Big Data Computing

Master's Degree in Computer Science

2025-2026

SAPIENZA
UNIVERSITÀ DI ROMA

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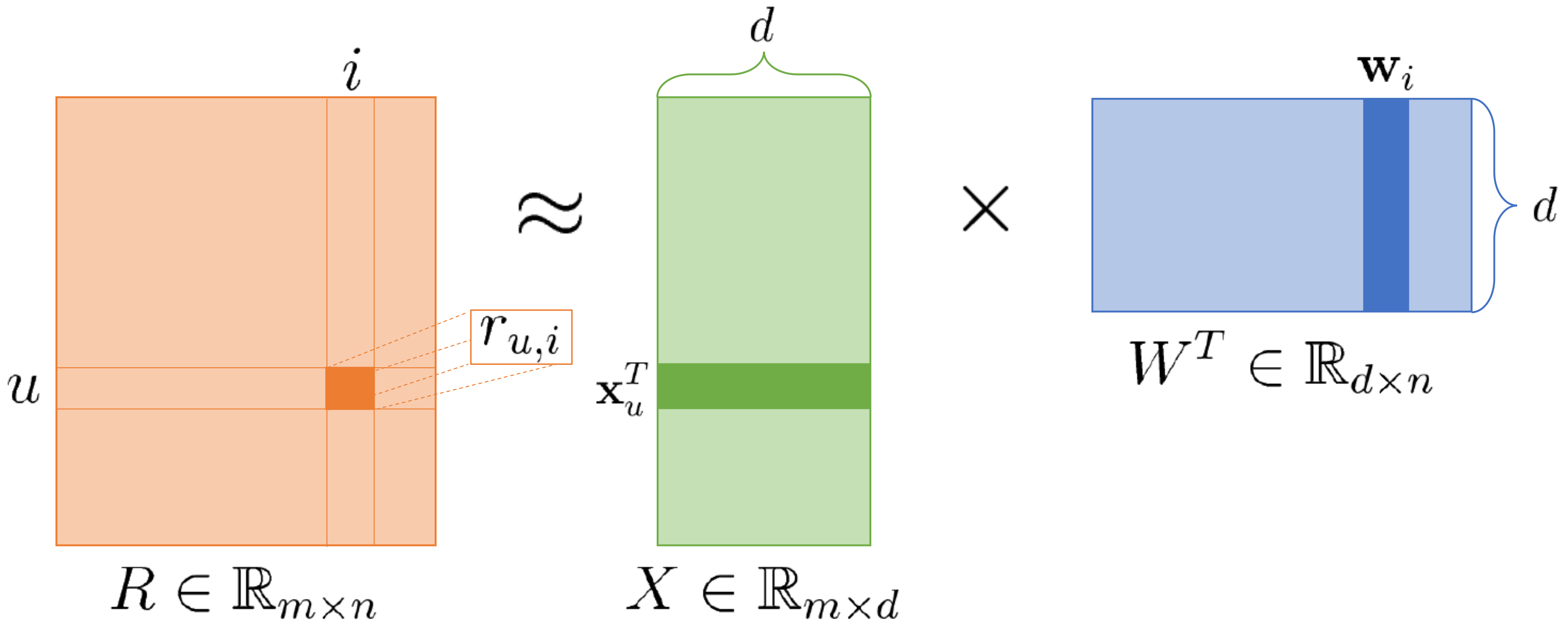
Matrix Factorization Framework


i
u
$r_{u,i}$
$R \in \mathbb{R}_{m\times n}$
$\approx$
d
$\mathbf{x}_u^T$
$X \in \mathbb{R}_{m\times d}$
$\times$
$\mathbf{w}_i$
d
$W^T \in \mathbb{R}_{d\times n}$

Approximate the user-item rating matrix R with the product of X x W^T

How Do We Learn X and W?

