Teoria degli Algoritmi

Corso di Laurea Magistrale in Matematica Applicata

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Lecture 2: Turing Machines

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Our Model of Computation: Turing Machines

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  - Hilbert wondered if it exists an "effective procedure" (i.e., our informal definition of algorithm) that decides whether any mathematical statement is true or false, in a finite number of steps
  - As a special case of this decision problem, Hilbert considered the validity problem for first-order logic (a.k.a. *entscheidungsproblem*)

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Turing Machines: An Informal Perspective

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Note

The linear nature of memory tape, as opposed to random access memory, is a limitation on computation speed but not power: a TM can find any memory location, i.e., tape cell, by sequentially scanning its tape

Turing Machines: A Formal Definition

Definition (Turing machine)

A Turing machine M is a 6-tuple $(Q, \Sigma, \delta_M, q_0, q_{\text{accept}}, q_{\text{reject}})$, where:

- Q is the finite set of **states**

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- $q_0 \in Q$ is the **start state**
- $q_{\text{accept}} \in Q$ is the **accept state**
- $q_{\text{reject}} \in Q$ is the **reject state**, s.t. $q_{\text{accept}} \neq q_{\text{reject}}$

Turing Machine: How Does It Work?

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- If M ever halts, it will leave the output string on the tape, i.e., $\sigma_{\text{out}} \in \Sigma^*$

Turing Machine: The Transition Function δ_M

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- δ_M is a **finite** function, which takes $|Q| \cdot |\Sigma \cup \{\varnothing\}|$ possible inputs and produces $3 \cdot |Q| \cdot |\Sigma \cup \{\varnothing\}|$ possible outputs
- The machine can compute an **infinite** function f_M that takes as input a string $\sigma_{\text{in}} \in \Sigma^*$ and produces another string $\sigma_{\text{out}} \in \Sigma^*$ as output, both of arbitrary lengths

Turing Machine: Example

Definition (Palindrome String)

Given an alphabet Σ, a **palindrome** is a string $x \in \Sigma^*$ that can be read exactly the same from left to right and from right to left. As an example: 0110, 1001001 are palindromes, 01100 is not.

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Definition (Test of Palindrome)

We define the *PAL* problem as follows:

- **input:** a binary string $x \in \Sigma^*$
- **output:** 1 if x is palindrome and 0 otherwise.

Turing Machine: Example

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- We now give a Turing Machine M that solves PAL
- M stops at an accepting state (q_{accept}) if the given input is a palindrome, and stops at a reject state (q_{reject}) if the input is not a palindrome
- For simplicity, M does not clean the tape nor write 1/0 if it accepts/rejects

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- Then M switches to state q_{match_σ} and compares the rightmost symbol with the leftmost σ read; if the two symbols are different, M rejects
- Otherwise, M deletes the rightmost symbol, goes all the way to the left (in state q_{reverse}), until reads $\varnothing$, goes one step right to the non-blank leftmost symbol, and goes back to state q_0

Turing Machine: Example

More formally, we define our M that solves PAL by specifying the alphabet Σ it operates on, the set of its states Q, and the transition function δ_M

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Note

You are welcome to visit https://turingmachinesimulator.com/, where you can build, debug, and learn more about Turing machines.

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From Turing Machines to Computable Functions

Definition (Computable Function)

Let $f : \Sigma^* \mapsto \Sigma^*$ be a (total) function and let M be a Turing machine. We say that M computes f if for every $x \in \Sigma^*$, $M(x) = f(x)$.
We say that a function f is computable if there exists a Turing machine M that computes it.

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Note

Defining a function "computable" if it can be computed by a Turing machine might seem incautious, but this is equivalent to being computable in virtually *any* reasonable model of computation.

The Church-Turing Thesis

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  - Or by any equivalent computational models proposed by Gödel (**recursive functions**) and Church (**λ-calculus**)
- The three formally-defined classes of computable functions coincide with the informal notion of an effectively calculable function
- Since the concept of effective calculability does not have a formal definition, the thesis, although it has near-universal acceptance, cannot be formally proven

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Definition

We define by $\mathcal{R}$ the set of **all** computable functions $f : \Sigma^* \mapsto \Sigma$

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Definition (Turing-decidable Language)

A language L is **Turing-decidable** (or simply **decidable**) if there is a Turing machine M that decides it

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- This is also the reason why the letter $\mathcal{R}$ is often used
- We stick to the term *functions* rather than *lanuguages*, although the following always holds:

$$f : \Sigma^* \mapsto \Sigma$$

$$L = \{x \in \Sigma^* \mid f(x) = 1\}$$

Infinite Loops and Partial Functions

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Definition

A partial function $f : A \mapsto B$ is a function that is only defined on a subset A' of A (i.e., $A' \subset A$). We can also think of such a function as mapping from A to $B \cup \{\perp\}$, where $\perp$ is a special "failure" symbol such that $f(a) = \perp$ indicates f is not defined on input a

Turing Machines Computing Partial Functions

Example

Consider the function $div : \mathbb{Z}^{0+} \times \mathbb{Z}^{0+} \mapsto \mathbb{Z}^{0+}$, defined as follows:

$$div(a, b) = \begin{cases} \lceil \frac{a}{b} \rceil, & \text{if } b > 0 \\ \perp, & \text{otherwise} \end{cases}$$

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  - If $a = b = 0$, M will output 0, which is also ok, since we do not care about what the machine outputs on inputs on which div is undefined

Computable Functions (Redefined)

Definition

Let f be a **total** or **partial** function, such that $f : \Sigma^* \mapsto \Sigma^*$ and let M be a Turing machine.

