Annulus Benchmarks

Underworld3

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Introduction

What are Annulus Benchmarks?

Fill information here later...

What is Underworld3?

Fill information here later...

What is Underworld3?

Fill information here later...

Solving for...

Incompressible Stokes flow in an annulus:

- Constant viscosity
- Gravity field that points towards the center of an annulus
- Density field that depends on the spatial coordinates

Which FEM cells are used? Why?

Fill information here later...

What is Penalty Method?

Fill information here later...

Annulus Benchmark: Thieulot

Analytical Equations

This benchmark is based on a manufactured solution in which an analytical solution to the isoviscous incompressible Stokes equations is derived in an annulus geometry. The velocity and pressure fields are as follows:

$$v_\theta(r,\theta) = f(r)\cos(k\theta)$$
$$v_r(r,\theta) = g(r)k\sin(k\theta)$$
$$p(r,\theta) = kh(r)\sin(k\theta) + \rho_0 g_r(R_2 - r)$$
$$\rho(r,\theta) = m(r)k\sin(k\theta) + \rho_0$$

with

$$f(r) = Ar + \frac{B}{r}$$
$$g(r) = \frac{A}{2}r + \frac{B}{r}\ln r + \frac{C}{r}$$

Analytical Equations (Continued)

$$h(r) = \frac{2g(r) - f(r)}{r}$$

$$m(r) = g''(r) - \frac{g'(r)}{r} - \frac{g(r)}{r^2}(k^2 - 1) + \frac{f(r)}{r^2} + \frac{f'(r)}{r}$$

$$A = -C\frac{2(\ln R_1 - \ln R_2)}{R_2^2 \ln R_1 - R_1^2 \ln R_2}$$

$$B = -C\frac{R_2^2 - R_1^2}{R_2^2 \ln R_1 - R_1^2 \ln R_2}$$

The parameters A and B are chosen so that $v_r(R_1, \theta) = v_r(R_2, \theta) = 0$ for all $\theta \in [0, 2\pi]$, i.e. the velocity is tangential to both inner and outer surfaces. The gravity vector is radial inward and of unit length.

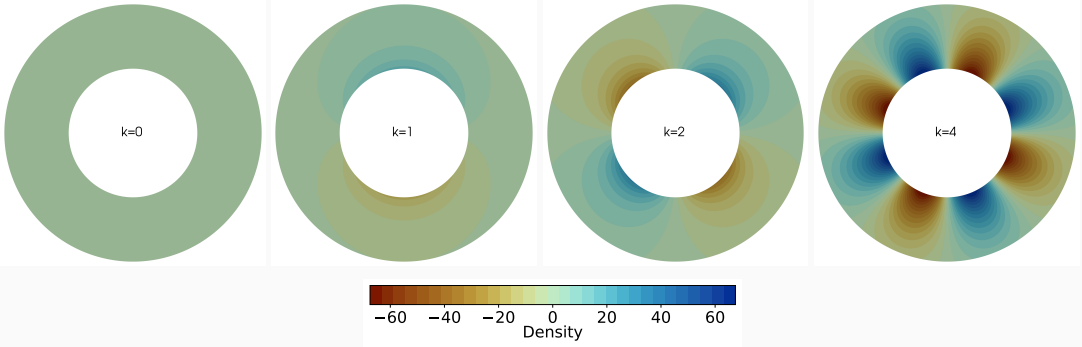
The parameter k controls the number of convection cells present in the domain.

In the present case, we set $R_1 = 1, R_2 = 2$ and $C = -1$.

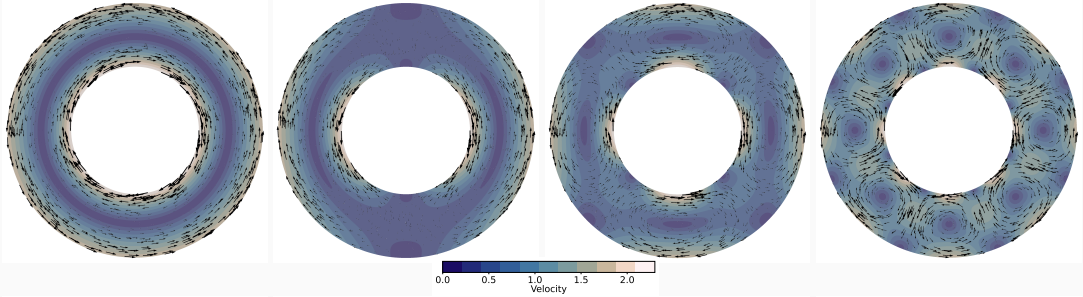
Boundary Condition

```
# boundary condition
v_diff =  v_uw.sym - v_ana.sym
stokes.add_natural_bc(vel_penalty*v_diff, "Upper")
stokes.add_natural_bc(vel_penalty*v_diff, "Lower")
```

