Regression Analysis Of Population Drinking Dataset

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Contents

1 Introduction

About The Data

Here we have 46 observations from different places of the following quantiles :-

Urban Population
Late Births
Wine Consumption Per Capita
Liquor Consumption Per Capita
Cirrhosis Death Rate

Here our response variable is Cirrhosis Death Rate and others are all covariates. To get an overview of the data, we first perform the exploratory data analysis.

2 Exploratory Data Analysis

Loading the Dataset

To get an overview of the data, we first load it in R and print first few values:-

```
library(MASS)
library(lattice)
library(olsrr)
library(car)
library(L1pack)
X  =  read.csv(file  =  "D:\\PG  files\\Projects\\Regression-Analysis-Project-main\\population_drinking1.txt",
header=TRUE,sep = "\t")
names(X) <- c("Ind","Ind_1","Urban population", "Late births", "Wine consumption per capita", "Liquor consumption per cap
head(X)
```

```
  Ind Ind_1 Urban population Late births Wine consumption per capita
1   1     1               44        33.2                           5
2   2     1               43        33.8                           4
3   3     1               48        40.6                           3
4   4     1               52        39.2                           7
5   5     1               71        45.5                          11
6   6     1               44        37.5                           9
 .iquor consumptio n per capita Cirrhosis death rate
1                            30                 41.2
2                            41                 31.7
3                            38                 39.4
4                            48                 57.5
5                            53                 74.8
6                            65                 59.8
```

Here we have “Cirrhosis death rate” as the response and “Urban population”, “Late births”, “Wine consumption per capita”, “Liquor consumption per capita” as the covariates.

Type of the covariates

To know type of each covariates, we use the str() function :-

```
str(X)
```

```
'data.frame':	46 obs. of  7 variables:
 $ Ind                          : int  1 2 3 4 5 6 7 8 9 10 ...
 $ Ind_1                        : int  1 1 1 1 1 1 1 1 1 1 ...
 $ Urban population             : int  44 43 48 52 71 44 57 34 70 54 ...
 $ Late births                  : num  33.2 33.8 40.6 39.2 45.5 37.5 44.2 31.9 45.6 45.9 ...
 $ Wine consumption per capita  : int  5 4 3 7 11 9 6 3 12 7 ...
 $ Liquor consumption per capita: int  30 41 38 48 53 65 73 32 56 57 ...
 $ Cirrhosis death rate         : num  41.2 31.7 39.4 57.5 74.8 59.8 54.3 47.9 77.2 56.6 ...
```

So the dataset contains no factor covariate hence we can perform multiple linear regression here. For ease of indexing, we name the columns as “I”, “1”, “A1”, “A2”, “A3”, “A4”, “Y”.

```
names(X) <- c("I","1","A1","A2","A3","A4","Y")
```

5-number Summary of Covariates

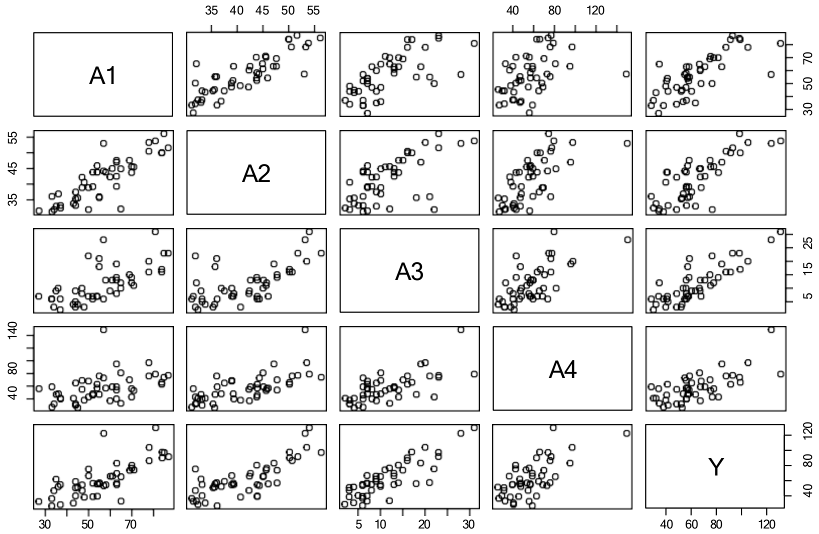
To get an idea of the values of each covariate we calculate the 5-number summary for each of them :-

```
summary(X[,-c(1,2)])

       A1              A2              A3              A4        
 Min.   :27.00   Min.   :31.20   Min.   : 2.00   Min.   : 26.00  
 1st Qu.:44.25   1st Qu.:35.62   1st Qu.: 6.25   1st Qu.: 41.50  
 Median :55.00   Median :42.25   Median :10.00   Median : 56.00  
 Mean   :56.26   Mean   :41.48   Mean   :11.59   Mean   : 57.50  
 3rd Qu.:65.00   3rd Qu.:45.83   3rd Qu.:15.75   3rd Qu.: 68.75  
 Max.   :87.00   Max.   :56.10   Max.   :31.00   Max.   :149.00  
       Y         
 Min.   : 28.00  
 1st Qu.: 48.90  
 Median : 57.65  
 Mean   : 63.49  
 3rd Qu.: 75.70  
 Max.   :129.90  
```

Pairwise Scatterplots

To get an idea of the relationship between covariates and response, we make pairwise scatterplots using pairs() function :-

```
pairs(X[,-c(1,2)])
```



This plot clearly indicates linear relationship between the covariates and response also. This might lead to the problem of multicollinearity which we will formally diagnose.

Correlation Between Covariates

We calculate the correlation between the covariates and response to get even better idea of linear dependence between them :-

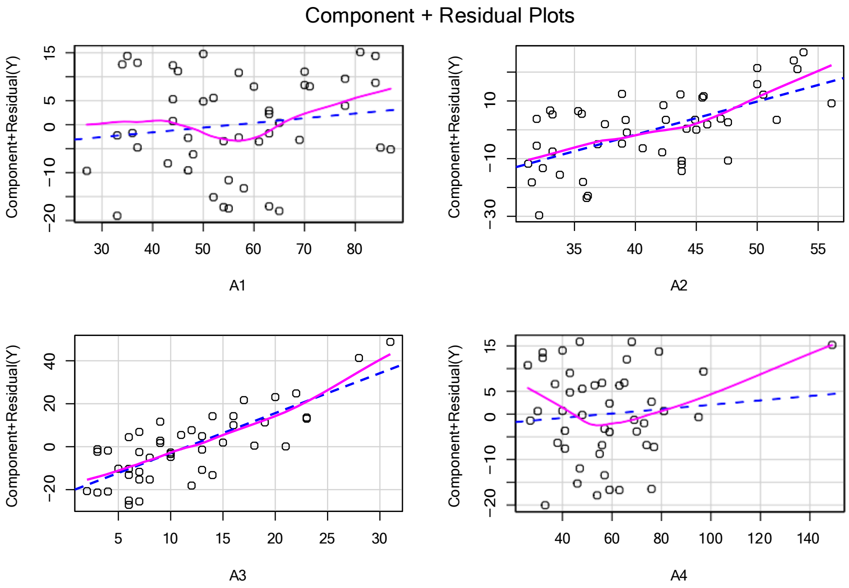
```
          A1        A2        A3        A4
A1 1.0000000 0.8432812 0.6786230 0.4402957
A2 0.8432812 1.0000000 0.6398407 0.6863643
A3 0.6786230 0.6398407 1.0000000 0.6759206
A4 0.4402957 0.6863643 0.6759206 1.0000000
```

We can see that the correlations are high between many predictors which can lead to problem of multicollinearity.

