Teoria degli Algoritmi

Corso di Laurea Magistrale in Matematica Applicata

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Lecture 3: Decidability

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1. Computability
2. Diagonalization
3. The Halting Problem
4. Beyond Undecidability
5. Summary

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    - **partial** computable functions are those **recognized** by a TM (i.e., the machine may never halt)
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- In other words, we identify the set of **computable functions** with those calculated by a Turing machine:
    - **partial** computable functions are those **recognized** by a TM (i.e., the machine may never halt)
    - **total** computable functions are those **decided** by a TM (i.e., the machine always halts)
- We also refer to (**Turing-**)**recognizable** or **recursively-enumerable** languages and (**Turing-**)**decidable** or **recursive** languages

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To put it differently, yet equivalently:

is there any function that cannot be computed by **any** *Turing machine?*

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  - To realize they must **simplified** first, before searching for an algorithmic solution to them
  - To stimulate your imagination!

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- For finite sets, the answer is of course straightforward: just count the elements of each A and B!
- The same does not work for infinite set as we will never finish counting!

The Size of Sets: Example

Example

Consider the set of **even** natural $\mathbb{E} = \{n \in \mathbb{N} \mid n \mod 2 = 0\}$.
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Question

Of course, both $\mathbb{E}$ and Σ^* are infinite (thus, larger than any finite set).
However, is one of the two larger than the other? Can we figure this out?

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- This method compares the size of sets **without** resorting to counting!
- Interestingly enough, this approach extends also to **infinite sets**

The Size of Sets: Cantor's Solution

Definition (Same Size Sets)

Suppose we have two sets A and B, and a function $f : A \mapsto B$.
We say that f is **one-to-one** (or **injective**) if it never maps two different elements to the same place, i.e., $\forall a, a' \in A,\ a \neq a' \Rightarrow f(a) \neq f(a')$

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We say that A and B are **same size sets** if there is a one-to-one, onto function $f : A \mapsto B$; such a function is called a **bijection** or **correspondence**.
In a correspondence between A and B, every element of A maps to a unique element of B and every element of B has a unique element of A that maps to it.

The Size of the Set of Even Natural Numbers

Example (The size of $\mathbb{N}$ vs. the size of $\mathbb{E}$)

Let $\mathbb{N}$ be the set of natural numbers, i.e., $\mathbb{N} = \{1, 2, 3, \ldots\}$, and let $\mathbb{E}$ be the set of **even** natural numbers, i.e., $\mathbb{E} = \{2, 4, 6, \ldots\}$.
Using Cantor's argument, prove that $\mathbb{N}$ and $\mathbb{E}$ have the same size.

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- Intuitively, this sounds odd: $\mathbb{E}$ seems "smaller" than $\mathbb{N}$, as the former is a proper subset of the latter
- However, we can find a correspondence $f : \mathbb{N} \mapsto \mathbb{E}$ which maps each element of $\mathbb{N}$ to an element of $\mathbb{E}$:

$$\forall n \in \mathbb{N},\ f(n) = 2n \in \mathbb{E}$$

Countable Set

Definition (Countable Set)

A set A is **countable** if either it is finite or it has the same size of $\mathbb{N}$

The Size of the Set of Rational Numbers

Example (The size of $\mathbb{Q}$ vs. the size of $\mathbb{N}$)

Let $\mathbb{Q}$ be the set of (positive) rational numbers, i.e., $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{N}\}$. Using Cantor's argument, prove that $\mathbb{Q}$ is countable as there exists a correspondence with $\mathbb{N}$.

