Big Data Computing

Master's Degree in Computer Science

2025-2026

SAPIENZA
UNIVERSITÀ DI ROMA

Gabriele Tolomei

Department of Computer Science

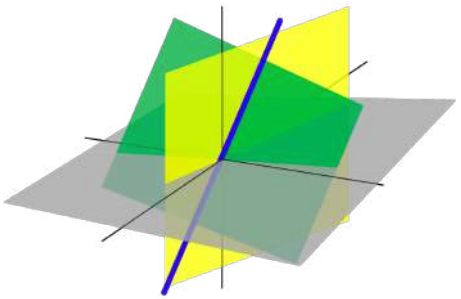
Sapienza Università di Roma

tolomei@di.uniroma1.it

PageRank's Interpretations

2 main perspectives

Linear Algebra



Probabilistic

Random Walk Interpretation of Page Rank

Imagine a random surfer navigating through the pages of the Web graph

Random Walk Interpretation of Page Rank

Initially, at time t=0 the surfer can be on any web page

www.donuts.com

www.krustyburger.com

www.duffbeer.com

…

www.moes.com

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Each web page has equal probability 1/N to be chosen as starting point

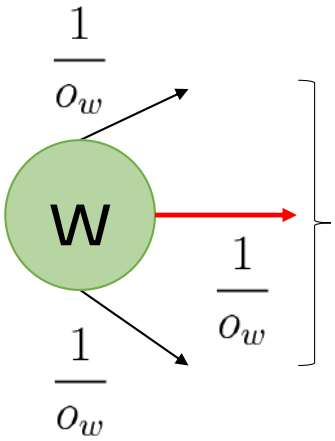
Random Walk Interpretation of Page Rank

At any given time t, the surfer is on some web page w

w

Random Walk Interpretation of Page Rank

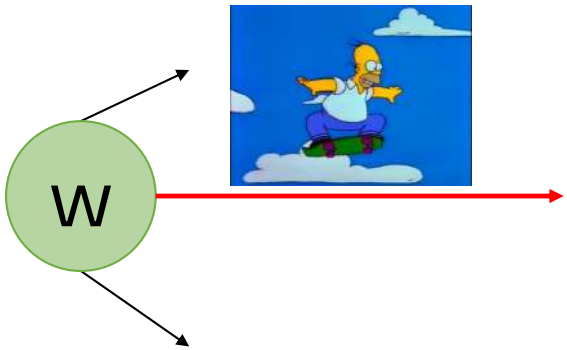
At time t+1, the surfer follows one of the outgoing links from web page w, chosen uniformly at random



O_w out-degree of node w

Random Walk Interpretation of Page Rank

The surfer ends up into some other web page v pointed by w



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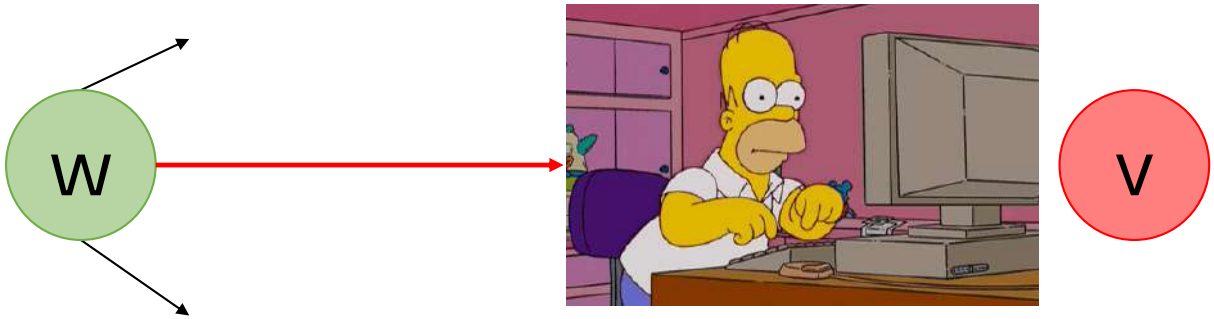

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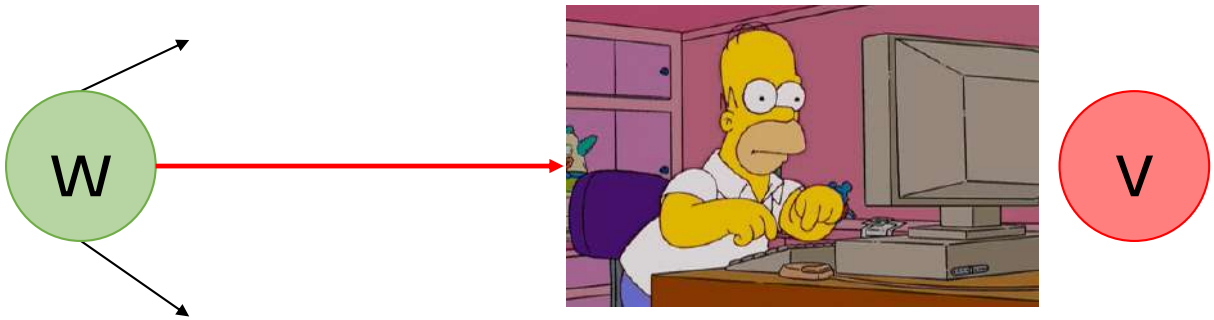

This process repeats indefinitely and is known as random walk

Transition Matrix M

$$\mathbf{M}_{N\times N} \quad m_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ 0 & \text{otherwise} \end{cases}$$ Column stochastic matrix

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Such a matrix describes a Markov chain over the finite state space V of nodes (i.e., pages) of the Web graph

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X Discrete-Valued Random Variable taking on |V| = N possible values