Partial Residual Plots

To get an idea of the nature of relationship between the covariates and response, we make the partial residual plot for all the covariates :-

```
crPlots(lm(Y~A1+A2+A3+A4,data = X))
```



Conclusion

The plot indicates the linear relationship between the covariates and response.

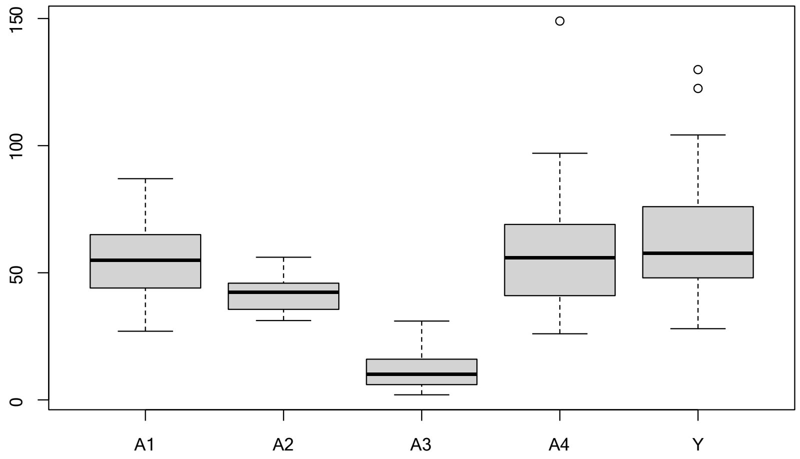
Hence, we will fit the usual multiple linear regression model with no higher order polynomial terms.

Later we will see if other models with interaction terms are better or not.

Boxplots of Covariates and Response

We draw the boxplots for differnet covariates to get idea of presence of outlier / high leavarage points :-

```r
boxplot(X[,-c(1,2)])
```



Here also we get some indication of possible presence of those points.

3 Regression Analysis

3.1 Fitting a Linear Model

Fitting a Linear Model to the Dataset

We fit a linear model of the form :-

$$\boldsymbol{Y}^{n\times1} = \boldsymbol{X}^{n\times p}\boldsymbol{\beta}^{p\times1} + \boldsymbol{\epsilon}^{n\times1}$$

where $n = 46$ (total number of observed responses) and $p = 5$ where columns of $\boldsymbol{X} = \mathbf{1}_n \;\; \boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \boldsymbol{x}_3 \;\; \boldsymbol{x}_4$ and $\boldsymbol{\beta} = \beta_0 \;\; \ldots \;\; \beta_4$ each corresponding to the 4 different covariates.

We fit a linear model based on the given dataset in R and then verify the different assumptions of it.

Features of the fitted model

We fit the linear model specified before in the dataset using lm() function and to get an idea about the estimates we use the summary() function :-

```
colnames(X)=c("I","1","A1","A2","A3","A4","Y")
attach(X)
reg <- lm(Y~A1+A2+A3+A4)
summary(reg)


Call:
lm(formula = Y ~ A1 + A2 + A3 + A4)

Residuals:
     Min       1Q   Median       3Q      Max
-18.8723  -6.7803   0.1507   7.3252  16.4419

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -13.96310   11.40035  -1.225   0.2276
A1            0.09829    0.24407   0.403   0.6893
A2            1.14838    0.58300   1.970   0.0556 .
A3            1.85786    0.40096   4.634 3.61e-05 ***
A4            0.04817    0.13336   0.361   0.7198
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 10.61 on 41 degrees of freedom
Multiple R-squared:  0.8136,Adjusted R-squared:  0.7954
F-statistic: 44.75 on 4 and 41 DF,  p-value: 1.951e-14
```

Explanation of the fitted model

As we can see only the coefficients β_2, β_3 for covariates “A2”,“A3” are statistically significant.

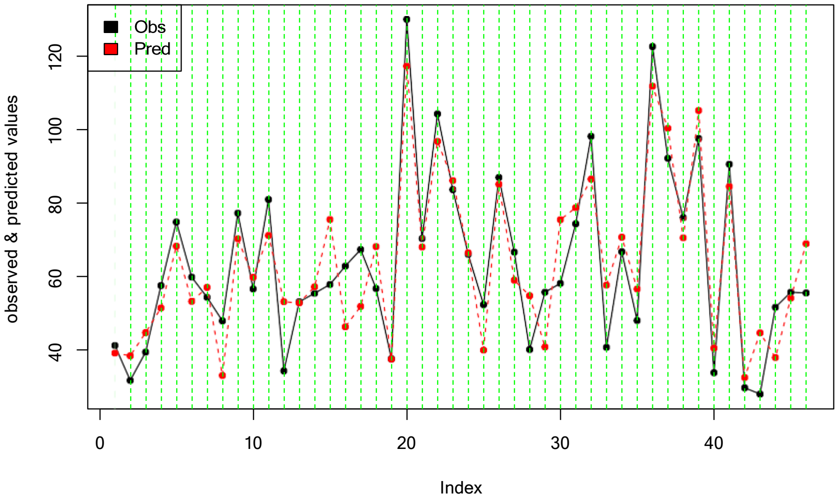
This does not imply that other covariates are insignificant since there maybe many problems that are hidden in the model.

So, before concluding anything we verify all the assumptions of a linear model.

Obs vs Fitted Values

We plot the observed vs fitted values to get some idea about prediction :-

```
plot(1:nrow(X),X$Y,type = "o",pch = 20,ylab = "observed & predicted values",xlab = "Index")
lines(1:nrow(X),reg$fitted.values,type = "o",pch = 20,col = "red",lty = 2)
abline(v = 1:nrow(X),lty = 2,col = rgb(0,1,0,alpha = 0.3))
legend("topleft",legend = c("Obs","Pred"),fill = c("black","red"))
```



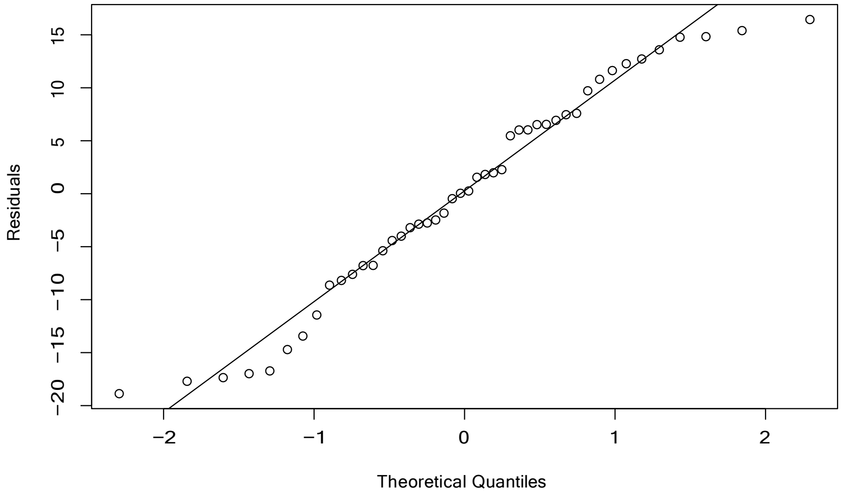
The fit is good except a few observations.
There may be many reasons for this which we will eventually look into.

3.2 Checking Model Assumptions

QQ-plot of residuals

We now plot the sorted residuals (quantiles) against the population quantiles of a normal distribution :-

```
resi<-residuals(reg)
qqnorm(resi,ylab="Residuals",main="")
qqline(resi)
```



We can see the qq-plot indicates ligh tailed residuals with possible deviation from normality.