Assuming we have access to a dataset of observed ratings

The matrix R is partially known and filled with those observations

To actually learn the latent factor representations x_u and w_i we minimize the following loss function

$$L(X,W) = \sum_{(u,i)\in\mathcal{D}} \underbrace{\left(r_{u,i} - \mathbf{x}_u^T \cdot \mathbf{w}_i\right)^2}_{\text{squared error term}} + \underbrace{\lambda\left(\sum_{u\in\mathcal{D}} ||\mathbf{x}_u||^2 + \sum_{i\in\mathcal{D}} ||\mathbf{w}_i||^2\right)}_{\text{regularization term}}$$

Training set of observed ratings

Matrix Factorization: Optimization

$$X^*, W^* = \text{argmin}_{X,W} L(X, W)$$

$$X^*, W^* = \text{argmin}_{X,W} \sum_{(u,i)\in\mathcal{D}} \left(r_{u,i} - \mathbf{x}_u^T \cdot \mathbf{w}_i\right)^2 + \lambda\Big(\sum_{u\in\mathcal{D}} ||\mathbf{x}_u||^2 + \sum_{i\in\mathcal{D}} ||\mathbf{w}_i||^2\Big)$$

$$X^*, W^* = \text{argmin}_{X,W} \left\{\frac{1}{2} \sum_{(u,i)\in\mathcal{D}} \left(r_{u,i} - \mathbf{x}_u^T \cdot \mathbf{w}_i\right)^2 + \lambda\Big(\sum_{u\in\mathcal{D}} ||\mathbf{x}_u||^2 + \sum_{i\in\mathcal{D}} ||\mathbf{w}_i||^2\Big)\right\}$$

Still, how do we solve this?

Learning Algorithms

$\boxed{\text{2 main optimization methods}}$

Learning Algorithms

2 main optimization methods

Stochastic Gradient Descent (SGD)

Learning Algorithms

$$\begin{tikzcd}
 & \boxed{\text{2 main optimization methods}} \arrow[dl] \arrow[dr] & \\
\text{Stochastic Gradient Descent (SGD)} & & \text{Alternating Least Squares (ALS)}
\end{tikzcd}$$

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$$\nabla L(\mathbf{x}_u; \mathbf{w}_i) = \frac{1}{2}\Big[-2(r_{u,i} - \mathbf{x}_u^T \cdot \mathbf{w}_i)\mathbf{w}_i + 2\lambda\mathbf{x}_u\Big] = -(r_{u,i} - \mathbf{x}_u^T \cdot \mathbf{w}_i)\mathbf{w}_i + \lambda\mathbf{x}_u$$

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We know that the updating strategy for SGD is as follows:

$$\mathbf{x}_u^{(t+1)} \leftarrow \mathbf{x}_u^{(t)} - \eta \nabla L(\mathbf{x}_u^{(t)}; \mathbf{w}_i^{(t)})$$

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At each iteration, both user and item latent vectors are updated by a magnitude proportional to η in the opposite direction of the gradient

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We define the prediction error associated with each training instance (u, i)

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Stochastic Gradient Descent (SGD)

- **Key Properties:**
  - Incremental → updates occur sample by sample
  - Easy to implement
  - Scales well when data is sparse
  - Order matters → the sequence of updates influences convergence
  - Learning rate (η) must be tuned carefully
  - Converges to a local optimum, but sometimes slowly

Stochastic Gradient Descent (SGD)

- **Good for:**
  - Very large, sparse datasets (Netflix-like settings)
  - Online/streaming updates (new ratings can be incorporated incrementally)
  - Systems where simplicity and speed per update matter

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- SGD has been shown to work well optimizing MF models
- However, it is not a popular choice if the dimensionality of the original rating matrix R is high
- Indeed, there are $d(m+n)$ parameters to optimize
- In real life problems, this number can get very large quite often, requiring both a parallelization mechanism or an alternative optimizer

Alternating Least Squares (ALS)

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- When one latent vector is fixed, the objective becomes quadratic (i.e., convex) and therefore can be solved optimally
- Each alternating iteration reduces to traditional least squares and can be solved using OLS or its regularized variant (e.g., pseudo-inverse)

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We want to set this to $\boxed{-\sum_{i\in\mathcal{D}}(r_{u,i} - \mathbf{x}_u^T \cdot \mathbf{w}_i)\mathbf{w}_i + \lambda\mathbf{x}_u = 0}$