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Probability distribution over web pages at time t

Random Walks as Markov Chains

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The transition probability of moving to the next state depends only on the present state and not on the previous states

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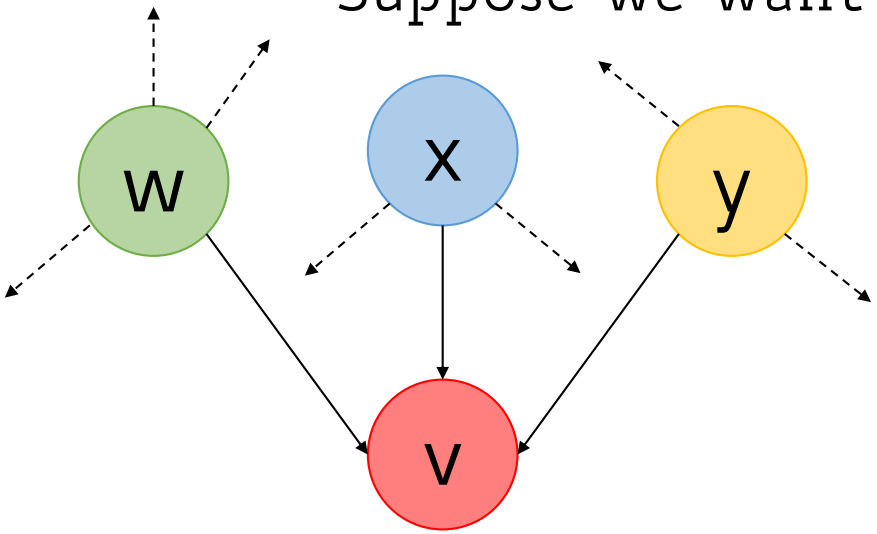
The probability that the random surfer will be on page v at time t+1 depends only on where the surfer was at time t

Random Walk Interpretation of Page Rank

Where is the random surfer at time t+1 knowing where he was at time t?

Suppose we want to estimate $P(X_{t+1} = v)$



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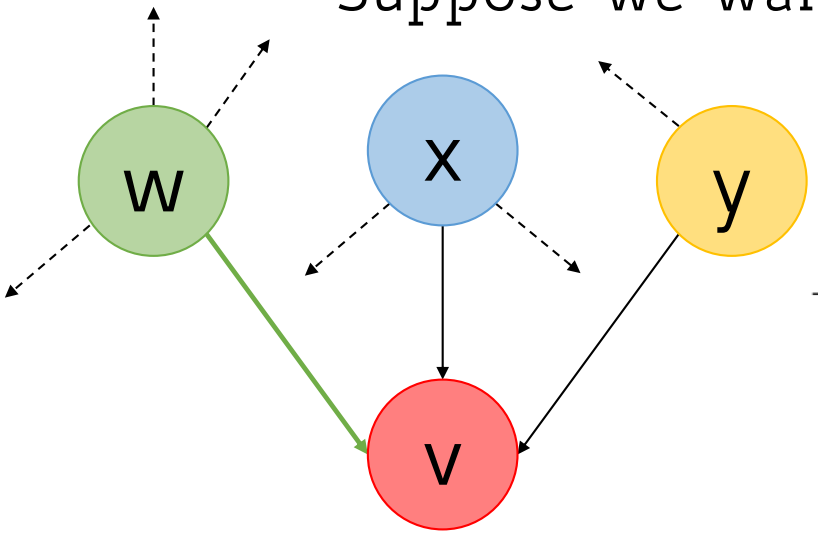

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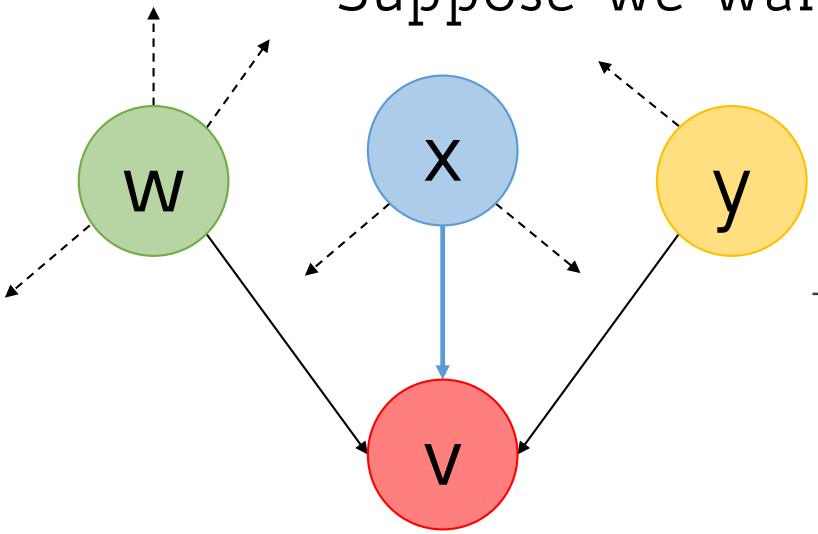