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The Size of the Set of Rational Numbers

- An easy way to build the correspondence $f : \mathbb{Q} \mapsto \mathbb{N}$ is to
  - list **all** the elements of $\mathbb{Q} : \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{1}, \frac{2}{2}, \ldots$
  - pair the first element of the list with the number 1 from $\mathbb{N}$, the second with 2, and so on and so forth
  - ensure that every member of $\mathbb{Q}$ appears exactly once (e.g., $\frac{1}{1} = \frac{2}{2} = \frac{3}{3} \ldots$)

The Infinite Matrix of Rational Numbers

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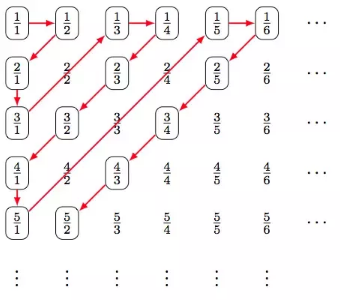
The Infinite Matrix of Rational Numbers

- To build the list of all the elements of $\mathbb{Q}$, just make an infinite matrix containing all the positive rational numbers
- The i-th row contains all the numbers whose numerator is equal to i
- The j-th column contains all the numbers whose denominator is equal to j
- In other words, the rational number $q = \frac{i}{j}$ is located on the i-th row and the j-th column of the matrix

The Infinite Matrix of Rational Numbers



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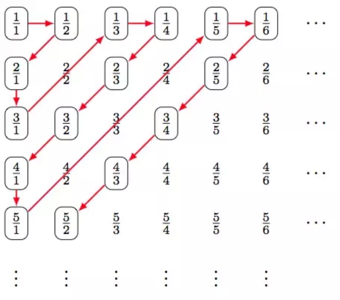

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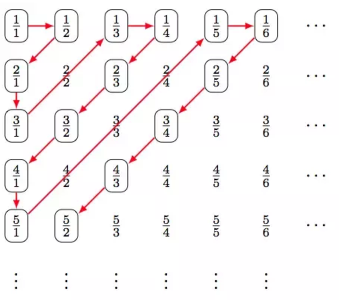

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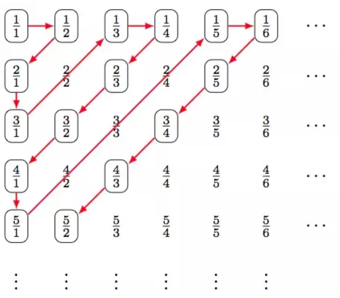

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- Instead, we list the elements following the diagonals, starting from the top-left corner

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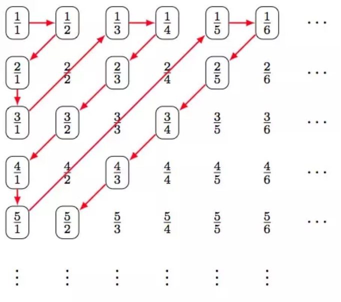

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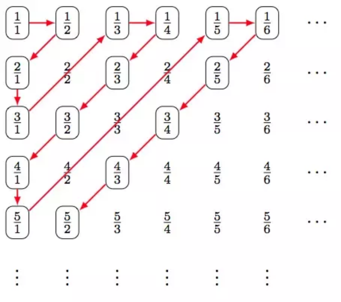

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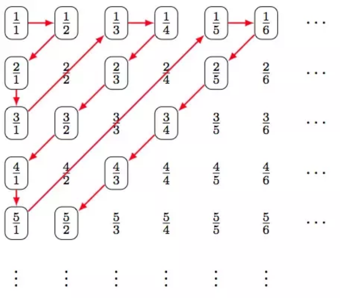

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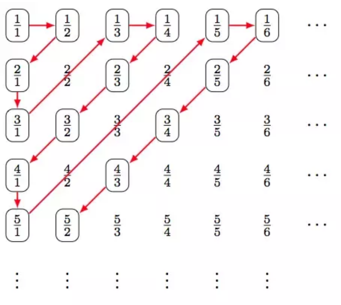

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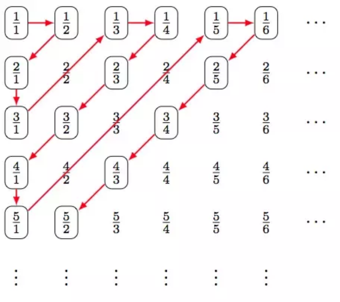

