

Teoria degli Algoritmi

Corso di Laurea Magistrale in Matematica Applicata a.a. 2020-21

Gabriele Tolomei

Dipartimento di Informatica
Sapienza Università di Roma
tolomei@di.uniroma1.it

Lecture 3: Decidability

Table of Contents

- 1 Computability
- 2 Diagonalization
- 3 The Halting Problem
- 4 Beyond Undecidability
- 5 Summary

Decidability & Recognizability

- We introduced the Turing Machine (TM) as a model of general-purpose computation

Decidability & Recognizability

- We introduced the Turing Machine (TM) as a model of general-purpose computation
- We defined the (formal) notion of algorithm in terms of TMs leveraging the Church-Turing thesis

Decidability & Recognizability

- We introduced the Turing Machine (TM) as a model of general-purpose computation
- We defined the (formal) notion of algorithm in terms of TMs leveraging the Church-Turing thesis
- In other words, we identify the set of **computable functions** with those calculated by a Turing machine:
 - **partial** computable functions are those **recognized** by a TM (i.e., the machine may never halt)
 - **total** computable functions are those **decided** by a TM (i.e., the machine always halts)

Decidability & Recognizability

- We introduced the Turing Machine (TM) as a model of general-purpose computation
- We defined the (formal) notion of algorithm in terms of TMs leveraging the Church-Turing thesis
- In other words, we identify the set of **computable functions** with those calculated by a Turing machine:
 - **partial** computable functions are those **recognized** by a TM (i.e., the machine may never halt)
 - **total** computable functions are those **decided** by a TM (i.e., the machine always halts)
- We also refer to (**Turing-**)**recognizable** or **recursively-enumerable** languages and (**Turing-**)**decidable** or **recursive** languages

Uncomputable Functions

Question

A natural question arises:

*is there any problem (i.e., function) that cannot be solved by **any** algorithm?*

Uncomputable Functions

Question

A natural question arises:

*is there any problem (i.e., function) that cannot be solved by **any** algorithm?*

To put it differently, yet equivalently:

*is there any function that cannot be computed by **any** Turing machine?*

Uncomputable Functions: Why Bother?

- Why should we care about problems that cannot be algorithmically solved?

Uncomputable Functions: Why Bother?

- Why should we care about problems that cannot be algorithmically solved?
- After all, showing that a problem is unsolvable does not appear to be of any interest

Uncomputable Functions: Why Bother?

- Why should we care about problems that cannot be algorithmically solved?
- After all, showing that a problem is unsolvable does not appear to be of any interest
- Two reasons why we should bother of problems that cannot be solved by an algorithm:
 - To realize they must **simplified** first, before searching for an algorithmic solution to them

Uncomputable Functions: Why Bother?

- Why should we care about problems that cannot be algorithmically solved?
- After all, showing that a problem is unsolvable does not appear to be of any interest
- Two reasons why we should bother of problems that cannot be solved by an algorithm:
 - To realize they must **simplified** first, before searching for an algorithmic solution to them
 - To stimulate your imagination!

The Size of Sets

- A clever technique used to proof the existence of uncomputable functions/undecidable languages

The Size of Sets

- A clever technique used to proof the existence of uncomputable functions/undecidable languages
- This was discovered by [Georg Cantor](#) in 1873, who was concerned with the problem of measuring the size of (infinite) sets

The Size of Sets

- A clever technique used to proof the existence of uncomputable functions/undecidable languages
- This was discovered by [Georg Cantor](#) in 1873, who was concerned with the problem of measuring the size of (infinite) sets

Question

If we have two sets A and B , how can we tell if one is larger than the other or if they are of the same size?

The Size of Sets

- A clever technique used to proof the existence of uncomputable functions/undecidable languages
- This was discovered by [Georg Cantor](#) in 1873, who was concerned with the problem of measuring the size of (infinite) sets

Question

If we have two sets A and B , how can we tell if one is larger than the other or if they are of the same size?

- For finite sets, the answer is of course straightforward: just count the elements of each A and B !

The Size of Sets

- A clever technique used to proof the existence of uncomputable functions/undecidable languages
- This was discovered by [Georg Cantor](#) in 1873, who was concerned with the problem of measuring the size of (infinite) sets

Question

If we have two sets A and B , how can we tell if one is larger than the other or if they are of the same size?

- For finite sets, the answer is of course straightforward: just count the elements of each A and B !
- The same does not work for infinite set as we will never finish counting!

The Size of Sets: Example

Example

Consider the set of **even** natural $\mathbb{E} = \{n \in \mathbb{N} \mid n \bmod 2 = 0\}$.
Then, consider the set of all possible binary strings of any (finite) length
 $\Sigma^* = \{0, 1\}^*$

The Size of Sets: Example

Example

Consider the set of **even** natural $\mathbb{E} = \{n \in \mathbb{N} \mid n \bmod 2 = 0\}$.
Then, consider the set of all possible binary strings of any (finite) length
 $\Sigma^* = \{0, 1\}^*$

Question

Of course, both \mathbb{E} and Σ^* are infinite (thus, larger than any finite set).
However, is one of the two larger than the other? Can we figure this out?

The Size of Sets: Cantor's Solution

- Cantor proposed a brilliant solution to the problem posed before

The Size of Sets: Cantor's Solution

- Cantor proposed a brilliant solution to the problem posed before
- He observed that two finite sets A and B have the same size if the elements of one set can be **paired** with the elements of the other set

The Size of Sets: Cantor's Solution

- Cantor proposed a brilliant solution to the problem posed before
- He observed that two finite sets A and B have the same size if the elements of one set can be **paired** with the elements of the other set
- This method compares the size of sets **without** resorting to counting!

The Size of Sets: Cantor's Solution

- Cantor proposed a brilliant solution to the problem posed before
- He observed that two finite sets A and B have the same size if the elements of one set can be **paired** with the elements of the other set
- This method compares the size of sets **without** resorting to counting!
- Interestingly enough, this approach extends also to **infinite sets**

The Size of Sets: Cantor's Solution

Definition (Same Size Sets)

Suppose we have two sets A and B , and a function $f : A \mapsto B$.

We say that f is **one-to-one** (or **injective**) if it never maps two different elements to the same place, i.e., $\forall a, a' \in A, a \neq a' \Rightarrow f(a) \neq f(a')$

The Size of Sets: Cantor's Solution

Definition (Same Size Sets)

Suppose we have two sets A and B , and a function $f : A \mapsto B$.

We say that f is **one-to-one** (or **injective**) if it never maps two different elements to the same place, i.e., $\forall a, a' \in A, a \neq a' \Rightarrow f(a) \neq f(a')$

Moreover, we say that f is **onto** (or **surjective**) if $\forall b \in B \exists a \in A$ s.t. $f(a) = b$.

The Size of Sets: Cantor's Solution

Definition (Same Size Sets)

Suppose we have two sets A and B , and a function $f : A \mapsto B$.

We say that f is **one-to-one** (or **injective**) if it never maps two different elements to the same place, i.e., $\forall a, a' \in A, a \neq a' \Rightarrow f(a) \neq f(a')$

Moreover, we say that f is **onto** (or **surjective**) if $\forall b \in B \exists a \in A$ s.t. $f(a) = b$.

We say that A and B are **same size sets** if there is a one-to-one, onto function $f : A \mapsto B$; such a function is called a **bijection** or **correspondence**.

The Size of Sets: Cantor's Solution

Definition (Same Size Sets)

Suppose we have two sets A and B , and a function $f : A \mapsto B$.

