Teoria degli Algoritmi

Corso di Laurea Magistrale in Matematica Applicata
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Lecture 6: Complexity Classes

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Complexity Relationships

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- On the other hand, we showed that at most an **exponential** difference exists between the time complexity of problems decided by deterministic vs. non-deterministic TMs
- Let's now try to classify (decision) problems on top of this distinction

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- $n = 100$; $n^3 = 1,000,000; 2^n \approx 1.26 \times 10^{30}$ (for larger n, huge difference!)

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- Polynomial time algorithms are fast enough for many purposes, but exponential time algorithms are rarely useful in practice

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Given a number x such that $n = \log x$ (bits), a trivial algorithm to find all prime factors of x would require to loop from 2 to $x - 1$ (or, more cleverly, up to $\sqrt{x}$).
Anyway, it requires a number of $O(x) = 2^{O(n)}$ iterations!

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- Indeed, we refer to a broader notion that includes models that closely approximate running times on actual computers
- For example, we showed that single-tape and multi-tape TMs are polynomially equivalent

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Note

Our goal is to present the fundamental properties of **computation**, rather than properties of Turing machines or any other specific model

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- After all, real programmers spend a lot of time trying to speedup their programs by that polynomial factors (or even less, for that matters!)
- We already disregarded constant factors when we introduced asymptotic notation
- Now, we propose to disregard the much grater polynomial differences, e.g., between n and n^3

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- On the contrary, the difference between an algorithm whose running time is n and another one n^3 is significant
- However, "core" questions like the polynomial vs. non-polynomial solvability of a problem is somewhat more important

The Class P: Definition

Definition (The Class P)

P is the class of languages that are decidable in polynomial time by a deterministic single-tape Turing machine. More formally:

$$P = \bigcup_k TIME(n^k)$$

The Class P

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- Item 1. indicates that P is mathematically robust, as it is not affected by specific peculiarities or nuances of the model of computation used
- Item 2. says that P is relevant from a practical perspective

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- For example, a running time of n^{100} is rare to be of any practical use!
- Still, setting the "threshold of practical solvability" to the class of polynomials has proven useful
- Once a polynomial time algorithm is found for a problem that formerly appeared to be solvable only in exponential time, we can get insights on the complexity of other problems as well (through reductions...)

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- In this way, we can focus on the important aspects and disregard tedious details like head movements or tape contents
- To analyze the polynomiality of an algorithm we describe it in terms of **number of stages**
- The notion of **stage** of an algorithm is similar to that of a **step** of a TM
- In general, though, implementing one stage of an algorithm will require many steps of a TM

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When both tasks are done, we can conclude that the algorithm runs in polynomial time because it runs for a polynomial number of stages, each one running in polynomial time

Examples of Problems in P: Encoding

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- We stick to the usual notation $\langle \cdot \rangle$ to denote a reasonable encoding of one or more objects into a string
- A reasonable encoding is one that allows for polynomial time encoding and decoding of objects into natural internal representations

Examples of Problems in P: Graph Encoding

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- The former is a list of nodes along with the list of edges
- The latter uses a matrix, where the entry $(i, j) = 1$ if there is an edge connecting node i with node j, or $(i, j) = 0$ otherwise

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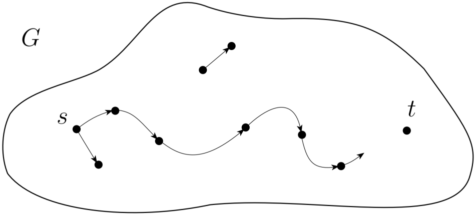
- When we analyze algorithms on graphs, the running time may be computed in terms of the number of nodes instead of the size of the graph representation
- This is because the size of reasonable graph representations is a polynomial in the number of nodes
- Thus, if we find that the running time of an algorithm on a graph is polynomial (exponential) in the number of nodes, we know that it is also polynomial (exponential) in the size of the graph representation

Examples of Problems in P: $PATH$

Definition (The $PATH$ problem)

Let $G = (V, E)$ be a **directed graph** containing two nodes $s, t \in V$ as shown below.
The $PATH$ problem is to determine whether a directed path exists from s to t:

$$PATH = \{\langle G, s, t\rangle \mid G \text{ is s directed graph with a directed path from } s \text{ to } t\}$$



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We must design a polynomial time algorithm that decides *PATH*.

Before sketching the proof, let's see that:

- a **brute-force** solution exists (i.e., *PATH* is actually decidable);

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We must design a polynomial time algorithm that decides $PATH$.
Before sketching the proof, let's see that:

- a **brute-force** solution exists (i.e., $PATH$ is actually decidable);
- the brute-force solution is not fast enough.