There maybe some outlier points present which we will verify later.

Shapiro-Wilk Test

We test the following hypothesis H_0 : residuals are normally distributed against $H_1 : H_0$ is false using Shapiro-Wilk test in R as :-

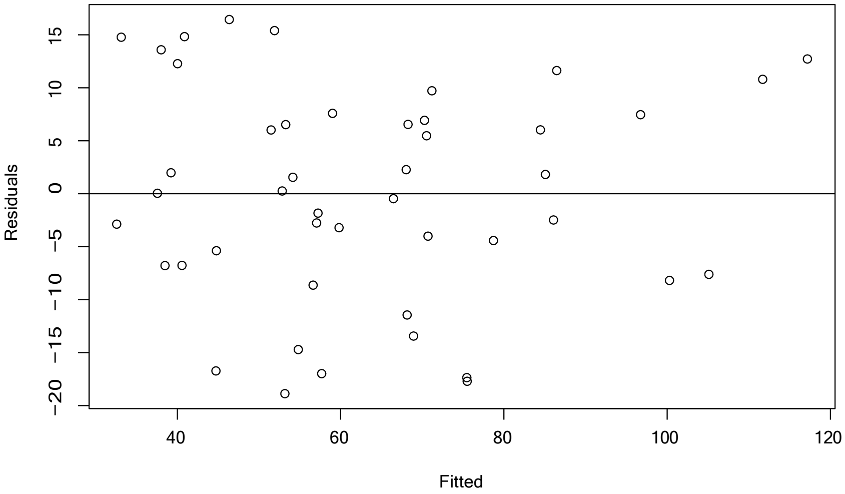
```
shapiro.test(resi)


Shapiro-Wilk normality test

data:   resi
W  =  0.95987,  p-value  =  0.1133
```

Though the p-value is more than 0.1 but this doesn't give strong evidence in favour of H_0 so we will further check for presence of correlation between the errors and other issues also.

Checking Homoskedasticity Assumptions

First to check homoskedasticity assumption, we make the residuals ($\hat{\varepsilon}$) vs fitted ($\hat{y}$) plots :-

```
plot(fitted(reg),residuals(reg),xlab="Fitted",ylab="Residuals")
abline(h=0)
```



We can see the plot doesn't give indication of presence of heteroskedasticity, hence we will perform confirmatory tests.

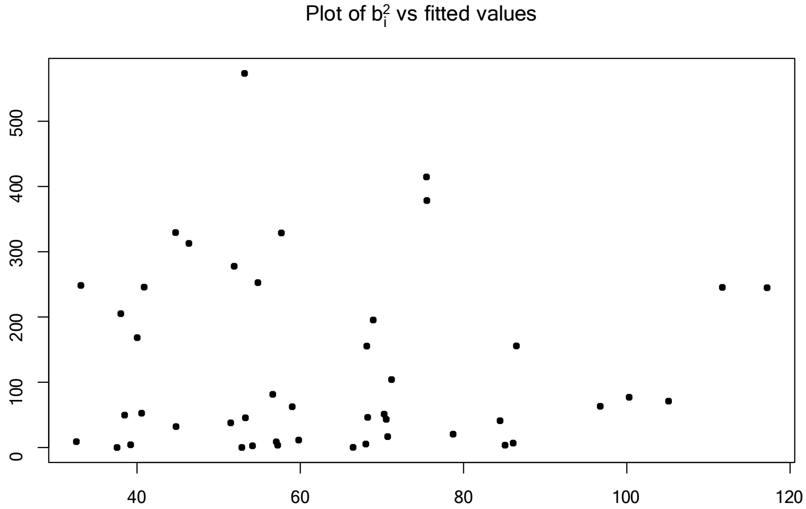
https://online.stat.psu.edu/stat462/node/117/#:~:text=When%20conducting%20a%20residual%

b_i vs $\hat{y}_i$ plot

A standard technique to detect presence of heteroskedasticity is to plot the quantities $b_i = \frac{e_i^2}{1-h_i}$ against the fitted values $\hat{y}_i$.

We make the plot using R :-

```
A = as.matrix(X[,-1])
H = A%*%solve(t(A)%*%A)%*%t(A)
H_i = diag(H)
e_i = residuals(reg)
b_i = e_i^2/(1-H_i)
plot(fitted(reg),b_i,pch = 20,main = bquote("Plot of" ~ b[i]^2 ~ "vs fitted values"
```



This plot gives no indication of any heteroskedasticity present in the residuals.

Breusch-Pagan Test

We perform the Breusch–Pagan test for testing the homoskedasticity assumptions using R :-

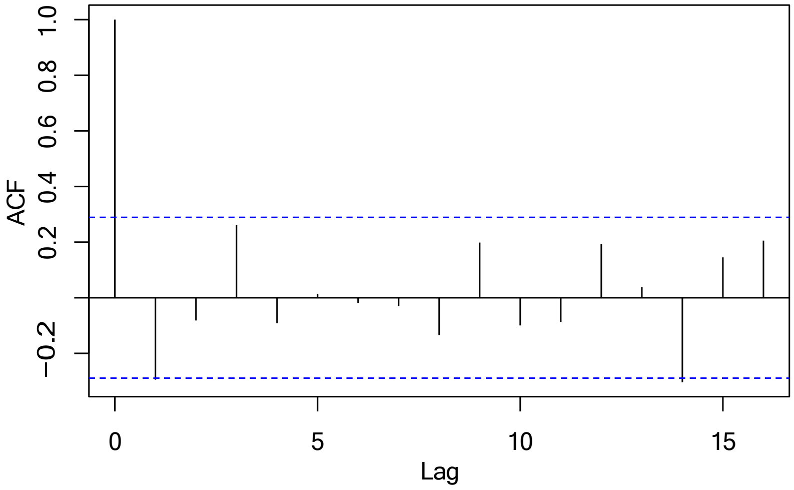
```
library(lmtest)
bptest(reg)


studentized Breusch-Pagan test

data:   reg
BP = 2.6929, df = 4, p-value = 0.6105
```

We can see that the p-value of the outcome is satisfactorily high so we can safely assume the error variances to be equal.

ACF plot

If the errors in the model are truely independent, then we will expect the sample autocorrelation coefficients for different lags k to be insignificant.

```
acf(resi,ylab = "",xlab = "",main = "")
title(xlab="Lag", ylab="ACF", line=2)
```

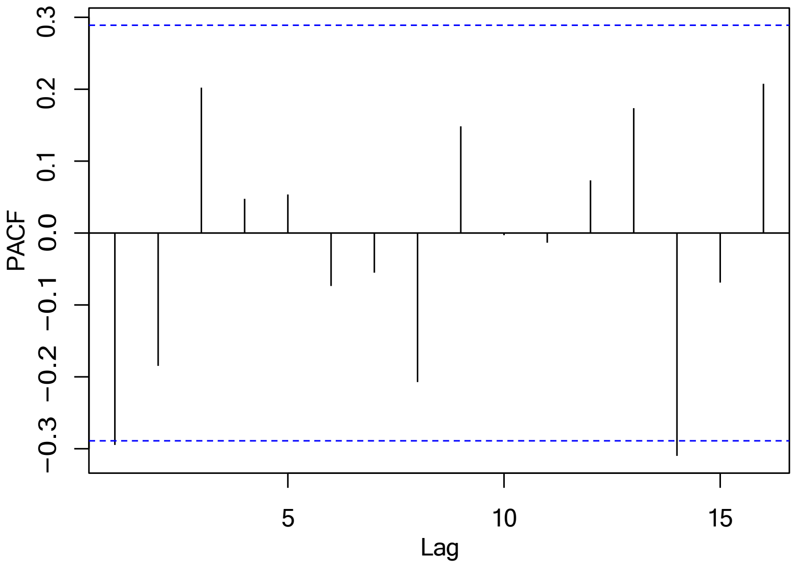


This plot clearly gives indication of no presence of any type of correlation between the residuals.