We say that f is **one-to-one** (or **injective**) if it never maps two different elements to the same place, i.e., $\forall a, a' \in A, a \neq a' \Rightarrow f(a) \neq f(a')$

Moreover, we say that f is **onto** (or **surjective**) if $\forall b \in B \exists a \in A$ s.t. $f(a) = b$.

We say that A and B are **same size sets** if there is a one-to-one, onto function $f : A \mapsto B$; such a function is called a **bijection** or **correspondence**.

In a correspondence between A and B , every element of A maps to a unique element of B and every element of B has a unique element of A that maps to it.

The Size of the Set of Even Natural Numbers

Example (The size of \mathbb{N} vs. the size of \mathbb{E})

Let \mathbb{N} be the set of natural numbers, i.e., $\mathbb{N} = \{1, 2, 3, \dots\}$, and let \mathbb{E} be the set of **even** natural numbers, i.e., $\mathbb{E} = \{2, 4, 6, \dots\}$.

Using Cantor's argument, prove that \mathbb{N} and \mathbb{E} have the same size.

The Size of the Set of Even Natural Numbers

Example (The size of \mathbb{N} vs. the size of \mathbb{E})

Let \mathbb{N} be the set of natural numbers, i.e., $\mathbb{N} = \{1, 2, 3, \dots\}$, and let \mathbb{E} be the set of **even** natural numbers, i.e., $\mathbb{E} = \{2, 4, 6, \dots\}$.

Using Cantor's argument, prove that \mathbb{N} and \mathbb{E} have the same size.

- Intuitively, this sounds odd: \mathbb{E} seems “smaller” than \mathbb{N} , as the former is a proper subset of the latter

The Size of the Set of Even Natural Numbers

Example (The size of \mathbb{N} vs. the size of \mathbb{E})

Let \mathbb{N} be the set of natural numbers, i.e., $\mathbb{N} = \{1, 2, 3, \dots\}$, and let \mathbb{E} be the set of **even** natural numbers, i.e., $\mathbb{E} = \{2, 4, 6, \dots\}$.

Using Cantor's argument, prove that \mathbb{N} and \mathbb{E} have the same size.

- Intuitively, this sounds odd: \mathbb{E} seems “smaller” than \mathbb{N} , as the former is a proper subset of the latter
- However, we can find a correspondence $f : \mathbb{N} \mapsto \mathbb{E}$ which maps each element of \mathbb{N} to an element of \mathbb{E} :

$$\forall n \in \mathbb{N}, f(n) = 2n \in \mathbb{E}$$

Countable Set

Definition (Countable Set)

A set A is **countable** if either it is finite or it has the same size of \mathbb{N}

The Size of the Set of Rational Numbers

Example (The size of \mathbb{Q} vs. the size of \mathbb{N})

Let \mathbb{Q} be the set of (positive) rational numbers, i.e., $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{N}\}$. Using Cantor's argument, prove that \mathbb{Q} is countable as there exists a correspondence with \mathbb{N} .

The Size of the Set of Rational Numbers

Example (The size of \mathbb{Q} vs. the size of \mathbb{N})

Let \mathbb{Q} be the set of (positive) rational numbers, i.e., $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{N}\}$. Using Cantor's argument, prove that \mathbb{Q} is countable as there exists a correspondence with \mathbb{N} .

- Intuitively, this sounds even stranger than the example before: \mathbb{Q} seems “much larger” than \mathbb{N}

The Size of the Set of Rational Numbers

Example (The size of \mathbb{Q} vs. the size of \mathbb{N})

Let \mathbb{Q} be the set of (positive) rational numbers, i.e., $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{N}\}$. Using Cantor's argument, prove that \mathbb{Q} is countable as there exists a correspondence with \mathbb{N} .

- Intuitively, this sounds even stranger than the example before: \mathbb{Q} seems “much larger” than \mathbb{N}
- However, we can find a correspondence $f : \mathbb{Q} \mapsto \mathbb{N}$ which maps each element of \mathbb{Q} to an element of \mathbb{N}

The Size of the Set of Rational Numbers

- An easy way to build the correspondence $f : \mathbb{Q} \mapsto \mathbb{N}$ is to
 - list **all** the elements of $\mathbb{Q} : \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{2}{1}, \frac{2}{2}, \dots$
 - pair the first element of the list with the number 1 from \mathbb{N} , the second with 2, and so on and so forth
 - ensure that every member of \mathbb{Q} appears exactly once (e.g., $\frac{1}{1} = \frac{2}{2} = \frac{3}{3} \dots$)

The Infinite Matrix of Rational Numbers

- To build the list of all the elements of \mathbb{Q} , just make an infinite matrix containing all the positive rational numbers

The Infinite Matrix of Rational Numbers

- To build the list of all the elements of \mathbb{Q} , just make an infinite matrix containing all the positive rational numbers
- The i -th row contains all the numbers whose numerator is equal to i

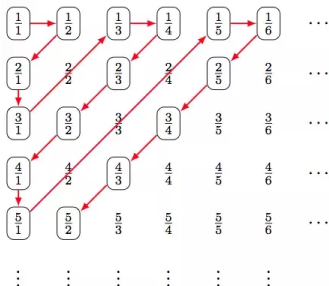
The Infinite Matrix of Rational Numbers

- To build the list of all the elements of \mathbb{Q} , just make an infinite matrix containing all the positive rational numbers
- The i -th row contains all the numbers whose numerator is equal to i
- The j -th column contains all the numbers whose denominator is equal to j

The Infinite Matrix of Rational Numbers

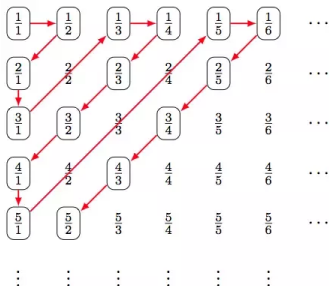
- To build the list of all the elements of \mathbb{Q} , just make an infinite matrix containing all the positive rational numbers
- The i -th row contains all the numbers whose numerator is equal to i
- The j -th column contains all the numbers whose denominator is equal to j
- In other words, the rational number $q = \frac{i}{j}$ is located on the i -th row and the j -th column of the matrix

The Infinite Matrix of Rational Numbers



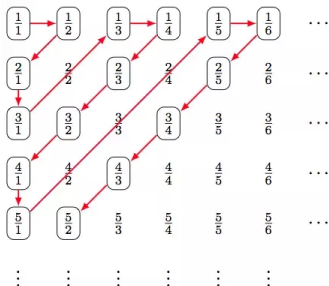
- We must turn the infinite matrix into a **list**

The Infinite Matrix of Rational Numbers



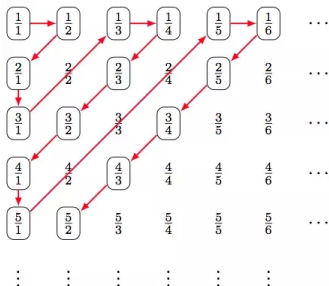
- We must turn the infinite matrix into a **list**
- A first (bad) attempt of doing this would be to begin the list with all the elements in the first row

The Infinite Matrix of Rational Numbers



- We must turn the infinite matrix into a **list**
- A first (bad) attempt of doing this would be to begin the list with all the elements in the first row
- That would not work as the first row is infinite and we would never get to the second row!