We say that M **computes** f if for every $x \in \Sigma^*$ on which f is defined, $M(x) = f(x)$.

We say that a (partial or total) function f is **computable** if there is a Turing machine that computes it.

A Clarification on the Role of $\perp$

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- As such, one might be tempted to think that M halts on x if and only if f is defined on x
- However, for a Turing machine M to compute a partial function f it is **not** necessary to enter an infinite loop on inputs x outside the domain of f
- All that is needed is for M to output $f(x)$ on $x \in domain(f)$: on any other input it is OK for M to output an arbitrary value or not to halt at all

Functions vs. Languages

- A Turing machine M **recognizes** a language L if for **every** input $x \in \Sigma^*$, $M(x)$ outputs 1 if and only if $x \in L$

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Definition (Turing-recognizable Language)

A language L is **Turing-recognizable** (or simply **recognizable** or **semi-decidable**) if there is a Turing machine M that recognizes it

A Note on the Terminology

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- This is also the reason why the letter $\mathcal{RE}$ is often used
- We stick to the term *functions* rather than *lanuguages*, although the following always holds:

$$f : \Sigma^* \mapsto \Sigma$$

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- Interestingly enough, the original computational model and its variants have all the same power
- They all compute the same functions/recognize the same set of languages

Multi-tape Turing Machines

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- The transition function δ_M is changed to allow for reading, writing, and moving the heads on some or all of the tapes, simultaneously
- Formally, the transition function of a k-tape Turing machine is defined as follows:

$$\delta_M : Q \times (\Sigma \cup \{\varnothing\})^k \mapsto Q \times (\Sigma \cup \{\varnothing\})^k \times \{-1, 0, +1\}^k$$

Multi-tape Turing Machines: Example

Consider a k-tape Turing Machine, then the expression

$$\delta_M(q_i, \sigma_1, \sigma_2, \ldots, \sigma_k) = (q_j, \sigma'_1, \sigma'_2, \ldots, \sigma'_k, +1, 0, \ldots, -1)$$

means that, if the machine is in state q_i and heads 1 through k are reading symbols σ_1 through σ_k, then it goes to state q_j, writes symbols σ'_1 through σ'_k and moves each head to the left (-1) or to the right (+1) of the current position, or leaves it where it is (0)

Equivalence Between Single- and Multi-Tape TMs

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- In fact, it can be proven that those two models of computations are indeed equivalent (i.e., they both recognize the same languages)
- To sketch the idea of the proof, consider two Turing machines: S, M
  - The former is a single-tape machine, whilst the latter is multi-tape
  - The key idea is to simulate M using S
  - We can lay down the content of the k tapes of M on the single tape of S, using a special symbol as delimiter (e.g., $\#$)
  - Add another extra symbol (e.g., $\bullet$) on top of the current symbol to mimic the head position on each tape

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where $\mathcal{P}(A)$ stands for the **power set** of A, i.e., the set of all subsets of A

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- If some branch leads to the accept state (q_{accept}), the machine accepts its input

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- In fact, it can be proven that those two models of computations are indeed equivalent (i.e., they both recognize the same languages)
- To sketch the idea of the proof, consider two Turing machines: D, N
  - The former is a deterministic machine, whilst the latter is non-deterministic
  - The key idea is to simulate N using D by letting D try **all** the possible branches of N's non-deterministic computation
  - If D ever reaches the accept state on one of these branches, D accepts; otherwise D's simulation may run forever

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- This guarantees that D will visit every node in the tree until it encounters an accepting configuration

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- We have already seen that we can use the same binary string encoding to represent virtually **any** object
- As a special case, we can therefore encode **any** Turing machine M together with **any** of its input x

Universal Turing Machine

Definition (Universal Turing Machine)

There exists a Turing machine U, such that on every string M which encodes a Turing machine, and $x \in \Sigma^*$:

$$U(M, x) = M(x)$$

If the machine M halts on x and outputs some $y \in \Sigma^*$ (i.e., $M(x) = y$), then:

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If M does **not** halt on x (i.e., $M(x) = \perp$) then:

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- The desired program U is an **interpreter** for Turing machines
- U gets a representation of the machine M (e.g., source code), and some input x, and simulates the execution of M on x

The Existence of a Universal Turing Machine

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- Translating the interpreter above into the corresponding Turing machine is "easy"

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- The existence of even a *single* such machine is already fundamental to computer science
- The idea of a "universal program" is of course not limited to theory
- The most famous practical example is represented by **compilers** (for programming languages), which are often used to compile themselves!

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- There exists few variants of standard TM like multi-tape or non-deterministic TMs yet they all have the same power
- The existence of a special Universal Turing Machine (UTM) allows us to design an algorithm that can run any other algorithm