

$$\|G\| = (2\,\|H\| + 4\,\|J\|)/(|G| - 2) = (2 \cdot 6 + 4 \cdot 5)/4 = 8,$$

and G has degree sequence $2, 2, 3, 3, 3, 3$. Let $x \in V$ with $G - x \cong H$. Say $V(G - x) = \{w_1, w_2, w_3, w_4, y\}$ and $E = \{w_1w_2, w_2w_3, w_3w_4, w_4w_1, yw_2, yw_3\}$. Then $d_G(x) = 8 - \|H\| = 2$. As $\Delta(G) = 3$ and $d_{G-x}(w_2) = 3 = d_{G-x}(w_3)$, we have $w_2, w_3 \notin N_G(x)$. Thus $N(x) \subseteq \{w_1, w_4, y\}$. If $xy \in E$ then $G - y$ contains no cycle with three vertices, so $H, J \ncong G - y$, a contradiction. Thus $N(x) = \{w_1, w_4\}$, and $E(G) = E(G - x) \cup \{xw_1, xw_4\}$. So

$$x \mapsto u, y \mapsto v, w_1 \mapsto z_1, \ldots, w_4 \mapsto z_4$$

is an isomorphism from G to Q. □

HW 5. (B) Give a (small) example (with proof) of two graphs that have the same degree sequence, but are not isomorphic. [Hint: One such example arises in the previous solution.]

HW 6. (*) Find a graph G such that the graphs shown in Figure 1.4.3 form a deck $\mathcal{G}$ of G. Prove that if $\mathcal{G}$ is also the deck of H then $G \cong H$.

HW 7. (*) A graph is *regular* if all its vertices have the same degree. Prove that if two regular graphs with at least three vertices have the same deck then they are isomorphic.

1.5. Some Important Graphs and Graph Constructions

A *path* is a graph $P = (V, E)$ such that V can be ordered as $v_1, \ldots, v_{|P|}$ (and thus there are no repetitions) so that $E = \{v_iv_{i+1} : i \in [|P|-1]\}$. The *length* of the path P is $\|P\|$. A path with only one vertex is possible; such paths are said to be *trivial.* Clearly, any two paths with the same length are isomorphic. We use the notation P_n to denote a fixed path

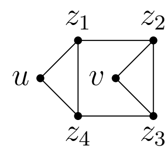


FIGURE 1.4.4. Q

with n vertices (and $n-1$ edges). If P is a path with n vertices, we say that P is a copy of P_n, or more carelessly $P = P_n$. We write $v_1 \dots v_n$ (without commas) to denote a copy of P_n whose edge set is $\{v_i v_{i+1} : i \in [n-1]\}$. For a path $P = v_1 \dots v_n$ set $\mathring{P} = v_2 \dots v_{n-1}$ and

$$\begin{aligned} v_i P &= v_i \dots v_n & v_i P v_j &= v_i \dots v_j & P v_j &= v_1 \dots v_j \\ \mathring{v}_i P &= v_{i+1} \dots v_n & v_i \mathring{P} v_j &= v_{i+1} \dots v_{j-1} & P \mathring{v}_j &= v_1 \dots v_{j-1}. \end{aligned}$$

The vertices v_1 and v_n are called the *ends* of P. The other vertices are called *inner* or *internal* vertices. Suppose G is a graph with $A, B \subseteq V(G)$ and $P, X \subseteq G$. Then P is an A, B-path if $V(P) \cap A = \{v_1\}$ and $V(P) \cap B = \{v_n\}$, and P is an X-path if $n \geq 2$ and $P \cap X = \{v_1, v_n\}$. An $\{a\}, B$-path is also called an a, B-path, etc. The *distance* $d(x, y)$ between x and y is the length of the shortest x, y-path, if such a path exists; else it is *infinity*.

A *cycle* is a graph C that has an edge $v_1 v_n$ such that $C - v_1 v_n$ is a path $v_1 \dots v_n$. The length of C is $\|C\|$. Clearly any two cycles with the same length are isomorphic. We use the notation C_n for a fixed cycle of length n and write $v_1 \dots v_n v_1$ to denote a copy of C_n whose edge set is $\{v_i v_{i \oplus 1} : i \in [n]\}$, where $\oplus$ denotes addition modulo n. The *girth* of a graph G is the length of its shortest cycle $C \subseteq G$, if there is one; else it is *infinity*. The *circumference* of G is the length of the longest cycle $C \subseteq G$, if there is one; else it is zero.

A *complete graph* is a graph $K = (V, E)$ such that $xy \in E$ for all $x, y \in V$. We use the notation K_n for a fixed complete graph with n vertices. Notice that the vertices of a complete graph are a clique. Then $\overline{K}_n$ is a graph with n vertices and no edges, and the vertices of $\overline{K}_n$ are a coclique. We call $\overline{K}_n$ the *empty graph on n vertices*. Now we introduce some notation not in the text: $K(A, B)$ denotes the graph (V, E) such that $V = A \cup B$ and $E = \{ab : a \neq b \wedge a \in A \wedge b \in B\}$. Then $K(A, A)$ denotes a complete graph whose vertex set is A; we abbreviate this by $K(A)$. Finally, for $a, b \in \mathbb{Z}^+$, let $K_{a,b}$ denote a graph of the form $K(A, B)$, where $|A| = a$, $|B| = b$, and $A \cap B = \emptyset$. Such a graph is called a *complete bipartite graph*. (We will have more to say about bipartite graphs shortly.)

Let $G = (V, E)$ and $H = (W, F)$ be graphs. Define the *sum* of G and H by

$$G + H := (V \cup W, E \cup F);$$

this may also be denoted by $G \cup H$. Define the *join* of G and H by

$$G \vee H := (G + H) + K(V, W).$$

The k-th power of G is the graph $G^k = (V, E^k)$, where $E^k = \{uv : d(u, v) \leq k\}$.

The *Petersen* graph has the form $(\binom{[5]}{2}, \{AB : A \cap B = \emptyset, A, B \in \binom{[5]}{2}\})$.

HW 8. (*) Prove that every graph G with $|G| \leq \|G\|$ contains a cycle. [Hint: Use induction, and consider a longest path $P \subseteq G$.]

1.6. Connection in graphs

DEFINITION 14 (1.2.2). A *walk* is a triple $W = (V, E, \sigma)$ such that (V, E) is a graph, $\sigma = (v_1, \dots, v_n)$ is a sequence with $V = \{v_1, \dots, v_n\}$ (possibly $|V| < n$), and $E = \{v_i v_{i+1} : i \in [n-1]\}$. Let $G(W) = (V, E)$, $V(W) = V$, $E(W) = E$ and $\sigma(W) = \sigma$. We denote the walk W by $v_1 \dots v_n$. If σ is one-to-one then this notation also denotes a path. If $v_1 = v_n$ then W is *closed*; otherwise it is *open*. If W is open then v_1 and v_n are its *ends*, and all other vertices are *inner* or *internal*. If all the edges $v_i v_{i+1}$ are distinct then W is a *trail*. The *length* of W is $n-1$. The walk W is a u, v-walk (trail, path) if $u = v_1$ and $v = v_n$. The trivial walk v_1 of length 0 is closed since then $n = 1$.

Let $G = G(W)$, and suppose $A, B \subseteq V$. An *A, B-walk* W is a walk whose first vertex is in A, whose last vertex is in B and whose interior vertices are in neither A nor B. If $A = \{a\}$ ($B = \{b\}$), we may say that W is an a, B-walk (A, b-walk). If $H \subseteq G$ then W is an H-walk if W is open with ends in H, but no edge and no inner vertex of W is in H.

We say that W contains a walk W', or W' is a subwalk of W, if W' is a walk such that $G(W') \subseteq G(W)$ and $\sigma(W')$ is a subsequence of $\sigma(W)$. We use the notation Wv_j, v_iWv_j, and v_iW to indicate the *subwalks* $v_1 \dots v_j$, $v_i \dots v_j$, *and* $v_i \dots v_n$. Also $W^* := v_n v_{v-1} \dots v_1$. Consider walks $W = v_1 \dots v_n$ and $U = u_1 \dots u_m$ in G, $i \le n$ and $h \le m$; then

$$v_1 W v_i u_j U_m = \begin{cases} v_1 \dots v_i u_{j+1} \dots u_m & \text{if } v_i = u_j \\ v_1 \dots v_i u_j \dots u_m & \text{if } v_i u_j \in G \\ \text{undefined} & \text{else.} \end{cases}$$

DEFINITION 15 (1.2.6). Let $G = (V, E)$ be a graph. Vertices $x, y \in V$ are *connected* (regardless of whether $xy \in E$) if there is an x, y-walk in G. The graph G is *connected* if all vertices x and y are connected. The *connection relation* is the set of ordered pairs (x, y) of G such that x and y are connected.

PROPOSITION 16. *The connection relation is an equivalence relation.*

PROOF. Clearly, the connection relation is reflexive and symmetric. It is transitive by concatenating walks. $\square$

LEMMA 17 (1.2.5). *Let $W = v_1 \dots v_n$ be an u, v-walk. Then $G(W)$ contains a u, v-path.*

PROOF. Argue by induction on the length l of W. If $G(W)$ is a path then we are done. Else, there exist $i, j \in [n]$ with $i < j$ such that $v_i = v_j$. Then $v_i v_{j+1} = v_j v_{j+1}$ and $W' = v_1 \dots v_i v_{j+1} \dots v_n$ is a shorter u, v-walk contained in W since $v_i v_{j+1} = v_j v_{j+1} \in E(W)$. By induction $G(W')$ contains a u, v-path P, and so P is contained in $G(W)$. $\square$

HW 9. (B,L) Let G be a graph with $x, y, z \in V(G)$. Prove that if G contains an x, y-path and a y, z-path then it contains an x, z-path. Be careful; it is not entirely trivial.

DEFINITION 18 (1.2.8). A *component* of G is a subgraph $H = G[X]$ induced by an equivalence class X of the connection relation.

The following is an easy but useful observation:

(1.6.1) If $G := (V, E)$ is connected then $E(S, V \smallsetminus S) \neq \emptyset$ for all $\emptyset \subset S \subset V$.

HW 10. (B) Let $P \subseteq G$ be an x, y-path. Prove that $G[P]$ contains an x, y-path Q with $Q = G[Q]$; in other words, G contains an induced x, y-path Q with $V(Q) \subseteq V(P)$.

HW 11. (*) Prove: Any two paths P and Q with maximum length in a connected graph have a common vertex.

In 1966 Gallai [**9**] asked if every connected graph has a vertex v such that every path with maximum length contains v. In 1969 Walther [**23**] gave an example on 12 vertices to show that this is false. Let $\tau(G)$ be the size of the smallest set $S \subseteq V(G)$ such that every maximum path meets S. For Walther's example τ is two. Grünbaum [**11**] constructed a connected graph on 324 vertices with $\tau = 3$. In 2014, Rautenbach & Sereni [**19**] proved that $\tau(G) \leq \lceil \frac{n}{4} - \frac{n^{2/3}}{90} \rceil$, where $n = |G|$. Long, Milans & Munaro [**14**] have recently shown that $\tau(G) \leq 8n^{3/4}$. In 2022, Eric Ren (an MAT 416 student) and I improved this to $\tau \leq 5n^{2/3}$. Can you find better lower or upper bounds?

1.7. Digraphs

Definition 19. A *directed graph* or *digraph* is a binary relation $G := (V, E)$. Elements of G are called *directed edges* or *diedges*. So edges are ordered pairs of vertices. It is possible that $(x, x) \in E$ or that both $(x, y) \in E$ and $(y, x) \in E$. We will write xy for (x, y) when it is known that G is a digraph. Sometimes we emphasize that G is a digraph and xy is a diedge by writing $\vec{G}$ and $\vec{xy}$. Let

$$N^+_{\vec{G}}(x) := \{y \in V : \vec{xy} \in E\}, E^+_{\vec{G}}(x) := \{\vec{xy} : \vec{xy} \in E\}, d^+_{\vec{G}}(x) := |E^+_{\vec{G}}(x)|,$$
$$\delta^+(\vec{G}) := \min\{d^+_{\vec{G}}(x) : x \in V\} \text{ and } \Delta^+(\vec{G}) := \max\{d^+_{\vec{G}}(x) : x \in V\}.$$

Similarly, let

$$N^-_{\vec{G}}(y) := \{x \in V : \vec{xy} \in E\}, E^-_{\vec{G}}(y) := \{\vec{xy} : \vec{xy} \in E\}, d^-_{\vec{G}}(y) := |E^-_{\vec{G}}(y)|,$$
$$\delta^-(\vec{G}) = \min\{d^-_{\vec{G}}(y) : y \in V\} \text{ and } \Delta^-(\vec{G}) := \max\{d^-_{\vec{G}}(y) : y \in V\}.$$

For $X, Y \subseteq V$, let $\vec{E}(X, Y) := \{\vec{xy} \in E : x \in X \text{ and } y \in Y\}$.

A *dipath* is a digraph of the form $\vec{P} := (V, \vec{E})$, where $V := \{v_1, \dots, v_n\}$ and $\vec{E} := \{\vec{v_i v_{i+1}} : i \in [n-1]\}$; its length is $n = |\vec{E}|$. For $x, y \in V$, an *x, y-dipath* is a dipath with $d^-(x) = 0 = d^+(y)$. The *didistance* $\vec{d}(x, y)$ between x, y is the minimum number of edges in an x, y-dipath, if there is an x, y-dipath; else it is ∞. A *dicycle* is a digraph of the form $\vec{C} := (V, \vec{E})$, where $V := \{v_1, \dots, v_n\}$ and $\vec{E} := \{\vec{v_i v_{i \oplus 1}} : i \in [n]\}$. A digraph is *acyclic* if it has no dicycle.

Two vertices x and y in a digraph G are *strongly connected* if G has both an x, y-diwalk and a y, x-diwalk; G is strongly connected if any two vertices of G are strongly connected. The strongly-connected relation $\approx$ is clearly an equivalence relation. Its equivalence classes are called *strong components* of G.

When we are working only in the context of digraphs, we usually suppress the arrows.

HW 12. (L) Let $G = (V, E)$ be a digraph. Prove: (i) The strongly-connected relation $\approx$ is an equivalence relation; (ii) if H and H' are distinct strong components then $E(H, H') = \emptyset$ or $E(H', H) = \emptyset$; and (iii) G has a strong component H such that $N^+(v) \subseteq H$ for all $v \in V(H)$. [Hint: For (iii), consider a dipath in G that meets as many strong components as possible].

HW 13. (B) Prove that a digraph G has a dicycle if $\delta^+(G) \geq 1$.

1.8. Bipartite graphs

Definition 20. A graph $G = (V, E)$ is *bipartite* if it has a *bipartition*, that is a partition of V into one or two cocliques. This means that $E = \emptyset$ or there exists a partition $\{A, B\}$ of V (with $V = A \cup B$, $A \cap B = \emptyset$) such that both A and B are independent; equivalently, $E = E_G(A, B)$. An A, B-*bigraph* is a bipartite graph with bipartition $\{A, B\}$. Notice that a graph is bipartite if and only if it is a subgraph of a complete bipartite graph.

Many theorems in graph theory assert the existence of some special structure in a graph—say a bipartition. To show that a particular graph has such a structure it is enough to make a lucky guess, and check that your guess provides the structure. In general, it is much harder to show that a graph does not have the desired structure. Typically this would require an exhaustive search of exponentially many possibilities—say all $2^{|G|}$ partitions of the vertices into at most two parts. However for some structures we can prove the existence of *obstructions* with the property that every graph either has the structure or it has an obstruction, but not both. In this case, a lucky guess of an obstruction provides a proof that the structure does not exist. Theorem 23 is an example of this phenomenon.

Definition 21. A path, cycle, trail, walk W is *even* (*odd*) if its length is even (odd).

Lemma 22. *If $W = v_1 \dots v_n v_1$ is an odd closed walk then $G(W)$ contains an odd cycle.*

Proof. Argue by induction on the length of W. If $G(W)$ is a cycle we are done. Else, as $\|W\| \geq 3$, there exist integers $1 \leq i < j \leq n$ with $v_i = v_j$. Then $v_i v_{j+1} = v_j v_{j+1} \in E$ and $W' := v_1 \dots v_i v_{j+1} \dots v_n v_1$ and $W'' := v_i v_{i+1} \dots v_j$ are shorter closed walks, whose lengths sum to the length of W. So one of them must be odd. By the induction hypothesis, the graph of the odd subwalk contains an odd cycle; this cycle is also contained in $G(W)$. $\square$

HW 14. (B) Prove: Every closed trail contains a cycle. Find a (short) closed walk that does not contain a cycle.

HW 15. (L) Let G be a digraph. Prove: (i) Every odd closed diwalk contains an odd dicycle; and (ii) if x is strongly connected to y and $\vec{d}(x, y) = \vec{d}(y, x) + 1 \mod 2$ then G contains an odd dicycle.

Theorem 23 (1.2.18). *A graph $G = (V, E)$ is bipartite if and only if it contains no odd cycle.*

Proof. First suppose G is bipartite with bipartition $\{A, B\}$. It suffices to show that if $C \subseteq G$ is a cycle then it is even. As G is bipartite, $E(C) \subseteq E(A, B)$. So each edge $e \in E(C)$ has exactly one end in A. Also each vertex of C is incident to two edges of C. Using (1.1.1), the length of C is the even number:

$$\|C\| = \sum_{e \in E(C)} 1 = \sum_{e \in E(C)} \sum_{v \in A \cap C} \iota_C(v, e) = \sum_{v \in A \cap C} \sum_{e \in E(C)} \iota_C(v, e) = \sum_{v \in A \cap C} 2 = 2|A \cap C|.$$

Now suppose G is not bipartite. Then some component $H \subseteq G$ is not bipartite (why?). Let $x \in V(H)$. Set

$$A := \{v \in V(H) : \text{ there exists an even } x, v\text{-walk in } H\} \text{ and}$$
$$B := \{v \in V(H) \setminus A : \text{ there exists an odd } x, v\text{-walk in } H\}.$$

Since H is a component of G, it is connected, so $A \cup B = V(H)$; by definition $A \cap B = \emptyset$. As H is not bipartite there is an edge $uv \in E(H[A]) \cup E(H[B])$. Thus there are walks xPu and xQv with the same parity (odd if $u, v \in A$, even if $u, v \in B$). So $W := xPuvQ^*x$ is an odd closed walk. By Lemma 22 there is an odd cycle $C \subseteq G(W)$, and $C \subseteq H \subseteq G$. □

Definition 24. A *decomposition* of a graph G is a set of subgraphs such that each edge of G appears in exactly one subgraph of the set.

Example 25. K_4 can be decomposed into two P_4's; it can also be decomposed into three P_3's, and into $\{K_3, K_{1,3}\}$.

HW 16. (*) Prove: A 3-regular graph G decomposes into $K_{1,3}$'s if and only if G is bipartite.

HW 17. (B) Prove that a graph G is bipartite if and only if $\alpha(H) \geq \frac{1}{2}|H|$ for all $H \subseteq G$.

Definition 26. A *kernel* of a digraph $D := (V, A)$ is an coclique $S \subseteq V$ such that for every $x \in V$ there exists $y \in N^+[x] \cap S$; in other words $V = S \cup N^-(S)$.

The 5-cycle $G = (V, E) = v_1v_2v_3v_4v_5v_1$ has several orientations. Let

$$E_1 = \{v_1\vec{v}_2, v_2\vec{v}_3, v_3\vec{v}_4, v_4\vec{v}_5, v_5\vec{v}_1\} \text{ and } E_2 = \{v_1\vec{v}_2, v_3\vec{v}_2, v_3\vec{v}_4, v_5\vec{v}_4, v_5\vec{v}_1\}.$$

Then $\vec{G}_1 := (V, \vec{E}_1)$ does not have a kernel since $\Delta^-(\vec{G}_1) = 1$, $\alpha(G) = 2$, and $|\vec{G}| = 5 > \alpha(G) \cdot (\Delta^-(\vec{G}_1) + 1)$, but $\{v_2, v_4\}$ is a kernel of $\vec{G}_2 := (V, \vec{E}_2)$.

HW 18. (L,+) Prove: If a digraph G has no odd dicycle then it has a kernel. [Hint: Argue by induction; first use HW 15 to find a kernel S in the strong component of G furnished by HW 12(iii); finally consider a kernel of $G' := G - X$, where $X = S \cup \bigcup_{x \in S} N^-(x)$.

1.9. Trees

A *cut-vertex* vertex in a graph G is a vertex such that $G - v$ has more components than G. So v is a cut-vertex of G if and only if $H - v$ is nonempty and not connected, where H is the component of G (maybe $H = G$) containing v.

A *cut-edge* in a graph G is an edge e such that $G - e$ has more components than G. So e is a cut-edge of G if and only if $H - e$ is not connected, where H is the component of G containing e. Thus e is not a cut-edge if and only if its ends are connected in $G - e$.

Proposition 27 (1.2.14.). *An edge $e := xy$ in G is not a cut-edge if and only if it belongs to a cycle.*

Proof. First suppose e is not a cut-edge. Then there exists an x, y-path P in $G - e$. So $xPyx$ is a cycle in G. Now suppose $e \in E(C)$ for some cycle $C \subseteq G$. Then $x(C - e)y$ is a path connecting the ends of e, and so e is not a cut-edge. □

Proposition 28. *Let v be a cut vertex. Then it is not the end of a maximal path.*

Proof. Let H be the component containing v, and consider any path $P := u \dots v$ ending in v. Then $P\mathring{v}$ is contained in a component of $H - v$. As v is a cut vertex, it has a neighbor w in another component of $H - v$. Thus Pvw witnesses that P is not maximal. □

DEFINITION 29. A graph is *acyclic* if it contains no cycle. Acyclic graphs are also called *forests*. A connected, acyclic graph is called a *tree*. A *leaf* is a vertex v with $d(v) = 1$. We say that a graph G satisfies (A) if it is acyclic, (C) if it is connected, and (E), if $|G| = \|G\| + 1$.

LEMMA 30. *Suppose a graph $G := (V, E)$ with $|G| \geq 2$ satisfies at least two of (A), (C) and (E). Then G has at least two leaves.*

PROOF. If (E) holds then $\|G\| = |G| - 1 \geq 1$; else (C) holds and G has distinct vertices that are connected. Anyway G has an edge.

Suppose (A) holds. Let $P = v_1 \dots v_t$ be a maximum path in G. As $\|G\| \geq 1$, $v_1 \neq v_t$. As P is maximum and G is acyclic, $N(v_1) = \{v_2\}$ and $N(v_t) = \{v_{t-1}\}$. So v_1 and v_t are leaves.

Otherwise G satisfies (C) and (E). Let L be the set of leaves in G. Since G is connected and has at least two vertices, $\delta(G) \geq 1$. Since G satisfies (E)

$$2|G| - |L| \leq |L| + 2|G - L| \leq \sum_{v \in L} d(v) + \sum_{v \in V \smallsetminus L} d(v) = \sum_{v \in V} d(v) = 2\|G\| =_{(E)} 2|G| - 2$$

$$2 \leq |L|.$$

□

LEMMA 31. *Suppose G is a graph with a leaf l and $G' = G - l$. Then each condition (A), (C), (E) is satisfied by G iff it is satisfied by G'.*

PROOF. Suppose G is acyclic. Since removing a vertex cannot create a cycle, G' is acyclic. Now suppose G' is acyclic. Since every vertex in a cycle has degree 2, adding a leaf l cannot create a cycle, and so G is acyclic.

Suppose G' is connected. Since l has a neighbor in $V(G')$, G is connected. Now suppose G is connected. Since $d(l) = 1$, l is not a cut-vertex, and so G' is connected.

Since $|G| = |G'| + 1$ and $\|G\| = \|G'\| + 1$, G satisfies (E) iff G' does. □

THEOREM 32 (2.1.4). *If a graph G satisfies at least two of the conditions (A), (C), and (E) then it satisfies all three.*

PROOF. Argue by induction on $|G|$. If $|G| = 1$ then G satisfies all of (A), (C) and (E) by inspection. Else $|G| \geq 2$. By Lemma 30, G has a leaf l. Let $G' = G - l$. By Lemma 31, G' satisfies the two conditions that G does. By induction G' satisfies the remaining condition. By Lemma 31, G does too. □

In particular, Theorem 32 implies that every tree T satisfies $|T| = \|T\| + 1$.

THEOREM 33. *Let $H := (V, E)$ be a spanning subgraph of a connected graph G. Then the following are equivalent:*

(1) H is a tree;
(2) H is acyclic and $H + xy$ has a unique cycle for all $xy \in E(G) \smallsetminus E$;
(3) For all $x, y \in V$ there is a unique x, y-path in H.

PROOF. Assume (1). Then H is acyclic and connected. Consider any $xy \in E(G) \smallsetminus E$. Now H has an x, y-path P, and $C := xPyx$ is a cycle in $H + xy$. Suppose $D \subseteq H + xy$ is a cycle. For any $e \in E(C \cup D)$, Proposition 27 implies $H^* := H + xy - e$ is connected. By Theorem 32, $|H| = \|H\| + 1$. As $|H| = |H^*|$ and $\|H\| = \|H^*\|$, we have $|H^*| = \|H^*\| + 1$. By Theorem 32, H^* is acyclic. Thus $e \in C \cap D$. As e was arbitrary, $C = D$. Thus (2) holds.

Assume (2). Consider any vertices $x, y \in V$. There is exactly one x, x-path, so assume $x \neq y$. If $xy \in E$ then, since H is acyclic, xy is the only x, y-path. Else $xy \in E(G) \smallsetminus E$.

By (2), $H + xy$ has a unique cycle C. Then $xy \in C$, and $P := x(C - xy)y$ is an xy-path in H. If $Q \subseteq H$ is an x, y-path then $D := xQyx$ is a cycle. By (2), $C = D$, so $P = C - xy = D - xy = Q$. Thus (3) holds.

Assume (3). Then H is connected. Since distinct vertices in a cycle are connected by two paths, H is acyclic. So (1) holds. $\square$

By Theorem 33, any two vertices x and y of a tree T are connected by a unique path. We denote this path by xTy.

A subgraph H of a graph G spans G if $V(H) = V(G)$. In this case we say that H is a spanning subgraph of G. A *spanning tree of G is a spanning subgraph of G* that is a tree.

Corollary 34. *Every connected graph G contains a spanning tree T.*

Proof. Let T be a maximally acyclic (or a minimally connected) subgraph of G. $\square$

HW 19. (B) Let tree T be a tree with $|T| \geq 2$; then T has a bipartition $\{A, B\}$ with $|A| \geq |B|$. Prove: A contains a leaf; indeed, A contains at least $|A| - |B| + 1$ leaves. [Hint: Count edges in a way reminiscent of the proof of Lemma 30.]

HW 20. (B) Let T be a tree such that exactly $2k$ of its vertices have odd degree, where $k \in \mathbb{Z}^+$. Prove: T decomposes into k paths. Is it easier to prove this for forests? [Hint: Induction.]