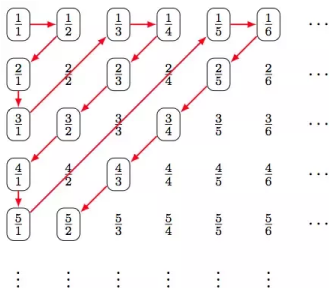
The Infinite Matrix of Rational Numbers



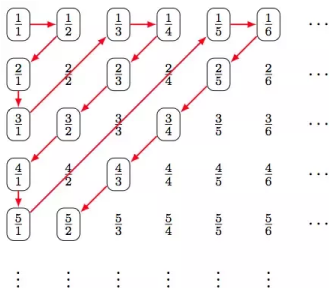
- We must turn the infinite matrix into a **list**
- A first (bad) attempt of doing this would be to begin the list with all the elements in the first row
- That would not work as the first row is infinite and we would never get to the second row!
- Instead, we list the elements following the diagonals, starting from the top-left corner

The Infinite Matrix of Rational Numbers

- The first diagonal contains the single element $\frac{1}{1}$

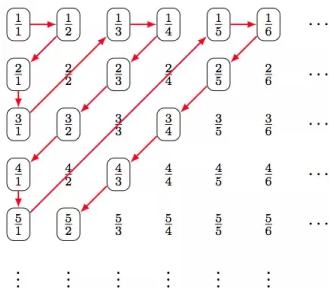


The Infinite Matrix of Rational Numbers



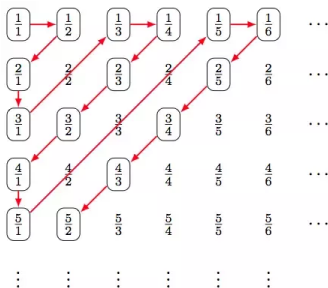
- The first diagonal contains the single element $\frac{1}{1}$
- The second diagonal, instead, contains two elements: $\frac{1}{2}$ and $\frac{2}{1}$

The Infinite Matrix of Rational Numbers



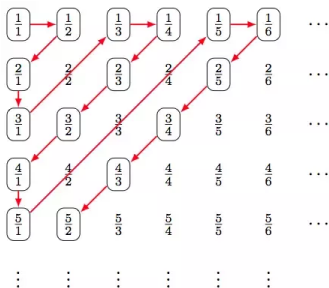
- The first diagonal contains the single element $\frac{1}{1}$
- The second diagonal, instead, contains two elements: $\frac{1}{2}$ and $\frac{2}{1}$
- The first three elements of our list are:
 $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}$

The Infinite Matrix of Rational Numbers



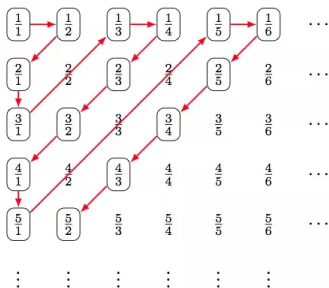
- The first diagonal contains the single element $\frac{1}{1}$
- The second diagonal, instead, contains two elements: $\frac{1}{2}$ and $\frac{2}{1}$
- The first three elements of our list are: $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}$
- In the third diagonal, things get a little bit more complicated as this contains: $\frac{1}{3}, \frac{2}{2}, \frac{3}{1}$

The Infinite Matrix of Rational Numbers



- The first diagonal contains the single element $\frac{1}{1}$
- The second diagonal, instead, contains two elements: $\frac{1}{2}$ and $\frac{2}{1}$
- The first three elements of our list are:
 $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}$
- In the third diagonal, things get a little bit more complicated as this contains: $\frac{1}{3}, \frac{2}{2}, \frac{3}{1}$
- We can't add those to our list as we would repeat $\frac{1}{1} = \frac{2}{2}$, so we add only $\frac{3}{1}$ and $\frac{1}{3}$

The Infinite Matrix of Rational Numbers



- The first diagonal contains the single element $\frac{1}{1}$
- The second diagonal, instead, contains two elements: $\frac{1}{2}$ and $\frac{2}{1}$
- The first three elements of our list are:
 $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}$
- In the third diagonal, things get a little bit more complicated as this contains: $\frac{1}{3}, \frac{2}{2}, \frac{3}{1}$
- We can't add those to our list as we would repeat $\frac{1}{1} = \frac{2}{2}$, so we add only $\frac{3}{1}$ and $\frac{1}{3}$
- If we keep going this way we will obtain the list of all the elements of \mathbb{Q}

Uncountable Sets

- We may be tempted to conclude that **any** two infinite sets have the same size...

Uncountable Sets

- We may be tempted to conclude that **any** two infinite sets have the same size...
- After all, we just need to show that a **correspondence** between the two exists

Uncountable Sets

- We may be tempted to conclude that **any** two infinite sets have the same size...
- After all, we just need to show that a **correspondence** between the two exists
- However, for some infinite sets no correspondence (with the set \mathbb{N}) exists

Uncountable Sets

- We may be tempted to conclude that **any** two infinite sets have the same size...
- After all, we just need to show that a **correspondence** between the two exists
- However, for some infinite sets no correspondence (with the set \mathbb{N}) exists
- We call those sets **uncountable**

The Set of Real Numbers \mathbb{R}

- The set of real numbers \mathbb{R} is an example of an uncountable set

The Set of Real Numbers \mathbb{R}

- The set of real numbers \mathbb{R} is an example of an uncountable set
- A real number is one that has a **decimal representation**
 - For example: $\pi = 3.1415926\dots$ or $\sqrt{2} = 1.4142135\dots$ are real numbers

The Set of Real Numbers \mathbb{R}

- The set of real numbers \mathbb{R} is an example of an uncountable set
- A real number is one that has a **decimal representation**
 - For example: $\pi = 3.1415926 \dots$ or $\sqrt{2} = 1.4142135 \dots$ are real numbers
- Cantor proved that \mathbb{R} is uncountable, and to do so he introduced the so-called **diagonalization method**

\mathbb{R} is Uncountable: Cantor's Proof

- To show that \mathbb{R} is uncountable, we must show that no correspondence exists between \mathbb{N} and \mathbb{R}

\mathbb{R} is Uncountable: Cantor's Proof

- To show that \mathbb{R} is uncountable, we must show that no correspondence exists between \mathbb{N} and \mathbb{R}
- The proof works by contradiction, i.e., assuming that a correspondence $f : \mathbb{N} \mapsto \mathbb{R}$ exists

\mathbb{R} is Uncountable: Cantor's Proof

- To show that \mathbb{R} is uncountable, we must show that no correspondence exists between \mathbb{N} and \mathbb{R}
- The proof works by contradiction, i.e., assuming that a correspondence $f : \mathbb{N} \mapsto \mathbb{R}$ exists
- Eventually, we want to get to an absurd!

\mathbb{R} is Uncountable: Cantor's Proof

- To show that \mathbb{R} is uncountable, we must show that no correspondence exists between \mathbb{N} and \mathbb{R}
- The proof works by contradiction, i.e., assuming that a correspondence $f : \mathbb{N} \mapsto \mathbb{R}$ exists
- Eventually, we want to get to an absurd!
- For f to be a correspondence it has to pair **all** the elements of \mathbb{N} with **all** the elements of \mathbb{R}

\mathbb{R} is Uncountable: Cantor's Proof

- To show that \mathbb{R} is uncountable, we must show that no correspondence exists between \mathbb{N} and \mathbb{R}
- The proof works by contradiction, i.e., assuming that a correspondence $f : \mathbb{N} \mapsto \mathbb{R}$ exists
- Eventually, we want to get to an absurd!
- For f to be a correspondence it has to pair **all** the elements of \mathbb{N} with **all** the elements of \mathbb{R}
- The idea is to find an element $x \in \mathbb{R}$ that is not paired with any element of \mathbb{N}

\mathbb{R} is Uncountable: Cantor's Proof

- To find the counterexample $x \in \mathbb{R}$ that leads to a contradiction, we use an example