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- We can't add those to our list as we would repeat $\frac{1}{1} = \frac{2}{2}$, so we add only $\frac{3}{1}$ and $\frac{1}{3}$

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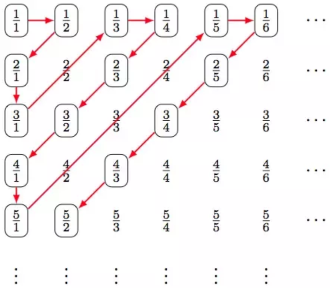

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- We can't add those to our list as we would repeat $\frac{1}{1} = \frac{2}{2}$, so we add only $\frac{3}{1}$ and $\frac{1}{3}$
- If we keep going this way we will obtain the list of all the elements of $\mathbb{Q}$

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- After all, we just need to show that a **correspondence** between the two exists
- However, for some infinite sets no correspondence (with the set $\mathbb{N}$) exists
- We call those sets **uncountable**

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- Cantor proved that $\mathbb{R}$ is uncountable, and to do so he introduced the so-called **diagonalization method**

$\mathbb{R}$ is Uncountable: Cantor's Proof

- To show that $\mathbb{R}$ is uncountable, we must show that no correspondence exists between $\mathbb{N}$ and $\mathbb{R}$

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- For f to be a correspondence it has to pair **all** the elements of $\mathbb{N}$ with **all** the elements of $\mathbb{R}$
- The idea is to find an element $x \in \mathbb{R}$ that is not paired with any element of $\mathbb{N}$

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  - $f(3) = 0.12345\ldots$
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- Now, we construct our counterexample x by giving its decimal representation as follows:
  - It is a number between 0 and 1, so all its significant digits are decimals
  - Our goal is to ensure that $\forall n \in \mathbb{N}, x \neq f(n)$

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- To ensure that $x \neq f(1)$, we let the **first** digit of x to be anything different from the **first** decimal digit of $f(1)$ (i.e., 1): let it be **4**

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- To ensure that $x \neq f(3)$, we let the **third** digit of x to be anything different from the **third** decimal digit of $f(3)$ (i.e., 3): let it be **7**
- We keep doing this all the way down the "diagonal" of the table for f

$\mathbb{R}$ is Uncountable: Cantor's Proof

- To ensure that $x \neq f(1)$, we let the **first** digit of x to be anything different from the **first** decimal digit of $f(1)$ (i.e., 1): let it be **4**
- To ensure that $x \neq f(2)$, we let the **second** digit of x to be anything different from the **second** decimal digit of $f(2)$ (i.e., 5): let it be **6**
- To ensure that $x \neq f(3)$, we let the **third** digit of x to be anything different from the **third** decimal digit of $f(3)$ (i.e., 3): let it be **7**
- We keep doing this all the way down the "diagonal" of the table for f
- Eventually, we build $x = \mathbf{0.467}\ldots$ which is not equal to **any** $f(n)$, as it differs from it in its n-th decimal digit

$\mathbb{R}$ is Uncountable: Implications

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- The fact that $\mathbb{R}$ is uncountable has profound implications also for the theory of computation
- Indeed, it tells us that there exist some numbers that cannot be **computed**
- In other words, there exist some languages that are **not decidable** (**nor even recognizable**)

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2. Diagonalization
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The Halting Problem

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  - There is no such an algorithm (i.e., Turing machine) that can **decide** if another Turing machine will ever halt and accept on a given input
- This result has profound philosophical and practical implications, showing that computers are inherently limited in a fundamental way

The Halting Problem: Formal Definition (1)

Definition (The Halting Problem)

We aim to find a **total computable function** $HALT_{ACC} : \Sigma^* \mapsto \Sigma$ such that for every string representation of a Turing machine along with its input $\langle M, x \rangle \in \Sigma^*$:

$$HALT_{ACC}(\langle M, x \rangle) = \begin{cases} 1 & \text{if } M(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$

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$$HALT_{ACC}(\langle M, x \rangle) = \begin{cases} 1 & \text{if } M(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Note

