Teoria degli Algoritmi

Corso di Laurea Magistrale in Matematica Applicata
a.a. 2020-21

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Lecture 7: NP Completeness

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- They found that certain problems in *NP* whose complexity is related to that of the **entire** class
- If a polynomial time algorithm exists for *any* of these problems, then **all** problems in *NP* would be solvable in polynomial time
- These problems are called *NP*-**complete**

NP-complete Problems: Why Do We Care About Them?

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- In an attempt to prove that $P = NP$, one would "only" needs to find a polynomial time algorithm for an NP-complete problem
- Vice versa, trying to prove $P \neq NP$ would require to find at least a problem that is in NP but not in P; in particular this is true for an NP-complete problem

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- The existence of *NP*-complete problems are fundamental both from the theoretical and the practical perspective
- In an attempt to prove that $P = NP$, one would "only" needs to find a polynomial time algorithm for an *NP*-complete problem
- Vice versa, trying to prove $P \neq NP$ would require to find at least a problem that is in *NP* but not in *P*; in particular this is true for an *NP*-complete problem
- On the practical side, even if we still don't know if $P \neq NP$, showing that a problem is *NP*-complete is a strong evidence of its non-polynomiality (as most people indeed think $P \neq NP$)

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- Let's consider variables that can take on only **two** possible values: 1 (TRUE) or 0 (FALSE)
- Those are normally called **boolean** or **binary** variables
- On those variables, we define **3 operators**:
  - AND: $x \wedge y$
  - OR: $x \vee y$
  - NOT: $\neg x$ (also denoted as $\overline{x}$)

The First *NP*-complete Problem: Boolean Satisfiability

Just to remind how those operators work:

$0 \wedge 0 = 0$	$0 \vee 0 = 0$	$\overline{0} = 1$
$0 \wedge 1 = 0$	$0 \vee 1 = 1$	$\overline{1} = 0$
$1 \wedge 0 = 0$	$1 \vee 0 = 1$	
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- A boolean formula is **satisfiable** if some assignment of 0s and 1s to the variables makes the formula evaluate to 1 (i.e., `TRUE`)
- The boolean formula ϕ of the example above is satisfiable because the assignment $x = 0$, $y = 1$, and $z = 0$ makes it `TRUE`

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Definition (The Satisfiability Problem)

The Satisfiability Problem is to test whether a given boolean formula ϕ is satisfiable, namely whether it exists an assignment that satisfies it.

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We now introduce the **Cook-Levin theorem**, which links the complexity of *SAT* with that of **all** problems in *NP*

The Cook-Levin Theorem

Theorem (Cook-Levin)

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We (re-)introduce a technique that is crucial to prove the Cook-Levin theorem

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- We already defined the concept of reducing one problem to another
- When problem A reduces to problem B ($A \leq B$), a solution to B can be used to solve A
- We now extend this idea by taking into account computational efficiency
- Intuitively, when a problem A is **efficiently** reducible to a problem B, an efficient solution to B can be used to solve A also efficiently

Polynomial Time Reducibility

Definition (Polynomial Time Computable Function)

A function $f : \Sigma^* \mapsto \Sigma^*$ is a **polynomial time computable function** if some polynomial time Turing machine M exists that halts with just $f(x)$ on its tape, when it is given x as input

Polynomial Time Reducibility

Definition (Polynomial Time Mapping Reduction)

Language A is **polynomial time mapping reducible** to language B (denoted by $A \leq_P B$) if a polynomial time computable function $f : \Sigma^* \mapsto \Sigma^*$ exists, such that for every x:

$$x \in A \Leftrightarrow f(x) \in B$$

The function f is called the **polynomial time reduction** of A to B

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- Polynomial time reducibility is the efficient-analog to **mapping reducibility**
- As with any other mapping reduction, a polynomial time reduction of A to B is a way to convert membership testing in A to that in B
- Big plus: the conversion is now done "efficiently" (i.e., in polynomial time)
- If one language is polynomial time reducible to another language B, which we already know a polynomial time solution for, then we obtain an overall polynomial time solution for A

Polynomial Time Reducibility

Theorem (Polynomial Time Reducibility)

If $A \leq_P B$ and $B \in P$ then $A \in P$

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Let M_B be the polynomial time algorithm deciding B and f be the polynomial time reduction from A to B. We can describe a polynomial time algorithm M_A that decides A as follows:

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Proof.