\mathbb{R} is Uncountable: Cantor's Proof

- To find the counterexample $x \in \mathbb{R}$ that leads to a contradiction, we use an example
- Suppose (again) that f exists and some of its values are:
 - $f(1) = \pi = 3.14159\dots$
 - $f(2) = 55.55555\dots$
 - $f(3) = 0.12345\dots$
 - $f(4) = 0.50000\dots$
 - \dots

\mathbb{R} is Uncountable: Cantor's Proof

- To find the counterexample $x \in \mathbb{R}$ that leads to a contradiction, we use an example
- Suppose (again) that f exists and some of its values are:
 - $f(1) = \pi = 3.14159\dots$
 - $f(2) = 55.55555\dots$
 - $f(3) = 0.12345\dots$
 - $f(4) = 0.50000\dots$
 - \dots
- Now, we construct our counterexample x by giving its decimal representation as follows:
 - It is a number between 0 and 1, so all its significant digits are decimals

\mathbb{R} is Uncountable: Cantor's Proof

- To find the counterexample $x \in \mathbb{R}$ that leads to a contradiction, we use an example
- Suppose (again) that f exists and some of its values are:
 - $f(1) = \pi = 3.14159\dots$
 - $f(2) = 55.55555\dots$
 - $f(3) = 0.12345\dots$
 - $f(4) = 0.50000\dots$
 - \dots
- Now, we construct our counterexample x by giving its decimal representation as follows:
 - It is a number between 0 and 1, so all its significant digits are decimals
 - Our goal is to ensure that $\forall n \in \mathbb{N}, x \neq f(n)$

\mathbb{R} is Uncountable: Cantor's Proof

- To ensure that $x \neq f(1)$, we let the **first** digit of x to be anything different from the **first** decimal digit of $f(1)$ (i.e., 1): let it be **4**

\mathbb{R} is Uncountable: Cantor's Proof

- To ensure that $x \neq f(1)$, we let the **first** digit of x to be anything different from the **first** decimal digit of $f(1)$ (i.e., 1): let it be **4**
- To ensure that $x \neq f(2)$, we let the **second** digit of x to be anything different from the **second** decimal digit of $f(2)$ (i.e., 5): let it be **6**

\mathbb{R} is Uncountable: Cantor's Proof

- To ensure that $x \neq f(1)$, we let the **first** digit of x to be anything different from the **first** decimal digit of $f(1)$ (i.e., 1): let it be **4**
- To ensure that $x \neq f(2)$, we let the **second** digit of x to be anything different from the **second** decimal digit of $f(2)$ (i.e., 5): let it be **6**
- To ensure that $x \neq f(3)$, we let the **third** digit of x to be anything different from the **third** decimal digit of $f(3)$ (i.e., 3): let it be **7**

\mathbb{R} is Uncountable: Cantor's Proof

- To ensure that $x \neq f(1)$, we let the **first** digit of x to be anything different from the **first** decimal digit of $f(1)$ (i.e., 1): let it be **4**
- To ensure that $x \neq f(2)$, we let the **second** digit of x to be anything different from the **second** decimal digit of $f(2)$ (i.e., 5): let it be **6**
- To ensure that $x \neq f(3)$, we let the **third** digit of x to be anything different from the **third** decimal digit of $f(3)$ (i.e., 3): let it be **7**
- We keep doing this all the way down the “diagonal” of the table for f

\mathbb{R} is Uncountable: Cantor's Proof

- To ensure that $x \neq f(1)$, we let the **first** digit of x to be anything different from the **first** decimal digit of $f(1)$ (i.e., 1): let it be **4**
- To ensure that $x \neq f(2)$, we let the **second** digit of x to be anything different from the **second** decimal digit of $f(2)$ (i.e., 5): let it be **6**
- To ensure that $x \neq f(3)$, we let the **third** digit of x to be anything different from the **third** decimal digit of $f(3)$ (i.e., 3): let it be **7**
- We keep doing this all the way down the “diagonal” of the table for f
- Eventually, we build $x = \mathbf{0.467} \dots$ which is not equal to **any** $f(n)$, as it differs from it in its n -th decimal digit

\mathbb{R} is Uncountable: Implications

- The fact that \mathbb{R} is uncountable has profound implications also for the theory of computation

\mathbb{R} is Uncountable: Implications

- The fact that \mathbb{R} is uncountable has profound implications also for the theory of computation
- Indeed, it tells us that there exist some numbers that cannot be **computed**

\mathbb{R} is Uncountable: Implications

- The fact that \mathbb{R} is uncountable has profound implications also for the theory of computation
- Indeed, it tells us that there exist some numbers that cannot be **computed**
- In other words, there exist some languages that are **not decidable (nor even recognizable)**

The Halting Problem

- This is one of the most fundamental problem in the theory of computation

The Halting Problem

- This is one of the most fundamental problem in the theory of computation
- Informally, it aims to find an algorithm that can tell whether a Turing machine M **halts and accepts** or not when this is given an input x

The Halting Problem

- This is one of the most fundamental problem in the theory of computation
- Informally, it aims to find an algorithm that can tell whether a Turing machine M **halts and accepts** or not when this is given an input x
- Using the same Cantor's diagonalization argument, we will prove that the halting problem is algorithmically **undecidable**:
 - There is no such an algorithm (i.e., Turing machine) that can **decide** if another Turing machine will ever halt and accept on a given input

The Halting Problem

- This is one of the most fundamental problem in the theory of computation
- Informally, it aims to find an algorithm that can tell whether a Turing machine M **halts and accepts** or not when this is given an input x
- Using the same Cantor's diagonalization argument, we will prove that the halting problem is algorithmically **undecidable**:
 - There is no such an algorithm (i.e., Turing machine) that can **decide** if another Turing machine will ever halt and accept on a given input
- This result has profound philosophical and practical implications, showing that computers are inherently limited in a fundamental way

The Halting Problem: Formal Definition (1)

Definition (The Halting Problem)

We aim to find a **total computable function** $HALT_{ACC} : \Sigma^* \mapsto \Sigma$ such that for every string representation of a Turing machine along with its input $\langle M, x \rangle \in \Sigma^*$:

$$HALT_{ACC}(\langle M, x \rangle) = \begin{cases} 1 & \text{if } M(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$

The Halting Problem: Formal Definition (1)

Definition (The Halting Problem)

We aim to find a **total computable function** $HALT_{ACC} : \Sigma^* \mapsto \Sigma$ such that for every string representation of a Turing machine along with its input $\langle M, x \rangle \in \Sigma^*$:

$$HALT_{ACC}(\langle M, x \rangle) = \begin{cases} 1 & \text{if } M(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Note

$\langle M, x \rangle$ can be thought of as the concatenation of two encodings: the Turing machine $\langle M \rangle$ and its input $\langle x \rangle$.

