Teoria degli Algoritmi

Corso di Laurea Magistrale in Matematica Applicata

a.a. 2020-21

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SAPIENZA UNIVERSITÀ DI ROMA

Recap from Previous Lecture

- We want to find an effective way to measure the trustworthiness of a page within the Web graph

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- We want to find an effective way to measure the trustworthiness of a page within the Web graph
- More generally, we want to assign a score which indicates the importance of a node in a graph
- Derive such a score from the structural properties of the graph only (i.e., via link analysis)
- Exploit the fact that the Web is an example of a scale-free network

Computing Node Importance

Several link analysis approaches to compute web page importance

PageRank

Hubs and Authorities (HITS)

Personalized PageRank

Web Spam Detection

PageRank

One Slide PageRank

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One Slide PageRank

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- Assigns a numerical score to each web page with the purpose of indicating its relative importance within the whole collection

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PageRank's Intuition: Links as Votes

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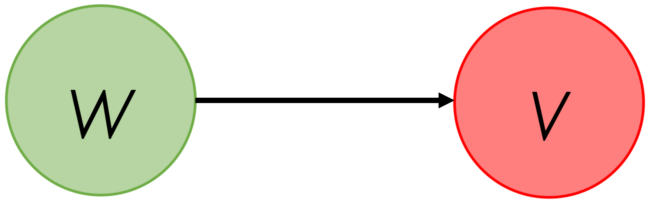

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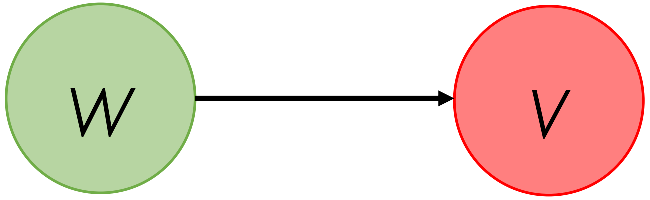

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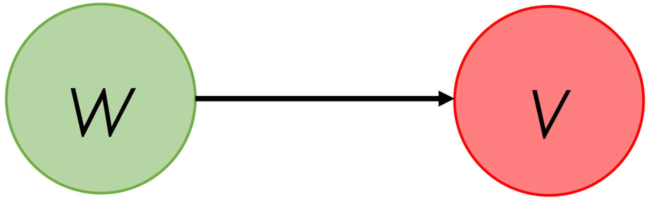

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Different web pages have different in-degree (scale-free network)

www.stanford.edu has more than 23K in-links

www.uniroma1.it/~tolomei has one or two in-links!

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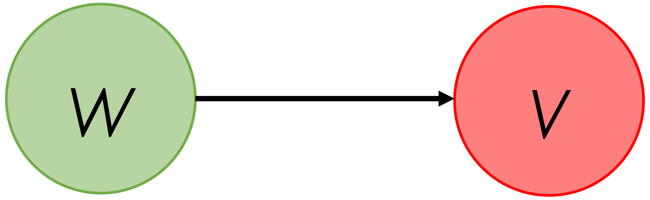

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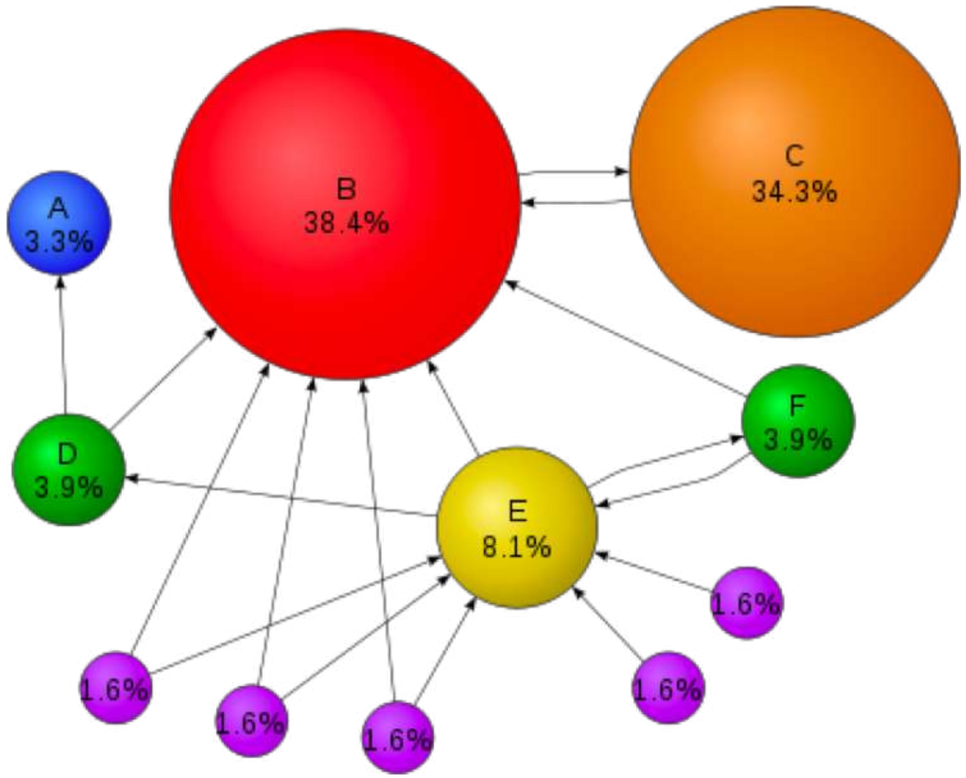
Recursive definition

PageRank Scores: Example



Circle size proportional to the node importance

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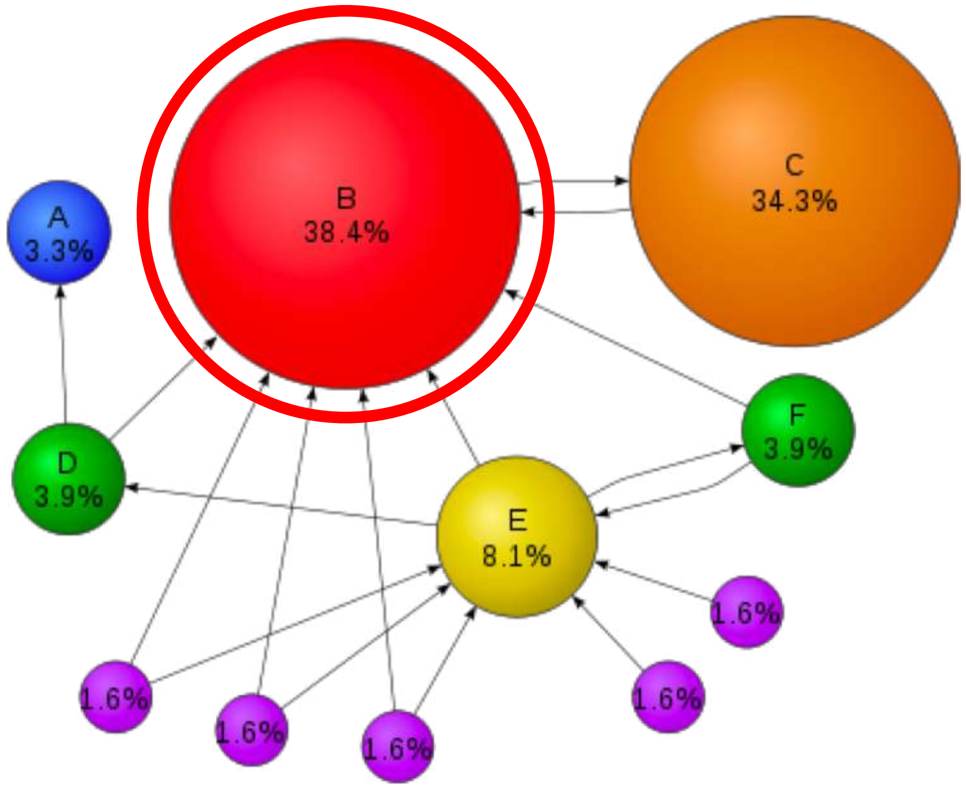

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B has a high score since many nodes point to it

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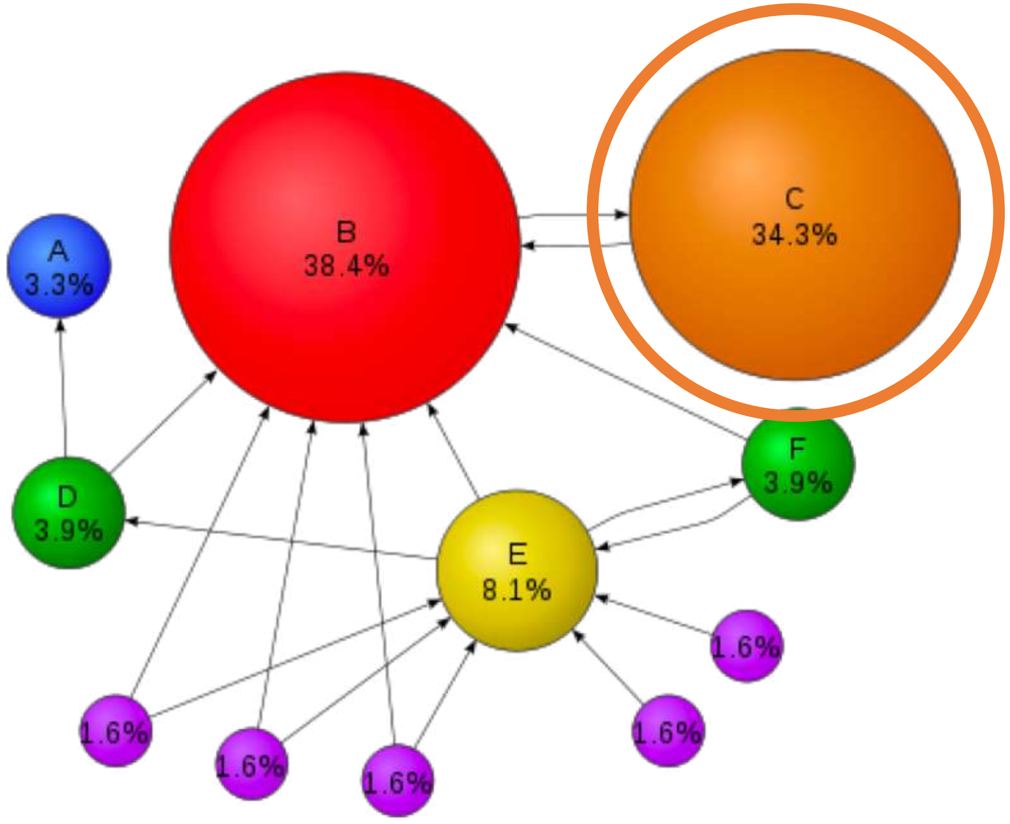

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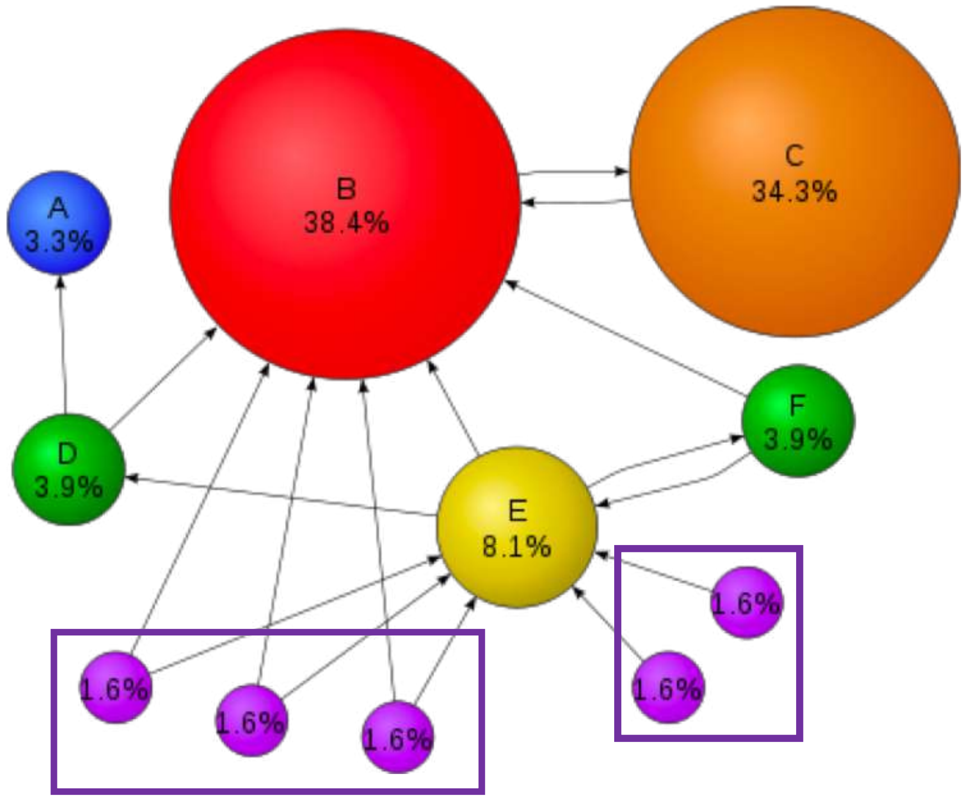

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Many other less important nodes

PageRank: Prelminaries

$G = (V, E)$ The Web Graph $|V| = N$ Number of Nodes (pages)

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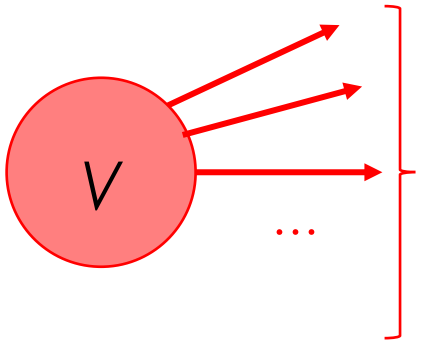
$|O_v| = o_v$ Out-degree of node *v*



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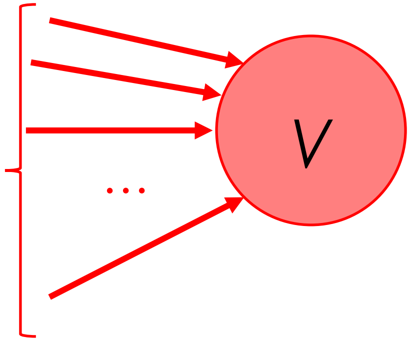

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Each link's vote to a page *v* is proportional to the importance of the source page *w*, which the link comes from

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If a page w has importance r_w and out-degree o_w, each out-link will get an **equal proportion** of the importance, i.e., r_w/o_w

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Each page v's importance can be computed just as the **sum of votes** of all its **incoming links** (i.e., in-degree)

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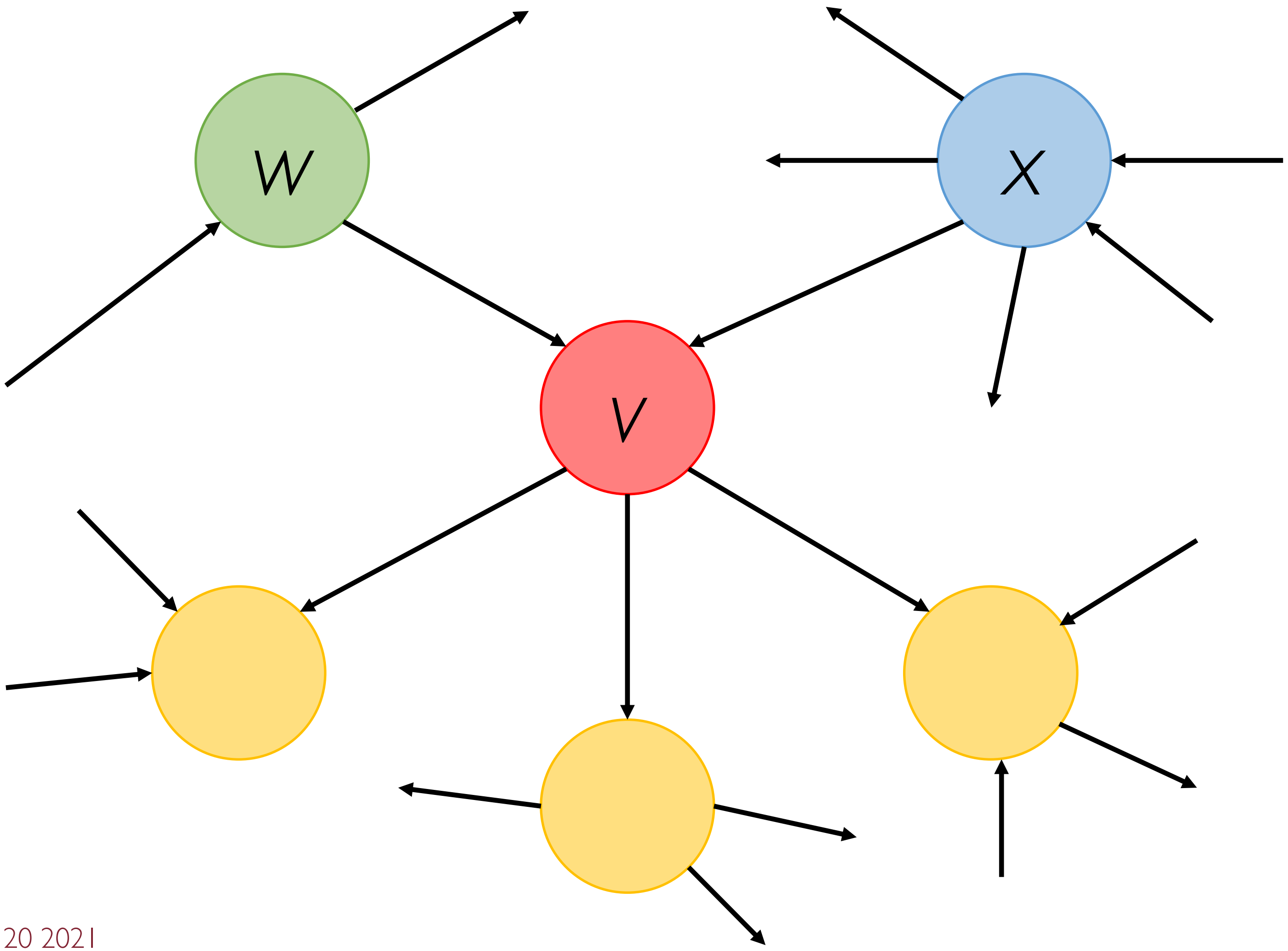