PACF plot

Similarly, we plot the sample partial autocorrelation coefficients for different lags and got the same kind of observations indicating no presence of correlations.

```
pacf(resi,ylab = "",xlab = "",main = "")
title(xlab="Lag", ylab="PACF", line=2)
```



Durbin-Watson Test

To test the null hypothesis H_0 : errors are uncorrelated against H_1 : errors are correlated, we perform Durbin-Watson test which gives the following results :-

```
require(lmtest)
dwtest(Y~A1+A2+A3+A4,data=X)


Durbin-Watson test

data:  Y ~ A1 + A2 + A3 + A4
DW = 2.5494, p-value = 0.9734
alternative hypothesis: true autocorrelation is greater than 0
```

Since the test gives high p-value we can accept H_0 hence the assumption of uncorrelated residuals can be assumed to be satishfied.

Breusch–Godfrey test

To check whether residuals are uncorrelated for higher orders, we perform the Breusch–Godfrey test upto order 20.

```
require(lmtest)
bgtest(reg,order = 20)


Breusch-Godfrey test for serial correlation of order up to 20

data:  reg
LM test = 24.951, df = 20, p-value = 0.2033
```

Here also the p-value is fairly high favouring the null assumption.

3.3 Detecting Influential Points

Hat Matrix Diagonals

To detect high leverage points, we compute the hat matrix diagonals h_i of the matrix $\boldsymbol{H} = \boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T$:-

```
hat_d <- hatvalues(reg)
head(sort(hat_d,dec=TRUE))

       36        20        12        38        30        39 
0.4994692 0.3042241 0.3015417 0.2998329 0.2078749 0.1721364 
```

We find out if there is any diagonal element with value $> \frac{2p}{n}$ as they should be looked at more closely.

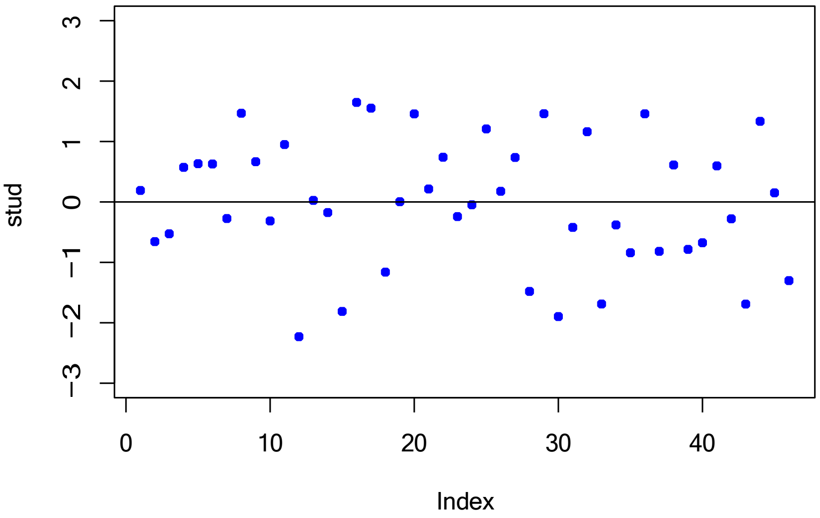
```
n = nrow(X);p = 5
hat_d[hat_d > 2*(p/n)]

       12        20        36        38 
0.3015417 0.3042241 0.4994692 0.2998329 
```

Hence we will apply other procedures also to confirm whether these points are influential or not.

Externally Studentized Residuals

We plot the externally studentized residuals using the formula $t_i^2 = r_i^2\left(\frac{n-p-1}{n-p-r_i^2}\right)$:-

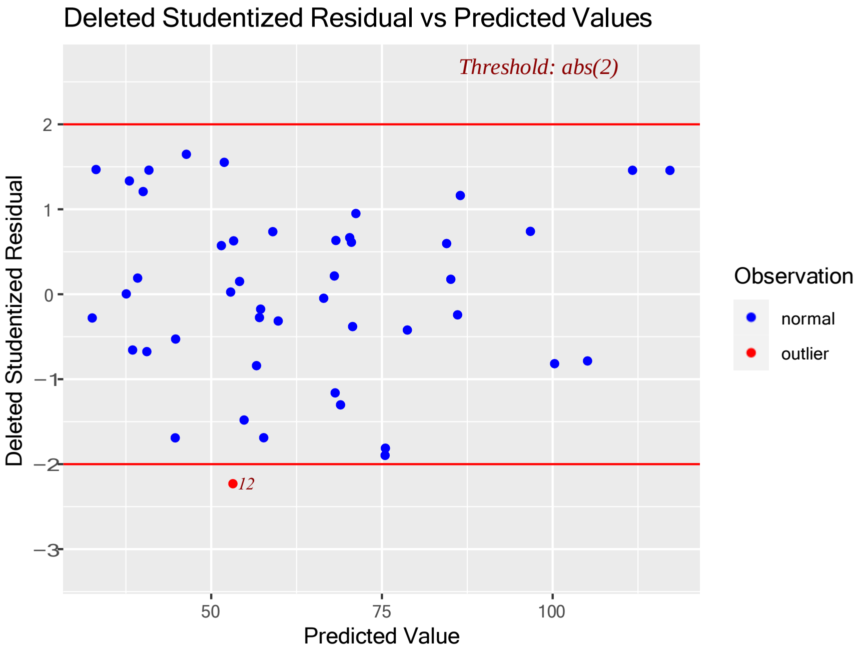
```
stud <- rstudent(reg)
plot(stud,ylim = c(-3,3),pch=20,col = "blue")
abline(h=c(0))
```



If the assumptions are correct i.e. $\boldsymbol{\epsilon} \sim N_n(\boldsymbol{0}, \sigma^2\boldsymbol{I})$ then we should get that $t_i \sim t_{n-p-1}$.

Hence the significant externally studentized residuals will have values $|t_i| > t_{n-p-1;\frac{\alpha}{2}} \iff t_i^2 > F_{n-p-1;\alpha}$:-

```
ols_plot_resid_stud_fit(reg)
```



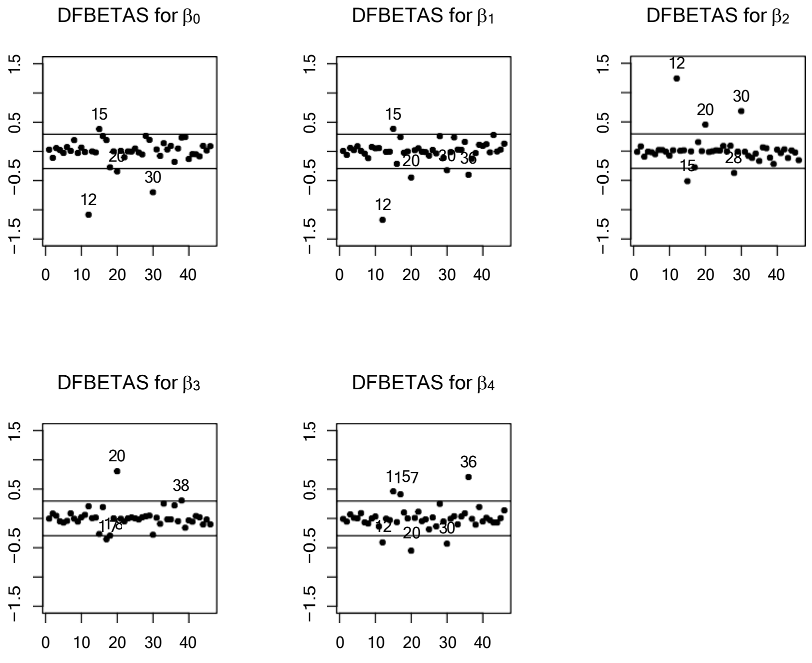
We can see from the plot that one residual is significant hence we treat that as an outlier.

