Teoria degli Algoritmi

Corso di Laurea Magistrale in Matematica Applicata

a.a. 2020-21

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Analysis of (Large) Graphs

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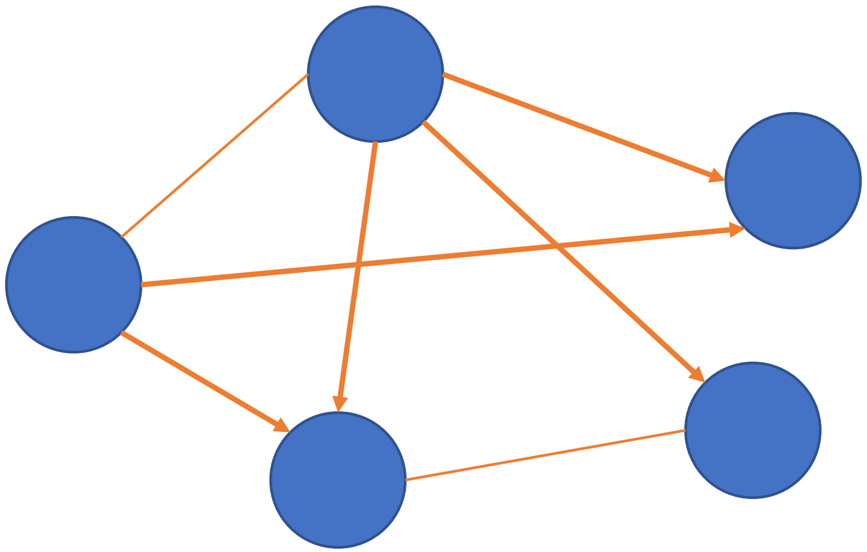
  - Maps (i.e., the set of cities and roads connecting them)
  - Social Networks (i.e., the set of social connections between people)
  - …

What is a Graph?

Informally, a set of vertices (nodes) connected by a set of edges (links)



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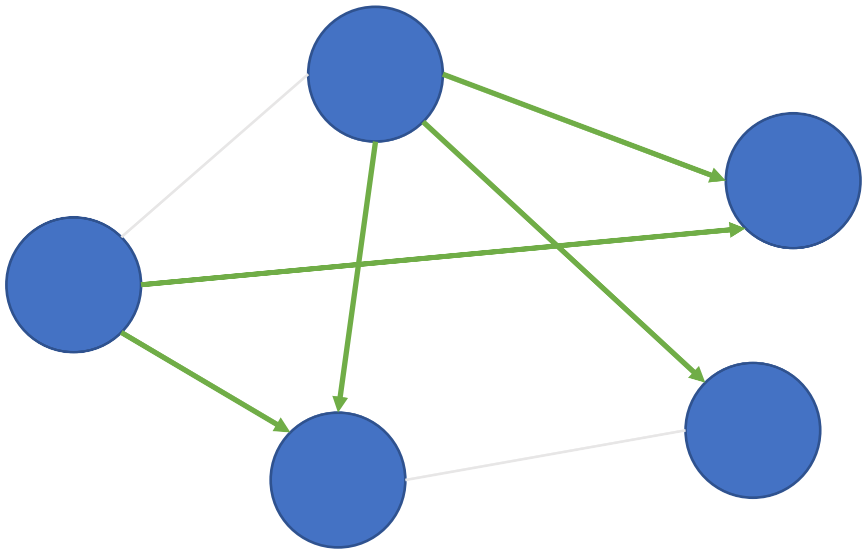
edges may be directed



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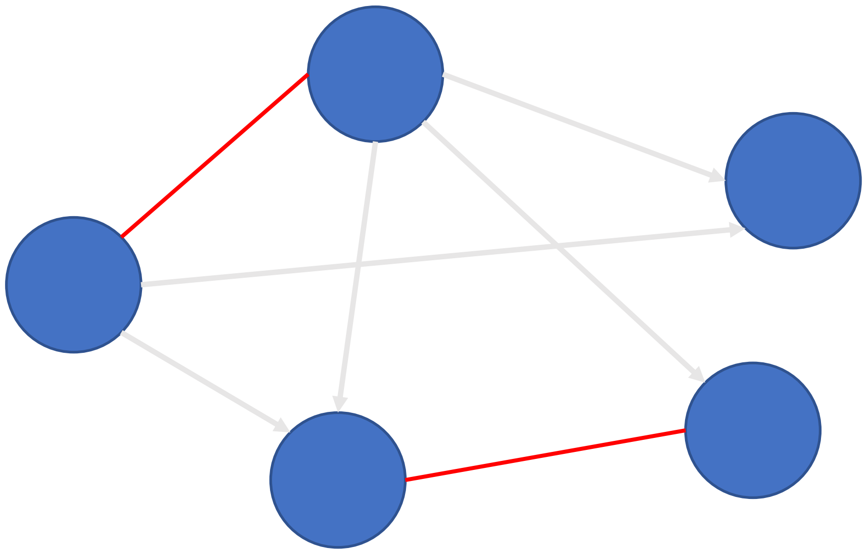

Directed vs. Undirected

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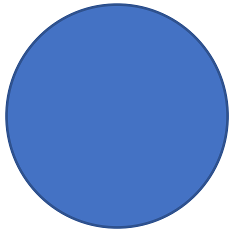

vertex (node)

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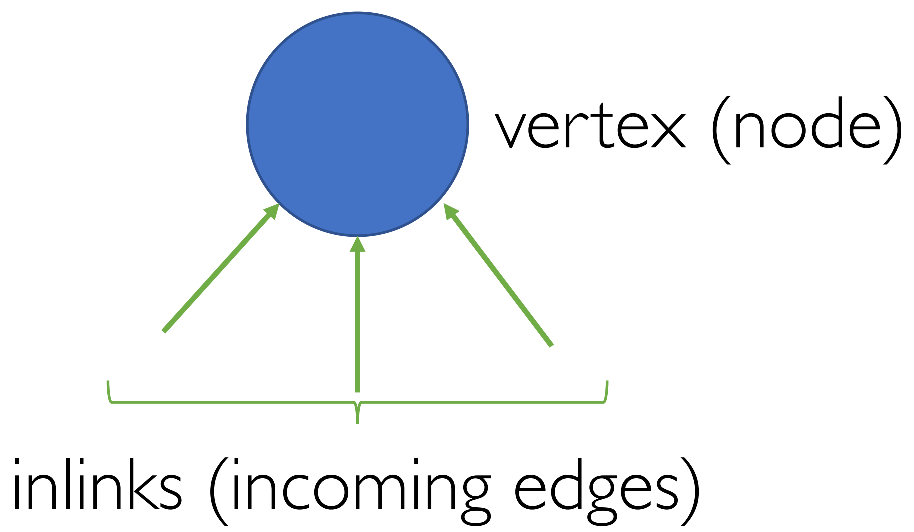