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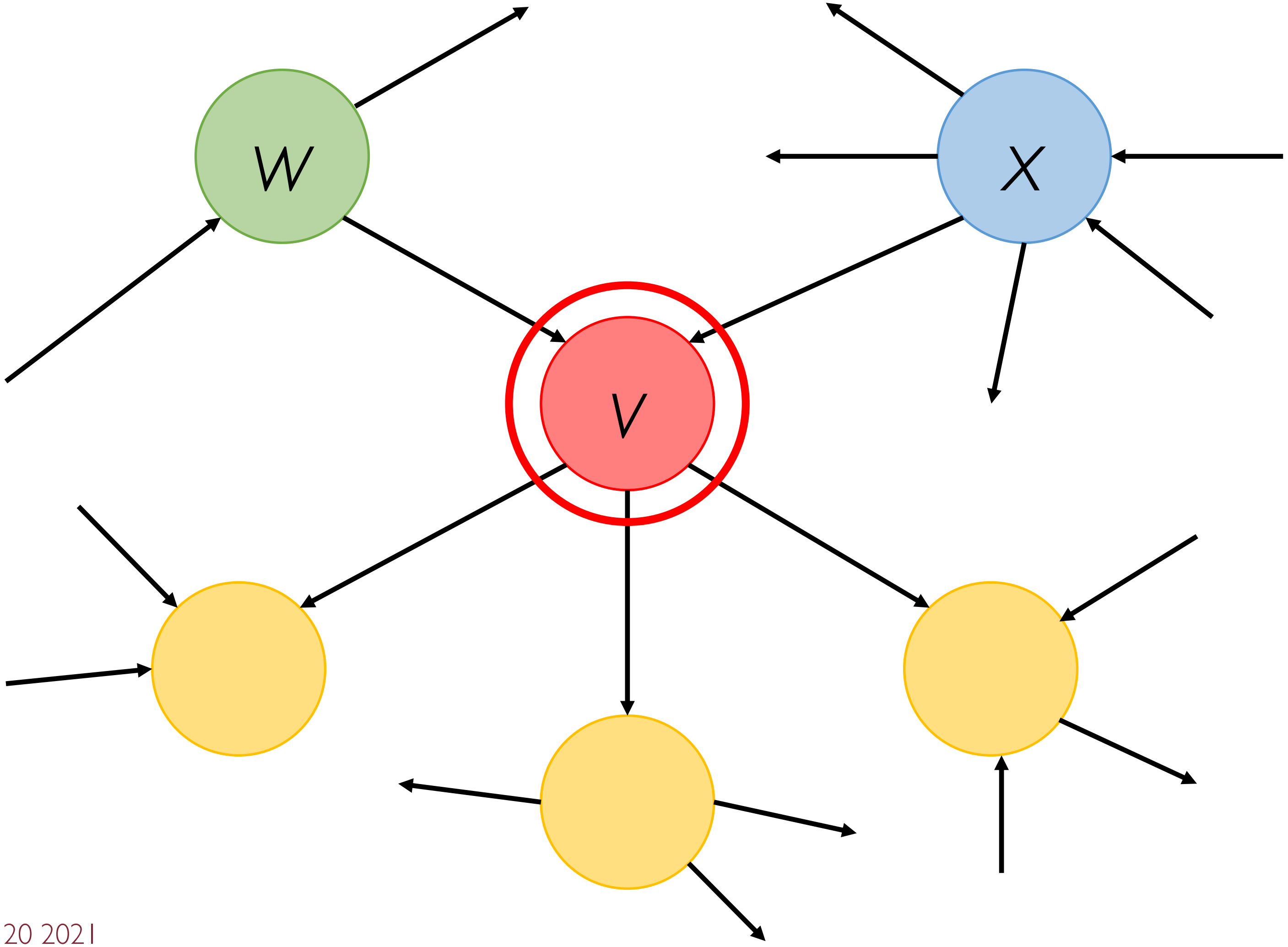
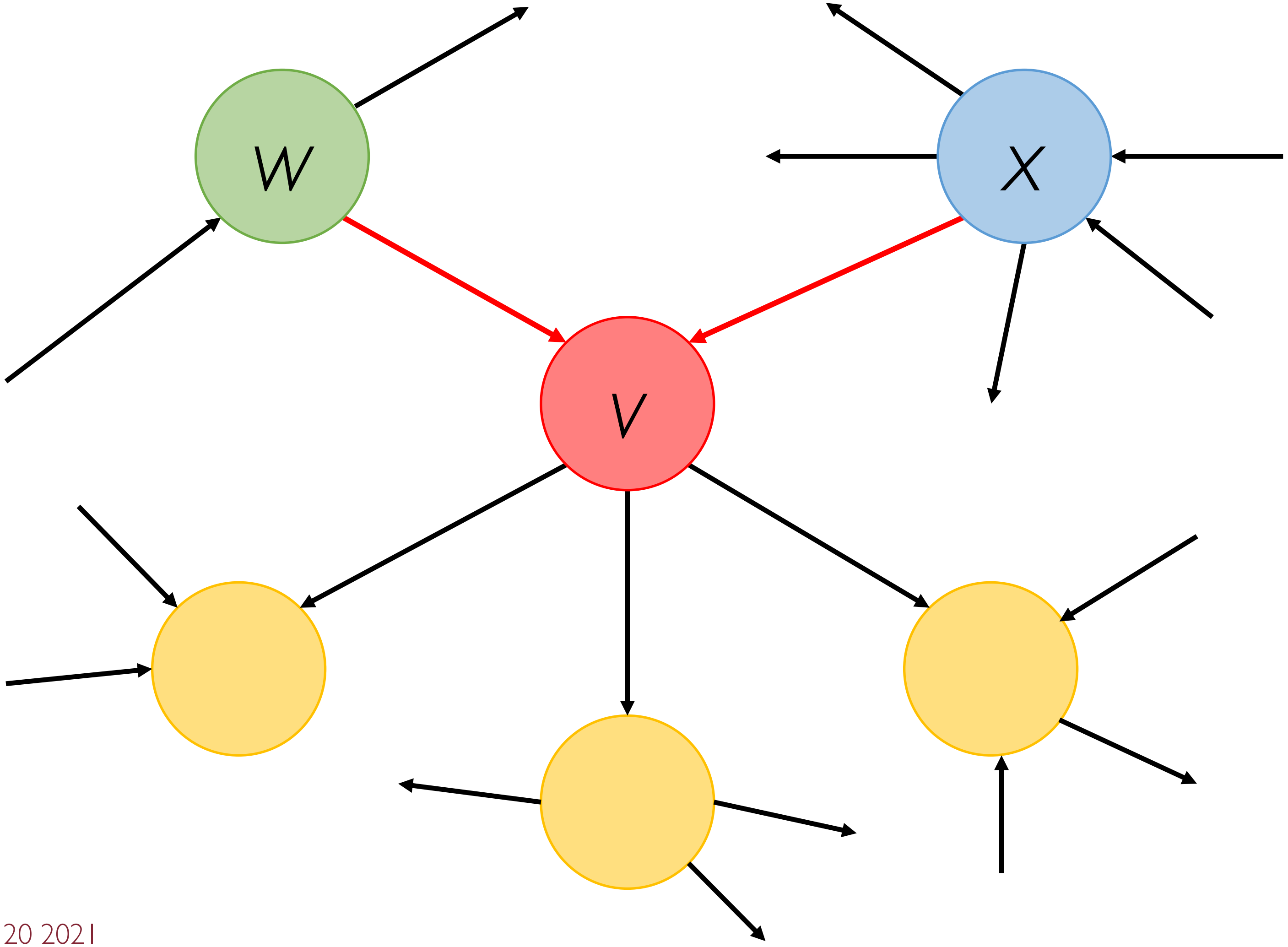


What is r_v?

PageRank: First Simple Recursive Formulation



Suppose *v* has only 2 in-links coming from *w* and *x*

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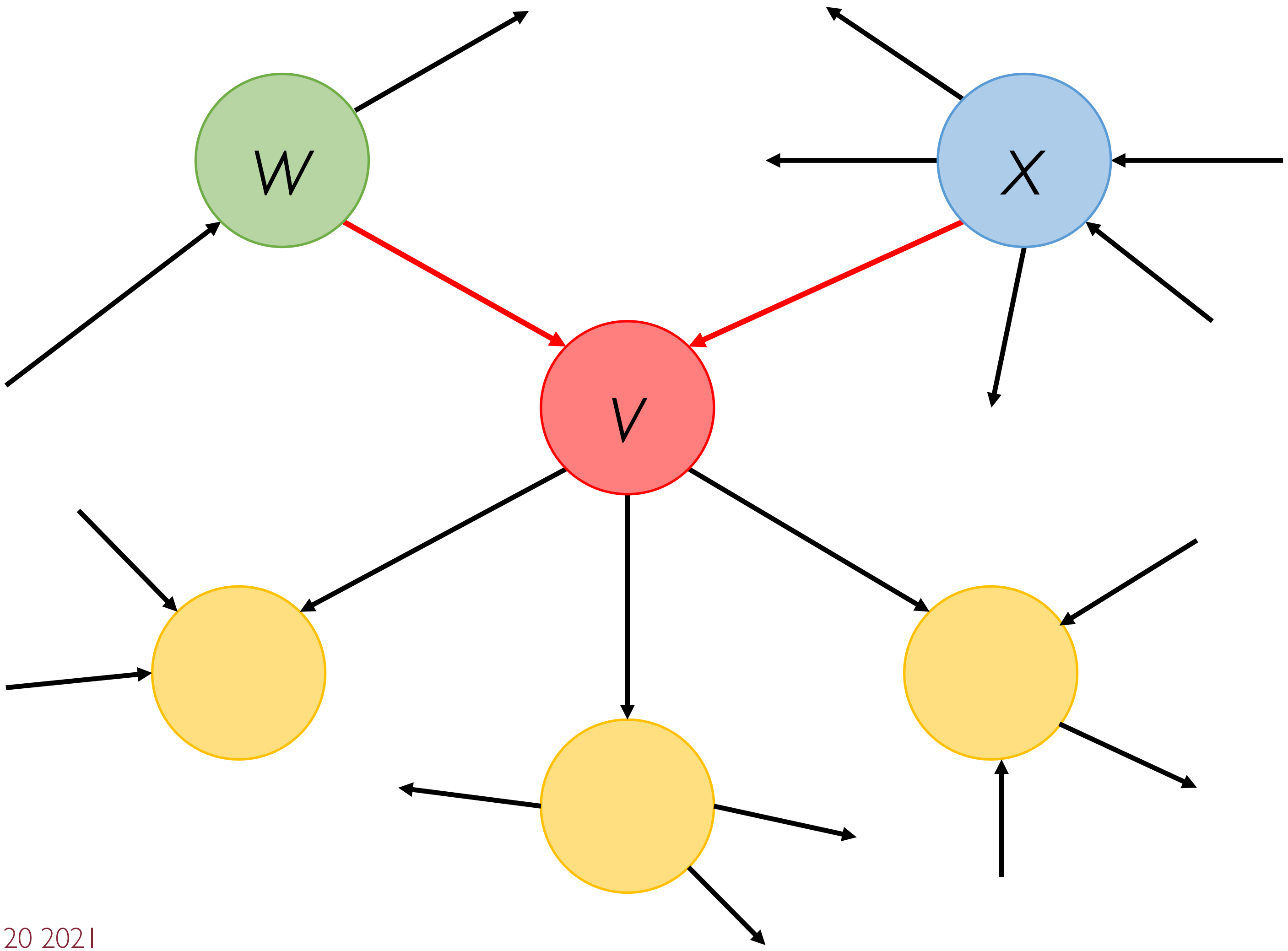
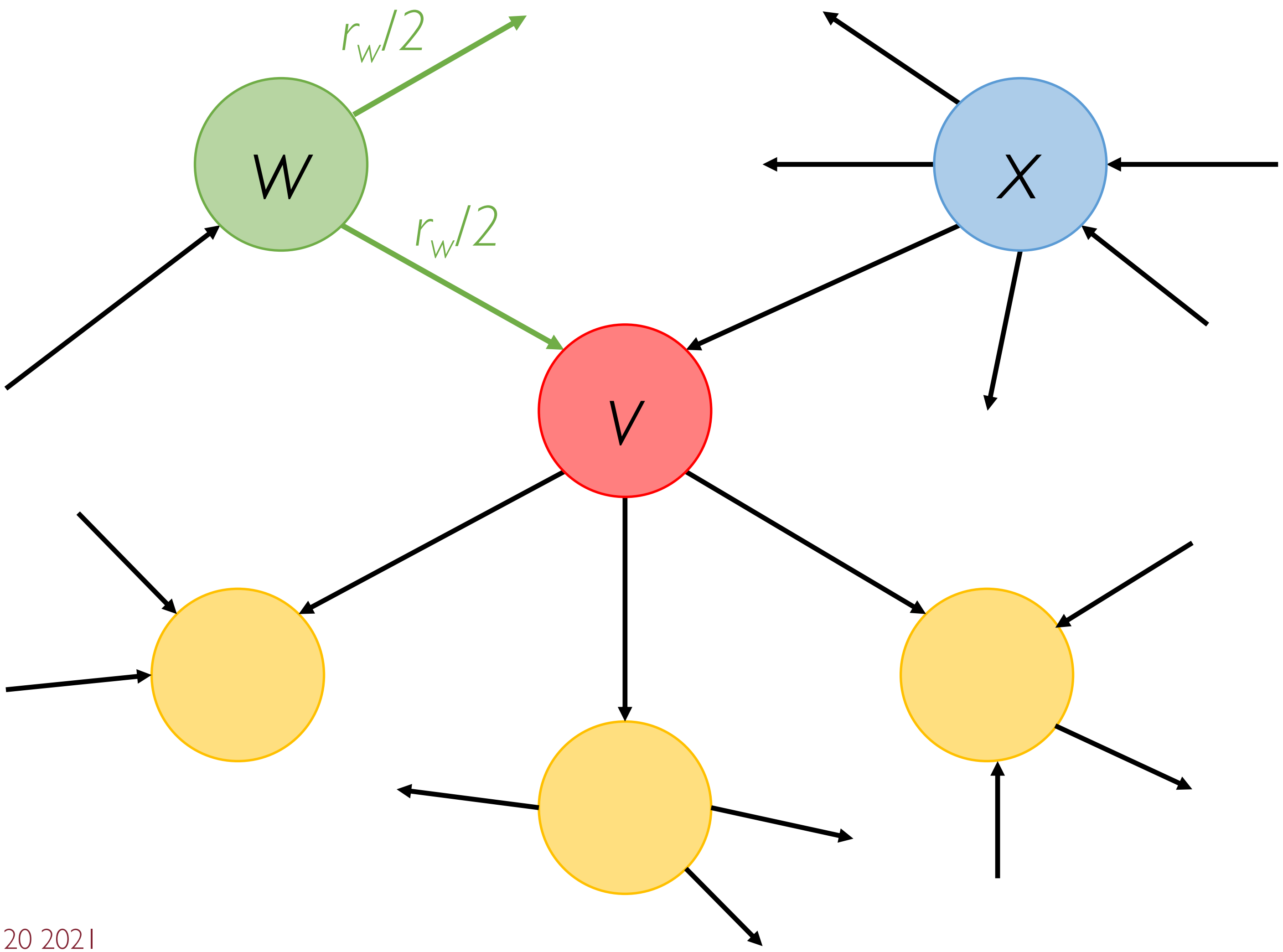


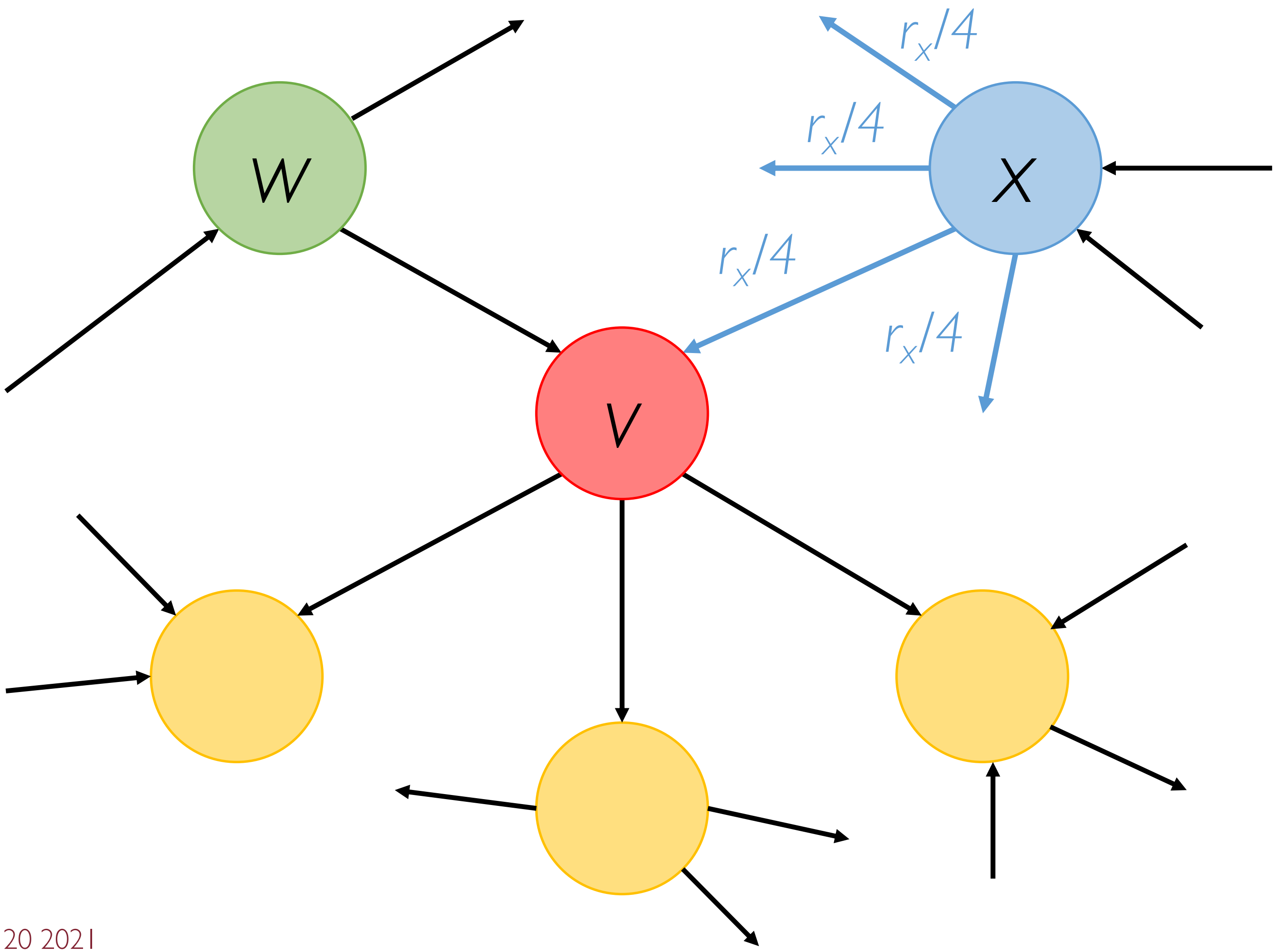
We must compute the in-link's **vote** from *w* and from *x*

PageRank: First Simple Recursive Formulation



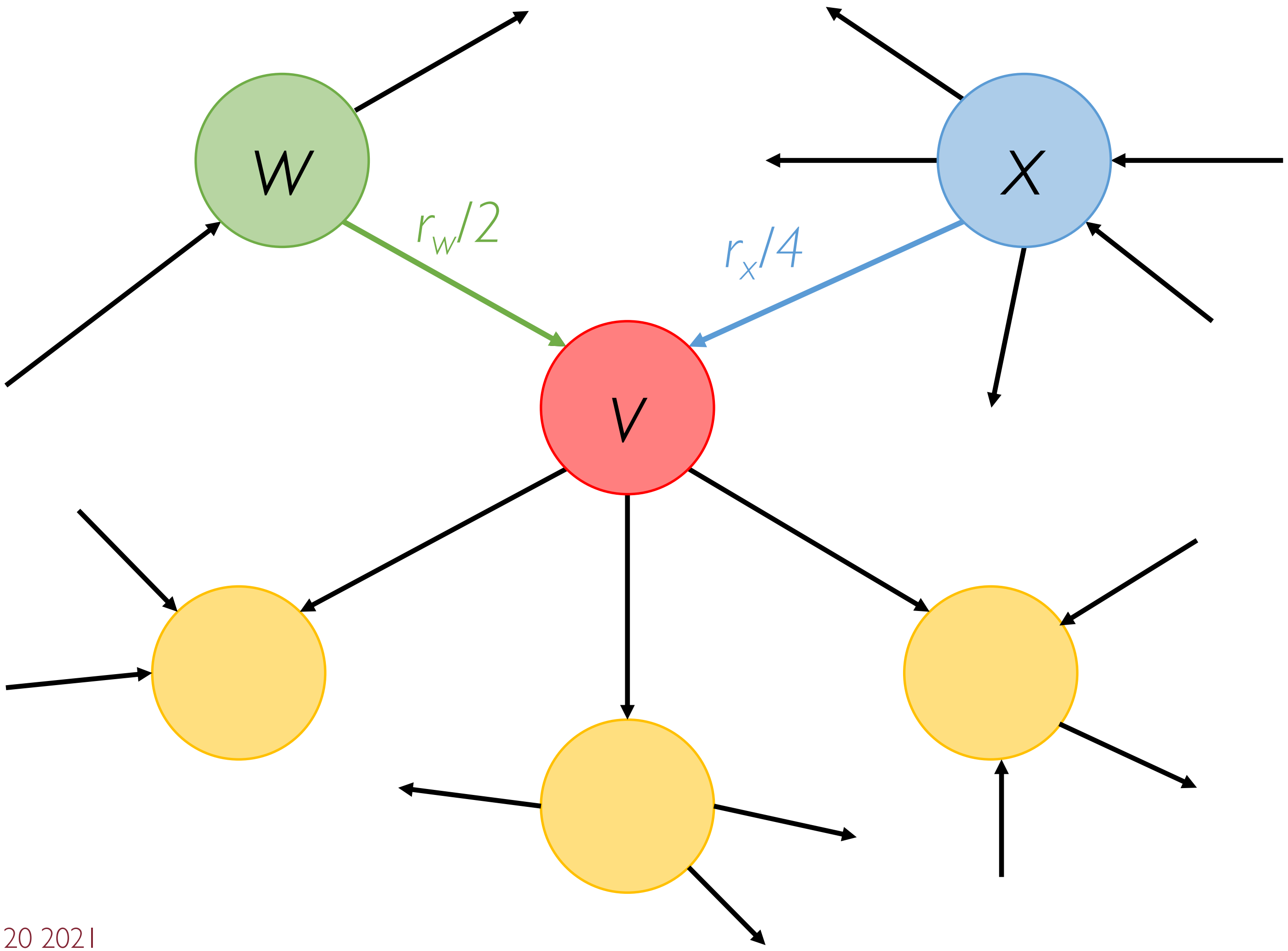
The importance of page w (r_w) is distributed across each of its 2 outgoing links