$HALT_{ACC}$ is the boolean function that outputs 1 if M halts (and accepts) x , i.e., $M(x) = 1$, or 0 otherwise (i.e., if $M(x) = 0$ **or** $M(x) = \perp$)

The Halting Problem: Formal Definition (2)

Definition (The Halting Problem)

Consider the **language** $A_{TM} : \{\langle M, x \rangle \in \Sigma^* \mid M \text{ is a TM and } M(x) = 1\}$.
We can also define:

$$HALT_{ACC}(\langle M, x \rangle) = \begin{cases} 1 & \text{if } \langle M, x \rangle \in A_{TM} \\ 0 & \text{if } \langle M, x \rangle \notin A_{TM} \end{cases}$$

The Halting Problem: Formal Definition (2)

Definition (The Halting Problem)

Consider the **language** $A_{TM} : \{\langle M, x \rangle \in \Sigma^* \mid M \text{ is a TM and } M(x) = 1\}$. We can also define:

$$HALT_{ACC}(\langle M, x \rangle) = \begin{cases} 1 & \text{if } \langle M, x \rangle \in A_{TM} \\ 0 & \text{if } \langle M, x \rangle \notin A_{TM} \end{cases}$$

Note

Finding whether the **total computable function** $HALT_{ACC}$ exists is equivalent to determine whether the language A_{TM} is **decidable**

The Halting Problem: Clarification

- The actual definition of the halting problem is subtly different from the one given above

The Halting Problem: Clarification

- The actual definition of the halting problem is subtly different from the one given above
- Our definition asks the function to output 1 if and only if the input TM M halts **and** accepts its input x

The Halting Problem: Clarification

- The actual definition of the halting problem is subtly different from the one given above
- Our definition asks the function to output 1 if and only if the input TM M halts **and** accepts its input x
- However, a more accurate definition would be provided by the following $HALT$ function, which outputs 1 whenever M halts on x (no matter if it accepts or rejects):

$$HALT(\langle M, x \rangle) = \begin{cases} 1 & \text{if } M(x) = 1 \vee M(x) = 0 \\ 0 & M(x) = \perp \end{cases}$$

- We will see that the two definitions are equivalent, i.e., both $HALT_{ACC}$ and $HALT$ are undecidable

The Halting Problem is not Decidable

Theorem (The Halting Problem is not Decidable)

*The function $HALT_{ACC}$ as defined above is **not total and computable**.
Equivalently, the language A_{TM} is **not decidable**.*

The Halting Problem is not Decidable

Theorem (The Halting Problem is not Decidable)

*The function $HALT_{ACC}$ as defined above is **not total and computable**. Equivalently, the language A_{TM} is **not decidable**.*

Note

Before getting into the proof, let us first observe that $HALT_{ACC}$ is **partial and computable** (i.e., A_{TM} is **recognizable**)

The Halting Problem is Recognizable

Theorem (The Halting Problem is Recognizable)

The following Turing machine U computes a partial function HALT_{ACC} (i.e., recognizes A_{TM}):

$U =$ On input $\langle M, x \rangle$, where M is a TM and x an input to it:

- ① Simulate M on input x ;*
- ② If M ever halts and accepts x (i.e., $M(x) = 1$) then return 1; if M ever halts and rejects x (i.e., $M(x) = 0$) then return 0*

The Halting Problem is Recognizable

Theorem (The Halting Problem is Recognizable)

The following Turing machine U computes a partial function $HALT_{ACC}$ (i.e., recognizes A_{TM}):

$U =$ On input $\langle M, x \rangle$, where M is a TM and x an input to it:

- ① Simulate M on input x ;*
- ② If M ever halts and accepts x (i.e., $M(x) = 1$) then return 1; if M ever halts and rejects x (i.e., $M(x) = 0$) then return 0*

Note

The Turing machine U above loops on input $\langle M, x \rangle$ if M loops on x . This is why U computes a partial but **not** a total function (i.e., U recognizes but **not** decides A_{TM})

The Halting Problem: Considerations

- The halting problem is not just a purely theoretical exercise!

The Halting Problem: Considerations

- The halting problem is not just a purely theoretical exercise!
- For example, think about managing the Apple's App Store or Google's Play Store: given some app code, you would like to know whether this gets into an infinite loop!

The Halting Problem: Considerations

- The halting problem is not just a purely theoretical exercise!
- For example, think about managing the Apple's App Store or Google's Play Store: given some app code, you would like to know whether this gets into an infinite loop!
- If U had some way to determine that M would loop forever on x , it could output 0

The Halting Problem: Considerations

- The halting problem is not just a purely theoretical exercise!
- For example, think about managing the Apple's App Store or Google's Play Store: given some app code, you would like to know whether this gets into an infinite loop!
- If U had some way to determine that M would loop forever on x , it could output 0
- Unfortunately, no algorithm exists that can make this determination **for every possible TMs and their inputs**

The Halting Problem is not Decidable

Theorem (The Halting Problem is not Decidable)

Consider the **language** $A_{TM} : \{\langle M, x \rangle \in \Sigma^* \mid M \text{ is a TM and } M(x) = 1\}$.
We can also define:

$$HALT_{ACC}(\langle M, x \rangle) = \begin{cases} 1 & \text{if } \langle M, x \rangle \in A_{TM} \\ 0 & \text{if } \langle M, x \rangle \notin A_{TM} \end{cases}$$

The function $HALT_{ACC}$ as defined above is **not total and computable**.
Equivalently, the language A_{TM} is **not decidable**.

The Halting Problem is not Decidable: Proof

- We assume that A_{TM} is decidable and obtain a contradiction

The Halting Problem is not Decidable: Proof

- We assume that A_{TM} is decidable and obtain a contradiction
- Let H be a decider for A_{TM} , such that on input $\langle M, x \rangle$, where M is a TM and x is a (binary) string:
 - H **halts and outputs 1** if M outputs 1 on x
 - H **halts and outputs 0** if M fails to output 1 on x (i.e., either outputs 0 or loop forever)

The Halting Problem is not Decidable: Proof

- We assume that A_{TM} is decidable and obtain a contradiction
- Let H be a decider for A_{TM} , such that on input $\langle M, x \rangle$, where M is a TM and x is a (binary) string:
 - H **halts and outputs 1** if M outputs 1 on x
 - H **halts and outputs 0** if M fails to output 1 on x (i.e., either outputs 0 or loop forever)
- In other words:

$$H(\langle M, x \rangle) = \begin{cases} 1 & \text{if } M(x) = 1 \\ 0 & \text{if } M(x) = 0 \vee M(x) = \perp \end{cases}$$

The Halting Problem is not Decidable: Proof

- Let's construct another TM called D , which uses H as a subroutine

The Halting Problem is not Decidable: Proof

- Let's construct another TM called D , which uses H as a subroutine
- This new TM calls H to determine what M does when the input to M is its own description $\langle M \rangle$

The Halting Problem is not Decidable: Proof

- Let's construct another TM called D , which uses H as a subroutine
- This new TM calls H to determine what M does when the input to M is its own description $\langle M \rangle$
- Once D has determined this information, it does the **opposite**, i.e., it outputs 0 if $H(\langle M, \langle M \rangle \rangle) = 1$ or 1 if $H(\langle M, \langle M \rangle \rangle) = 0$

$$D(\langle M \rangle) = \begin{cases} 1 & \text{if } H(\langle M, \langle M \rangle \rangle) = 0 \Leftrightarrow M(\langle M \rangle) = 0 \vee M(\langle M \rangle) = \perp \\ 0 & \text{if } H(\langle M, \langle M \rangle \rangle) = 1 \Leftrightarrow M(\langle M \rangle) = 1 \end{cases}$$

The Halting Problem is not Decidable: Proof

What happens if we run D with its own description $\langle D \rangle$ as input?

$$D(\langle D \rangle) = \begin{cases} 1 & \text{if } H(\langle D, \langle D \rangle \rangle) = 0 \Leftrightarrow D(\langle D \rangle) = 0 \vee D(\langle D \rangle) = \perp \\ 0 & \text{if } H(\langle D, \langle D \rangle \rangle) = 1 \Leftrightarrow D(\langle D \rangle) = 1 \end{cases}$$

The Halting Problem is not Decidable: Proof

What happens if we run D with its own description $\langle D \rangle$ as input?

$$D(\langle D \rangle) = \begin{cases} 1 & \text{if } H(\langle D, \langle D \rangle \rangle) = 0 \Leftrightarrow D(\langle D \rangle) = 0 \vee D(\langle D \rangle) = \perp \\ 0 & \text{if } H(\langle D, \langle D \rangle \rangle) = 1 \Leftrightarrow D(\langle D \rangle) = 1 \end{cases}$$

Note

No matter what D does, it is forced to do the opposite, which leads us to a contradiction! As such, neither D nor H can exist!