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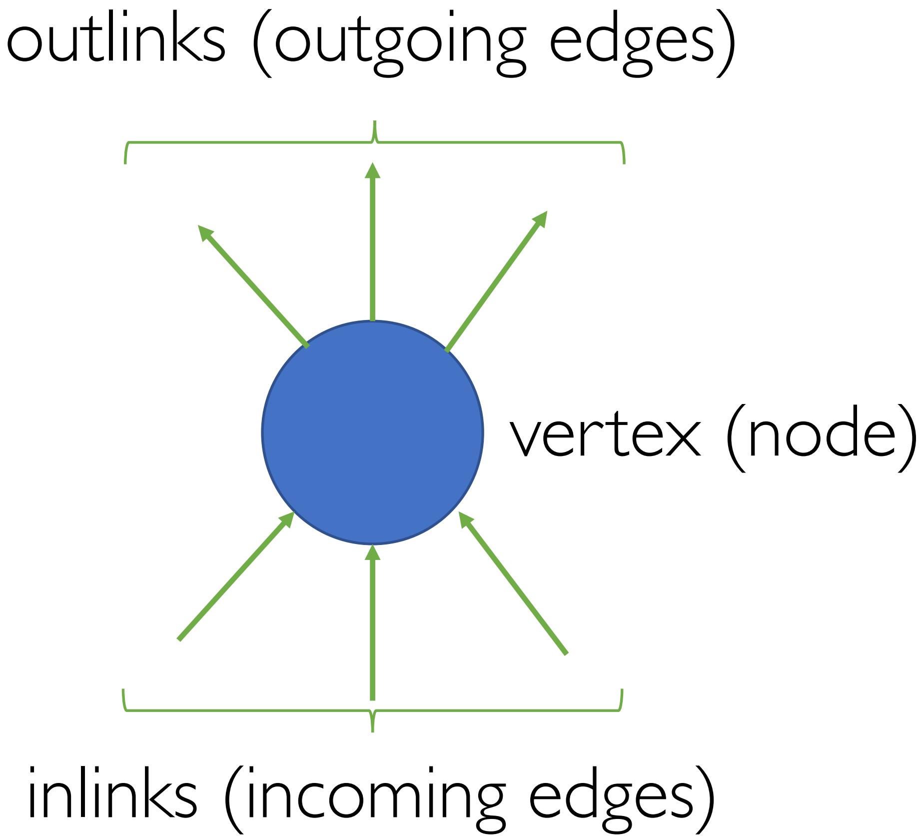

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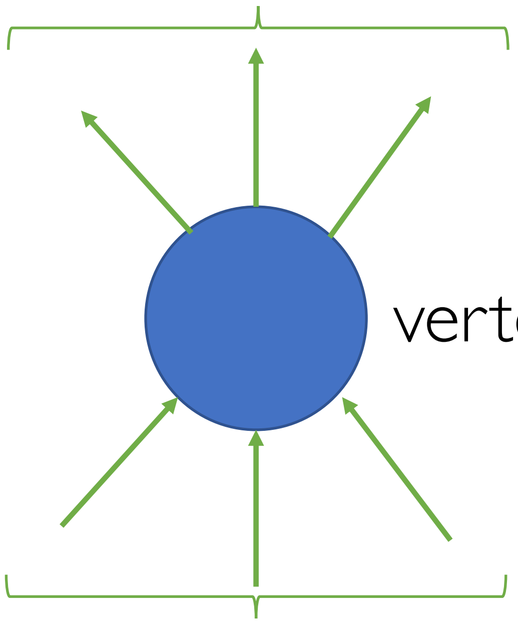
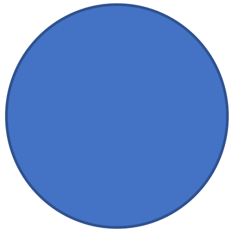


Undirected



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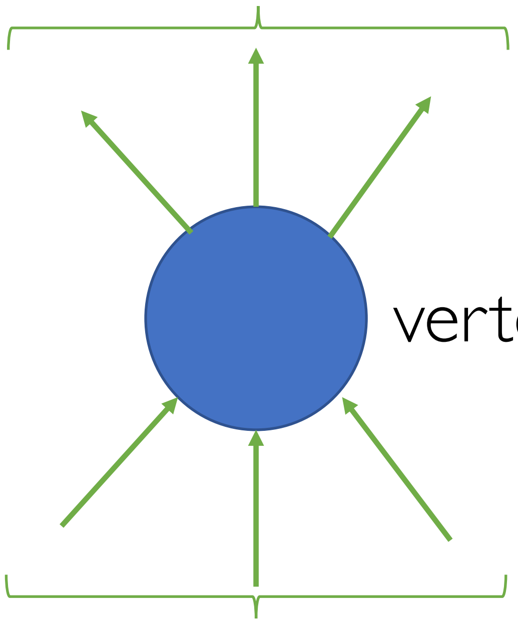

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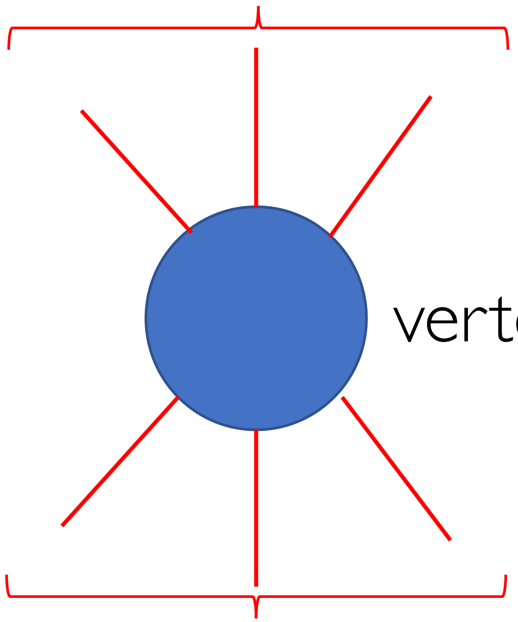
Undirected

incident edges



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Graph Formalism

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Note that an undirected graph is just a special case of a directed graph where the set of edges contain symmetric pairs of vertices

Node's Degree

Intuitively, the number of inbound/incident links to a node

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To be more explicit, in the case of a directed graph sometimes we distinguish between in-degree and out-degree

$$\text{in-deg}(v) = |\{u \in V | (u, v) \in E\}|$$

$$\text{out-deg}(v) = |\{u \in V | (v, u) \in E\}|$$

How Do We Represent Graphs?