HW 21. (B) Let G be a connected graph with $|G| \geq 3$ and $\|G\| \geq 2|G| - 2$. Prove: G contains two cycles with the same length. [Hint: Upper bound the number of cycle lengths; lower bound the number of cycles. Consider adding edges to a spanning tree of G and using the pigeonhole principle.]

HW 22. (L) Prove: If H is a subgraph of a connected graph G, and the ends of every edge of G are connected in H, then H is connected. [Hint: Prove by induction on k that all vertices $x, y \in H$ with $d_G(x, y) \leq k$ are connected in H].

HW 23. (+) Let T be a tree with even order. Prove: T has exactly one spanning subgraph such that every vertex has odd degree.

HW 24. (+) Let $d_1, \dots, d_n$ be positive integers with $n \geq 2$. Prove: There exists a tree T with n vertices $v_1, \dots, v_n$ such that $d_T(v_1) = d_1, \dots, d_T(v_n) = d_n$ iff $\sum d_i = 2n - 2$. [Hint: In the backward direction you are only given numbers—$d_1, \dots, d_n$; you must construct the tree. Try induction.]

HW 25. (+) A *root* of a graph G is a special vertex r. A spanning tree T of a graph with root r is normal if every edge $xy \in E(G)$ satisfies either $x \in rTy$ or $y \in rTx$. Prove: Every connected graph with root r has a normal spanning tree [Hint: Prove the stronger statement that every path P with end r is contained in a normal spanning tree.]

HW 26. (+) Let $\mathcal{T}$ be a set of subtrees of a tree G such that $T \cap T' \neq \emptyset$ for all $T, T' \in \mathcal{T}$. Prove: $\bigcap \mathcal{T} \neq \emptyset$. [Hint: Argue by induction on $|G|$.]

HW 27. (+) Prove: For every tree T there exists a vertex v such that for every edge e the component of $T - e$ containing v has at least as many vertices as the component not containing v. [Hint: Consider orienting some of the edges of T.]

1.10. Dirac's Theorem

A path in a graph G is *maximal* if it is not a subpath of a longer path in G. It is *maximum* if there is no longer path in G.

An *embedding* of H into G is an isomorphism from H to a subgraph of G. If there exists an embedding of H in G then we say that H can be *embedded* in G, or that H is embeddable in G. Recall that a subgraph $H \subseteq G$ is said to be a *spanning* subgraph of G if $V(H) = V(G)$. A spanning cycle (path) of G is called a *Hamilton* cycle (path). If G contains a Hamilton cycle, G is said to be *hamiltonian*.

Many questions in graph theory have the following form: Given two graphs G and H with $|H| = |G|$ what "local" conditions on G ensure that H is embeddable in G? If G is complete then trivially H is embeddable in G. This is guaranteed by the local condition $\delta(G) = |G| - 1$, but in many cases we can do much better. Corollaries 37 and 36 below are examples.

We have seen that not only can the question of whether a graph is bipartite be answered positively with proof by a lucky guess—the bipartition, it can also be answered negatively with proof by a lucky guess—an odd cycle. The question of whether a graph is hamiltonian can also be answered positively with proof by a lucky guess—the Hamilton cycle. However there is no "efficient" guessing method known for deciding with proof that a graph has no Hamiltonian cycle, and it is strongly believed that there is no such method. In particular, the problem of deciding whether a graph is hamiltonian is NP-complete.

Intuitively, if a graph has enough edges—for instance if it is complete—then it is hamiltonian. Here are some ways of quantifying what "enough" means.

Theorem 35. *Every connected graph $G := (V, E)$ with $|G| \geq 3$ contains a Hamilton cycle or a path of length $l := \min\{d(x) + d(y) : xy \in \overline{E}\}$.*

Proof. Let $P := v_1 \dots v_t$ be a maximum path in G. It exists since $|G|$ is finite and a trivial path is a candidate. We are done if $\|P\| \geq l$; else $l \geq \|P\| + 1$. We first prove:

$$G \text{ contains a cycle } C \text{ with } V(C) = V(P). \tag{1.10.1}$$

As $|G| \geq 3$ and G is connected, $t \geq 3$ (G is complete or has two nonadjacent connected vertices). Thus, if $v_1 v_t \in E$ then $C := v_1 P v_t v_1$ is the desired cycle; else $v_1 v_t \notin E$, and so:

$$d(v_1) + d(v_t) \geq l \geq \|P\| + 1 = |P| = t. \tag{1.10.2}$$

Since P is maximum, $N(v_1), N(v_t) \subseteq P$. Let

$$X := \{i \in [t-1] : v_1 v_{i+1} \in E\} \text{ and } Y := \{i \in [t-1] : v_t v_i \in E\}.$$

Then $|X| = d(v_1)$ and $|Y| = d(v_t)$. By $X, Y \subseteq [t-1]$, inclusion-exclusion and (1.10.2):

$$t - 1 \geq |X \cup Y| = |X| + |Y| - |X \cap Y| = d(v_1) + d(v_t) - |X \cap Y| \geq t - |X \cap Y|$$
$$|X \cap Y| \geq 1.$$

Let $i \in X \cap Y$ (Figure 1.10.1). Then $C := v_1 v_{i+1} P v_t v_i P^* v_1$ spans P, proving (1.10.1).

Let $C =: u_1 \dots u_t u_1$. For all $i \in [t]$, the path $u_i \dots u_t u_1 \dots u_{i \ominus 1}$ is maximum, and so $N(u_i) \subseteq C$. Thus $\|C, V \smallsetminus C\| = 0$. As G is connected, (1.6.1) implies $V(C) = V$. Thus C is a Hamilton cycle. $\square$

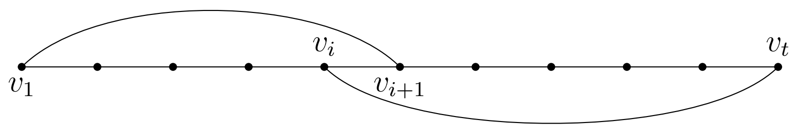


FIGURE 1.10.1. Hamilton cycle

Here is a condition that ensures a graph is hamiltonian.

COROLLARY 36 (Ore's Theorem 1960 [**16**] 7.2.9). *Every graph G with $|G| \geq 3$ and $d(x) + d(y) \geq |G|$ for all $xy \in$ is hamiltonian.*

PROOF. By Theorem 35, it suffices to show that G is connected. Consider any vertices $x, y \in V(G)$. We must find an x, y-path. If $xy \in E$ we are done; else $xy \notin E$. Then

$$|G| \geq |N[x] \cup N[y]| = |N[x]| + |N[y]| - |N[x] \cap N[y]| \geq |G| + 2 - |N[x] \cap N[y]|$$
$$|N[x] \cap N[y]| \geq 2,$$

so x is connected to y by a path of length at most 2. □

COROLLARY 37 (Dirac's Theorem 1952 [**7**] 7.2.8). *Every graph G with $\delta(G) \geq \frac{1}{2}|G| > 1$ is hamiltonian.*

PROOF. The hypothesis ensures that the hypothesis of Corollary 36 holds. □

Pósa (when he was in high school) posed a conjecture extending Dirac's Theorem:

CONJECTURE 38 (Pósa 1963). *If $\delta(G) \geq \frac{2}{3}|G|$ then G contains the square (2-power) of a Hamilton cycle.*

The next Theorem answers a related question.

THEOREM 39 (Fan & Kierstead 1996 [**8**]). *If $\delta(G) \geq \frac{2|G|-1}{3}$ then G contains the square of a Hamilton path. The degree condition is best possible.*

The next conjecture generalizes Corollary 37 and Conjecture 38.

CONJECTURE 40 (Seymour 1974). *If $\delta(G) \geq \frac{k}{k+1}|G|$ then G contains the k-th power of a Hamilton cycle.*

Seymour's Conjecture was proved for sufficiently large graphs (even more vertices than the number of electrons in the known universe when $k \geq 2$).

THEOREM 41 (Komlós, Sárközy & Szemerédi 1998 [**13**]). *For every integer k there exists an integer n such that Conjecture 40 is true for graphs G with $|G| \geq n$.*

The next theorem improves the bound on n when $k = 2$.

THEOREM 42 (Châu, DeBiasio & Kierstead 2011 [**5**]). *Conjecture 38 is true for graphs G with $|G| \geq 2 \times 10^8$.*

Bradt, DeBiasio and I now think we can lower this bound to 2×10^5.

HW 28. (B) Prove that $K_{a,a-1}$ is not hamiltonian for any integer $a > 1$. More generally, prove that if $\alpha(G) > \frac{1}{2}|G|$ then G is not hamiltonian. Determine $\delta(K_{a,a-1})$. [Hint: Try some small examples, say with $a = 2, 3, 4$. Then count the edges of a hypothetical cycle in two ways.]

HW 29. (B) Let $t \in \mathbb{Z}^+$. Prove that $G := \overline{K}_{t,t} \vee K_1$ is not hamiltonian. Determine $\delta(G)$. [Hint: Can a Hamilton cycle of G contain a cut-vertex?]

HW 30. (B) Prove that if $\delta(G) \geq \frac{|G|-1}{2}$ then G has a Hamilton path. [Hint: Use Theorem 35 or carefully add a vertex and use Corollary 37]

HW 31. (+) Let G be an X, Y-bigraph with $|X| = |Y| = k$ and $\delta(G) \geq \frac{k+1}{2} \geq 2$. Prove that G contains a Hamilton cycle. [Hint: First prove that if G contains a maximal path $P = P_t$ then $G[P]$ contains a cycle of length at least $2\lfloor \frac{t}{2} \rfloor$. Then show that G contains a Hamilton path.] For all k give an example to show that the bound on the minimum degree cannot be lowered.

1.11. Degree-even graphs and Euler's Theorem

Definition 43. A multigraph is *eulerian* if it has a closed trail containing all edges. (Recall that $T = v$ is closed.) Such a trail is said to be an Euler trail. A multigraph is *degree even* if every vertex has even degree. A component is *trivial* if it has no edges.

For $H \subseteq G$ and $v \in V(G) \smallsetminus V(H)$, set $d_H(v) := 0$. If T is a trail in a graph G then let $d_T(v) = d_{G(T)}(v)$.

Proposition 44. *Let $T := v_1 ... v_n v_1$ be a closed trail in a multigraph $G := (V, E)$. Then $G(T)$ is degree even.*

Proof. Argue by induction on n. If $E(T) = \emptyset$ then $n = 1$ and $d_T(v) = 0$. If T is a cycle then $d_T(v_i) = 2$ (even) for every $v \in V(T)$ (even if $E(T)$ consists of one loop or two parallel edges). Else, there are $1 \leq i < j < n$ with $v_i = v_j$. Let $T_1 := v_1 T v_i v_{j+1} T v_1$ and $T_2 := v_i T v_j (= v_i)$. Then T_1 and T_2 are both closed trails, shorter than T, and every edge of T is in exactly one of T_1 and T_2. By induction applied to T_1 and T_2, every $v \in V(T)$ satisfies:

$$d_T(v) = d_{T_1}(v) + d_{T_2}(v) \equiv 0 + 0 \equiv 0 \mod 2. \qquad \square$$

Corollary 45. *Let $T := v_1 ... v_n$ be an open trail in a multigraph $G := (V, E)$. Then for every $v \in V(T)$, the degree $d_T(v)$ of v is even if and only if $v \notin \{v_1, v_n\}$.*

Proof. As T is open, $v_1 \neq v_n$. Let $G' := G + v_n v_1$ and $T' := T + v_n v_1$, where $v_n v_1$ is a new edge (if $v_n v_1 \in E(T)$ we get a new parallel edge). As $v_n v_1$ is a new edge, T' is a closed trail. By Proposition 44, $d_{T'}(v)$ is even for all $v \in V(T)$. Since $d_T(v_i) = d_{T'}(v_i)$ if $v \in V(T) \smallsetminus \{v_1, v_n\}$, and $d_T(v) = d_{T'}(v) - 1$ if $v \in \{v_1, v_n\}$, we are done. $\square$

Theorem 46 (Euler (1736) 1.2.26). *A multigraph G is eulerian if and only if G is degree even and has at most one nontrivial component.*

Proof. First suppose G has a Euler trail T. As T is connected it only has edges from one component. As T contains all edges, there is at most one nontrivial component in G. As T is closed and contains all edges, every vertex has even degree by Proposition 44.

Now suppose G has at most one nontrivial component $H := (V, E)$, and every vertex has even degree. Let $T := v_1 \dots v_t$ be a maximum length trail in G; it exists because the trivial path is a candidate and the length of a trail is bounded by $\|G\|$. Then T is closed: Otherwise v_t is incident to an odd number of edges of T by Corollary 45. Since $d(v_t)$ is even, v_t is incident to some edge $v_t v$ that is not in T. So we can extend T to $T^+ := v_1 T v_t v$, contradicting the maximality of T.

It remains to show that $E \subseteq E(T)$. Otherwise there is an edge $ab \in E \smallsetminus E(T)$. If $a, b \in V(T)$, let $T' := abTv_tv_1Tb$. Else, as H is connected, (1.6.1) implies that there is an edge $v_i w \in E(V(T), V \smallsetminus V(T))$; let $T' := wv_iTv_tv_1Tv_i$. Anyway $\|T'\| > \|T\|$, a contradiction. $\square$

THEOREM 47 (1.2.33). *A connected graph G with exactly q vertices of odd degree decomposes into* $\max\{1, \frac{q}{2}\}$ *trails.*

PROOF. By Lemma 11, q is even. Let G^+ be the result of adding a new vertex v^+ to G so that $N_{G^+}(v^+) = \{x \in V : d_G(x) \equiv 1 \mod 2\}$. Since q is even, and every $v \in V(G)$ satisfies $d_{G^+}(v) \equiv d_G(v) + 1 \mod 2$ if and only if $d_G(v)$ is odd, G^+ is even. By Theorem 46, G^+ has an Eulerian Trail T. Removing v^+ partitions T into $\frac{q}{2}$ trails that decompose G. $\square$

Alternatively, we could have proved Theorem 47 by adding q edges linking disjoint pairs of odd degree vertices.

CHAPTER 2

Matchings

In this Chapter we will need the following easy proposition several times.

PROPOSITION 48. *Every connected multigraph G with $\Delta(G) \leq 2$ is a path or a cycle.*

PROOF. Let $P := v_0 \dots v_t$ be a maximum path in G. Then $N(v_0), N(v_t) \subseteq P$. As $\Delta(G) \leq 2$, $N(v_i) = \{v_{i-1}, v_{i+1}\} \subseteq P$ for all $i \in [t-1]$. Since G is connected, P spans G. So $P \subseteq G \subseteq P + v_0v_t$, i.e., G is a path or cycle. □

DEFINITION 49. Let $G := (V, E)$ be a graph. A *matching* M in G is a set of pairwise nonadjacent edges. A matching $M \subseteq E$ is *maximal* if there is no matching $M' \subseteq E$ with $M \subsetneq M'$. It is *maximum* if there is no matching $M' \subseteq E$ with $|M| < |M'|$. A vertex of G is said to be *M-covered* (or *M-saturated* or *M-matched)* if and only if it is incident to an edge of M; otherwise it is *M-uncovered* (or *M-unsaturated* or *M-unmatched)*. A set of vertices $X \subseteq V$ is said to be *M-covered* (or *M-saturated* or *M-matched)* if every $x \in X$ is M-covered. The matching M is *perfect* if every vertex of G is M-covered.

DEFINITION 50. Given a matching M in a graph $G := (V, E)$, a path or cycle H is M-alternating if $E(H) \subseteq M \cup M'$, where M' is a matching. Notice that if H is an M-alternating cycle with $E(H) = M \cup M'$ then $|M| = |M'|$. An alternating path P is M-augmenting if $|P| \geq 2$, and both ends of P are M-uncovered.

When M is clear from the context we may drop the prefix M- from covered, uncovered, alternating, augmenting, etc. If H is a graph with $\Delta(H) = 1$ then $E(H)$ is a matching. In this case, we may write H- instead of $E(H)$-.

THEOREM 51 (3.1.10 Berge [**1**] 1957). *A matching M in a graph $G := (V, E)$ is not maximum in G iff G has an M-augmenting path.*

PROOF. Suppose P is an M-augmenting path. Then

$$M' = M \triangle E(P) =_{def} (M \setminus E(P)) \cup (E(P) \setminus M)$$

is a larger matching.

Now suppose M is not maximum. Pick a matching M' with $|M'| > |M|$. Let $H := (V, F)$, where $F := M \triangle M'$. Then (*) $|F \cap M| < |F \cap M'|$. As each vertex is incident to at most one edge of each matching, $\Delta(H) \leq 2$. By Proposition 48, the components of H are cycles and alternating paths. Each cycle C satisfies $|C \cap M| = |C|/2 = |C \cap M'|$. By (*), some component of H has more edges from M' than M; it must be an M-augmenting path. □

HW 32. (*) Two players Alice and Bob play a game on a graph G. Alice begins the game by choosing any vertex. All other plays consist of the player, whose turn it is, choosing an unchosen vertex that is joined to the last chosen vertex. The winner is the last player

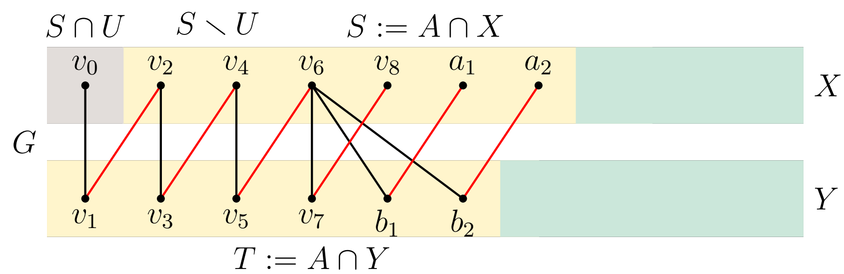


FIGURE 2.1.1. Hall's Theorem. Edges of the maximum matching M are red.

who can play legally. Prove that Alice has a winning strategy if G has no perfect matching, and Bob has a winning strategy if it does. [Hint: Use Theorem 51.]

HW 33. (+) Let M be a matching in a graph G with an M-uncovered vertex u. Prove that if G has no M-augmenting path starting at u then G has a maximum matching L such that u is L-uncovered. [Hint: Consider the set of ends of alternating paths starting at u.]

2.1. Bipartite matching

For $S \subseteq X$ set $N_G(S) := \bigcup_{v \in S} N(v) \setminus S$. For a function $f : A \to B$ and $S \subseteq A$, let $f(S) := \{y \in B : \exists x \in S(f(x) = y)\}$ be the range of f restricted to S.

THEOREM 52 (3.1.11 Hall [**12**] 1935). *An X, Y-bigraph $G := (V, E)$ has a matching that covers X iff*

$$|S| \leq |N(S)| \text{ for all } S \subseteq X. \tag{2.1.1}$$

PROOF. If M is a matching covering X and $S \subseteq X$ then $|S| = |E(S, Y) \cap M| \leq |N(S)|$; so (2.1.1) holds.

Arguing by contraposition, suppose M is a maximum matching that does not cover X. We must prove that (2.1.1) fails by constructing a set $S \subseteq X$ with $|N_G(S)| < |S|$.

Let U be the nonempty set of uncovered vertices in X, and let $A \subseteq V$ be the set of ends of alternating paths starting in U. Set $S = A \cap X$ and $T = A \cap Y$. Then $U \subseteq S$ (witnessed by trivial paths). Consider any alternating path $P = v_0 \dots v_n$ with $v_0 \in U$. If i is even, then $v_i \in S$ and $v_{i-1}v_i \in M$ when $i \neq 0$; else i is odd, $v_i \in T$ and $v_{i-1}v_i \notin M$. See Figure 2.1.1. We claim:

$$\text{(i) } N_G(S) \subseteq T \text{ and (ii) } T \subseteq T' := \{y \in V : xy \in M \text{ for some } x \in S \setminus U\}. \tag{2.1.2}$$

For (i), let $z \in S$ and $w \in N(z)$. Then there is an alternating path $Q := y_0 \dots y_{2k}$ with $y_0 \in U$, $y_{2k} = z$ and $y_{2k-1}y_{2k} \in M$. Thus $zw \notin M$. As G is bipartite, $w \in Y$. If $w \in V(Q)$ then $w \in T$; else Qzw is an alternating path witnessing $w \in T$. Anyway, (i) holds.

For (ii), let $w \in T$. Now there is an alternating path $P := y_0 \dots y_{2k+1}$ with $y_0 \in U$, $y_{2k+1} = w$ and $y_{2k}w \notin M$. As M is maximum, G has no augmenting path by Theorem 51. So w is covered; say $wx \in M$. Now Pwx is an alternating path; so $x \in S \setminus U$ and $w \in T'$, proving (ii).

Now $|N(S)| \leq_{(i)} |T| \leq_{(ii)} |T'| \leq |S \setminus U| < |S|$ since M is a matching and $U \neq \emptyset$. $\square$

A multigraph is *k-regular* if every vertex has degree k. A *k-factor* is a k-regular spanning subgraph. So a subgraph is a 1-factor if and only if its edge set is a perfect matching.

COROLLARY 53 (3.1.13). *For $k \in \mathbb{Z}^+$, every k-regular bipartite multigraph has a 1-factor.*

PROOF. Suppose G is an k-regular multigraph with bipartition $\{X, Y\}$. Now $|X| = |Y|$ since

$$k|X| = |E(X, Y)| = k|Y|.$$

Thus we are done if G has a matching that covers X. Note that (2.1.1) holds: For all $S \subseteq X$,

$$k|S| = |E(S, Y)| = |E(S, N(S))| \leq |E(X, N(S))| = k|N(S)|,$$

and so $|S| \leq |N(S)|$. By Hall's Theorem we are done. $\square$

Let $\mathcal{S} = \{S_i : i \in I\}$ be a family of sets. A *system of distinct representatives* (sdr) for $\mathcal{S}$ is a sequence $(a_i : i \in I)$ of distinct elements such that $a_i \in S_i$ for all $i \in I$. For example, let G be a graph, and let I be a coclique of G. Then $\mathcal{S} = \{N(v) : v \in I\}$ is a family of sets, and $(va_v : v \in I)$ is an sdr for $\mathcal{S}$ if and only if $\{va_v : v \in I\}$ is a matching.

HW 34. (B) Prove: If $\mathcal{S} = \{S_i : i \in I\}$ is a finite family of sets, then $\mathcal{S}$ has an sdr if and only if $|J| \leq |\bigcup_{j \in J} S_j|$ for all $J \subseteq I$. Find an infinite $\mathcal{S}$ for which this fails. [Hint: Use Theorem 52.]

HW 35. (B) Prove that there is an injection $f : \binom{[2k+1]}{k+1} \to \binom{[2k+1]}{k}$ such that $f(S) \subseteq S$ for all $S \in \binom{[2k+1]}{k+1}$. [Hint: Use Corollary 53.]

HW 36. (B) Let $G = (V, E)$ be an X, Y-bigraph with $|X| \leq |Y|$. Prove that if $|X| \leq d(x) + d(y)$ for all $x \in X$ and $y \in Y$ with $xy \notin E$ then G has a matching covering X.

HW 37. (*) Let $\mathcal{P} = \{P_1, \dots, P_t\}$ and $\mathcal{Q} = \{Q_1, \dots, Q_t\}$ be partitions of a set S into t parts of size k, where $k \in \mathbb{Z}^+$. Prove: $\mathcal{P}$ and $\mathcal{Q}$ have a common sdr, i.e., $(a_{\sigma(i)} : i \in [t])$ is an sdr of $\mathcal{Q}$ for some sdr $(a_i : i \in [t])$ of $\mathcal{P}$ and permutation σ of $[t]$.

HW 38. (+) Suppose G is an X, Y-bigraph with $\delta(G) \geq 1$ such that every edge xy with $x \in X$, satisfies $d(x) \geq d(y)$. Prove that G has a matching that covers every vertex in X. [Hint: Consider assigning each edge xy with $x \in X$ the weights $w(xy) = \frac{1}{d(x)}$ and $w'(xy) = \frac{1}{d(y)}$ and summing weights.]

HW 39. (+) Let G be an A, B-bigraph with $|A| = n = |B|$ and $\delta(G) \geq k$, where $k, n \in \mathbb{N}$. Prove: If $E(X, Y) \neq \emptyset$ for all $X \subseteq A, Y \subseteq B$ with $|X|, |Y| \geq k$ then G has a perfect matching. [Hint: Use Hall's Theorem; for $S \subseteq A$, consider cases based on $|S|$.]

HW 40. (+) For $k \in \mathbb{Z}^+$, let G be a A, B-bigraph with $\delta(G) \geq 1$ such that $d(b) \leq k\delta(G)$ for every vertex $b \in B$. Prove: There is $H \subseteq G$ such that $d_H(a) = 1$ for every $a \in A$ and $d_H(b) \leq k$ for every $b \in B$. [Hint: Modify G by "duplicating" the vertices of B.]

HW 41. (Q) Prove that for all partitions $\mathcal{P} = \{P_1, \dots, P_n\}$ and $\mathcal{Q} = \{Q_1, \dots, Q_n\}$ of a region S of area n into parts of area 1 there exists a bijection $\sigma : [n] \to [n]$ such that area $(P_i \cap Q_{\sigma(i)}) \geq f(n)$ for all $i \in [n]$, where

$$f(n) := \begin{cases} \frac{4}{(n+1)^2} & \text{if } n \text{ is odd} \\ \frac{4}{n(n+2)} & \text{if } n \text{ is even} \end{cases}.$$

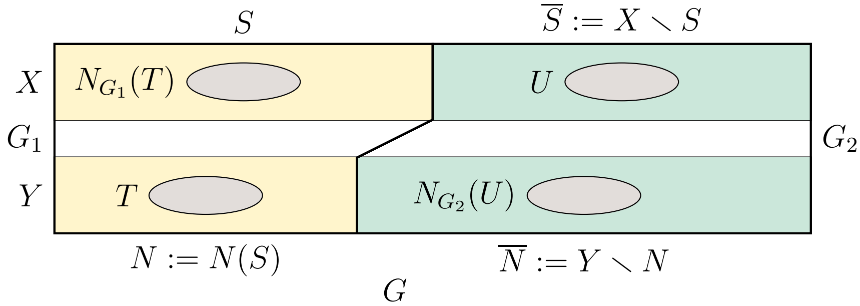


Figure 2.1.2. König-Egerváry Theorem: $W := N \cup \overline{S}$ is a cover; $M := M_1 \cup M_1$ is a matching, where $M_1 \subseteq G_1$ covers N and $M_2 \subseteq G_2$ covers $\overline{S}$.

Also show that the function f is optimal.

Definition 54. A *cover* of a graph G is a subset $Q \subseteq V(G)$ such that every edge of G is incident to a vertex of Q.