The Halting Problem is not Decidable: Proof

- Where is the **diagonalization argument** in this proof?

The Halting Problem is not Decidable: Proof

- Where is the **diagonalization argument** in this proof?
- We must examine the tables of behavior for the two TMs used: H and D

The Halting Problem is not Decidable: Proof

- Where is the **diagonalization argument** in this proof?
- We must examine the tables of behavior for the two TMs used: H and D
- We start first by showing a more general table of behavior of **all** TMs when they are input with the description of some (other) TM

The Halting Problem is not Decidable: Proof

- Where is the **diagonalization argument** in this proof?
- We must examine the tables of behavior for the two TMs used: H and D
- We start first by showing a more general table of behavior of **all** TMs when they are input with the description of some (other) TM
- In other words, given a generic TM M_i the table shows the behavior of M_i when this is input with $\langle M_j \rangle$

The Halting Problem is not Decidable: Proof

- Where is the **diagonalization argument** in this proof?
- We must examine the tables of behavior for the two TMs used: H and D
- We start first by showing a more general table of behavior of **all** TMs when they are input with the description of some (other) TM
- In other words, given a generic TM M_i the table shows the behavior of M_i when this is input with $\langle M_j \rangle$
- Without loss of generality, the output of M_i when input with $\langle M_j \rangle$ is either: 1 (*accept*), 0 (*reject*), or \perp (*does not halt*)

The Behavior of All Turing Machines

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$...	$\langle M_k \rangle$...
M_1	accept	reject	accept	doesn't halt	...	reject	...
M_2	accept	accept	accept	accept	...	doesn't halt	...
M_3	reject	doesn't halt	doesn't halt	reject	...	accept	...
M_4	accept	accept	reject	reject	...	doesn't halt	...
...
M_k	doesn't halt	reject	accept	accept		accept	...
...

Entry i, j contains the output of TM M_i on input $\langle M_j \rangle$

The Behavior of H

The table below shows the behavior of TM H on inputs from the table above: if M_i accepts $\langle M_j \rangle$ so does H , if M_i either rejects or doesn't halt on $\langle M_j \rangle$ then H rejects

The Behavior of H

The table below shows the behavior of TM H on inputs from the table above: if M_i accepts $\langle M_j \rangle$ so does H , if M_i either rejects or doesn't halt on $\langle M_j \rangle$ then H rejects

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$...	$\langle M_k \rangle$...
M_1	accept	reject	accept	reject	...	reject	...
M_2	accept	accept	accept	accept	...	reject	...
M_3	reject	reject	reject	reject	...	accept	...
M_4	accept	accept	reject	reject	...	reject	...
...
M_k	reject	reject	accept	accept		accept	...
...

The Behavior of D

Assuming H exists, so does D ! As such, D must be located somewhere in the list of **all** TMs

The Behavior of D

Assuming H exists, so does D ! As such, D must be located somewhere in the list of **all** TMs

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$...	$\langle D \rangle$...
M_1	<u>accept</u>	reject	accept	reject	...	reject	...
M_2	accept	<u>accept</u>	accept	accept	...	reject	...
M_3	reject	reject	<u>reject</u>	reject	...	accept	...
M_4	accept	accept	reject	<u>reject</u>	...	reject	...
...
D	reject	reject	accept	accept		?	...
...

The Behavior of D

Since D computes the opposite of the diagonal entries, a contradiction occurs in correspondence of column $\langle D \rangle$, where the entry must be the opposite of itself!

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$...	$\langle D \rangle$...
M_1	<u>accept</u>	reject	accept	reject	...	reject	...
M_2	accept	<u>accept</u>	accept	accept	...	reject	...
M_3	reject	reject	<u>reject</u>	reject	...	accept	...
M_4	accept	accept	reject	<u>reject</u>	...	reject	...
...
D	reject	reject	accept	accept		?	...
...

Turing-unrecognizable Languages

- We have shown that the halting problem (i.e., the language it defines A_{TM}) is **undecidable** (or semi-decidable)

Turing-unrecognizable Languages

- We have shown that the halting problem (i.e., the language it defines A_{TM}) is **undecidable** (or semi-decidable)
- We know that A_{TM} is **recognizable**

Turing-unrecognizable Languages

- We have shown that the halting problem (i.e., the language it defines A_{TM}) is **undecidable** (or semi-decidable)
- We know that A_{TM} is **recognizable**
- However, there exist languages that are not even recognized by a TM

Turing-unrecognizable Languages

- We have shown that the halting problem (i.e., the language it defines A_{TM}) is **undecidable** (or semi-decidable)
- We know that A_{TM} is **recognizable**
- However, there exist languages that are not even recognized by a TM
- We will prove that using the same diagonalization argument

Turing-unrecognizable Languages

Corollary (Turing-unrecognizable Languages)

- *There are **countably** many Turing machines*

Turing-unrecognizable Languages

Corollary (Turing-unrecognizable Languages)

- *There are **countably** many Turing machines*
- *There are **uncountably** many languages*

Turing-unrecognizable Languages

Corollary (Turing-unrecognizable Languages)

- There are **countably** many Turing machines
- There are **uncountably** many languages
- Each Turing machine can **recognize** a single language

Turing-unrecognizable Languages

Corollary (Turing-unrecognizable Languages)

- There are **countably** many Turing machines
- There are **uncountably** many languages
- Each Turing machine can **recognize** a single language



Some languages are not Turing-recognizable

Turing-unrecognizable Languages: Proof

- To prove the corollary above we need to show that:

Turing-unrecognizable Languages: Proof

- To prove the corollary above we need to show that:
 - 1 The set of **all** Turing machines is **countable**

Turing-unrecognizable Languages: Proof

- To prove the corollary above we need to show that:
 - 1 The set of **all** Turing machines is **countable**
 - 2 The set of **all** languages is **uncountable**

Turing-unrecognizable Languages: Proof

- To prove the corollary above we need to show that:
 - 1 The set of **all** Turing machines is **countable**
 - 2 The set of **all** languages is **uncountable**
- Let's start with 1!

1) The Set of All TMs is Countable

- We first observe that for any finite alphabet Σ the (infinite) set of all the finite-length strings Σ^* is countable

1) The Set of All TMs is Countable

- We first observe that for any finite alphabet Σ the (infinite) set of all the finite-length strings Σ^* is countable
- We can indeed make a list of all the elements of Σ^* by noticing that there are finitely many strings of each specific length:
 - We enumerate all the strings of length 0, then those of length 1, length 2, etc.

