Teoria degli Algoritmi

Corso di Laurea Magistrale in Matematica Applicata

a.a. 2020-21

SAPIENZA
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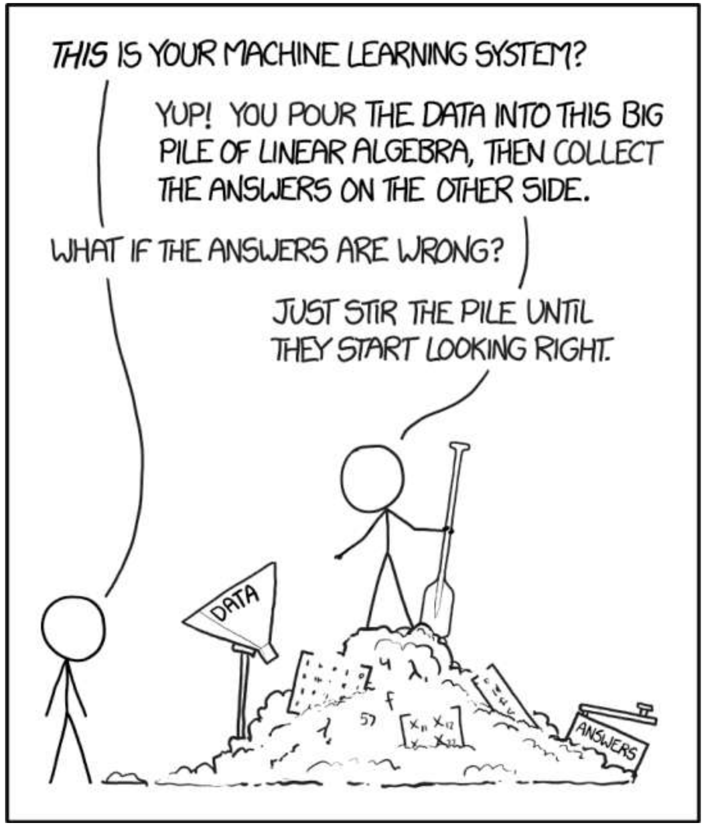
How Much Data Do We Need?

In general, the more data we have the better we learn



source: https://xkcd.com/1838/

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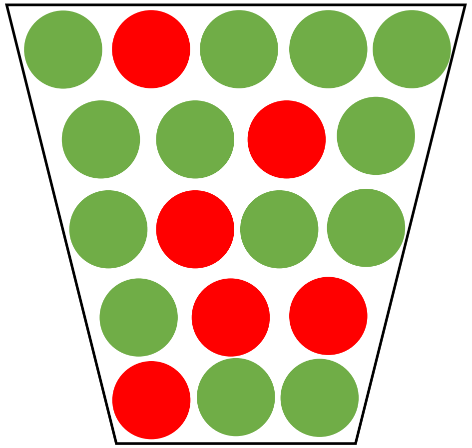
- We only dispose of a finite data sample (i.e., the training set) where we know the value of the unknown function
- Outside of that, the function may take on any value!
- **Question:** Can we use our finite sample to learn something outside of it?

A Related Experiment

Consider a bin with red and green marbles



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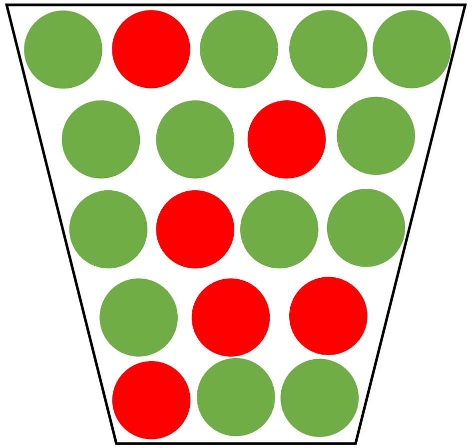

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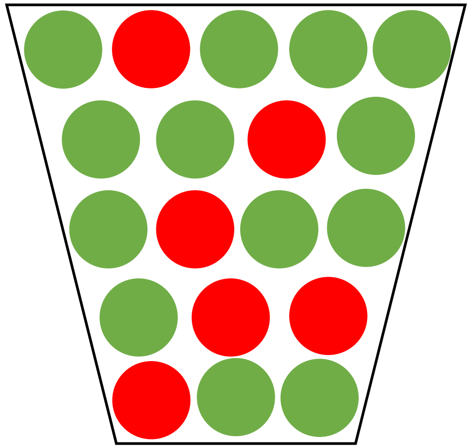

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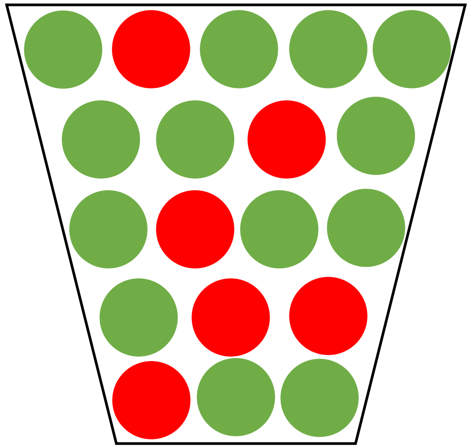

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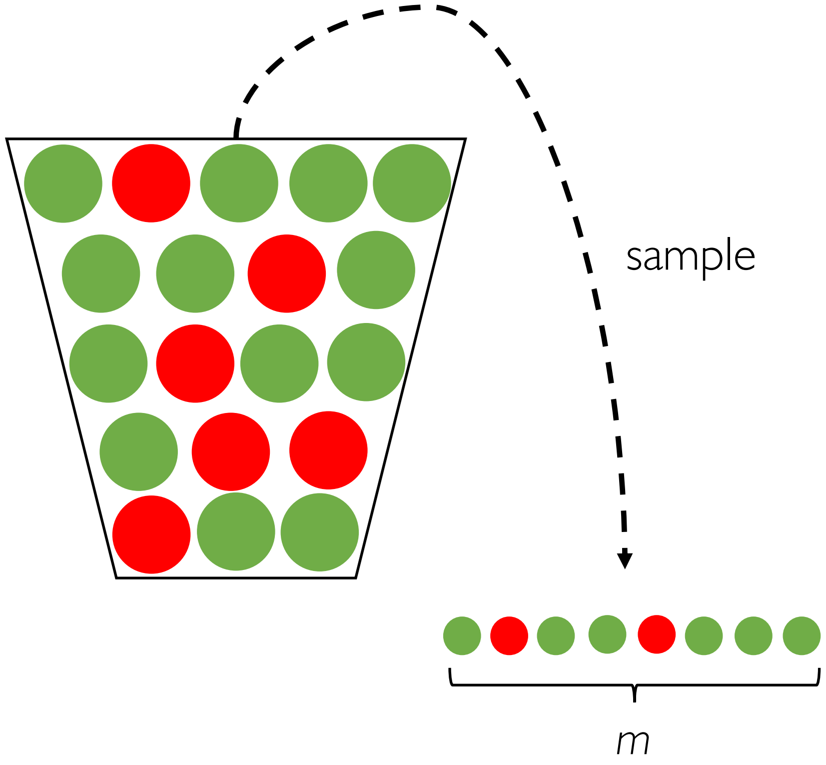