ALS: Item Vector Fixed

$$-\sum_{i \in \mathcal{D}} (r_{u,i} - \mathbf{x}_u^T \cdot \mathbf{w}_i)\mathbf{w}_i + \lambda \mathbf{x}_u = 0$$

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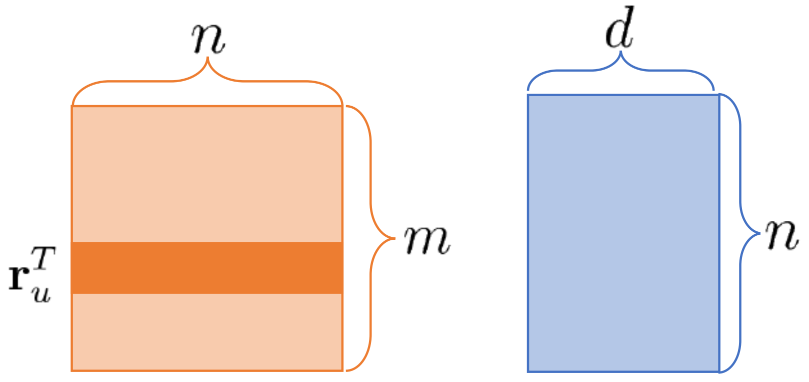

user-item rating matrix R item-matrix W

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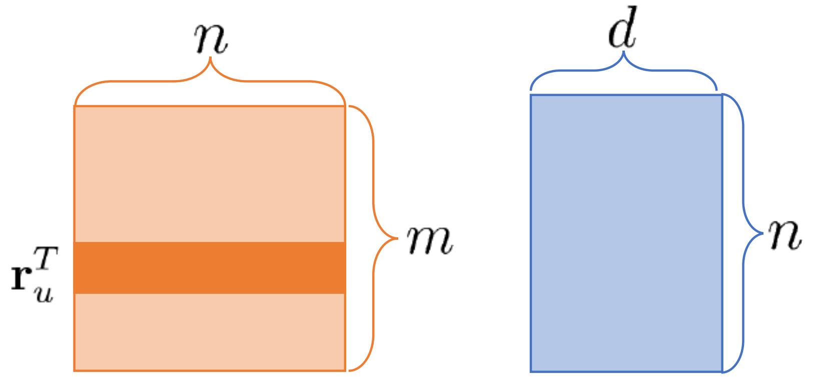

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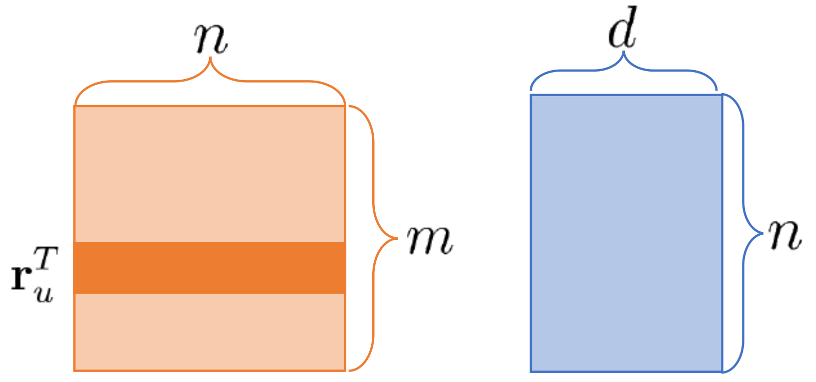