1) The Set of All TMs is Countable

- We first observe that for any finite alphabet Σ the (infinite) set of all the finite-length strings Σ^* is countable
- We can indeed make a list of all the elements of Σ^* by noticing that there are finitely many strings of each specific length:
 - We enumerate all the strings of length 0, then those of length 1, length 2, etc.
- The set of all Turing machines is countable because each TM has a finite (binary) string encoding $\langle M \rangle \in \Sigma^*$, where $\Sigma = \{0, 1\}$

1) The Set of All TMs is Countable

- We first observe that for any finite alphabet Σ the (infinite) set of all the finite-length strings Σ^* is countable
- We can indeed make a list of all the elements of Σ^* by noticing that there are finitely many strings of each specific length:
 - We enumerate all the strings of length 0, then those of length 1, length 2, etc.
- The set of all Turing machines is countable because each TM has a finite (binary) string encoding $\langle M \rangle \in \Sigma^*$, where $\Sigma = \{0, 1\}$
- Therefore, we can list all those encodings $\langle M_1 \rangle, \langle M_2 \rangle, \dots$ as above

1) The Set of All TMs is Countable

- We first observe that for any finite alphabet Σ the (infinite) set of all the finite-length strings Σ^* is countable
- We can indeed make a list of all the elements of Σ^* by noticing that there are finitely many strings of each specific length:
 - We enumerate all the strings of length 0, then those of length 1, length 2, etc.
- The set of all Turing machines is countable because each TM has a finite (binary) string encoding $\langle M \rangle \in \Sigma^*$, where $\Sigma = \{0, 1\}$
- Therefore, we can list all those encodings $\langle M_1 \rangle, \langle M_2 \rangle, \dots$ as above
- In other words, we can easily find a correspondence between the set of natural numbers \mathbb{N} and the (infinite) set of **all** Turing machines (encoded as binary strings)

2) The Set of All Languages is Uncountable

- Two approaches to show that this statement is true:
 - 1 using the countable set of all the finite-length binary strings and Cantor's diagonalization
 - 2 using the uncountable set of all the infinite-length binary strings

2) The Set of All Languages is Uncountable

- Two approaches to show that this statement is true:
 - ① using the countable set of all the finite-length binary strings and Cantor's diagonalization
 - ② using the uncountable set of all the infinite-length binary strings
- Let's start from 1!

2) The Set of All Languages is Uncountable: First Proof

- Let $\Sigma^* = \{\sigma_1, \sigma_2, \dots\}$ be the set of all the finite-length binary strings

2) The Set of All Languages is Uncountable: First Proof

- Let $\Sigma^* = \{\sigma_1, \sigma_2, \dots\}$ be the set of all the finite-length binary strings
- We already shown that Σ^* is **countable**

2) The Set of All Languages is Uncountable: First Proof

- Let $\Sigma^* = \{\sigma_1, \sigma_2, \dots\}$ be the set of all the finite-length binary strings
- We already shown that Σ^* is **countable**
- Any language L can be seen as a subset of Σ^*

2) The Set of All Languages is Uncountable: First Proof

- Let $\Sigma^* = \{\sigma_1, \sigma_2, \dots\}$ be the set of all the finite-length binary strings
- We already shown that Σ^* is **countable**
- Any language L can be seen as a subset of Σ^*
- We must show that the set \mathcal{L} of **all** languages is **uncountable**

2) The Set of All Languages is Uncountable: First Proof

- Suppose that \mathcal{L} is, instead, countable

2) The Set of All Languages is Uncountable: First Proof

- Suppose that \mathcal{L} is, instead, countable
- Therefore, there exists a correspondence $f : \mathbb{N} \mapsto \mathcal{L}$

2) The Set of All Languages is Uncountable: First Proof

- Suppose that \mathcal{L} is, instead, countable
- Therefore, there exists a correspondence $f : \mathbb{N} \mapsto \mathcal{L}$
- We can define $f(k)$ as the k -th element of \mathcal{L} , denoted by L_k
(**Remember:** each element of \mathcal{L} is actually a subset of Σ^*)

2) The Set of All Languages is Uncountable: First Proof

- Suppose that \mathcal{L} is, instead, countable
- Therefore, there exists a correspondence $f : \mathbb{N} \mapsto \mathcal{L}$
- We can define $f(k)$ as the k -th element of \mathcal{L} , denoted by L_k
(**Remember:** each element of \mathcal{L} is actually a subset of Σ^*)
- Now, we apply Cantor's diagonalization to build a new element $L_j \in \mathcal{L}$ as follows:
 - If $\sigma_k \in L_k$ then $\sigma_k \notin L_j$
 - If $\sigma_k \notin L_k$ then $\sigma_k \in L_j$

2) The Set of All Languages is Uncountable: First Proof

- Suppose that \mathcal{L} is, instead, countable
- Therefore, there exists a correspondence $f : \mathbb{N} \mapsto \mathcal{L}$
- We can define $f(k)$ as the k -th element of \mathcal{L} , denoted by L_k
(**Remember:** each element of \mathcal{L} is actually a subset of Σ^*)
- Now, we apply Cantor's diagonalization to build a new element $L_j \in \mathcal{L}$ as follows:
 - If $\sigma_k \in L_k$ then $\sigma_k \notin L_j$
 - If $\sigma_k \notin L_k$ then $\sigma_k \in L_j$
- Then, we have found an element $L_j \in \mathcal{L}$, such that there is no $j \in \mathbb{N}$, $f(j) = L_j$

2) The Set of All Languages is Uncountable: Second Proof

- In this second proof, instead, we start from a different perspective

2) The Set of All Languages is Uncountable: Second Proof

- In this second proof, instead, we start from a different perspective
- We first observe that the set of all **infinite** binary sequences is uncountable

2) The Set of All Languages is Uncountable: Second Proof

- In this second proof, instead, we start from a different perspective
- We first observe that the set of all **infinite** binary sequences is uncountable
- An infinite binary sequence is a never-ending sequence of binary symbols (0s and 1s)

2) The Set of All Languages is Uncountable: Second Proof

- In this second proof, instead, we start from a different perspective
- We first observe that the set of all **infinite** binary sequences is uncountable
- An infinite binary sequence is a never-ending sequence of binary symbols (0s and 1s)
- Let \mathcal{B} be the set of all such infinite binary sequences: we can show \mathcal{B} is uncountable using Cantor's diagonalization

Note

The infinite set of finite-length binary strings $\Sigma^* = \{0, 1\}^*$ is **not equal** to the set of infinite-length binary strings \mathcal{B}

2) The Set of All Languages is Uncountable: Second Proof

- To show that \mathcal{B} is uncountable we assume it is, in fact, countable

2) The Set of All Languages is Uncountable: Second Proof

- To show that \mathcal{B} is uncountable we assume it is, in fact, countable
- If that is the case, we will be able to list all the infinite-length binary sequences

2) The Set of All Languages is Uncountable: Second Proof

- To show that \mathcal{B} is uncountable we assume it is, in fact, countable
- If that is the case, we will be able to list all the infinite-length binary sequences
- In other words, we can create a correspondence $f : \mathbb{N} \mapsto \mathbb{B}$, e.g.:
 - $f(1) = 00000 \dots$
 - $f(2) = 11010 \dots$
 - $f(3) = 00101 \dots$
 - \dots

2) The Set of All Languages is Uncountable: Second Proof

- To show that \mathcal{B} is uncountable we assume it is, in fact, countable
- If that is the case, we will be able to list all the infinite-length binary sequences
- In other words, we can create a correspondence $f : \mathbb{N} \mapsto \mathbb{B}$, e.g.:
 - $f(1) = 00000 \dots$
 - $f(2) = 11010 \dots$
 - $f(3) = 00101 \dots$
 - \dots
- However, we can build another binary sequence b such that its i -th bit is the opposite of the i -th bit of each $f(i)$. In the example above:
 $b = 100 \dots$

2) The Set of All Languages is Uncountable: Second Proof

- To show that \mathcal{B} is uncountable we assume it is, in fact, countable
- If that is the case, we will be able to list all the infinite-length binary sequences
- In other words, we can create a correspondence $f : \mathbb{N} \mapsto \mathbb{B}$, e.g.:
 - $f(1) = 00000 \dots$
 - $f(2) = 11010 \dots$
 - $f(3) = 00101 \dots$
 - \dots
- However, we can build another binary sequence b such that its i -th bit is the opposite of the i -th bit of each $f(i)$. In the example above: $b = 100 \dots$
- We have found a sequence b which is not paired with any natural number listed $\implies \mathcal{B}$ is **uncountable**