DFBETAS

After outliers , we check for presence of high leavarage points, which can be detected using DFBETAS measure for different parameters $DFBETAS_{ij} = \frac{\hat{\beta}_j - \hat{\beta}_{(i)j}}{S(i)\sqrt{\sum_i c^2_{j+1,i}}}$ where $\boldsymbol{C} = ((c_{ij})) = \left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}$.

We will consider the points for which $|DFBETAS_{ij}| > \frac{2}{\sqrt{n}}$ In the next slide we plot the values for all the 5 coefficients $\beta_i, i = 0, ..., 4$.

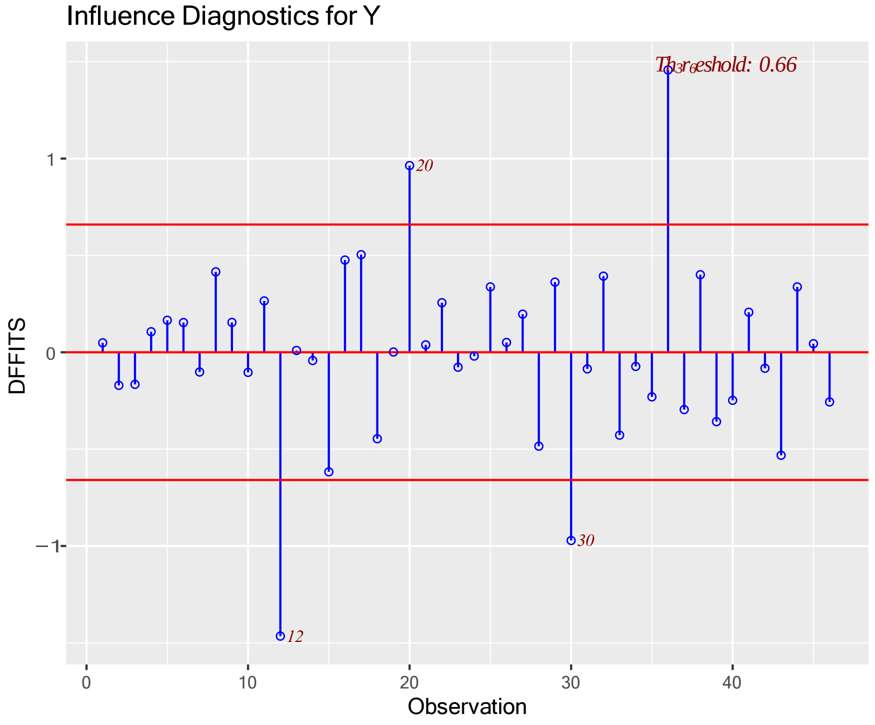
```
par(mfrow = c(2,3))
DFBETAS = dfbetas(reg)
for(i in 1:5)
{
  plot(DFBETAS[,i],main=bquote("DFBETAS for" ~ beta[.(i-1)]),ylab="",ylim=c(-1.5,1.5),xlab="",pch=
  abline(h=c(-2/sqrt(n),2/sqrt(n)))
  ind = which(abs(DFBETAS[,i]) > 2/sqrt(n)) # beta_0
  text = text(ind,DFBETAS[ind,i],pos = 3,labels = ind)
}
```



DFFITS

To notice the change in fitted values, we plot the DFFITS values for all the points where $DFFITS_i = t_i \left(\frac{h_i}{1-h_i}\right)^{1/2}$ and we will check for the points for which $|DFFITS_i| > 2\sqrt{\frac{p}{n}}$.

```
ols_plot_dffits(reg)
```



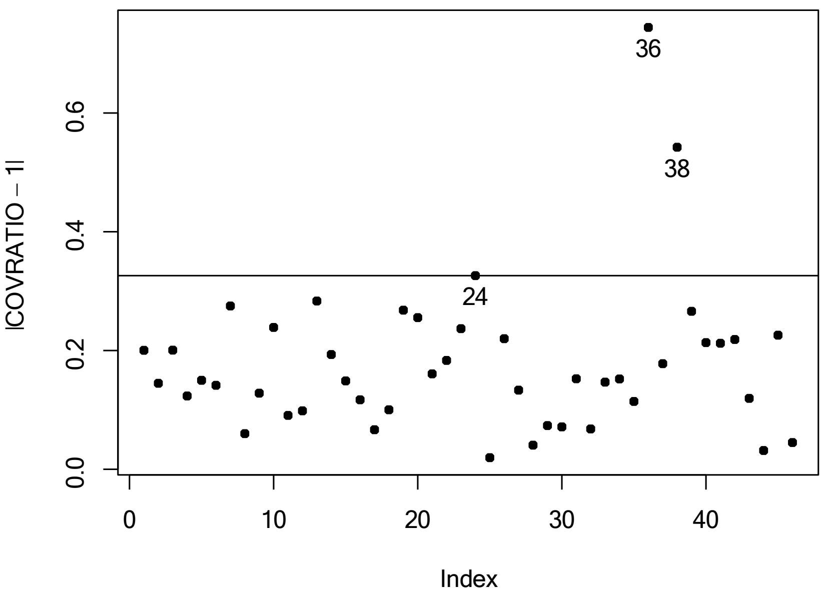
COVRATIO

We also plot the COVRATIO values which are defined as

$$COVRATIO_i = {}^{A}\,\frac{n-p-1}{n-p} + \frac{t_i^2}{n-p}\,{}^{B_{-p}}\,(1-h_i)^{-1}$$

and we consider the points to have high fluence for which $|COVRATIO - 1| > \frac{3p}{n}$.

```
COVRATIO  =  covratio(reg)
plot(abs(COVRATIO-1),ylab=expression(abs(COVRATIO-1)),pch  =  20)
abline(h  =  3*p/n)
ind  =  which(abs(COVRATIO-1)  >=  3*p/n)
text(ind,abs(COVRATIO[ind]-1),pos  =  1,labels  =  ind)
```



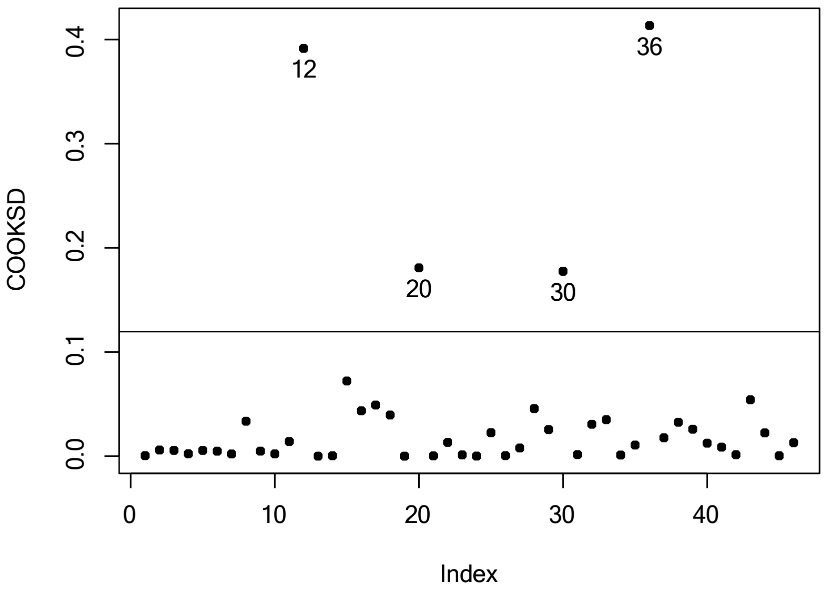
Cook's D

Lastly, we calculate the Cook's Distance $D_i = \frac{(\hat{\beta}_{(i)}-\hat{\beta})^T X^T X(\hat{\beta}_{(i)}-\hat{\beta})}{pS^2} = r_i^2 \frac{h_i}{p(1-h_i)}$ for all the n points.