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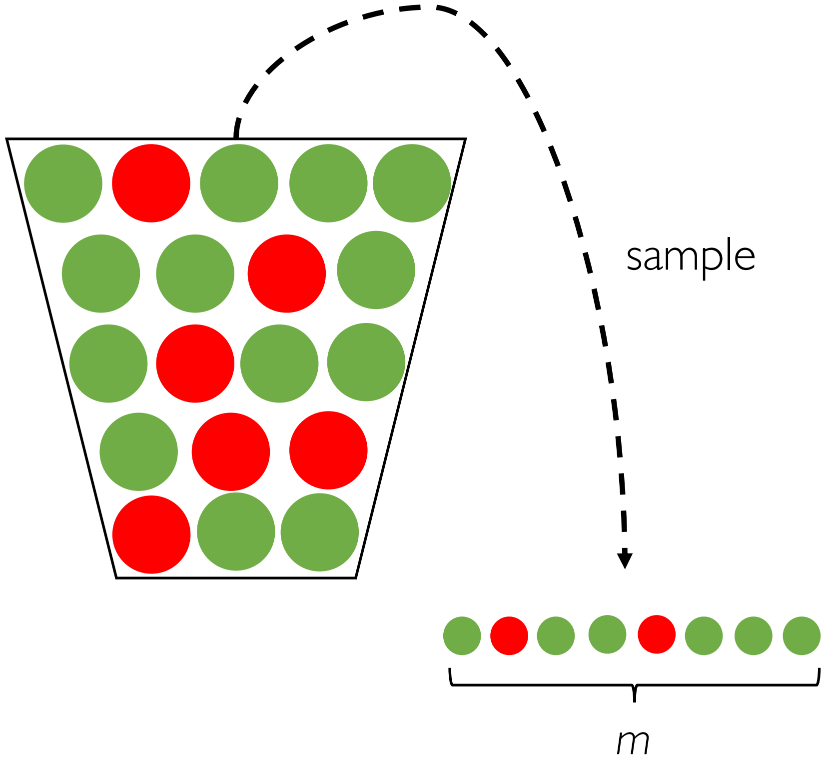

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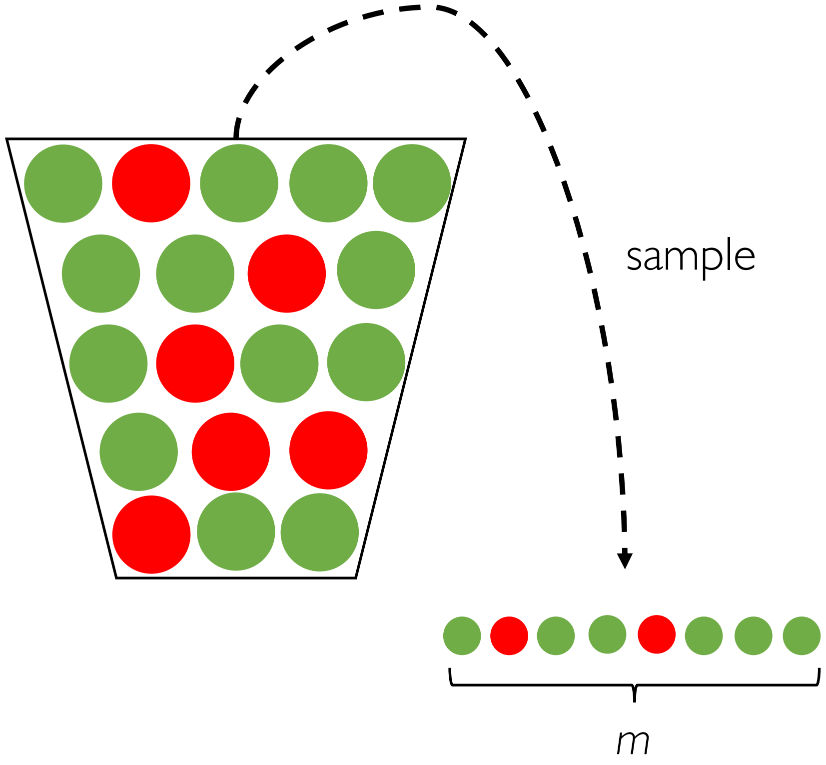
Note:
The bin can be considered either infinite or the sampling being done with replacement

A Related Experiment

Does p' say something about p?



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Short Answer: NO!

Our sample could be made of all green marbles even though the bin mostly contains red ones

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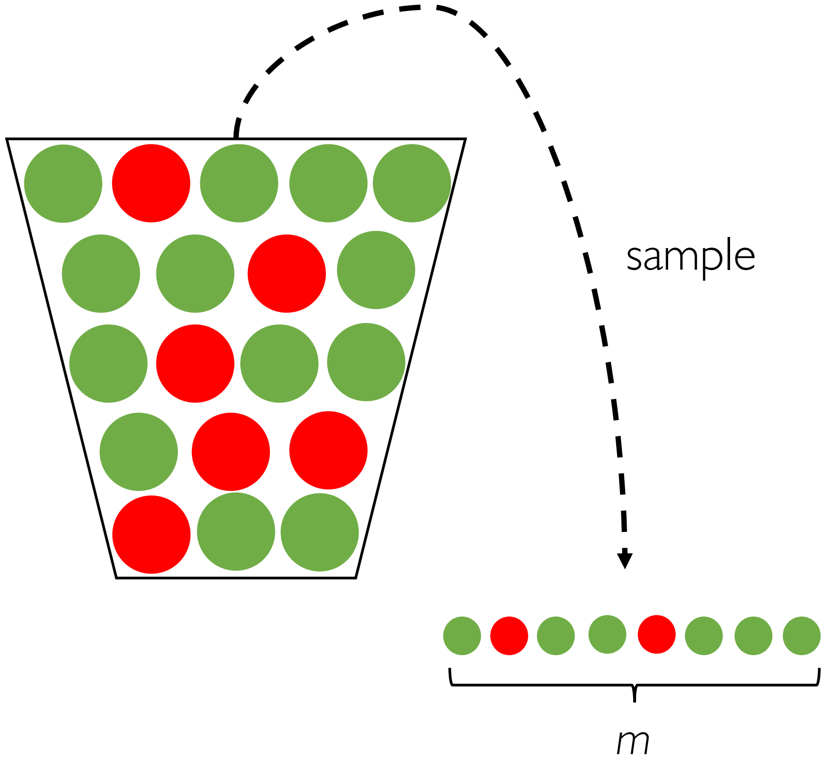

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If the sample is "big enough" (m is "large"), sample frequency p' is likely close to the true bin frequency p

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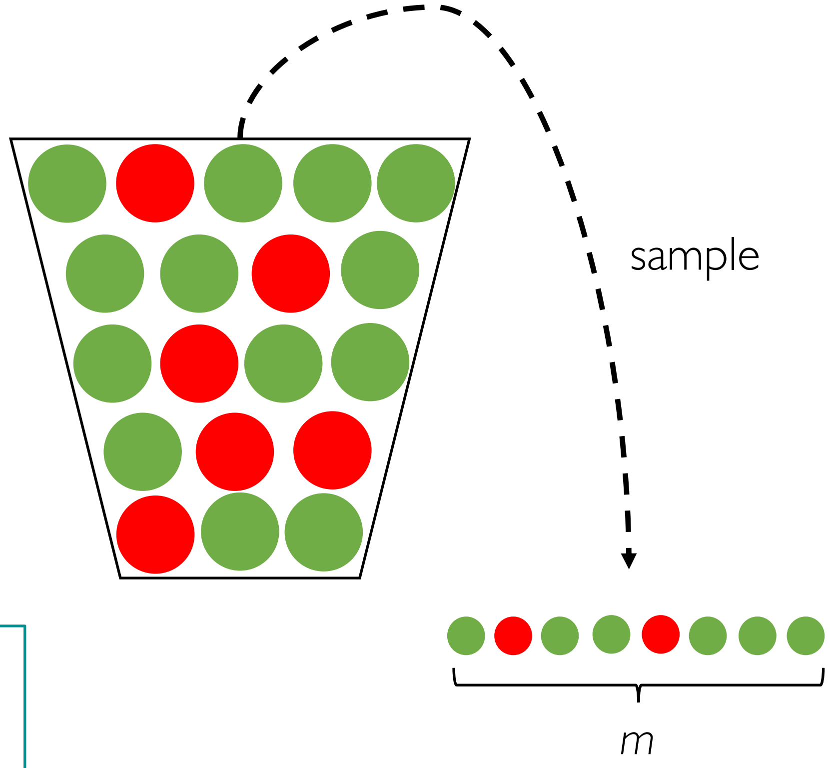