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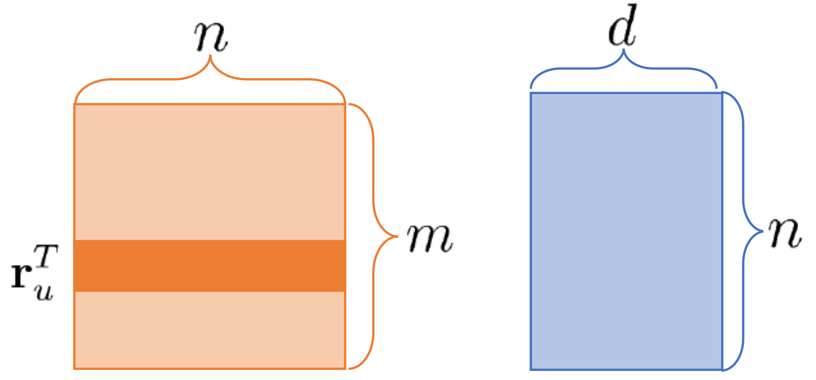
$$= (W^T W + \lambda I)^{-1} \cdot W^T \cdot \mathbf{r}_u = \mathbf{x}_u(W^T W + \lambda I) \cdot (W^T W + \lambda I)^{-1}$$



user-item rating matrix R item-matrix W

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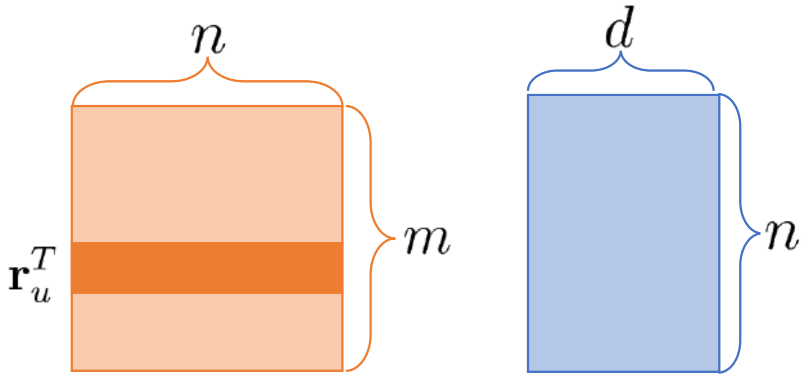

user-item rating matrix R item-matrix W

ALS: User Vector Fixed

$$-\sum_{u\in\mathcal{D}}(r_{u,i} - \mathbf{x}_u^T \cdot \mathbf{w}_i)\mathbf{x}_u + \lambda\mathbf{w}_i = 0$$

$$= -X^T(\mathbf{r}_i - X \cdot \mathbf{w}_i) - \lambda\mathbf{w}_i = 0$$

$$= X^T \cdot \mathbf{r}_i = X^T X \cdot \mathbf{w}_i + \lambda\mathbf{w}_i$$

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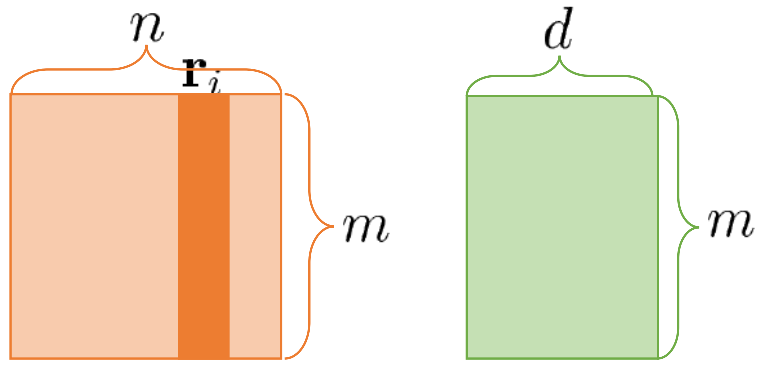

user-item rating matrix R user-matrix X

$$= (X^T X + \lambda I)^{-1} \cdot X^T \cdot \mathbf{r}_i = \mathbf{w}_i\cancel{(X^T X + \lambda I)} \cdot \cancel{(X^T X + \lambda I)}^{-1}$$

$$\boxed{\mathbf{w}_i = (X^T X + \lambda I)^{-1} \cdot X^T \cdot \mathbf{r}_i}$$

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1. Initialize all the user latent vectors X and all the item latent vectors W randomly
2. Fix all the item vectors W and solve for X (users)
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4. Repeat step 2 and 3 until convergence

Convergence is guaranteed because in each step the loss function either decreases or stays unchanged, never increases