Let G be a graph with a cover Q and a matching M. Every edge of M is incident to a vertex of Q, and no vertex of Q is incident to two edges of M, so $|M| \leq |Q|$. Can we choose choose Q and M so that $|Q| = |M|$? Well, suppose $G = C_{2k+1}$. Since G is 2-regular, every $v \in Q$ is incident to 2 edges of G. Thus $2|Q| \geq 2k+1$, and so $|Q| \geq k+1$. On the other hand, every matching M in C has $2|M|$ ends. Thus $2|M| \leq 2k+1$. So $|M| \leq k < |Q|$.

Theorem 55 (3.1.16 König, Egerváry [1931]). *If G is a graph with matching M and cover W then $|M| \leq |W|$; if G is bipartite then M and W can be chosen with $|M| = |W|$.*

Proof. Order $V := V(G)$ as $v_1 \prec \cdots \prec v_{|G|}$. Since W is a cover, every edge of M is incident to some vertex of W (possibly two). Define a function $g : M \to W$ by $g(e)$ is the least $w \in e \cap W$. Since M is a matching, no vertex of W can be incident two edges of M. So g is an injection. Thus $|M| \leq |W|$.

Now suppose G is an X,Y-bigraph. Pick a set $S \subseteq X$ so that (*) $t := |S| - |N(S)|$ is maximum. Let $\overline{S} = X \setminus S$, $N = N(S)$, $\overline{N} = Y \setminus N$ and $W = N \cup \overline{S}$. Then W is a cover since $E(S, \overline{N}) = \emptyset$. It remains to find a matching M with $|M| = |W|$. We will show:

(1) $G' := G[S \cup N]$ has a matching M_1 covering N; and
(2) $G'' := G[\overline{S} \cup \overline{N}]$ has a matching M_2 covering $\overline{S}$.

As $G' \cap G'' = \emptyset$, this implies that $M := M_1 \cup M_2$ is a matching with

$$|M| = |M_1| + |M_2| = |N| + |\overline{S}| = |W|.$$

Checking (2.1.1) for (1), consider any $T \subseteq N$. Since (i) $N_G(S \setminus N_{G'}(T)) \subseteq N \setminus T$,

$$\begin{aligned} |S| - |N| &\geq_{(*)} |S \setminus N_{G'}(T)| - |N_G(S \setminus N_{G'}(T))| \geq_{(i)} |S \setminus N_{G'}(T)| - |N \setminus T| \\ &\geq |S| - |N_{G'}(T)| - |N| + |T|. \end{aligned}$$

Thus $|N_{G'}(T)| \geq |T|$. So (1) holds by Theorem 52.

Checking (2.1.1) for (2), consider any $U \subseteq \overline{S}$. Then

$$|S| - |N| \geq_{(*)} |S \cup U| - |N_G(S \cup U)| \geq |S| + |U| - |N| - |N_{G''}(U)|.$$

Thus $|N_{G''}(U)| \geq |U|$. So (2) holds by Theorem 52. $\square$

Alternatively, we could have proved the second part of Theorem 55 by continuing the proof of Theorem 52 as follows.

Second proof. Now suppose G is an X,Y-bigraph, and let M be a maximum matching. We will construct a cover W with $|M| = |W|$. If M saturates X then $W = X$ works. Otherwise, we follow the proof of Theorem 52, using the same notation. Set $\overline{S} = X \smallsetminus S$, $\overline{T} = Y \smallsetminus T$ and $W = \overline{S} \cup T$. Then W is a cover: Suppose $xy \in E$ with $x \in X$. If $x \in S$ then $y \in T \subseteq W$ by 2.1.2(i); else $x \in \overline{S} \subseteq W$.

To show $|W| \leq |M|$, we prove g (defined for M and W) is onto. Let $w \in W$. If $w \in T$ then there is $x \in S$ with $xw \in M$ by 2.1.2(ii). As $x \notin W$, $g(xw) = w$. If $w \in \overline{S}$ then there is y with $wy \in M$ since $U \subseteq S$. As M is a matching, 2.1.2(ii) implies $y \in \overline{T}$. So $g(wy) = w$. $\square$

HW 42. (B) Prove: Every bipartite graph has a matching of size at least $\frac{\|G\|}{\Delta(G)}$. [Hint: Use Theorem 55. How small can a cover be?]

2.2. General matching

Notice that if H is a component of a graph G and $|H|$ is odd then G does not have a perfect matching.

Definition 56. Let $\mathcal{C}_G$ be the set of components of the graph G. A component with an odd number of vertices is said to be an *odd component*. Let $\mathcal{O}_G$ be the set of odd components of G and $o(G) = |\mathcal{O}_G|$.

Theorem 57 (3.3.3 Tutte [**21, 22**] 1947). *A graph $G := (V, E)$ has a 1-factor if and only if*

$$o(G - S) \leq |S| \text{ for all } S \subseteq V. \tag{2.2.1}$$

Proof. Suppose G has a 1-factor. For all $S \subseteq V$, all odd components of $G - S$ must have vertices matched to distinct vertices in S. So $o(G - S) \leq |S|$.

Now suppose (2.2.1). Argue by induction on $\|\overline{G}\|$. Applying (2.2.1) to $S := \emptyset$ shows that G has no odd component; so $|G|$ is even. Let

$$U := \{v \in V : N[v] = V\}.$$

First suppose $G - U$ is empty or consists of complete components. By (2.2.1), $|U| \geq o(G - U)$; so distinct vertices of U can be added to each $H \in \mathcal{O}_{G-U}$ to form disjoint even cliques. The remaining vertices of $G - U$ form disjoint even cliques. As $|G|$ is even, the remaining vertices of U form an even clique. Each of these even cliques has a 1-factor. Combining them yields a 1-factor of G.

Else some $H \in \mathcal{C}_{G-U}$ is not complete. Thus there is an induced path $aba' \subseteq H$. As $b \notin U$, there exists $b' \in V$ with $bb' \notin E$. Consider $G_e = G + e$ for $e \in \{aa', bb'\} \subseteq \overline{G}$. By induction, the theorem holds for both G_e. Note that G_e satisfies (2.2.1): while adding e to G may merge two components of $G - S$ into one, it never creates more odd components. So $G + aa'$ and $G + bb'$ have perfect matchings M^+ and L^+, respectively.

Set $M := M^+ - aa'$ and $L := L^+ - bb'$. If $M = M^+$ or $L = L^+$ then G has a 1-factor. Else a, a' are the only M-uncovered vertices, and b, b' are the only L-uncovered vertices. So the component of $M \triangle L$ containing a is an M, L-alternating path $P = a \dots x$ with $x \in \{a', b, b'\}$

and no inner vertex in $\{a', b, b'\}$. If $x = a'$ then P is M-augmenting; if $x = b$ then $aPxa'$ is M-augmenting; else $x = b'$ and $baPx$ is L-augmenting. Anyway, by Theorem 51, G has a matching M^* with $|M^*| = |M^+| = |L^+|$. So (V, M^*) is a 1-factor. $\square$

The next theorem is a generalization of Tutte's Theoerm. We will give two proofs. The first builds on the previous proof. The second starts from scratch.

THEOREM 58 (3.3.7 Berge [**2**] 1958). *The number of uncovered vertices of a maximum matching in a graph $G = (V, E)$ is equal to*

$$d := \max_{S \subseteq V}(o(G - S) - |S|). \tag{2.2.2}$$

FIRST PROOF. Say $d = o(G - S_0) - |S_0|$ and M is a matching. Every odd component of $G - S_0$ contains an M-uncovered vertex or a vertex x with $xy \in M$ for some $y \in S_0$. Thus the number of M-uncovered vertices is at least d.

Now assume (2.2.2). Using $S = \emptyset$ shows $d \geq 0$. Let $G^+ := G \vee K_d(Q)$, where Q is a set of d new vertices (if $d = 0$ then $G^+ = G$). Then Q is a clique in G^+. It suffices to show $o(G^+ - T) - |T| \leq 0$ for all $T \subseteq V(G^+)$: Then by Theorem 57, G^+ has a perfect matching M^+, and $M := M^+ \smallsetminus E(Q, V)$ is a matching in G with at most $|Q| = d$ uncovered vertices.

First note that $|G^+|$ is even:

$$|G^+| = |G| + d \equiv (o(G - S_0) + |S_0|) + (o(G - S_0) - |S_0|) \equiv 2o(G - S_0) \equiv 0 \mod 2.$$

As $|G^+|$ is even, $o(G^+ - T) \equiv |T| \mod 2$. If $Q \nsubseteq T$ then $G - T$ is connected. So $o(G^+ - T) \leq 1$, and by parity, $o(G^+ - T) - |T| \leq 0$. Otherwise $Q \subseteq T$, and so

$$o(G^+ - T) - |T| = o(G - (T \smallsetminus Q)) - |T \smallsetminus Q| - |Q| \leq d - d \leq 0. \qquad \square$$

COROLLARY 59. *Let $G := (V, E)$ be a graph, $d := \max_{S \subseteq V}(o(G - S) - |S|)$, and S_0 satisfy $d = o(G - S_0) - |S_0|$. Then every maximum matching covers S_0.*

PROOF. Let M be a maximum matching. Every odd component of $G - S_0$ contains an M-uncovered vertex or a vertex x that is matched to a vertex $y \in S_0$. As M has d M-uncovered vertices, every vertex of S_0 is M-covered. $\square$

Here is the second proof of Theorem 58. First some definitions. A graph G is *factor critical* if $G - v$ has a perfect matching for every vertex $v \in V(G)$. A set S is *matchable* into $\mathcal{O}_{G-S}$ if there exists a matching M that covers S such that each edge $e \in M$ has one end in S and one end in an odd component of $G - S$, and at most one vertex of each odd component is covered.

SECOND PROOF OF THEOREM **??**. For any set $S \subseteq V$ and matching M, there are at least $o(G - S) - |S|$ uncovered vertices: each odd component $H \subseteq G - S$ has an M-uncovered vertex, unless $M \cap E(S, V(H)) \neq \emptyset$, and there are at most $|S|$ such edges in M. So it suffices to show that there exists a set $S \subseteq V$ and a matching M with exactly $o(G - S) - |S|$ uncovered vertices.

Argue by induction on $|G|$. For the base step $|G| = 1$, let $S = \emptyset$. Then $o(G - S) - |S| = 1$ and the only vertex of G is uncovered by any matching. Now consider the induction step.

Choose a set $S \subseteq V$ so that $o(G - S) - |S|$ is maximum, and subject to this, $|S|$ is also maximum. We first prove the following three claims:

CLAIM (1). Every component of $G - S$ is odd.

Proof. Suppose $H \in \mathcal{C}_{G-S}$ with $|H|$ even. Choose a non-cut vertex x (end of a maximal path) of H, and set $S' = S + x$. Then

$$\mathcal{O}_{G-S'} = \mathcal{O}_{G-S} + (H - x) \text{ and } |S'| = |S| + 1.$$

Thus $o(G - S) - |S| = o(G - S') - |S'|$, contradicting the choice of S, since $|S| < |S'|$. $\square$

Claim (2). Every odd component of $G - S$ is factor critical.

Proof. Consider any $H \in \mathcal{O}_{G-S}$ and any vertex $x \in V(H)$. We must show that $H' = H - x$ has a perfect matching. By the induction hypothesis, it suffices to show that $o(H' - T) - |T| \leq 0$ for all $T \subseteq V(H')$. So consider any such T, and set $S' = S \cup T + x$. Then $|S'| = |S| + |T| + 1 > |S|$, and so by the choice of S

$$o(G - S) - |S| > o(G - S') - |S'|.$$

Since $T + x \subseteq V(H)$,

$$\mathcal{O}_{G-S'} = (\mathcal{O}_{G-S} - H) \cup \mathcal{O}_{H'-T}.$$

So

$$o(G - S) - |S| > o(G - S') - |S'| = o(G - S) - 1 + o(H' - T) - |S| - |T| - 1$$
$$2 > o(H' - T) - |T|.$$

Moreover, by Claim (1), H is an odd component, and so $|H'|$ is even. Thus

$$0 \equiv |H'| \equiv |H' - T| + |T| \equiv o(H' - T) + |T| \equiv o(H' - T) - |T| \mod 2.$$

Hence $1 \neq o(H' - T) - |T|$, and so $0 \geq o(H' - T) - |T|$. $\square$

Claim (3). *S is matchable into $\mathcal{O}_{G-S}$.*

Proof. Let H be the $S, \mathcal{O}_{G-S}$-bigraph with edge set

$$F := \{xD : x \in S, D \in \mathcal{O}_{G-S} \text{ and } N(x) \cap V(D) \neq \emptyset\}.$$

It suffices to show that H has a matching that covers S. For this we apply Hall's Theorem.

Consider any set $T \subseteq S$. Let $S' := S - T$. By the choice of S

$$o(G - S) - |S| \geq o(G - S') - |S'| \geq o(G - S) - N_H(T) - |S| + |T|$$
$$|N_H(T)| \geq |T|.$$

$\square$

Finally, we obtain a maximum matching M as follows. By Claim (3) there is a matching M_0 that covers S and one vertex of $|S|$ odd components. For each $H \in \mathcal{O}_{G-S}$ choose a vertex v_H, and if possible, choose v_H so that it is M_0-covered. Next use Claim (2) to obtain matchings M_H of $H - v_H$ for every odd component $H \in \mathcal{O}_{G-S}$. Then

$$M := M_0 \cup \bigcup_{H \in \mathcal{O}_{G-S}} M_H$$

is matching of G. Using Claim (1), it covers every vertex of G except those $o(G - S) - |S|$ vertices v_H that are not covered by M_0. $\square$

We have actually proved a stronger statement.

Theorem 60. *Let $G = (V, E)$ be a graph and $S^* \subseteq V$ be a set of vertices such that for all $S \subseteq V$*

(1) $o(G-S)-|S|\le o(G-S^*)-|S^*|$; *and*
(2) *if equality holds in (1) then* $|S|\le|S^*|$.

Then every component of $G-S^*$ *is odd and factor-critical; and every maximum matching covers* S^*, *matches* S^* *into* $\mathcal{O}_{G-S^*}$, *and leaves at most one vertex of each component of* $G-S^*$ *uncovered.*

An isomorphism from a graph to itself is called an automorphism. A graph $G=(V,E)$ is transitive if for all $x,y\in V$ there is an automorphism φ of G with $\varphi(x)=y$. For example C_n, K_n, $K_{n,n}$ and the Petersen graph are all transitive, but P_n with $n\ge 3$ is not transitive.

HW 43. (*) Prove that a transitive graph does not have a cut vertex.

HW 44. (+) Prove that if G is a connected, transitive graph with $|G|$ even then G has a perfect matching. [Hint: Use Corollary 59 to show that if G does not have a perfect matching then some, but not all, of its vertices have the property that they are covered in every maximum matching. Where does your argument use that G is connected and $|G|$ is even?] (Lovasz has conjectured that every connected transitive graph has a hamiltonian path; there are only four known examples of such graphs that have no hamiltonian cycle.)

2.3. Applications of Matching Theorems

A cut-edge is sometimes called a *bridge*. A *bridgeless* graph is a graph without cut-edges. It need not be connected.

Theorem 61 (3.3.8 Petersen [**17**] 1891). *Every bridgeless cubic graph* $G=(V,E)$ *contains a* 1*-factor.*

Proof. By Tutte's Theorem, it suffices to show that $o(G-S)\le|S|$ for every subset $S\subseteq V$. Fix any such S and consider any $H\in\mathcal{O}_{G-S}$. Since G is cubic and $|H|$ is odd,

$$\|H,S\|\equiv 2\|H\|+\|H,S\|=\sum_{v\in V(H)}d(v)=3|H|\equiv 1 \mod 2.$$

It follows that $\|H,S\|$ is odd, and since G is bridgeless, $\|H,S\|\ge 3$. Thus

$$3o(G-S)\le\|S,V\|\le 3|S|,$$

and so $o(G\setminus S)\le|S|$. □

Theorem 62 (3.3.9 Petersen [**17**] 1891). *Let* $k\in\mathbb{Z}^+$. *Then every* $2k$*-regular graph has a* 2*-factor.*

Proof. Suppose $G=(V,E)$ is $2k$-regular with $k\in\mathbb{Z}^+$. It suffices to show that each component of G has a 2-factor, so we may assume G is connected. By Euler's Theorem 46, G has an Eulerian trail $T=v_1\dots v_nv_1$. Let $V'=\{v':v\in V\}$ and $V''=\{v'':v\in V\}$ be disjoint sets of new vertices, where $v\mapsto v'$ and $v\mapsto v''$ are bijections. Let H be the V',V''-bigraph defined by $E(H)=\{v'_iv''_{i\oplus 1}:i\in[n]\}$. Since each vertex v of G is incident to $2k$ edges, it appears k times in T. Say $v=v_{i_1}=\dots=v_{i_k}$. Then

$$E(v)=\{v_{i_1-1}v_{i_1},v_{i_1}v_{i_1+1},\dots,v_{i_k-1}v_{i_k},v_{i_k}v_{i_k+1}\}$$

and

$$N_H(v')=\{v''_{i_1+1},\dots,v''_{i_k+1}\}\text{ and }N_H(v'')=\{v'_{i_1-1},\dots,v'_{i_k-1}\}.$$

So H is k-regular. By the Corollary 53, H has a perfect matching M. Let $F = \{xy \in E : x'y'' \in M\}$. Then (V, F) is a 2-factor of G: for each $y \in V$ there exists a unique x such that $x'y'' \in M$ and a unique z such that $y'z'' \in M$. Since $xy, yz \in T$ and T is a trail, $xy \neq yz$. $\square$

HW 45. (B,*) Prove that a 3-regular graph has a 1-factor if and only if it decomposes into copies of P_4.

HW 46. (*) Let G be a k-regular graph with $|G|$ even that remains connected when any $k-2$ edges are removed. Prove: G has a 1-factor. [Hint: Use ideas from the proof of Theorem 61]

HW 47. Suppose G is a graph on $2k$ vertices with $k \geq 3$, whose complement $\overline{G}$ does not have a 1-factor. Let S be the set whose existence is guaranteed by Tutte's Theorem (applied to $\overline{G}$). Prove:

(1) If $|S| = 0$ then G contains $K_{c,2k-c}$ for some odd c.
(2) If $|S| \geq k-1$ then G contains K_{k+1}.
(3) If $1 \leq |S| \leq k-2$ then $\Delta(G) \geq k+1$. [Hint: $\frac{x+1}{x+2}(2k-x) > k$ when $1 \leq x \leq k-2$.]

HW 48. (+) A graph is *claw-free* if it does not contain an induced $K_{1,3}$. Prove: A connected, claw-free graph of even order has a 1-factor. Find (easy) a small counter example if the graph is not connected.

CHAPTER 3

Connectivity

Definition 63. A *separating set* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ is disconnected. The *connectivity* $\kappa(G)$ is the minimum size of a vertex set S such that $G - S$ has more than one component or only one vertex. A graph G is k-connected if $k \leq \kappa(G)$. Two vertices x and y are *separated* by S if they are in different components of $G - S$. In particular, they are **not** in S.

Note that it is not possible to disconnect a complete graph by removing vertices, so $\kappa(K_n) = n - 1$. Also $\kappa(G) = 0$ iff $|G| = 1$ or G is disconnected.

Definition 64. Let G be a graph with $|G| > 1$. A *disconnecting set* of edges in G is a set $F \subseteq E(G)$ such that $G - F$ has more than one component. The *edge-connectivity* $\kappa'(G)$ of G is the minimum size of a set $F \subseteq E(G)$ such that $G - F$ is disconnected. The graph G is k-edge-connected if $k \leq \kappa'(G)$. Two vertices x and y are *separated* by F if they are in different components of $G - F$.

Following West, we may write $[S, T]$ for $E(S, T)$. An *edge cut* in G is a set of edges of the form $[S, \overline{S}]$, where $\emptyset \neq S \neq V(G)$ and $\overline{S}$ denotes $V(G) \smallsetminus S$. Then, by notation, $|[S, \overline{S}]| = \|S, \overline{S}\|$. Note that a minimal disconnecting set of edges is an edge cut.

3.1. Basics

Theorem 65 (Whitney [1932] 4.1.9). *Every graph $G := (V, E)$ satisfies*

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Proof. Choose a vertex $v \in V$ with $d(v) = \delta(G)$. Then $E(v)$ is a disconnecting set of edges of size $\delta(G)$, and so $\kappa'(G) \leq \delta(G)$.

For the first inequality, let $[S, \overline{S}]$ be a minimum edge cut; so $\|S, \overline{S}\| = \kappa'(G)$. If every vertex in S is adjacent to every vertex in $\overline{S}$, then

$$\kappa'(G) = \|S, \overline{S}\| = |S||\overline{S}| = |S|(|G| - |S|) \geq |G| - 1 \geq \kappa(G).$$

Else there is $x \in S$ and $y \in \overline{S}$ with $xy \notin E$. Define $f : [S, \overline{S}] \to V$ by $f(e) = z$ if $e = xz$; else $f(e) \in e \cap (S - x)$. So $f(e) \in e \smallsetminus \{x, y\}$. Every x, y-path P contains an edge $e \in [S, \overline{S}]$, and

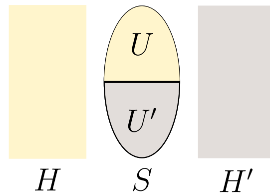


FIGURE 3.1.1. Proof of Theorem 66

$f(e)$ is an interior vertex of P. So range(f) separates x from y. Thus

$$\kappa'(G) = \|S, \overline{S}\| \geq |\text{range}(f)| \geq \kappa(G). \qquad \square$$

THEOREM 66. *Every 3-regular graph $G := (V, E)$ satisfies $\kappa(G) = \kappa'(G)$.*

PROOF. Put $\kappa := \kappa(G)$ and $\kappa' := \kappa'(G)$. By Theorem 65, $\kappa \leq \delta(G) = 3$. If $\kappa = 3$ then $3 = \kappa \leq \kappa' \leq \delta(G) = 3$, and so $\kappa = \kappa'$. Now suppose $0 \leq \kappa \leq 2$. Note that $|G| \geq |N[v]| = 4$ for any $v \in V$. Thus G has a minimum separating set S with $|S| = \kappa$. Let H, H' be two components of $G - S$. Pick a minimal $U \subseteq S$ subject to $\|x, H \cup U\| \leq 1$ for all $x \in U' := S \smallsetminus U$. See Figure 3.1.1 Then for all $x \in U'$, $\|x, H \cup U\| \leq 1$; moreover, for all $x \in U$, $\|x, H \cup U\| \geq 2$, and so $\|x, H' \cup U'\| \leq 1$. Thus

$$\kappa' \leq \|H \cup U, H' \cup U'\| \leq \|U, H' \cup U'\| + \|U', H \cup U\| \leq |U \cup U'| = |S| = \kappa. \qquad \square$$

HW 49. (B,L) If $G := (V, E)$ is a graph with a minimum separating set S then $\|x, U\| \neq 0$ for every $x \in S$ and component U of $G - S$.

HW 50. (B,L) If G is k-connected and G' is obtained from G by adding a new vertex x with at least k neighbors in G then G' is k-connected.

HW 51. (B,L) Let G_1 and G_2 be k-connected graphs with $|G_1 \cap G_2| \geq k$. Prove: $G := G_1 \cup G_2$ is k-connected.

HW 52. Prove: For $r \in \mathbb{Z}^+$, if G is an r-connected graph with $|G|$ even that has no induced subgraph isomorphic to $K_{1,r+1}$, then G has a 1-factor. Find an r-edge-connected graph H with $|H|$ even that has no induced subgraph isomorphic to $K_{1,r+1}$ and no 1-factor.

3.2. Low Connectivity

Now we provide characterizations of 2-connected graphs (Theorem 68) and 3-connected graphs (Lemma 71). Both provides ways to make induction arguments, and both are needed for the proof of Kuratowski's Theorem (121) characterizing planar graphs.

DEFINITION 67. Suppose H is a subgraph of G. Recall that a path $P \subseteq G$ is an H-path if it has *two* ends in H, but no edges or inner vertices in H. Let $\mathcal{Q} := Q_0, Q_1, \dots, Q_t$ be a sequence of subgraphs of G, and set $G_i := Q_0 \cup \dots \cup Q_{i-1}$. Then $\mathcal{Q}$ is a 2-*witness* for G if Q_0 is a cycle, $G_{t+1} = G$, and Q_i is a G_i-path in G for all $i \in [t]$.

THEOREM 68 (Whitney [1932] 4.2.8). *A graph $G := (V, E)$ is 2-connected iff it has a 2-witness.*

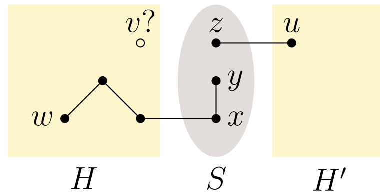


FIGURE 3.2.1. Proof of Lemma 71

PROOF. First suppose G is 2-connected. Then $\delta(G) \geq 2$, and so G contains a cycle C. Let $H \subseteq G$ be a maximal subgraph such that H has a 2-witness $Q_0, \ldots, Q_t$. It exists because C is a candidate. It suffices to show that $H = G$.

Suppose $V(G - H) \neq \emptyset$. By (1.6.1), there is $xy \in E(H, G - H)$ with x in H. As G is 2-connected, $G - x$ is connected, so there is a y, H-path P in $G - x$. Now $Q_{t+1} := xyP$ is an H-path in $H + Q_{t+1}$, contradicting the maximality of H. Thus H spans G.

Now suppose $xy \in E(G - H)$. Then xy is an H-path of $H + xy$, contradicting the maximality of H. So H is an induced, spanning subgraph of G. Thus $G = H$.

Now suppose G has a 2-witness $Q_0, \ldots, Q_t$. Argue by induction on t that G is 2-connected. If $t = 0$ then G is 2-connected since $G = Q_0$ is a cycle. Else $t \geq 1$. By induction, $H := \bigcup_{i=0}^{t-1} Q_i$ is 2-connected. Let $Q_t := v_1 \ldots v_s$, and pick a v_s, v_1-path $P \subseteq H$. Then $C := v_1 P v_s Q_t v_1$ is a cycle with $G = H \cup C$. Now H and C are both 2-connected, and $|H \cap C| \geq 2$, so G is 2-connected by 51 ($(G - x = (H - x) \cup (C - x)$ is connected for all $x \in V(G)$). $\square$

DEFINITION 69. Let $e = xy$ be an edge in a graph G, and fix a new vertex v_e. The graph $G \cdot e$ obtained by *contracting* e is defined by

$$G \cdot e := (G \cup K(v_e, N_G(\{x, y\}))) - x - y.$$

Note that if P' is a path in $G \cdot e$ then either P' is a path in G or $v_e \in V(P')$. In the latter case we can obtain a path in G by replacing v_e by one of x, y, xy, yx. If P is a path in G then either P is a path in $G \cdot e$ or one or both of x, y are in $V(P)$. In the latter case we can obtain a path P' in $G \cdot e$ by replacing one of x, y, xPy, yPx by v_e.