$\langle M, x \rangle$ can be thought of as the concatenation of two encodings: the Turing machine $\langle M \rangle$ and its input $\langle x \rangle$.
$HALT_{ACC}$ is the boolean function that outputs 1 if M halts (and accepts) x, i.e., $M(x) = 1$, or 0 otherwise (i.e., if $M(x) = 0$ **or** $M(x) = \perp$)

The Halting Problem: Formal Definition (2)

Definition (The Halting Problem)

Consider the **language** $A_{TM} : \{\langle M, x\rangle \in \Sigma^* \mid M \text{ is a TM and } M(x) = 1\}$.
We can also define:

$$HALT_{ACC}(\langle M, x\rangle) = \begin{cases} 1 & \text{if } \langle M, x\rangle \in A_{TM} \\ 0 & \text{if } \langle M, x\rangle \notin A_{TM} \end{cases}$$

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Note

Finding whether the **total computable function** $HALT_{ACC}$ exists is equivalent to determine whether the language A_{TM} is **decidable**

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- However, a more accurate definition would be provided by the following $HALT$ function, which outputs 1 whenever M halts on x (no matter if it accepts or rejects):

$$HALT(\langle M, x \rangle) = \begin{cases} 1 & \text{if } M(x) = 1 \lor M(x) = 0 \\ 0 & M(x) = \perp \end{cases}$$

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$$HALT(\langle M, x \rangle) = \begin{cases} 1 & \text{if } M(x) = 1 \vee M(x) = 0 \\ 0 & M(x) = \perp \end{cases}$$

- We will see that the two definitions are equivalent, i.e., both $HALT_{ACC}$ and $HALT$ are undecidable

The Halting Problem is not Decidable

Theorem (The Halting Problem is not Decidable)

The function $HALT_{ACC}$ as defined above is **not total and computable**. *Equivalently, the language A_{TM} is* **not decidable**.

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Note

Before getting into the proof, let us first observe that $HALT_{ACC}$ is **partial and computable** (i.e., A_{TM} is **recognizable**)

The Halting Problem is Recognizable

Theorem (The Halting Problem is Recognizable)

The following Turing machine U computes a partial function $HALT_{ACC}$ (i.e., recognizes A_{TM}):

U = On input $\langle M, x \rangle$, where M is a TM and x an input to it:

1. *Simulate M on input x;*
2. *If M ever halts and* accepts *x (i.e., $M(x) = 1$) then return 1; if M ever halts and* rejects *x (i.e., $M(x) = 0$) then return 0*

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Note

The Turing machine U above loops on input $\langle M, x\rangle$ if M loops on x.
This is why U computes a partial but **not** a total function (i.e., U recognizes but **not** decides A_{TM})

The Halting Problem: Considerations

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- The halting problem is not just a purely theoretical exercise!
- For example, think about managing the Apple's App Store or Google's Play Store: given some app code, you would like to know whether this gets into an infinite loop!
- If U had some way to determine that M would loop forever on x, it could output 0
- Unfortunately, no algorithm exists that can make this determination **for every possible TMs and their inputs**

The Halting Problem is not Decidable

Theorem (The Halting Problem is not Decidable)

Consider the **language** $A_{TM} : \{\langle M, x\rangle \in \Sigma^* \mid M \text{ is a TM and } M(x) = 1\}$.
We can also define:

$$HALT_{ACC}(\langle M, x\rangle) = \begin{cases} 1 & \text{if } \langle M, x\rangle \in A_{TM} \\ 0 & \text{if } \langle M, x\rangle \notin A_{TM} \end{cases}$$

The function $HALT_{ACC}$ as defined above is **not total and computable**.
Equivalently, the language A_{TM} is **not decidable**.