ALS: Pseudocode

- <u>Key Properties:</u>
  - Deterministic updates → each alternating step is a guaranteed minimizer for the constrained objective (closed-form)
  - No learning rate → much easier to tune
  - Parallelizable because each user (or item) update is independent
  - Typically converges in fewer iterations (but each iteration is heavier computationally)
  - Solves large linear systems → uses more memory and computational power

ALS: Pseudocode

- Good for:
  - Distributed systems (Spark, Hadoop) where parallel computation is essential
  - Dense or moderately sparse datasets
  - Scenarios where stable, monotonic convergence is important

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SGD vs. ALS

- In general, SGD is easier and faster than ALS
- However, ALS is favorable in at least 2 cases:
  - **Parallelization:** each $\mathbf{x}_u$ and $\mathbf{w}_i$ is computed independently of user/item factors
  - **Implicit Data:** the training set is dense and looping over each single instance - as SGD does - would be unfeasible

SGD vs. ALS

Aspect	SGD	ALS
Update style	Incremental (rating by rating)	Block-wise (solve full least-squares problems)
Convergence	Potentially slow, noisy	Faster in iterations, monotonic
Parallelism	Hard to parallelize safely	Highly parallelizable
Hyperparameters	Learning rate must be tuned	Only regularization (λ)
Implementation complexity	Very simple	More complex (matrix inversions)
Dataset size	Very large-scale sparse	Large-scale, especially distributed
Adaptability	Good for online/streaming	Batch-oriented
Stability	Sensitive to η	Very stable, no oscillations
Hardware	Works on a single CPU/GPU easily	Thrives on clusters / distributed systems

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- The basic learning framework tries to capture the interactions between users and items that produce the different rating values

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- However, much of the observed variation in ratings depends on **biases** associated with users or items, independent of any interactions
- For example, some users systematically tend to give higher ratings than others, and some items receive higher ratings than others

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$$b_{u,i} = \mu + b_u + b_i$$

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Overall avg. rating

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Observed deviations of user u from the avg.

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Observed deviations of item i from the avg.

Including Biases: Example

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$\boxed{b_{\text{Joe}} = -0.3}$ Joe is a critical user who tends to give 0.3 less stars than avg.

$$\boxed{b_{\text{Joe,Titanic}} = 3.7 - 0.3 + 0.5 = 3.9}$$

Bias term

Including Bias into the Optimization

$$\hat{r}_{u,i} = \underbrace{\mathbf{x}_u^T \cdot \mathbf{w}_i}_{\text{latent factors}} + \underbrace{\mu + b_u + b_i}_{\text{bias}}$$

$$\boxed{\text{The estimated rating of an item i for the user u is now made of 2 components}}$$

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Latent factor term

models user-item interaction

Including Bias into the Optimization

$$\hat{r}_{u,i} = \underbrace{\mathbf{x}_u^T \cdot \mathbf{w}_i}_{\text{latent factors}} + \underbrace{\mu + b_u + b_i}_{\text{bias}}$$

The estimated rating of an item i for the user u is now made of 2 components

Latent factor term

models user-item interaction

Bias term

models global average, user and item bias

Including Bias into the Optimization

Overall, the original optimization problem becomes as follows

$$X^*, W^* = \operatorname{argmin}_{X,W} \left\{ \frac{1}{2} \sum_{(u,i)\in\mathcal{D}} \left[r_{u,i} - (\mathbf{x}_u^T \cdot \mathbf{w}_i + \mu + b_u + b_i) \right]^2 + \lambda \left(\sum_{u\in\mathcal{D}} ||\mathbf{x}_u||^2 + \sum_{i\in\mathcal{D}} ||\mathbf{w}_i||^2 + \sum_{u\in\mathcal{D}} b_u^2 + \sum_{i\in\mathcal{D}} b_i^2 \right) \right\}$$

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Can still be solved using ALS

Collaborative Filtering: Drawbacks

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sparsity

the vast majority of items are not rated by users

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- Netflix is a good example of hybrid recommender systems