PageRank: First Simple Recursive Formulation



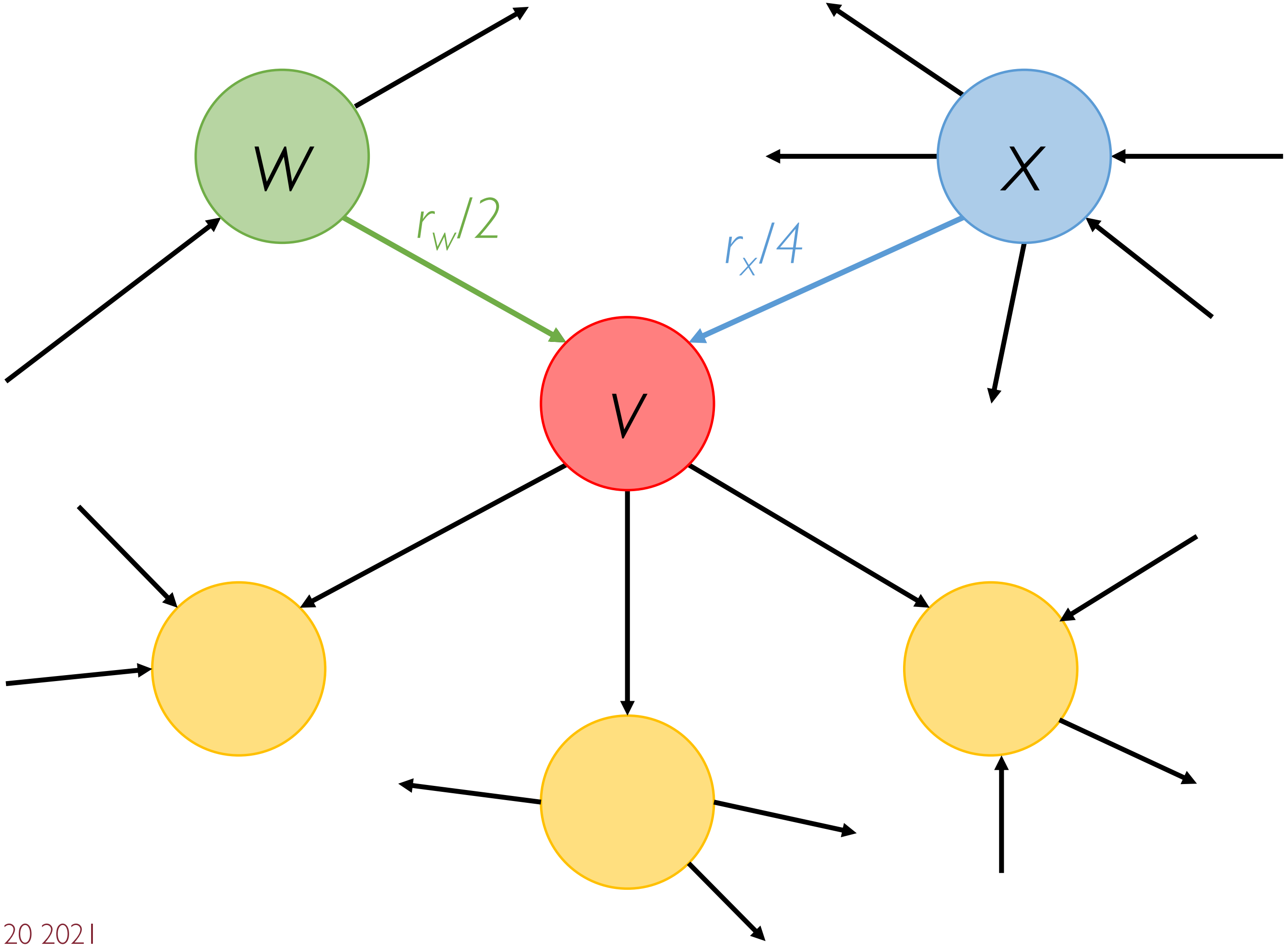
The importance of page x (r_x) is distributed across each of its 4 outgoing links

PageRank: First Simple Recursive Formulation



The importance of page v (r_v) is just the **sum** of its incoming links' votes

PageRank: First Simple Recursive Formulation



The importance of page v (r_v) is just the **sum** of its incoming links' votes

$$\boxed{r_v = r_w/2 + r_x/4}$$

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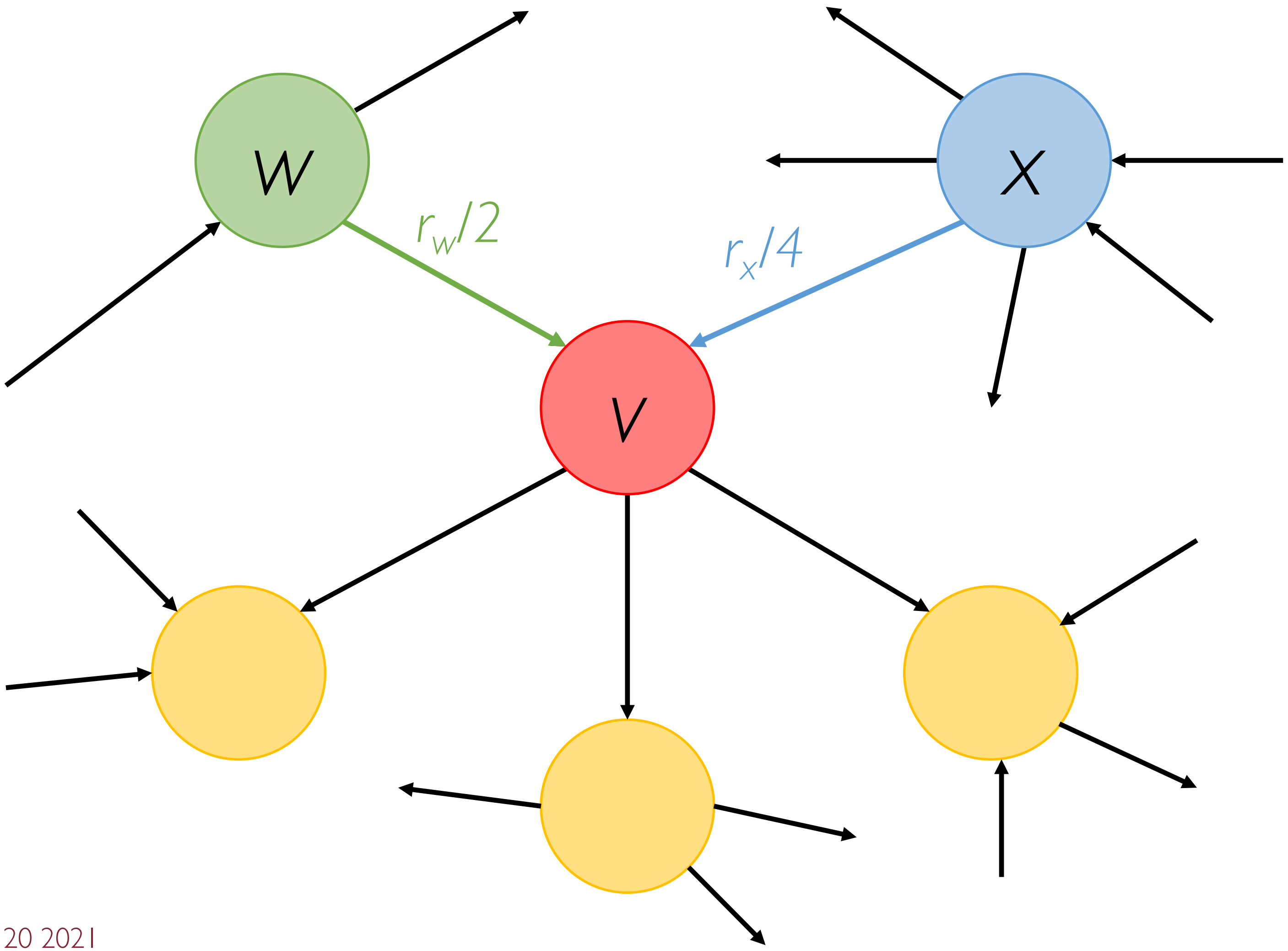

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$$\boxed{r_v = \sum_{u \in I_v} \frac{r_u}{o_u}}$$

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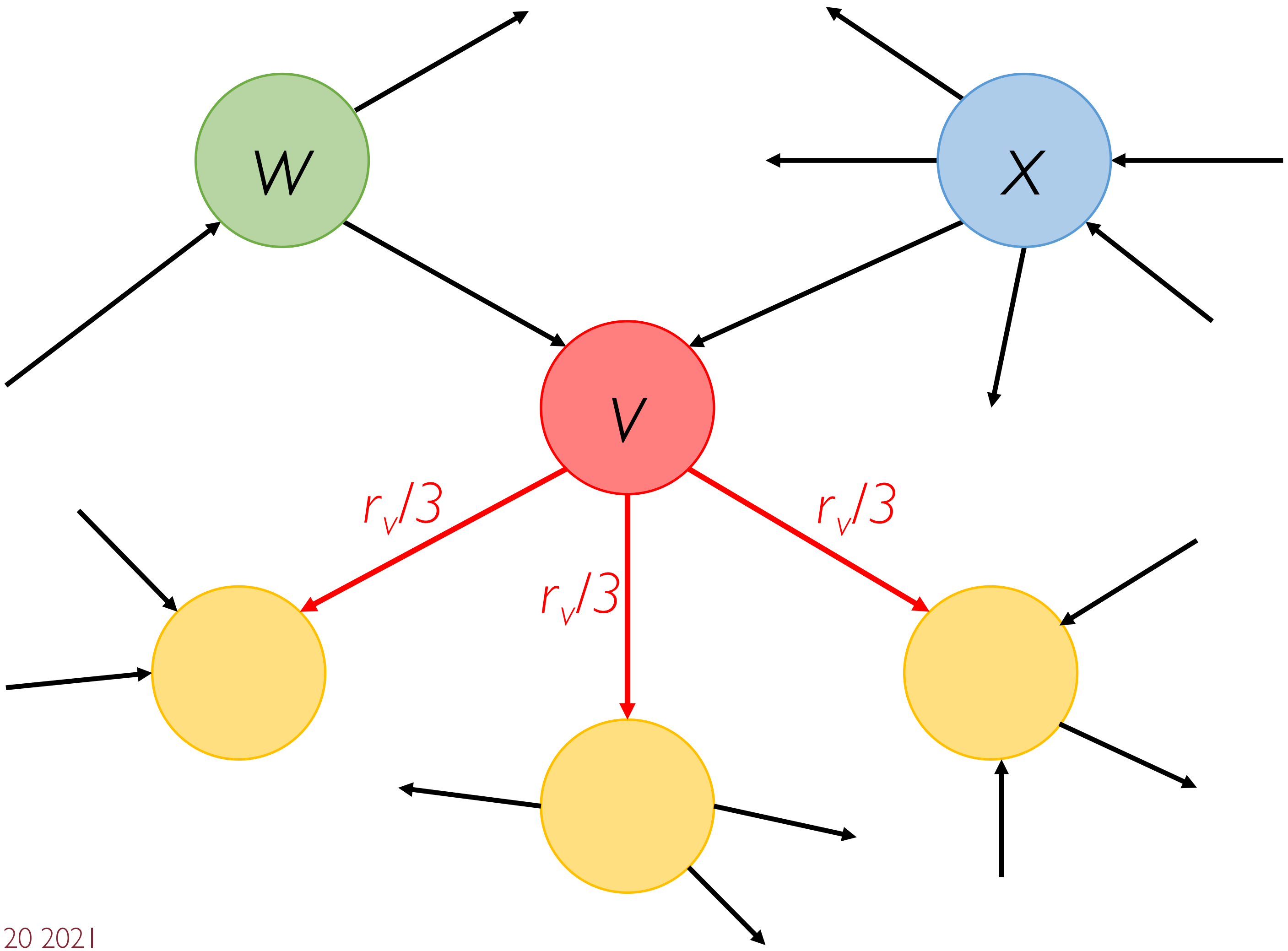

Similarly, page v **uniformly** distributes its importance r_v to its outgoing links

PageRank's Interpretations

$\boxed{\text{2 main perspectives}}$

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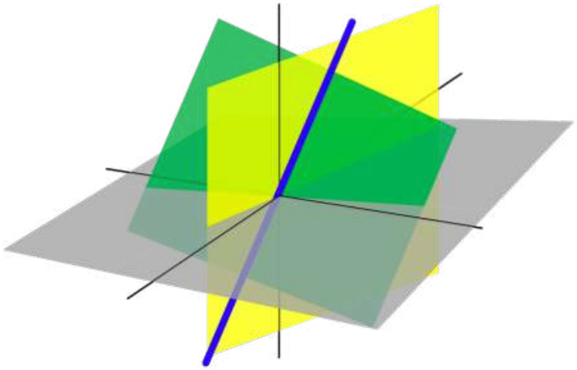
2 main perspectives

Linear Algebra



PageRank's Interpretations

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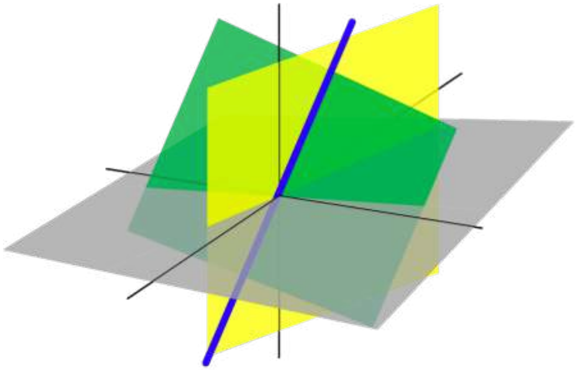

Probabilistic

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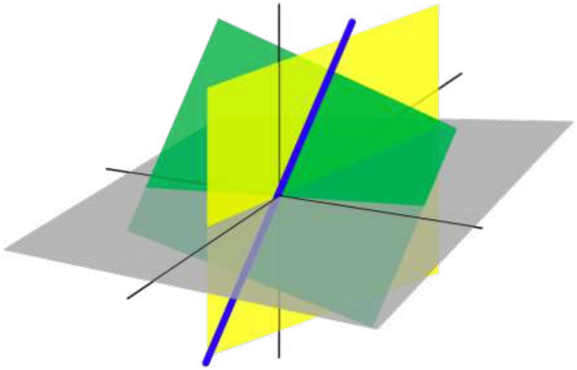
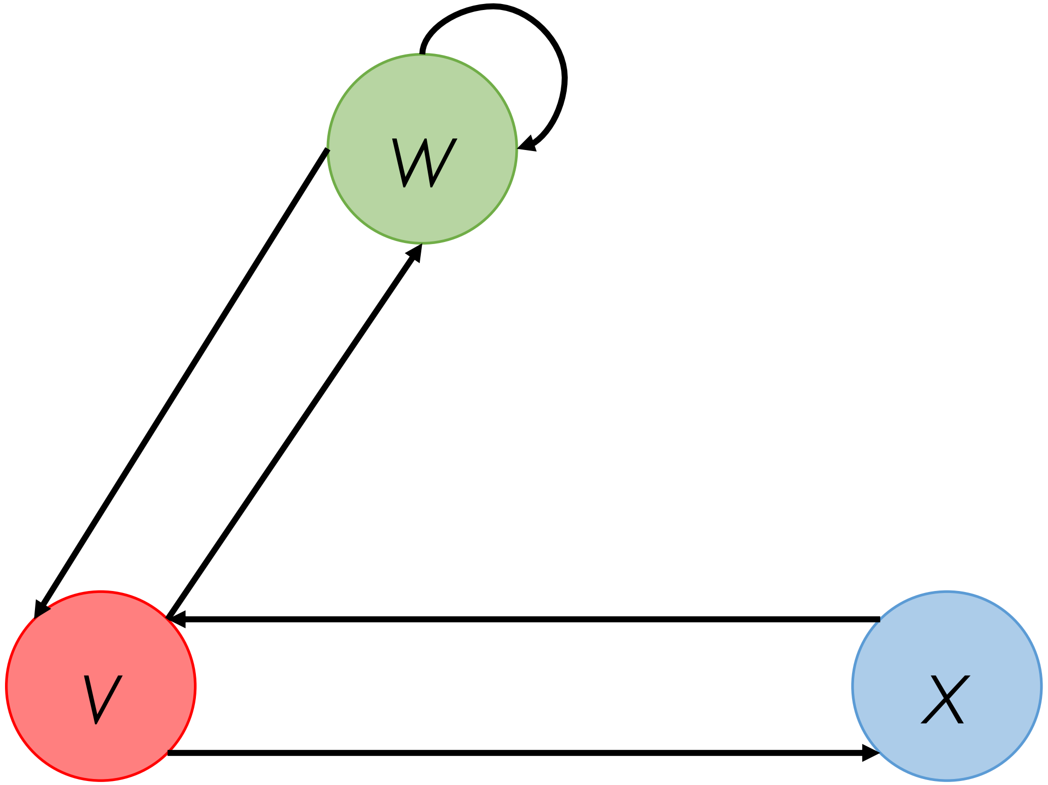


Probabilistic

PageRank: The "Flow" Model



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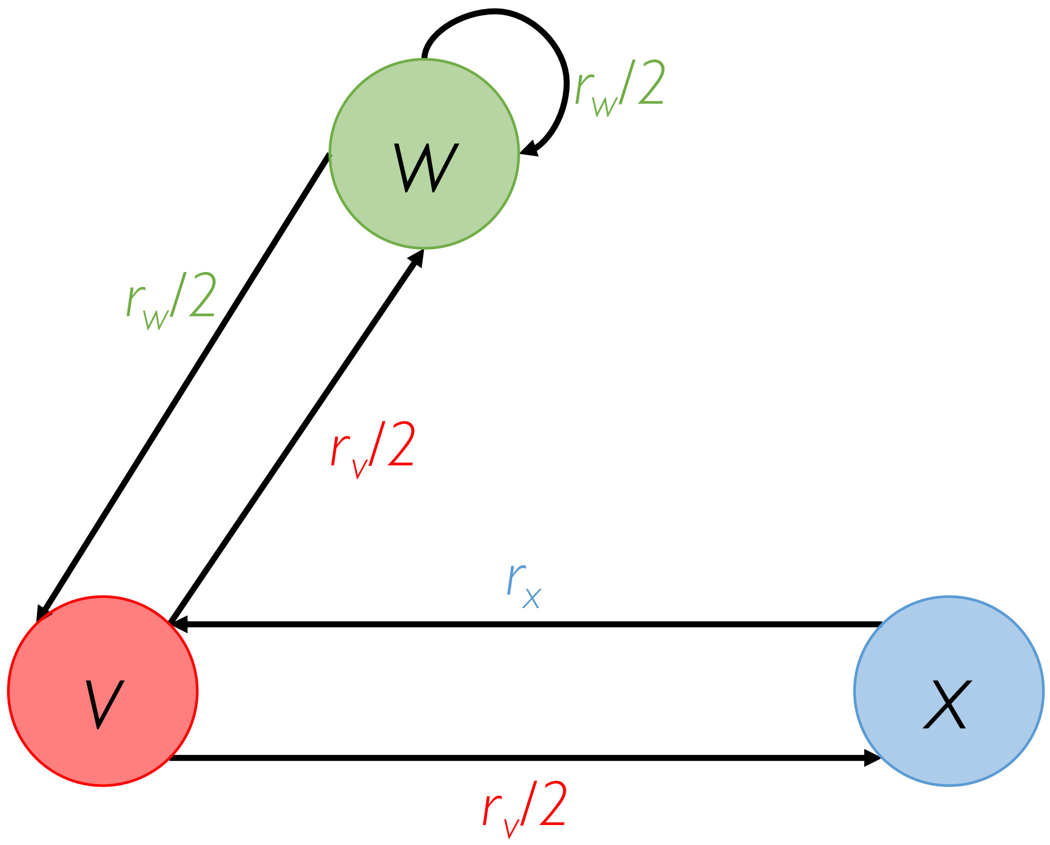

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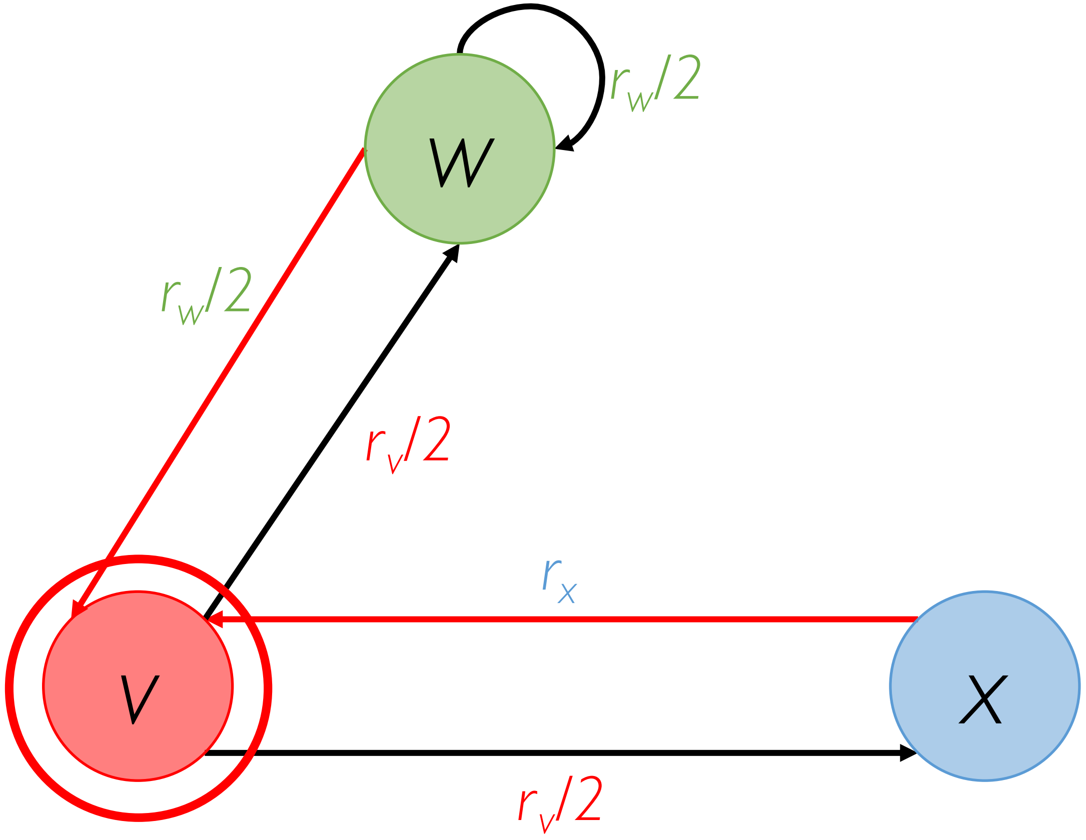