Let M_B be the polynomial time algorithm deciding B and f be the polynomial time reduction from A to B. We can describe a polynomial time algorithm M_A that decides A as follows:

$M_A =$ "On input x:

1. Compute $f(x)$;
2. Run M_B on $f(x)$ and output whatever M_B does."

□

Polynomial Time Reducibility

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- 3*SAT* assumes the formula ϕ which we must test the satisfiability of has a particular form
- We call a **literal** any boolean variable (x) or its negated $(\overline{x})$
- A **clause** is several literals connected with $\vee$s, e.g., $(x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4)$

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- A **clause** is several literals connected with $\vee$s, e.g., $(x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4)$
- A boolean formula is in **conjunctive normal form** (CNF) if it comprises several clauses connected with $\wedge$s:

$$(x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4) \wedge (x_3 \vee \overline{x_5} \vee x_6) \wedge (x_3 \vee \overline{x_6})$$

3SAT

Definition (3SAT)

Let ϕ be a 3-CNF boolean formula, i.e., a CNF boolean formula where each clause has exactly 3 literals, e.g.:

$$\phi = (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_3 \vee \overline{x_5} \vee x_6) \wedge (x_3 \vee \overline{x_6} \vee x_4) \wedge (x_4 \vee x_5 \vee x_6)$$

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We define 3*SAT* as the problem of testing whether a 3-CNF formula is satisfiable, i.e.:

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We define $3SAT$ as the problem of testing whether a 3-CNF formula is satisfiable, i.e.:

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Note

In a satisfiable CNF formula, each clause must contain **at least one** literal whose assignment equals to 1 (`TRUE`)

Polynomial Time Reducibility: $3SAT \leq_P CLIQUE$

Theorem ($3SAT \leq_P CLIQUE$)

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Proof.

A sketch of the proof can be the following.
The polynomial time reduction f that we look for must convert 3-CNF boolean formulas to graphs. Graphs are constructed so as cliques of a specified size correspond to satisfying assignments of the formula. Structures within the graph ar designed to mimic the behavior of literals and clauses. □

$3SAT \leq_P CLIQUE$: Proof

- Let ϕ be a 3-CNF formula with k clauses as follows:

$$\phi = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \ldots \wedge (a_k \vee b_k \vee c_k)$$

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- The nodes of G are organized into k groups of three nodes each, called **triples**: $t_1, t_2, \ldots, t_k$

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- The nodes of G are organized into k groups of three nodes each, called **triples**: $t_1, t_2, \ldots, t_k$
- Each triple t_i represents a clause of the original formula ϕ, and each node in a triple is a literal of the associated clause

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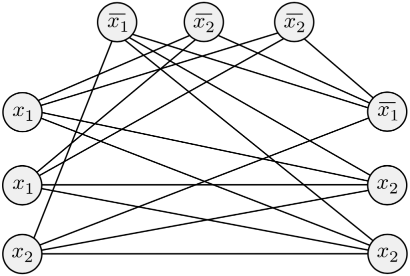

$$\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$$

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- The nodes just selected form a k-clique!

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- Therefore, G has a k-clique

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- Therefore, each of the k triples contains exactly one of the k nodes which the k-clique is made of
- We assign truth values to the variables of ϕ so that each literal labeling a clique node is set to `TRUE`

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- Hence, each clause contains at least a literal that is assigned `TRUE`
- Therefore, ϕ is satisfiable!

$3SAT \leq_P CLIQUE$: Considerations

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    - $|E| < \binom{3k}{2} = \frac{3k(3k-1)}{2} = O(k^2)$
- The size of the graph G is polynomial in the size of the 3-CNF formula ϕ

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- Polynomial time reducibility allows us to link their complexities
- In the following, we show how to link the complexities of an **entire** class of problems

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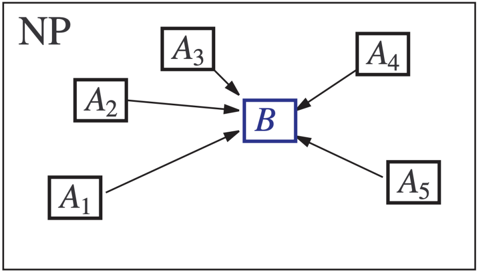

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2. Every $A \in NP$ is polynomial time reducible to B

Note

- *NP*-complete problems are the "most difficult" problems in NP
- If we omit first requirement (i.e., $B \in NP$), then we say that B is *NP*-**hard**

NP-completeness: Consequences

Theorem

If B is NP-complete and $B \in P$ then $P = NP$

NP-completeness: Consequences

Theorem

If B is NP-complete and $B \in P$ then $P = NP$

Proof.