Netflix's Hybrid Recommender System

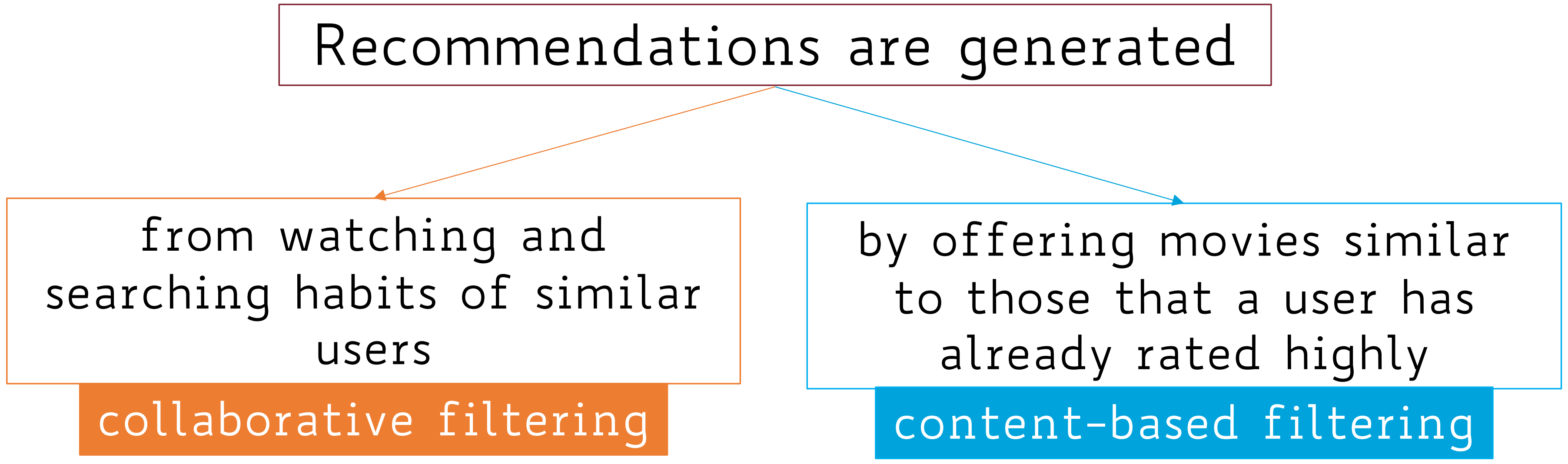
Recommendations are generated

Netflix's Hybrid Recommender System

$$\begin{tikzcd}
\boxed{\text{Recommendations are generated}} \arrow[d] \\
\boxed{\text{from watching and searching habits of similar users}}
\end{tikzcd}$$

collaborative filtering

Netflix's Hybrid Recommender System



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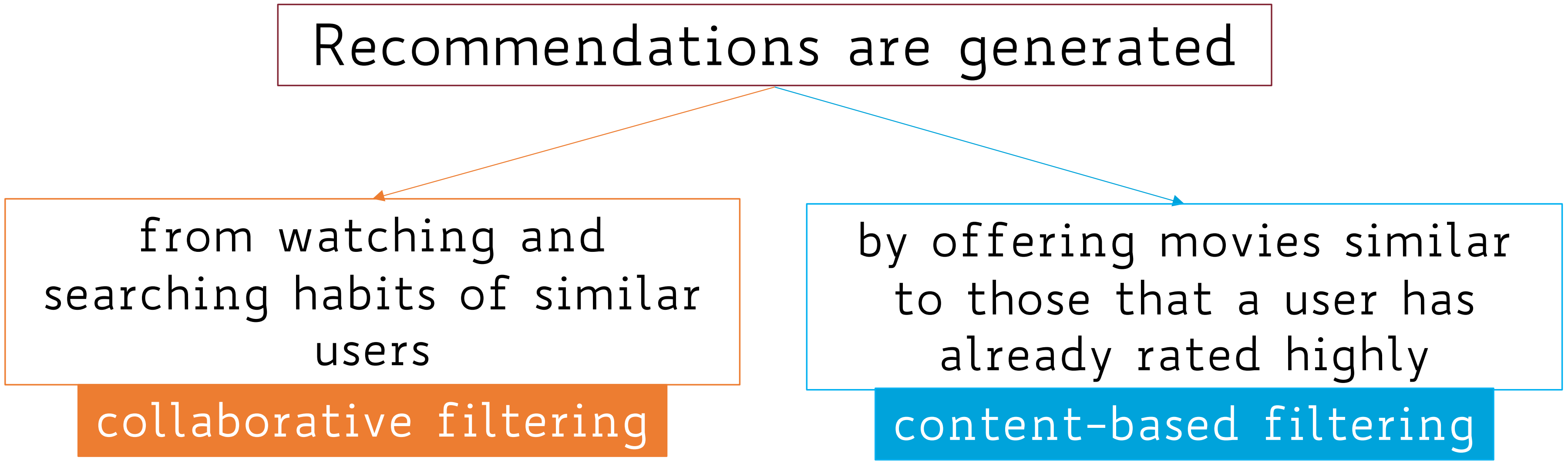
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[Netflix: What Happens When You Press Play?]