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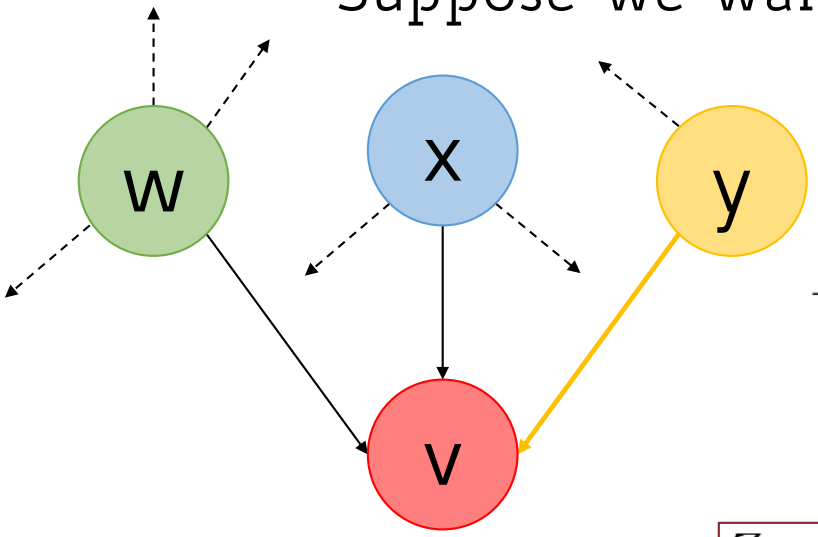

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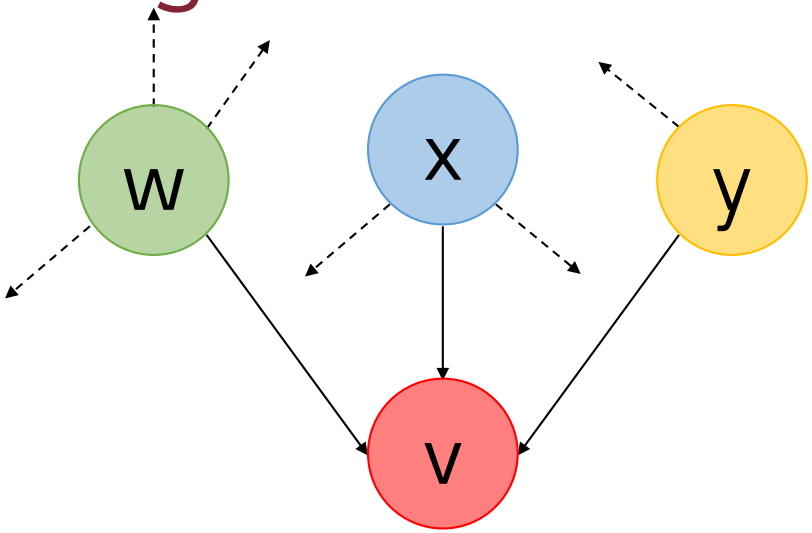
$$P(X_t = x, Z_x = v) +$$
$$P(X_t = y, Z_y = v)$$

$$Z_u \sim \text{Uniform}(1, o_u)$$

Random Walk Interpretation of Page Rank



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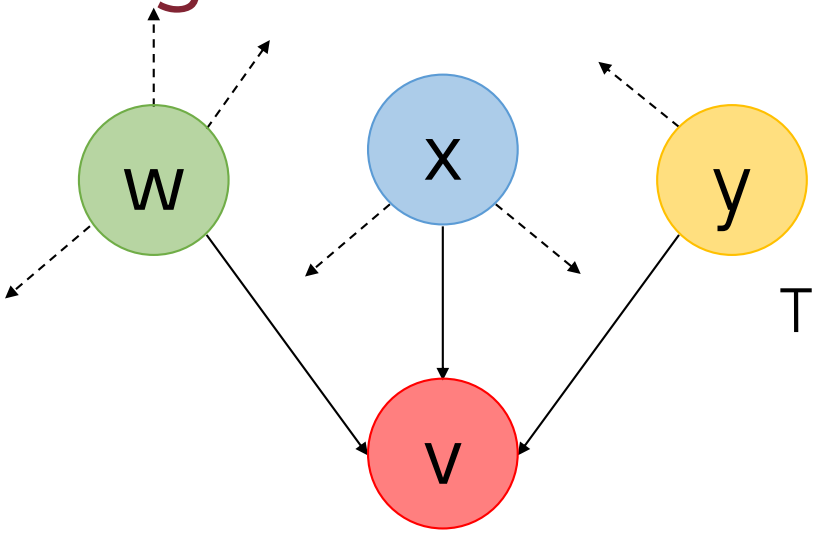

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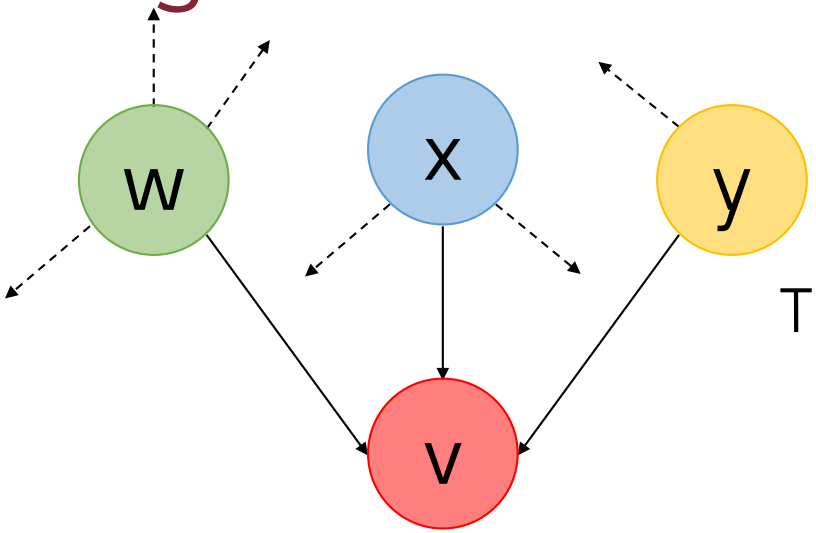

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This resembles our PageRank equation

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Solving the former is equivalent to solving the latter!