3 main ways of representing graphs

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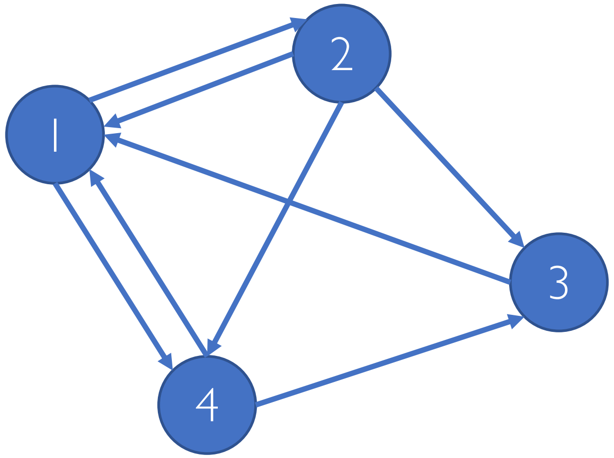

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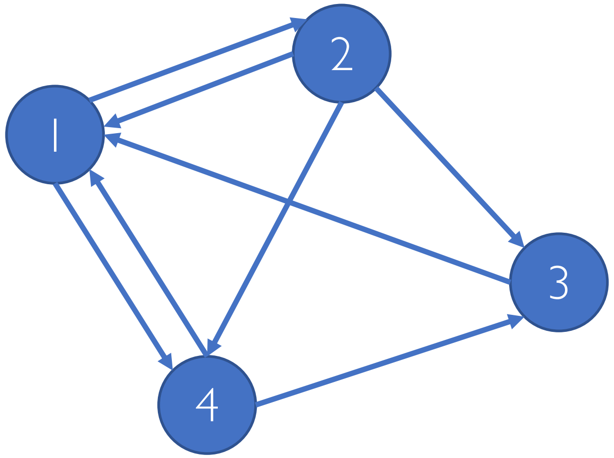

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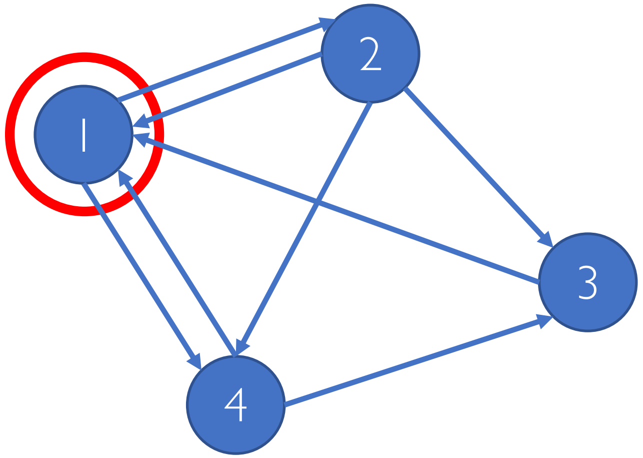

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- PROs:
  - Most intuitive representation
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- CONs:
  - Space inefficient (especially for loosely connected graphs, i.e., sparse matrices)
  - Easy to write yet hard to compute

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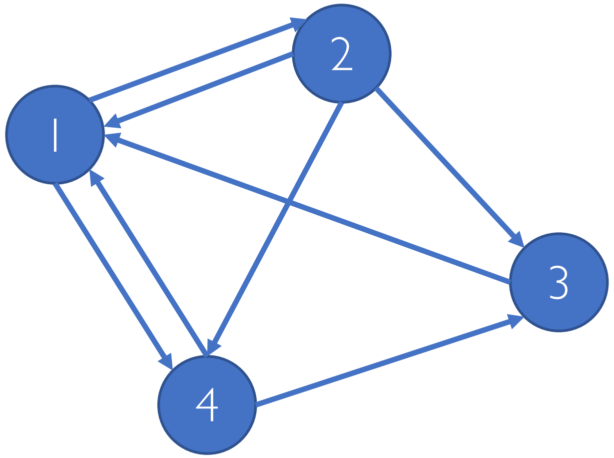

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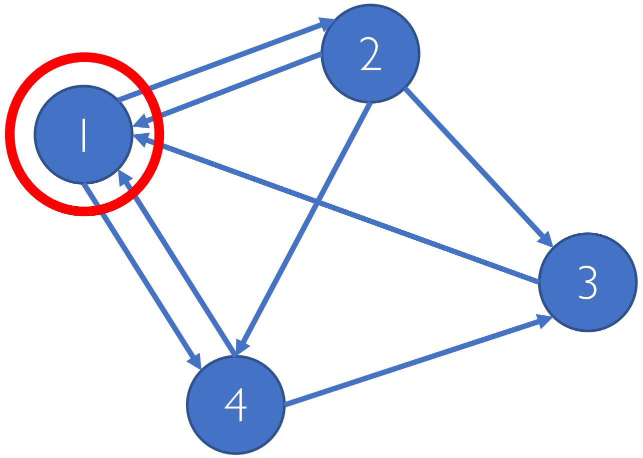

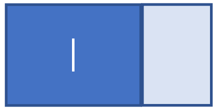

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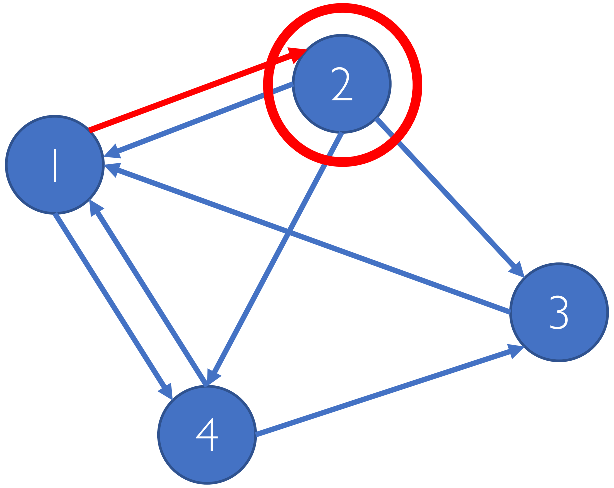