LEMMA 70. *Let $G := (V, E)$ be a graph with $xy \in E$. If S' separates $G' := G \cdot xy$ then S separates G, where $S = S'$ if $v_{xy} \notin S'$ and $S = S' - v_{xy} + x + y$ else.*

PROOF. Let U and W be distinct components of $G' - S'$ with $v_{xy} \notin U$. Choose $u \in U$ and $w \in V \cap W + x$ with $w \neq x$ if possible. We claim that S separates u from w. Consider any u, w-path $P \subseteq G$. Then there is a path $P' \subseteq G'$ with $P' = P$ or else P' is formed from P by replacing one of x, y, xPy, yPx by v_e. Now P' is a U, W-path, so it contains a vertex $s \in S'$. Either $s \neq v_{xy}$ or $\{x, y\} \cap P \cap S \neq \emptyset$. Anyway $P \cap S \neq \emptyset$. $\square$

LEMMA 71 (Thomassen [1980] 6.2.9). *Every 3-connected graph G with $|G| \geq 5$ has an edge e such that $G \cdot e$ is 3-connected.*

PROOF. Suppose not. Consider any edge xy. Then $G' := G \cdot xy$ is not 3-connected. As $|G'| \geq 4$, G' has a separating 2-set S'. As G is 3-connected, $S' \not\subseteq G$, so $S' = \{v_{xy}, z\}$ for some $z \in V(G)$. By Lemma 70, $S := \{x, y, z\}$ is a separating 3-set in G. Now pick the edge xy

and the 2-separating set S' so that $G-S$ has a component H with $|H|$ as large as possible. Put $H^+ = G[H \cup \{x,y\}]$. Then $H' := (G-S) - H \neq \emptyset$. See Figure 3.2.1.

As S is a minimal separating set, there is $u \in N(z) \cap H'$ by HW 49. Then $G \cdot uz$ has a separating set $\{v_{uz}, v\}$, and $\{u,v,z\}$ is a separating set for G. Since $\kappa(G) \geq 3$, for every $w \in H - v$ there is a $w, \{x,y\}$-path P in $G - \{v,z\}$. As H is a component of $G-S$, $\mathring{P} \subseteq H - v$. So $H^+ - v$ is connected. As $z, u \notin H^+$, $H^+ - v$ is contained in a component H^* of $G - \{u,v,z\}$ with $|H^*| > |H|$, contradicting the choice of xy, z, H. □

Definition 72. A sequence of graphs $G_0, \dots, G_s$ is a 3-*witness for* G iff

(1) $G_0 = K_4$ and $G_s = G$; and
(2) for each $i \in [s]$ there is an edge $xy \in E(G_i)$ such that $G_{i-1} = G_i \cdot xy$ and $d_{G_i}(x), d_{G_i}(y) \geq 3$.

Theorem 73. *A graph G is 3-connected iff it has a 3-witness.*

Proof. First suppose that G is 3-connected. Then $|G| \geq 4$. We show by induction on $|G|$ that G has a 3-witness. Suppose $|G| = 4$. If $xy \notin E(G)$ then $V(G) \smallsetminus \{x,y\}$ is a 2-set that separates x from y, a contradiction. So $G = K_4$, and $G_0 = K_4 = G$ is a 3-witness for G. Otherwise, $|G| \geq 5$. By Lemma 71, there exists an edge $xy \in E(G)$ such that $G \cdot xy$ is 3-connected. Since G is 3-connected, $d(x), d(y) \geq 3$. By induction, $G \cdot xy$ has a 3-witness $G_0, \dots, G_s$. So $G_0, \dots, G_s, G$ is a 3-witness for G.

Now suppose $G_0, \dots, G_s$ is a 3-witness for G. We show by induction on s that G is 3-connected. If $s = 0$ then $K_4 = G_0 = G$ is 3-connected. Otherwise, for some edge $xy \in E(G)$, both $G_{s-1} = G \cdot xy$ and $d_G(x), d_G(y) \geq 3$. By induction $G \cdot xy$ is 3-connected. Suppose for a contradiction that S is a 2-separator in G. If $S = \{x,y\}$ then v_{xy} is a cut vertex of $G \cdot xy$, a contradiction. So there is a component H of $G - S$ that contains at least one, say x, of x and y, and another component H'. If $y \in H$ then S separates v_{xy} from H' in $G \cdot xy$, a contradiction. Else $y \in S$. Since $N(x) \subseteq V(H) \cup S$ and $d_G(x) \geq 3 > |S|$, x has a neighbor v in H. Then $S' := S - y + v_{xy}$ separates v from H' in $G \cdot xy$, another contradiction. □

The last paragraph of the above proof is subtle. If $d_G(x) < 3$ then we could have $S = N(x)$, and $V(H) = \{x\}$. Now $H - x$ is not a component of $G \cdot xy - S'$ since it is empty.

Conjecture 74 (Lovasz). *There exists a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that for all $k \in \mathbb{Z}^+$ and $f(k)$-connected graphs G and all vertices $x, y \in V(G)$, there exists a partition $\{V_1, V_2\}$ of $V(G)$ such that $G[V_1]$ is an x,y-path and $G[V_2]$ is k-connected.*

HW 53. Show that Conjecture 74 is true in the case $k = 1$ with $f(1) := 3$. (Use Lemma 71; or modify the hint for HW 57, and use Theorem 76).

3.3. Menger's Theorem

Sometimes when proving a statement by induction it is easier to prove a stronger statement. This paradigm is called the *inventors paradox.* The reason this may work is that while more must be proved in the conclusion of the induction step, the induction hypothesis provides more on which to base the proof. The next proof is an excellent example of the inventors paradox.

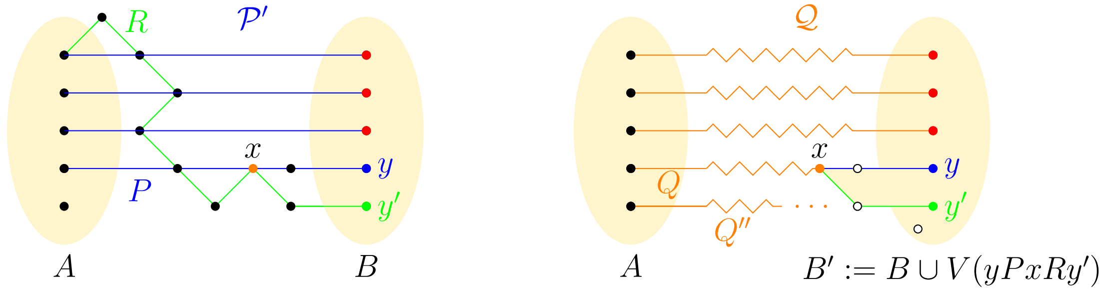


FIGURE 3.3.1. Left: A, B-connector $\mathcal{P}'$ and path R; Right: A, B'-connector $\mathcal{Q}$.

DEFINITION 75. Let $G = (V, E)$ be a graph and A and B be subsets of V. An A, B-*path* is a path P with exactly one end in A and exactly one end in B (possibly $|P| = 1$), and whose internal vertices are in neither A nor B. An A, B-*connector* is a set of disjoint A, B-paths. Let $l(A, B)$ be the maximum size of an A, B-connector. For an A, B-connector $\mathcal{P}$, let $\operatorname{end}(\mathcal{P}) := B \cap \bigcup \mathcal{P}$. An A, B-*separator* is a set of vertices S such that $G - S$ has no A, B-path. Here we allow S to contain vertices of $A \cup B$; for example, A is an A, B-separator. Let $k(A, B)$ be the minimum cardinality of an A, B-separator.

Here is another fundamental theorem.

THEOREM 76 (Menger [**15**] 1927 4.2.17). *Let $G := (V, E)$ be a graph with $A, B \subseteq V$. Then the size $l := l(A, B)$ of a maximum A, B-connector is equal to the size $k := k(A, B)$ of a minimum A, B-separator.*

PROOF. Let $\mathcal{P}'$ be any A, B-connector, and let S be any A, B-separator.

Since S contains at least one vertex of each path in $\mathcal{P}'$, we can define a function $f : \mathcal{P}' \to S$ by setting $f(P)$ equal to the first $x \in S \cap V(P)$. As the paths in $\mathcal{P}'$ are disjoint, each vertex of S is on at most one path in $\mathcal{P}'$. Thus f is an injection; so $|\mathcal{P}'| \leq |S|$. Hence $l \leq k$.

To show $k \geq l$, we assume $|\mathcal{P}'| < k$, and show that $\mathcal{P}'$ is not maximum; thus a maximum A, B-connector $\mathcal{P}$ satisfies $l = |\mathcal{P}| \geq k$. Using the Inventors Paradox, we actually prove more:

(*) if $|\mathcal{P}'| < k$ then there is an A, B-connector $\mathcal{P}$ such that (i) $|\mathcal{P}| = |\mathcal{P}'| + 1$ and (ii) $\operatorname{end}(\mathcal{P}') \subseteq \operatorname{end}(\mathcal{P})$.

Argue by induction on $|G - B|$. If $|G - B| = 0$ then $A \subseteq B$, so A is a minimum A, B-separator. Thus there is $a \in A \setminus \bigcup \mathcal{P}' \neq \emptyset$. Now $\mathcal{P} := \mathcal{P}' + (\{a\}, \emptyset)$ witnesses (*).

Otherwise $|G - B| > 0$. Now $\operatorname{end}(\mathcal{P}')$ is not an A, B-separator, since $|\operatorname{end}(\mathcal{P}')| = |\mathcal{P}'| < k$; so there is an A, B-path $R =: Ry'$ in $G - \operatorname{end}(\mathcal{P}')$. If $R \cap \bigcup \mathcal{P}' = \emptyset$ then $\mathcal{P} := \mathcal{P}' + R$ witnesses (*). Else, let x be the last vertex of R in $\bigcup \mathcal{P}'$; say $x \in P \in \mathcal{P}'$ and $P =: Py$. Then $y, y' \in B$ and $x \notin B$. See Figure 3.3.1. Put $B' := B \cup V(yPxRy')$. Then $|G - B'| < |G - B|$. As $\mathring{x}Ry' \cap \bigcup \mathcal{P} = \emptyset$, $\mathcal{Q}' := \mathcal{P}' - P + Px$ is an A, B'-connector with $\operatorname{end}(\mathcal{Q}') = \operatorname{end}(\mathcal{P}') - y + x$. As every A, B-path contains an A, B'-path, every A, B'-separator is an A, B-separator, so $k = k(A, B) \leq k(A, B')$.

By induction, there are an A, B'-connector $\mathcal{Q}$ and vertex $y'' \in B'$ such that $|\mathcal{Q}| - 1 = |\mathcal{Q}'| = |\mathcal{P}'|$ and $\operatorname{end}(\mathcal{Q}) = \operatorname{end}(\mathcal{Q}') + y''$. So $y'' \notin \operatorname{end}(Q')$. Let $Q, Q'' \in \mathcal{Q}$ with $Q := Qx$ and $Q'' := Q''y''$. Set $\mathcal{P}_0 := \mathcal{Q} - Q - Q''$. If $y'' \in xPy$ then set $\mathcal{P} := \mathcal{P}_0 + QxRy' + Q''y''Py$; if

$y'' \in xRy'$ then set $\mathcal{P} := \mathcal{P}_0 + QxPy + Q''y''Ry'$; else set $\mathcal{P} := \mathcal{P}_0 + QxPy + Q''$. Evidently, $\mathcal{P}$ witnesses (*). □

3.4. Applications of Menger's Theorem

DEFINITION 77. Let $G := (G, V)$ be a graph with distinct vertices a and b. A set $S \subseteq V \smallsetminus \{a, b\}$ *separates a from b* if there are no a,b-paths in $G - S$. For distinct nonadjacent vertices a, b, let $\kappa(a, b)$ be the minimum size of a set $S \subseteq V \smallsetminus \{a, b\}$ that separates a from b. Two a,b-paths are *independent* or *internally disjoint* if they have no inner vertices in common. Let $\lambda(a, b)$ be the maximum size of a set of independent a,b-paths.

If $ab \in E$ then no $S \subseteq V$ separates a from b. This is why $\kappa(a, b)$ is only defined when $ab \notin E$.

COROLLARY 78 (4.2.17). *If a and b are distinct nonadjacent vertices of a graph $G := (V, E)$ then $\lambda(a, b) = \kappa(a, b)$.*

PROOF. As a and b are nonadjacent, if $\mathcal{Q}$ is a set of independent a,b-paths then $\{\mathring{Q} : Q \in \mathcal{Q}\}$ is an $N(a), N(b)$-connector, and if $\mathcal{P}$ is an $N(a), N(b)$-connector then $\{aPb : P \in \mathcal{P}\}$ is a set of independent a,b-paths; also $S \subseteq V \smallsetminus \{a, b\}$ separates a from b if and only if S is an $N(a), N(b)$-separator. Thus by Theorem 76,

$$\lambda(a, b) = l(N(a), N(b)) = k(N(a), N(b)) = \kappa(a, b).$$ □

THEOREM 79 (4.2.21). *All graphs $G = (V, E)$ satisfy $\kappa(G) = t := \min_{a,b \in V, a \neq b} \lambda(a, b)$.*

PROOF. Choose a, b with $t = \lambda(a, b)$. If G is complete then

$$t = \lambda(a, b) = 1 + (|G| - 2) = |G| - 1 = \kappa(G),$$

since ab is an a,b-path, and acb is also an a,b-path for all $c \in V - a - b$.

Otherwise G has a separating set S with $|S| = \kappa(G)$. Let x, y be vertices in distinct components of $G - S$. Then

$$t = \lambda(a, b) \leq_{\text{def}} \lambda(x, y) \leq |S| = \kappa(G). \tag{3.4.1}$$

Now we show $t \geq \kappa(G)$. If $ab \notin E$ then by Corollary 78, there is a set U separating a from b with $t = |U| \geq \kappa(G)$. Else, $ab \in E$; set $G' = G - ab$. As ab is an a,b-path,

$$\lambda_G(a, b) = \lambda_{G'}(a, b) + 1 = \kappa_{G'}(a, b) + 1 \geq \kappa(G') + 1.$$

We will show $\kappa(G') + 1 \geq \kappa(G)$.

Pick a separating set W in G' with $|W| = \kappa(G')$. If $W + a$ or $W + b$ is a separating set of G then we are done; else each component of $G' - W$ contains an end of ab and has size 1. Now $V = S \cup \{a, b\}$, so

$$\kappa(G) \leq |G| - 1 = |W| + 1 = \kappa(G') + 1 = t.$$ □

DEFINITION 80 (4.2.22). Let $G = (V, E)$ be a graph with $x \in V$ and $U \subseteq V$. An x,U-*fan* is a set $\mathcal{F}$ of x,U-paths such that $|\mathcal{F}| = |U|$ and $F \cap F' = \{x\}$ for all distinct $F, F' \in \mathcal{F}$.

THEOREM 81 (4.2.23). *A graph $G := (V, E)$ is k-connected if and only if (*) $|G| \geq k + 1$ and G has an x,U-fan for all $x \in V$ and all k-sets $U \subseteq V - x$.*

Proof. Suppose G is k-connected. By HW 50, $G^+ := G + K(z, U)$ is k-connected, where z is a new vertex. By Corollary 78, there are k-independent x, z-paths. Deleting z from each gives the desired fan.

Now suppose (*) holds. Then $\delta(G) \geq k$ (why?), and for all distinct x and y, there exist k independent x, y-paths (why?). Thus $\lambda(x, y) \geq k$. By Theorem 79,

$$\kappa(G) = \min_{x,y \in V, x \neq y} \lambda(x, y) \geq k. \qquad \square$$

Definition 82 (4.2.18). The line graph $H := L(G)$ of a graph $G := (V, E)$ with $E \neq \emptyset$ is defined by

$$V(H) := E \text{ and } E(H) := \left\{ ee' \in \binom{E}{2} : e \cap e' \neq \emptyset \right\}.$$

Definition 83. Let x and y be distinct vertices in a graph G. An x, y-edge cut is a set of edges F such that there are no x, y-paths in $G - F$; let $\kappa'(x, y)$ be the size of a minimum x, y-edge cut. Two x, y-paths are edge-disjoint if they have no common edges; let $\lambda'(x, y)$ be the maximum size of a set of edge-disjoint x, y-paths.

Theorem 84 (4.2.19). *Let $G := (V, E)$ be a graph with distinct vertices $x, y \in V$. Then $\kappa'(x, y) = \lambda'(x, y)$.*

Proof. Set $G' := G + x' + xx' + y' + yy'$, where x' and y' are new vertices. Then $\kappa'_G(x, y) = \kappa'_{G'}(x, y)$ and $\lambda'_G(x, y) = \lambda'_{G'}(x, y)$. A set of edges disconnects x from y in G' iff the corresponding set of vertices separates xx' from yy' in $L(G')$. Moreover, edge disjoint x, y-paths in G' correspond to independent xx', yy'-paths in $L(G')$. Thus Corollary 78 yields $\kappa_{L(G')}(xx', yy') = \lambda_{L(G')}(xx', yy')$, and so

$$\kappa'_G(x, y) = \kappa'_{G'}(x, y) = \kappa_{L(G')}(xx', yy') = \lambda_{L(G')}(xx', yy') = \lambda'_{G'}(x, y) = \lambda'_G(x, y). \qquad \square$$

Theorem 85 (HW 4.2.24). *Let $G := (V, E)$ be a k-connected graph with $k \geq 2$. Then for any k-set $S \subseteq V$ there is a cycle $C \subseteq G$ with $S \subseteq V(C)$.*

Proof. Let $C \subseteq G$ be a cycle containing as many vertices of S as possible. It exists because $\delta(G) \geq \kappa(G) \geq 2$. We claim that $S \subseteq V(C)$. Otherwise, let $v \in S \setminus V(C)$. Then $|S \cap V(C)| < k$. Arguing by contradiction, it suffices to find a cycle containing $S \cap V(C) + v$.

Orient C cyclically as $\vec{C}$. Let $t = \min\{k, |C|\}$. As $\kappa(G) \geq k$, we also have $k(C, N(v)) \geq t$. By Menger's Theorem there is a set $\mathcal{P}$ of t disjoint $C, N(v)$-paths; together with v they form a fan $\mathcal{F}$. Set $F = \bigcup \mathcal{F}$, let $x_1, \ldots, x_t$ be a sequence of the leaves of F in cyclic order around $\vec{C}$, and set $\vec{P_i} = x_i \vec{C} x_{i+1}$. Then there exists $i \in [k]$ such that $\vec{P_i}$ contains no internal vertices from S: If $t = |C|$ this is true for all $i \in [k]$; otherwise it follows by the pigeonhole principle, since $t = k > |S \cap V(C)|$. So $D = x_{i+1} \vec{C} x_i F v F x_{i+1}$ is a cycle containing $S \cap V(C) + v$. $\square$

HW 54. (B) Prove: Every 2-connected graph G has a cycle C with $|C| \geq \min\{|G|, 2\kappa(G)\}$.

HW 55. (*) If G is a 2-connected graph with $\alpha(G) \leq \kappa(G)$ then G is hamiltonian.

HW 56. (+) Let G be a 2-connected graph that does not induce $K_{1,3}$. Then G has a cycle of length at least $\min\{|G|, 4\kappa(G)\}$.

HW 57. (Q) Prove Conjecture 74 is true in the case $k = 2$ with $f(2) := 5$ (or if you want you may use some other $f(2) \leq 10$). [Hint: Choose an induced x, y-path P so that $G - P$ contains the biggest 2-connected subgraph possible.]

HW 58. (B,*) Let $G = (V, E)$ be a graph with $x \in V$ and $Y, Z \subseteq V$ and $k = |Y| = |Z| - 1$. Suppose $\mathcal{Q} = \{Q_y : y \in Y\}$ is an x, Y-fan in G, where each Q_y is an x, y-path. Similarly, suppose $\mathcal{R} = \{R_z : z \in Z\}$ is an x, Z-fan in G, where each R_z is an x, z-path. Prove: There exists an $x, (Y + z)$-fan in G for some $z \in Z$. [Hint: Add new vertices w, whose neighborhood is Z, and v, whose neighborhood is $Y + w$; then Menger's Theorem.]

CHAPTER 4

Graph coloring

Definition 86. Let $G = (V, E)$ be a graph and C be a set (of colors). A *proper C-coloring* of G is a function $f : V \to C$ such that for all vertices $x, y \in V$ if $xy \in E(G)$ then $f(x) \neq f(y)$. For $k \in \mathbb{N}$, a *proper k-coloring* is a proper coloring $f : V \to C$ with $|C| = k$. The *chromatic number* $\chi(G)$ is the least k such that G has a proper k-coloring. When $\chi(G) = k$, we say that G is *k-chromatic*; when $\chi(G) \leq k$, we say that G is *k-colorable*. For $i \in C$, $f^{-1}(i)$ is called a color class. Notice that $f^{-1}(i)$ is a coclique. If f is a $[k]$-coloring of G then $\{f^{-1}(i) : i \in [k]\}$ is a partition of G into cocliques. Moreover, if $\{U_i : i \in k\}$ is a partition of V into cocliques then setting $f(x) = i$ if $x \in U_i$ yields a proper k-coloring of G. In this chapter we will assume that all colorings are proper unless otherwise stated.

Proposition 87 (HW). *Every graph G satisfies $\omega(G), \frac{|G|}{\alpha(G)} \leq \chi(G) \leq \Delta(G) + 1$.*

HW 59. (B) Prove Proposition 87.

4.1. Examples

Clearly, $\chi(K_t) = t$. Thus $\omega(G) \leq \chi(G)$ for all graphs G. However, the following example shows that $\chi(G)$ cannot be upper bounded in terms of $\omega(G)$.

Example 88 (5.2.3 Mycielski [1955]). For every $k \in \mathbb{N}$ with $k \geq 2$ there exists a graph G_k with $\omega(G_k) = 2$ and $\chi(G_k) = k$.

Proof. First we recursively construct $G_2, G_3, G_4 \dots$. Let $G_2 = K_2$. For $k \geq 3$, suppose $G_{k-1} = (V_{k-1}, E_{k-1})$ has been constructed. Define $G_k = (V_k, E_k)$ as follows: Let $V'_{k-1} = \{v' : v \in V_{k-1}\}$ be a set of new vertices, x_k be a new vertex, and put

$$V_k = V_{k-1} \cup V'_{k-1} + x_k \text{ and}$$

$$E_k = E_{k-1} \cup \{uv' : uv \in E_{k-1}\} \cup \{x_k v' : v' \in V'_{k-1}\}.$$

So $N(v') \cap V_{k-1} = N(v) \cap V_{k-1}$ for all $v \in V_{k-1}$. See Figure 4.1.1.

Suppose $\omega(G_k) \geq 3$, and choose $Q = K_3 \subseteq G_k$. Then $k \geq 3$. Since $N(x_k) = V'_{k-1}$ is independent, $x_k \notin Q$, and $|V'_{k-1} \cap Q| \leq 1$. As $\omega(G_{k-1}) =_{i.h.} 2$, there is at least one $v' \in V'_{k-1} \cap Q$. Now $N(v') \cap Q = N(v) \cap Q$, so $Q - v' + v$ is a K_3 in G_{k-1}, a contradiction.

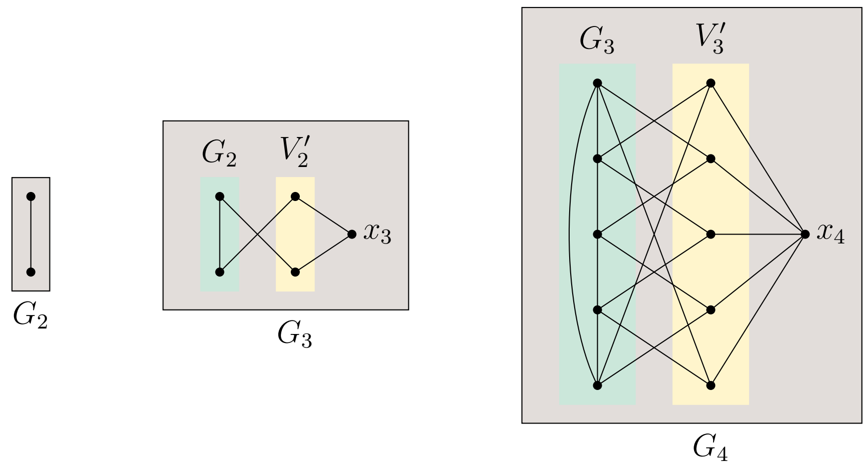


FIGURE 4.1.1. Mycielski's graphs: $\omega(G_i) = 2$ and $\chi(G_i) = i$.

Now $\chi(G_k) \leq k$: If $k = 2$ this is obvious; else G_{k-1} has a $(k-1)$-coloring f' by induction. Extend f' to a k-coloring f of G_k by setting $f(v') = k$ (new color) for all $v' \in V'$, and $f(x_k) = 1$.

Finally, $\chi(G_k) \geq k$: If $k = 2$ this is obvious. If $k \geq 3$ then it suffices to show that every $(k-1)$-coloring g of $G_k - x_k$ satisfies $g(V'_{k-1}) = [k-1]$, since then x_k will require a new color. Suppose not. After possibly renaming color classes, suppose $k - 1 \in [k-1] \smallsetminus g(V'_{k-1})$. For a contradiction, we construct a $(k-2)$-coloring h of G_{k-1}. Define $h : V_{k-1} \to [k-2]$ by:

$$h(v) = \begin{cases} g(v) & \text{if } g(v) \neq k-1 \\ g(v') & \text{if } g(v) = k-1 \end{cases}.$$

If $uv \in E_{k-1}$ then $g(u) \neq g(v)$ since g is proper; if $g(u) \neq k-1 \neq g(v)$ then

$$h(u) = g(u) \neq g(v) = h(v);$$

else exactly one of u, v is colored with $k-1$ by g; say $g(v) = k-1$. Since $uv' \in E_k$,

$$h(u) = g(u) \neq g(v') = h(v). \qquad \square$$

HW 60. (*) For $k \geq 2$, let G_k be the graph in Example 88. Prove: G_k is critical, i.e., $\chi(G_k - e) < \chi(G_k)$ $(= k)$ for all $e \in E_k$. [Hint: Using induction, consider three types of edges.]

HW 61. (*) Let $P = \{v_1, \ldots, v_n\}$ be a path, and suppose $G = (V, E)$ is a graph such that V is a subset of the set of subpaths of P, and $E = \{RQ \in V : R \cap Q \neq \emptyset\}$. Prove: $\chi(G) = \omega(G)$. [Hint: Order V so that if $Q, R \in V$ (subpaths of P), the first vertex of Q is q, the first vertex of R is r and $q < r$, then Q precedes R. Then color V in this order.]

HW 62. (*) Let $G = (V, E)$ be a k-colorable graph, and let P be a set of vertices such that (*) the distance $d_G(x, y)$ between any two vertices $x, y \in P$ is at least 4. Prove: Any $[k+1]$-coloring of $G[P]$ can be extended to a $[k+1]$-coloring of G. Give a counterexample with four vertices if in (*) 4 is replaced by 2.

4.2. Brooks' Theorem

In this section we work hard to improve the easy upper bound of Proposition 87 by one.