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More generally, the probability of visiting any web page after t steps is:

$$\boxed{\mathbf{p}(t) = \mathbf{M}^t\mathbf{p}(0)}$$

Random Walk Interpretation of Page Rank

$$\mathbf{p}(0) = (\ \underbrace{1/N}_{P(X_0=1)}\ , \ldots ,\ \underbrace{1/N}_{P(X_0=w)}\ , \ldots ,\ \underbrace{1/N}_{P(X_0=N)}\ )^T$$

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Random Walk Interpretation of Page Rank

$\{\mathbf{p}(t)\}_{t=0,1,\ldots,T}$

Discrete Stochastic Process

Markov chain

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p^* is the stationary distribution of the random walk

Equivalence between Formulations

Linear Algebra

Probabilistic

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System of linear "flow" equations

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So the PageRank vector r^* corresponds to the stationary distribution p^* for the random walk on the graph encoded by M!

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Intuitively, the PageRank vector indicates for each web page the probability that a random surfer will eventually get to that page

Hang on a Second…

Linear Algebra

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How do we know that the power iteration method always converge to r^*?

existence

How do we know that r^* is unique?

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existence and uniqueness of r^* (p^*) are guaranteed under certain conditions on the matrix M

Existence and Uniqueness of PageRank

If M is a column stochastic matrix with all positive entries:

- $\lambda = 1$ is an eigenvalue of M with multiplicity one
- $\lambda = 1$ is the largest eigenvalue of M
- There exists a unique (right) eigenvector r^* associated with the eigenvalue $\lambda = 1$ with the sum of its entries equal to 1

Perron-Frobenius theorem (circa 1910)

Existence and Uniqueness of PageRank

If M is a column stochastic matrix with all positive entries, then M has a unique steady-state vector p^* such that for any p(0)

$$\mathbf{p}(t) = \mathbf{M}^t\mathbf{p}(0) \quad \text{converges to } \mathbf{p}^* \text{ as } t \to \infty$$

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The steady-state vector is the unique eigenvector associated with the largest eigenvalue $\lambda = 1$

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$$\mathbf{M}_1 = \begin{bmatrix} 0.6 & 0.5 & 0 \\ 0.4 & 0.3 & 1 \\ 0 & 0.2 & 0 \end{bmatrix} \qquad \mathbf{M}_2 = \begin{bmatrix} 0.6 & 0.5 & 0.1 \\ 0.2 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0.5 \end{bmatrix}$$

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Both M_1 and M_2 are column stochastic, but only M_2 is positive

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We show how they fixed the issues with the original definition of M to accommodate for the heterogeneity of the Web graph

By doing so, we know that a solution to our PageRank problem exists and is unique!

Google's PageRank

Problems with Original PageRank Formulation

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Then we discuss how Brin and Page fixed this in their seminal paper which sets up the rising of Google

Problems with Original PageRank Formulation

2 main issues to solve:

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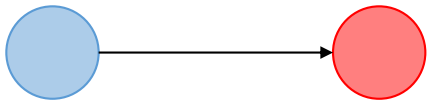
Dead End

Pages with no outlinks cause PageRank to leak out



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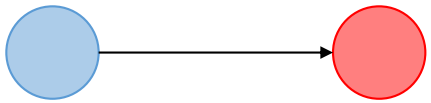
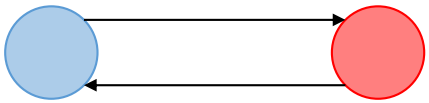


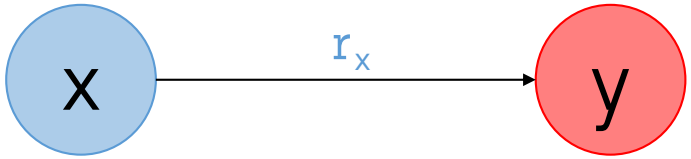
Spider Trap

Not every node is reachable and PageRank gets eventually absorbed by a few pages



The "Dead End" Problem (Dangling Nodes)

Example:



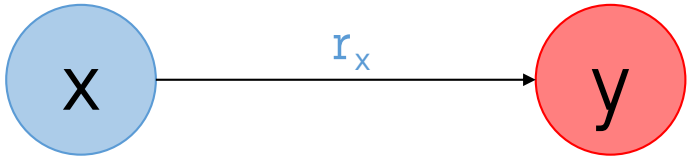
$$r_y = r_x$$

$\mathbf{M}$

0	0
1	0

The "Dead End" Problem (Dangling Nodes)

Example:



$$r_y = r_x$$

$\mathbf{M}$

0	0
1	0

When a web page has no outgoing links (dangling node) the resulting column vector in the matrix M is not stochastic anymore!

Previously, we assumed each web page has at least one outgoing link, and therefore M was stochastic

The "Dead End" Problem (Dangling Nodes)

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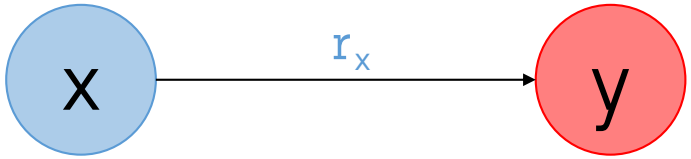

$$r_y = r_x$$

$\mathbf{M}$

0	0
1	0

Assume the following initialization for r:

$$\mathbf{r}(0) = \begin{bmatrix} r_x^{(0)} \\ r_y^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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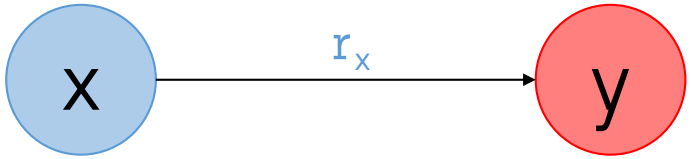