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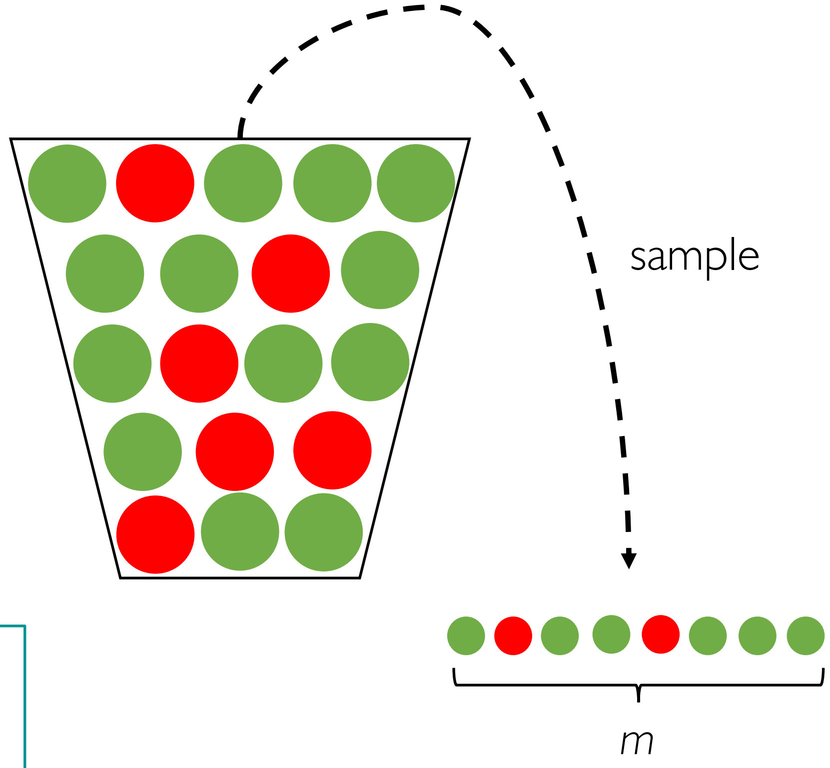
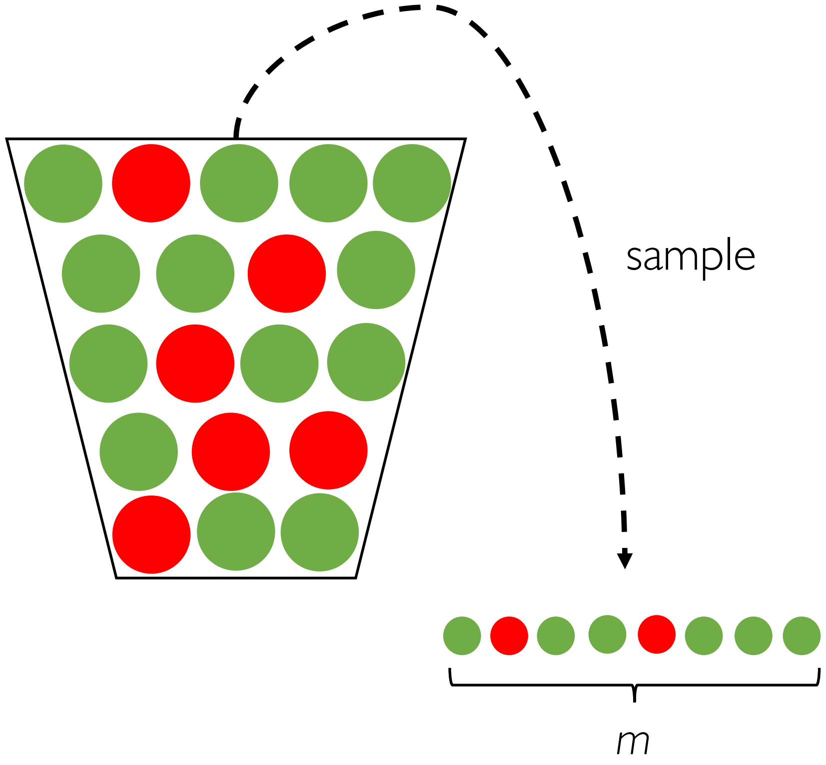


But what does p' say about p, exactly?

A Related Experiment

In a big sample (large *m*), *p*' is probably close to *p* (within *epsilon*)



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Hoeffding's Inequality

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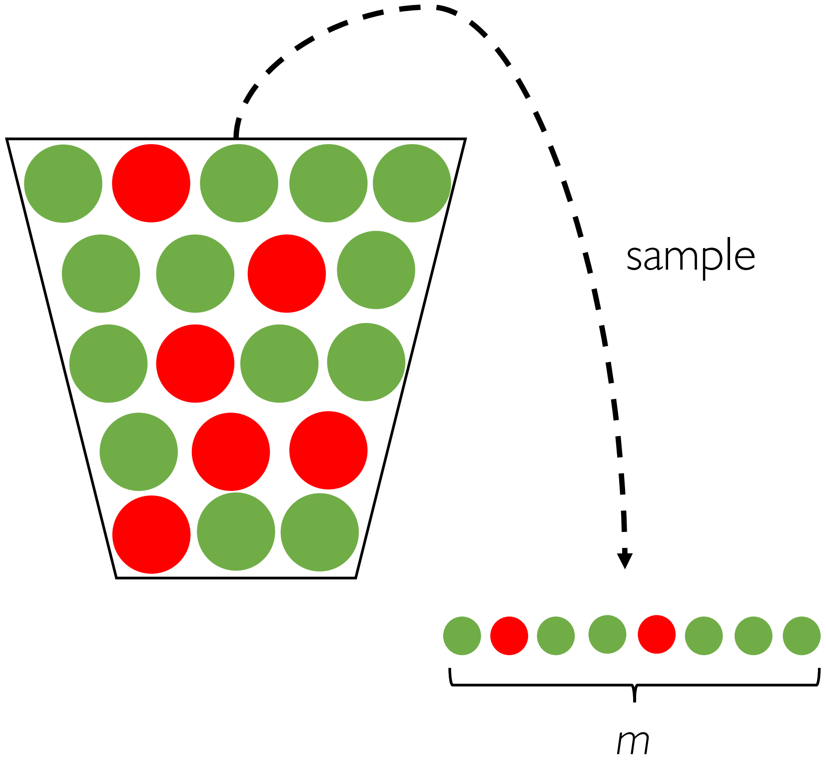

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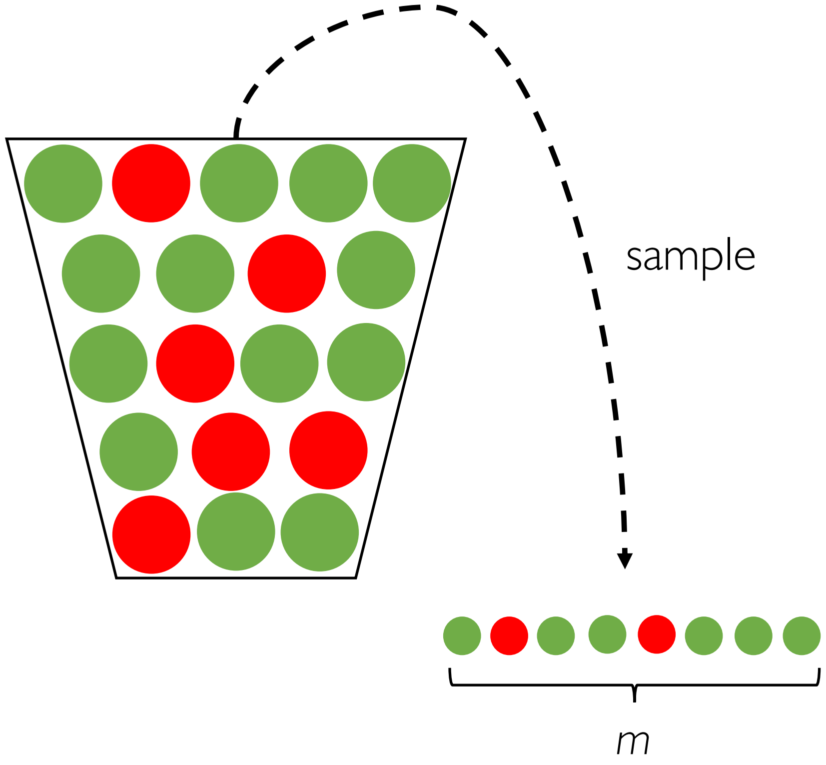