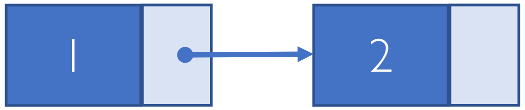

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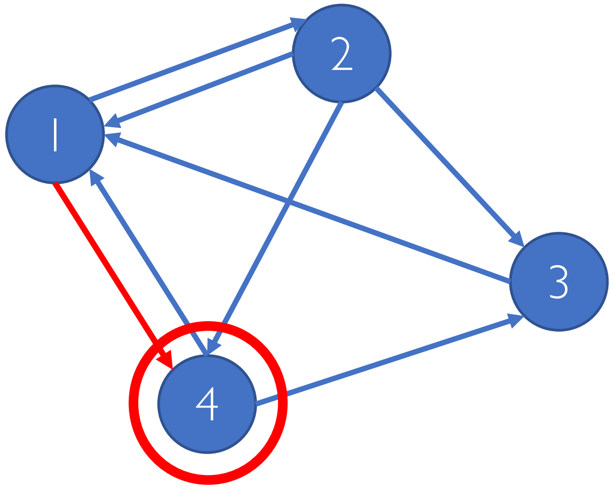

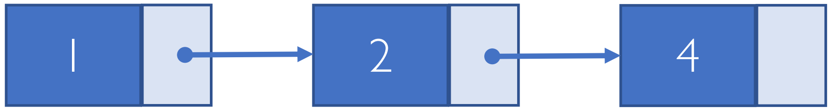

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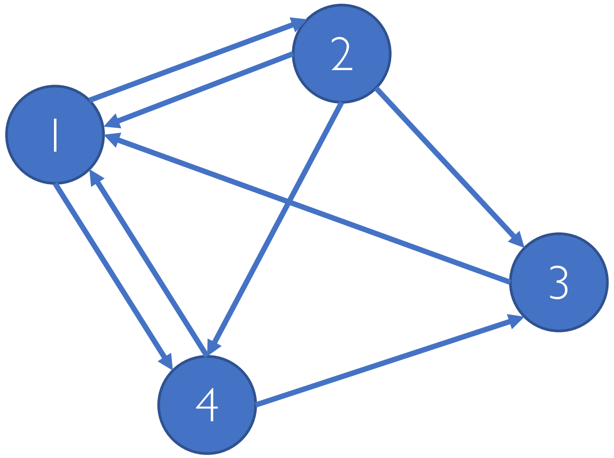

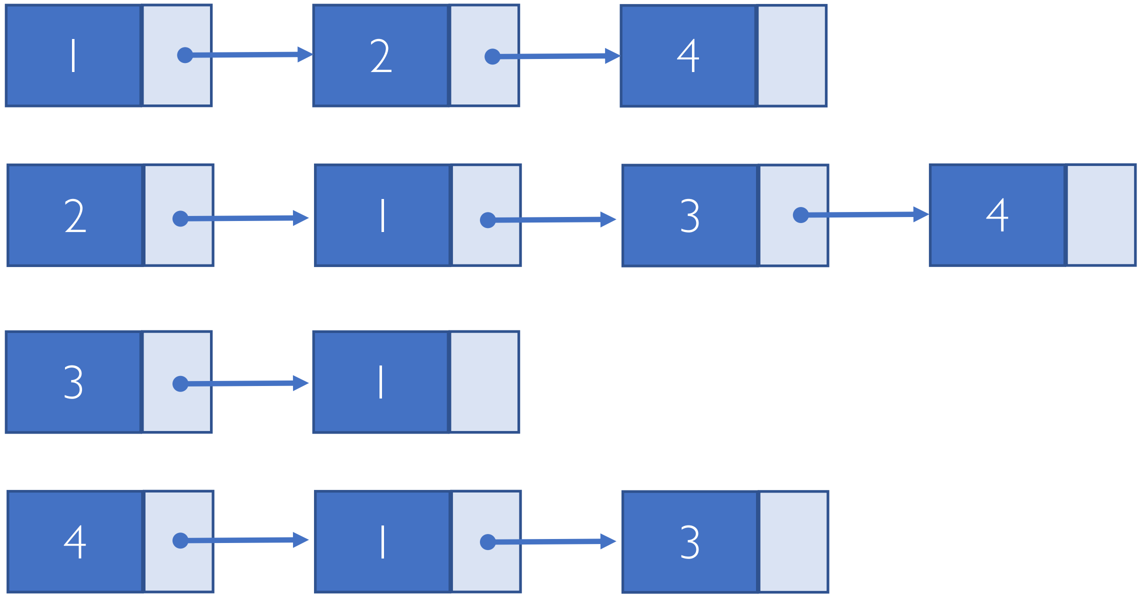

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Note that with adjacency matrix, any computation over incoming (outgoing) links reduces to a column (row) scan of the matrix

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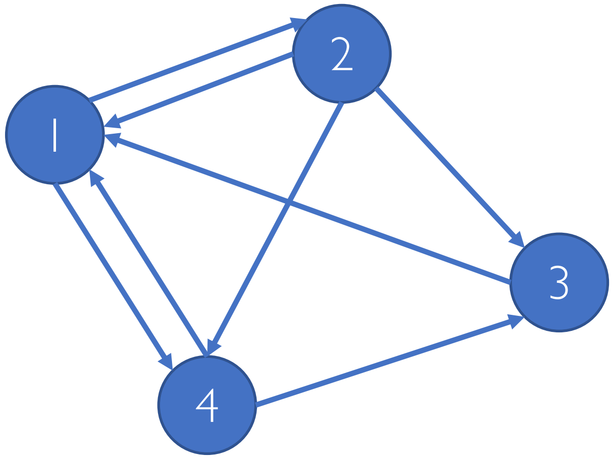

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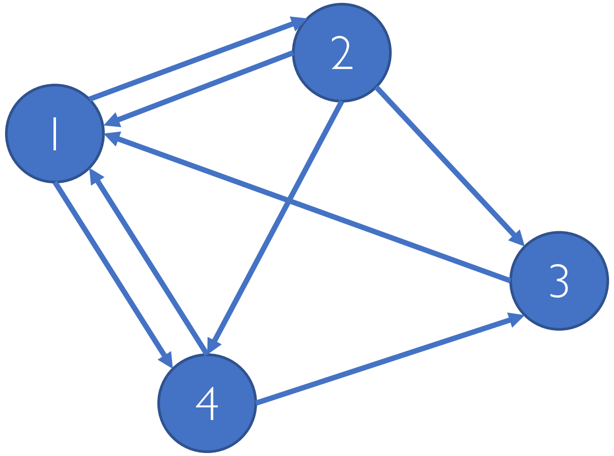

