Teoria degli Algoritmi

Corso di Laurea Magistrale in Matematica Applicata

a.a. 2020-21

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Lecture 5: Complexity

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- Even though a problem is decidable, and thus computationally solvable in theory, it may not be solvable **in practice**
- The reason of such **intractability** is related to the extraordinary amount of resources (mostly, time and space/memory) required by a solution to the problem
- We now delve into the realm of **computational complexity theory**, which investigates the time, space/memory, and any other resource needed to solve a computational problem

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- After that, we discuss the possibility that certain decidable problems need a huge amount of time therefore making them intractable in practice
- Finally, we try to determine when we are facing with such intractable problems

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- Obviously, A is a **decidable language**, i.e., we can build a TM that decides A
- How much time does a single-tape TM need to decide A?
- To answer the question above, let us describe a TM M_A that decides A, so that we can count the **number of steps** it takes

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Example (A decider for $A = \{0^k1^k \mid k \geq 0\}$)

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- For the sake of simplicity, we compute the **running time** of an algorithm purely as a function of the string representing the input
- In **worst-case analysis**, which we focus on here, we consider the longest running time of **all** inputs of a fixed size
- In **average-case analysis**, instead, we consider the average of all the running times of **all** inputs of a fixed size

The Definition of Running Time

Definition (Running Time (Time Complexity))

Let M be a deterministic Turing machine that halts on all inputs (i.e., a decider).

The **running time** (or **time complexity**) of M is the function:

$$f : \mathbb{N} \mapsto \mathbb{N}$$

where $f(n)$ is the **maximum** number of steps that M needs to perform in order to halt on **any** input of size n.

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- We do not need to be extremely accurate, and an estimation of the running time is sufficient
- A convenient form of estimation is called **asymptotic analysis**
- Such an estimation tries to capture the running time of an algorithm when this is input with **large size** inputs

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Example

Let $f(n) = 6n^3 + 2n^2 + 20n + 45$. This is a 4-term polynomial expression, and the highest order term is $6n^3$. Disregarding the coefficient 6, we say that $f(n)$ is **asymptotically** at most n^3

Big-O: Definition

- We formalize the **asymptotic notation** (or **Big-O notation** as follows:

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Definition (Big-O)

Let f, g be two functions such that $f, g : \mathbb{N} \mapsto \mathbb{R}_{\geq 0}$. We say that $\mathbf{f(n) = O(g(n))}$ if there exist $c, n_0 \in \mathbb{Z}^+$, such that for every $n \geq n_0$:

$$f(n) \leq cg(n)$$

When $f(n) = O(g(n))$ we say that $g(n)$ is an **upper bound** for $f(n)$, or more precisely, that $g(n)$ is an **asymptotic upper bound** for $f(n)$, to stress the fact that we are not considering any constant factor.

Big-O: Intuition

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- We can think of O as representing a suppressed constant
- In practice, most functions f that we will encounter have an obvious highest order term

Big-O: Examples

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Let $f(n) = 5n^3 + 2n^2 + 22n + 6$.

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Thus, selecting the highest order term ($5n^3$) and disregarding its coefficient (5) gives us $f(n) = O(n^3)$.
Let's verify that this result satisfies our formal definition of asymptotic upper bound. We must find c and n_0, such that:

$$\underbrace{5n^3 + 2n^2 + 22n + 6}_{f(n)} \leq cn^3 \ \forall n \geq n_0$$

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For $c = 6$ and $n_0 = 10$, it holds that:

$$5n^3 + 2n^2 + 22n + 6 \leq 6n^3 \ \forall n \geq 10$$

Big-O: Observations

Note

In the example above, it also holds (trivially) that $f(n) = O(n^4)$, because $n^4 \geq n^3$ and so it is still an asymptotic upper bound of f.

Conversely, $f(n) \neq O(n^2)$: regardless of the values we assign to c and n_0, the condition we seek for according to the definition remains unsatisfied

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- When we write $f(n) = O(\log n)$ we don't need to specify the base as constant factors are suppressed anyway

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- Similarly, when O occurs at the exponent like $f(n) = 2^{O(n)}$ the same idea applies
- Here, the expression represents an upper bound of 2^{cn} for some constant c

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- Notice that $n = 2^{\log_2 n}$, and therefore $n^c = 2^{c \log_2 n}$
- As such, $2^{O(\log n)}$ represents an upper bound for n^c for some constant c
- Finally, the expression $f(n) = O(1)$ is a "convention" to represent a value that is never more than a constant (i.e., it does not depend on n)

Big-O vs. small-o

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- Big-O notation has a companion called **small-o**
- Intuitively, Big-O notation says that one function is **asymptotically no greater than** another
- Instead, we use small-o to indicate that a function is **asymptotically less than** another
- The difference between Big-O and small-o is the same as the difference between $\leq$ and $<$

small-o: Definition

Definition (small-o)

Let f, g be two functions such that $f, g : \mathbb{N} \mapsto \mathbb{R}_{\geq 0}$. We say that $\mathbf{f(n) = o(g(n))}$ if:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$

In other words, $f(n) = o(g(n))$ means that, for any real number $c > 0$ there exists a number n_0, such that $f(n) < cg(n)$ for all $n \geq n_0$.