From the definition above, if B is NP-complete it means that $B \in NP$ and **every** other language/problem $A \in NP$ is polynomially reducible to B.

NP-completeness: Consequences

Theorem

If B is NP-complete and $B \in P$ then $P = NP$

Proof.

From the definition above, if B is *NP*-complete it means that $B \in NP$ and **every** other language/problem $A \in NP$ is polynomially reducible to B. Now, if we know that a solver for B exists and it runs in polynomial time (i.e., $B \in P$) then we can solve **every** other problem $A \in NP$ by:

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NP-completeness: Consequences

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Since the process above is a composition of polynomial time algorithms and it holds for all $A \in NP$, we can state that
$\forall A \in NP, A \in P \Leftrightarrow P = NP$. □

NP-completeness: Consequences

Theorem

If B is NP-complete and $B \leq_P C$ for $C \in NP$, then C is NP-complete

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We know that $C \in NP$, so we must show that every other problem in NP is polynomial time reducible to C.

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Instead of proving this for every possible $A \in NP$, we can take advantage of knowing that B is NP-complete itself!
Because B is NP-complete, every other $A \in NP$ is polynomial time reducible to it, i.e., $A \leq_P B,\ \forall A \in NP$.
Plus, we know from the hypothesis that $B \leq_P C$, and due to composition of polynomial time reductions $A \leq_P B\ \forall A \in NP$ and $B \leq_P C$ then $A \leq_P C\ \forall A \in NP$. Therefore C is NP-complete! □

NP-completeness: Consequences

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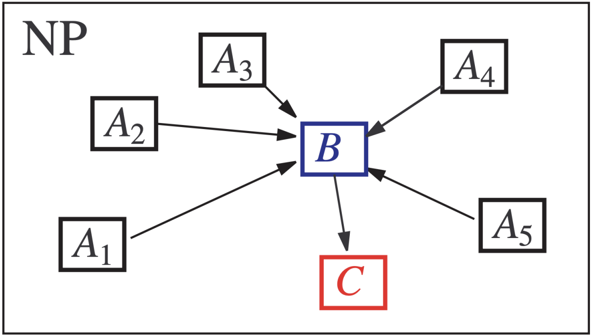
If B is NP-complete and $B \leq_P C$ for $C \in NP$, then C is NP-complete



NP-completeness: The Cook-Levin Theorem

- Once we have **one** *NP*-complete problem, we may obtain others by polynomial time reduction from it

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NP-completeness: The Cook-Levin Theorem

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- However, finding such first *NP*-complete problem is really hard!
- Historically, we do so by showing that our original problem of boolean satisfiability (SAT) is *NP*-complete

NP-completeness: The Cook-Levin Theorem

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SAT is NP-complete

NP-completeness: The Cook-Levin Theorem

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In order to show that *SAT* is *NP*-complete we must prove that:

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NP-completeness: The Cook-Levin Theorem

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In order to show that SAT is NP-complete we must prove that:

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Proving 1 is straightforward:
A polynomial time NTM can guess assignment to a boolean formula ϕ and accept if that assignment satisfies ϕ

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Proving 1 is straightforward:
A polynomial time NTM can guess assignment to a boolean formula ϕ and accept if that assignment satisfies ϕ
In alternative, given ϕ along with a certificate (i.e., an assignment) we can design a polynomial time verifier that checks if the assignment satisfies ϕ □

NP-completeness: The Cook-Levin Theorem

Proof Sketch.

In order to show that SAT is NP-complete we must prove that:

1. $SAT \in NP$
2. $\forall A \in NP,\ A \leq_P SAT$

Proving 2 is harder! □

NP-completeness: The Cook-Levin Theorem

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  1. $w \in A \Leftrightarrow f(w) = \langle \phi \rangle \in SAT$
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- Let N be polynomial time NTM that decides A in time at most n^k, where $n = |w|$

NP-completeness: The Cook-Levin Theorem

Outline of the basic approach:

$w \in A \Leftrightarrow$ NTM N accepts input w

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Note

The basic intuition is to be able to show that **any** algorithm can be encoded as a boolean formula!