$$r_y = r_x$$

$\mathbf{M}$

0	0
1	0

$$\mathbf{r}(1) = \mathbf{M}\,\mathbf{r}(0)$$

0	=	0	0	1
1		1	0	0

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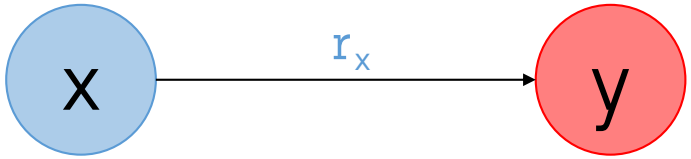

$$r_y = r_x$$

$$\mathbf{M}$$

0	0
1	0

$$\mathbf{r}(2) = \mathbf{M}\ \mathbf{r}(1)$$

0	=	0	0	0
0		1	0	1

The "Dead End" Problem (Dangling Nodes)

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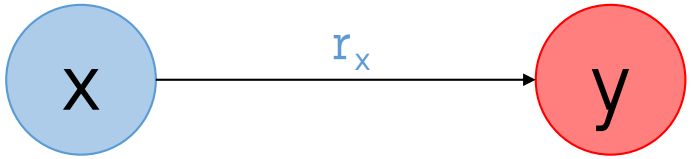


$$r_y = r_x$$

$\mathbf{M}$

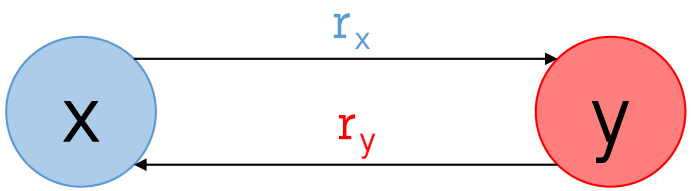
0	0
1	0

$\mathbf{r}(0)$	$\mathbf{r}(1)$	$\mathbf{r}(2)$	...	$\mathbf{r}(t-1)$	$\mathbf{r}(t)$
1	0	0		0	0
0	1	0		0	0

The PageRank vector vanishes to 0!

The "Spider Trap" Problem

Example:



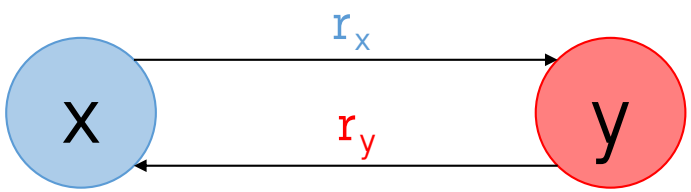
$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

$\mathbf{M}$

0	1
1	0

The "Spider Trap" Problem

Example:



$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

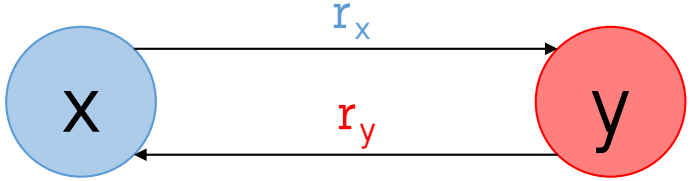
$\mathbf{M}$

0	1
1	0

M is column stochastic non-negative (but not strictly positive)
Does PageRank converge regardless of the initialization of r?

The "Spider Trap" Problem

Example:



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1	0

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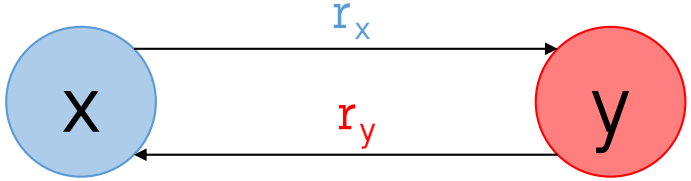

$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

$\mathbf{M}$

0	1
1	0

$$\mathbf{r}(1) = \mathbf{M}\,\mathbf{r}(0)$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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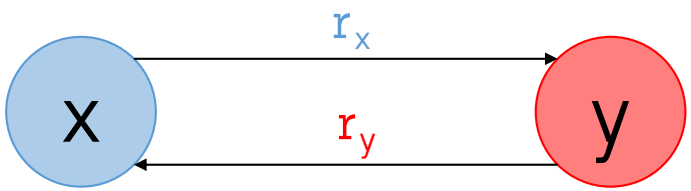
Example:



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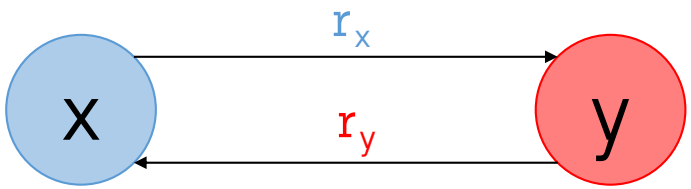

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$\mathbf{r}(0)$	$\mathbf{r}(1)$	$\mathbf{r}(2)$	...	$\mathbf{r}(t-1)$	$\mathbf{r}(t)$
1	0	1		0	1
0	1	0		1	0

The PageRank vector keeps alternating its components and never converges!