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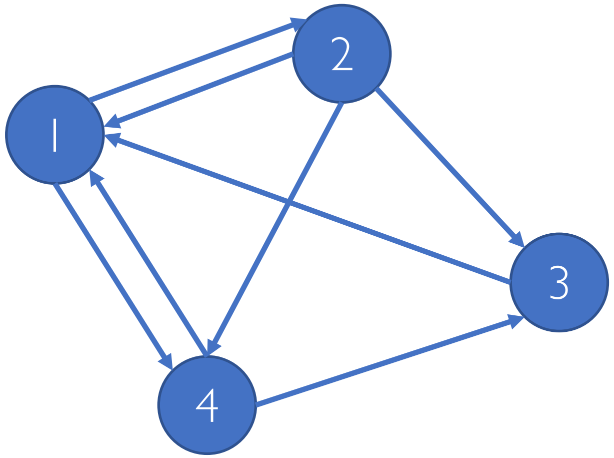

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Edge lookup may require scanning the whole list of edges

Some Famous Graph Problems

Problems

Applications

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- The Web graph is a great test bed for link analysis

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- Other attempts: DMOZ, LookSmart

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Web Search Engines

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- The list of top-*k* documents most similar to a query are returned (e.g., measuring cosine similarity between each query-document pair)

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The Web is huge and full of untrusted documents!

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Trustworthy web pages should point to each other

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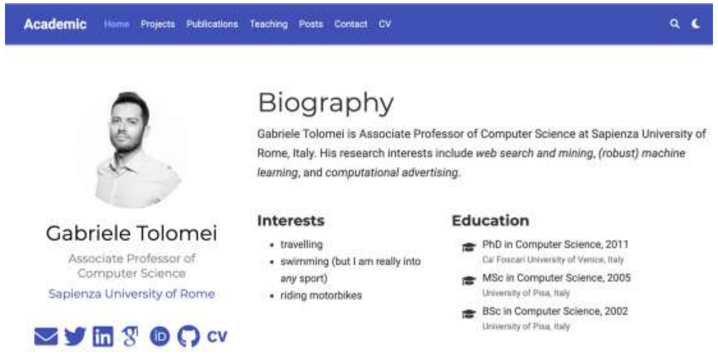

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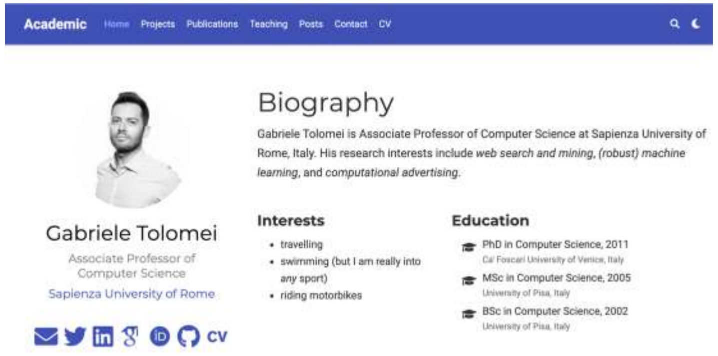

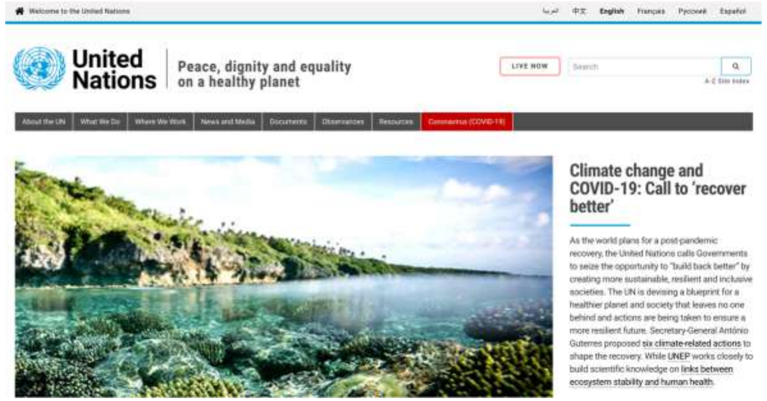

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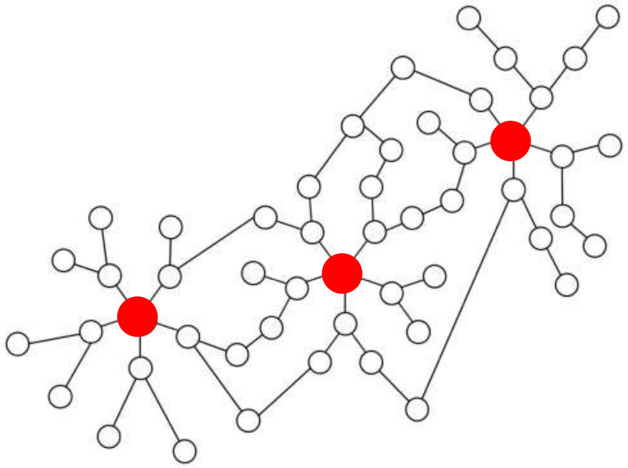

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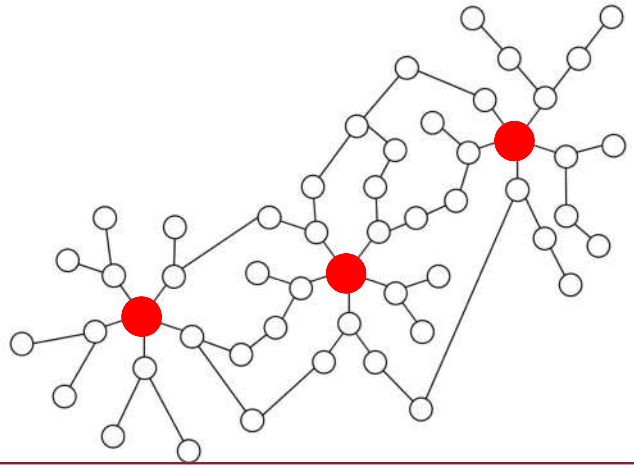