Definition 89. Call a graph Q a *b-barrier* if $Q = K_b$ or both $b = 3$ and Q is an odd cycle. Let (non-standard) $\omega^*(G)$ be the maximum b such that G has a b-barrier.

Every b-barrier is $(b-1)$-regular. Moreover, if Q is an $\omega^*(G)$-barrier then

$$\omega^*(G) = \omega^*(Q) = \chi(Q) \leq \chi(G) \leq \Delta(G) + 1. \tag{4.2.1}$$

Theorem 90 (Brooks [4] 1941). *For all graphs G, if $\omega^*(G) \leq \Delta(G)$ then $\chi(G) \leq \Delta(G)$.*

Proof. Set $\Delta = \Delta(G)$. Suppose $\omega^*(G) \leq \Delta$; then $\Delta \geq 2$. Argue by induction on $|G|$. If $\Delta = 2$ then $\omega^*(G) \leq 2$, so G has no odd cycle; by Theorem 23, $\chi(G) = 2$. Else $\Delta \geq 3$.

Suppose $D := K_{\Delta+1} - uv \subseteq G$. Set $D' := G - D$. Consider $x \in D \smallsetminus \{u, v\}$ and $w \in \{u, v\}$. Then $\|x, D\| = \Delta$, and $\|w, D\| = \Delta - 1$. As $\Delta(G) \leq \Delta$, $\|x, D'\| = 0$, and $\|w, D'\| \leq 1$. Thus there are $u', v' \in V(D')$ such that $[D, D'] \subseteq \{uu', vv'\}$. By induction (and Proposition 87), D' has a Δ-coloring f. As $\Delta \geq 3$, we can extend f to a Δ-coloring of G by first coloring u and v with $\alpha \in [\Delta] \smallsetminus \{f(u'), f(v')\}$, and then greedily coloring $D - u - v$.

Else (*) $K_{\Delta+1} - uv \nsubseteq G$. Let $M \subseteq V$ be a maximal coclique. Then $\|v, M\| \geq 1$ for all $v \in V \smallsetminus M$. Put $G' := G - M$. Now $\Delta(G') \leq \Delta - 1$ and $\chi(G) \leq \chi(G') + 1$. We are done if $\chi(G') \leq \Delta(G')$; else, by induction, there is a Δ-barrier $Q \subseteq G'$. As Q is $(\Delta - 1)$-regular, each $v \in V(Q)$ has a unique neighbor $v^* \in M$. If $V(Q) \subseteq N(u)$ for some $u \in M$, then $|Q| \leq \Delta$, so $Q = K_\Delta$; but then $G[Q + u] = K_{\Delta+1}$, a contradiction. Thus there are $x, y \in Q$ with $x^* \neq y^*$. As Q is connected, we can pick x and y with $xy \in E(Q)$. As Q is a clique or cycle, Q has a spanning x, y-path P.

Put $G^* := G + x^*y^* - Q$. Now $\Delta(G^*) \leq \Delta$: x^* and y^* have gained and lost one neighbor. By (*), $\omega(G^*) \leq \Delta$. By induction (and Proposition 87), G^* has a Δ-coloring f; note that $f(x^*) \neq f(y^*)$. Extend f to a Δ-coloring of G by first coloring x with $f(y^*)$, and then coloring greedily along P. This is possible since each inner vertex has an uncolored neighbor when it is colored, and y has two neighbors x and y^* with the same color. □

Conjecture 91 (Borodin & Kostochka [3] 1977). *If a graph G satisfies $8, \omega(G) < \Delta(G)$ then $\chi(G) < \Delta(G)$.*

Reed [20] used sophisticated probabilistic methods to prove the conjecture for $\Delta(G) > 10^{14}$.

HW 63. (*) For a graph G let $\theta(G) = \max_{uv \in E(G)}(d(u) + d(v))$. Prove: For $r \in \mathbb{N}$, if $\theta(G) \leq 2r + 1$ then $\chi(G) \leq r + 1$. Also prove that if $\theta(G) \leq 2r$ and $\omega^*(G) \leq r$ then $\chi(G) \leq r$. [Hint: Using induction, what vertex should you delete?]

4.3. Turán's Theorem

Let $n, s \in \mathbb{Z}^+$. In this section we determine the number of edges a graph on n vertices must have to ensure it contains K_s. In other words, how many edges can we put into a graph on n vertices without getting K_s.

Definition 92. A graph is said to be *r-partite* if it is r-colorable. Saying r-partite instead of r-colorable tends to emphasize the partition into r independent sets provided

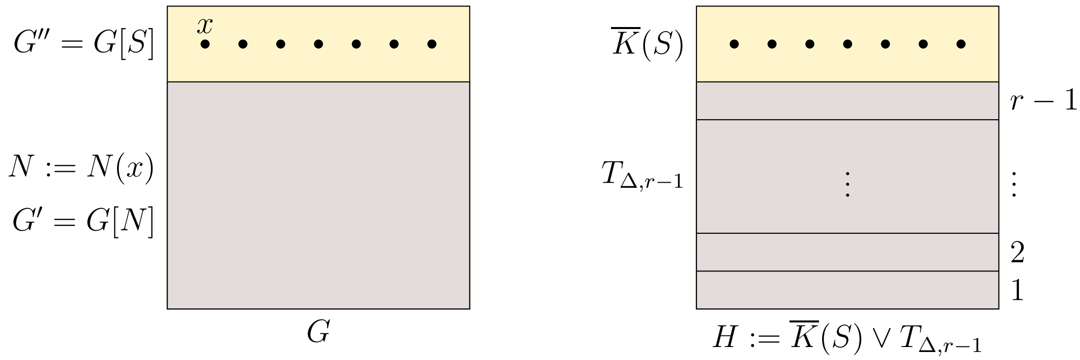


Figure 4.3: Turán's Theorem

by the r-coloring. These independent sets are called *parts.* The *complete r-partite graph* $K_{n_1,\ldots,n_r}$ is the r-partite graph with r parts of sizes $n_1, \ldots, n_r$ such that any two vertices in different parts are adjacent. The *Turán graph* $T_{n,r}$ is the complete r-partite graph on n vertices such that any two parts differ in size by at most one.

By Proposition 87, every r-partite graph G satisfies $\omega(G) \leq r$.

Lemma 93 (5.2.8). *The Turán graph $T_{n,r}$ has more edges than any other r-partite graph on n vertices.*

Proof. Let G be an r-partite graph on n vertices with as many edges as possible; say $\mathcal{X}$ is an r-partition of G. Clearly, G is a complete r-partite graph. So, if $G \neq T_{n,r}$ then there exist parts $X, Y \in \mathcal{X}$ with $|X| - |Y| \geq 2$ and $x \in X$. Let G' be the complete r-partite graph with r-partition $\mathcal{X}' := \mathcal{X} - X - Y + (X - x) + (Y + x)$. Then

$$E(G') \supseteq E(G) - \{xy : y \in Y\} + \{xx' : x' \in X - x\}.$$

Thus

$$\|G'\| \geq \|G\| - |Y| + |X| - 1 \geq \|G\| + 1,$$

a contradiction. So $G \cong T_{n,r}$. $\square$

Theorem 94 (5.2.9 Turán [1941]). *The Turán graph $T_{n,r}$ has more edges than any other graph G with $|G| \leq n$ and $\omega(G) \leq r$.*

Proof. We will show by induction on r that if $G = (V, E)$ is a graph with $|G| = n$, $\omega(G) \leq r$, and $\|G\| \geq \|T_{n,r}\|$ then $G \cong T_{n,r}$. If $r = 1$ then $G \cong \overline{K}_n \cong T_{n,1}$; so suppose $r > 1$.

Choose $x \in V$ with $d(x) = \Delta := \Delta(G)$. Set $N := N(x)$, $G' := G[N]$, $S := V - N$ and $G'' := G[S]$. Then $|G'| = \Delta$, and $\omega(G') \leq r - 1$, since $K + x$ is a clique in G for every clique

K in G'. Set $H := T_{\Delta,r-1} \vee \overline{K}(S)$. Then H is an r-partite graph on n vertices. So

$$\begin{aligned}
\|G\| &= \|G'\| + \|G''\| + \|N, S\| \\
&= \|G'\| + \sum_{v\in S} d_G(v) - \|G''\| && \text{(double counting)} \\
&\le \|T_{\Delta,r-1}\| + \sum_{v\in S} d_G(v) - \|G''\| && \text{(induction)} \quad (4.3.1) \\
&\le \|T_{\Delta,r-1}\| + \Delta|S| && \text{(maximum degree)} \quad (4.3.2) \\
&= \|H\| \\
&\le \|T_{n,r}\| && \text{(Lemma 93)} \quad (4.3.3)
\end{aligned}$$

Inequality (4.3.1) is strict unless $G' \cong T_{\Delta,r-1}$. Inequality (4.3.2) is strict unless $G'' = \overline{K}(S)$ and $G = G' \vee G''$. Inequality (4.3.3) is strict unless $H \cong T_{n,r}$. If $\|G\| \ge \|T_{n,r}\|$ then all three inequalities are tight, and so

$$G \cong T_{\Delta,r-1} \vee \overline{K}(S) = H \cong T_{n,r}. \qquad \square$$

HW 64. (B) Prove that if $\omega(G) \le r$ then $\|G\| \le (1-1/r)|G|^2/2$. [Hint: Set $|G| =: n =: rs + l$, $l := n \mod r$. The case $l = 0$ is easy (why?), so suppose $l > 0$. Let $T = (W, F) := T_{n',r}$, where $n' := r(s+1)$, Q be an r-clique in T, and $Q' \subseteq Q$ with $|Q'| = r - l$. Consider $\mu : W \to \mathbb{R}^+$; for $wx \in F$ let $\mu(wx) = \mu(w)\mu(x)$, and put $\mu(T) = \sum_{e\in F} \mu(e)$. Prove:

$$\text{If } \mu(w) = \begin{cases} 1 & \text{if } w \in W \setminus Q \\ l/r & \text{if } w \in Q \end{cases}, \text{ then } \mu(T) = (1-1/r)|G|^2/2;$$

$$\text{if } \mu(w) = \begin{cases} 1 & \text{if } w \in W \setminus Q' \\ 0 & \text{if } w \in Q' \end{cases}, \text{ then } \mu(T_{n,r}) = \|T_{n,r}\|.$$

Using Turan's Theorem and cancelation, prove that $\|G\| \le \mu(T_{n,r}) \le \mu(T)$.]

HW 65 (*). Let G be a graph with $\frac{|G|+1}{2} \le \delta(G) \le \Delta(G) \le |G| - 2$. Prove: (1) Every edge of G is contained in a K_3; (2) G contains two disjoint K_3's; and (3) for all n there is a graph H with $n \le \frac{|H|+1}{2} \le \delta(H)$ that has no two disjoint K_3's. [Hints: (1) Use $\delta(G)$. For (2), consider the following cases: (i) $|G| \le 6$; (ii) G is regular—use HW 64; and (iii) G is not regular: consider a high-degree vertex x and a vertex $y \notin N(x)$; similarly to the argument for (1), show that x and y are in disjoint triangles. For (3), you must construct a graph that does not satisfy the hypothesis of (2); use Turán's Theorem for guidance.]

4.4. Edge Coloring

Definition 95. Let $G = (V, E)$ be a graph. A proper k-edge-coloring of G is a function $f : E \to C$ with $|C| = k$ such that $f^{-1}(i)$ is a matching for all $i \in C$. The *chromatic index* $\chi'(G)$ of G is the least k such that G has a proper k-edge-coloring. In this section we will assume that all edge colorings are proper. Note that this is not the case when we consider Ramsey Theory.

HW 66. (B) Let P be the Petersen graph and $v \in V(P)$. Determine $\chi'(P - v)$. [Hint: This requires matching lower and upper bounds.]

THEOREM 96 (7.1.17 König [1916]). *Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$.*

PROOF. Clearly $\Delta(G) \leq \chi'(G)$. Now we show by induction on $\Delta := \Delta(G)$ that $\chi'(G) \leq \Delta$. The base step $\Delta = 0$ is trivial since G has no edges to color, so suppose $\Delta \geq 1$. It suffices to find a Δ-regular, bipartite multigraph H with $G \subseteq H$: Then H has a perfect matching M by Corollary 53. Color all edges in $M \cap E(G)$ with color Δ, and set $G' = G - M$. Then $\Delta(G') \leq \Delta(H - M) = \Delta - 1$, and so by induction, G' has a $(\Delta - 1)$-edge-coloring. Together with M this gives a Δ-edge-coloring of G.

It remains to construct H. Suppose G is an A, B-bigraph with $|A| \leq |B|$. Let $A \subseteq A'$, where $A' \cap B = \emptyset$ and $|A'| = |B|$. Choose an A', B-bipartite multigraph $H \supseteq G$ with $\|H\|$ maximal subject to $\Delta(H) \leq \Delta$. It exists because $G \cup A'$ is a candidate. Now

$$\sum_{v \in A'} d_H(v) = \|H\| = \sum_{v \in B} d_H(v) \leq \Delta |B|.$$

If $\|H\| = \Delta|B|$ then H is Δ-regular. Else there are vertices $a \in A'$ and $b \in B$ with $d_H(a), d_H(b) < \Delta$. Then $H' := H + e$, where $e \in E(a, b)$ is a new, possibly parallel edge, contradicts the maximality of the multigraph H. □

Notice how useful it was to consider multigraphs instead of graphs in the previous proof. Instead, if we wanted to add an edge ab that already existed in H, we could have added a new $K_{\Delta,\Delta} - a'b'$ together with edges ab' and $a'b$. Would this process end? Yes, because each such step would reduce the sum $\sum_{v \in V}(\Delta - d_H(v))$. But the multigraph route is far simpler.

Next we consider edge coloring general graphs. First we introduce the important technique of Kempe swaps. Consider a graph $G := (V, E)$ with a proper, *partial,* edge coloring $f : E_0 \to C$ that we are trying to extend to all of G. We call an edge $e \in E$ an α-edge, if $f(e) = \alpha$. For colors $\alpha, \beta \in [k]$, let $G_{\alpha,\beta}$ be the graph formed from the α- and β-edges of G; thus $G_{\alpha,\beta} := (V, E_{\alpha,\beta})$, where $E_{\alpha,\beta} := f^{-1}(\{\alpha, \beta\})$. Now $\Delta(G_{\alpha,\beta}) \leq 2$. Suppose H is a component of $G_{\alpha,\beta}$. By Proposition 48, H is a path or cycle. Let $y \in V(H)$. Switching the colors α and β on the edges of H yields a new, proper, partial, k-edge-coloring called an α, β, x*-Kempe swap* that is denoted by $f_{\alpha,\beta,y}$.

Suppose the edge xy is uncolored, and we would like to extend f by coloring xy with $\alpha \in C$. This is only possible if (*) α appears on no edge incident to x or y. In other words, $\alpha \in \bar{f}(x) \cap \bar{f}(y)$, where $\bar{f}(v) := C \smallsetminus f(E(v))$ for all $v \in V$. Suppose (*) fails; then $\bar{f}(x) \subseteq f(E(y))$. If $\alpha \in \overline{f}(x)$ and $\beta \in \overline{f}(y)$, and x is not in the component of $G_{\alpha,\beta}$ containing y, then $\alpha \in \overline{f}_{\alpha,\beta,y}(x) \cap \overline{f}_{\alpha,\beta,y}(y)$, so we can extend $f_{\alpha,\beta,y}$ by coloring xy with α.

When $\alpha \in f(E(y))$ and $\beta \in \bar{f}(y)$, the component of $G_{\alpha,\beta}$ containing y is a path $P = y \dots z$ starting at y. We call it the *Kempe path* of $f_{\alpha,\beta,y}$, and denote it by $P_{\alpha,\beta,y}$.

PROPOSITION 97. *Suppose $G = (V, E)$ is a graph with vertex y, f is a partial, k-edge-coloring of G, $\alpha \in f(E(y))$ and $\beta \in \overline{f}(y)$. Then $f_{\alpha,\beta,w}(e) = f(e)$ for all $e \in E$, except that $\{f_{\alpha,\beta,w}(e)\} = \{f(e)\} \triangle \{\alpha, \beta\}$ for $e \in P$; and $\bar{f}_{\alpha,\beta,w}(z) = \bar{f}(z)$ for all $z \in V$, except that $\bar{f}_{\alpha,\gamma,w}(z) = \bar{f}(z) \triangle \{\alpha, \gamma\}$ for $z \in \{w, t\}$.*

Now we prove Vizing's fundamental result, Theorem 99. The next lemma provides the induction step for its proof and has its own merit (see HW 68). For an element x and a set A, put $\iota(x, A) = 1$ if $x \in A$ and $\iota(x, A) = 0$ if $x \notin A$; write $\iota(x, y)$ for $\iota(x, \{y\})$.

LEMMA 98 (Vizing's Lemma). *Let $G = (V, E)$ be a loopless multigraph with $v \in V$ and $y \in N(v)$. For $k \in \mathbb{Z}^+$, if (i) $d(v) \le k$, (ii) $\chi'(G - v) \le k$ and (iii) $d(z) \le k - \mu(vz) + \iota(z, y)$ for all $z \in N(v)$, then $\chi'(G) \le k$.*

PROOF. Set $G' = (V', E') := G - v$, and argue by induction on k. By (i–iii), $\Delta(G) \le k$. If $k = 1$ then E is a matching, so $\chi'(G) \le 1$. Else $k > 1$. Let $f : V' \to C$ be a k-edge-coloring of G' with $|C| = k$. For $z \in N(v)$, $|\bar{f}(z)| = k - d_{G'}(z) = k - d_G(z) + \mu(vz)$; by cancelation,

$$d_G(z) \le k - \mu(vz) + \iota(z, y) \;\Leftrightarrow\; 2\mu(vz) - \iota(z, y) \le |\bar{f}(z)|. \tag{4.4.1}$$

By (iii) and 4.4.1, we can pick witness sets $\phi(z) \subseteq \bar{f}(z)$ with (iv) $|\phi(z)| = 2\mu(vz) - \iota(z, y)$; moreover the existence of these sets insures (iii). Set $\phi_\alpha := \{z \in N(v) : \alpha \in \phi(z)\}$. Then

$$\sum_{\gamma \in C} |\phi_\gamma| = \sum_{\gamma \in C} \sum_{z \in N(v)} \iota(\gamma, \phi(z)) = \sum_{z \in N(v)} \sum_{\gamma \in C} \iota(\gamma, \phi(z)) = \sum_{z \in N(v)} |\phi(z)|. \tag{4.4.2}$$

Case 1: There are $w \in N(v)$ and $\alpha \in C$ with $\emptyset \ne \phi_\alpha \subseteq \{y, w\}$. Pick $x \in \phi_\alpha$, preferring y, and $e \in E(v, x)$. Set $M := f^{-1}(\alpha) + e$ and $H := G - M$. Then M is a matching. So it suffices to show that $\chi'(H) \le k - 1$; this follows by induction if (i–iii) hold with G, y, k replaced by $H, y', k - 1$, where $y' = w$ if $x = y$ and $y' = y$ else. For (i), $e \in E(G) \smallsetminus E(H)$, so $d_H(v) \le k - 1$. For (ii), $f' := f \restriction H - v$ is a $(k - 1)$-coloring with range $C - \alpha$. For (iii), by 4.4.1, it suffices to check (*) $2\mu_H(vz) - \iota(z, y') \le |\phi(z) - \alpha|$, using $2\mu_G(vz) - \iota(z, y) \le |\phi(z)|$. If $z \notin \{y, w\}$ then $\alpha \notin \phi(z)$ and $\iota(z, y) = 0$, so (*) holds. If $z = x$ then $\mu_H(vx) = \mu_G(vx) - 1$, so (*) holds. Else $z = y'$. If $z = y' = y$, then $\alpha \notin \phi(z)$, and neither side of (*) changes; if $z = y' = w$ then $\iota(z, y') > \iota(z, y)$, so (*) holds.

Case 2: Not Case 1. Then $|\phi_\gamma| \ne 1$ for all $\gamma \in C$. Let $\beta \in \phi(y)$. There is a color $\alpha \in C$ with $\phi_\alpha = \emptyset$: else $2 \le |\phi_\gamma|$ for each of the k colors γ in C, which yields the following contradiction:

$$2d(v) \le_{(i)} 2k \le \sum_{\gamma \in C} |\phi_\gamma| =_{(4.4.2)} \sum_{z \in N(v)} |\phi(z)| =_{(iv)} \sum_{z \in N(v)} (2\mu(vz) - \iota(y, z)) = 2d(v) - 1.$$

Let g be the Kempe swap $f_{\alpha\beta, y}$ with Kempe path $P_{\alpha,\beta,y} = y \ldots z$. Then $|\phi(z) \cap \{\alpha, \beta\}| = 1$. After replacing $f, \phi(y), \phi(z)$ by $g, \phi(y) - \beta + \alpha, \phi(z) \triangle \{\alpha, \beta\}$, we are back to Case 1 with $y \in \phi_\alpha \subseteq \{y, w\}$, where $w = z$, if $z \in N(v)$, and w is any member of $N(v) - y$ else. $\square$

THEOREM 99 (Vizing (1964)). *Every multigraph G satisfies $\chi'(G) \le \Delta(G) + \mu(G)$.*

PROOF. Set $k := \Delta(G) + \mu(G)$, and argue by induction on $|G|$. If $|G| = 1$ then $\chi'(G) = 0 \le k$. Else pick $v \in V$. By induction, $\chi'(G - v) \le k$. As $\Delta(G) \le k - \mu(G)$, Lemma 98 implies $\chi'(G) \le k$. $\square$

HW 67 (B,L). Verify that in the proof of Lemma 98, $\Delta(G) \le k$. [Hint: Consider several cases of the degree of $x \in V = \{v\} \cup N(v) \cup (V' \smallsetminus N(v))$.]

HW 68 (B). Let $G = (V, E)$ be a graph, and set $X := \{v \in V : d(v) = \Delta(G)\}$. Prove: If $G[X]$ is acyclic then $\chi'(G) \le \Delta(G)$. [Hint: Use Lemma 98 and induction on $|X|$. What vertex should you remove?]

CONJECTURE 100 (Goldberg (1973), Seymour (1979)). *Every loopless multigraph M with $\chi'(M) \ge \Delta(M) + 2$ satisfies $\chi'(M) = \max_{H \subseteq M} \lceil \frac{\|H\|}{\lfloor |H|/2 \rfloor} \rceil$.*

Conjecture 100 may have just been proved; the paper is 75 pages. It has not yet been accepted for publication (added S 2022).

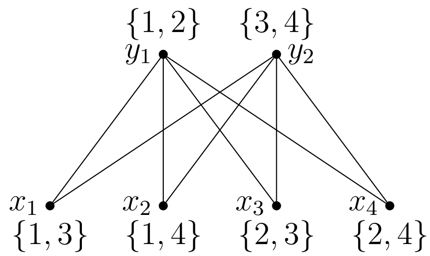


FIGURE 4.5.1. Example 104

HW 69 (B). Prove: Every loopless multigraph M satisfies $\chi'(M) \geq \max_{H\subseteq M}\lceil \frac{\|H\|}{\lfloor |H|/2 \rfloor}\rceil$.

DEFINITION 101 (4.2.18). The line graph $H = L(G)$ of a graph $G = (V, E)$ is defined by

$$V(H) = E \text{ and } E(H) = \{ee' : e \cap e' \neq \emptyset\}.$$

If H is the line graph (see Definition 82) of a simple graph G then H contains neither an induced copy of $K_{1,3}$ nor an induced copy of $K_5 - e$ (a K_5 missing one edge). Also, $\chi(H) = \chi'(G)$ and $\omega(H) = \Delta(G)$, unless $\Delta(G) = 2$ and $\omega(G) = 3$. So the following theorem (with an extra observation for the case $\Delta(G) = 2 < \omega(G)$) extends Vizing's Theorem for simple graphs.

THEOREM 102 (Kierstead & Schmerl 1983). *Every graph H that contains neither an induced copy of $K_{1,3}$ nor an induced copy of $K_5 - e$ satisfies $\chi(H) \leq \omega(H) + 1$.*

4.5. List Coloring

DEFINITION 103. Let $G = (V, E)$ be a graph and C a set of colors. We write 2^C for the power set of C. A *list assignment* for G is a function $L : V \to 2^C$. One should think of $L(v) \subseteq C$ as the set of colors that are available for coloring the vertex v. A *k-list assignment* is a list assignment L such that $|L(v)| = k$ for all $v \in V$. Given a list assignment L, an L-coloring is a proper coloring g such that $g(v) \in L(v)$ for all $v \in V$. In this case G is L-colorable. The graph G is *k-list-colorable* (also *k-choosable*) if for every k-list assignment L it is L-colorable. The *list-chromatic number* (also *choosability*, also *choice number*) $\chi_l(G)$ *of* G is the least k such that it is k-list colorable.

HW 70. (B) Prove: $\chi(G) \leq \chi_l(G) \leq \Delta(G) + 1$.

EXAMPLE 104. Let $G = K_{t,t^t}$. Then $\chi(G) = 2$, but $\chi_l(G) \geq t + 1$.

PROOF. Say $G = K(X, Y)$, where $|X| = t$. Let L be a t-list assignment for G such that the vertices of X have disjoint lists of size t, and for each $\sigma \in \prod_{x\in X} L(x)$ there exists $y_\sigma \in Y$ with $L(y_\sigma) = \text{range}(\sigma)$. Then for any L-coloring σ of $G[X]$, the vertex v_σ cannot be colored from the list $L(y_\sigma) = \text{range}(\sigma)$. □

HW 71. (*) Prove: For all graphs G, if $\omega^*(G) \leq \Delta(G)$ then $\chi_l(G) \leq \Delta(G)$. [Hint: Argue by induction on $|G|$. Assume G is connected (how?). Choose a vertex v and a component H of $G - v$ so that $|H|$ is minimum (if G has no cut vertex then $H = G - v$). Then H has

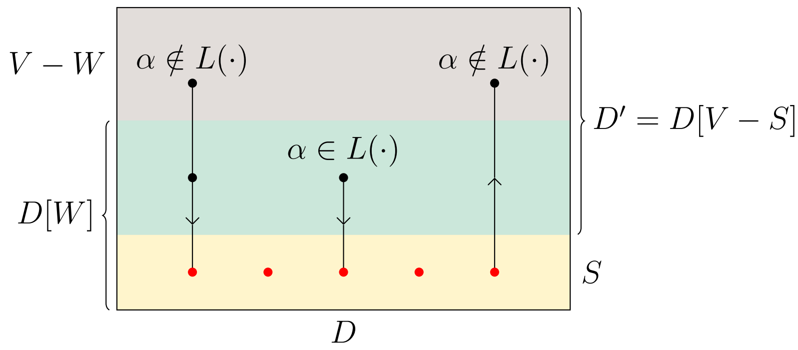


FIGURE 4.5.2. Lemma 4.5.2: $\alpha =$ red.

no cut vertex of G (why?). Fix a list assignment L, and consider two cases depending on whether or not there are two vertices $x, y \in H + v$ with $L(x) \neq L(y)$; in the first case find a good ordering of the vertices of H'; in the second case use Brooks' Theorem.]