Big-O vs. small-o: Example

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Let's go back to $f(n) = 5n^3 + 2n^2 + 22n + 6$.
We already know that $f(n) = O(n^3)$, i.e., we have found there **exist** $c = 6$ and $n_0 = 10$, such that:

$$f(n) = 5n^3 + 2n^2 + 22n + 6 \leq cn^3 \ \forall n \geq n_0$$

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We already know that $f(n) = O(n^3)$, i.e., we have found there **exist** $c = 6$ and $n_0 = 10$, such that:

$$f(n) = 5n^3 + 2n^2 + 22n + 6 \leq cn^3 \ \forall n \geq n_0$$

However, $f(n) \neq o(n^3)$ as we can find a value c for which **there is no** n_0 such that $f(n) < cg(n)$ for all $n \geq n_0$, e.g., $c = 5$:

$$5n^3 + 2n^2 + 22n + 6 \not< 5n^3 \ \forall n \geq n_0$$

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Let's analyze the algorithm implemented by the TM that decides the language $A = \{0^k 1^k \mid k \geq 0\}$

Example (A decider for $A = \{0^k 1^k \mid k \geq 0\}$)

$M_A =$ "On input string x:

1. Scan across the tape and if a 0 occurs to the right of a 1, **reject**;
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- In stage 1, the machine scans across the tape to verify that the input is of the form 0^*1^*

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- Using Big-O notation, that means $O(n)$ steps

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- In stages 2 the machine scans across the tape to verify if any 0s or 1s are left on the tape
- This is the test M_A needs to take to check whether it should enter the loop or not
- Since this involves a full scan of the tape, it costs n steps

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- Each iteration is composed by two full scans (i.e., one for testing the loop condition and the other for crossing off symbols): $2n = O(n)$
- Overall, $n/2 * O(n) = O(n^2)$ steps

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- In stage 4, the machine makes a full scan to decide whether it should accept or reject the input
- The time taken by this stage is again $O(n)$
- Thus, the total time of execution of M_A on an input x whose length is n is:

$$O(n) + O(n^2) + O(n) = O(n^2)$$

Time Complexity Class: A Formal Definition

Definition (The class TIME(t(n)))

Let $t : \mathbb{N} \mapsto \mathbb{R}_{\geq 0}$ be a function.
We define the **time complexity class TIME(t(n))** as the collection of all languages that are decidable by an $O(t(n))$ time Turing machine

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Is there a TM that decides the language A asymptotically more quickly?
In other words, is $A \in TIME(t(n))$, where $t(n) = o(n^2)$?

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- That would improve the running time by a factor of 2 since we would need to do $n/4$ iterations rather than $n/2$
- Unfortunately, this won't affect the asymptotic running time (Remember: constant factors do not count!)

Analyzing Algorithms: An Example

We can design another TM M'_A that still decides A, yet it shows that $A \in TIME(n \log n)$

Example (Another decider for $A = \{0^k 1^k \mid k \geq 0\}$)

M'_A = "On input string x:

1. Scan across the tape and if a 0 occurs to the right of a 1, **reject**;
2. Repeat as long as some 0s and 1s are left on the tape:
   - a. Scan across the tape, checking if the total number of 0s and 1s remaining is even or odd: if it is odd **reject**;
   - b. Scan again across the tape, crossing off every 0 (resp. 1) in alternating way, starting from the first 0 (resp. 1) encountered;
3. If no 0s (resp. 1s) are left on the tape, **accept**; otherwise, **reject**."

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- On every scan performed in stage 2.b, the number of 0s (resp., 1s) is cut in half and any reminder discarded

Analyzing Algorithms: An Example

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- The total time of stages 2, 2.a, and 2.b is: $(1 + \log_2 n) * O(n) = O(n \log n)$
- Putting everything together, the total running time of M'_A is: $O(n) + O(n \log n) = O(n \log n)$

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- Earlier on, we have shown that $A \in TIME(n^2)$ but now we have a better upper bound, i.e., $A \in TIME(n \log n)$
- This result **cannot** be improved on a single-tape TM
- However, we can decide the language A in $O(n)$ time (a.k.a. **linear time**) if the TM has a second tape

Analyzing Algorithms: An Example

The following two-tape TM M''_A decides A in $O(n)$ time

Example (Another decider for $A = \{0^k1^k \mid k \geq 0\}$)

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4. If we reached the end of the input and all 0s have been crossed off, **accept**; if any 0s remain, **reject**."

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- Each of the 4 stages uses $O(n)$ steps, so the total running time is $O(n)$
- Note that this is the best possible running time because n steps are necessary at least to read the input!