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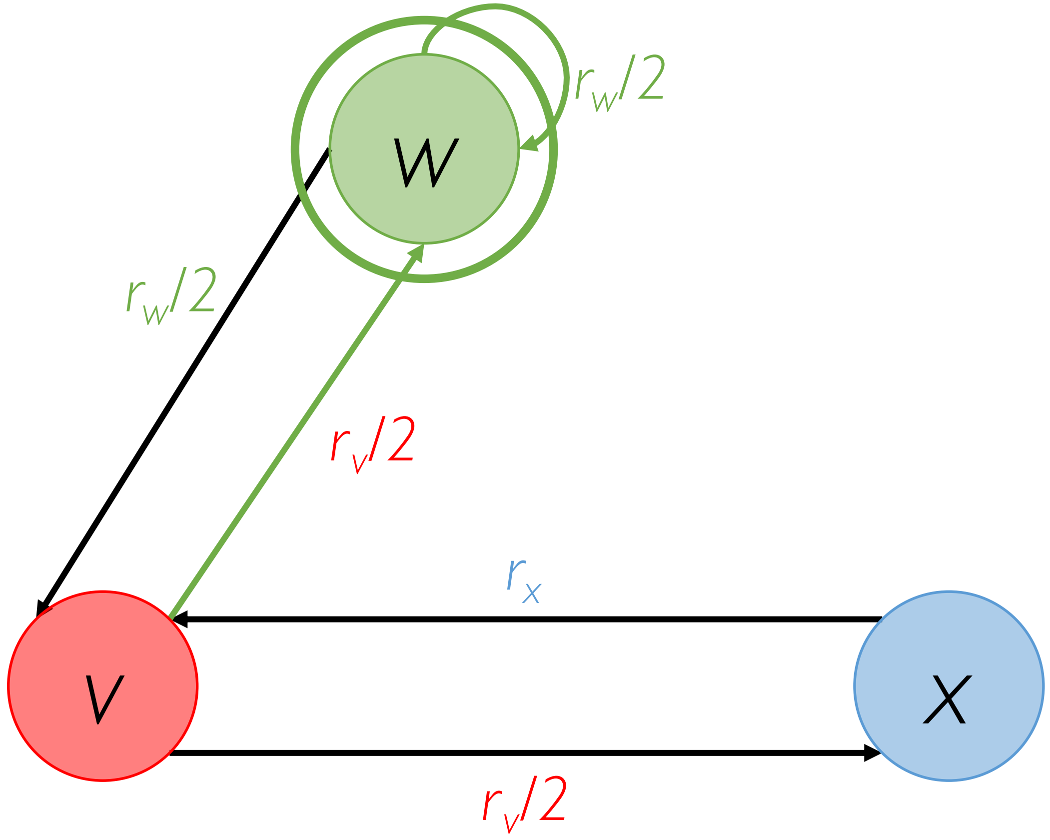

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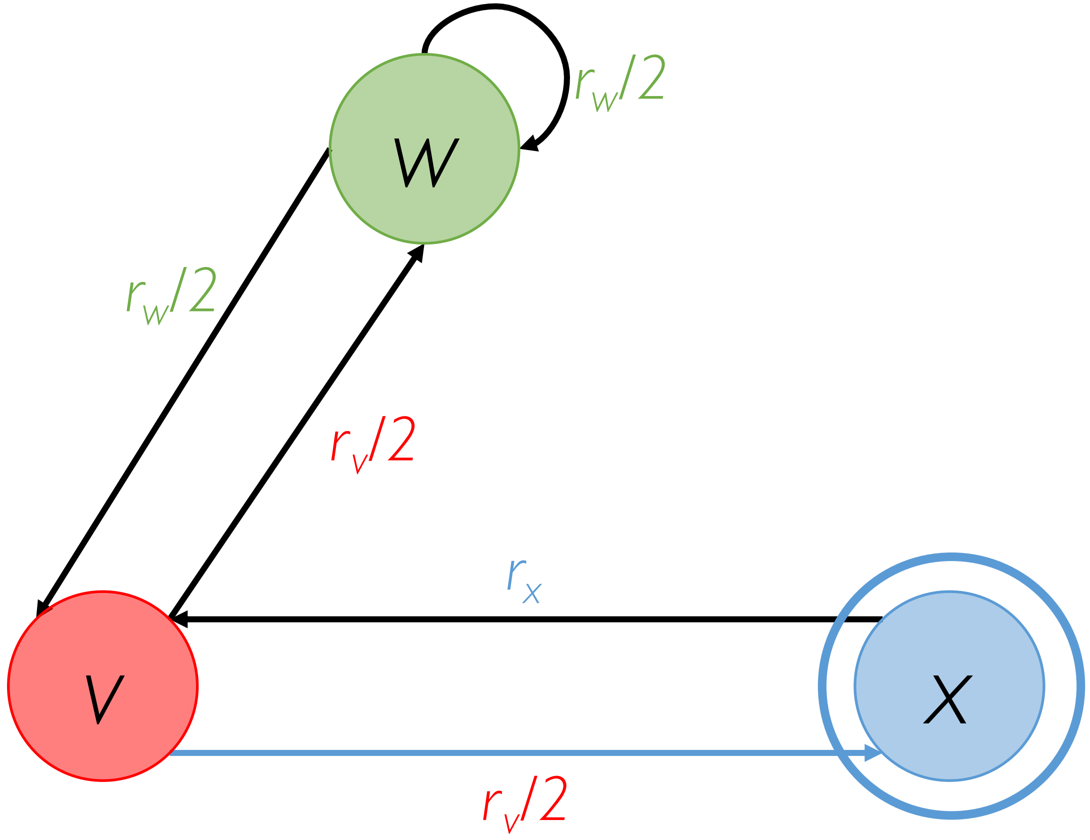

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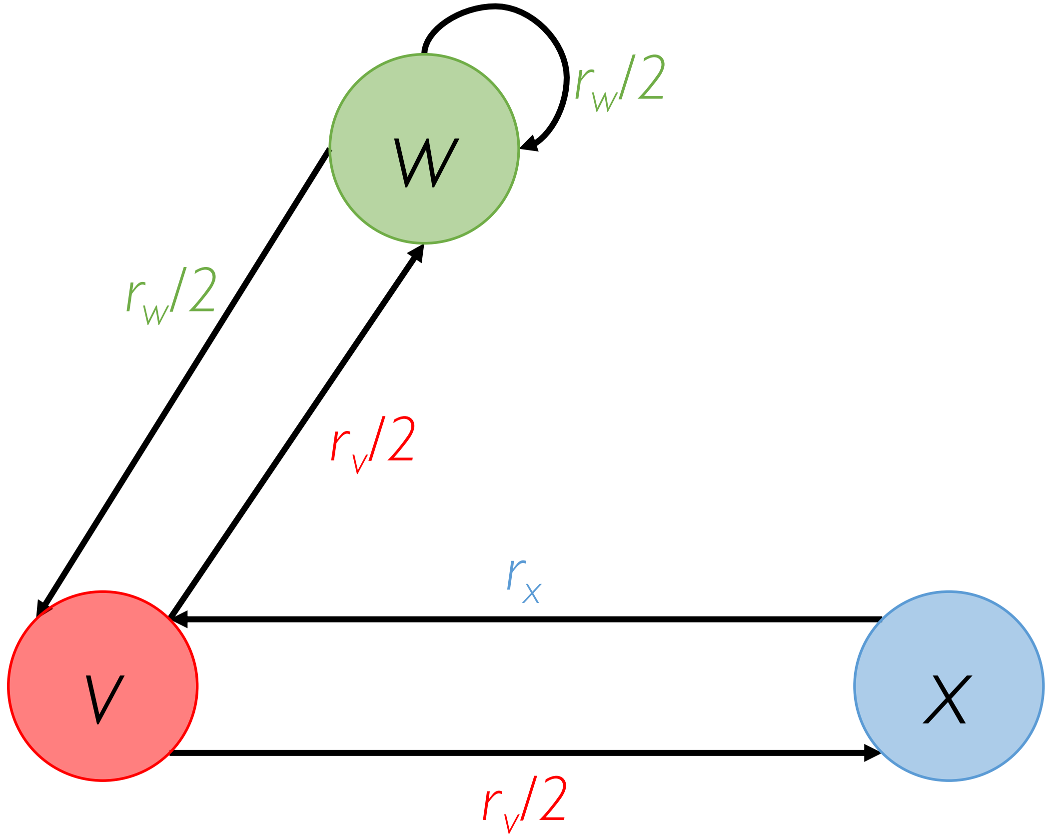

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"Flow" Equations

Solving the System of "Flow" Equations

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3 equations with 3 unknowns: r_v, r_w, and r_x

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No unique solution!

Infinitely many apart from a constant scale factor

Solving the System of "Flow" Equations

$$\begin{cases} r_v = r_w/2 + r_x \\ r_w = r_v/2 + r_w/2 \\ r_x = r_v/2 \\ \boxed{r_v + r_w + r_x = 1} \end{cases}$$

Additional constraint (equation) enforces the uniqueness of the solution

Solving the System of "Flow" Equations

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$$\boxed{r_v = r_w = \frac{2}{5} \quad r_x = \frac{1}{5}}$$

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This may work for very small systems of linear equations (e.g., using Gaussian elimination)

Solving the System of "Flow" Equations

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In the case of web pages we might have 100s of billions of equations!

We need a new formulation

PageRank: The Matrix Formulation

Represent the Web graph of documents $G=(V, E)$ s.t. $|V|=N$
as a column stochastic matrix M of size $N \times N$

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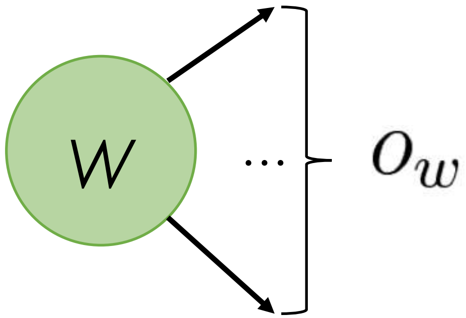

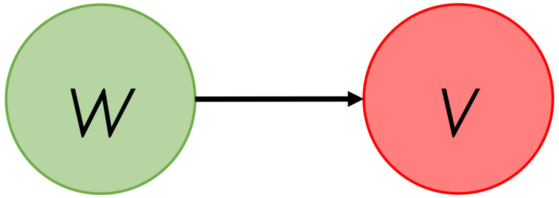

One of those o_w links
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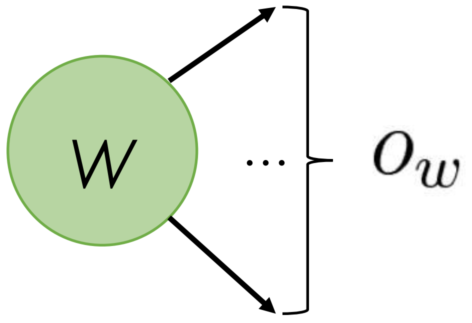
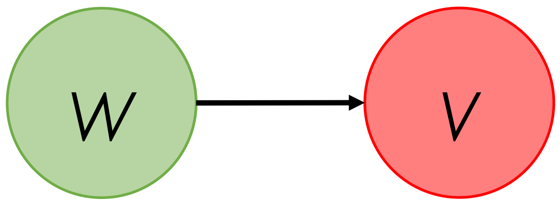
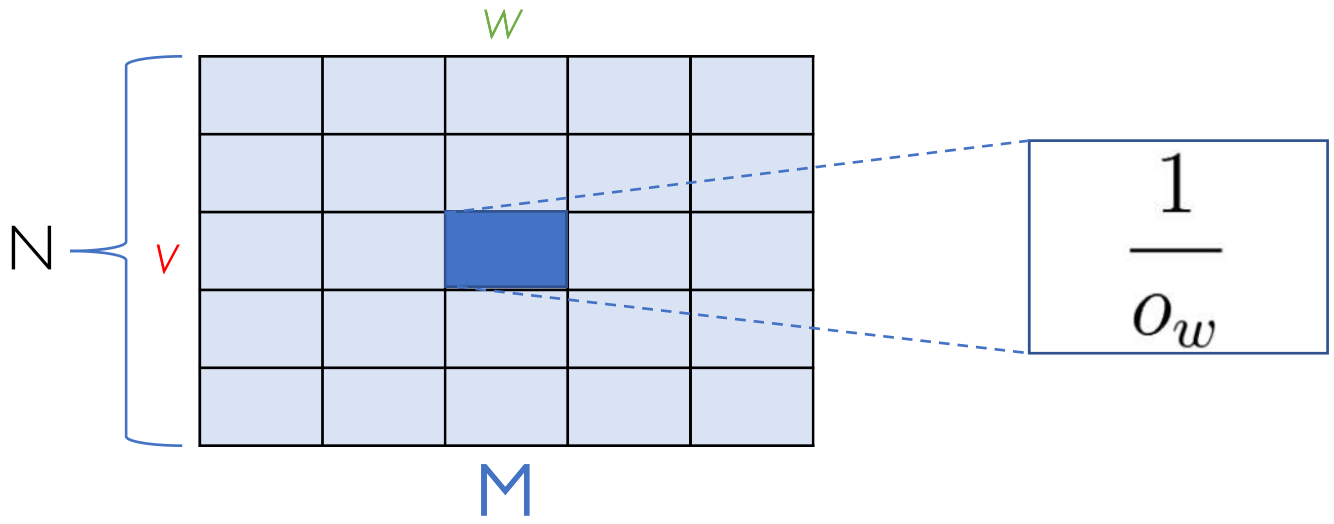




One of those o_w links points to page *v*



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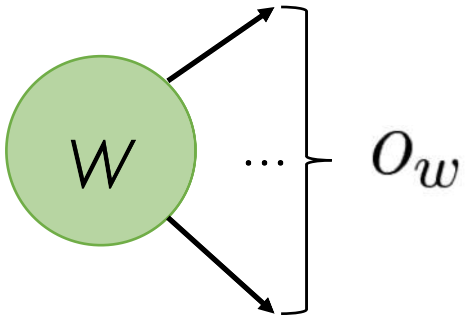
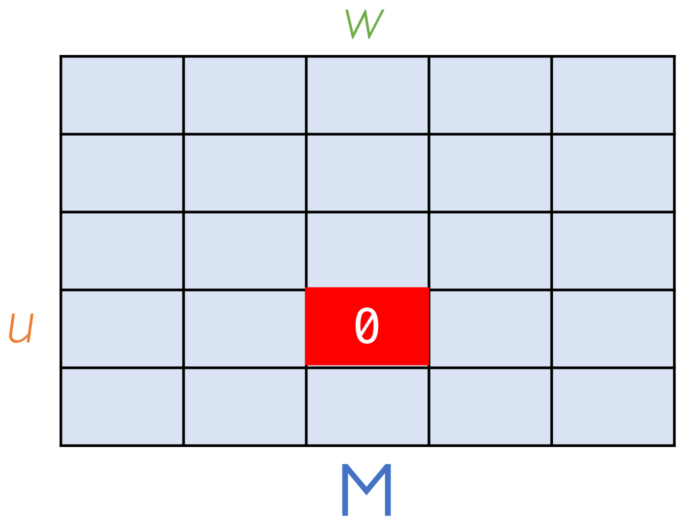


For any other page *u* which *w* is not pointing to M[*u*, *w*] = 0



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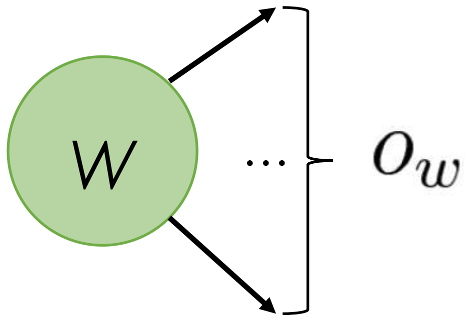

M is column stochastic because, by design, each of its column sums up to 1

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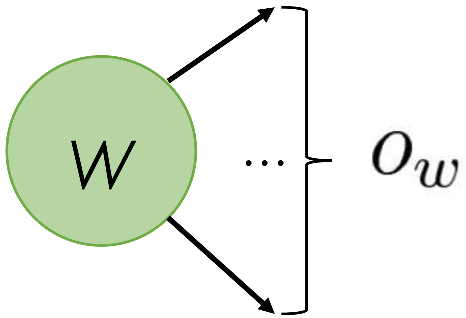

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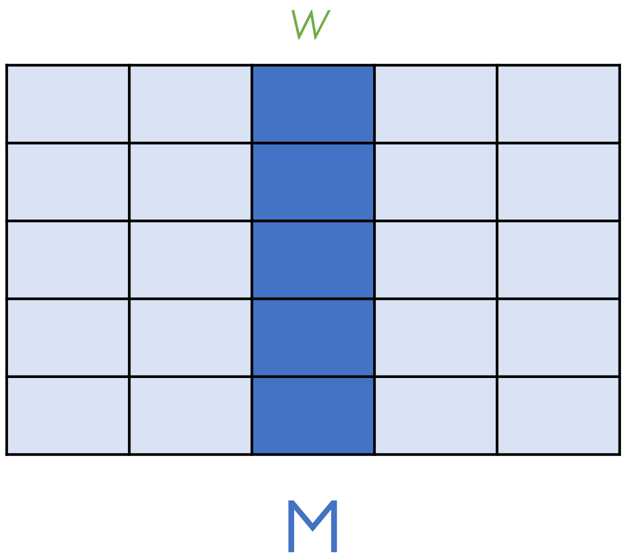
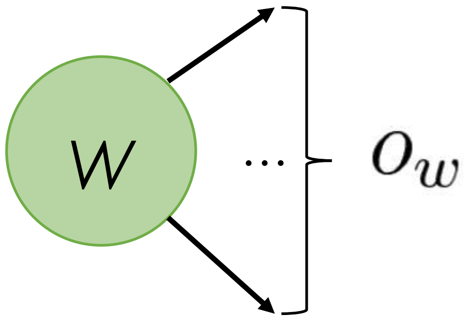


The w-th column will contain o_w <= N non-zero entries, each evaluating to $1/o_w$

$$\sum_{v=1}^{N} m_{v,w} = o_w \times \frac{1}{o_w} = 1$$

PageRank: The Matrix Formulation



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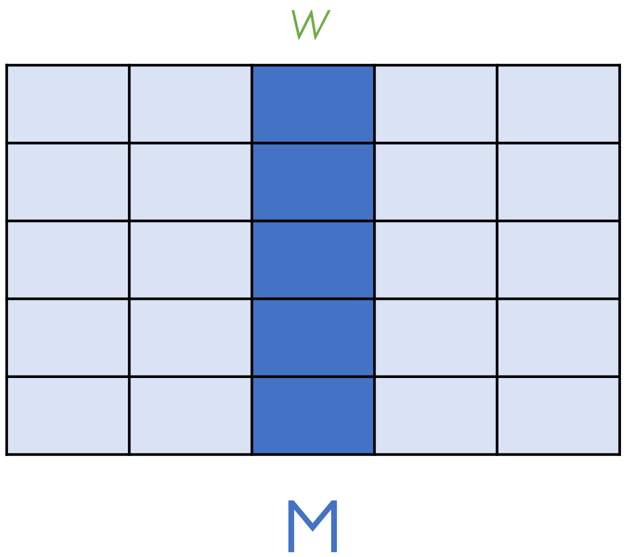

Note:
We are implicitly assuming there exists at least one outgoing link from each node

A Formal View of the Matrix M

$$\mathbf{A}_{N \times N} \quad a_{v,w} = \begin{cases} 1 & \text{if } w \in O_v \\ 0 & \text{otherwise} \end{cases}$$

Traditional adjacency matrix

A Formal View of the Matrix M

$$\mathbf{A}_{N\times N} \quad a_{v,w} = \begin{cases} 1 & \text{if } w \in O_v \\ 0 & \text{otherwise} \end{cases}$$

Traditional adjacency matrix

$$\mathbf{L}_{N\times N} \quad l_{v,w} = \begin{cases} o_v & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

Diagonal matrix of out-degrees

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Column stochastic matrix

$$\boxed{\mathbf{M} = (\mathbf{L}^{-1}\mathbf{A})^T}$$

PageRank: The Matrix Formulation

$\mathbf{r}$ Nx1 rank vector with an entry for each page

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All the rank scores must sum up to 1

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$\mathbf{r}$ Nx1 rank vector with an entry for each page