Rank nodes (i.e., assign them an importance score) on the basis of their connectivity

The Web as a "Scale-Free" Network

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They refer to graphs (i.e., networks) exhibiting such a behavior as scale-free networks

Barabási, A.L. and Albert,R. *Emergence of scaling in random networks*, Science, 286:509-512, October 15, 1999

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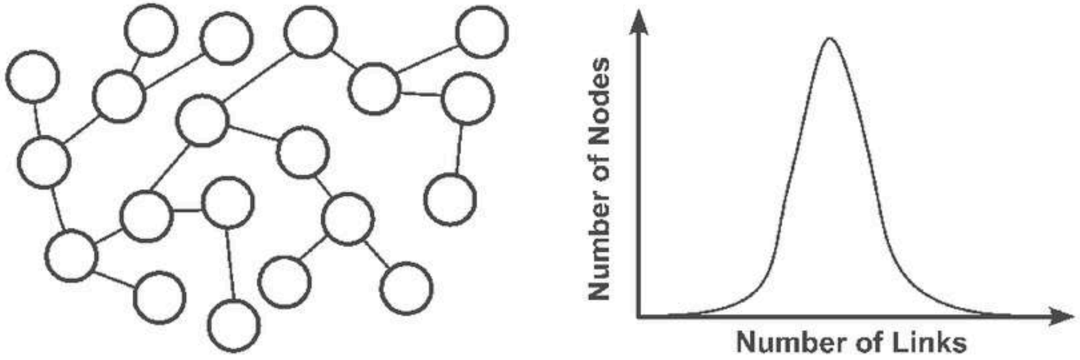

Random Graph

bell-shaped curve of the degree distribution
(Binomial/Poisson distribution)

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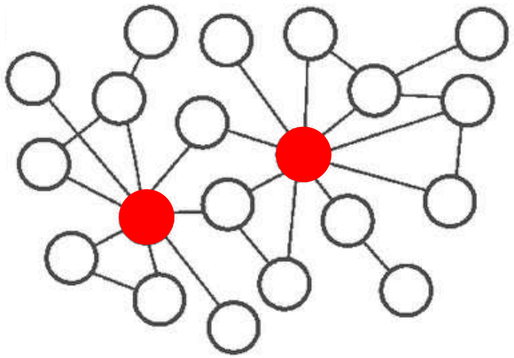
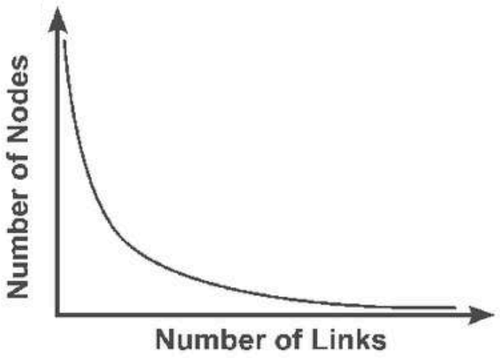
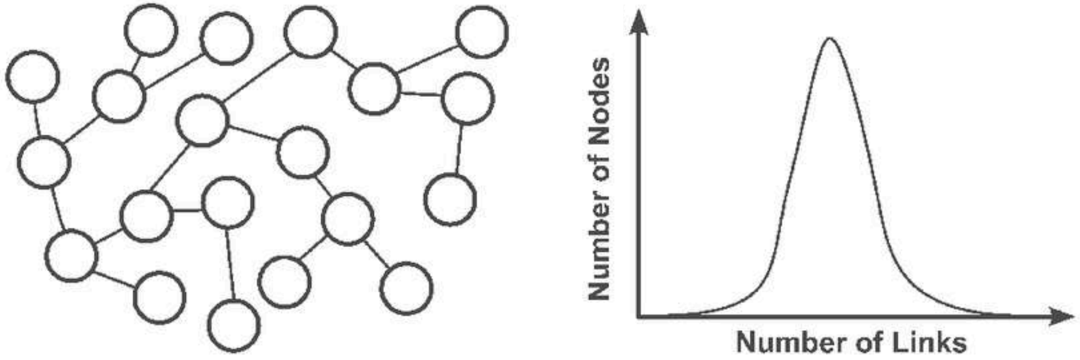




Scale-Free Graph

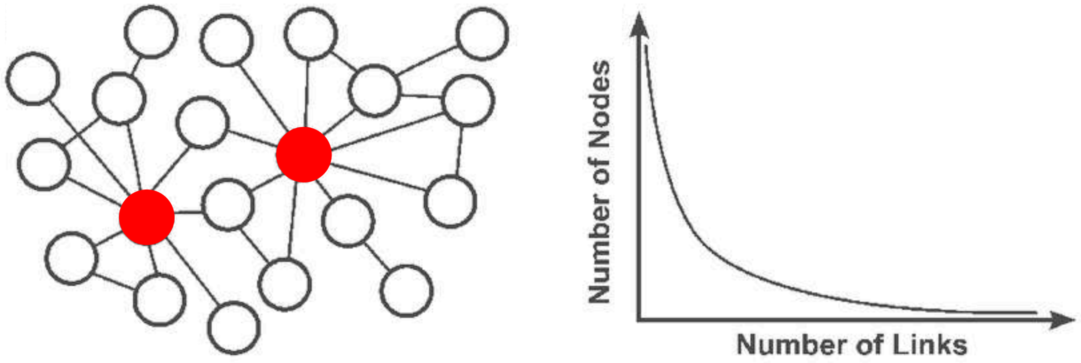
Most nodes have few links, and few nodes (i.e., red ones) have a large number of links, resulting into a power law degree distribution

The Web as a Scale-Free Network



Random Graph

bell-shaped curve of the degree distribution (Binomial/Poisson distribution)



Scale-Free Graph

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Tëmkin, I. and Eldredge N. *Networks and Hierarchies: Approaching Complexity in Evolutionary Theory*. February 2015

Scale-Free Networks

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The ratio of very connected nodes to the number of nodes in the rest of the network remains constant as the network changes in size