We flag the points as suspicious for which $D_i > \frac{4}{n}$ here $n = 46, p = 5$. Whose value equals to 0.087.

We plot the values and see if such suspicious points exists or not.

```
COOKSD  =  cooks.distance(reg)
plot(COOKSD,pch=20)
abline(h  =  3*mean(COOKSD))
ind  =  which(COOKSD  >  3*mean(COOKSD))  # beta_0
text  =  text(ind,COOKSD[ind],pos  =  1,labels  =  ind)
```



We can clearly notice that the points 12,20,30,36 have significant values of D_i. So we will investigate them further.

Conclusion

From all the diagnostics performed for finding influential observations, we can make the following table of our findings :-

Diagnostic Measures	Points Detected
h_i	12, 20, 36, 38
t_i	12
$DFBETAS$	12, 15, 17, 18, 20, 28, 30, 36
$DFFITS$	12, 20, 30, 36
$COV\ RATIO$	24, 36, 38
$Cook's\ D$	12, 20, 30, 36

Hence, from the table, we conclude the points 12, 20, 30, 36 to be influential points and we we will later remove them from the model and see the changes occuring in all aspects of the fitted linear models.

3.4 Remedies For Influential Points

Removing Influential Points

We remove the influential points and then again fit a linear model with all the covariates and write the summary output of the fitted model here :-

```
summary(lm(Y~A1+A2+A3+A4,data=X[-c(12,20,30,36),]))


Call:
lm(formula = Y ~ A1 + A2 + A3 + A4, data = X[-c(12, 20, 30, 36),
    ])
```

```
Residuals:
     Min        1Q    Median        3Q       Max
-16.6468   -5.0683   -0.3998    6.1885   16.7016

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 20.2021     11.7358   1.721 0.09353 .
A1            0.8783     0.2582   3.401  0.00162 **
A2           -0.7104     0.6205  -1.145  0.25960
A3            1.2890     0.3960   3.255  0.00243 **
A4            0.1489     0.1312   1.134  0.26392
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 8.672 on 37 degrees of freedom
Multiple R-squared:  0.8236,Adjusted R-squared:  0.8045
F-statistic: 43.18 on 4 and 37 DF,  p-value: 1.867e-13
```

This model has increased value of $R^2{}_{adj} = 0.8045$.

Improvements in the fitted model

From the output,we can see that the $R^2{}_{adj}$ value has increased from that of the full model which was = 0.7954.

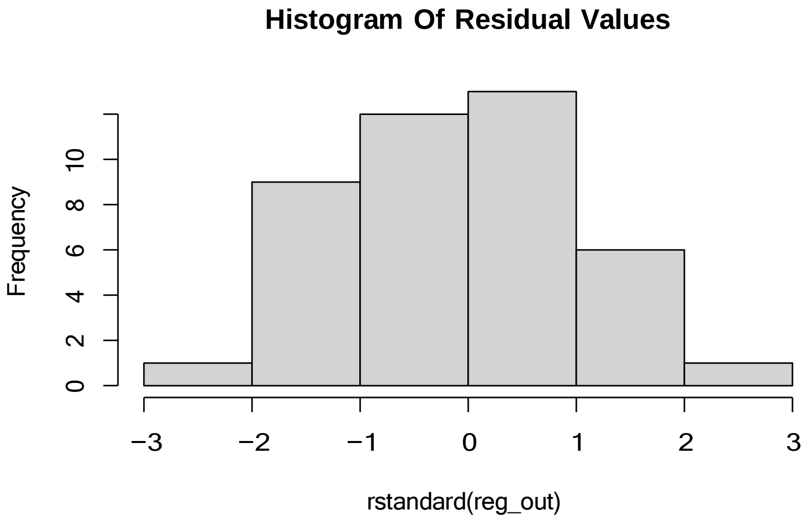
1 2 Also we can see that the model indicates the estimates of the intercept term $\hat{\beta}_0$, variables A1 & A3 1 2 $\hat{\beta}_1, \hat{\beta}_3$ to be signifcant.

Whether covariate A2 is signifcant or not, will be verified later.

Histogram of Residuals

We also plot the histogram of the residual values and can notice it's symmetric about 0 and seems to be normally distributed :-

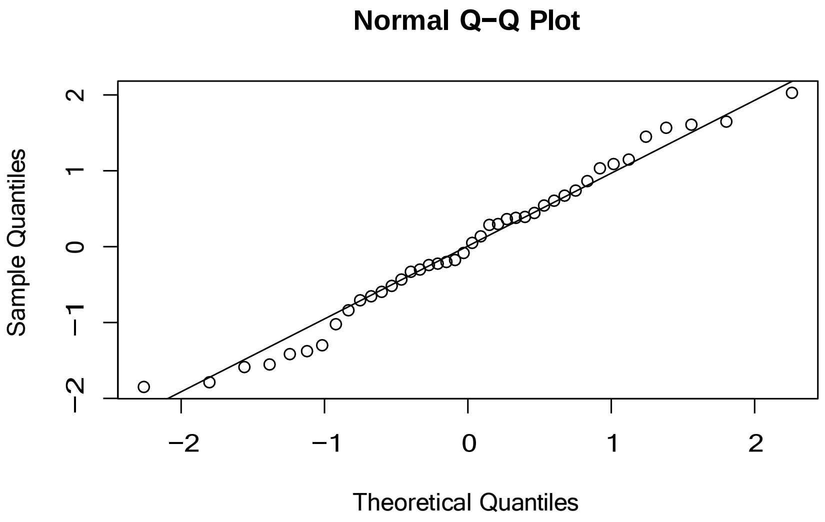
```
reg_out = lm(Y~A1+A2+A3+A4,data = X[-c(12,20,30,36),])
hist(rstandard(reg_out),breaks = 5,main = "Histogram Of Residual Values")
```



QQ-plot

For checking the assumptions for the residuals we again make the quantile-quantile plot of the standardized residuals :-

```
reg_out = lm(Y~A1+A3+A4,data = X[-c(12,20,30,36),])
qqnorm(rstandard(reg_out))
qqline(rstandard(reg_out))
```



Checking Other Assumptions

We again perform both the Shapiro-Wilk test and Durbin-Watson test for checking normality and presence of correlation between the residuals, respectively. We write down the observations in the following table :-

Tests	Model with influential points	Model without influential points
Shapiro-Wilk	0.1133	0.8429
Durbin-Watson	0.9734	0.6943
Breusch-Pagan	0.6105	0.5742
Breusch-Godfrey	0.2033	0.7973

Hence we can see considerable improvement in the normality assumptions of the residuals whereas the uncorrelated & homoskedasticity assumptions are more or less remains equally acceptable.

Hence, the model can be considered to be better than the previous model as a result of removing the influential points.