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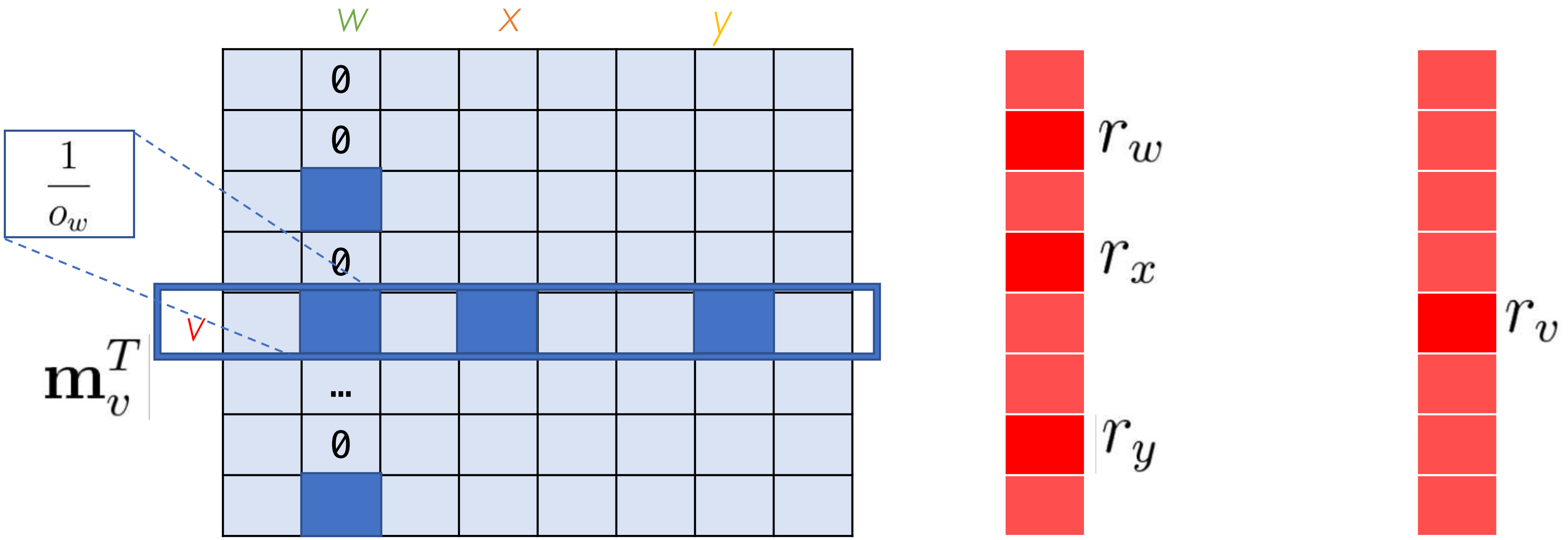
All the rank scores must sum up to 1

$$r_v = \sum_{w \in I_v} \frac{r_w}{o_w} \quad \Rightarrow \quad \mathbf{r} = \mathbf{M}\mathbf{r}$$

Flow equations in matrix form

PageRank: The Matrix Formulation



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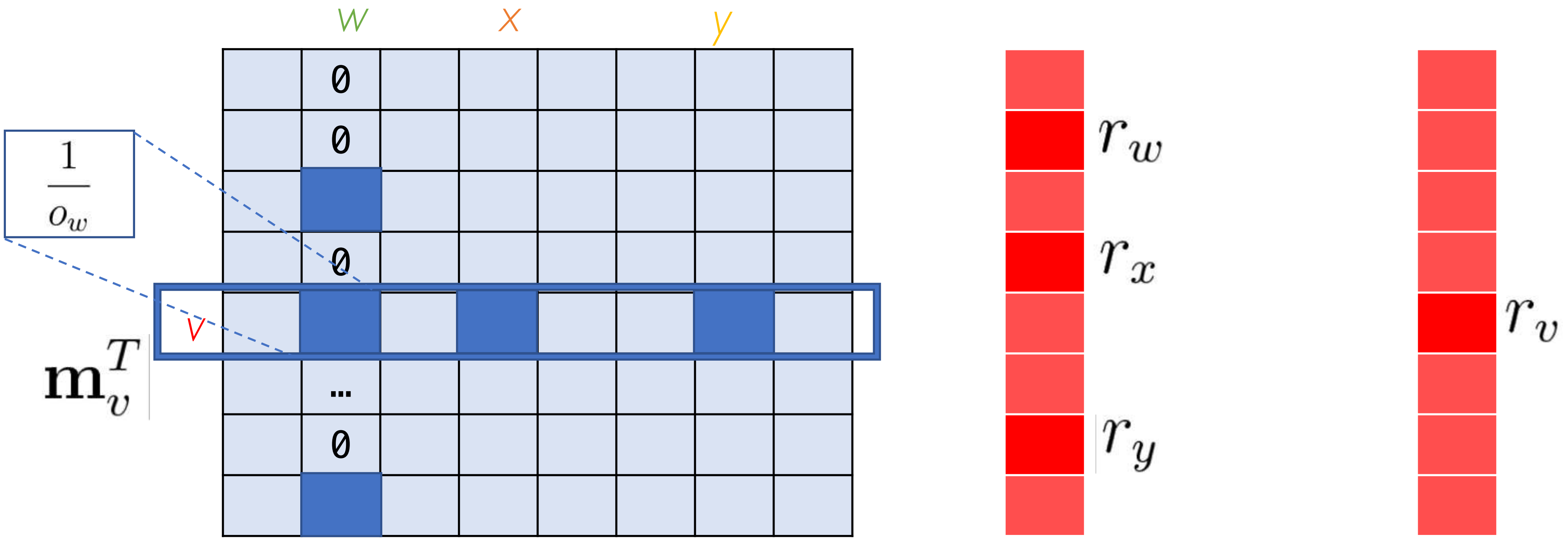

$$r_v = \mathbf{m}_v^T \cdot \mathbf{r} = \sum_{w=1}^{N} m_{v,w} \times r_w = \sum_{w=1}^{N} \frac{1}{o_w} \times r_w = \sum_{w=1}^{N} \frac{r_w}{o_w} = \sum_{w \in I_v} \frac{r_w}{o_w}$$

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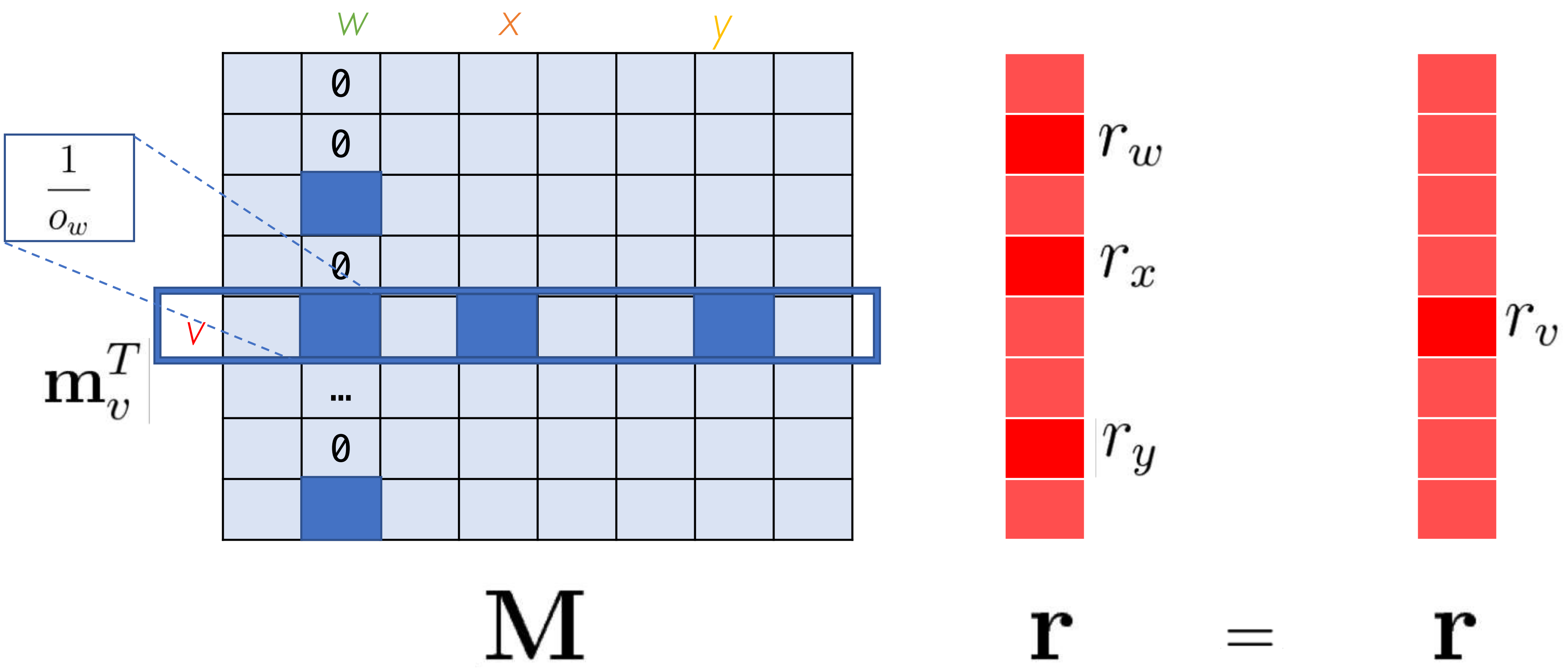

PageRank: The Eigenvector Formulation

$$\boxed{\mathbf{Mr} = \mathbf{r}}$$

Doesn't it look familiar?

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$$\mathbf{Ax} = \lambda \mathbf{x}$$

$\mathbf{x}$ is an eigenvector

λ is an eigenvalue

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So, the rank vector **r** is an eigenvector of the matrix M

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x is an eigenvector

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So, the rank vector **r** is an eigenvector of the matrix M

In fact, **r** is the eigenvector corresponding to the eigenvalue $\lambda = 1$

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For a fixed eigenvalue, eigenvectors are just scalar multiples of each other

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Since PageRank should reflect only the relative importance of the nodes, choose $\mathbf{r} = \mathbf{r}^*$ as the eigenvector whose entries sum up to 1

PageRank: The Eigenvector Formulation

$$\boxed{\mathbf{M}\mathbf{r} = \mathbf{r}}$$

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Since PageRank should reflect only the relative importance of the nodes, choose $\mathbf{r} = \mathbf{r}^*$ as the eigenvector whose entries sum up to 1

This may be referred to as the probabilistic eigenvector corresponding to the eigenvalue $\lambda = 1$

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We know from linear algebra theory that for any stochastic matrix M its largest eigenvalue is $\lambda = 1$

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Note:

So far, we have assumed that M is (column) stochastic yet this may not be the case for the general Web graph…

PageRank: Quick Recap

We start from "flow" equations

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We reformulate the system of linear equations using linear algebra
(i.e., stochastic matrix **M** and rank vector **r**)

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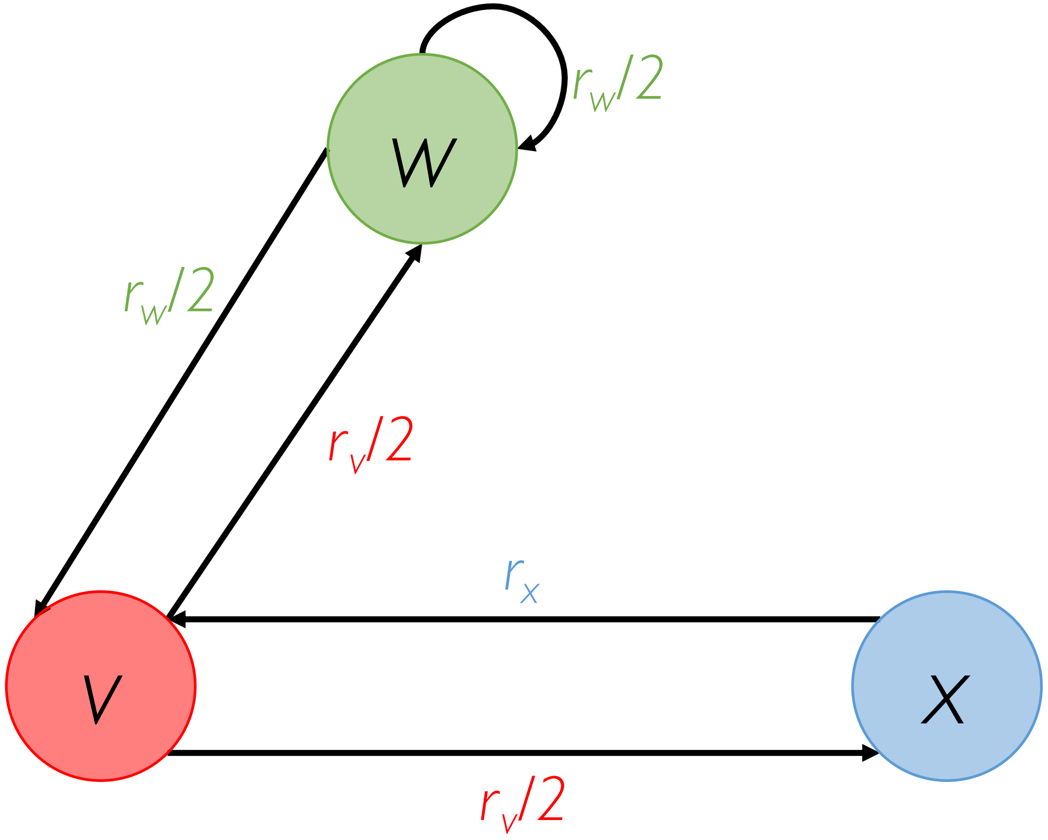
⇩

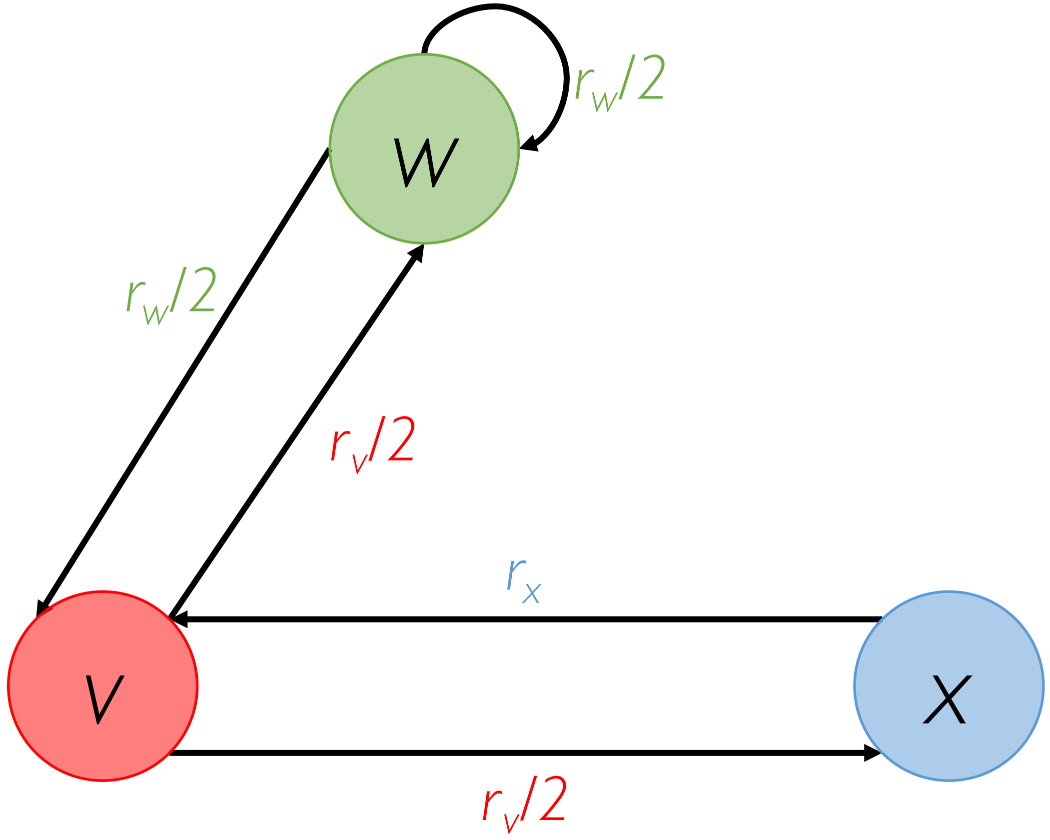
We know how to solve this efficiently using power iteration method

PageRank: The "Flow" Model



$$\begin{cases} r_v = r_w/2 + r_x \\ r_w = r_v/2 + r_w/2 \\ r_x = r_v/2 \end{cases}$$

PageRank: The "Flow" Model



$$\begin{cases} r_v = r_w/2 + r_x \\ r_w = r_v/2 + r_w/2 \\ r_x = r_v/2 \end{cases}$$

$$\mathbf{r} = \mathbf{M}\,\mathbf{r}, \qquad \mathbf{M} = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$$

PageRank: Power Iteration Method

At the beginning, we assume all pages have the same rank score, uniformly distributed across the N pages

$$\textbf{init:}\ \ t = 0; \mathbf{r}(t) = (1/N, 1/N, \ldots, 1/N)^T$$

PageRank: Power Iteration Method

Keep updating the rank vector r until convergence

$$\textbf{init:}\ \ t = 0; \mathbf{r}(t) = (1/N, 1/N, \ldots, 1/N)^T$$

$$\textbf{repeat:}$$

$$\mathbf{r}(t+1) = \mathbf{M}\mathbf{r}(t)$$

$$\textbf{until}\ \ \delta(\mathbf{r}(t+1), \mathbf{r}(t)) < \epsilon$$

$$\epsilon > 0$$

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$$\epsilon > 0$$

$$\left\{ \begin{array}{l} \delta(\mathbf{r}(t+1), \mathbf{r}(t)) = |\mathbf{r}(t+1) - \mathbf{r}(t)| \\ \text{or} \\ \delta(\mathbf{r}(t+1), \mathbf{r}(t)) = ||\mathbf{r}(t+1) - \mathbf{r}(t)|| \end{array} \right.$$

Power Iteration Method: Example

$$
\begin{cases}
r_v = r_w/2 + r_x \\
r_w = r_v/2 + r_w/2 \\
r_x = r_v/2
\end{cases}
$$

$$
\mathbf{r} = \mathbf{M} \, \mathbf{r}, \qquad \mathbf{M} = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}
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$$\mathbf{r}(0) = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Power Iteration Method: Example

$$\begin{cases} r_v = r_w/2 + r_x \\ r_w = r_v/2 + r_w/2 \\ r_x = r_v/2 \end{cases}$$

$$\mathbf{r} = \mathbf{M}\,\mathbf{r}$$

0	1/2	1
1/2	1/2	0
1/2	0	0

$$\mathbf{r}(0) = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \qquad \mathbf{r}(1) = \mathbf{M}\mathbf{r}(0) = \begin{bmatrix} 3/6 \\ 1/3 \\ 1/6 \end{bmatrix}$$

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$$\underset{\mathbf{r}(0)}{\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}} \quad \mathbf{r}(1) = \mathbf{M}\mathbf{r}(0) = \begin{bmatrix} 3/6 \\ 1/3 \\ 1/6 \end{bmatrix} \quad \mathbf{r}(2) = \mathbf{M}\mathbf{r}(1) = \begin{bmatrix} 1/3 \\ 5/12 \\ 3/12 \end{bmatrix}$$

Power Iteration Method: Example

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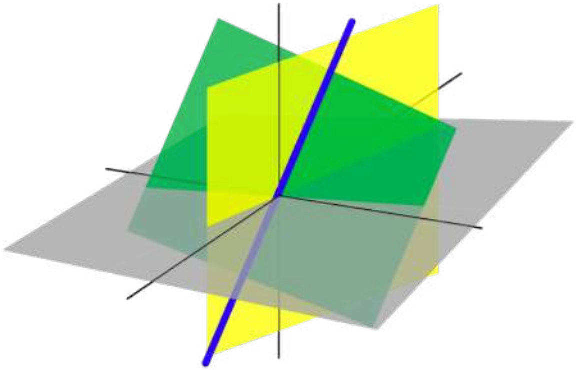
We came up with the same set of solutions for r_v, r_w, and r_x without explicitly solving the system of equations

PageRank's Interpretations

2 main perspectives

Linear Algebra



Probabilistic

Random Walk Interpretation of Page Rank

Imagine a random surfer navigating through the pages of the Web graph

Random Walk Interpretation of Page Rank

Initially, at time $t{=}0$ the surfer can be on **any** web page

www.donuts.com

www.krustyburger.com

www.duffbeer.com

…

www.moes.com

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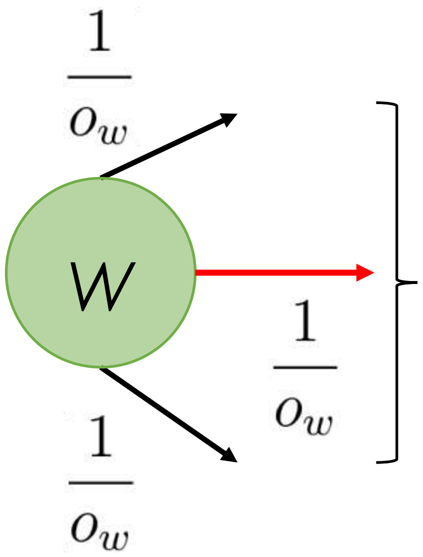
Each web page has equal probability $1/N$ to be chosen as starting point