Problems with Original PageRank Formulation

2 main issues to solve:

Dead End

Pages with no outlinks cause PageRank to leak out

Spider Trap

Not every node is reachable and PageRank gets eventually absorbed by a few pages

Problems with Original PageRank Formulation

2 main issues to solve:

Dead End

Pages with no outlinks cause PageRank to leak out

M is not column stochastic as some nodes have no outlinks

Spider Trap

Not every node is reachable and PageRank gets eventually absorbed by a few pages

Problems with Original PageRank Formulation

2 main issues to solve:

Dead End

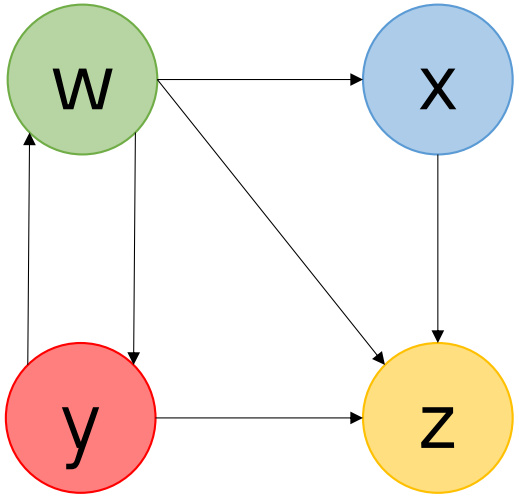
Pages with no outlinks cause PageRank to leak out

Spider Trap

Not every node is reachable and PageRank gets eventually absorbed by a few pages

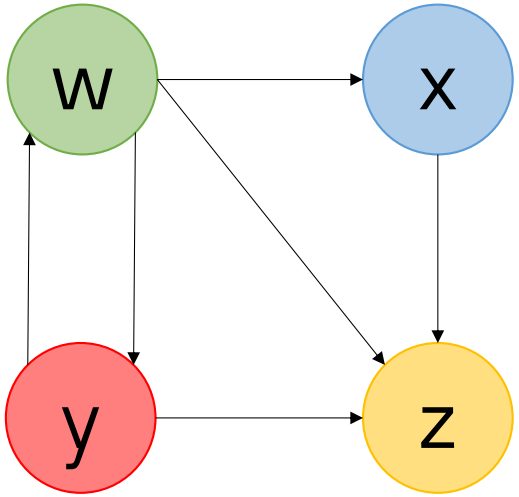
M is stochastic but not strictly positive

Deal with Dangling Nodes



$$\mathbf{M} = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1 & 1/2 & 0 \end{bmatrix} \end{array}$$

Deal with Dangling Nodes



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z is a dangling node

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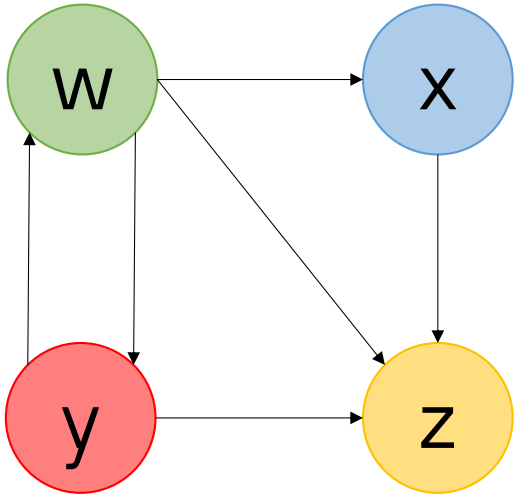

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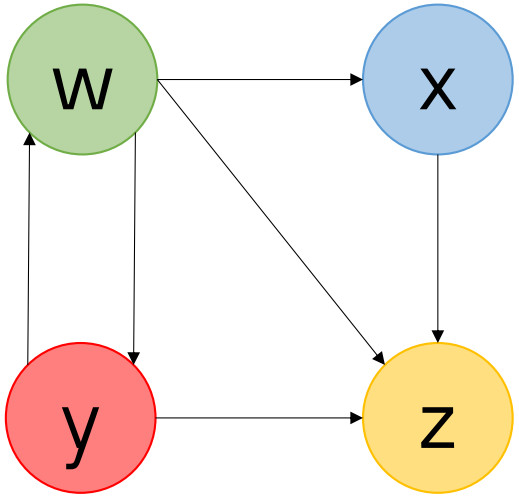
M is not (column) stochastic

Deal with Dangling Nodes



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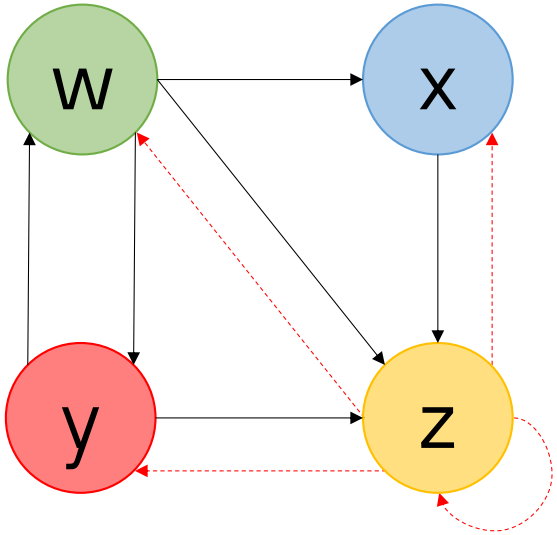
If we apply simplified PageRank to M the rank vector r will eventually vanish to 0

Deal with Dangling Nodes



$$\mathbf{M}' = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 1/2 & 1/4 \\ 1/3 & 0 & 0 & 1/4 \\ 1/3 & 0 & 0 & 1/4 \\ 1/3 & 1 & 1/2 & 1/4 \end{bmatrix} \end{array}$$

Solution: Teleporting

Create artificial links from any dangling node to any other node

Deal with Dangling Nodes: Teleporting

This adjustment is justified by modeling the behaviour of a web surfer

Deal with Dangling Nodes: Teleporting

This adjustment is justified by modeling the behaviour of a web surfer

After reading a page with no out-going link, jump to a page picked uniformly at random amongst the N