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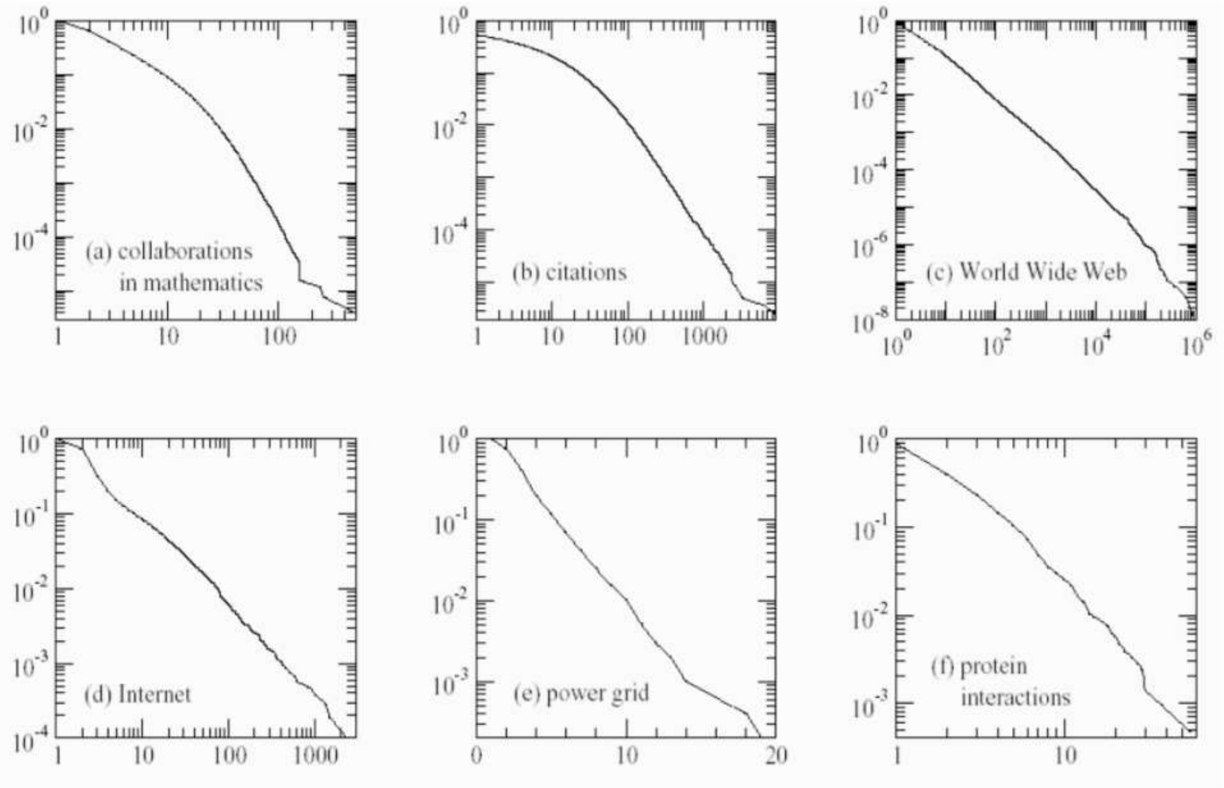
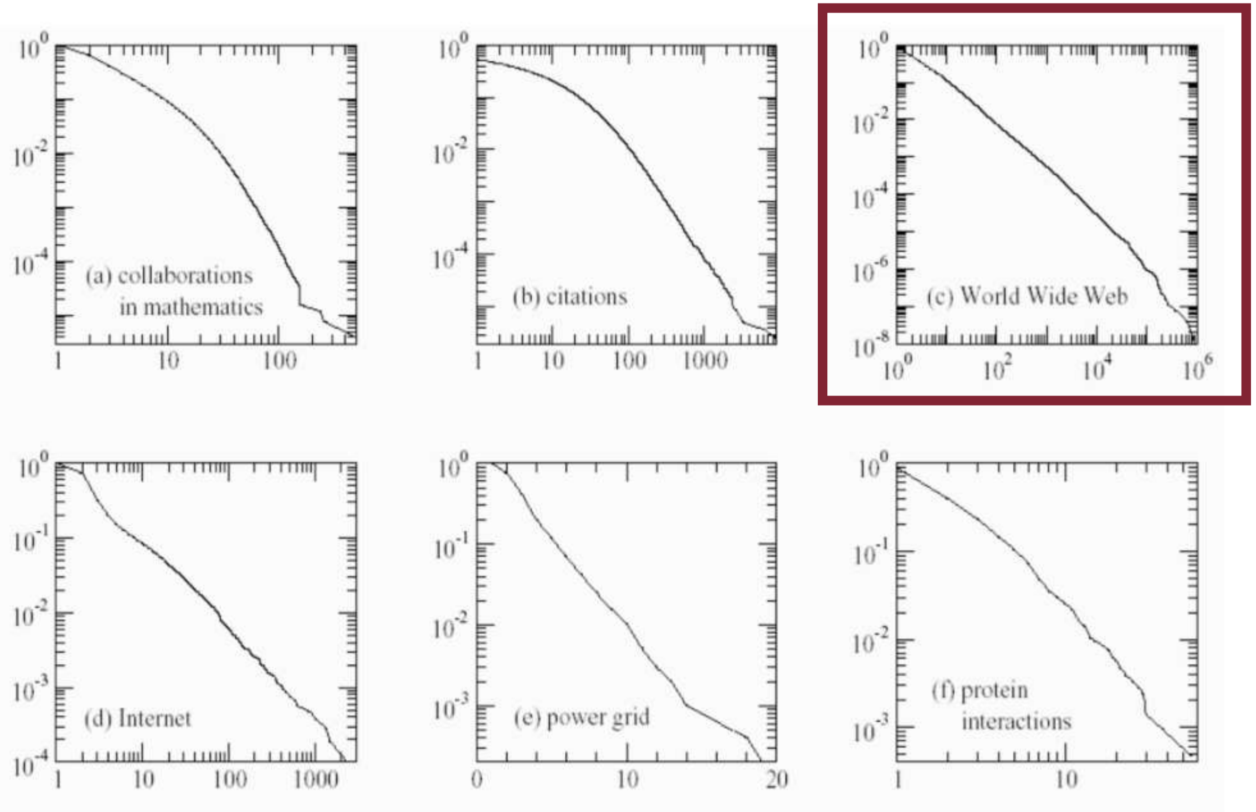
$$\boxed{p(\text{linking to node } i) \propto \frac{k_i}{\sum_j k_j}}$$