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PATH: Brute-Force Solution

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- The number of paths is, roughly, n^n, which is exponential in the number of nodes (i.e., each node is connected to any other node)
- The brute-force solution has **exponential** running time

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- Here, we successively mark all nodes in G that have been visited so far starting from s and going to the nodes directly reachable from it
- We can do so for length-1, length-2, up to length-n paths

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A polynomial time algorithm for *PATH*.

$M =$ "On input $\langle G, s, t \rangle$:

1. Mark node s as **visited**;

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   a. Scan all the edges of G, and if an edge (i, j) is found going from a marked node i to an unmarked node j, mark node j;

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3. If t is marked, **accept**; otherwise **reject**."

$\square$

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- Thus, the total number of stages used by M is at most $1 + 1 + n$, namely polynomial in the number of nodes of G and therefore on its size

Note

Stages 1 and 3 are easily implemented in polynomial time on any reasonable deterministic model. Stage 2.a involves scanning the input and test for marked nodes, which again can be implemented in polynomial time. Hence M is a polynomial time algorithm for *PATH*

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- As we observed in the case of *PATH*, we can sometimes avoid brute-force search to obtain polynomial time solutions
- However, for some other (interesting) problems, any attempt to avoid brute-force solutions has been unsuccessful
- For those problems, no polynomial time algorithms have been found (so far)

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- We don't know *exactly* why for some problems we were unsuccessful to find any polynomial time algorithm
- Perhaps, these problems have in fact polynomial time algorithms that solve them, yet they are still unknown
- Or, maybe, some of these problems simply cannot be solved in polynomial time as they are *intrinsically* difficult

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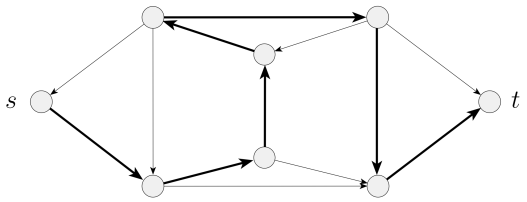
- A remarkable result, though, shows that the complexity of many problems are linked together
- A polynomial time algorithm for one such problem can be used to solve an entire class of problems
- Let's see this through an example, called *HAMPATH*

The *HAMPATH* Problem

Definition (*HAMPATH*)

Let $G = (V, E)$ be a **directed graph**.
We define a so-called **Hamiltonian path** a directed path that goes through each and every node of G **exactly once**.
The *HAMPATH* problem asks to find whether G contains a Hamiltonian path connecting two specific nodes, s and t, as shown below.

$$HAMPATH = \{\langle G, s, t\rangle \mid G \text{ contains a Hamiltonian path from } s \text{ to } t\}$$



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- This is just a slight modification of the brute-force algorithm given for *PATH*
- We enumerate all directed paths of G, check if there exists a path from s to t, and - if it does - test if this is a Hamiltonian path
- No one knows whether *HAMPATH* is solvable in polynomial time

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- Still, if someone claims that a Hamiltonian path exists and gives it to us, we can "easily" check if that is true
- In other words, **veryfing** the existence of a Hamiltonian path may be much easier than **finding** if it exists

The $\overline{HAMPATH}$ Problem

- Some problems may not be even polynomially verifiable

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- For example, the complement of the *HAMPATH* problem, i.e., $\overline{HAMPATH}$, is not polynomially verifiable
- Even if we could determine (somehow) that a graph does not contain a Hamiltonian path, we don't know how to give a "proof" that someone else can use to verify its non-existence
- The only (known) way to verify the non-existence would be to use the same exponential-time algorithm used for making the claim in the first place

Polynomial Verifiability: Definition

Definition (Polynomial Verifiability)

A **verifier** for a language A is an algorithm V, where:

$$A = \{x \mid V \text{ accepts } \langle x, c \rangle \text{ for some string } c\}$$

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A language A is **polynomially verifiable** if it has a polynomial time verifier.

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Example

For the *HAMPATH* problem, a certificate for the string $\langle G, s, t \rangle \in HAMPATH$ is just the Hamiltonian path from s to t.
The verifier can check in polynomial time that $\langle G, s, t \rangle \in HAMPATH$, given such certificate.

A Polynomial Verifier for *HAMPATH*

- Let's consider $\langle G, s, t\rangle \in HAMPATH$, where $G = (V, E)$, $|V| = n$, and $|E| \leq n^2 = O(n^2)$

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- Overall, verification takes $O(n^2)$ steps, which is clearly polynomial in n

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NP is the class of languages/problems that have polynomial time verifiers.