Random Walk Interpretation of Page Rank

At any given time t, the surfer is on some web page w

Random Walk Interpretation of Page Rank

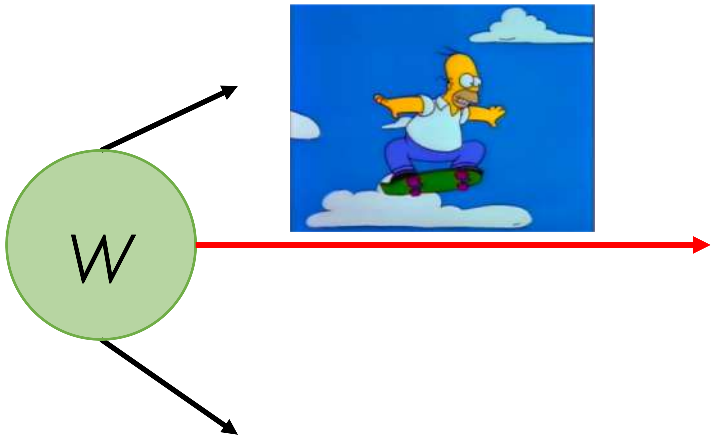
At time $t+1$, the surfer follows one of the outgoing links from web page *w*, chosen uniformly at random



o_w out-degree of node *w*

Random Walk Interpretation of Page Rank

The surfer ends up into some other web page *v* pointed by *w*



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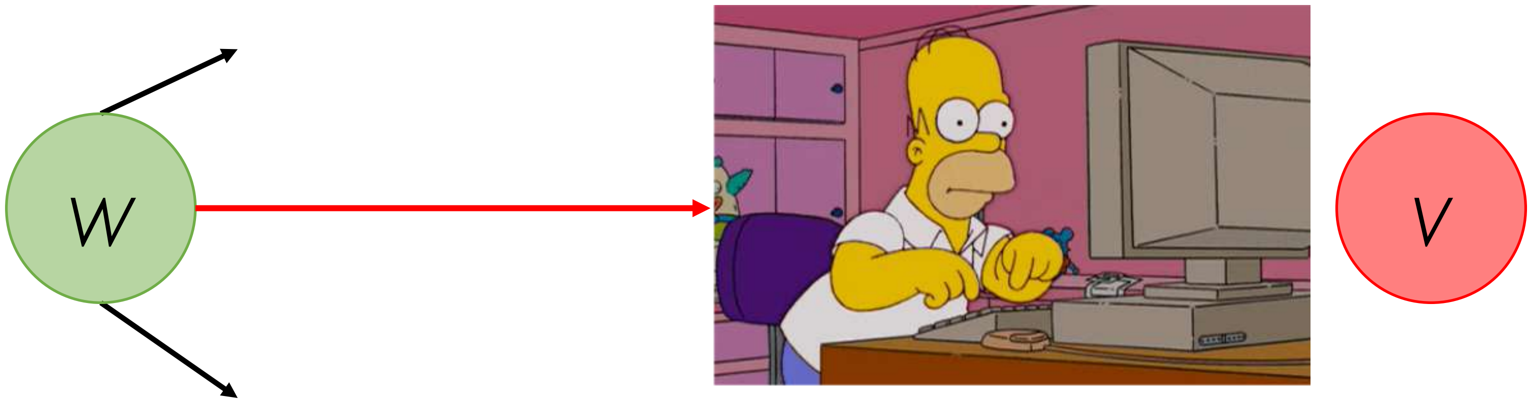

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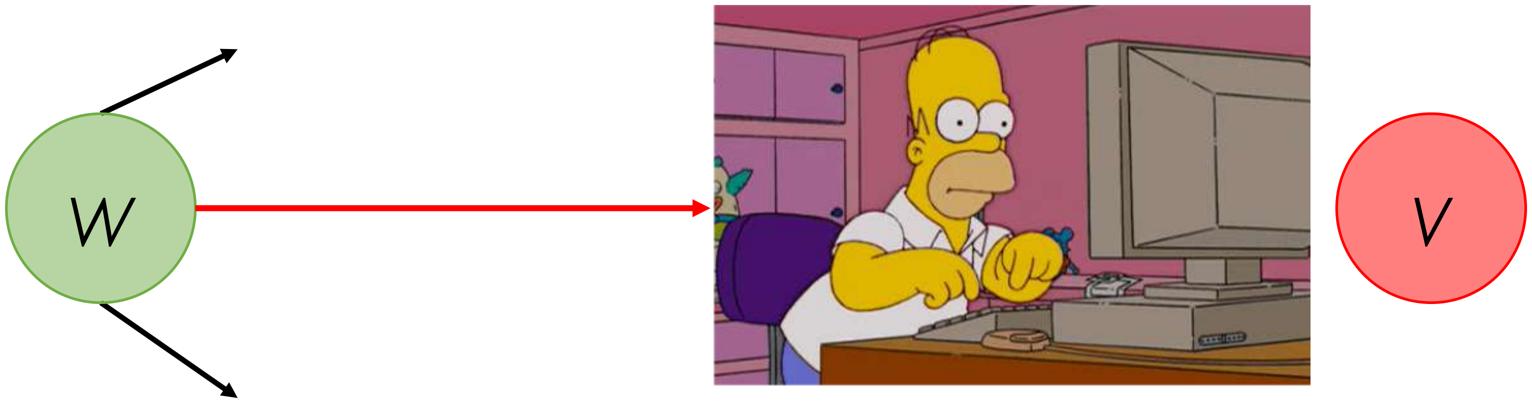

This process repeats indefinitely and is known as **random walk**

Transition Matrix M

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Column stochastic matrix

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Column stochastic matrix

The v-th, w-th entry of M indicates the probability of a random surfer moving from page w to page v

Such a matrix describes a Markov chain over the finite state space V of nodes (i.e., pages) of the Web graph

Random Walk Interpretation of Page Rank

X Discrete-Valued Random Variable taking on |V| = N possible values

Random Walk Interpretation of Page Rank

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$X = w$ Indicates a random surfer is on web page *w*

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N-dimensional stochastic (i.e., probability) vector associated with X

$$\mathbf{p} \subseteq \mathbb{R}^N = (P(X = 1), \ldots, P(X = w), \ldots, P(X = N))^T$$

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N-dimensional stochastic (i.e., probability) vector associated with X at time *t*

$$\mathbf{p}(t) \subseteq \mathbb{R}^N = (P(X_t=1), \ldots, P(X_t=w), \ldots, P(X_t=N))^T$$

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Probability distribution over web pages at time *t*

Random Walks as Markov Chains

Random Walks are also known as stochastic processes with Markov property (i.e., Markov chains)

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The transition probability of moving to the next state depends only on the present state and not on the previous states

$$P(X_{t+1} = v | X_1 = x_1, X_2 = x_2, \ldots, X_t = x_t) = P(X_{t+1} = v | X_t = x_t)$$

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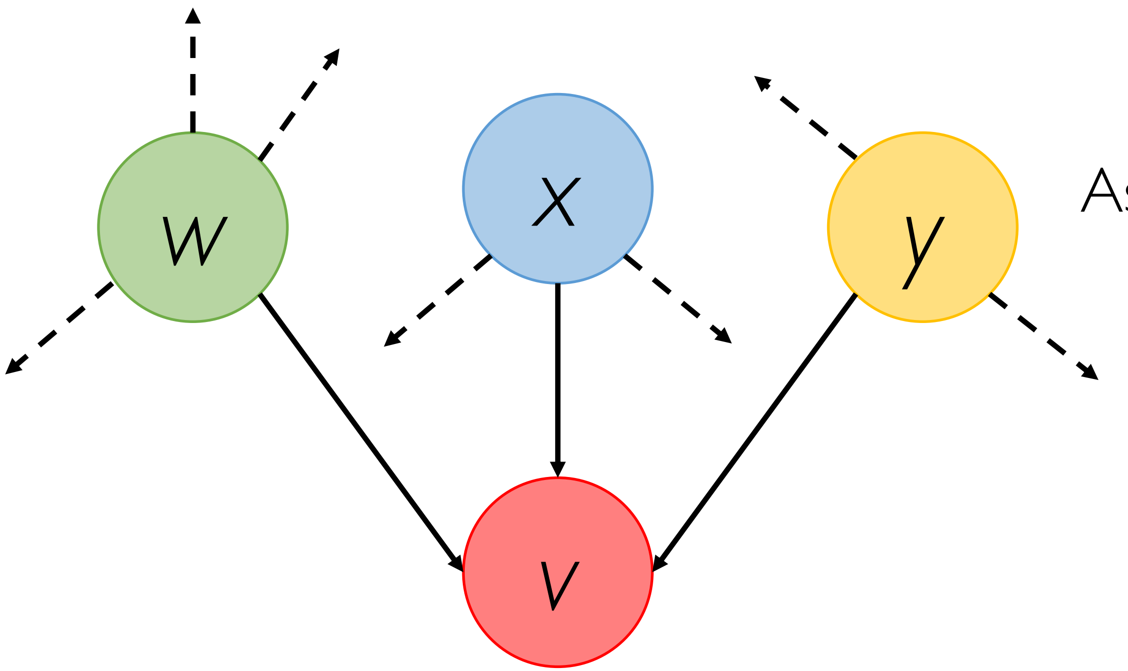
The probability that the random surfer will be on page *v* at time *t+1* depends only on where the surfer was at time *t*

Random Walk Interpretation of Page Rank

Where is the random surfer at time $t+1$ knowing where he was at time t?
Suppose we want to estimate $P(X_{t+1} = v)$



Assume v has only **3 incoming links** from w, x, and y

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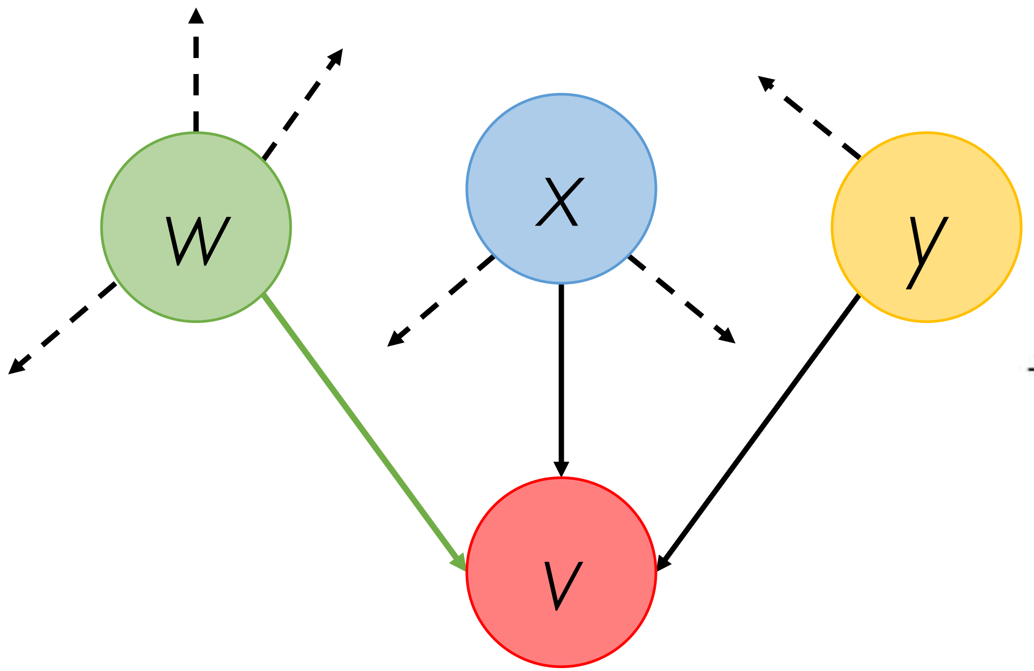

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$$P(X_{t+1} = v) = \boxed{P(X_t = w, Z_w = v) +}$$

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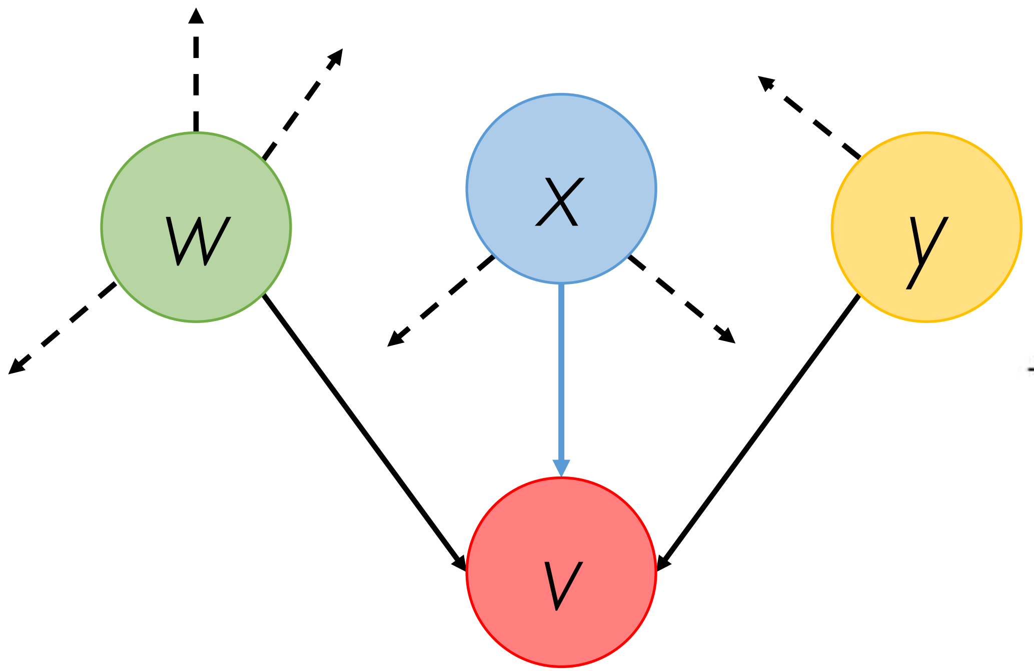

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$$\boxed{P(X_t = x, Z_x = v) +}$$

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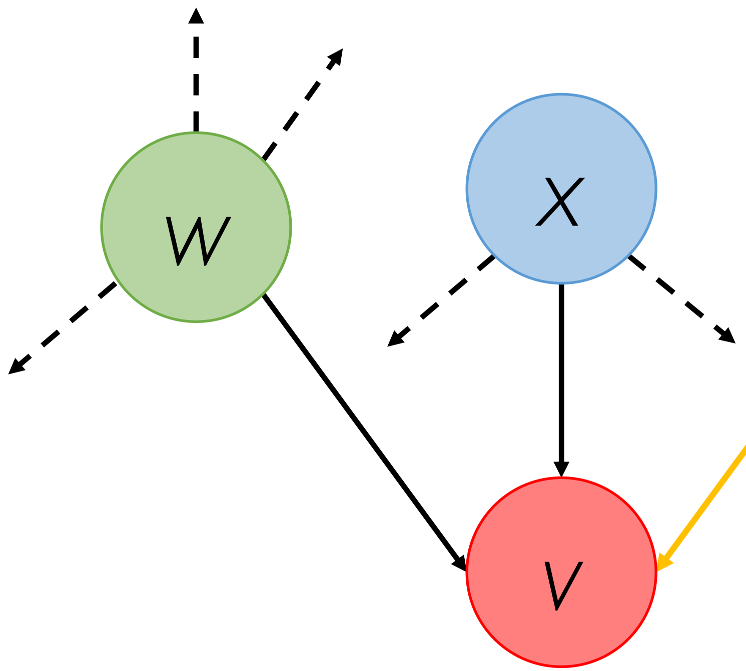
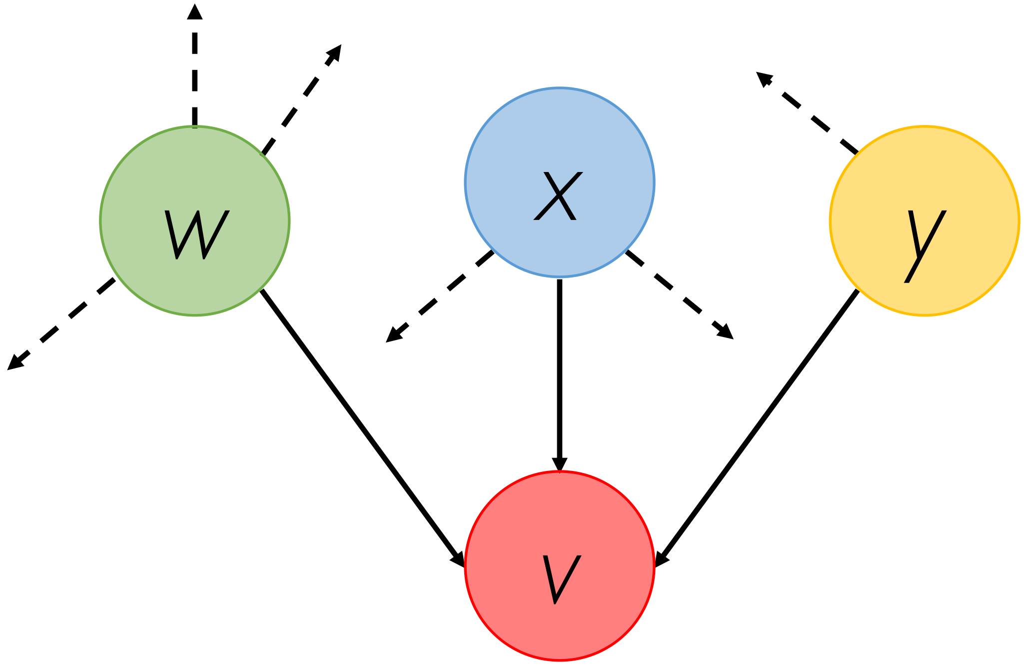


Assume v has only 3 incoming links from w, x, and y

$$P(X_{t+1} = v) = P(X_t = w, Z_w = v) + P(X_t = x, Z_x = v) + \boxed{P(X_t = y, Z_y = v)}$$

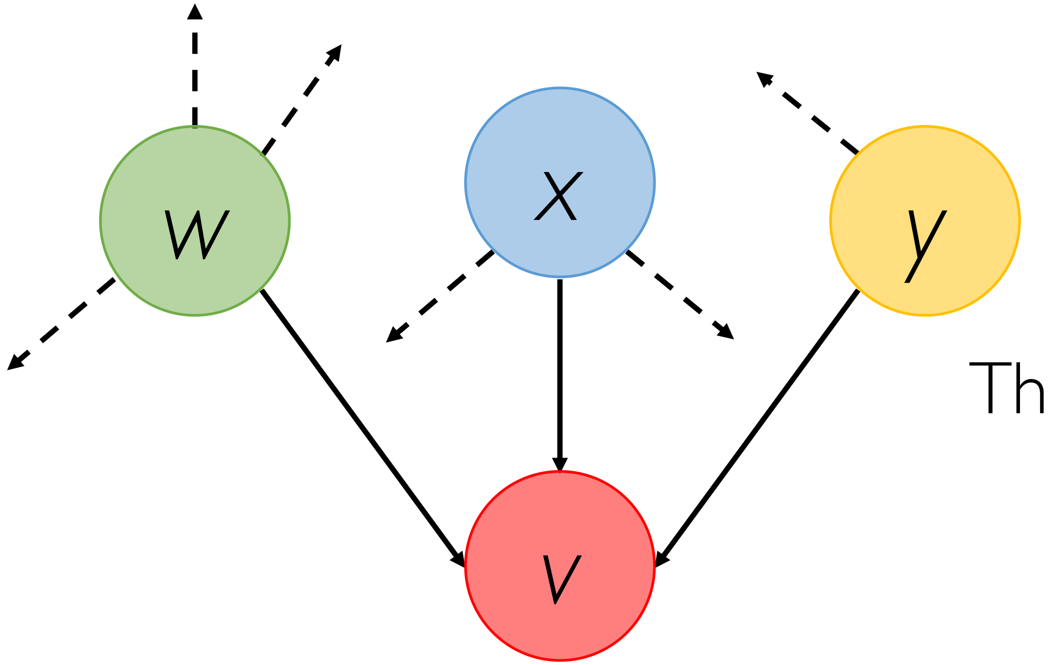
$$\boxed{Z_u \sim \text{Uniform}(1, o_u)}$$

Random Walk Interpretation of Page Rank



$$\boxed{\mathbf{p}(t+1) = \mathbf{M}\mathbf{p}(t)}$$

Random Walk Interpretation of Page Rank

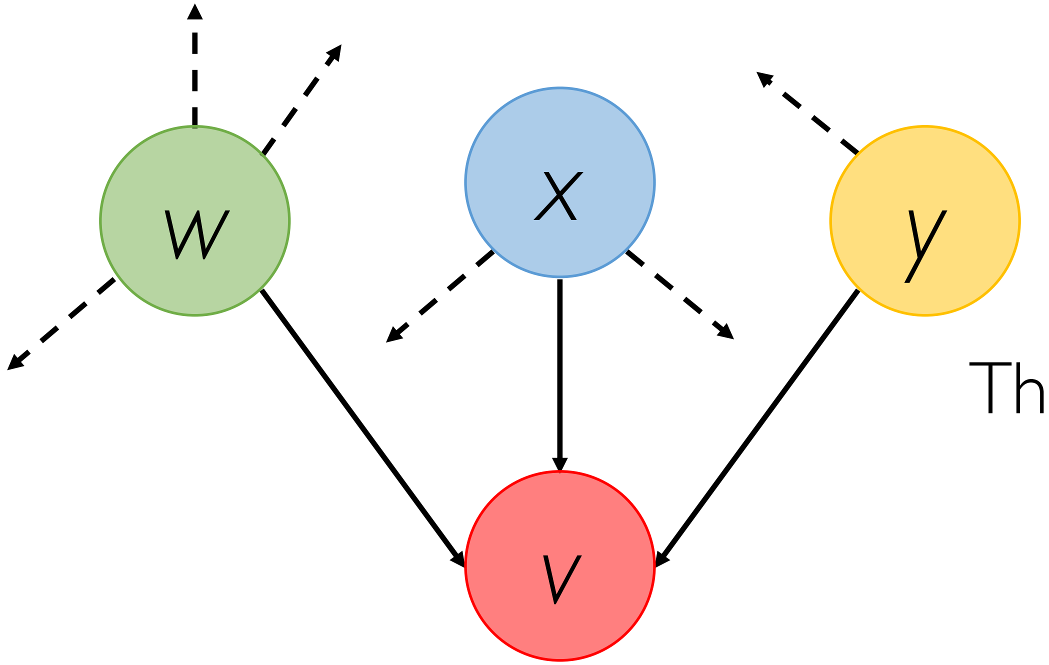


$$\boxed{\mathbf{p}(t+1) = \mathbf{M}\mathbf{p}(t)}$$

This resembles our PageRank equation

$$\boxed{\mathbf{r}(t+1) = \mathbf{M}\mathbf{r}(t)}$$

Random Walk Interpretation of Page Rank



$$\boxed{\mathbf{p}(t+1) = \mathbf{M}\mathbf{p}(t)}$$

This resembles our PageRank equation

$$\boxed{\mathbf{r}(t+1) = \mathbf{M}\mathbf{r}(t)}$$

Solving the former is equivalent to solving the latter!