DEFINITION 105. An *edge-list assignment* for G is a function $L : E \to 2^C$. One should think of $L(e) \subseteq C$ as the set of colors that are available for coloring the edge e. A *k-edge-list assignment* is an edge-list assignment L such that $|L(e)| = k$ for all $e \in E$. Given an edge-list assignment L, an L-coloring is a proper edge-coloring g such that $g(e) \in L(e)$ for all $e \in E$. In this case, G is L-list-colorable. The graph G is *k-edge-list-colorable* (also *k-edge-choosable*) if for every k-edge-list assignment L, it is L-colorable. The *list-chromatic index* (also *edge-choosability, edge-choice number*) $\chi'_l(G)$ *of* G is the least k such that it is k-edge-list colorable.

CONJECTURE 106. *Every graph G satisfies $\chi'_l(G) = \chi'(G)$.*

LEMMA 107 (8.4.29 Bondy & Boppana & Siegel). *Let $D = (V, A)$ be a digraph all of whose induced subgraphs have kernels. If L is a list assignment for D satisfying $d^+(v) < |L(v)|$ for all $v \in V$ then D has an L-coloring.*

PROOF. Argue by induction on $|D|$. Let $v_0 \in V$. Since $|L(v_0)| > d^+(v_0) \geq 0$, there is $\alpha \in L(v_0)$. Set $W = \{v \in V : \alpha \in L(v)\}$. Then $v_0 \in W$. By hypothesis $D[W]$ has a (nonempty) kernel S. Color every vertex in S with α. This is possible because S is independent and $S \subseteq W$. If $S = V$ we are done (base step).

Otherwise, it suffices to L-color $D' := D - S$ so that no vertex in $N_D(S)$ is colored α. For this purpose, let L' be the list assignment for D' defined by $L'(v) = L(v) - \alpha$. As $S \neq \emptyset$, we have $|D'| < |D|$, so by induction, it suffices to show $|L'(v)| > d^+_{D'}(v)$ for all $v \in V \smallsetminus S$.

If $v \notin W$ then $\alpha \notin L(v)$, and so

$$|L'(v)| = |L(v)| > d^+_D(v) \geq d^+_{D'}(v).$$

Else $v \in W$. Since S is a kernel of $D[W]$, there exists $w \in S = V \smallsetminus V(D')$ with $vw \in A$. So

$$|L'(v)| = |L(v) - \alpha| > d^+_D(v) - 1 \geq d^+_{D'}(v). \qquad \square$$

LEMMA 108 (Gale-Shapely). *Let $G = (V, E)$ be a nontrivial X, Y-bigraph. If $D = (E, A)$ is an orientation of the line graph of G such that for all $z \in V$, the subgraph $D[E_G(z)]$ induced by the edges incident to z is transitive, then D has a kernel.*

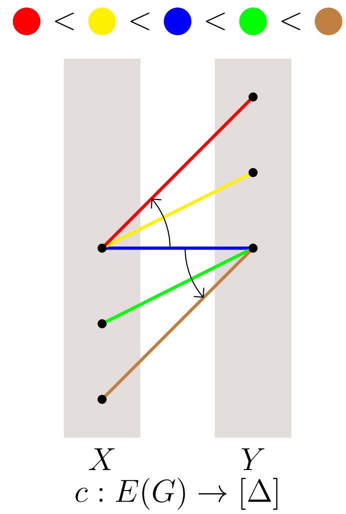


Figure 4.5.3. Galvin's Theorem: G is an X,Y-bigraph; the edge-coloring c is used to orient the edges of $H := L(G)$ so that each $D[E_G(z)]$ is transitive.

Proof. Argue by induction on $|D|$. For each $x \in X$ with $E_G(x) \neq \emptyset$, pick $e_x = xy \in E$ with $N_D^+(e_x) \cap E_G(x) = \emptyset$; this is possible since $D[E_G(x)]$ is transitive. Thus (*) $N_D^+(e_x) \subseteq E_G(y)$. Every $e \in E$ has an end $x \in X$, so $e_x \in N_D^+[e]$. If $Q := \{e_x : x \in X\}$ is independent then it is a kernel of D; else there are distinct $x, x' \in X$ with $e_x \cap e_{x'} \neq \emptyset$. As $x \neq x'$, there is $y \in Y$ with $e_x, e_{x'} \in E_G(y)$. Say $e_x e_{x'} \in A$. By induction, $D' := D - e_x$ has a kernel S. So there is $e^* \in N_D^+[e_{x'}] \cap S$. Then $e_{x'} = e^*$ or $e_{x'}e^* \in A$. Anyway, by (*), $e^* \in E_G(y)$, and as $D[E_G(y)]$ is transitive, $e^* \in N_D^+[e_x] \cap S$. So S is a kernel of D. $\square$

Theorem 109 (8.4.30 Galvin 1995). *Every X,Y-bigraph G satisfies $\chi'_l(G) = \Delta(G)$.*

Proof. Set $\Delta := \Delta(G)$, and let H be the line graph of G. Then $\chi'_l(G) = \chi_l(H)$. By Lemma 107, it suffices to find an orientation D of H with $\Delta^+(D) \leq \Delta - 1$ such that every induced subgraph of D has a kernel. By Theorem 96, there is a Δ-edge coloring $c : E(G) \to [\Delta]$. Each $e\vec{e}' \in E(H)$ satisfies $e \cap e' \subseteq X$ or $e \cap e' \subseteq Y$. Define an orientation $D = (E(G), A)$ of H by putting

$$e\vec{e}' \in A \text{ iff } (e \cap e' \subseteq X \wedge c(e) > c(e')) \vee (e \cap e' \subseteq Y \wedge c(e) < c(e')).$$

Then $D[E_G(z)]$ is transitive for all $z \in V(G)$. Every induced subgraph $D' \subseteq D$ is the line graph of a subgraph of G, and so has a kernel by Lemma 108. Each $e := xy \in E(G)$ satisfies $d_D^+(e) \leq \Delta - 1$, since all $e' \in N_D^+(e) \cap E_G(x)$ get distinct colors from $[c(e) - 1]$ and all $e'' \in N_D^+(e) \cap E_G(y)$ get distinct colors from $[\Delta] \smallsetminus [c(e)]$. Thus $\chi_l(H) = \Delta$ by Lemma 107. $\square$

CHAPTER 5

Planar graphs

We have been informally drawing graphs in the Euclidean plane $\mathbb{R}^2$ since the start of the semester. Now we formalize the definition of a drawing of a graph in $\mathbb{R}^2$.

5.1. Very Basic Topology of the Euclidean Plane

For $x \in \mathbb{R}^2$ the *open ball around x with radius r* is the set $B_r(x) := \{y \in \mathbb{R}^2 : \|x, y\| < r\}$. A set $U \subseteq \mathbb{R}^2$ is *open* if for all points $p \in U$ there exists $r > 0$ such that $B_r(p) \subseteq U$. In particular, $\mathbb{R}^2$ and $\emptyset$ are open. The complement of an open set is a *closed* set. The *frontier* of a set X is the set of all points $x \in \mathbb{R}^2$ such that $B_r(x) \cap X \neq \emptyset$ and $B_r(x) \setminus X \neq \emptyset$ for all $r > 0$. Note that if X is open, then its frontier lies in $\mathbb{R}^2 \setminus X$.

Let $p, q \in \mathbb{R}^2$. The p, q-line segment $L(p, q)$ is the subset of $\mathbb{R}^2$ defined by $L(p, q) := \{p + \lambda(q - p) : 0 \leq \lambda \leq 1\}$ and $\mathring{L}(p, q) := L(p, q) \setminus \{p, q\}$. For distinct points $p_0, \dots, p_k \in \mathbb{R}^2$, the union $A(p_0, \dots, p_k) := \bigcup_{i \in [k]} L(p_{i-1}, p_i)$ is a (polygonal) p_0, p_k-arc provided $L(p_{i-1}, p_i) \cap \mathring{L}(p_{j-1}, p_j) = \emptyset$ for all distinct $i, j \in [k]$; in this case, p_0 and p_k are *linked* by $A(p_0, \dots, p_k)$. If $A(p_0, \dots, p_k)$ is an arc and $A(p_0, \dots, p_k) \cap \mathring{L}(p_k, p_0) = \emptyset$ then $P(p_0, \dots, p_k, p_0) := A(p_0, \dots, p_k) \cup L(p_k, p_0)$ is a polygon. Note that arcs and polygons are closed in $\mathbb{R}^2$.

Let U be an open set. Two points $x, y \in U$ are linked in U if there exists an x, y-arc contained in U. The relation of being linked is an equivalence relation on U. Its equivalence classes are called *regions*. Regions are open: Suppose $R \subseteq U$ is a region and $x \in R$. Then there exists a $r > 0$ such that $B_r(x) \subseteq U$. Clearly every $y \in B_r(x)$ is linked to x in U, since $L(x, y) \subseteq B_r(x)$. So $B_r(x) \subseteq R$. A closed set X separates a region R if $R \setminus X$ has more than one region.

THEOREM 110 (Jordan Curve Theorem for Polygons). *For every polygon $P \subseteq \mathbb{R}^2$, the set $\mathbb{R}^2 \setminus P$ has exactly two regions. Each of these regions has the entire polygon as its frontier.*

PROOF. *To be continued ...* □

5.2. Graph Drawings

Recall that in this class all graphs are finite. Let $G = (V, E)$ be a (multi)graph. A *drawing* of G is a graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ such that $G \cong \widetilde{G}$, $\widetilde{V} \subseteq \mathbb{R}^2$, and each edge $\tilde{e} \in \widetilde{E}$ is an arc linking its ends. So edges are no longer just pairs of vertices, but have their own identity and structure (we need this anyway to formally deal with different edges linking the same two vertices). Clearly, every finite graph has a drawing. Moreover, by moving vertices slightly and readjusting edges, we can (and do) require the following additional properties for drawings without restricting the set of finite graphs that can be drawn.

(1) No three edges have a common internal point.
(2) The only vertices contained in an edge are its endpoints.

(3) No two edges are tangent.
(4) No two edges have more than one common internal point.

A crossing in a drawing of a (multi)graph is a point that is in the interior of two edges. A a drawing of a (multi)graph is *plane* if it has no crossing. A *plane* (multi)graph is a *plane* drawing of a (multi)graph. A *planar* (multi)graph is a (multi)graph that has a plane drawing.

5.3. Basic facts

Let $\widetilde{G} := (\widetilde{V}, \widetilde{E})$ be a plane (multi)graph. The faces of $\widetilde{G}$ are the regions of $\mathbb{R}^2 \smallsetminus (\widetilde{V} \cup \bigcup \widetilde{E})$. The frontier of a face f is called its *boundary*; it is denoted by $G[f]$. Let $F(\widetilde{G})$ be the set of faces of $\widetilde{G}$.

Fact 111 (B-Graph). *The boundary $\widetilde{G}[f]$ of a face of $\widetilde{G}$ is the union of some edges and vertices of $\widetilde{G}$—so $\widetilde{G}[f]$ is a subgraph of $\widetilde{G}$.*

Note that a plane cycle is a polygon. By the Jordan Curve Theorem we have:

Proposition 112 (B-Cycle). *A plane cycle is the boundary of exactly two faces.*

Fact 113 (B-Forest). *If there is a face $f \in F(\widetilde{G})$ with $\widetilde{G}[f]$ acyclic then $F(\widetilde{G}) = \{f\}$ and $\widetilde{G}[f] = \widetilde{G}$.*

Fact 114 (B-Edges). *Suppose $e \in E(\widetilde{G})$. If e is not a cut-edge then e is on the boundary of exactly two faces, so $|F(\widetilde{G} - e)| = |F(\widetilde{G})| - 1$. If e is a cut-edge then e is on the boundary of exactly one face, so $|F(\widetilde{G}) - e)| = |F(\widetilde{G})|$.*

For an edge e and face f, let $\iota(e, f) = 1$ if e is a non-cut-edge in $\widetilde{G}[f]$, $\iota(e, f) = 2$ if e is a cut-edge in $\widetilde{G}[f]$ and $\iota(e, f) = 0$ else. Define the *length* $l(f)$ of f by $l(f) = \sum_{e \in E(G[f])} \iota(e, f)$.

Proposition 115. $\sum_{f \in F(\widetilde{G})} l(f) = 2\|\widetilde{G}\|$.

Proof. By Fact B-Edges and the definition of ι, every edge $e \in E$ satisfies $\sum_{f \in F(\widetilde{G})} \iota(e, f) = 2$, so using the definition of length,

$$\sum_{f \in F(\widetilde{G})} l(f) = \sum_{f \in F(\widetilde{G})} \sum_{e \in E(\widetilde{G})} \iota(e, f) = \sum_{e \in E(\widetilde{G})} \sum_{f \in F(\widetilde{G})} \iota(e, f) = 2\|\widetilde{G}\|. \qquad \square$$

Fact 116 (+ Edges). *Let $f \in F(\widetilde{G})$, $Y := \{y_1, \ldots, y_t\} \subseteq V(\widetilde{G}[f]$ and $x \in f$.*

(1) $G' := G + y_i y_j$ is planar, and if $\widetilde{G}'$ is a drawing of G' then $|F(G')| = |\widetilde{G}| + i$ where $i = 0$ if if $y_i y_j$ is a cut-edge and $i = 1$ else.

(2) If $C := \widetilde{G}[f]$ is a cycle then we can draw $Q := K(x, Y)$ as $\widetilde{Q}$ so that $\mathring{\widetilde{Q}} \subseteq f$. In this case, the faces of $\widetilde{G} \cup \widetilde{Q}$ incident to v are bounded by cycles $v_i C v_{i \oplus 1} x v_i$.

Fact 117 (Contraction). *Suppose $e = xy$ is a link (non-loop-edge) of $\widetilde{G}$. Then there is a plane graph $\widetilde{G} \cdot e$ formed from $\widetilde{G}$ by deleting y and adding edges xv with $x\mathring{v}$ in the face of $\widetilde{G} - y$ containing y for all $v \in N(y) - x$, and $|F(\widetilde{G} \cdot e)| = |F(\widetilde{G})|$.*

Theorem 118 (6.1.21 Euler's Formula (1758)). *Every connected plane multigraph $\widetilde{G}$ satisfies*

$$|\widetilde{G}| - \|\widetilde{G}\| + |F(\widetilde{G})| = 2.$$

Proof. Argue by induction on $\|\widetilde{G}\|$. If $\|\widetilde{G}\| = 0$ then $|\widetilde{G}| = 1$ since G is connected. By Fact B-Forest, $|F(\widetilde{G})| = 1$, so

$$|\widetilde{G}| - \|\widetilde{G}\| + |F(\widetilde{G})| = 1 - 0 + 1 = 2.$$

Otherwise, $\|G\| \geq 1$. [Here are two proofs; the second one avoids contraction.] If $\in E(G)$ is a link then by Fact Contraction, there is a plane drawing $\widetilde{G'}$ of $G' := G \cdot e$ with $|F(\widetilde{G})| = |F(\widetilde{G'})|$. Also $|G| = |G'| + 1$ and $\|G\| = \|G'\| + 1$ (since we are doing contraction in multigraphs, some edges become parallel links of loops, but only e is lost). As $\widetilde{G'}$ is connected, induction yields:

$$|\widetilde{G}| - \|\widetilde{G}\| + |F(\widetilde{G})| = |\widetilde{G'}| + 1 - (\|\widetilde{G'}\| + 1) + |F(\widetilde{G'})| = |\widetilde{G'}| - \|\widetilde{G'}\| + |F(\widetilde{G'})| =_{ind} 2.$$

Else e is a loop. Then e is not a cut-edge. Now $\widetilde{G'} = \widetilde{G} - e$ is connected. By Fact B-Edges, $|F(\widetilde{G})| = |F(\widetilde{G'})| + 1$, so by induction

$$|\widetilde{G}| - \|\widetilde{G}\| + |F(\widetilde{G})| = |\widetilde{G'}| - (\|\widetilde{G'}\| + 1) + |F(\widetilde{G'})| + 1 = |\widetilde{G'}| - \|\widetilde{G'}\| + |F(\widetilde{G'})| =_{ind} 2. \quad \square$$

[Here is the second proof; I recommend this proof.] Argue by induction on $\|\widetilde{G}\|$. Suppose $\widetilde{G}$ is acyclic. By hypothesis, $\widetilde{G}$ is connected, so by Theorem 32, $|\widetilde{G}| = \|\widetilde{G}\| + 1$. Let f be a face of $\widetilde{G}$. Then $\widetilde{G}[f] \subseteq \widetilde{G}$ is acyclic. By Fact B-Forest, f is the only face of $\widetilde{G}$. Thus

$$(|G| - \|G\|) + |F(G)| = 1 + 1 = 2.$$

Else, there is a cycle $C \subseteq G$. Let $e \in C$. By Proposition 27, e is not a cut-edge, so $G - e$ is connected. By Fact B-Edges, $|F(\widetilde{G})| = |F(\widetilde{G'})| + 1$. By induction

$$|\widetilde{G}| - \|\widetilde{G}\| + |F(\widetilde{G})| = |\widetilde{G'}| - (\|\widetilde{G'}\| + 1) + |F(\widetilde{G'})| + 1 = |\widetilde{G'}| - \|\widetilde{G'}\| + |F(\widetilde{G'})| =_{ind} 2.$$

Theorem 119. *If G is a simple planar graph with $|G| \geq 3$ then $\|G\| \leq 3|G| - 6$. Moreover, if G has girth at least 4 then $\|G\| \leq 2|G| - 4$.*

Proof. Argue by induction on G (to avoid proving G is connected). If $|G| = 3$ then $G \subseteq K_3$, so $\|G\| \leq 3 = 3|G| - 6$. Suppose G is connected. As $|G| \geq 3$, $\|G\| \geq 2$. Let $\widetilde{G}$ be a drawing of G. By Fact B-Forest, either (i) every face boundary contains a cycle, or (ii) G has exactly one face whose boundary is G, and every edge is a cut edge. Let $f \in F(\widetilde{G})$. As G is simple, if (i) then $l(f) \geq \|\widetilde{G}[f]\| \geq 3$; if (ii) then and $l(f) = 2\|\widetilde{G}[f]\| = 2\,\|G\| \geq 4$. By Proposition 115,

$$2\|G\| = \sum_{f \in F(\widetilde{G})} l(f) \geq 3|F(\widetilde{G})|.$$

So $|F(\widetilde{G})| \leq \frac{2}{3}\|G\|$. As G is connected, Theorem 118 yields

$$2 = |G| - \|G\| + |F(\widetilde{G})| \leq |G| - \frac{1}{3}\|G\|,$$

and so $3|G| - 6 \geq \|G\|$.

Now suppose the girth of G is at least 4. Then $l(f) \geq 4$ for every face $f \in F(\widetilde{G})$. So

$$2\|G\| = \sum_{f \in F(\widetilde{G})} l(f) \geq 4|F(\widetilde{G})|.$$

Thus $|F(\widetilde{G})| \leq \frac{1}{2}\|G\|$. By Theorem 118,

$$2 = |G| - \|G\| + |F(\widetilde{G})| \leq |G| - \frac{1}{2}\|G\|,$$

and so $\|G\| \leq 2|G| - 4$. This completes the base step.

Finally, assume $|G| \geq 4$ and G is not connected; let H be a component of G. If there is $v \in V$ with $d(v) \leq 1$ then $|G - v| \geq 3$ and $\|G\| \leq \|G - v\| + 1$; else $\delta(G) \geq 2$, and so $|H| \geq 3$, $|G - H| \geq 3$ and $\|G\| = \|H\| + \|G - H\|$. Anyway, by induction, $\|G\| \leq 3|G| - 6$, and if the girth of G is at least 4 then $\|G\| \leq 2|G| - 4$. □

Corollary 120. *Neither K_5 nor $K_{3,3,}$ is planar.*

Proof. If K_5 is planar then by Theorem 119, we have the contradiction

$$10 = \|K_5\| \leq 3|K_5| - 6 = 9.$$

As $K_{3,3}$ is bipartite, its girth is at least 4. If it is planar then Theorem 119 yields the contradiction:

$$9 = \|K_{3,3}\| \leq 2|K_{3,3}| - 4 = 8.$$ □

HW 72. (B) Let G be a simple planar graph with girth at least k. Prove: $\|G\| \leq \frac{k}{k-2}(|G| - 2)$.

HW 73. (*) Prove: Every simple planar graph G with $|G| \geq 4$ has at least four vertices with degree less than six.

HW 74. (+) Prove: Every simple planar graph G with $\delta(G) = 5$ has a matching with at most $\frac{1}{5}|G|$ uncovered vertices. [Hint: Argue by induction on $|G|$, and use Tutte's Theorem 57, Corollary 119 & Corollary 120.]

HW 75. (B) Let $G = (V, E)$ be a simple planar graph. Prove: (i) $\delta(v) \leq 5$; and (ii) $\chi_l(G) \leq 6$. Now suppose G has girth at least 4. Prove: (iii) $\delta(G) \leq 3$; and $\chi_l(G) \leq 4$.

HW 76. (*) Prove: Every bipartite planar graph G satisfies $\chi_l(G) \leq 3$. [Hint: Use HW 18.]

5.4. Kuratowski's Theorem

A graph H is a *subdivision* of a graph G if H is formed by replacing some of the edges $xy \in E(G)$ by an x, y-path P whose internal vertices are not vertices of G and have degree 2 in H. The vertices of G are called branch vertices and the new vertices are called subdivision vertices. Observe that G is not a subgraph of a proper subdivision of itself. We write $H = TG$ to indicate that H is a subdivision of G, $H = TG(X)$ to indicate that X is the set of branch vertices of H and $H = TG(X, Y)$ to indicate the bipartition of the branch vertices of H, if G is bipartite. A *K-graph* (for Kuratowski) is a TK_5 or a $TK_{3,3}$.

Theorem 121 (6.2.2 Kuratowski (1930)). *A graph is planar iff it contains neither a subdivision of TK_5 nor a subdivision of $TK_{3,3}$ (that is, it does not contain a K-graph).*

We will divide the proof of Kuratowski's Theorem into several parts. The next Lemma proves necessity.

Lemma 122. *If G is planar then G does not contain a K-graph.*

Proof. First observe that if G is planar then all its subgraphs are planar. Arguing by contraposition, we assume there is a K-graph $Q \subseteq G$, and we show by induction on the number h of subdivision vertices in Q that Q is not planar. If $h = 0$ then this is Corollary 120. If $h > 0$, consider a subdivision vertex x and its two distinct neighbors y_1 and y_2. Contracting xy yields a K-graph with one less subdivision vertex. Thus by induction $Q \cdot xy$ is nonplanar. By Fact Contraction, Q is nonplanar, and so by the first observation, G is nonplanar. □

We will break the proof of sufficiency into two parts. Suppose G is a graph with no K-graph. First, we show that if G is 3-connected then G is planar; then we show that if G is not 3-connected then we can add edges to G until it is 3-connected without creating any K-graph. When G is 3-connected, we argue by induction. To apply the induction hypothesis, we contract a (special) edge $e := xy$ for which $G \cdot e$ is still 3-connected. The next lemma shows that $G \cdot e$ also has no K-graph.

Lemma 123. *Let $e := x_1x_2$ be an edge of a graph G. If G contains no K-graph then $G \cdot e$ contains no K-graph.*

Proof. Arguing by contraposition, suppose $Q \subseteq G \cdot e$ is a K-graph. If $Q \subseteq G$ we are done; else $v_e \in V(Q)$. Then $2 \leq d_Q(v_e) \leq 4$ and $N_Q(v_e) \subseteq N_G(x_1) \cup N_G(x_2)$. If there is $i \in [2]$ with $|N_Q(v_e) \smallsetminus N_G(x_i)| \leq 1$ then replacing v_e by x_i or the path x_1x_2 yields a subdivision of Q, and so a K-graph. Else, $d_Q(v_e) = 4$, $|N_Q(v_e) \smallsetminus N_G(x_1)| = 2 = |N_Q(v_e) \smallsetminus N_G(x_2)|$, and $Q = TK_5$; say $Q = TK_5(\{v_e, a, b, c, d\})$. Then $Q' := G[Q - v_e + x_1 + x_2)] \subseteq G$, and Q' contains a $TK_{3,3}(\{x_1, a, b\}, \{x_2, c, d\})$. □

In order to obtain a plane drawing of G from a plane drawing of $\widetilde{H}$ of $H := G \cdot e$, we observe that the 2-connected graph $\widetilde{H} - v_e$ is a drawing of $G - \{x, y\}$. We then use the next lemma to draw x, y, and their incident edges in the face of $\widetilde{H} - v_e$ that contains v_e.

Lemma 124. *Every face of a* 2-connected plane graph $\widetilde{G}$ is bounded by a cycle.

Proof. By Theorem 68, $\widetilde{G}$ has a plane 2-witness $\widetilde{P}_0, \ldots, \widetilde{P}_h$. Argue by induction on h. If $h = 0$, then $G := \widetilde{P}_0$ is a cycle. By Proposition 112, it is the face boundary of the only two faces of $\widetilde{G}$. Else $h \geq 1$. Let $\widetilde{P}_h$ have ends x, y, and note that $\mathring{\widetilde{P}}_h$ is contained in some face f' of $\widetilde{H} := \widetilde{G} - \mathring{\widetilde{P}}_h$. As $\widetilde{P}_0, \ldots, \widetilde{P}_{h-1}$ is a 2-witness, $\widetilde{H}$ is 2-connected. By induction, every face of $\widetilde{H}$ is bounded by a cycle. Let $\widetilde{C} = xv_1 \ldots v_a y v_{a+2} \ldots v_b x$ be the cycle bounding f'. By Fact + Edges, $F(\widetilde{G}) = F(\widetilde{H}) - f' + f_1 + f_2$ where f_1 and f_2 are bounded by cycles. □

Theorem 125. *Every* 3*-connected graph G that contains no K-graph is planar.*

Proof. Argue by induction on $|G|$. As K_4 is planar, assume $|G| \geq 5$. By Theorem 71, $H := G \cdot e$ is 3-connected for some edge $e = xy$. Put $X := N_G(x) - y$ and $Y := N_G(y) - x$. As $3 \leq \kappa(G), \kappa(H)$, we have $3 \leq d_H(v_e) \leq |X \cup Y|$, and (say) (*) $2 \leq |X| \leq |Y|$. By Lemma 123, H contains no K-graph. By induction, H has a plane drawing $\widetilde{H}$.