2) The Set of All Languages is Uncountable: Second Proof

- So far we have shown that \mathcal{B} is uncountable

2) The Set of All Languages is Uncountable: Second Proof

- So far we have shown that \mathcal{B} is uncountable
- Now we want to show that the set \mathcal{L} all languages over a finite alphabet Σ is uncountable as well

2) The Set of All Languages is Uncountable: Second Proof

- So far we have shown that \mathcal{B} is uncountable
- Now we want to show that the set \mathcal{L} all languages over a finite alphabet Σ is uncountable as well
- To do so, we build a correspondence $f : \mathcal{L} \mapsto \mathcal{B}$, thus showing they both have the same size

2) The Set of All Languages is Uncountable: Second Proof

- So far we have shown that \mathcal{B} is uncountable
- Now we want to show that the set \mathcal{L} all languages over a finite alphabet Σ is uncountable as well
- To do so, we build a correspondence $f : \mathcal{L} \mapsto \mathcal{B}$, thus showing they both have the same size
- Let $\Sigma^* = \{\sigma_1, \sigma_2, \dots\}$ the list of finite-length strings over Σ (e.g., $\Sigma = \{0, 1\}$)

2) The Set of All Languages is Uncountable: Second Proof

- So far we have shown that \mathcal{B} is uncountable
- Now we want to show that the set \mathcal{L} all languages over a finite alphabet Σ is uncountable as well
- To do so, we build a correspondence $f : \mathcal{L} \mapsto \mathcal{B}$, thus showing they both have the same size
- Let $\Sigma^* = \{\sigma_1, \sigma_2, \dots\}$ the list of finite-length strings over Σ (e.g., $\Sigma = \{0, 1\}$)
- Each language $L \in \mathcal{L}$ can be described as a unique sequence in \mathcal{B} , where the i -th bit of that sequence is 1 if $\sigma_i \in L$, or 0 if $\sigma_i \notin L$

Note

Without loss of generality, each language $L \in \mathcal{L}$ is indeed composed of infinitely many finite-length strings, therefore represented as an infinite binary sequence

2) The Set of All Languages is Uncountable: Second Proof

Definition (Characteristic Sequence)

We call the infinite binary sequence associated with each $L \in \mathcal{L}$ the **characteristic sequence** of L , denoted by χ_L

2) The Set of All Languages is Uncountable: Second Proof

Definition (Characteristic Sequence)

We call the infinite binary sequence associated with each $L \in \mathcal{L}$ the **characteristic sequence** of L , denoted by χ_L

Example

Suppose L is the language of all strings starting with a 0 over the alphabet $\Sigma = \{0, 1\}$. Then, its **characteristic sequence** χ_L will be as follows:

$$\Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, \dots\}$$

$$L = \{0, 00, 01, 000, 001, \dots\}$$

$$\chi_L = \{0, 1, 0, 1, 1, 0, 0, 1, 1, \dots\}$$

2) The Set of All Languages is Uncountable: Second Proof

- The function $f : \mathcal{L} \mapsto \mathcal{B}$, where $f(L) = \chi_L$ equals the characteristic sequence of L is one-to-one and onto, thus a correspondence

2) The Set of All Languages is Uncountable: Second Proof

- The function $f : \mathcal{L} \mapsto \mathcal{B}$, where $f(L) = \chi_L$ equals the characteristic sequence of L is one-to-one and onto, thus a correspondence
- Given that \mathcal{B} is uncountable and we are able to put it into a correspondence with \mathcal{L} then \mathcal{L} is **uncountable** as well

2) The Set of All Languages is Uncountable: Second Proof

- The function $f : \mathcal{L} \mapsto \mathcal{B}$, where $f(L) = \chi_L$ equals the characteristic sequence of L is one-to-one and onto, thus a correspondence
- Given that \mathcal{B} is uncountable and we are able to put it into a correspondence with \mathcal{L} then \mathcal{L} is **uncountable** as well
- We have shown (again) that the set of all languages \mathcal{L} cannot be put into correspondence with the set of all Turing machines

2) The Set of All Languages is Uncountable: Second Proof

- The function $f : \mathcal{L} \mapsto \mathcal{B}$, where $f(L) = \chi_L$ equals the characteristic sequence of L is one-to-one and onto, thus a correspondence
- Given that \mathcal{B} is uncountable and we are able to put it into a correspondence with \mathcal{L} then \mathcal{L} is **uncountable** as well
- We have shown (again) that the set of all languages \mathcal{L} cannot be put into correspondence with the set of all Turing machines
- We can conclude that some languages are **not** recognized by any Turing machine

The Set of All Languages is Uncountable: Extra Proof

Theorem (Cantor's Theorem)

For **any** set A , the set of all subsets of A - also known as the **power set** of A , denoted by $\mathcal{P}(A)$ - has a strictly greater cardinality than A itself:

$$|\mathcal{P}(A)| > |A|$$

The Set of All Languages is Uncountable: Extra Proof

Theorem (Cantor's Theorem)

For **any** set A , the set of all subsets of A - also known as the **power set** of A , denoted by $\mathcal{P}(A)$ - has a strictly greater cardinality than A itself:

$$|\mathcal{P}(A)| > |A|$$

Example (Countably Finite Sets)

If A is countable and finite, the relation trivially holds:

Let $|A| = n$, then $|\mathcal{P}(A)| = 2^n$

The Set of All Languages is Uncountable: Extra Proof

Theorem (Cantor's Theorem)

For **any** set A , the set of all subsets of A - also known as the **power set** of A , denoted by $\mathcal{P}(A)$ - has a strictly greater cardinality than A itself:

$$|\mathcal{P}(A)| > |A|$$

Example (Countably Infinite Sets)

If A is countable and infinite, the relation still holds:

Let $A = \mathbb{N}$, then $|\mathbb{N}| = \aleph_0$ and $|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$

In particular, the power set of the set of natural numbers is uncountably infinite and has the same size as the set of real numbers, whose cardinality $\mathfrak{c} = 2^{\aleph_0}$ is referred to as the **cardinality of the continuum**:

$$|\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = \mathfrak{c} = 2^{\aleph_0} > \aleph_0 = |\mathbb{N}|$$

Summary

- We ask ourselves whether there exists any problem that is not algorithmically solvable

Summary

- We ask ourselves whether there exists any problem that is not algorithmically solvable
- We consider as a reference example the well-known **Halting Problem**

Summary

- We ask ourselves whether there exists any problem that is not algorithmically solvable
- We consider as a reference example the well-known **Halting Problem**
- We proved the Halting Problem is **undecidable** (or semi-decidable), although it is of course **recognizable**

Summary

- We ask ourselves whether there exists any problem that is not algorithmically solvable
- We consider as a reference example the well-known **Halting Problem**
- We proved the Halting Problem is **undecidable** (or semi-decidable), although it is of course **recognizable**
- The proof is based on Cantor's diagonalization argument used to compare the size of (infinite) sets

Summary

- We ask ourselves whether there exists any problem that is not algorithmically solvable
- We consider as a reference example the well-known **Halting Problem**
- We proved the Halting Problem is **undecidable** (or semi-decidable), although it is of course **recognizable**
- The proof is based on Cantor's diagonalization argument used to compare the size of (infinite) sets
- Diagonalization allows us to prove that there are functions/languages that cannot be even recognized (i.e., we have countably many TMs yet uncountably many languages)