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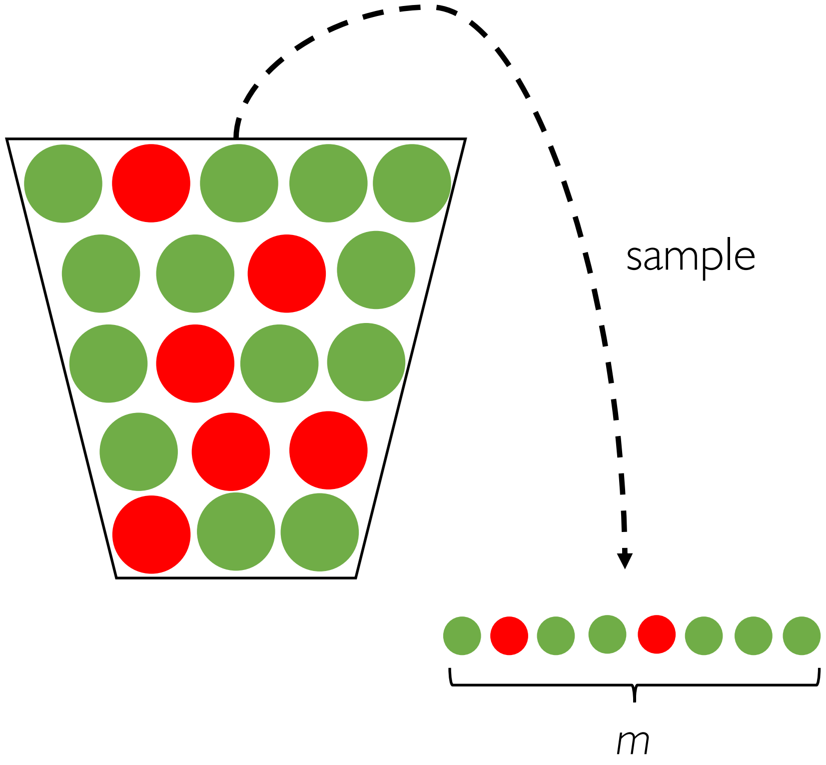

The "bad event" is *p*' deviating more than *epsilon* from the true *p*

We want the probability of such bad event to be small!

Hoeffding's Inequality

$$P(|p' - p| > \epsilon) \leq 2e^{\boxed{-2m}\epsilon^2}$$

The presence of *m* as a negative exponent contributes to keep the right-hand expression small (as *m* increases)

Hoeffding's Inequality

$$P(|p' - p| > \epsilon) \leq 2e^{-2m\boxed{\epsilon^2}}$$

Wait! *m* is multiplied by *epsilon* squared and therefore its effect as negative exponent gets diluted as *epsilon* gets smaller (i.e., the closer we want *p*' to the real *p*)

Hoeffding's Inequality

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There is a tradeoff between the sample size (*m*), the tolerance (*epsilon*), and the bound

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$$p' \overset{\text{PAC}}{=} p$$

The statement above is Probably Approximately Correct

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- It is valid for every positive integer *m* and every *epsilon* > 0
- Bound does not depend on *p* because it is an unknown quantity, and *epsilon* just represents our tolerance

Connection to the Learning Problem

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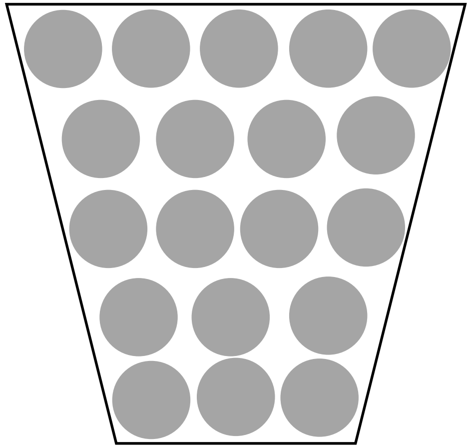
- How does PAC statement guaranteed by the Hoeffding's inequality help us to determine whether learning is feasible?
- We must make a link between the bin example and learning
- In the bin example, the unknown quantity we want to estimate is a single number p, i.e., the frequency of red marbles in the bin
- In the learning problem, the unknown quantity we want to estimate is a full-fledged target function f: X → Y

Connection to the Learning Problem

- The bin can be seen as the whole input space X
- Each marble is a single data point x in X



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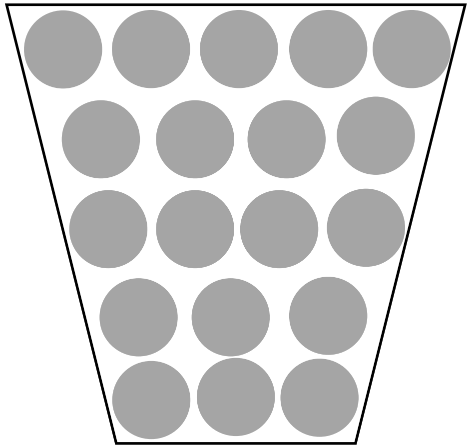
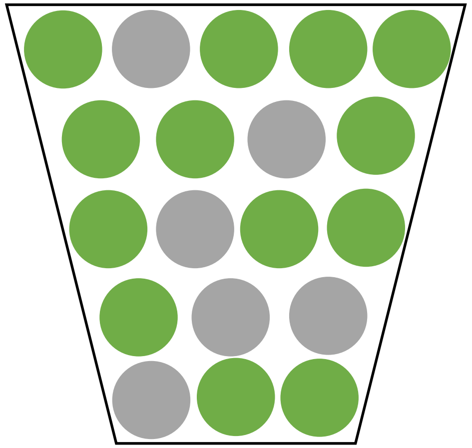


How do we color each marble?

Connection to the Learning Problem

green marbles: correspond to data points where a given hypothesis h agrees with the true unknown target function f



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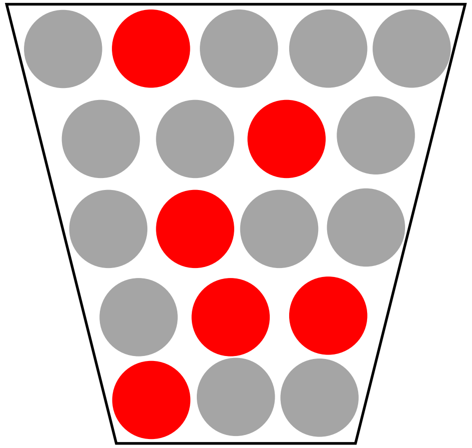
red marbles: correspond to data points where a given hypothesis h disagrees with the true unknown target function f