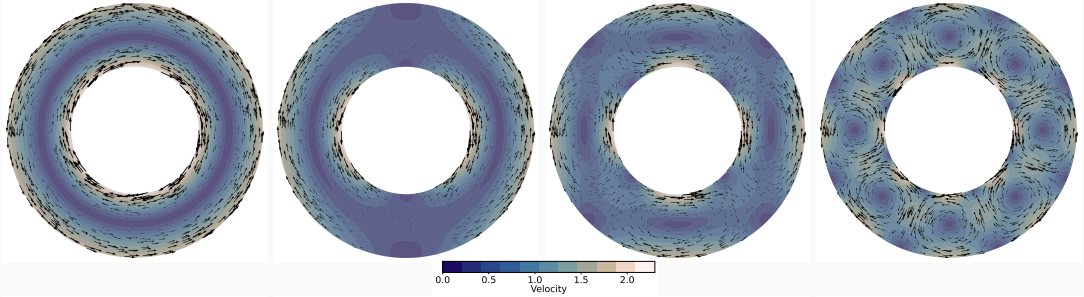
Density Distribution



Analytical Solution

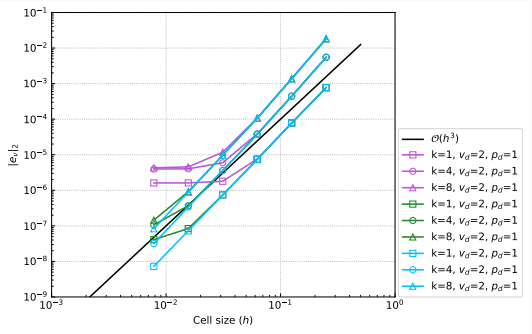


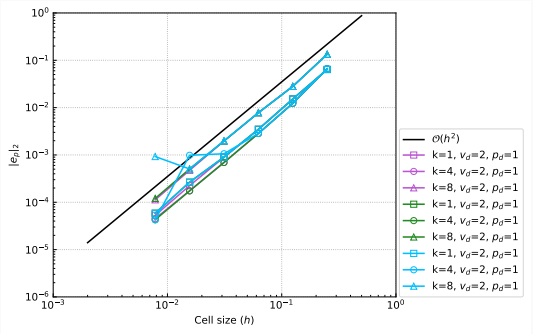
Numerical Solution



Error Convergence

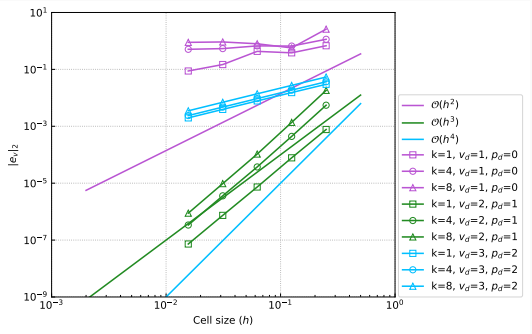
Velocity penalty: 2.5e6 (magenta), 1e8 (green), 1e10 (skyblue)
Stokes tolerance: 1e-10
Stokes element: P_2P_1

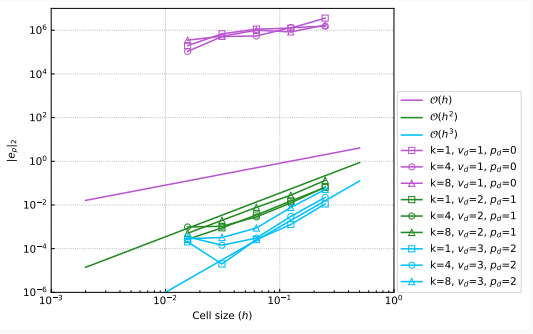




Error Convergence

Stokes elements: $P_1P_0(disc.pressure), P_2P_1, P_3P_2$
Velocity penalty: 2.5e6, 1e10, 1e10
Stokes tolerance: 1e-7, 1e-10, 1e-10





Conclusions

Annulus Benchmark: Kramer

Analytical Equations

Detailed derivation of the analytical equations are found:

Analytical solutions for mantle flow in cylindrical and spherical shells

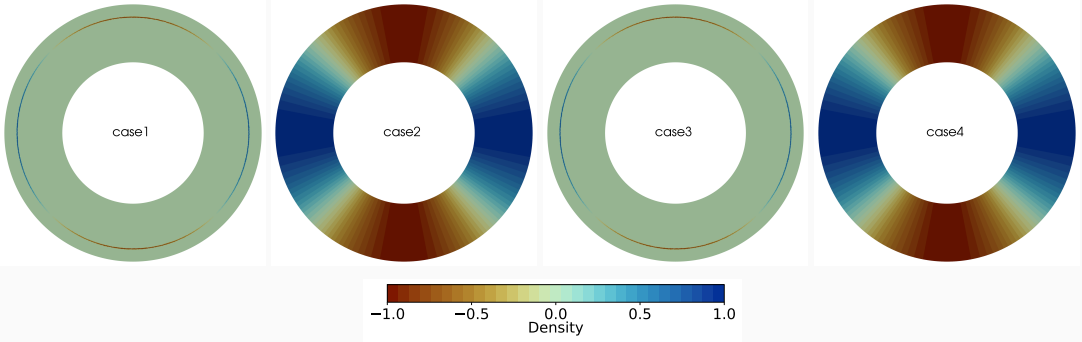
Python package to compute analytical solutions:

Assess

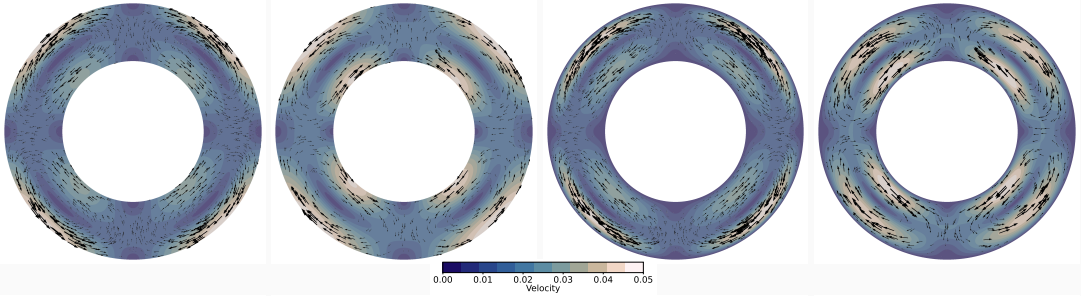
Boundary Condition

```
# boundary condition
v_diff =  v_uw.sym - v_ana.sym
stokes.add_natural_bc(vel_penalty*v_diff, "Upper")
stokes.add_natural_bc(vel_penalty*v_diff, "Lower")
```

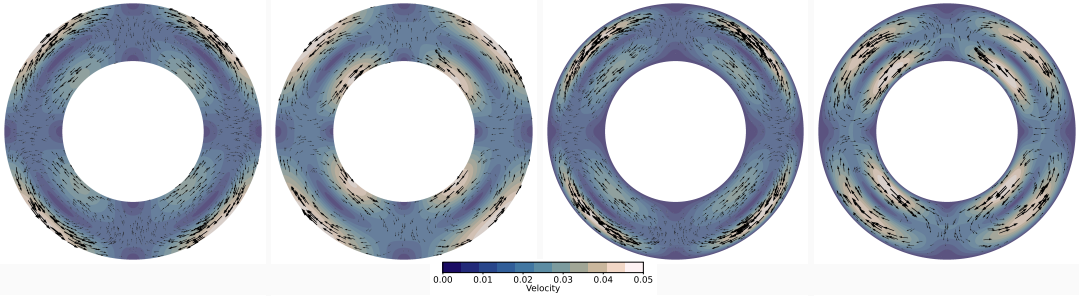
Density Distribution



Analytical Solution

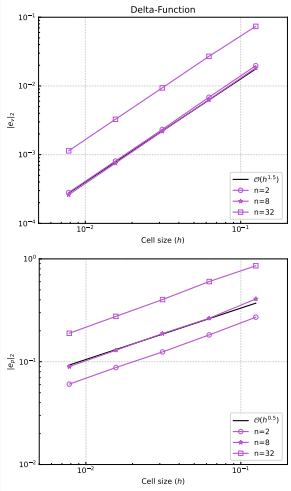


Numerical Solution



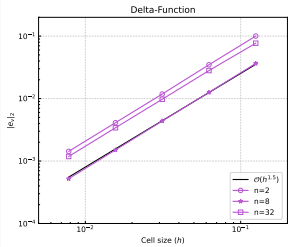
Error Convergence

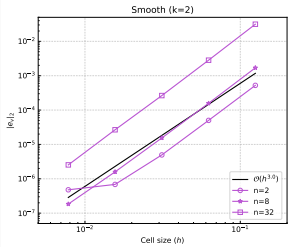
Free-Slip

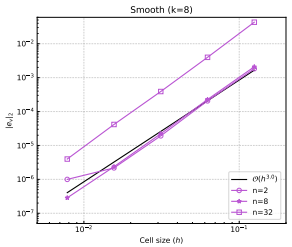


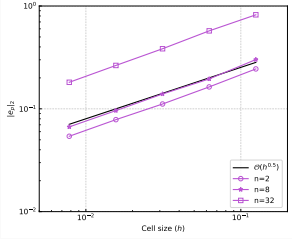
Error Convergence

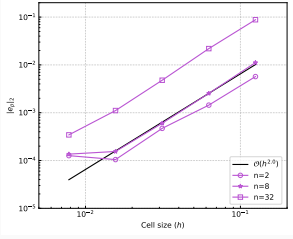
Zero-Slip

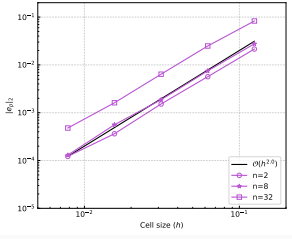












Conclusions