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- The class NP is crucial because it contains many problems of practical interest
- For example, we have shown that $HAMPATH \in NP$
- The term "NP" comes from **non-deterministic polynomial time**, and is derived from an alternative definition that makes use of non-deterministic Turing machines

A Non-Deterministic TM Solving *HAMPATH*

We can design a non-deterministic TM $N_{HAMPATH}$ to decide *HAMPATH*

Example (A non-deterministic decider for *HAMPATH*)

$N_{HAMPATH}$ = "On input string $\langle G, s, t\rangle$:

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3. Check if $s = p_1$ and $t = p_n$; if either fails, **reject**.
4. For each $1 \leq i < n$, check whether (p_i, p_{i+1}) is an edge of G; if any is not, **reject**, otherwise **accept**."

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- Stages 2 and 3 are simple polynomial-time checks, i.e., $O(n^2) + O(1) = O(n^2)$ time
- Finally, also stage 4 runs in polynomial time, as we must check if each of the n pairs is an actual edge, thereby needing $O(n^2)$ time

NP and NTM

Theorem (NP and NTM)

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Proof.

The idea of the proof is based on converting a polynomial time verifier to an equivalent polynomial time NTM, and vice versa.
The NTM simulates the verifier by guessing the certificate.
The verifier simulates the NTM by using the accepting branch as the certificate. $\square$

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- ($\Leftarrow$) Assume A is decided by a NTM N, we can construct a polynomial time verifier V as follows:
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NP: Two Definitions

- So far, we have given **two definitions** of the class NP:
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- So far, we have given **two definitions** of the class *NP*:
  1. The class of problems whose solution can be verified in polynomial time by a polynomial time verifier;
  2. The class of problems that can be decided by a polynomial time non-deterministic TM.
- Also, we showed that the two definitions above are **equivalent**

$NTIME(t(n))$

Analogously to the deterministic time complexity class $TIME(t(n))$, we can define the non-deterministic time complexity class $NTIME(t(n))$ as follows:

Definition ($NTIME(t(n))$)

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Corollary ($NP = \bigcup_k NTIME(n^k)$)

The class NP is insensitive to the choice of "reasonable" non-deterministic computational model, as all such models are polynomially equivalent.

Examples of Problems in NP: $CLIQUE$

Definition (The $CLIQUE$ Problem)

A **clique** in an undirected graph is a subgraph, where every two nodes are connected by an edge.

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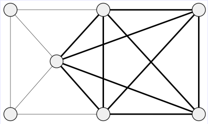
A k-clique is a clique that contains k nodes (e.g., a 5-clique is shown in the picture below).



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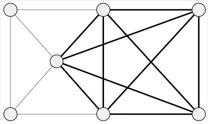

The $CLIQUE$ problem is to determine whether a graph contains a clique of a specified size:

$$CLIQUE = \{\langle G, k \rangle \mid G \text{ is undirected graph with a } k\text{-clique}\}$$

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Let $x_1, \ldots, x_k$ be a collection of integers, i.e., $x_i \in \mathbb{Z}\ \forall i \in \{1, \ldots, k\}$, and $t \in \mathbb{Z}$ a **target**.

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The $SUBSET\_SUM$ problem is to determine if the collection contains a subcollection whose sum is exactly t.

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Note that S and S' are, in fact, considered **multisets** and so repetitions are allowed.

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- If $P = NP$, any polynomially verifiable problem would be also polynomially decidable

P vs. NP

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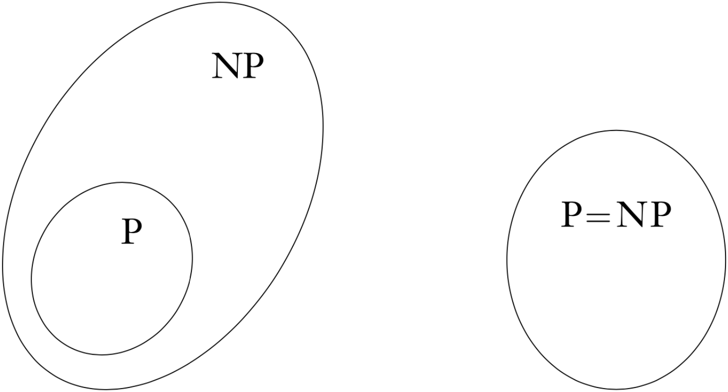
- To check for a proof of $P = NP$, many people have tried to design a polynomial time decider for some well-known problems in NP, without success
- On the other hand, proving that $P \neq NP$ would need to show that no polynomial time algorithm exists to replace brute-force search deciders

P vs. NP



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- Unfortunately, we don't know whether NP is contained in a "smaller" deterministic time complexity class

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- NP is the class of languages for which membership can be **verified** in polynomial time
  (Equivalently, NP is the class of languages that are decidable in polynomial time by an NTM)
- We loosely refer to "polynomial time" as "quick" and "exponential time" as "intractable"
- The P vs. NP problem is still open and a proof of $P = NP$ (resp., $P \neq NP$) would have a tremendous impact on the computational complexity hierarchy