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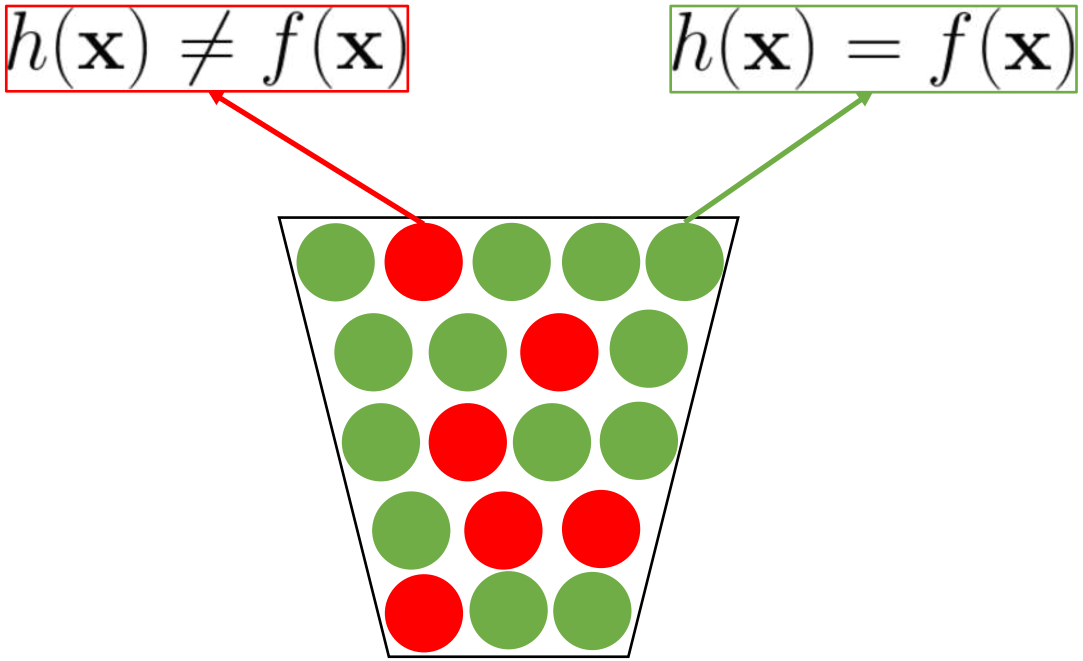
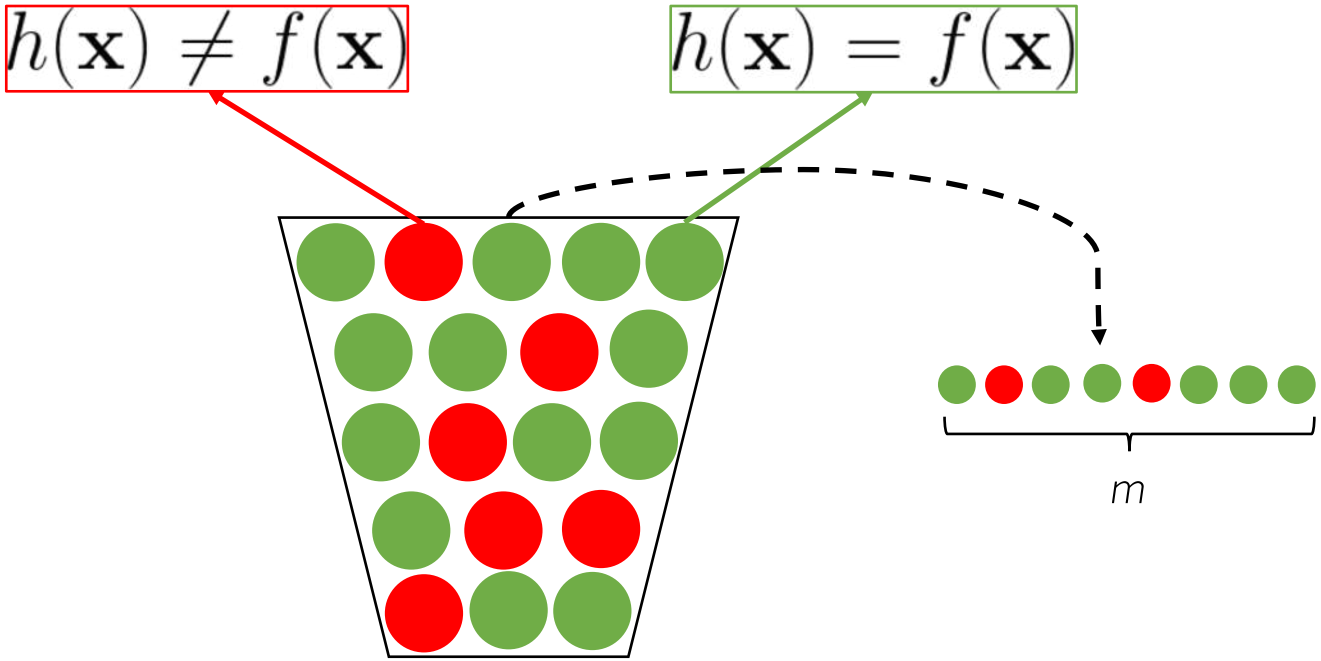


We introduce a probability distribution P over the input space X
There is no need to know what P is and no restriction on P

Are We Done?



For this specific h, p' (in-sample error) is PAC equivalent to p (out-of-sample error)

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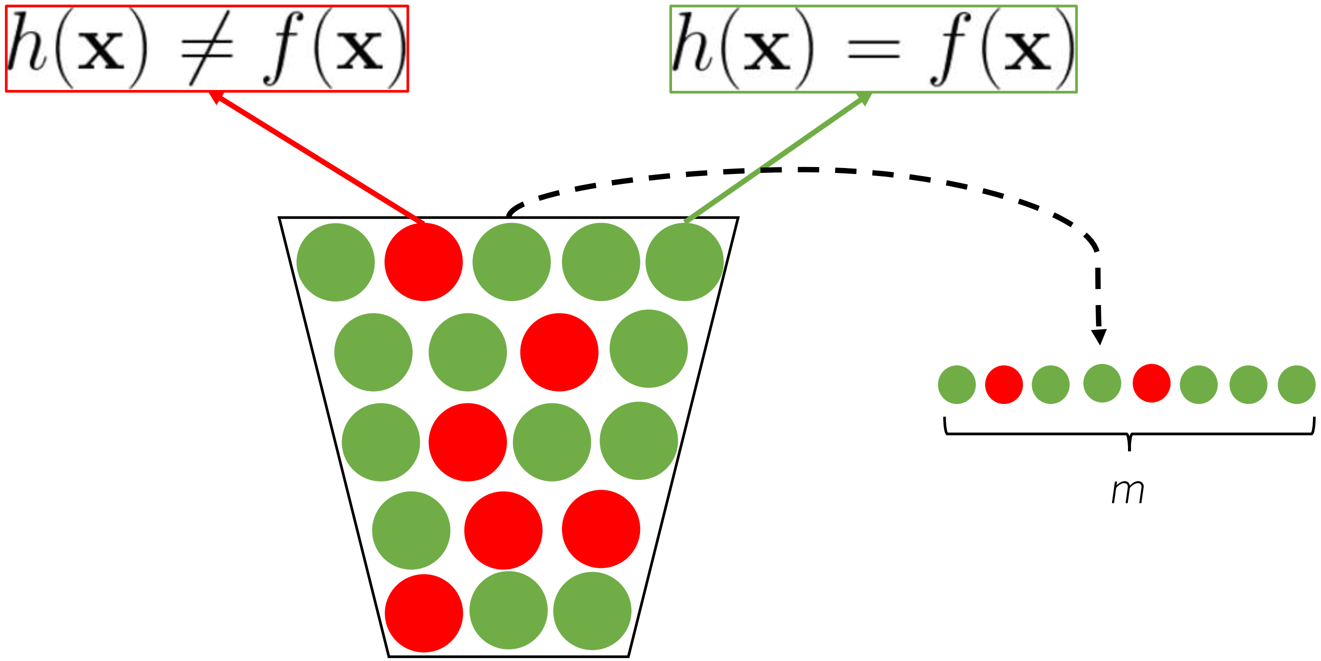
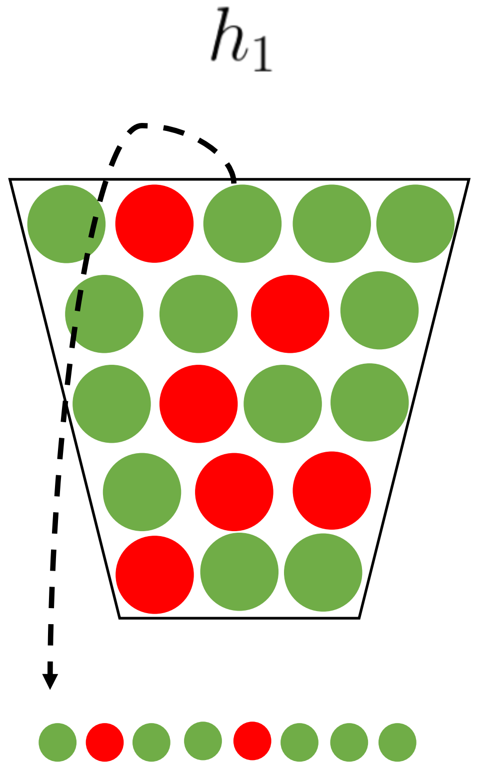