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- We started from a single-tape TM M_A that decides A in $O(n^2)$ time
- Then, we provide an improved version of the TM above, M'_A, which is able to decide A in $O(n \log n)$ time
- Finally, we exhibit a two-tape TM M''_A which decides A in $O(n)$ time (i.e., linear time)
- It turns out that the time complexity of A depends on the characteristics of the model of computation

Complexity vs. Computability

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Complexity vs. Computability

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- In computability theory, the Church-Turing thesis guarantees that all *reasonable* models of computation are **equivalent**, i.e., they all decide the same class of languages
- In complexity theory, the choice of the actual model of computation affects the time complexity of languages
- Languages may be decidable in, say, linear time using one model but not necessarily on other models

Table of Contents

Complexity Relationships Among Models

- We examine how the choice of computational model may affect the time complexity of languages

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- We examine how the choice of computational model may affect the time complexity of languages
- To achieve that, we consider **3 models of computation**:
  1. single-tape TM
  2. multi-tape TM
  3. non-deterministic TM

Single- vs. Multi-Tape TM Time Complexity

Theorem

Let $t(n)$ be a function where $t(n) \geq n$. Then every $t(n)$ multi-tape TM has an equivalent $O(t^2(n))$ time single-tape TM.

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Proof.

We already showed how to convert any multi-tape TM M into a corresponding single-tape TM S that simulates it. The sketch of the proof is to demonstrate that simulating each step of M on S requires at most $O(t(n))$ steps. Hence, the total time used by S is $O(t^2(n))$ □

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- Initially, S puts its tape into the format that represents all the k tapes of M, then starts simulating M's steps

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- Basically, S has to shift a portion of its tape to the right

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- Each of these active portions has length at most $t(n)$ because M uses $t(n)$ tape cells in $t(n)$ steps if the head just moves rightward at every single step (**Remember:** M runs in $t(n)$ steps)

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- A scan of the active portion of S's tape takes $O(t(n))$ steps

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- In total, we have $O(n) + O(t^2(n))$ steps; since we have assumed $t(n) \geq n$, the running time of S is $O(t^2(n))$

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- Remember that a non-deterministic TM is a decider if **all** its computation branches halt on all inputs.

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Definition

Let N be a non-deterministic TM that is a decider. The **running time** of N is given by the function $f : \mathbb{N} \mapsto \mathbb{N}$, where $f(n)$ is the **maximum number of steps** that N uses on **any** branch of its computation on any input of length n.

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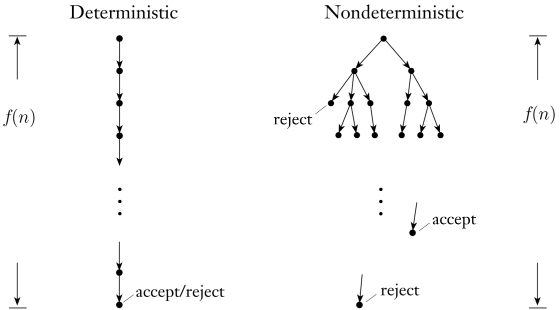

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- It follows that the total number of leaves of the tree is at most $b^{t(n)}$

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Deterministic vs. Non-Deterministic TM Time Complexity: Proof

Internal Nodes vs. Leaves

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The expression above is a finite geometric series $\sum_{i=0}^{h-1} ar^i$, where $a = 1$ and $r = b$, whose closed-form solution is $n = \frac{(1-b^h)}{(1-b)}$

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Since $b >= 2$ the total number of internal nodes in is less than twice the number of leaves

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- Therefore, the running time of D is $O(t(n)b^{t(n)})$
- Notice that $b^x = (\underbrace{c^{\log_c b}}_{b})^x$; if we set $x = t(n)$ and $c = 2$, we obtain:

$$b^{t(n)} = (2^{\log_2 b})^{t(n)} = 2^{\log_2 b * t(n)} = 2^{k*t(n)}, \text{ where } k = \log_2 b$$

$$b^{t(n)} = 2^{O(t(n))}$$

Deterministic vs. Non-Deterministic TM Time Complexity: Proof

- The time for starting from the root and traveling down to a node is $O(t(n))$
- Therefore, the running time of D is $O(t(n)b^{t(n)})$
- Notice that $b^x = (\underbrace{c^{\log_c b}}_{b})^x$; if we set $x = t(n)$ and $c = 2$, we obtain:

$$b^{t(n)} = (2^{\log_2 b})^{t(n)} = 2^{\log_2 b * t(n)} = 2^{k * t(n)}, \text{ where } k = \log_2 b$$

$$b^{t(n)} = 2^{O(t(n))}$$

- Overall, the running time complexity of D is $O(t(n)) * 2^{O(t(n))} = 2^{O(t(n))}$

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$$(2^{O(t(n))})^2 = 2^{O(2t(n))} = 2^{O(t(n))}$$

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- It is for this reason that, for larger problems and larger inputs, the difference between using a single-tape TM and a multi-tape TM is negligible
- All reasonable deterministic computational models are **polynomially equivalent**; this means that any one of them can simulate another with only a polynomial increase in running time