The Halting Problem is not Decidable: Proof

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- Let H be a decider for A_{TM}, such that on input $\langle M, x \rangle$, where M is a TM and x is a (binary) string:
  - H **halts and outputs 1** if M outputs 1 on x
  - H **halts and outputs 0** if M fails to output 1 on x (i.e., either outputs 0 or loop forever)

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  - H **halts and outputs 0** if M fails to output 1 on x (i.e., either outputs 0 or loop forever)
- In other words:

$$H(\langle M, x\rangle) = \begin{cases} 1 & \text{if } M(x) = 1 \\ 0 & \text{if } M(x) = 0 \vee M(x) = \perp \end{cases}$$

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- Let's construct another TM called D, which uses H as a subroutine
- This new TM calls H to determine what M does when the input to M is its own description $\langle M \rangle$
- Once D has determined this information, it does the **opposite**, i.e., it outputs 0 if $H(\langle M, \langle M \rangle \rangle) = 1$ or 1 if $H(\langle M, \langle M \rangle \rangle) = 0$

$$D(\langle M \rangle) = \begin{cases} 1 & \text{if } H(\langle M, \langle M \rangle \rangle) = 0 \Leftrightarrow M(\langle M \rangle) = 0 \vee M(\langle M \rangle) = \perp \\ 0 & \text{if } H(\langle M, \langle M \rangle \rangle) = 1 \Leftrightarrow M(\langle M \rangle) = 1 \end{cases}$$

The Halting Problem is not Decidable: Proof

What happens if we run D with its own description $\langle D \rangle$ as input?

$$D(\langle D \rangle) = \begin{cases} 1 & \text{if } H(\langle D, \langle D \rangle \rangle) = 0 \Leftrightarrow D(\langle D \rangle) = 0 \vee D(\langle D \rangle) = \perp \\ 0 & \text{if } H(\langle D, \langle D \rangle \rangle) = 1 \Leftrightarrow D(\langle D \rangle) = 1 \end{cases}$$

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Note

No matter what D does, it is forced to do the opposite, which leads us to a contradiction! As such, neither D nor H can exist!

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- In other words, given a generic TM M_i the table shows the behavior of M_i when this is input with $\langle M_j \rangle$
- Without loss of generality, the output of M_i when input with $\langle M_j \rangle$ is either: 1 (*accept*), 0 (*reject*), or $\perp$ (*does not halt*)

The Behavior of All Turing Machines

	$<M_1>$	$<M_2>$	$<M_3>$	$<M_4>$	…	$<M_k>$	…
M_1	accept	reject	accept	doesn't halt	…	reject	…
M_2	accept	accept	accept	accept	…	doesn't halt	…
M_3	reject	doesn't halt	doesn't halt	reject	…	accept	…
M_4	accept	accept	reject	reject	…	doesn't halt	…
…	…	…	…	…	…	…	…
M_k	doesn't halt	reject	accept	accept		accept	…
…	…	…	…	…	…	…	…

Entry i, j contains the output of TM M_i on input $\langle M_j \rangle$

The Behavior of H

The table below shows the behavior of TM H on inputs from the table above: if M_i accepts $\langle M_j \rangle$ so does H, if M_i either rejects or doesn't halt on $\langle M_j \rangle$ then H rejects

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	<M_1>	<M_2>	<M_3>	<M_4>	…	<M_k>	…
M_1	accept	reject	accept	reject	…	reject	…
M_2	accept	accept	accept	accept	…	reject	…
M_3	reject	reject	reject	reject	…	accept	…
M_4	accept	accept	reject	reject	…	reject	…
…	…	…	…	…	…	…	…
M_k	reject	reject	accept	accept		accept	…
…	…	…	…	…	…	…	…

The Behavior of D

Assuming H exists, so does D! As such, D must be located somewhere in the list of **all** TMs

The Behavior of *D*

Assuming *H* exists, so does *D*! As such, *D* must be located somewhere in the list of **all** TMs

	<M_1>	<M_2>	<M_3>	<M_4>	…	<D>	…
M_1	<u>accept</u>	reject	accept	reject	…	reject	…
M_2	accept	<u>accept</u>	accept	accept	…	reject	…
M_3	reject	reject	<u>reject</u>	reject	…	accept	…
M_4	accept	accept	reject	<u>reject</u>	…	reject	…
…	…	…	…	…	…	…	…
D	reject	reject	accept	accept		?	…
…	…	…	…	…	…	…	…

The Behavior of D

Since D computes the opposite of the diagonal entries, a contradiction occurs in correspondence of column $\langle D \rangle$, where the entry must be the opposite of itself!