The problem here is that we fixed h

We have verified h rather than learning it from many different hypotheses

Generalize to Multiple Bins



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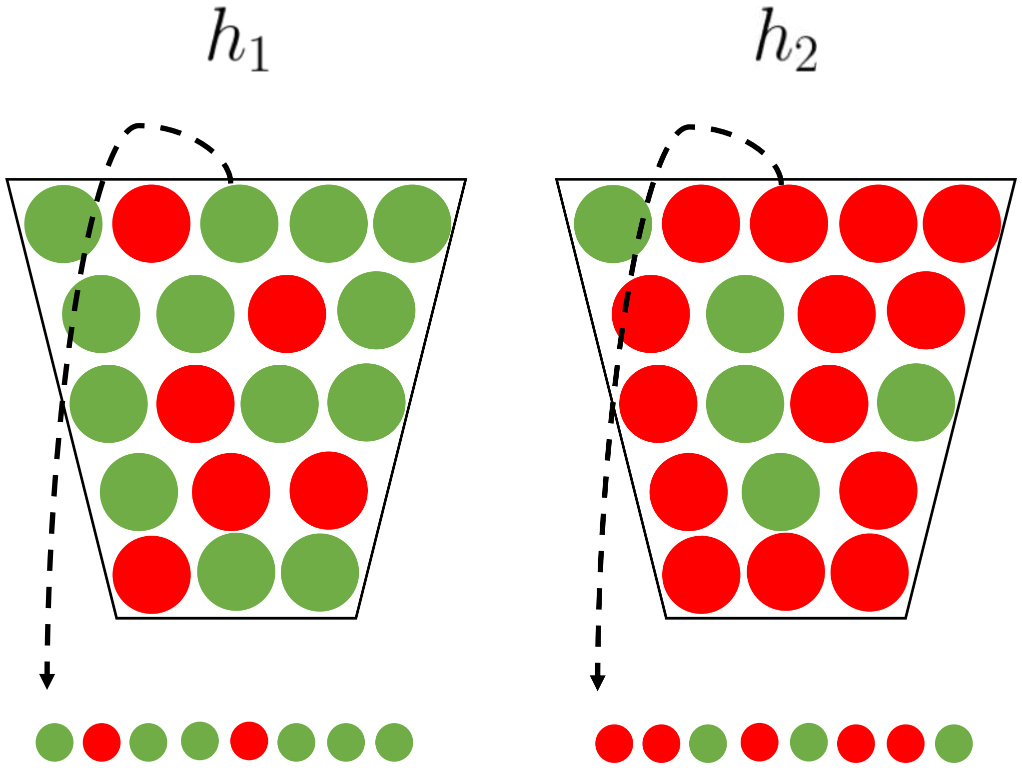

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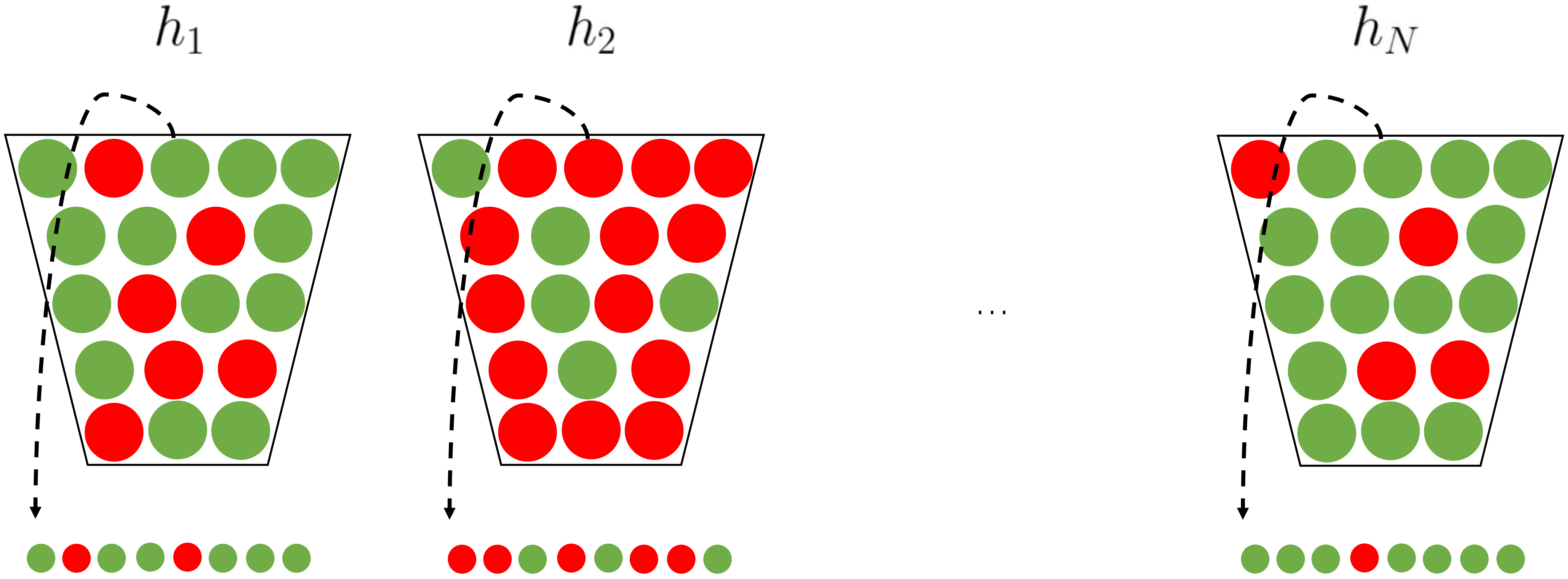

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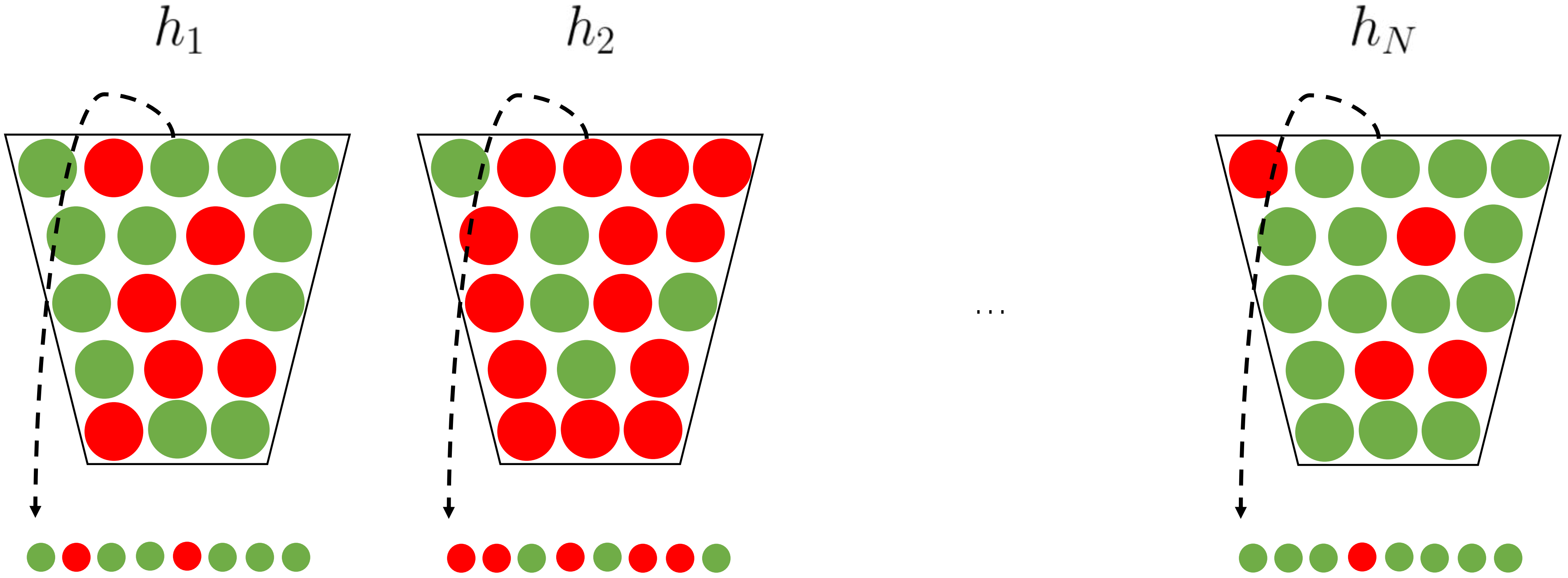
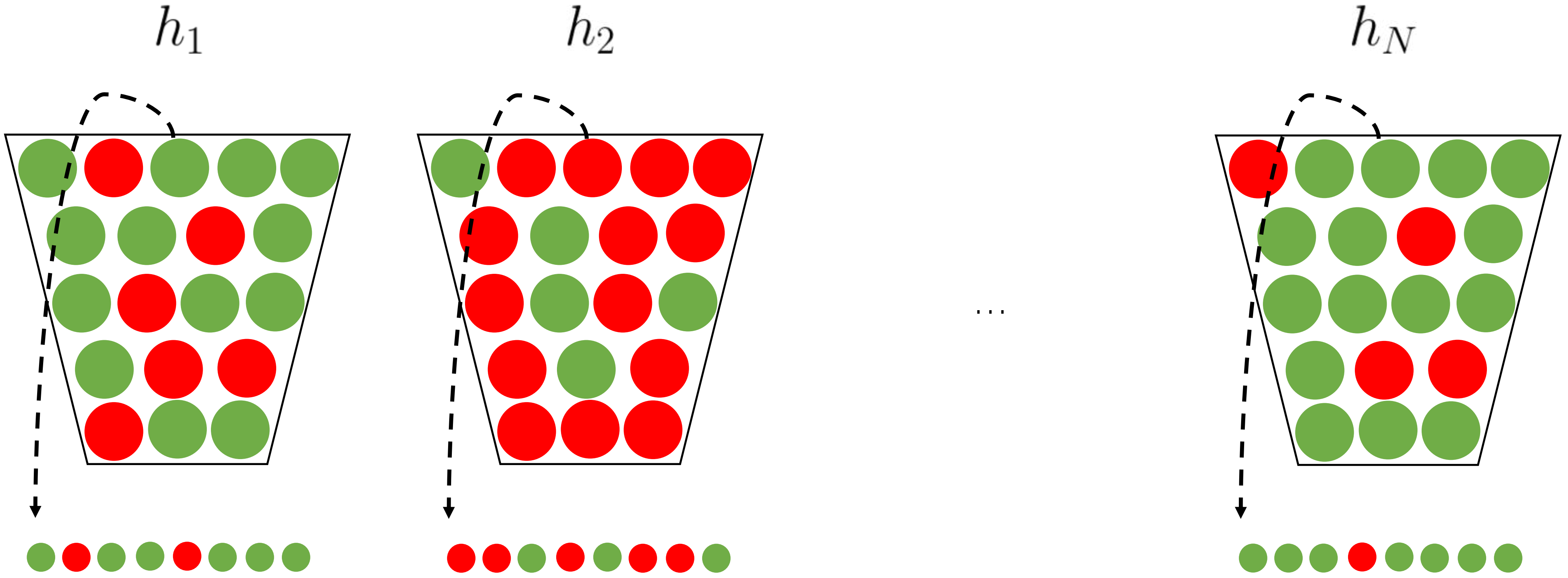