3.5 Collinearity

Multicollinearity

Next we consider the problem of multicollinearity that may be present in our dataset as suspected from the pairwise scatterplots.

We calculate the condition number for the scaled and centred model matrix $\boldsymbol{X}^*$ which is $\kappa(\boldsymbol{X}^*) = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}}$ where λ_i's are the eigenvalues of $\boldsymbol{X}^{*T}\boldsymbol{X}^* = \boldsymbol{R}_{xx}$. We calculate $\kappa(\boldsymbol{X}^*)$ using R :-

```
X_mdl = model.matrix(reg)[,-1]
kappa(scale(X_mdl))

[1]  8.624355
```

Hence, the square of condition number $\kappa^2(\boldsymbol{X}^*) \approx 74.379$ is an upper bound for the VIFs which is quite large !

VIF

Now, to determine whether some covariate with corresponding column $\boldsymbol{x}_j$, can be predicted accurately using other covariates or not, we compute the variance inflation factors $\text{VIF}_j = \frac{1}{1-R_j^2}$ where R_j^2 is the coefficient of determination of the regression of $\boldsymbol{x}^{*(j)}$ on the columns of $\boldsymbol{X}^{*(j)}$.

We calculate the VIF_j values for $j = 1, 2, 3, 4$ in R :-

```
require("faraway")
vif(lm(Y~A1+A2+A3+A4,data = X[-c(12,20,30,36),]))#high implies collinearity

       A1        A2        A3        A4
9.261059  9.360894  2.857795  2.751179
```

For the variables A1 and A2, we can see that the VIF values are greater than 5 and even close to 10! So we can interpret this as "the standard error of $\hat{\beta}_1$ and $\hat{\beta}_2$ would be $\sqrt{9.26} \approx 3.043$ and $\sqrt{9.361} \approx 3.059$ times more (respectively) than it would have been without the presence of collinearity".

Effect of Influential Points on Collinearity

This is a very interesting observation that we have made in the dataset.

If we remove the influential points and then calculate the VIF values, we get :-

```
require("faraway")
vif(lm(Y~A1+A2+A3+A4,data = X[-c(12,20,30,36),]))#high implies collinearity

       A1        A2        A3        A4
9.261059  9.360894  2.857795  2.751179
```

But if we do the same without removing those points, we get :-

```
require("faraway")
vif(lm(Y~A1+A2+A3+A4,data = X[]))#high implies collinearity

       A1        A2        A3        A4
5.910333  6.748416  3.080737  3.488172
```

So, as we can see the VIF values increased after removal of the influential points.

This is intuitive from the fact that actual linear dependence between the covariates was being slightly nullified by the presence of such influential points.

Demonstrating Effect of Collinearity

To demonstrate how collinearity can affect the estimates badly, we deliberately introduce some random noise in the response observations ($\epsilon \sim N(0, 1)$) and then fit a linear model and see the changes in the estimate.

```
set.seed(2124)
lmod_per = lm(Y+10*rnorm(nrow(X)-4,s = 5) ~ A1+A2+A3+A4,data = X[-c(12,20,30,36),])
reg$coefficients

 (Intercept)           A1           A2           A3           A4
-13.96310010   0.09828590   1.14837707   1.85786103   0.04817018

lmod_per$coefficients

(Intercept)          A1          A2          A3          A4
   1.304038    2.353402   -2.501280   -4.056390    1.437409
```

Hence, we can clearly see the how the estimates change a lot for introducing random noise in the response.

3.6 Remedies For Collinearity

Dealing with Collinearity

To deal with the collinearity present in the dataset, we first try to remove one of the correlated covariates "A1" or "A2" and see what improvements are observed in the variation inflation factors :-

Model	Condition Number (κ)	R^2_{adj}
$Y = \beta_0 + \beta_1 A1 + \beta_2 A2 + \beta_3 A3 + \beta_4 A4$	8.512	0.795
$Y = \beta_0 + \beta_2 A2 + \beta_3 A3 + \beta_4 A4$	5.625	0.755
$Y = \beta_0 + \beta_1 A1 + \beta_3 A3 + \beta_4 A4$	3.951	0.803

We also check for other assumptions between the models :-

Tests	Full Model	Without "A1"	Without "A2"
Shapiro-Wilk	0.1133	0.563	0.2461
Durbin-Watson	0.9734	0.7376	0.7638
Breusch-Pagan	0.6105	0.3742	0.9423

Dealing with Collinearity

Hence, the model with covariates "A1", "A3", "A4" seems to be a much better model in terms of both prediction and accuracy of the estimates of $\boldsymbol{\beta}$. Also if we calculate the VIF values for the last model, we get them to be considerably small :-

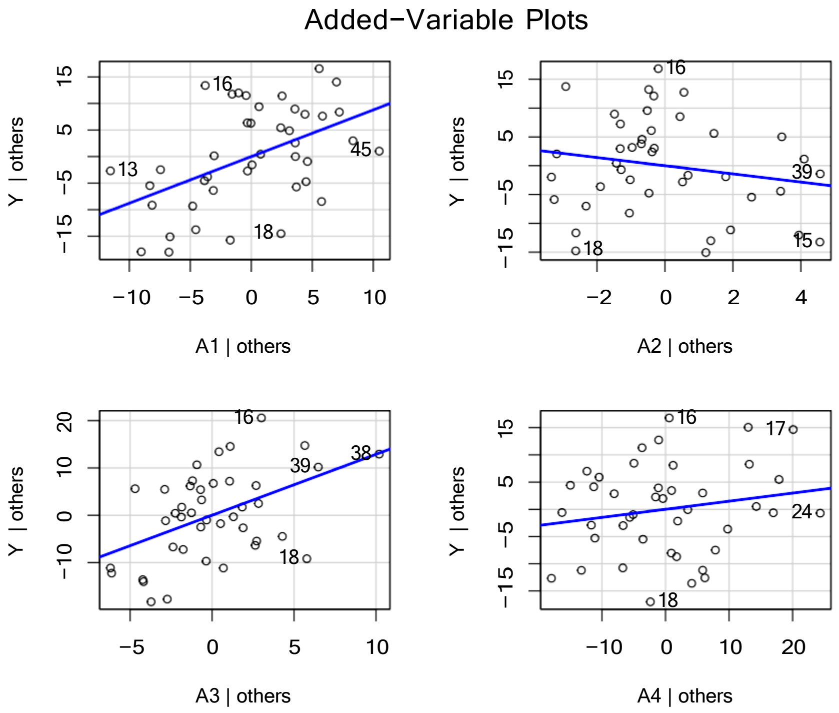
```
vif(reg_out)

      A1       A3       A4
2.360578 2.491333 1.677148
```

This is also relatable from the fact that initially covariate "A2" had the maximum VIF value and the condition number also decreased significantly due to its removal.

Added Variable Plot

For better understanding the contribution of a covariate in the regression model, we make a scatter plot of $\boldsymbol{e}^{(i)} = (\boldsymbol{I} - \boldsymbol{P}_i)\,\boldsymbol{Y}$ against $(\boldsymbol{I} - \boldsymbol{P}_i)\,\boldsymbol{x}^{(i)}$ where $\boldsymbol{e}^{(i)}$ are the residuals of the model with variable Ai excluded and $\boldsymbol{x}^{(i)}$ is the column of observations of Ai. This is also called the added variable plot.

```
avPlots(lm(Y~A1+A2+A3+A4,data = X[-c(12,20,30,36),]))
```



Conclusion

From the 4 plots, we can see that slopes of the fitted lines for the added variable plots of A1 & A3 are much more signifcant than other two plots for A2 & A4.

We can make some important conclusions from here.