Scale-Free Networks: Examples



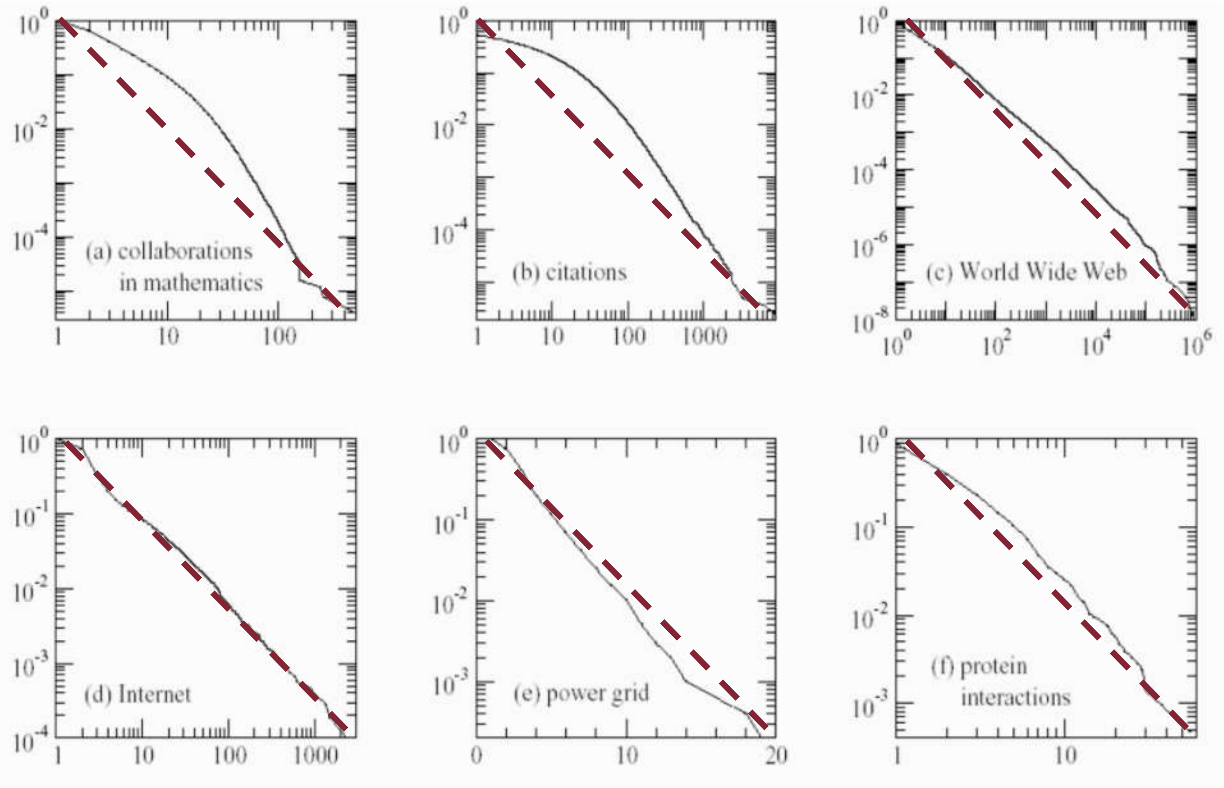
Many real-world networks are scale-free

Scale-Free Networks: Examples



The Web is one of those!

Scale-Free Networks: Examples



On log-log scale power law distributions look like straight lines

$$\log(p(k)) = \log(\alpha k^{-\gamma}) = \underbrace{\log(\alpha)}_{\text{constant } q} + \log(k^{-\gamma}) = q - \gamma \log(k)$$

Computing Node Importance

Several link analysis approaches to compute web page importance

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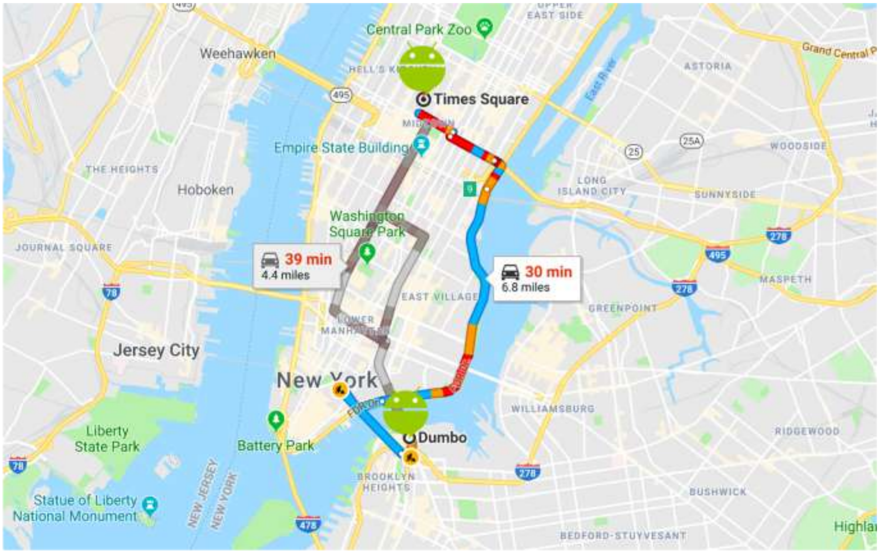

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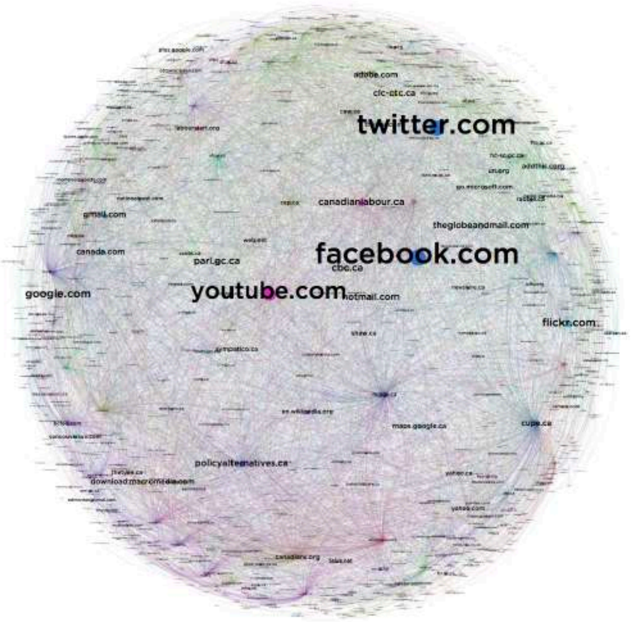

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  - Computing the importance of a page in the Web graph
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- Working with large-scale graphs may require specific tools/frameworks

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- Idea: Use node's connectivity to determine the importance of a node