Set $\widetilde{H}' := \widetilde{H} - v_e$, and let f be the face of $\widetilde{H}'$ that contains v_e. Now $\widetilde{H}'$ is 2-connected. By Lemma 124, $C := \widetilde{H}'[f]$ is a cycle containing $X \cup Y$. Let $C := x_1P_1x_2P_2x_3 \ldots x_nP_nx_1$ where each P_i is an X-path in C with ends $x_i, x_{i \oplus 1}$ and $n = d_G(x) - 1 \geq 2$. Let $G' := G - y$, and obtain a drawing $\widetilde{G}'$ of G by starting with $\widetilde{H}$, identifying x with v_e, and deleting the interiors of all v_e, z-arcs with $z \in Y \setminus X$. By Fact + Edges, The faces of $\widetilde{G}'$ that are contained in f have the form f_i where $\widetilde{G}'[f_i] = xx_iP_ix_{i\oplus 1}x$, $i \in [n]$.

By (*), if $Y \subseteq X$ then $X = Y$, so G contains $TK_5(\{x, y, x_1, x_2, x_3\})$, a contradiction. Thus there is $y' \in Y \cap \mathring{P}_i$ with $i \in [n]$. If $Y \setminus X \subseteq P_i + x$ then by Fact + Edges, $\widetilde{G}'$ can be extended to a drawing of G with y and all edges in $E(y) \setminus E(x) + xy$ drawn in $f_i \cup \widetilde{G}'[f]$. Else, there is $y'' \in Y \setminus \widetilde{G}'[f_i]$. Now G contains $TK_{3,3}(\{y, x_i, x_{i+1}\}, \{x, y', y''\})$, a contradiction. □

A graph G is *K-edge-maximal* if G has no K-graph, but $G + e$ has a K-graph for all $e \in E(\overline{G})$. (So K_i with $i \in [4]$ is K-edge-maximal. We will show by induction on $|G|$ that if G is not 3-connected then we can add edges to G to get a 3-connected graph G' with no K-graph. The next lemma is the first step. We will refer to a path whose ends are branch vertices and whose inner vertices are subdivision vertices as a *subdivided edge*.

Lemma 126. *Let $G := (V, E)$ be a graph with a minimum separating set S, where $|S| \le 2$. Suppose $V = V_1 \cup V_2$, $S = V_1 \cap V_2$, $V_1, V_2 \ne S$ and $E(V_1 \setminus S, V_2 \setminus S) = \emptyset$. If G is K-edge-maximal then so are $G_1 := G[V_1]$ and $G_2 := G[V_2]$, and $G[S] = K_2$.*

Proof. Let $S := \{u^i : i \in [|S|]\}$. As S is a minimum separating set, HW 49 implies that for all $u^i \in S$ and $j \in [2]$, there are $v^i_j \in N(u^i) \cap (V_j \setminus S)$ such that if $u^i, u^h \in S$ then v^i_j and v^h_j are in the same component of $G - S$. Let $w_i \in V_i \setminus S$ for $i \in [2]$. By maximality, $G + e$ contains a K-graph H_e with $e \in E(H_e)$ for all $e \in E(\overline{G})$. As K_5 and $K_{3,3}$ are 3-connected, (*) H_e has 3 independent paths between any two branch vertices.

First we show that $G[S] = K_2$. If $S = \emptyset$ then $e := w_1w_2$ is a cut-edge in $G + e$, so the subdivided edge in H_e containing e has a branch vertex in both V_1 and V_2. As all V_1, V_2-paths contain e, this contradicts (*). Now suppose $|S| = 1$. Let $e = v^1_1v^1_2$. By (*) and $|S + e| = 2$, all the branch vertices of H_e are contained in some G_i, $i \in [2]$. As $e \in H_e$, there is a subdivided edge containing $P := v^1_i v^1_{3-i} \dots u^1$ with $H_e[V_{3-i} + v^1_i] = P$. Replacing P with $v^1_i u^1$ yields a K-graph in G, a contradiction. So $|S| = 2$. Suppose $e := u^1u^2 \notin E$. As S is a 2-separator of $G + e$ and $e \in H_e$, there is $i \in [2]$ with the branch vertices of H_e contained in G_i. By the choice of v^1_{3-i}, v^2_{3-i}, there is a u^1, u^2-path P in G_{3-i}: else $\{u_1\}$ would separate $u^1v^1_{3-i}$ from u^2. Replacing e by P yields a K-graph in G, a contradiction. So $G[S] = K_2$.

Finally, we show that each G_i is maximal. Let $e \in E(\overline{G}_i)$. By (*), the branch vertices of H_e are contained in some G_j, $j \in [2]$. If they are contained in G_{3-i}, then $P := G_i \cap H_e$ is a u^1, u^2-path containing e. Replacing P with u^1u^2 yields a K-graph in $G_{3-i} \subseteq G$, a contradiction. So the branch vertices of H_e are contained in G_i. If $H_e \cap G_{3-i} - S \ne \emptyset$ then $H_e \cap G_{3-i}$ is a u^1, u^2 path P. Replacing P by u^1u^2 yields a K-graph in $G_i + e$, so $H_e \subseteq G_i$. Thus G_i is maximum. □

Now we finish the proof of Kuratowski's Theorem by proving that K-edge-maximal graphs are 3-connected.

Theorem 127. *Every K-edge-maximal graph is 3-connected or is complete.*

Proof. Let $G = (V, E)$ be K-edge-maximal, and argue by induction on $|G|$. Let S be a minimum separating set. If $|S| \ge 3$ then we are done; else there are sets $V_1, V_2 \ne S$ with $V = V_1 \cup V_2$, $S = V_1 \cap V_2$, and $E(V_1 \setminus S, V_2 \setminus S) = \emptyset$. By Lemma 126, $G[S] = K_2$ and each $G_i := G[V_i]$ is K-edge-maximal; say $S = \{x, y\}$. By induction each G_i is either 3-connected or complete, so Theorem 125 implies each G_i is planar. For each G_i, let $\widetilde{G}_i$ be a plane drawing of G_i. For $i \in [2]$, let $f_i \in F(\widetilde{G}_i)$ be a face whose boundary contains xy and a third vertex z_i. Then $G_i + xz_i + yz_i$ is planar by Fact + Edges, and so contains no K-graph. Now $xz_i, yz_i \in G_i$ by the maximality of G_i. Let v_i be a new vertex and $G_i^+ := G_i \cup K(\{v_i\}, \{x, y, z_i\})$. Using Fact + Edges to draw v and its incident edges in f_i shows that G_i^+ is planar.

As $e := z_1z_2 \in E(\overline{G})$, $G' := G + e$ contains a K-graph H with $e \in H$. If all the branch vertices of H are contained in the same G_i then there is a path $P = z_i z_{3-i} \dots u$ in G', where $u \in S$. Replacing P by uz_i yields a K-graph in G_i, a contradiction. Thus H has branch vertices in each G_i. As $\{x, y, z_1\}$ is an V_1, V_2-separator in G', (*) there are at most three independent V_1, V_2-paths. As K_5 is 4-connected, $H = TK_{3,3} =: TK(X, Y)$. Some G_i, $i \in [2]$, has two (branch) vertices from X. By (*), G_{3-i} does not have two vertices of Y, and so G_i does. Thus by (*), G_{3-i} does not have the remaining vertex of X and the remaining vertex of Y. So G_{3-i} has has exactly one branch vertex v. Thus $H \cap G'[V_{3-i} + z_i]$ is a $v, \{x, y, z_i\}$-fan F. So H with F replaced by $K(v_1, \{x, y, z_i\})$ is a subdivision of $K_{3,3}$ in G_i^+. As G_i^+ is planar, this contradicts Lemma 122. □

Proof of Kuratowski's Theorem 121. First suppose G contains a K-graph. Then by Lemma 122, G is not planar. Now suppose G contains no K-graph. Then G is a spanning subgraph of a graph G' that is K-edge-maximal. By Theorem 127, G' is 3-connected or $|G'| \leq 3$. By Theorem 125, G' is planar, so G is planar. □

A simple planar graph is *maximally planar* if it is not a proper spanning subgraph of any simple planar graph.

Corollary 128. *Every maximally planar graph G is either 3-connected or complete.*

Proof. Suppose G is a maximally planar graph. By Theorem 121, G contains no K-graph. Moreover, for every edge $e \in \overline{G}$, $G + e$ is nonplanar, and so by Theorem 121, contains a K-graph. Thus G is K-edge-maximum. By Theorem 127, if G is not 3-connected then $|G| \leq 4$. As K_3 is planar, G is complete. □

HW 77. (*) Prove: Every 3-connected, non-planar graph G with $|G| \geq 6$ contains a subdivision of $K_{3,3}$.

5.5. List coloring planar graphs

In 1852, Francis Guthrie observed that he could (properly) color a map of the counties of England with four colors so that any two bordering counties received distinct colors. Francis asked his brother Fredrick Gutherie whether this was always the case. To visualize this question in our terms, suppose each county has a county seat, designated by a dot on the map, and any two bordering counties have a road between their county seats (and these roads do not cross). Then the county seats and roads form a planar graph. Any (proper) coloring of the counties induces a proper coloring to the county seats, where a seat receives the same color as its county, and vice versa. So in our terms Francis asked whether every planar graph is 4 colorable.

Fredrick brought the question to his advisor, Augustus De Morgan, who had also been Francis's advisor. De Morgan wrote,

> "A student of mine [Guthrie] asked me today to give him a reason for a fact which I did not know was a fact—and do not yet. He says that if a figure be

> any how divided and the compartments differently colored so that figures with any portion of common boundary line are differently colored—four colors may be wanted but not more—the following is his case in which four colors are wanted. Query cannot a necessity for five or more be invented. . . ”

Francis’s question became known as the 4-color problem. In 1879 and 1880 two false proofs were given, but their falsity was not discovered until 1890 and 1891. Finally, in 1976, Appel and Haken proved the 4-color Theorem: $\chi(G) \leq 4$ for every planar graph. Their extremely long proof was computer generated. Indeed, it was the first proof of a major theorem to rely on a computer program. In 1997 Robertson, Sanders, Seymour, and Thomas, gave a cleaner proof based on a rewritten program.

The rest of this section is devoted to list-coloring planar graphs. We will prove Thomassen’s Theorem that every planar graph is 5-list-colorable, and (eventually) show an example, essentially due to Vaughn, of a planar graph that is not 4-edge-colorable. First we need some structure results for planar graphs.

Recall that a simple planar graph is *maximally planar* if it is not a proper spanning subgraph of any simple planar graph. A simple plane graph $\widetilde{G}$ is *maximally plane* if it is not a proper spanning subgraph of any simple plane graph. A *triangulation* is a simple plane graph $\widetilde{G}$ with $\widetilde{G}[f] = C_3$ for all $f \in F(\widetilde{G})$. The following problem implies that every plane graph is a spanning subgraph of a triangulation.

PHW 1. Let G be a planar graph with $|G| \geq 3$ and plain drawing $\widetilde{G}$. Prove: The following are equivalent: (i) $\widetilde{G}$ is a triangulation; (ii) $\|G\| = 3|G| - 6$; (iii) G is maximally planar; (iv) $\widetilde{G}$ is maximally plane. [Hint: If $\widetilde{G}$ is maximally planar then use + Edges to show that $V(\widetilde{G}[f])$ is a clique for every face f. If $V(\widetilde{G}[f])$ is a k-clique with $k \geq 4$, then use Fact + Edges to draw a plane 5-clique, a contradiction.]

A *weak triangulation* is a simple plane graph $\widetilde{G}$ with faces are bounded by cycles that has at most one face whose boundary is not C_3. Let f_0 be a (possibly arbitrary) face with $|C|$ maximum, where $C := \widetilde{G}[f_0]$. Call f_0 the *outer face* of $\widetilde{G}$ and C *the long cycle* of $\widetilde{G}$.

PHW 2. (-) Give an example of a *small* planar graph that has two plane drawings, one a weak triangulation and one not.

PHW 3. (-) Prove that every weak triangulation is 2-connected.

Since weak triangulations are triangulations, PHW 1 implies that every plane graph is contained in a weak triangulation. The next two problems show how to do induction on weak triangulations.

PHW 4. Let G be a planar graph with a plane drawing $\widetilde{G}$ such that $\widetilde{G}$ is a weak triangulation with outer face f_0 and long cycle $C := v_1 \dots v_s v_1$. Prove: (i) $2\|\widetilde{G}\| = 3|F(\widetilde{G})| + s - 3$. Now suppose v_s is not incident to a chord of C. Set $G^+ = G + K(v_s, V(v_2 \dots v_{s-2}))$, and let $\widetilde{G}^+$ be a plane drawing of G^+ with the new edges in f_0. Prove: (ii) $\|G^+\| = 3|G^+| - 6$; and conclude (iii) $\widetilde{G} - v_s$ is a weak triangulation with long cycle C', where $V(C') = V(C - v_s) \cup N(v_s)$. [Hint: Use PHW 1, Corollary 128, and Lemma 124.]

PHW 5. Let G be a planar graph with a plane drawing $\widetilde{G}$ such that $\widetilde{G}$ is a weak triangulation with outer face f_0 and long cycle $C := v_1 \dots v_s v_1$. Prove: If v_s is incident to a chord $v_s v_i$ of C then there are weak triangulations $\widetilde{G}_1$ and $\widetilde{G}_2$ such that $\widetilde{G} = \widetilde{G}_1 \cup \widetilde{G}_2$ and $\widetilde{G}_1 \cap \widetilde{G}_2 = K(v_1 v_s)$ with long cycles $C_1 := v_i v_{i\oplus 1} \dots v_j v_i$ and $C_2 := v_j v_{j\oplus 1} \dots v_i v_j$, respectively.

Theorem 129 (8.4.32 Thomassen (1994)). *All planar graphs $G := (V, E)$ are 5-list colorable.*

Proof. We will prove a stronger claim by induction on $|G|$ (Inventor's Paradox).

Claim. Suppose $\widetilde{G}$ is a weak triangulation of G. Pick a face f_0 with $l(f_0)$ maximum (it is unique unless G is a triangulation), and set $C := v_1 v_2 \dots v_s v_1 := \widetilde{G}[f_0]$, $x = v_1$ and $y = v_2$. If L is a list assignment for G such that

(1) $L(x) = \{\alpha\}$, $L(y) = \{\beta\}$, and $\alpha \neq \beta$,
(2) $|L(v)| = 3$ for all vertices $v \in \{v_3, \dots, v_s\}$, and
(3) $|L(v)| = 5$ for all vertices $v \in V(G - C)$,

then G has an L-coloring.

The claim implies the theorem since adding edges and vertices to G, and deleting colors from some lists of L does not make it easier to L-color G. Thus we can assume that we start with a triangulation.

Proof of Claim. Argue by induction on $|G|$. Note that $|G| \geq |C| \geq 3$. First consider the base step $|G| = 3$. Color x with α and y with β. The last vertex z has three colors in its list, so it can be colored with a color distinct from α and β.

Now consider the induction step $|G| > 3$. There are two cases. See Figure 5.5.1.

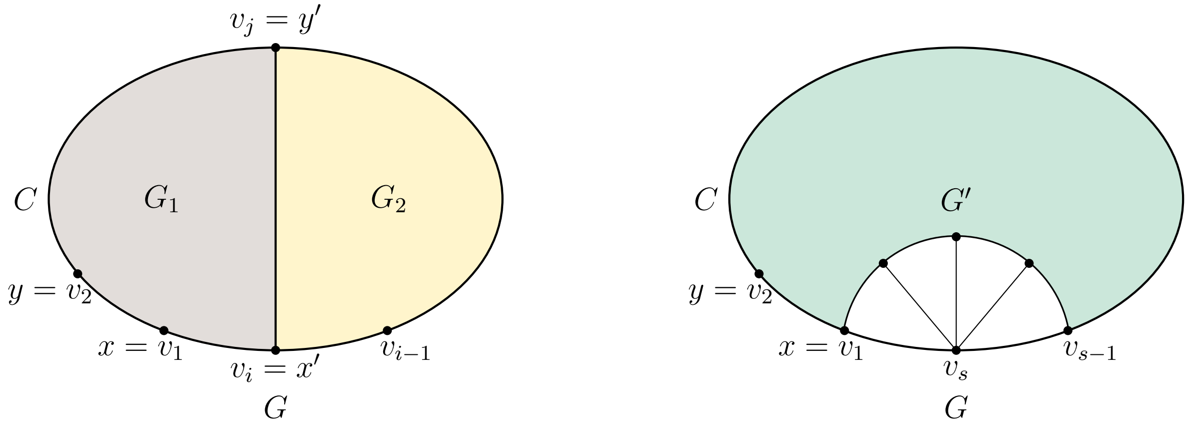
Case 1: C has a chord $v_i v_j$. By PHW 5, there are weak triangulations $\widetilde{G}_1$ and $\widetilde{G}_2$ with $\widetilde{G} = \widetilde{G}_1 \cup \widetilde{G}_2$ and $\widetilde{G}_1 \cap \widetilde{G}_2 = K(v_i v_j)$ whose long cycles are $C_1 := v_i v_{i\oplus 1} \dots v_1 v_2 \dots v_j v_i$ and $C_2 := v_j v_{j\oplus 1} \dots v_i v_j$, respectively. By induction there is an L-coloring g_1 of $\widetilde{G}_1$. Set $x' := v_i$, $\alpha' := g_1(x')$, $y' := v_j$, $\beta' := g_1(y')$, $L'(x') := \{\alpha'\}$, $L'(y') := \{\beta'\}$ and $L'(v) := L(v)$ for all vertices of $\widetilde{G}_2 - x' - y'$. By induction there is an L'-coloring g_2 of $\widetilde{G}_2$. As the chord $v_i v_j$ separates G and $g_1 \restriction \{v_i, v_j\} = g_2 \restriction \{v_i, v_j\}$, setting $f := g_1 \cup g_2$ yields an L-coloring of G.

Case 2: C has no chord. By PHW 4, $\widetilde{G'} := \widetilde{G} - v_s$ is weakly triangulated with long cycle C', and $V(C) \cup N(v_s) \subseteq V(C')$. Let $\gamma, \delta \in L(v_s) - \alpha$ be distinct colors. Define a list assignment L' for G' by

$$L'(v) = \begin{cases} L(v) - \gamma - \delta & \text{if } v \in N(v_s) - x - v_{s-1} \\ L(v) & \text{else} \end{cases}$$

(and shrinking oversized lists). By induction $\widetilde{G'}$ has an L'-coloring f'. Pick $\varepsilon \in \{\gamma, \delta\}$ with $\varepsilon \neq f'(v_{s-1})$. Finally, extend f' to an L-coloring f of G by setting $f(v_s) = \varepsilon$. □

This completes the proof of the claim and the Theorem. □



Case 1: C has a chord. Case 2: C has no chord.

FIGURE 5.5.1. Thomassen's Theorem

CHAPTER 6

Extras

6.1. Lower Bounds on Ramsey's Theorem

THEOREM 130. *For every integer $k \geq 2$ there exists a graph G such that $\omega(G) < k$, $\alpha(G) < k$, and $|G| \geq \lfloor 2^{k/2-1/2} \rfloor$. In other words,* $\mathrm{Ram}(k,k) \geq 2^{k/2-1/2}$.

PROOF. Fix $k \geq 2$, and set $n = \lfloor 2^{k/2-1/2} \rfloor$. Let V be a set of n vertices, and $\mathcal{G}$ be the set of all graphs G with $V(G) = V$. So $G = (V,E) \in \mathcal{G}$ if and only if $E \subseteq \binom{V}{2}$. Since there are $2^{\binom{n}{2}}$ choices for E,

$$|\mathcal{G}| = 2^{\binom{n}{2}} =: N. \tag{6.1.1}$$

For $X \subseteq V$ with $|X| = k$, let $\mathcal{G}_X$ be the set of graphs in $\mathcal{G}$ such that X is a clique or co-clique. So if $G := (V,E) \in \mathcal{G}$ then $G \in \mathcal{G}_X$ iff $E \cap \binom{X}{2} \in \{\emptyset, \binom{X}{2}\}$ and $E \smallsetminus \binom{X}{2} \subseteq \binom{V}{2} \smallsetminus \binom{X}{2}$. There are two possibilities for the first conjunct and $2^{\binom{n}{2}-\binom{k}{2}}$ possibilities for the second. Thus

$$|\mathcal{G}_X| = 2 \cdot 2^{\binom{n}{2}-\binom{k}{2}} = 2 \cdot 2^{-(k^2-k)/2} N. \tag{6.1.2}$$

Any graph G in

$$\mathcal{G} \smallsetminus \bigcup_{X \in \binom{V}{k}} \mathcal{G}_X,$$

satisfies $\omega(G), \alpha(G) < k$ and $|G| = n$. So it suffices to prove (*) $|\bigcup_{X \in \binom{V}{k}} \mathcal{G}_X| < |\mathcal{G}|$. Since (a) $|\bigcup_{X \in \binom{V}{k}} \mathcal{G}_X| \leq \binom{n}{k} |\mathcal{G}_X|$, (b) $\binom{n}{k} < \frac{n^k}{k!}$, and (c) $\frac{n}{2^{k/2-1/2}} \leq 1$, (*) follows from:

$$\begin{aligned}
|\mathcal{G}| - |\bigcup_{X \in \binom{V}{k}} \mathcal{G}_X| &\geq N - \binom{n}{k} \cdot 2 \cdot 2^{-(k^2-k)/2} N && (6.1.1), (a), (6.1.2) \\
&> N\left(1 - 2\frac{n^k}{k!} 2^{-(k-1)k/2}\right) && (b) \\
&\geq N\left(1 - \frac{2}{k!}\left(\frac{n}{2^{k/2-1/2}}\right)^k\right) \geq 0. && (c)
\end{aligned}$$

□

6.2. Equitable Coloring

DEFINITION 131. An equitable k-coloring of a graph $G = (V,E)$ is a proper coloring $f : V \to [k]$ such that difference $||f^{-1}(i)| - |f^{-1}(j)||$ in the sizes of the the i-th and j-th color classes is at most 1 for all $i, j \in [k]$. In particular, every color is used if $|G| \leq k$.

THEOREM 132 (Hajnal & Szemerédi Theorem (1976)). *Every graph G with maximum degree at most r has an equitable $(r+1)$-coloring.*

The proof was long and complicated, and did not provide a polynomial time algorithm. Kierstead and Kostochka found a much simpler and shorter proof. This better understanding has led to many new results, several of which are stated below.

Let $\theta(G) = \max\{d(x) = d(y) : xy \in E(G)\}$.

THEOREM 133 (Kierstead & Kostochka (2008)). *For every $r \geq 3$, each graph G with $\theta(G) \leq 2r+1$ has an equitable $(r+1)$-coloring.*

THEOREM 134 (Kierstead, Kostochka, Mydlarz & Szemerédi). *There is an algorithm that constructs an equitable k-coloring of any graph G with $\Delta(G)+1 \leq k$, using time $O(r|G|^2)$.*

PROBLEM 135. Find a polynomial time algorithm for constructing the coloring in Theorem (133).

One might hope to prove an equitable version of Brooks' Theorem, but the following example shows that the statement would require special care: Let $r \in \mathbb{Z}^+$ be odd. Then $K_{r,r}$ satisfies $\Delta(K_{r,r}) = r$ and $\omega(K_{r,r}) = 2$, but has no r-equitable coloring. Chen, Lih and Wu [?] proposed the following common strengthening of Theorem 132 and Brooks' Theorem.

CONJECTURE 136. *Let G be a connected graph with $\Delta(G) \leq r$. Then G has no equitable r-coloring if and only if either (a) $G = K_{r+1}$, or (b) $r = 2$ and G is an odd cycle, or (c) r is odd and $G = K_{r,r}$.*

Kierstead and Kostochka have proved the conjecture for $r \leq 4$, and also for $r \geq \frac{1}{4}|G|$.

PROOF OF THEOREM 132. Let G be a graph with $\Delta(G) \leq r$. We may assume that $|G|$ is divisible by $r+1$: If $|G| = s(r+1) - p$, where $p \in [r]$ then set $G' := G + K_p$. Then $|G'|$ is divisible by $r+1$ and $\Delta(G') \leq r$. Moreover, the restriction of any equitable $(r+1)$-coloring of G' to G is an equitable $(r+1)$-coloring of G. So assume $|G| = (r+1)s$.

We argue by induction on $\|G\|$. The base step $\|G\| = 0$ is trivial, so consider the induction step. Let u be a non-isolated vertex. By the induction hypothesis, there exists an equitable $(r+1)$-coloring of $G - E(u)$. We are done unless some color class V contains an edge uv. Since $\Delta(G) \leq r$, some color class W contains no neighbors of u. Moving u to W yields an $(r+1)$-coloring of G with all classes of size s, except for one *small* class $V^- := V - u$ of size $s-1$ and one *large* class $V^+ := W + u$ of size $s+1$. Such a coloring is called *nearly equitable.*

Given a nearly equitable $(r+1)$-coloring, define an auxiliary digraph $\mathcal{H}$, whose vertices are the color classes, so that UW is a directed edge if and only if some vertex $y \in U$ has no neighbors in W. In this case we say that y *witnesses* UW. Let $\mathcal{A}$ be the set of classes from which V^- can be reached in $\mathcal{H}$, $\mathcal{B}$ be the set of classes not in $\mathcal{A}$ and $\mathcal{B}'$ be the set of classes reachable from V^+ in $\mathcal{H}[\mathcal{B}]$. Set $a := |\mathcal{A}|$, $b := |\mathcal{B}|$, $b' := |\mathcal{B}'|$, $A := \bigcup \mathcal{A}$, $B := \bigcup \mathcal{B}$ and $B' := \bigcup \mathcal{B}'$. Then $r+1 = a+b$. Since every vertex $y \in B$ has a neighbor in every class of $\mathcal{A}$ and every vertex $z \in B'$ also has a neighbor in every class of $\mathcal{B} \smallsetminus \mathcal{B}'$,

$$(*) \qquad \|y, A\| \geq a \text{ for all } y \in B \text{ and } \|z, A \cup B \smallsetminus B'\| \geq a + b - b' \text{ for all } z \in B'.$$

Case 0: $V^+ \in \mathcal{A}$. Then there exists a V^+, V^--path $\mathcal{P} = V_1, \ldots, V_k$ in $\mathcal{H}$. Moving each witness y_j of $V_j V_{j+1}$ to V_{j+1} yields an equitable $(r+1)$-coloring of G. $\square$

We now argue by a secondary induction on b, whose base step $b = 0$ holds by Case 0. Also $|A| = as - 1$ and $|B| = bs + 1$. Now consider the secondary induction step.

A class $W \in \mathcal{A}$ is *terminal*, if every $U \in \mathcal{A} - W$ can reach V^- in $\mathcal{H} - W$. Let $\mathcal{A}'$ be the set of terminal classes, $a' := |\mathcal{A}'|$ and $A' := \bigcup \mathcal{A}'$. An edge $wz \in E(G)$ is *solo* if $w \in W \in \mathcal{A}'$, $z \in B$ and $N(z) \cap W = \{w\}$. Ends of solo edges are *solo vertices* and *solo neighbors* of each other.

Order $\mathcal{A}$ as $V^-, X_1, \ldots, X_{a-1}$ so that each X_i has a previous out-neighbor. Let i be the largest index such that X_i is nonterminal.Then X_j is terminal when $i+1 \leq j \leq a-1$.