NP-completeness: The Cook-Levin Theorem

Key Idea:

"Satisfying assignment of ϕ" $\Leftrightarrow$ "Accepting computation history of N on input w"

NP-completeness: The Cook-Levin Theorem

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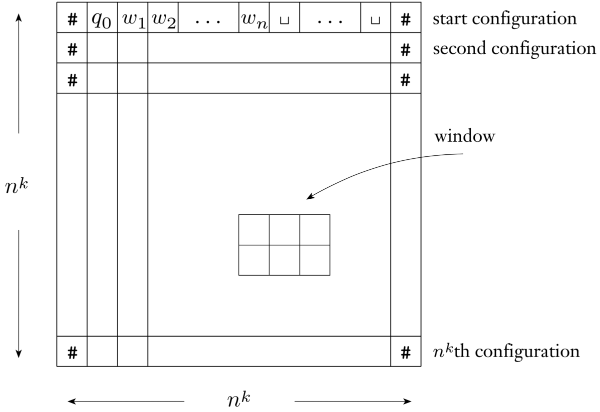
- Every accepting tableau for N on w corresponds to an accepting computation branch of N on w
- The problem of determining whether N accepts w is **equivalent** to finding if an accepting tableau for N on w exists

NP-completeness: The Cook-Levin Theorem



NP-completeness: The Cook-Levin Theorem

Step 1: Describe computations of NTM N on w by boolean variables using the tableau

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- $x_{i,j,s} = 1$ means "cell (i, j) contains s"

NP-completeness: The Cook-Levin Theorem

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Note

We design ϕ so that a satisfying assignment to its variables $x_{i,j,s}$ corresponds to an accepting tableau for N on w

NP-completeness: The Cook-Levin Theorem

Step 2: Express conditions for an accepting sequence of configurations of NTM N on w by a boolean formula ϕ as the AND of four parts:

$$\phi = \phi_{\text{cell}} \wedge \phi_{\text{start}} \wedge \phi_{\text{move}} \wedge \phi_{\text{accept}}$$

1. ϕ_{cell} = "for each cell (i, j), there is **exactly one** $s \in C$ with $x_{i,j,s} = 1$" (cell is well defined)

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NP-completeness: The Cook-Levin Theorem

$$\phi_{\text{cell}} = \bigwedge_{1 \leq i,j \leq n^k} \left[\left(\bigvee_{s \in C} x_{i,j,s} \right) \wedge \left(\bigwedge_{\substack{s,t \in C \\ s \neq t}} (\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}}) \right) \right].$$

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- The second inner component says that in each pair of variables, at least one is turned off
- The accepting tableau must meet condition 1

NP-completeness: The Cook-Levin Theorem

$$\begin{aligned}\phi_{\text{start}} = \; & x_{1,1,\#} \wedge x_{1,2,q_0} \wedge \\ & x_{1,3,w_1} \wedge x_{1,4,w_2} \wedge \ldots \wedge x_{1,n+2,w_n} \wedge \\ & x_{1,n+3,\sqcup} \wedge \ldots \wedge x_{1,n^k-1,\sqcup} \wedge x_{1,n^k,\#}\,.\end{aligned}$$

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- This formula ensures that the first row of the tableau is the starting configuration of N on w by explicitly stipulating that the corresponding variables are on

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- The accepting tableau must meet condition 2

NP-completeness: The Cook-Levin Theorem

$$\phi_{\text{move}} = \bigwedge_{1 \leq i < n^k,\ 1 < j < n^k} \big(\text{the } (i, j)\text{-window is legal}\big).$$

$$\bigvee_{\substack{a_1, \ldots, a_6 \\ \text{is a legal window}}} \big(x_{i,j-1,a_1} \wedge x_{i,j,a_2} \wedge x_{i,j+1,a_3} \wedge x_{i+1,j-1,a_4} \wedge x_{i+1,j,a_5} \wedge x_{i+1,j+1,a_6}\big)$$

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- This formula ensures that each 2x3 window is legal according to *N*'s transition function (proof omitted here)

NP-completeness: The Cook-Levin Theorem

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- The accepting tableau must meet condition 3

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$$\phi_{\text{accept}} = \bigvee_{1 \le i,j \le n^k} x_{i,j,q_{\text{accept}}}.$$

NP-completeness: The Cook-Levin Theorem

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- This formula ensures that an accepting configuration occurs in the tableau

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- The accepting tableau must meet condition 4

NP-completeness: The Cook-Levin Theorem

Putting all together:

NP-completeness: The Cook-Levin Theorem

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- Given a non-deterministic Turing machine N and some input w we have shown that we can build a propositional formula ϕ:

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- The subformulas encode the 4 conditions needed there be an accepting tableau for the computation of N on input w
- It only remains to show that the reduction from w to ϕ is computable in polynomial time

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- SAT in NP-complete!