Deal with Dangling Nodes: Teleporting

Initially, we set $\mathbf{M}_{N\times N}$ $m_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ 0 & \text{otherwise} \end{cases}$

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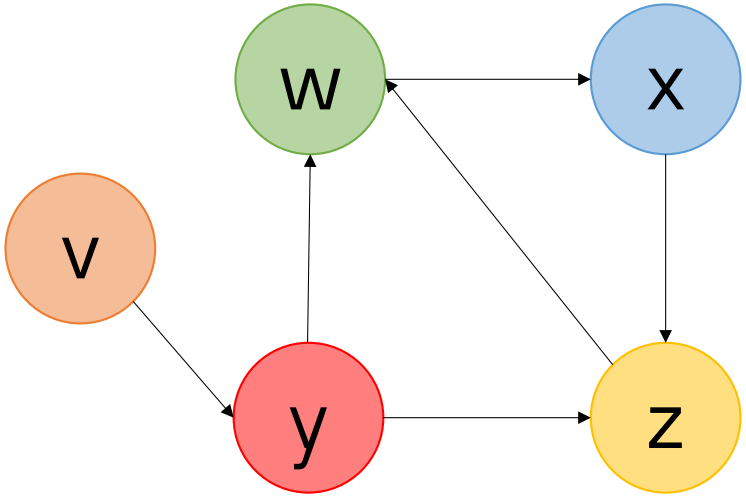
$$\mathbf{M} \rightsquigarrow \mathbf{M}'$$

This transformation allows M' to be column stochastic

Deal with Spider Traps



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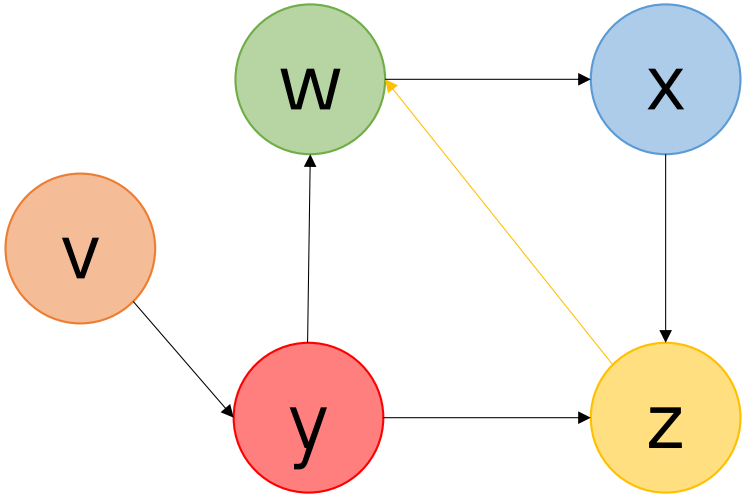
Deal with Spider Traps



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z is not a dangling node anymore

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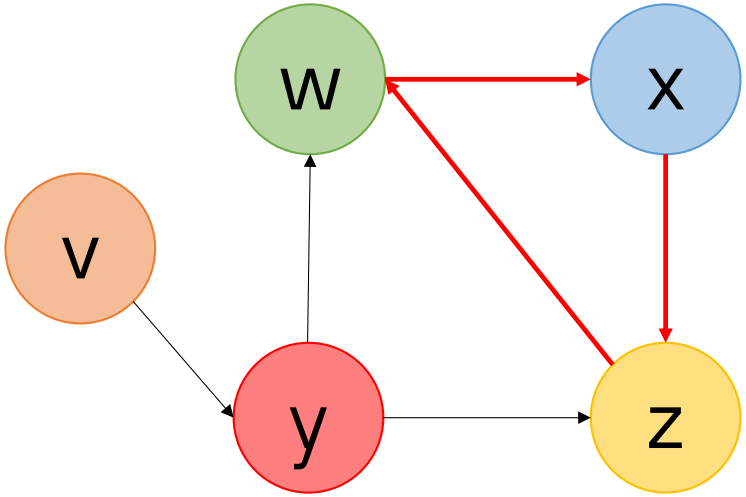
Deal with Spider Traps



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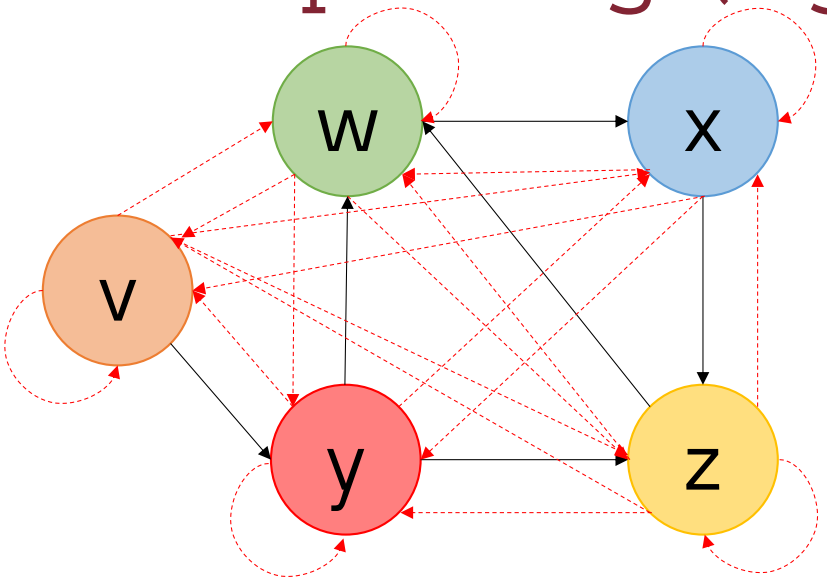
If we apply simplified PageRank to M some entries of the rank vector r will eventually drop to 0, as we get stuck in w, x, z