Case 1: $a - b \leq i$. Then there is $j > i$ such that X_j cannot reach V^- in $\mathcal{H} - X_i$ and $X_j \in \mathcal{A}'$. Then X_j has no out-neighbors before X_i, so $\|w, A\| \geq i \geq a - b$ and $\|w, B\| < 2b$ for each $w \in X_j$. Let S be the set of solo vertices in X_j, and $D := X_j \smallsetminus S$. If $v \in B \smallsetminus N(S)$ then v has no solo neighbor in X_j, and so has at least two neighbors in D. Thus $(2b-1)\,|D| \geq 2\,|B \smallsetminus N(S)|$. Using $|S| + |D| = s$ and $r\,|S| \geq \|S, A\| + |N(S) \cap B|$,

$$bs + (a-1)\,|S| = b\,|D| + r\,|S| > |B \smallsetminus N(S)| + \|S, A\| + |N(S) \cap B| > bs + \|S, A\|.$$

Thus $(a-1)\,|S| > \|S, A\|$, and so there is $w \in S$ with $\|w, A\| \leq a - 2$. Thus w witnesses some edge $X_j X \in E(\mathcal{H}[A])$. Since $w \in S$, it has a solo neighbor $y \in B$.

Move w to X and y to X_j. This yields a nearly equitable colorings of $G[A+y]$. Since X_j is terminal, $X + w$ can still reach V^-. Thus by Case 0, $G[A+y]$ has an equitable a-coloring. By (*), $\Delta(G[B-y]) \leq b-1$. So by the primary induction hypothesis $G[B-y]$ has an equitable b-coloring. After combining these equitable colorings we are done.

Case 2: $i < a - b$. Then $X_{a-b}, \ldots, X_{a-1}$ are terminal. Then $a' \geq a - (i+1) \geq b$. For $y \in B'$, let $\sigma(y)$ be the number of solo neighbors of y. Similarly to (*), every $y \in B'$ satisfies:

$$r \geq d(y) \geq a + b - b' + \|y, B'\| + a' - \sigma(y) \geq r + 1 + \|y, B'\| + a' - b' - \sigma(y).$$

So $\sigma(y) \geq a' - b' + \|y, B'\| + 1$. Let I be a maximal independent set with $V^+ \subseteq I \subseteq B'$. Then $\sum_{y \in I}(\|y, B'\| + 1) \geq |B'| = b's + 1$. Since $a' \geq b$,

$$\sum_{y \in I} \sigma(y) \geq \sum_{y \in I}(a' - b' + \|y, B'\| + 1) \geq s(a' - b') + b's + 1 > a's = |A'|.$$

So some vertex $w \in W \in \mathcal{A}'$ has two solo neighbors y_1 and y_2 in the independent set I.

Since the class Y of y_1 is reachable from V^+, we can equitably b-color $G[B - y_1]$. Let Y' be the new class of y_2. If w witnesses an edge WX of $G[A]$ then we are done as in Case 1. Otherwise, move w to some class $U \subseteq B - y_1$. Replacing w with y_1 in W to get W^* and moving w to U yields a new nearly equitable $(r+1)$-coloring of G. Now at least $a+1$ classes, W^*, Z', and all $X \in \mathcal{A}' - W$, can reach V^-, so we are done by the secondary induction. $\square$

APPENDIX A

Notation

A.1. Standard Notation

$V(G)$: vertices of G
$E(G)$: edge set of G
$G[X]$: subgraph of G induced by X
$\overline{E}(G) := \binom{V(G)}{2} \setminus E(\overline{G})$
$\overline{G} := (V(G), \overline{E}(G))$
$N_G(v)$: neighbors of v in G
$N_G[v] := N_G(v) + v$
$N_G[S] := \bigcup_{v \in S} N[v]$, where $S \subseteq V$
$N_G(S) := N[S] \setminus S$, where $S \subseteq V$
$d_G(v) := |N(v)|$, where G is a graph
$\delta(G)$: minimum degree
$\Delta(G)$: maximum degree
$\kappa(G)$: connectivity
$\kappa'(G)$: edge-connectivity
$\omega(G)$: size of maximum clique
$\alpha(G)$: size of maximum coclique
$\cong$: isomorphic
$G \cup H := (V \cup W, E \cup F)$
$G + H+ = G \cup H$
$G \vee H := (G + H) + K(V, W)$
$\mathbb{N}$: set of natural numbers (including 0)
$\mathbb{Z}$: set of integers
$\mathbb{Z}^+$: set of positive integers

A.2. Exceptional Notation

$d_G(x, y)$: length of the shortest x, y-path in G
$E_G(v) := \{vu : vu \in E(G)\}$, edge set
$E_G(A, B) := \{ab \in E(G) : a \neq b \wedge (a, b) \in A \times B\}$, edge set
$\|a, B\|_G := |E_G(\{a\}, B)|$, number
$\|A, B\|_G := \sum_{a \in A} \|a, B\|_G$, number (edges in $A \cap B$ count twice)
$|G| := |V(G)| = n(G)$, number
$\|G\| := |E(G)| = e(G)$, number
$K(A, B) := (A \cup B, \{ab : a \neq b \wedge (a, b) \in A \times B\})$, graph
$K(A) := K(A, A)$, graph

$\binom{S}{k}$: set of k-subsets of S
$\{a, \dots, b\} := \{n \in \mathbb{N} : a \leq n \leq b\}$
$[a] := [1, a]$
$S + x := S \cup \{x\}$
$S - x := S \setminus \{x\}$
$E^k := \{uv : d(u, v) \leq k\}$
$G^k := (V, E^k)$: k-th power of G
$\omega^*(G)$: maximum size of a barrier

APPENDIX B

Standards

The syllabus for Midterm 1 is in red; The syllabus for Midterm 2 is in blue; the syllabus for the final is everything in red, blue, or black.

B.0.1. MAT 416–Level C.

(1) Theorem 23 (Characterization of bipartite graphs). You may use Lemma 22.
(2) Theorem 46 (Euler's Theorem). You may use Proposition 44.
(3) Theorem 32 (Tree Theorem). Provide proofs for required parts of Lemma 30 and Lemma 31.
(4) Theorem 51 (Berge's theorem on maximum matchings).
(5) Corollary 53. You may use Hall's Theorem.
(6) Theorem 61 Petersens's Matching Theorem.
(7) Theorem 68 (Whitney's Theorem).
(8) Lemma 107 (Kernel Lemma).
(9) Euler's Formula Theorem 118
(10) Theorem 119.
(11) Theorem 121 (using Theorems 125 and 127).

B.0.2. MAT 416–Level A. All MAT 416–Level C and HWA #1–4, HWA #5–7, HW 8 except (+) and:

(1) Theorem 6 (Ramsey's Theorem for graphs).
(2) Corollary 37 (Dirac's Theorem). Provide proofs for required parts of Theorem 35.
(3) Theorem 52 (Hall's Theorem) Do not use Theorem 55.
(4) Theorem 55 (König–Egerváry Theorem). Use Theorem 52, Hall's Theorem.
(5) Theorem 57 (Tutte's Theorem).
(6) Theorem 62 (Petersen's 2-Factor Theorem). Use Theorem 52, Hall's Theorem.
(7) Theorem 71 (Thomassen's Contraction Theorem).
(8) Theorem 76 (Menger's Theorem).
(9) Theorem 90 (Brooks' Theorem).
(10) Theorem 94 (Turan's Theorem). You may assume Lemma 93, but be prepared to prove it if asked.
(11) Theorem 99 (Vizing's Theorem). This includes proving Lemma 98.
(12) Lemma 108
(13) Theorem 109 (Galvin's Theorem). You may use Lemmas 107 and 108.
(14) Theorem 125 (3-connected case of Kuratowski's Theorem). You may assume: 123, 122 and 124.
(15) Theorem 127 (using Lemma 126).
(16) Thomassen's 5-Choosability Theorem 129.

(17) Lower bound on $\mathrm{Ram}(k,k)$ Theorem 130.

B.0.3. MAT–513. All MAT 416–Level A, HWA #1–4, HWA #5,6, HWA#7,8. Note: HWA #1–7 consists of HW #1–73. The syllabus for the qualifier is the syllabus for the final together with all old qualifiers.

B.1. Sample MAT 416/513 Midterm 1

Directions: Use one sheet per problem. Order the sheets by problem before submitting.

MAT **513 Students:** Do the **last 4** problems. You may replace some of these with problems from the first four for half credit.

MAT **416 Students:** Do **any 5** problems; one will be treated as extra credit. The first four problems are intended to be somewhat easier. Only the five chosen problems should be turned in.

(1) Prove: If an acyclic graph G satisfies $|G| = \|G\| + 1$ then it is connected.
(2) Prove: Every k-regular bigraph G has a perfect matching.
(3) Let $P \subseteq G$ be an x, y-path. Prove: $G[P]$ contains an x, y-path Q with $Q = G[Q]$.
(4) Let T be a forest (acyclic graph) such that $2k$ of its vertices have odd degree. Prove that T decomposes into k paths.

****

(5) Prove: A graph is bipartite if and only if it contains no odd cycle.
(6) Prove: A graph $G = (V, E)$ has a perfect matching if $o(G - S) \leq |S|$ for all $S \subseteq V$.
(7) Let $n \geq 2$ and $d_1, \ldots, d_n \in \mathbb{Z}^+$. Prove: If $\sum_{i=1}^{n} d_i = 2n - 2$ then there is a tree with vertices $v_1, \ldots, v_n$ such that $d(v_i) = d_i$ for all $i \in [n]$.
(8) For $k, n \in \mathbb{N}$, let G be an A, B-bigraph with $|A| = n = |B|$ such that $\delta(G) \geq k$, and for all $X \subseteq A, Y \subseteq B$, if $|X|, |Y| \geq k$ then $|E(X, Y)| \neq \emptyset$. Prove: G has a perfect matching.

APPENDIX C

Graph Theory Qualifier Syllabus

Be prepared to prove the following theorems.

(1) Theorem 6 (Ramsey's Theorem for graphs).
(2) Theorem 46 (Euler's Theorem). You may use Lemma 44.
(3) Corollary 37 (Dirac's Theorem). Provide proofs for required parts of Theorem 35.
(4) Theorem 51 (Berge's theorem on maximum matchings).
(5) Theorem 52 (Hall's Theorem) Do not use Theorem 55.
(6) Corollary 53. You may use Hall's Theorem.
(7) Theorem 55 (König–Egerváry Theorem) You may use Hall's Theorem.
(8) Theorem 57 (Tutte's Theorem).
(9) Theorem 62 (Petersen's 2-Factor Theorem).
(10) Theorem 71 (Thomassen's Contraction Theorem).
(11) Theorem 76 (Menger's Theorem).
(12) Example 88 Mycielski's Construction.
(13) Theorem 90 (Brooks' Theorem).
(14) Theorem 94 (Turan's Theorem). You may assume Lemma 93, but be prepared to prove it if asked.
(15) Theorem 99 (Vizing's Theorem). This includes proving Lemma 98.
(16) Lemma 107 (Kernel Lemma).
(17) Lemma 108 (Gale-Shapely Lemma)
(18) Theorem 109 (Galvin's Theorem).
(19) Theorem 118 (Euler's Formula) and Theorem 119.
(20) Kuratowski's Theorem 121. You may assume: 123, 122 and 124. You should be able to prove Theorem 125 and also Theorem 121, using Theorems 125 and 127 (easy).
(21) Theorem 129 (Thomassen's 5-Choosability Theorem).
(22) Theorem 130 (Lower bound on $\mathrm{Ram}(k,k)$).

Be prepared to solve homework problems HW #1–75 and old qualifier problems as well as variations and/or similar problems.

APPENDIX D

Matching card trick52

Consider a deck of $2k+1$ cards numbered $1,\dots,2k+1$, and denoted by $[2k+1]$. The class chooses a *hand* H consisting of $k+1$ of these cards, and gives them to Professor A. Professor A looks at them, puts one of them in his pocket, and then has a student spread the remaining k cards face-up on a table. Professor B, who has observed none of this transaction, now enters the room, looks at the cards on the table and identifies the one in Professor A's pocket. How is this done?

Solution. Our arithmetic is done modulo $k+1$, and we use $k+1$ instead of 0 for the representative of its equivalence class. Arrange the cards of H in order as $c_1 < \cdots < c_{k+1}$. Let $x = \sum_{c \in H} c \mod k+1$. Professor A hides card c_x. When Professor B arrives, he sees that the cards $d_1 < \cdots < d_{k+1}$ in $[2k+1] \setminus (H - c_x)$ are missing, and he calculates $y := \sum_{c \in H-x} = x - c_x$. The class is holding $c_x - 1 - (x-1) = -y$ cards less than c_x and Professor A is holding c_x. It follows that $c_x = d_{1-y}$, and Professor B can calculate the rhs.

Another way of saying this is that Professor B knows the missing cards $\overline{d}_1 > \cdots > \overline{d}_{k+1}$. Then $c_x = \overline{d}_{k+2-(1-y)} = \overline{d}_y$.

APPENDIX E

Alternative proofs

E.1. Euler's Theorem

SECOND PROOF OF THEOREM 46 (SUFFICIENCY). Suppose G is even and has at most one nontrivial component G'. We argue by induction on $\|G'\|$. If G' is a cycle then the cycle in its natural order is the Eulerian trail; if $\|G'\| = 0$ then any vertex is an Eulerian trail.

Otherwise, by Corollary **??**, G' contains a cycle C. Let H be a nontrivial component of $G' - E(C)$ (maybe $H = G' - E(C)$), and set $H' = G' - H + C$. Both H and H' are even. Also H' is connected since all components of $G' - E(C)$ that are contained in H' are connected to each other in H' by edges of C. Moreover, $\|H\| \leq \|G'\| - \|C\| < \|G'\|$ and $\|H'\| = \|G'\| - \|H\| < \|G'\|$. So H and H' are nonempty even connected graphs with $\|G'\| = \|H\| + \|H'\|$. By the induction hypothesis H and H' contain Eulerian trails T and T'. Moreover, T contains a vertex $v_1 \in C$ and T' contains all vertices of C. Choose notation so that $T = v_1 \dots v_n v_1$ and $T' = v_1 u_2 \dots u_m v_1$. Then $v_1 T v_n v_1 T' u_m v_1$ is an Eulerian trail in G', and G. $\square$

E.2. Hall's Theoerm

PROOF OF THEOREM 52. If M is a matching covering X and $S \subseteq X$ then $|S| = |E(S,Y) \cap M| \leq |N(S)|$; so (**??**) holds.

Suppose (**??**) holds for some X,Y-bigraph with no matching saturating X; among such counterexamples choose G with $|G|$ minimal, and subject to this $\|G\|$ maximal. By minimality, (1) $N(X) = Y$: if $y \in Y \smallsetminus N(X)$ then $G - y$ is a smaller counterexample. Also (2) all $a \in X$ satisfy $N(a) \neq Y$: else, since $G - a$ has a matching saturating $X - a$ and there is an unsaturated vertex $b \in Y$ by (**??**), G has a matching saturating X.

Let $a \in X$; by (2) there is $b \in Y$ with $ab \notin E$. Since $G + ab$ satisfies (**??**), maximality implies $G + ab$ has a matching M^+ saturating X. By (1) there is $a' \in X$ with $a'b \in E$; by (2) there is $b' \in Y$ with $a'b' \notin E$. Again by maximality, $G + a'b'$ has a matching L^+ saturating X. Set $M := M^+ - ab$ and $L := L^+ - a'b'$.

Let H be the spanning submultigraph of G with edge set $M \cup L + a'b$, where edges in $M \cap L$ have multiplicity 2. (Figure E.2.1). Then $\Delta(H) \leq 2$. As G is bipartite, Proposition **??** implies the components of H are paths and even cycles. Each cycle has a perfect matching. As $d_H(a) = 1$ and $d_H(x) = 2$ for all $x \in X - a$, each path P has an end $y \in Y$ and a matching saturating $V(P) - y$. Combining these matchings yields a matching saturating X.

Here is an alternative way through the last paragraph.

Let H be the spanning subgraph of G with $E(H) = M \triangle L$. (Figure E.2.1). Then $\Delta(H) \leq 2$ and $d_H(b), d_H(b') \leq 1 = d_H(a) = d_H(a') < d_H(x)$ for all $x \in X \smallsetminus \{a, a'\}$. So the component of H containing a is an alternating a,v- path P. If $v \in Y$ then P is an

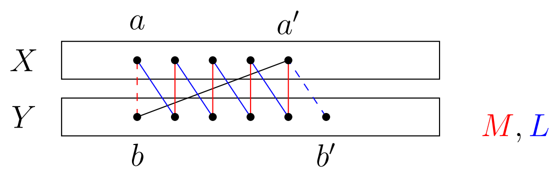


FIGURE E.2.1. X, Y-bigraph $H \subseteq G$

M-augmenting path. Else $v = a'$, and $aPvb$ is an M-augmenting path. Anyway, G has a matching saturating X. $\square$

E.3. König, Egerváry Theorem

PROOF OF THEOREM ??. Order V as $v_1 \prec \cdots \prec v_{|G|}$. Since W is a cover, every edge is incident to some vertex of W (possibly two). Define a function $g : M \to W$ by $g(e)$ is the least $w \in e \cap W$. Since M is a matching, no vertex of W can be incident to two edges of M. So g is an injection. Thus $|M| \leq |W|$.

Now suppose G is an X, Y-bigraph. Let U be the set of unsaturated vertices in X. Set $m = \{(x, y) \in X \times Y : xy \in M\}$. Since the ends of M in X are distinct, m is a function with domain $X \smallsetminus U$. Since the ends of m in Y are distinct, m is an injection.

If $U = \emptyset$ then X is a cover with $|W| \leq |X| = |M| \leq |W|$; so suppose $U \neq \emptyset$. Letting $A \subseteq V(G)$ be the set of ends of alternating paths starting in U, set $S := A \cap X$, $\overline{S} := X \smallsetminus S$, $T := A \cap Y$, and $\overline{T} := Y \smallsetminus T$. Then $U \subseteq S$ (witnessed by trivial paths). Consider any alternating path $P = v_0 \ldots v_n$ with $v_0 \in U$. If i is even then $v_i \in S$, and if also $i \neq 0$ then $v_{i-1}v_i \in M$; if i is odd then $v_i \in T$. We first show:

$$\text{(i) } N(S) \subseteq T \text{ and (ii) } T \subseteq m(S \smallsetminus U). \tag{E.3.1}$$

(i) Let $z \in N(S)$; say $wz \in E(S, z)$. Then there is an alternating path $Q = y_0 \ldots y_{2k} (= w)$ with $y_0 \in U$ and $y_{2k-1}w \in M$. Either $z \in V(Q)$ or Qwz is an alternating path starting in U. Regardless, $z \in T$, proving (E.3.1.i).

(ii) Let $z \in T$. Then there is an alternating path $P = y_0 \ldots y_{2k+1} (= z)$ with $y_0 \in U$ and $y_{2k}z \notin M$. Since M is maximum, G has no augmenting path. So z is saturated; say $zx \in M$. Then Pzx is an alternating path; so $x \in S \smallsetminus U$, and $z \in m(S \smallsetminus U)$, proving (E.3.1.ii).[1]

The set $W = \overline{S} \cup T$ is a vertex cover of G: Suppose $xy \in E$ with $x \in X$. If $x \in S$ then $y \in T \subseteq W$ by (i); else $x \in \overline{S} \subseteq W$.

It remains to show that $|W| \leq |M|$. For this it suffices to prove that g is onto. Consider $w \in W$. If $w \in T$ then there is $x \in S$ with $xw \in M$ by (ii). As $x \notin W$, $g(xw) = w$. If $w \in \overline{S}$ then there is y with $wy \in M$, since $U \subseteq S$. As M is a matching, (ii) implies $y \in \overline{T}$. So $g(wy) = w$. $\square$

[1](i) and (ii) imply the sufficiency of Hall's Criteria (??): $|N(S)| \leq |S \smallsetminus U|$, so (??) fails if $U \neq \emptyset$.

E.4. Menger's Theorem

THEOREM 137 (Menger 1927 4.2.17). *Let $G = (V, E)$ be a graph, and suppose $A, B \subseteq V$. Then the size $l := l(A, B)$ of a maximum set of disjoint A, B-paths is equal to the size $k := k(A, B)$ of a minimal A, B-separating set.*

PROOF 1. ($l \leq k$) If $\mathcal{P}$ is a set of disjoint A, B-paths and S is an A, B-separator then S must contain at least one vertex of each path, and each vertex of S is on at most one path of $\mathcal{P}$. Thus the function $f : \mathcal{P} \to S$ defined by setting $f(P)$ equal to the first $x \in S \cap V(P)$ is an injection; so $|\mathcal{P}| \leq |S|$. Choosing $\mathcal{P}$ maximum and S minimum yields the inequality.

($k \leq l$) For a set of A, B-paths $\mathcal{P}$ let $\mathrm{end}(\mathcal{P})$ denote the set of ends in B of paths in $\mathcal{P}$. It suffices to show (*) if $\mathcal{P}'$ is a set of disjoint A, B-paths with $|\mathcal{P}'| < k$ then there exists a set $\mathcal{P}$ of disjoint A, B-paths such that $|\mathcal{P}| = |\mathcal{P}'| + 1$ and $end(P') \subseteq end(\mathcal{P})$. Argue by induction on $|G'|$, where $G' := G - B$. If $|G'| = 0$ then $A \subseteq B$. So the A, B-paths are exactly the paths consisting of a single vertex of A, and $k = |A|$. Thus (*) holds.

Suppose $|G'| > 0$, and fix a set $\mathcal{P}'$ of disjoint A, B-paths with $|\mathcal{P}'| < k$. Since $|\mathrm{end}(\mathcal{P}')| = |\mathcal{P}'| < k$, there is an A, B-path $R = Ry'$ in $G - \mathrm{end}(\mathcal{P}')$. If $R \cap \bigcup \mathcal{P}' = \emptyset$ then put $\mathcal{P} := \mathcal{P}' + R$. If not, let x be the last vertex of R that is in $\bigcup \mathcal{P}'$; say $x \in P \in \mathcal{P}'$ and y is the end of P in B. Note that $x \notin B$, since $y \neq y'$ and $V(P) \cap B = y$. Put $B' := B \cup V(xRy' \cup xPy)$ and $\mathcal{Q}' := \mathcal{P}' - P + Px$. Then $\mathrm{end}(\mathcal{Q}') = \mathrm{end}(\mathcal{P}') - y + x$. Since every A, B-path contains an A, B'-path, $k = k(A, B) \leq k(A, B')$. Since $B + x \subseteq B'$, $|G - B'| < |G - B|$. So by induction, there exists a set $\mathcal{Q}$ of A, B'-paths and $y'' \in B'$ such that $|\mathcal{Q}| - 1 = |\mathcal{Q}'| = |\mathcal{P}'|$ and $\mathrm{end}(\mathcal{Q}) = \mathrm{end}(\mathcal{Q}') + y''$. So $x \neq y''$. Let x, y'' be the ends of $Q, Q'' \in \mathcal{Q}$. Set $\mathcal{P}_0 = \mathcal{Q} - Q - Q''$. If $y'' \in xPy$ then set $\mathcal{P} := \mathcal{P}_0 + QxRy' + Q''y''Py'$; if $y'' \in xRy'$ then set $\mathcal{P} := \mathcal{P}_0 + QxPy + Q''y''Ry'$; else set $\mathcal{P} := \mathcal{P}_0 + QxPy + Q''$. Evidently, $\mathcal{P}$ witnesses (*). □

PROOF 2 ($k \leq l$). So it suffices to show $k \leq l$. Argue by induction on $\|G\|$.
Base Step: $\|G\| = 0$. Then every A, B-path is trivial. So $A \cap B$ is the maximum set of disjoint A, B-paths and the minimum A, B-separating set. Thus $l = |A \cap B| = k$.
Induction Step: $\|G\| \geq 1$. Let $e = xy \in E(G)$, and put $G' = G \cdot e$. For any $U \subseteq V$, define

$$U' = \begin{cases} U - \{x, y\} + v_e & \text{if } U \cap \{x, y\} \neq \emptyset \\ U & \text{otherwise} \end{cases},$$

and note that for every $T \subseteq V(G')$ there exists $S \subseteq V$ with $T = S'$. Using Lemma **??** and the discussion before, every set $\mathcal{P}'$ of disjoint A', B'-paths corresponds to a set $\mathcal{P}$ of disjoint A, B-paths with $|\mathcal{P}| = |\mathcal{P}'|$ (but not vice versa). So

$$l_{G'}(A', B') \leq l.$$

Also, if S is an A, B-separator in G if and only if S' is an A', B'-separator in G'. So

$$k_{G'}(A', B') \leq k \leq k_{G'}(A', B') + 1.$$

Choose a minimum A', B'-separator T in G'. If $k_{G'}(A', B') = k$ then by the induction hypothesis applied to G' we have:

$$k = k_{G'}(A', B') \leq l_{G'}(A', B') \leq l,$$

and we are done. Otherwise, $k = k_{G'}(A', B') + 1$. In this case $v_{xy} \in T$, and $T = S'$, where $S := T - v_{xy} + x + y$. In particular $xy \in G[S]$.

Set $G'' = G - e$. Since $e \in G[S]$,

$$k_G(A, S) = k_{G''}(A, S) \text{ and } k_G(B, S) = k_{G''}(B, S) \tag{E.4.1}$$

Since S separates A from B in G, every A, S-separator in G separates A from B, and so has size at least $|S|$, and a similar statement holds for B. So we have

$$k_G(A, S), k_G(S, B) \geq k. \tag{E.4.2}$$

Thus

$$\begin{aligned}
&|S| \geq l_G(A, S) \geq l_{g''}(A, S) =_{i.h.} k_{G''}(A, S) =_{(\text{E.4.1})} k_G(A, S) \geq_{(\text{E.4.2})} k = |S| \text{ and} \\
&|S| \geq l_G(B, S) \geq l_{g''}(B, S) =_{i.h.} k_{G''}(B, S) =_{(\text{E.4.1})} k_G(B, S) \geq_{(\text{E.4.2})} k = |S|.
\end{aligned}$$

Let $\mathcal{K}_A$ be a collection of $|S| = k$ disjoint A, S-paths and $\mathcal{K}_B$ be a collection of $|S|$ disjoint S, B-paths. Then for each $z \in S$ there is a unique A, z-path P_z and a unique z, B-path Q_z. If $v \in V(P_w) \cap V(Q_z)$ then $v \in S$, since otherwise PvQ is an A, B-walk in $G - S$, contradicting the fact that S is an A, B-separator. Thus $w = v = z$, and so $\{P_z z Q_z : z \in S\}$ is a collection of $|S| = k$ disjoint A, B-paths. $\square$

Definition 138 (4.2.18). The line graph $H = L(G)$ of a graph $G = (V, E)$ is defined by

$$V(H) = E \text{ and } E(H) = \{ee' : e \cap e' \neq \emptyset\}.$$

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