	<M1>	<M2>	<M3>	<M4>	…	<D>	…
M1	**accept**	reject	accept	reject	…	reject	…
M2	accept	**accept**	accept	accept	…	reject	…
M3	reject	reject	**reject**	reject	…	accept	…
M4	accept	accept	reject	**reject**	…	reject	…
…	…	…	…	…	…	…	…
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- However, there exist languages that are not even recognized by a TM
- We will prove that using the same diagonalization argument

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Corollary (Turing-unrecognizable Languages)

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$$\Downarrow$$

Some languages are not Turing-recognizable

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  2. The set of **all** languages is **uncountable**
- Let's start with 1!

1) The Set of All TMs is Countable

- We first observe that for any finite alphabet Σ the (infinite) set of all the finite-length strings Σ^* is countable

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- Therefore, we can list all those encodings $\langle M_1 \rangle, \langle M_2 \rangle, \ldots$ as above
- In other words, we can easily find a correspondence between the set of natural numbers $\mathbb{N}$ and the (infinite) set of **all** Turing machines (encoded as binary strings)

2) The Set of All Languages is Uncountable

- Two approaches to show that this statement is true:
  1. using the countable set of all the finite-length binary strings and Cantor's diagonalization
  2. using the uncountable set of all the infinite-length binary strings

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- Any language L can be seen as a subset of Σ^*
- We must show that the set $\mathcal{L}$ of **all** languages is **uncountable**

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  - If $\sigma_k \in L_k$ then $\sigma_k \notin L_j$
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  - If $\sigma_k \notin L_k$ then $\sigma_k \in L_j$
- Then, we have found an element $L_j \in \mathcal{L}$, such that there is no $j \in \mathbb{N}, f(j) = L_j$

2) The Set of All Languages is Uncountable: First Proof

- To find the counterexample $L_j \in \mathcal{L}$ that leads to a contradiction, let us assume that f exists and some of its values are:
  - $f(1) = L_1 = \{\epsilon, 010, 11101, \ldots\}$
  - $f(2) = L_2 = \{1, 00, 101, 00010, \ldots\}$
  - $f(3) = L_3 = \{10, 11, 000, 10101, \ldots\}$
  - $f(4) = L_4 = \{\epsilon, 1, 00, 01010, \ldots\}$
  - $\ldots$

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  - $\ldots$
- Suppose also that $\sigma_1 = \epsilon$, $\sigma_2 = 0$, $\sigma_3 = 1$, $\sigma_4 = 00, \ldots$

2) The Set of All Languages is Uncountable: First Proof

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  - $f(1) = L_1 = \{\epsilon, 010, 11101, \ldots\}$
  - $f(2) = L_2 = \{1, 00, 101, 00010, \ldots\}$
  - $f(3) = L_3 = \{10, 11, 000, 10101, \ldots\}$
  - $f(4) = L_4 = \{\epsilon, 1, 00, 01010, \ldots\}$
  - $\ldots$
- Suppose also that $\sigma_1 = \epsilon$, $\sigma_2 = 0$, $\sigma_3 = 1$, $\sigma_4 = 00, \ldots$
- σ_i will be included in L_j iff $\sigma_i \notin f(i) = L_i$

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  - $\ldots$
- $\sigma_1 = \epsilon \notin L_j$ as $\sigma_1 \in L_1$, therefore $L_j = \{\} \neq f(1)$
- $\sigma_2 = 0 \in L_j$ as $\sigma_2 \notin L_2$, therefore $L_j = \{0\} \neq f(2)$
- $\sigma_3 = 1 \in L_j$ as $\sigma_3 \notin L_3$, therefore $L_j = \{0, 1\} \neq f(3)$
- $\sigma_4 = 00 \notin L_j$ as $\sigma_4 \in L_4$, therefore $L_j = \{0, 1\} \neq f(4)$
- $\ldots$