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- Rather than using *SAT*, typically a reduction is shown from one of its variant, i.e., 3*SAT* or 3-CNF formulas
- Before being able to do that, we need to show that 3*SAT* is also *NP*-complete

$3SAT$ is NP-complete

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- Instead, we slightly adapt the proof we used to show that SAT is NP-complete to achieve this

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- Thus, ϕ_{cell} is an AND of **clauses**, therefore already in CNF

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- Using the **distributive law**, however, we can transform any OR of ANDs into an equivalent AND of ORs (i.e., CNF)

Converting to CNF

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  3. distributive law:
     $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$; $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

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- In each clause containing less than 3 literals, we just replicate one of the literals until getting a 3-literal clause
- In each clause that has more than 3 literals, we need to split them into multiple 3-literal clauses preserving the satisfiability

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Example

Suppose our clause is made of 4 literals:

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Here, z is a new variable (literal) and if some assignment of the a_i's satisifies c we can also find a setting of z that satisfies c'.

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Example

More generally, if the clause contains ℓ literals:

$$c = (a_1 \vee a_2 \vee \ldots \vee a_\ell)$$

We can replace it with $\ell - 2$ clauses as follows:

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  2. C is NP-hard, i.e., $\forall A \in NP,\ A \leq_P C$
- Recall that we proved that if B is NP-complete and $B \leq_P C$ then C is NP-complete

Proving *NP*-completeness: Summary

How to prove that a problem C is NP-complete?

- We therefore need to show that:
  1. $C \in NP$
  2. a well-known NP-complete problem B polynomial time reduces to C (e.g., $SAT \leq_P C$ or $3SAT \leq C$)
  3. the reduction actually takes polynomial time

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- We showed that $3SAT \leq_P CLIQUE$
- We showed that $3SAT$ is NP-complete
- Thus, $CLIQUE$ is NP-complete!

Karp's 21 *NP*-Complete Problems (1972)

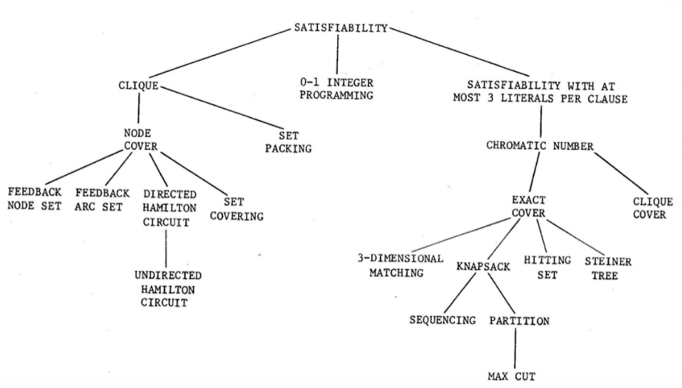

Figure: Karp's 21 *NP*-complete Problems

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- Optimization version of *NP*-complete problems are *NP*-hard

Problem	Decision Version	Optimization Version
CLIQUE	Does a graph G have a clique of size k ?	Find largest clique
ILP	Does $\exists$ integer vector y such that $Ay \leq b$?	Find integer vector y to max $d^\top y$ s.t. $Ay \leq b$
TSP	Does a graph G have tour of length $\leq d$?	Find min length tour
Scheduling	Given set of tasks and constraints, can we finish all tasks in time d ?	Find min time schedule

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- How to deal with an *NP*-complete or *NP*-hard problem?
  - Approximation algorithm
  - Probabilistic algorithm
  - Special cases
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- NP includes all problems that are in P plus, for example, $HAMPATH$, $CLIQUE$, $SUBSET\text{-}SUM$, $3SAT$, etc.
- P vs. NP question:
  1. We know that $P \subseteq NP$
  2. We **don't** know whether $P \neq NP$ or $P = NP$

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- If B is *NP*-complete and $B \leq_P C$ for $C \in NP$, then C is *NP*-complete
- **Cook-Levin Theorem:** *SAT* is *NP*-complete
- 3*SAT*, *CLIQUE*, *SUBSET-SUM*, *HAMPATH*, etc. are all *NP*-complete (via polynomial time reduction)