We must inspect samples generated under every hypothesis and pick the most "favorable" one

Generalize to Multiple Bins



Intuitively, this means scanning through all the *N* samples and selecting the one with the smallest value of p' (sample frequency of red marbles)

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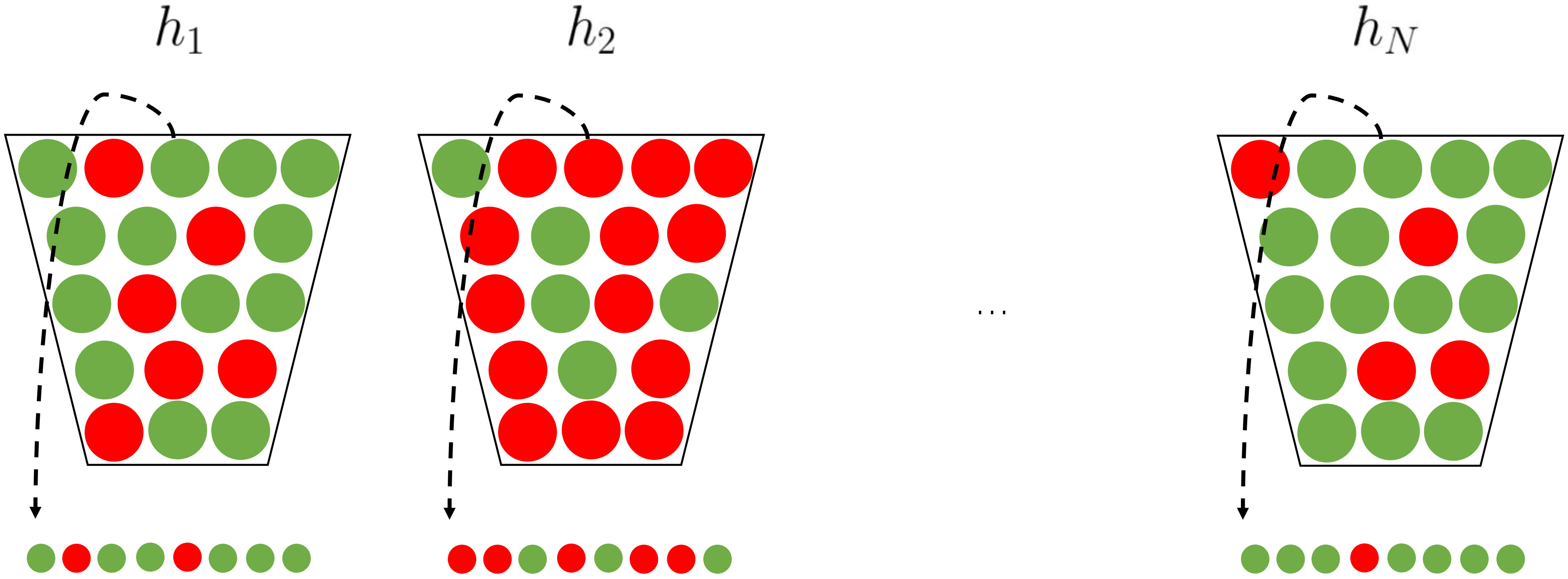
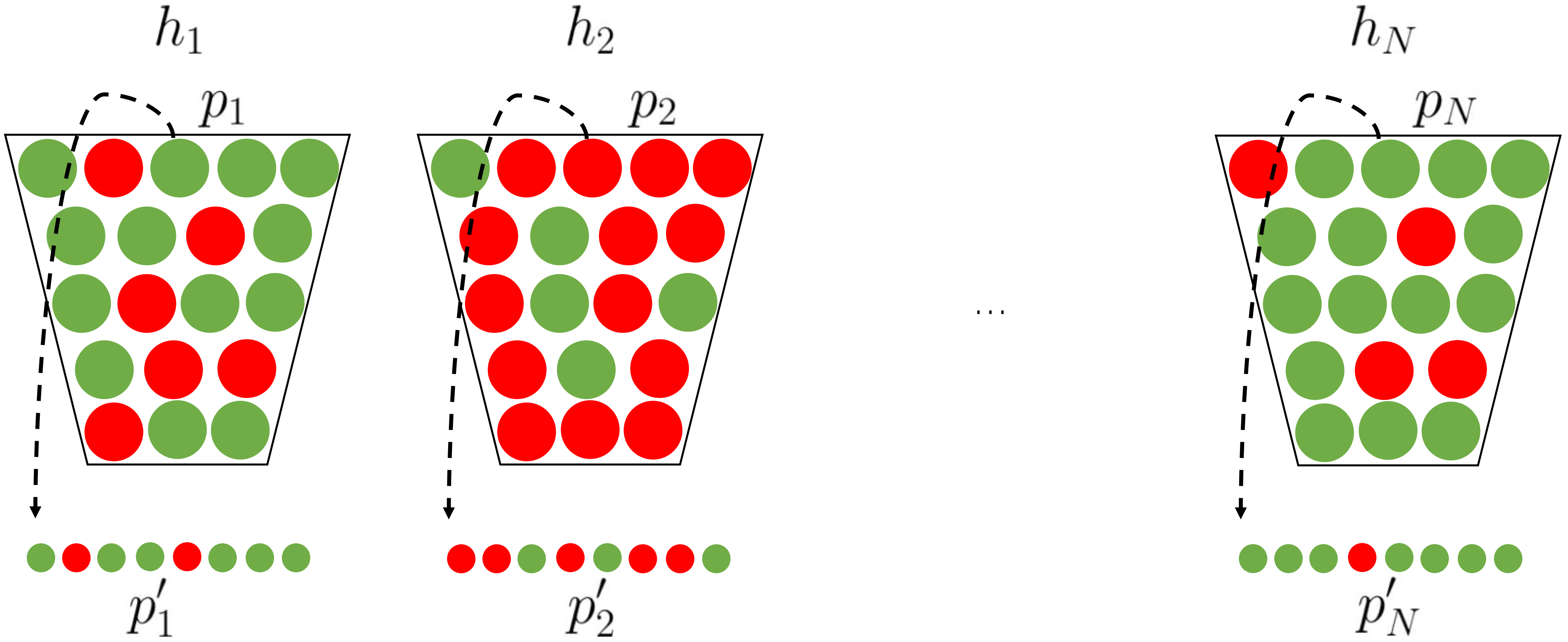