For more details about how Netflix actually works

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- Participating teams submit predicted ratings for a test set of approximately 3M ratings

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- The first team that can improve on the Netflix algorithm's RMSE performance by 10% or more wins a $1 million prize

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- According to the contest website, more than 48,000 teams from 182 different countries have downloaded the data

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A combination of 100 different predictor sets, mostly factorization models

Evaluation Metrics

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Online

A/B testing measuring CTR, ROI, and other "live" metrics

Evaluation Metrics: RMSE

$$\mathrm{RMSE} = \frac{1}{|\mathcal{D}_{\text{test}}|} \sqrt{\sum_{(u,i)\in\mathcal{D}_{\text{test}}} (r_{u,i} - \hat{r}_{u,i})^2}$$

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The RMSE might penalize a method that does well for high ratings and badly for others

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For a binary classifier predicting a condition (y = 1) or not, we define

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Mapping of binary classification terminology to recommender systems

binary classifier	recommender system
# with condition (y = 1)	# of all possible relevant items for a user
# predicted positive (TP + FP)	# of recommended items
# correct positives (TP)	# of recommended items that are relavant

Evaluation Metrics: Precision & Recall

For a recommender system, we can therefore define

$$P = \frac{\#\text{ relevant item recommendations}}{\#\text{ items recommended}} \qquad R = \frac{\#\text{ relevant item recommendations}}{\#\text{ items actually relevant}}$$

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A recommender system generates k=5 items to recommend

There are only 3 relevant items

The success/failure of our recommendations: [0, 1, 1, 0, 0] 0=not relevant/1=relevant

$$P = \frac{2}{5} \qquad R = \frac{2}{3}$$

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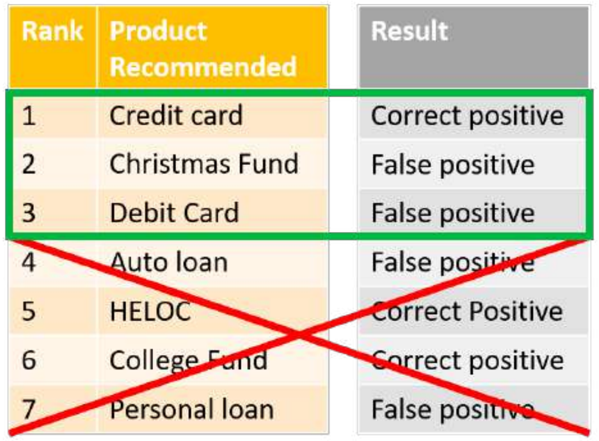
- Imagine taking our list of N recommendations and considering only the first element, then only the first two, then only the first three, and so on
- P@k and R@k are simply the precision and recall calculated only from the subset of the first k recommendations