Deal with Spider Traps: Teleporting (Again!)



$$\mathbf{M}' = \begin{array}{c} \\ v \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccccc} v & w & x & y & z \end{array} \\ \begin{bmatrix} 0.03 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.455 & 0.88 \\ 0.03 & 0.88 & 0.03 & 0.03 & 0.03 \\ 0.88 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.88 & 0.455 & 0.03 \end{bmatrix} \end{array}$$

Solution: Probabilistic Teleporting

Create artificial links from each node to every other node and follow each of it with probability (1-d)/N

Deal with Spider Traps: Probabilistic Teleporting

To avoid the surfer to get stuck in a spider trap

Deal with Spider Traps: Probabilistic Teleporting

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On each page w the surfer will either follow one of its outgoing links with probability d or jump to another page with probability (1-d)

Deal with Spider Traps: Probabilistic Teleporting

To avoid the surfer to get stuck in a spider trap

On each page w the surfer will either follow one of its outgoing links with probability d or jump to another page with probability (1-d)

d is called damping factor

d = 0.85 in the original Google formulation

The Google's PageRank Formulation

$$\mathbf{M}_{N\times N} \quad m_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ 0 & \text{otherwise} \end{cases}$$

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Ensure the matrix is stochastic

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$$\mathbf{G} = d\mathbf{M}' + \frac{1-d}{N} \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}}_{\mathbf{1}_{N\times N}}$$

$\mathbf{M} \rightsquigarrow \mathbf{M}'$

Ensure the matrix is stochastic

$\mathbf{M}' \rightsquigarrow \mathbf{G}$

Ensure the matrix is strictly positive

Why Does Teleporting Solve Our Problem?

$$\mathbf{G} = d\mathbf{M}' + \frac{1-d}{N}\underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}}_{\mathbf{1}_{N\times N}}$$

The matrix G so modified is (column) stochastic and strictly positive

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The Perron-Frobenius theorem now applies to G and guarantees the existence (convergence) and uniqueness of the steady-state eigenvector $\mathbf{r}^*$

$$\mathbf{r}(t) = \mathbf{G}^t\mathbf{r}(0)$$

$$\mathbf{r} \rightsquigarrow \mathbf{r}^* \text{ as } t \to \infty$$

How Do We Actually Compute PageRank?

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Problem:
G represents a fully-connected graph with a huge number of nodes (web pages)

G is a dense matrix

How Do We Actually Compute PageRank?

Assuming the number of web pages in the graph is $N=10^9$

G will have N^2 entries = 10^{18}

Say each entry is stored using a 32-bit integer
(i.e., 4 bytes per entry)

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Note: The Web contains far more than $N=10^9$ pages!

Re-Arrange the Equation

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$$r_v = \sum_{w=1}^{N} d\mathbf{M}'_{v,w} \times r_w + \sum_{w=1}^{N} \frac{1-d}{N} \times r_w \qquad r_v = \sum_{w=1}^{N} d\mathbf{M}'_{v,w} \times r_w + \frac{1-d}{N} \underbrace{\sum_{w=1}^{N} r_w}_{=1}$$

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We can work with M' rather than G

Re-Arrange the Equation

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At each iteration we can compute PageRank vector as follows:

1. $\mathbf{r}(t+1) = d\mathbf{M}'\mathbf{r}(t)$
2. $\mathbf{r}(t+1) = \mathbf{r}(t+1) + \left[\frac{1-d}{N}\right]_{N\times 1}$

Add the constant (1-d)/N to each component of r(t+1)

PageRank: Pseudocode

Algorithm: PageRank

Input : A directed Web graph $G = (V, E)$, where $|V| = N$ and its associated matrix $\mathbf{M}_{N\times N}$ defined as follows: $\mathbf{M}_{v,w} = \frac{1}{o_w}$ if w points to v, 0 otherwise ($o_w = |O_w|$ where $O_w = \{x \in V : (w, x) \in E\}$);
A *damping factor* $d \in (0, 1)$;
A *tolerance* $\epsilon > 0$.

Output: The PageRank vector $\mathbf{r}^*_{N\times 1}$

Init : $t \leftarrow 0; \mathbf{r}(t) \leftarrow \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$;

repeat
  $t \leftarrow t + 1$;
  /* Compute the temporary PageRank score of every page v */
  for $i \leftarrow 1$ **to** N **do**
    $r_v^{\text{tmp}}(t) \leftarrow \sum_{w\in I_v} \frac{r_w(t-1)}{o_w}$; /* $r_v^{\text{tmp}}(t) = 0$ if v has no in-links */
  end
  /* Adjust the PageRank score of each page v with *teleporting* */
  for $i \leftarrow 1$ **to** N **do**
    $r_v(t) \leftarrow d \times r_v^{\text{tmp}}(t) + \frac{1-d}{N}$;
  end
until $|\mathbf{r}(t) - \mathbf{r}(t-1)| < \epsilon$
return $\mathbf{r}^* = \mathbf{r}(t)$;

Take-Home Message of Today

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- Represent the Web graph as a matrix M, where a link from page *w* to *v* is a **vote** from *w* to *v*

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- We present an example of **link analysis** algorithm: **PageRank**
- **Goal:** Find an **importance score** for each web page
- Represent the Web graph as a matrix M, where a link from page *w* to *v* is a **vote** from *w* to *v*
- 2 different yet equivalent approaches:
  - **Linear Algebra** → Matrix eigenvector
  - **Probabilistic** → Stationary distribution of Markov chain (**random walk**)

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- The Web graph is **disconnected** and may contain **no-exit loops**
- Google solution: **probabilistic teleport links**
- Still efficiently computable from the original, sparse matrix M