Note that p' is the in-sample error and it depends on a specific h

In other words, we will have a different in-sample error for every h

Generalize to Multiple Bins



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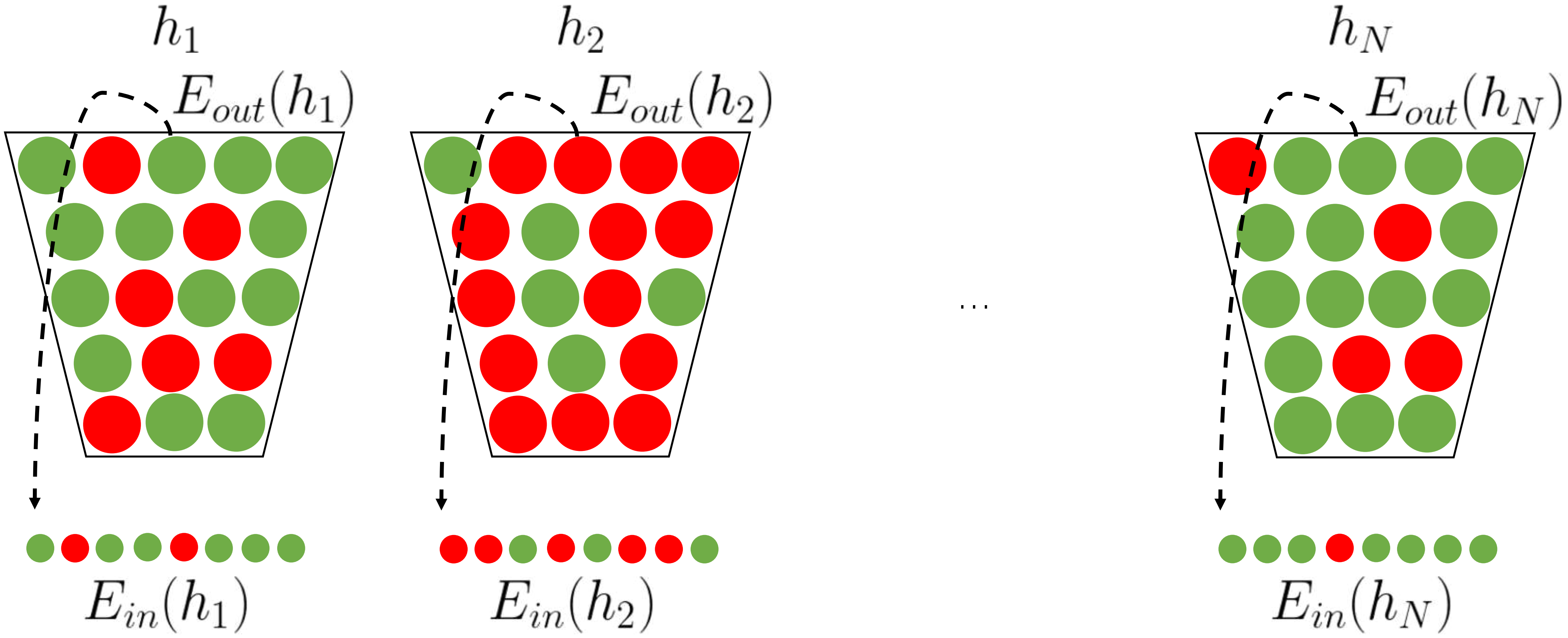

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Hoeffding's inequality using our new notation

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The probability that, for a given *h*, the in-sample error deviates from the true out-of-sample error by more than *epsilon* is less than or equal to a hopefully small quantity

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Why?

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The coin is fair!

p is the parameter of the Bernoulli distribution
(i.e., the probability of "success", e.g., getting a head)

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A very rare event!

Coin Analogy

We can also see this as an example of a binomial random variable

$$Y = X_1 + \ldots + X_{10}$$
$$Y \sim \text{Binomial}(n, p) = \text{Binomial}(10, 1/2)$$

$$P(Y = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$P(Y = 10) = \binom{10}{10} p^{10} (1 - p)^{10-10} = p^{10} = \left(\frac{1}{2}\right)^{10}$$

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$$q = 1 - p^{10} = 1 - \left(\frac{1}{2}\right)^{10} \approx 99.9\%$$

Coin Analogy

Q2: Suppose now we have 1,000 fair coins and we toss each of them 10 times, what is the probability that some coin will get 10 heads?

Since coin tosses are i.i.d. events, the probability that no coins (out of 1,000 coins) gets 10 heads is:

$$q^{1000}$$

Coin Analogy

Q2: Suppose now we have 1,000 fair coins and we toss each of them 10 times, what is the probability that some coin will get 10 heads?

A2: The probability that at least one coin comes up 10 heads is:

$$1 - q^{1000} = 1 - \left[1 - \left(\frac{1}{2}\right)^{10}\right]^{1000} =$$

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A2: The probability that at least one coin comes up 10 heads is:

$$1 - q^{1000} = 1 - \left[1 - \left(\frac{1}{2}\right)^{10}\right]^{1000} = 1 - \left(\frac{1023}{1024}\right)^{1000} \approx 62.4\%$$

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A2: The probability that at least one coin comes up 10 heads is:

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Not rare at all!

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- We can formulate the problem above as a sequence of $n = 1{,}000$ repeated experiments (i.e., one for each coin)

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- Each experiment is itself a sequence of 10 Bernoulli trials, where the probability of "success" is equal to getting 10 heads ($p = 2^{-10}$)
- The total number of success is given by another random variable Z

$$Z \sim \text{Binomial}(n, p),\ \ n = 1,000; p = \left(\frac{1}{2}\right)^{10}$$

Coin Analogy

$$Z \sim \text{Binomial}(n, p),\ \ n = 1,000; p = \left(\frac{1}{2}\right)^{10}$$

We therefore ask the following:

$$P(Z \geq 1) = 1 - P(Z = 0) =$$

$$= 1 - \binom{n}{0} p^0 (1-p)^{n-0} = 1 - (1-p)^{1000}$$

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- In the coin example, each hypothesis is the same (as the coins are fair)
- It is actually likely that we pick an unlucky sample, even though the true out-of-sample error is 1/2 (fair coin)
- Plain "vanilla" Hoeffding's inequality bound doesn't apply anymore when we have multiple hypotheses

A New Bound

Let's go back to our 1,000 coins example

$$B_i = \begin{cases} 1 & \text{if coin } i \text{ comes with 10 heads} \\ 0 & \text{otherwise} \end{cases}$$

$$B_i \sim \text{Bernoulli}(p), \;\; p = \left(\frac{1}{2}\right)^{10}$$

$$C = B_1 + B_2 \ldots + B_{1000}$$

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If we considered $n = 1{,}024$ coins we would obtain the trivial bound

$$\boxed{P(C \geq 1) \leq 1}$$

Union Bound for the Learning Problem

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Assuming a finite set of N hypotheses

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We can apply Hoeffding's inequality to each of them

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&\ldots \\
&P(|E_{in}(h_N) - E_{out}(h_N)| > \epsilon) \boxed{\leq \sum_{i=1}^{N} 2e^{-2m\epsilon^2} = 2Ne^{-2m\epsilon^2}}
\end{aligned}$$

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- Thanks to the Boole's inequality we can give an adjusted bound to the probability of picking a "bad" hypothesis
- A bad hypothesis is one whose in-sample performance deviates from out-of-sample performance by more than a tolerance *epsilon*
- Note, though, that the bound we came up with is not tight at all as it assumes the worst case scenario
    - Each event of choosing a "bad" hypothesis is disjoint from each other

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Take-home message
Learning is feasible in a **probabilistic sense**!