P@k: Example

$$k = 3$$

$$P@3 = \frac{1}{3}$$



Rank	Product Recommended	Result
1	Credit card	Correct positive
2	Christmas Fund	False positive
3	Debit Card	False positive
~~4~~	~~Auto loan~~	~~False positive~~
~~5~~	~~HELOC~~	~~Correct Positive~~
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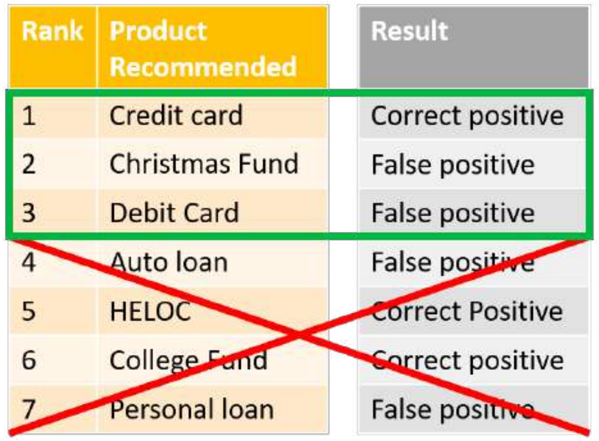

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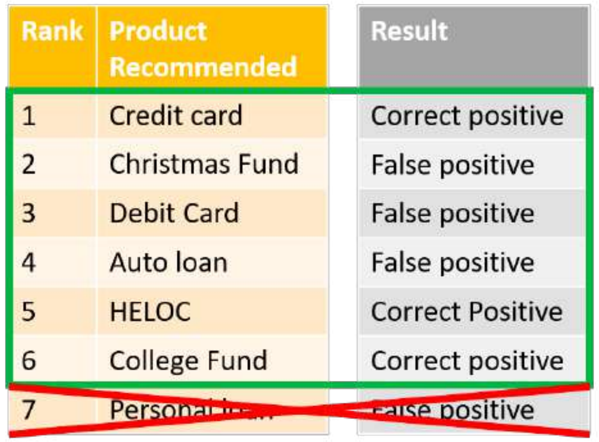

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$$k = 6$$

$$P@6 = \frac{3}{6}$$

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indicator function $\mathbf{1}_{\mathrm{Rel}}(k) = \begin{cases} 1 & \text{if item } k \in \mathrm{Rel} \\ 0 & \text{otherwise} \end{cases}$

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$$MAP@N = \frac{1}{|\mathcal{U}|}\sum_{u=1}^{|\mathcal{U}|} AP@N(u) = \frac{1}{|\mathcal{U}|}\sum_{u=1}^{|\mathcal{U}|}\frac{1}{|\mathrm{Rel}|}\sum_{k=1}^{N} P@k(u) \times \mathbf{1}_{\mathrm{Rel}}(k,u)$$

Personalization

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Intuitively, a high personalization score indicates the recommender system is able to provide a highly personalized experience to the users

Personalization

Suppose 3 users are recommended the following lists of items

u_1 = [A, B, C, D] u_2 = [A, B, C, E] u_3 = [A, B, F, G]

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	A	B	C	D	E	F	G
u_1	1	1	1	1	0	0	0
u_2	1	1	1	0	1	0	0
u_3	1	1	0	0	0	1	1

Personalization

Compute the 3-by-3 triangular matrix containing the cosine similarity between each pair of user's recommendation binary vector

$$M_{i,j} = \text{cosine}(\mathbf{u}_i, \mathbf{u}_j)$$

	u_1	u_2	u_3
u_1	1	0.75	0.58
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~0.64

Take the average of the upper triangle of the matrix M above

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$\boxed{\text{~}0.64}$

$\boxed{\text{Personalization} = 1 - 0.64 = 0.36}$

Take-Home Message of Today

- 2 main approaches:
  - Content-based (explicitly creating user and item profiles)
  - Collaborative-filtering (extract patterns from past observed ratings)

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  - Content-based (explicitly creating user and item profiles)
  - Collaborative-filtering (extract patterns from past observed ratings)
- Hybrid approaches combining both usually work better in practice
- New Neural-Network-based approaches have been proposed recently

Recommended Readings and Information :)

- A huge body of work on recommender systems is available out there!
- Surveys:
  - Adomavicius & Tuzhilin [2005]
  - Koren & Volinsky [2009]
  - Bobadilla *et al*. [2013]
  - Zhang *et al*. [2019]
- Well-renowned series of Conferences: RecSys, KDD, SIGIR, TheWebConf