Random Walk Interpretation of Page Rank

Initially, the stochastic vector **p**(0) is a uniform probability distribution

Random Walk Interpretation of Page Rank

Initially, the stochastic vector p(0) is a uniform probability distribution

The probability that page *v* will be visited after one step corresponds to the *v*-th entry of p(1), obtained as follows:

$$\mathbf{p}(1) = \mathbf{M}\mathbf{p}(0)$$

Random Walk Interpretation of Page Rank

Initially, the stochastic vector p(0) is a uniform probability distribution

The probability that page *v* will be visited after one step corresponds to the *v*-th entry of p(1), obtained as follows:

$$\mathbf{p}(1) = \mathbf{M}\mathbf{p}(0)$$

More generally, the probability of visiting *any* web page after *t* steps is:

$$\boxed{\mathbf{p}(t) = \mathbf{M}^t\mathbf{p}(0)}$$

Random Walk Interpretation of Page Rank

$$\mathbf{p}(0) = (\ \underbrace{1/N}_{P(X_0=1)}\ , \ldots, \ \underbrace{1/N}_{P(X_0=w)}\ , \ldots, \ \underbrace{1/N}_{P(X_0=N)}\ )^T$$

Random Walk Interpretation of Page Rank

$$\mathbf{p}(0) = (\underbrace{1/N}_{P(X_0=1)}, \ldots, \underbrace{1/N}_{P(X_0=w)}, \ldots, \underbrace{1/N}_{P(X_0=N)})^T$$

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Random Walk Interpretation of Page Rank

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Random Walk Interpretation of Page Rank

$\{\mathbf{p}(t)\}_{t=0,1,\ldots,T}$

Discrete
Stochastic Process

Markov chain

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$\mathbf{p}^*$ is the stationary distribution of the random walk

Equivalence between Formulations

Linear Algebra

Probabilistic

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System of linear "flow" equations

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So the PageRank vector $\mathbf{r}^*$ corresponds to the stationary distribution $\mathbf{p}^*$ for the random walk on the graph encoded by $\mathbf{M}$!

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So the PageRank vector $\mathbf{r}^*$ corresponds to the stationary distribution $\mathbf{p}^*$ for the random walk on the graph encoded by $\mathbf{M}$!

Intuitively, the PageRank vector indicates for each web page the probability that a random surfer will eventually get to that page

Hang on a second…

Linear Algebra

Probabilistic

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How do we know that the power iteration method always converge to $\mathbf{r}^*$?

existence

Probabilistic

How do we know that a Markov chain always converge to a steady-state $\mathbf{p}^*$?

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Linear Algebra

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existence and uniqueness of $\mathbf{r}^*$ ($\mathbf{p}^*$) are guaranteed under certain conditions on the matrix $\mathbf{M}$

Existence and Uniqueness of PageRank

If M is a column stochastic matrix with all positive entries:

- $\lambda = 1$ is an eigenvalue of M with multiplicity one
- $\lambda = 1$ is the largest eigenvalue of M
- There exists a unique (right) eigenvector $\mathbf{r}^*$ associated with the eigenvalue $\lambda = 1$ with the sum of its entries equal to 1

Perron-Frobenius theorem (circa 1910)

Existence and Uniqueness of PageRank

If M is a column stochastic matrix with all positive entries, then M has a unique steady-state vector p^* such that for any $p(0)$

$$\mathbf{p}(t) = \mathbf{M}^t \mathbf{p}(0) \quad \text{converges to} \quad \mathbf{p}^* \quad \text{as} \quad t \to \infty$$

Perron-Frobenius theorem (circa 1910)

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The steady-state vector is the unique eigenvector associated with the largest eigenvalue $\lambda = 1$

Are We Done, Then?

Problem: We cannot apply the Perron-Frobenius theorem to the matrix M as we originally defined it

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$$\mathbf{M}_1 = \begin{bmatrix} 0.6 & 0.5 & 0 \\ 0.4 & 0.3 & 1 \\ 0 & 0.2 & 0 \end{bmatrix} \qquad \mathbf{M}_2 = \begin{bmatrix} 0.6 & 0.5 & 0.1 \\ 0.2 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0.5 \end{bmatrix}$$

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Both M_1 and M_2 are column stochastic, but only M_2 is positive

So? Should We Give Up?

Here is where Brin and Page, in fact Google, comes in!

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We show how they fixed the issues with the original definition of M to accommodate for the heterogeneity of the Web graph

By doing so, we know that a solution to our PageRank problem exists and is unique!

Google's PageRank

Problems with Original PageRank Formulation

We cannot directly apply the Perron-Frobenius theorem to the original Web graph matrix M

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Then we discuss how Brin and Page fixed this in their seminal paper which sets up the rising of Google

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2 main issues to solve:

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Dead End

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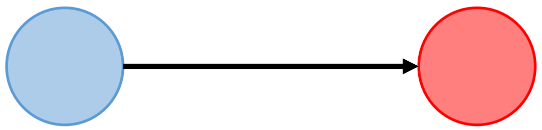
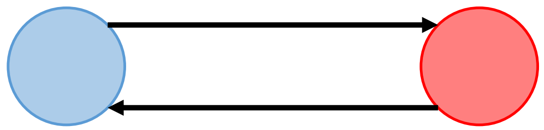


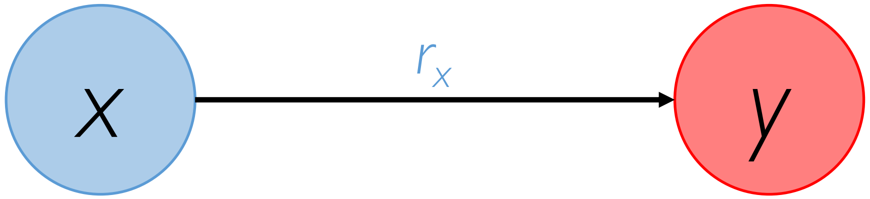
Spider Trap

Not every node is reachable and PageRank gets eventually absorbed by small group of pages



The "Dead End" Problem (Dangling Nodes)

Example:



$$r_y = r_x$$

M

0	0
1	0

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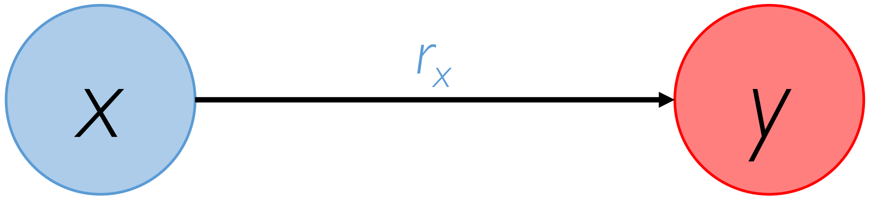

$$r_y = r_x$$

$$\mathbf{M} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

When a web page has no outgoing links (dangling node) the resulting column vector in the matrix M is not stochastic anymore!

Previously, we assumed each web page has ***at least one*** *outgoing link, and therefore M was stochastic*

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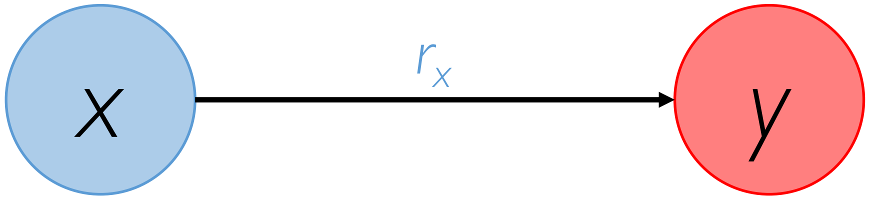

$$r_y = r_x$$

M

0	0
1	0

Assume the following initialization for **r**:

$$\mathbf{r}(0) = \begin{bmatrix} r_x^{(0)} \\ r_y^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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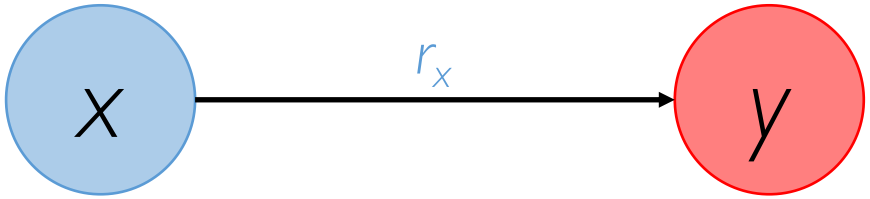

$$r_y = r_x$$

$$\mathbf{M} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{r}(1) = \mathbf{M} \quad \mathbf{r}(0)$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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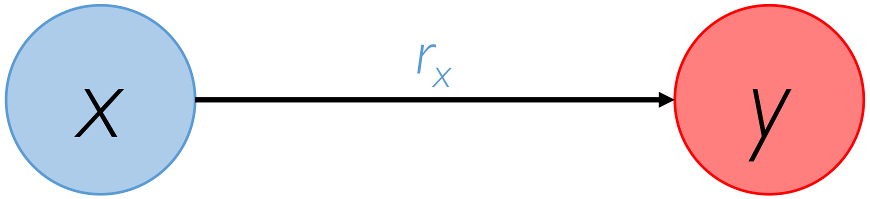

$$r_y = r_x$$

$$\mathbf{M} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{r}(2) = \mathbf{M} \; \mathbf{r}(1)$$

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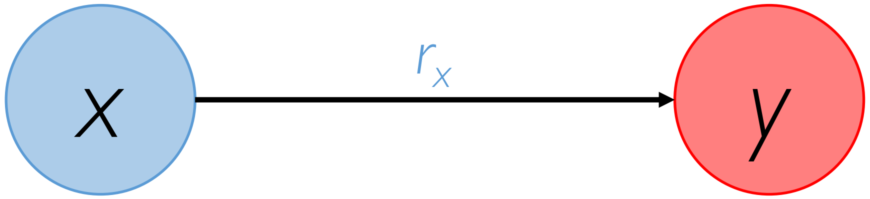


$$r_y = r_x$$

M

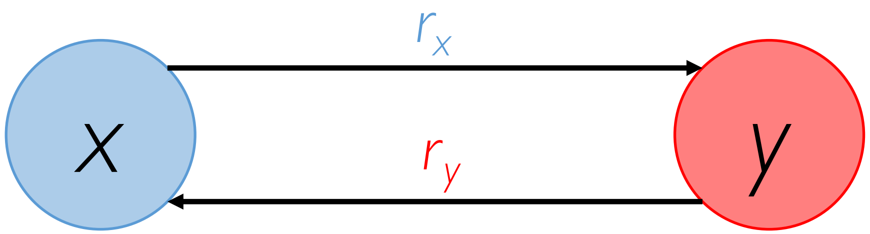
0	0
1	0

$\mathbf{r}(0)$	$\mathbf{r}(1)$	$\mathbf{r}(2)$		$\mathbf{r}(t-1)$	$\mathbf{r}(t)$
1	0	0		0	0
0	1	0	…	0	0

The PageRank vector vanishes to 0!

The "Spider Trap" Problem

<u>Example:</u>



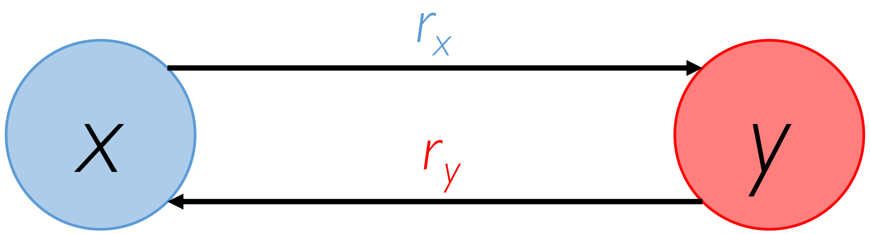
$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

$\mathbf{M}$

0	1
1	0

The "Spider Trap" Problem

Example:



$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

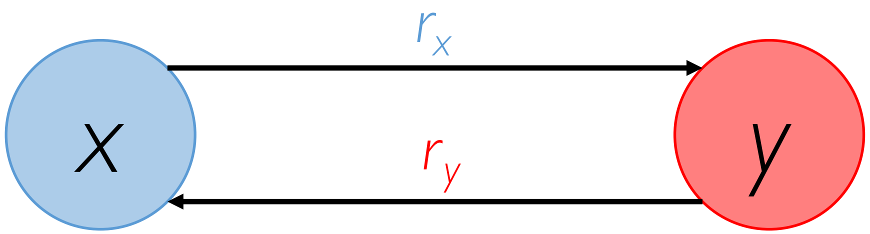
M

0	1
1	0

M is column stochastic non-negative (but not strictly positive)
Does PageRank converge regardless of the initialization of r?

The "Spider Trap" Problem

Example:



$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

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0	1
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Assume the same initialization as before for **r**:

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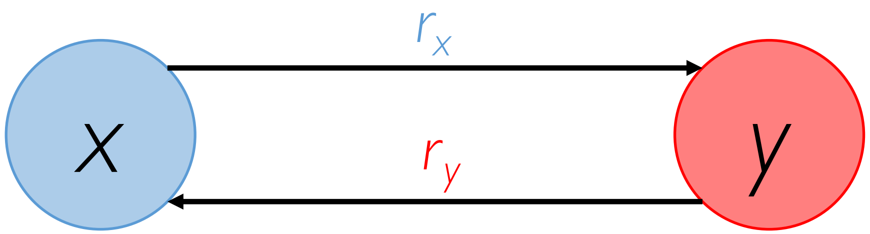

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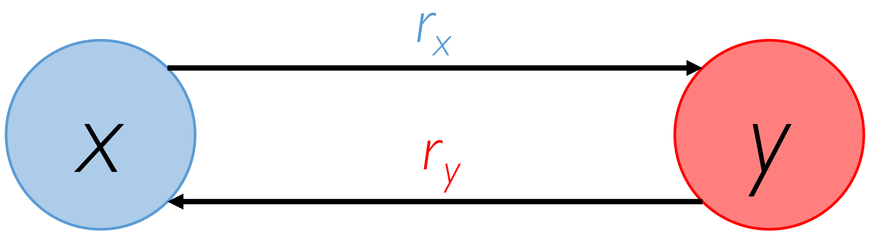

$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

M

0	1
1	0

$$\mathbf{r}(2) = \mathbf{M} \quad \mathbf{r}(1)$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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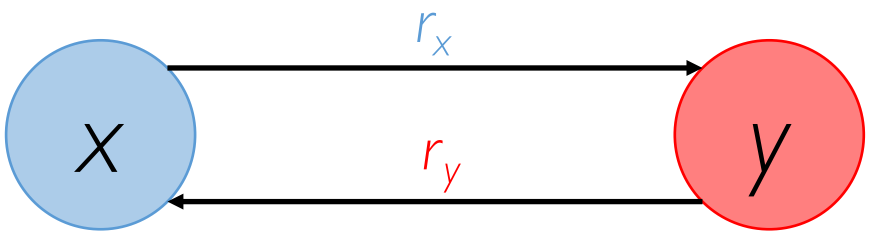


$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

$\mathbf{M}$

0	1
1	0

$\mathbf{r}(0)$	$\mathbf{r}(1)$	$\mathbf{r}(2)$	...	$\mathbf{r}(t-1)$	$\mathbf{r}(t)$
1	0	1		0	1
0	1	0		1	0

The PageRank vector keeps alternating its components and **never** converges!

Problems with Original PageRank Formulation

2 main issues to solve:

Dead End	Spider Trap
Pages with no outlinks cause PageRank to leak out	Not every node is reachable and PageRank gets eventually absorbed by small group of pages

Problems with Original PageRank Formulation

2 main issues to solve:

Dead End

Pages with no outlinks cause PageRank to leak out

M is **not column stochastic** as some nodes have no outlinks

Spider Trap

Not every node is reachable and PageRank gets eventually absorbed by small group of pages

Problems with Original PageRank Formulation

2 main issues to solve:

Dead End

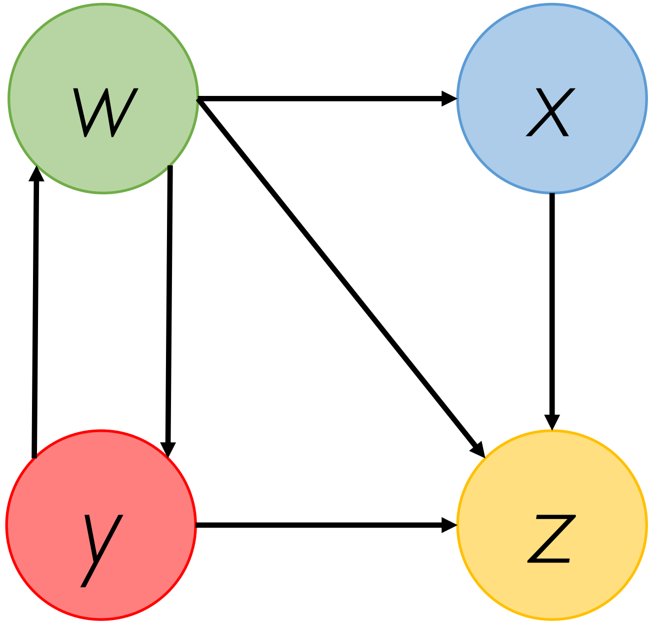
Pages with no outlinks cause PageRank to leak out

Spider Trap

Not every node is reachable and PageRank gets eventually absorbed by small group of pages

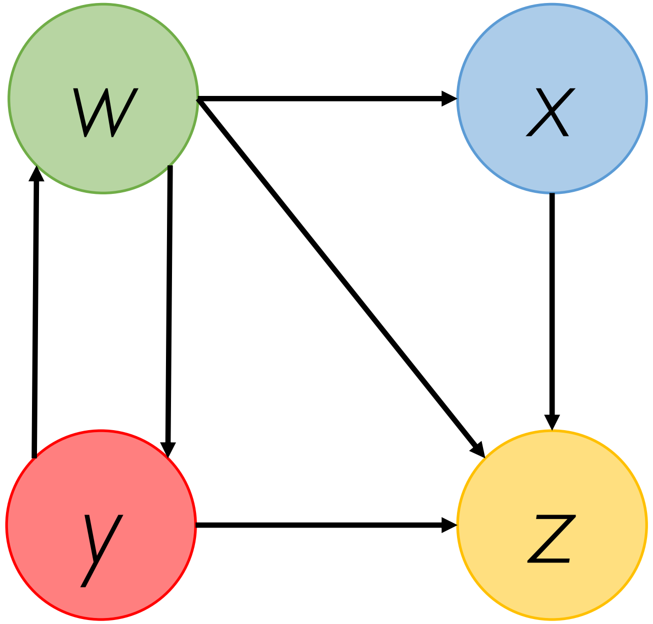
M is stochastic but **not strictly positive**

Deal with Dangling Nodes



$$\mathbf{M} = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1 & 1/2 & 0 \end{bmatrix} \end{array}$$