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- We first observe that the set of all **infinite** binary sequences is uncountable
- An infinite binary sequence is a never-ending sequence of binary symbols (0s and 1s)
- Let $\mathcal{B}$ be the set of all such infinite binary sequences: we can show $\mathcal{B}$ is uncountable using Cantor's diagonalization

Note

The infinite set of finite-length binary strings $\Sigma^* = \{0, 1\}^*$ is **not equal** to the set of infinite-length binary strings $\mathcal{B}$

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- However, we can build another binary sequence b such that its i-th bit is the opposite of the i-th bit of each $f(i)$. In the example above: $b = 100\ldots$
- We have found a sequence b which is not paired with any natural number listed $\Longrightarrow \mathcal{B}$ is **uncountable**

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- Let $\Sigma^* = \{\sigma_1, \sigma_2, \ldots\}$ the list of finite-length strings over Σ (e.g., $\Sigma = \{0, 1\}$)
- Each language $L \in \mathcal{L}$ can be described as a unique sequence in $\mathcal{B}$, where the i-th bit of that sequence is 1 if $\sigma_i \in L$, or 0 if $\sigma_i \notin L$

Note

Without loss of generality, each language $L \in \mathcal{L}$ is indeed composed of infinitely many finite-length strings, therefore represented as an infinite binary sequence

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Definition (Characteristic Sequence)

We call the infinite binary sequence associated with each $L \in \mathcal{L}$ the **characteristic sequence** of L, denoted by χ_L

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Example

Suppose L is the language of all strings starting with a 0 over the alphabet $\Sigma = \{0, 1\}$. Then, its **characteristic sequence** χ_L will be as follows:

$$\Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots\}$$

$$L = \{0, 00, 01, 000, 001, \ldots\}$$

$$\chi_L = \{0, 1, 0, 1, 1, 0, 0, 1, 1, \ldots\}$$

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- The function $f : \mathcal{L} \mapsto \mathcal{B}$, where $f(L) = \chi_L$ equals the characteristic sequence of L is one-to-one and onto, thus a correspondence

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- We have shown (again) that the set of all languages $\mathcal{L}$ cannnot be put into correspondence with the set of all Turing machines
- We can conclude that some languages are **not** recognized by any Turing machine

The Set of All Languages is Uncountable: Extra Proof

Theorem (Cantor's Theorem)

For **any** *set A, the set of all subsets of A - also known as the* **power set** *of A, denoted by $\mathcal{P}(A)$ - has a strictly greater cardinality than A itself:*

$$|\mathcal{P}(A)| > |A|$$

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Example (Countably Finite Sets)

If A is countable and finite, the relation trivially holds:
Let $|A| = n$, then $|\mathcal{P}(A)| = 2^n$

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Example (Countably Infinite Sets)

If A is countable and infinite, the relation still holds:
Let $A = \mathbb{N}$, then $|\mathbb{N}| = \aleph_0$ and $|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$
In particular, the power set of the set of natural numbers is uncountably infinite and has the same size as the set of real numbers, whose cardinality $\mathfrak{c} = 2^{\aleph_0}$ is referred to as the **cardinality of the continuum**:

$$|\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = \mathfrak{c} = 2^{\aleph_0} > \aleph_0 = |\mathbb{N}|$$

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- We ask ourselves whether there exists any problem that is not algorithmically solvable
- We consider as a reference example the well-known **Halting Problem**
- We proved the Halting Problem is **undecidable** (or semi-decidable), although it is of course **recognizable**
- The proof is based on Cantor's diagonalization argument used to compare the size of (infinite) sets
- Diagonalization allows us to prove that there are functions/languages that cannot be even recognized (i.e., we have countably many TMs yet uncountably many languages)