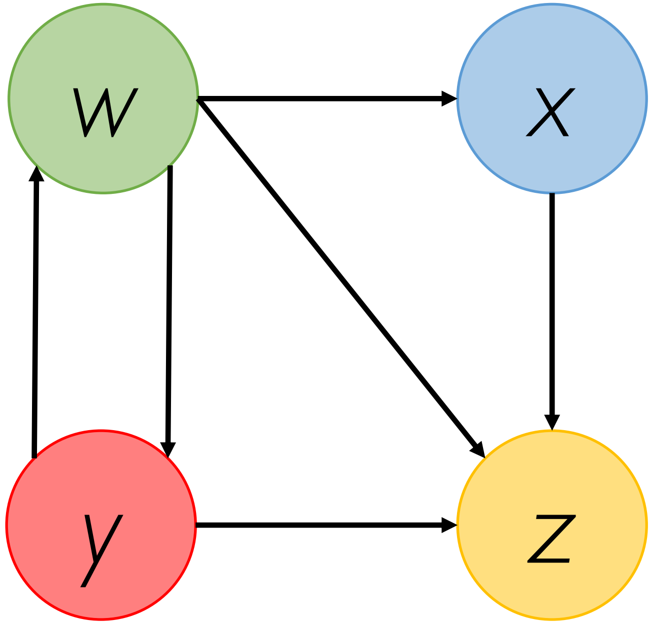
Deal with Dangling Nodes



z is a dangling node

$$\mathbf{M} = \begin{array}{c c} & \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1 & 1/2 & 0 \end{bmatrix} \end{array}$$

Deal with Dangling Nodes

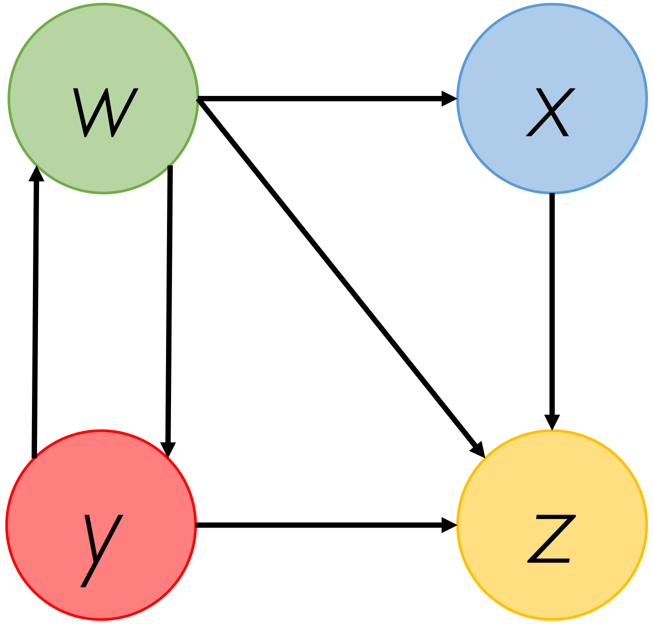


z is a dangling node

$$\mathbf{M} = \begin{array}{c|cccc} & w & x & y & z \\ \hline w & 0 & 0 & 1/2 & 0 \\ x & 1/3 & 0 & 0 & 0 \\ y & 1/3 & 0 & 0 & 0 \\ z & 1/3 & 1 & 1/2 & 0 \end{array}$$

M is not
(column) stochastic

Deal with Dangling Nodes



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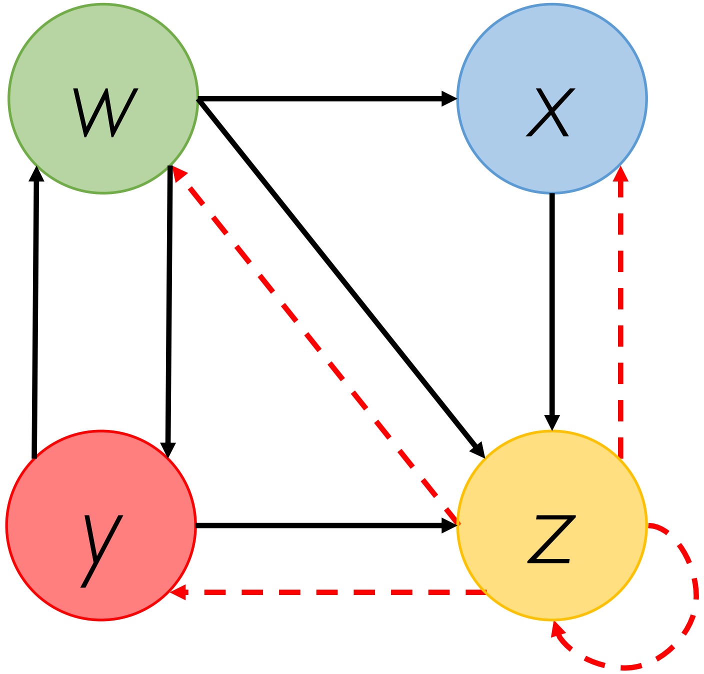
If we apply simplified PageRank to **M** the rank vector **r** will eventually vanish to **0**

Deal with Dangling Nodes



$$\mathbf{M}' = \begin{array}{cc} & \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{bmatrix} 0 & 0 & 1/2 & 1/4 \\ 1/3 & 0 & 0 & 1/4 \\ 1/3 & 0 & 0 & 1/4 \\ 1/3 & 1 & 1/2 & 1/4 \end{bmatrix} \end{array}$$

<u>Solution:</u> Teleporting

Create artificial links from any dangling node to any other node

Deal with Dangling Nodes: Teleporting

This adjustment is justified by modeling the behaviour of a web surfer

Deal with Dangling Nodes: Teleporting

This adjustment is justified by modeling the behaviour of a web surfer

After reading a page with no out-going link, jump to a page picked uniformly at random amongst the N

Deal with Dangling Nodes: Teleporting

Initially, we set $\mathbf{M}_{N \times N}$ $m_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ 0 & \text{otherwise} \end{cases}$

Deal with Dangling Nodes: Teleporting

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Now we change it to $\mathbf{M}'_{N\times N}$ $m'_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ \frac{1}{N} & \text{if } \sum_{v=1}^{N} m_{v,w} = 0 \\ 0 & \text{otherwise} \end{cases}$

Deal with Dangling Nodes: Teleporting

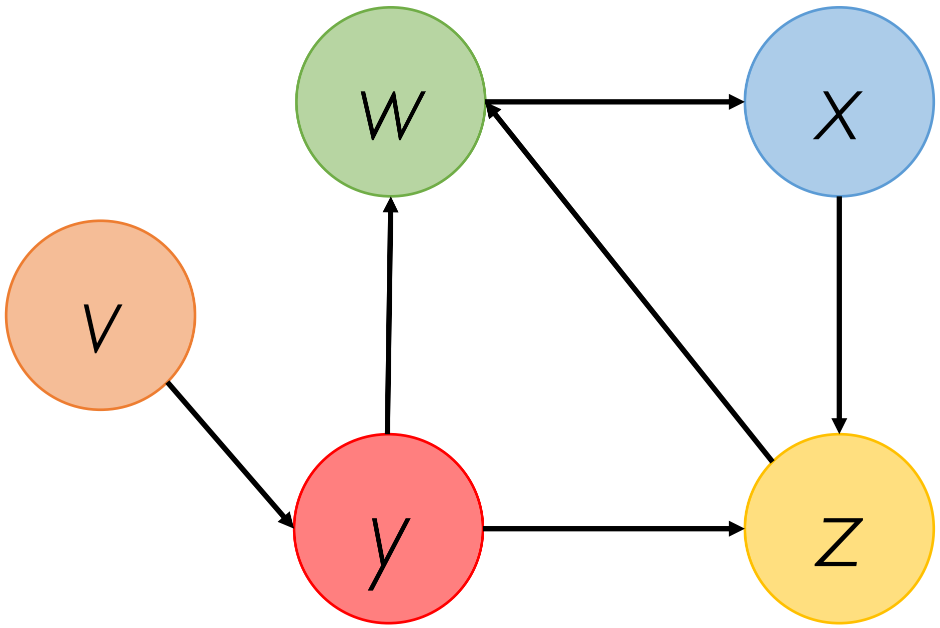
Initially, we set $\mathbf{M}_{N\times N}$ $m_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ 0 & \text{otherwise} \end{cases}$

Now we change it to $\mathbf{M}'_{N\times N}$ $m'_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ \frac{1}{N} & \text{if } \sum_{v=1}^{N} m_{v,w} = 0 \\ 0 & \text{otherwise} \end{cases}$

$$\boxed{\mathbf{M} \rightsquigarrow \mathbf{M}'}$$

This transformation allows M' to be column stochastic

Deal with Spider Traps



$$\mathbf{M} = \begin{array}{c|c} & \begin{array}{ccccc} v & w & x & y & z \end{array} \\ \begin{array}{c} v \\ w \\ x \\ y \\ z \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 0 \end{bmatrix} \end{array}$$

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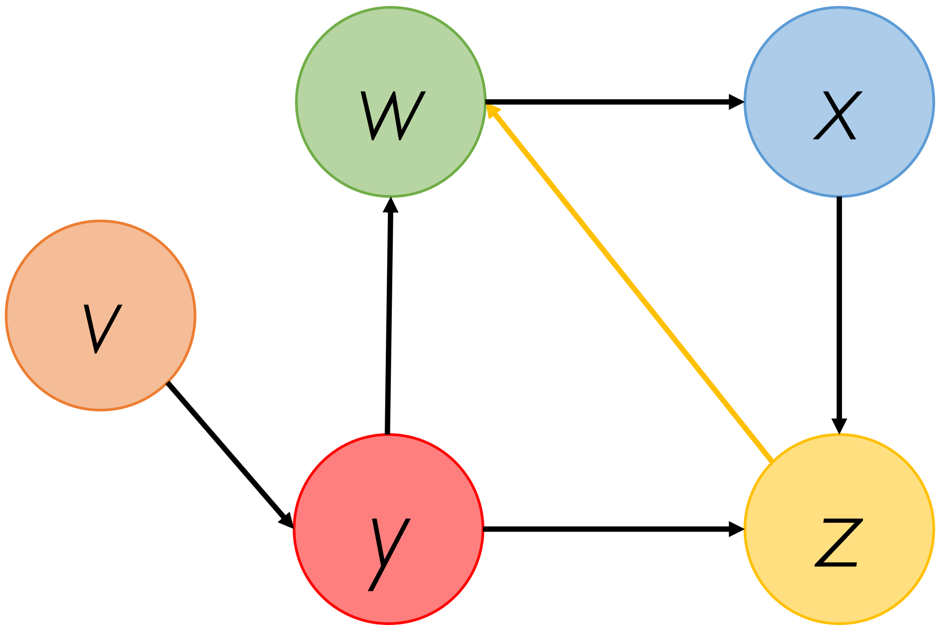

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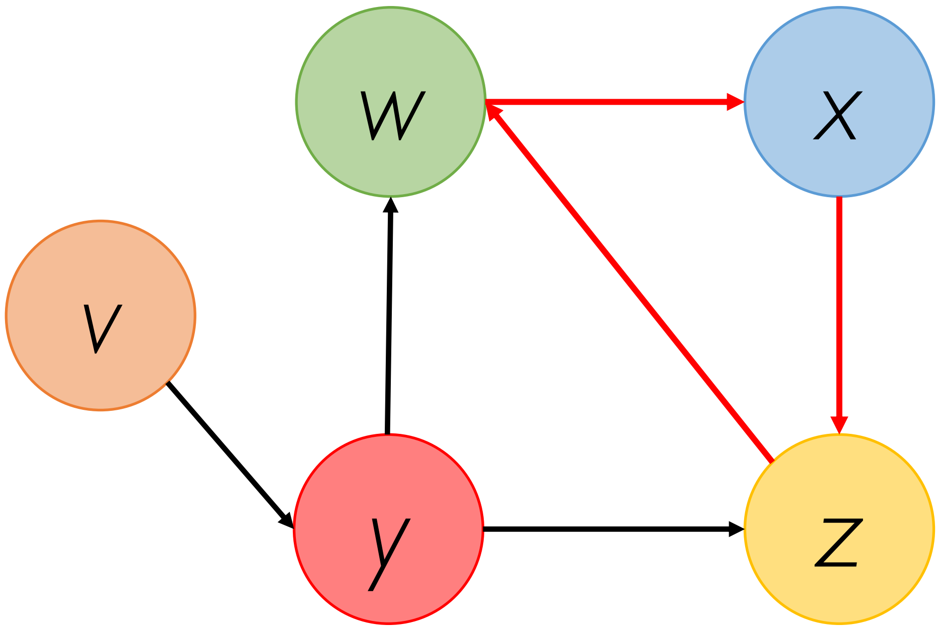
$$

z is not a dangling node anymore

M is (column) stochastic

Deal with Spider Traps



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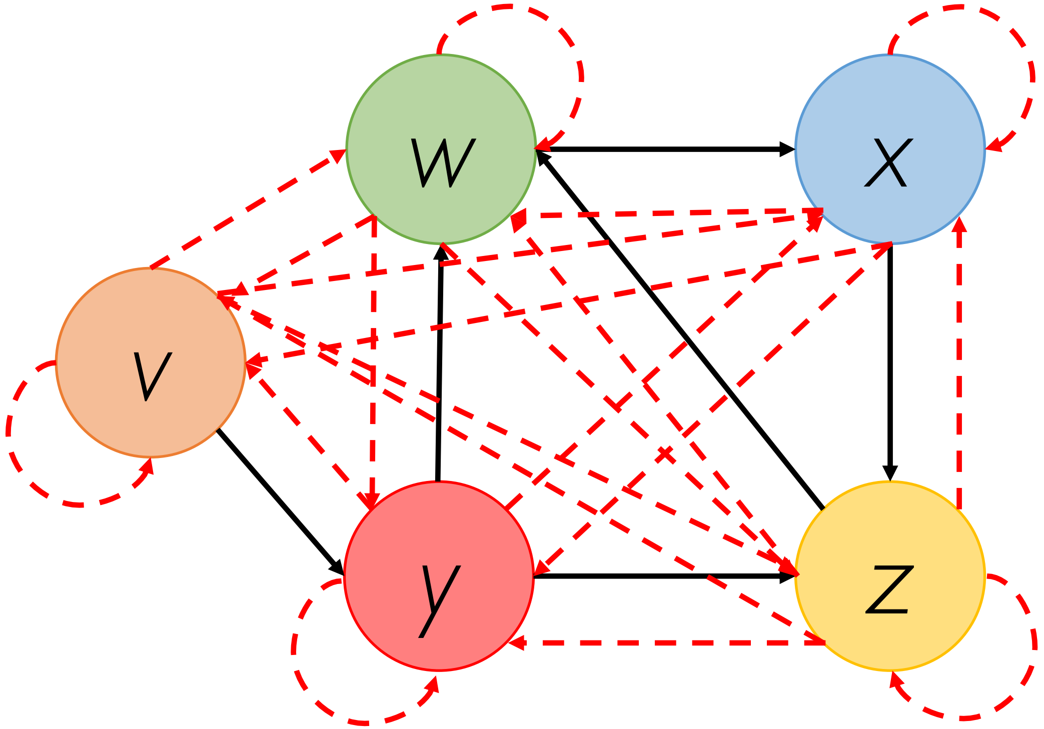
$$

If we apply simplified PageRank to M some entries of the rank vector **r** will eventually drop to 0, as we get stuck in *w*, *x*, *z*

Deal with Spider Traps: Teleporting (Again!)



$$\mathbf{M}' = \begin{array}{c} \\ v \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccccc} v & w & x & y & z \end{array} \\ \begin{bmatrix} 0.03 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.455 & 0.88 \\ 0.03 & 0.88 & 0.03 & 0.03 & 0.03 \\ 0.88 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.88 & 0.455 & 0.03 \end{bmatrix} \end{array}$$

Solution: Probabilistic Teleporting

Create artificial links from each node to every other node and follow each of it with probability (1-d)/N

Deal with Spider Traps: Probabilistic Teleporting

To avoid the surfer to get stuck in a spider trap

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On each page *w* the surfer will either follow one of its outgoing links with probability *d* or jump to another page with probability *(1-d)*

Deal with Spider Traps: Probabilistic Teleporting

To avoid the surfer to get stuck in a spider trap

On each page *w* the surfer will either follow one of its outgoing links with probability *d* **or** jump to another page with probability *(1-d)*

d is called **damping factor**

d = 0.85 in the original Google formulation

The Google's PageRank Formulation

$$\mathbf{M}_{N\times N} \quad m_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ 0 & \text{otherwise} \end{cases}$$

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$$\mathbf{G} = d\mathbf{M}' + \frac{1-d}{N} \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}}_{\mathbf{1}_{N\times N}}$$

$\boxed{\mathbf{M} \rightsquigarrow \mathbf{M}'}$

Ensure the matrix is stochastic

$\boxed{\mathbf{M}' \rightsquigarrow \mathbf{G}}$

Ensure the matrix is strictly positive

Why Does Teleporting Solve Our Problem?

$$\mathbf{G} = d\mathbf{M}' + \frac{1-d}{N} \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}}_{\mathbf{1}_{N \times N}}$$

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The Perron-Frobenius theorem now applies to G and guarantees the existence (convergence) and uniqueness of the steady-state eigenvector $\mathbf{r}^*$

$$\mathbf{r}(t) = \mathbf{G}^t\mathbf{r}(0)$$

$$\mathbf{r} \rightsquigarrow \mathbf{r}^* \text{ as } t \to \infty$$

How Do We Actually Compute PageRank?

$$\mathbf{r}(t+1) = \mathbf{G}\mathbf{r}(t)$$

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Easy if we have enough memory to store **G**, **r**(t+1), and **r**(t)

Problem:

G represents a fully-connected graph with a huge number of nodes (web pages)

G is a **dense** matrix

How Do We Actually Compute PageRank?

Assuming the number of web pages in the graph is $N=10^9$

G will have N^2 entries $= 10^{18}$

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Note: The Web contains far more than $N=10^9$ pages!

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$$\mathbf{r} = \mathbf{G}\mathbf{r} \qquad\qquad \mathbf{G}_{v,w} = d\mathbf{M}'_{v,w} + \frac{1-d}{N}$$

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$$r_v = \sum_{w=1}^{N} d\mathbf{M}'_{v,w} \times r_w + \sum_{w=1}^{N} \frac{1-d}{N} \times r_w$$

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$$\boxed{\mathbf{r} = d\mathbf{M}'\mathbf{r} + \left[\frac{1-d}{N}\right]_{N\times 1}} \qquad \begin{bmatrix} \frac{1-d}{N} \\ \frac{1-d}{N} \\ \vdots \\ \frac{1-d}{N} \end{bmatrix}$$

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Approximately 10 links per web page reduces the amount of memory required to store M' by a factor of 8 w.r.t. G (10^{10} vs. 10^{18} entries)

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We can work with M' rather than G

Re-Arrange the Equation

$$\boxed{\mathbf{r} = d\mathbf{M}'\mathbf{r} + \left[\frac{1-d}{N}\right]_{N\times 1}}$$

At each iteration we can compute PageRank vector as follows:

1. $\mathbf{r}(t+1) = d\mathbf{M}'\mathbf{r}(t)$
2. $\mathbf{r}(t+1) = \mathbf{r}(t+1) + \left[\frac{1-d}{N}\right]_{N\times 1}$

Add the constant *(1-d)/N* to each component of **r**(t+1)

PageRank: Pseudocode

Algorithm: PageRank

Input : A directed Web graph $G = (V, E)$, where $|V| = N$ and its associated matrix $\mathbf{M}_{N \times N}$ defined as follows: $\mathbf{M}_{v,w} = \frac{1}{o_w}$ if w points to v, 0 otherwise ($o_w = |O_w|$ where $O_w = \{x \in V : (w, x) \in E\}$);
A *damping factor* $d \in (0, 1)$;
A *tolerance* $\epsilon > 0$.

Output: The PageRank vector $\mathbf{r}^*_{N \times 1}$

Init : $t \leftarrow 0; \mathbf{r}(t) \leftarrow \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$;

repeat

- $t \leftarrow t + 1$;
- /* Compute the temporary PageRank score of every page v */
- **for** $i \leftarrow 1$ **to** N **do**
  - $r_v^{\mathrm{tmp}}(t) \leftarrow \sum_{w \in I_v} \frac{r_w(t-1)}{o_w}$; /* $r_v^{\mathrm{tmp}}(t) = 0$ if v has no in-links */
- **end**
- /* Adjust the PageRank score of each page v with *teleporting* */
- **for** $i \leftarrow 1$ **to** N **do**
  - $r_v(t) \leftarrow d \times r_v^{\mathrm{tmp}}(t) + \frac{1-d}{N}$;
- **end**

until $|\mathbf{r}(t) - \mathbf{r}(t-1)| < \epsilon$

return $\mathbf{r}^* = \mathbf{r}(t)$;

Take-Home Message of Today

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- We present an example of link analysis algorithm: PageRank
- Goal: Find an importance score associated with each web page
- Represent the Web graph as a matrix M, where a link from page *w* to *v* is a vote from *w* to *v*
- 2 different yet equivalent approaches:
  - Linear Algebra → Matrix eigenvector
  - Probabilistic → Stationary distribution of Markov chain (random walk)

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- The existence (convergence) and uniqueness of PageRank is guaranteed only for certain matrices M (Perron-Frobenius theorem)
- The Web graph is disconnected and may contain no-exit loops
- Google solution: probabilistic teleport links
- Still efficiently computable from the original, sparse matrix M