This indicates that once predictor A1, A3, A4 is included, A2 can be excluded from the model for the high collinearity present between them.

Similar can be said for A4.

Now, for much better conclusions, we perform further model selection procedures based on several criterias.

3.7 Model Selection

Stepwise Selection

We perform the stepwise selection algorithm which performs a forward selection (FS) followed by a backward elimination (BE) using the AIC criterion and get to an optimum model.

We get the following sequence of models in the selection procedure :-

Model	AIC Value
$Y = \beta_0$	372.1803
$Y = \beta_0 + \beta_3 A3$	319.7608
$Y = \beta_0 + \beta_1 A1 + \beta_3 A3$	305.0896

We give the final model as an output we get in R :-

```
Stepwise  Selection  Method
-------------------------

Candidate  Terms:

1 . A1
2 . A2
3 . A3
4 . A4

 Step  0:  AIC  =  372.1803
 Y ~ 1


Variables  Entered/Removed:

                                   Enter New Variables
-----------------------------------------------------------------------------------------
Variable        DF       AIC           Sum Sq           RSS          R-Sq        Adj.  R-Sq
-----------------------------------------------------------------------------------------
A1               1     319.761      11455.113       4316.877      0.726           0.719
A3               1     324.405      10950.375       4821.615      0.694           0.687
A2               1     340.276       8736.385       7035.605      0.554           0.543
A4               1     355.432       5678.829      10093.161      0.360           0.344
-----------------------------------------------------------------------------------------

- A1 added


 Step 1 : AIC = 319.7608
 Y ~ A1

                                    Enter New Variables
---------------------------------------------------------------------------------------
Variable        DF       AIC           Sum Sq          RSS          R-Sq         Adj.  R-Sq
---------------------------------------------------------------------------------------
A3               1     305.090      12869.415      2902.575      0.816           0.807
A4               1     318.972      11732.403      4039.587      0.744           0.731
A2               1     321.033      11529.316      4242.673      0.731           0.717
---------------------------------------------------------------------------------------

- A3 added


 Step  2 : AIC =   305.0896
 Y ~ A1 + A3

                               Remove  Existing  Variables
---------------------------------------------------------------------------------------
Variable        DF       AIC           Sum Sq          RSS          R-Sq         Adj.  R-Sq
---------------------------------------------------------------------------------------
A3               1     319.761      11455.113      4316.877      0.726           0.719
A1               1     324.405      10950.375      4821.615      0.694           0.687
---------------------------------------------------------------------------------------

                                    Enter New Variables
---------------------------------------------------------------------------------------
Variable        DF       AIC           Sum  Sq         RSS          R-Sq         Adj.  R-Sq
---------------------------------------------------------------------------------------
A2               1     306.749      12892.877      2879.113      0.817           0.803
A4               1     306.775      12891.077      2880.913      0.817           0.803
---------------------------------------------------------------------------------------


No  more  variables  to  be  added  or  removed.
```

```
Final Model Output
------------------

                              Model Summary
-------------------------------------------------------------------
R                             0.903        RMSE              8.627
R-Squared                     0.816        Coef. Var        14.066
Adj. R-Squared                0.807        MSE              74.425
Pred R-Squared                0.788        MAE               6.755
-------------------------------------------------------------------
 RMSE: Root Mean Square Error
 MSE: Mean Square Error
 MAE: Mean Absolute Error

                                    ANOVA

                      Sum of
                     Squares        DF    Mean Square      F          Sig.
---------------------------------------------------------------------------
Regression         12869.415         2       6434.707    86.459      0.0000
Residual            2902.575        39         74.425
Total              15771.990        41
---------------------------------------------------------------------------

                                  Parameter Estimates

      model      Beta    Std. Error    Std. Beta      t        Sig      lower      upper
-----------------------------------------------------------------------------------------
(Intercept)     9.924         5.106                 1.944    0.059     -0.403     20.251
         A1     0.640         0.126        0.521    5.078    0.000      0.385      0.895
         A3     1.516         0.348        0.447    4.359    0.000      0.813      2.219
-----------------------------------------------------------------------------------------


                                 Stepwise   Summary
-----------------------------------------------------------------------------------
Variable      Method       AIC        RSS        Sum Sq        R-Sq       Adj. R-Sq
-----------------------------------------------------------------------------------
A1           addition    319.761    4316.877    11455.113    0.72629      0.71945
A3           addition    305.090    2902.575    12869.415    0.81597      0.80653
-----------------------------------------------------------------------------------
```

Hence, this method gives the model containing covariates “A1”, “A3” as the optimum one.

Best Subset Selection

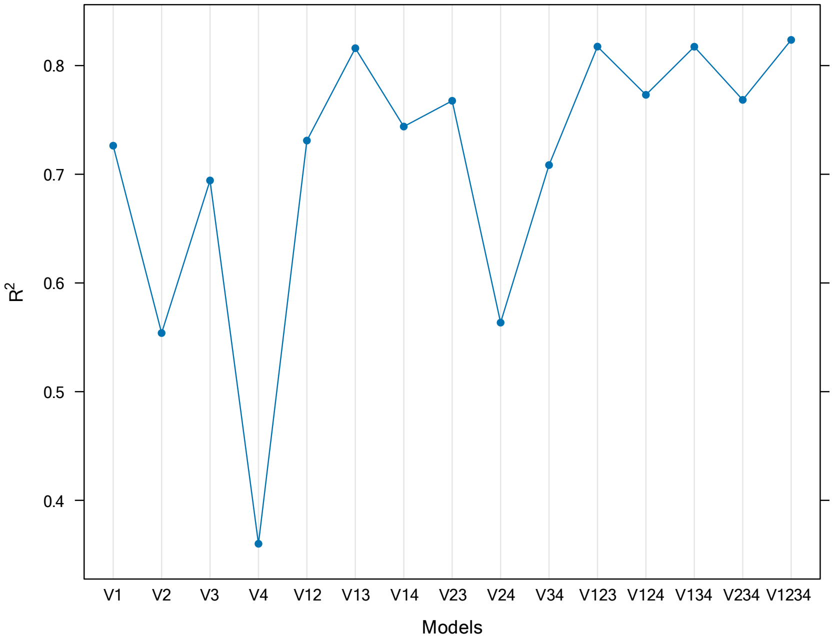
We use different criterions for chosing optimal model among all the 15 possible linear models and plot the diagrams for all of them one by one.

```
X_m = X[,-c(1,2)]
X1 <- X_m[-c(12,20,30,36),]
names(X1) <- c("V1","V2","V3","V4","V5")
models<-list()
models[["V1"]]<-lm(V5~V1,X1)
models[["V2"]]<-lm(V5~V2,X1)
models[["V3"]]<-lm(V5~V3,X1)
models[["V4"]]<-lm(V5~V4,X1)
models[["V12"]]<-lm(V5~V1+V2,X1)
models[["V13"]]<-lm(V5~V1+V3,X1)
models[["V14"]]<-lm(V5~V1+V4,X1)
models[["V23"]]<-lm(V5~V2+V3,X1)
models[["V24"]]<-lm(V5~V2+V4,X1)
models[["V34"]]<-lm(V5~V3+V4,X1)
models[["V123"]]<-lm(V5~V1+V2+V3,X1)
```

```
models[["V124"]]<-lm(V5~V1+V2+V4,X1)
models[["V134"]]<-lm(V5~V1+V3+V4,X1)
models[["V234"]]<-lm(V5~V2+V3+V4,X1)
models[["V1234"]]<-lm(V5~V1+V2+V3+V4,X1)

mnames<-factor(names(models),levels = names(models))
```

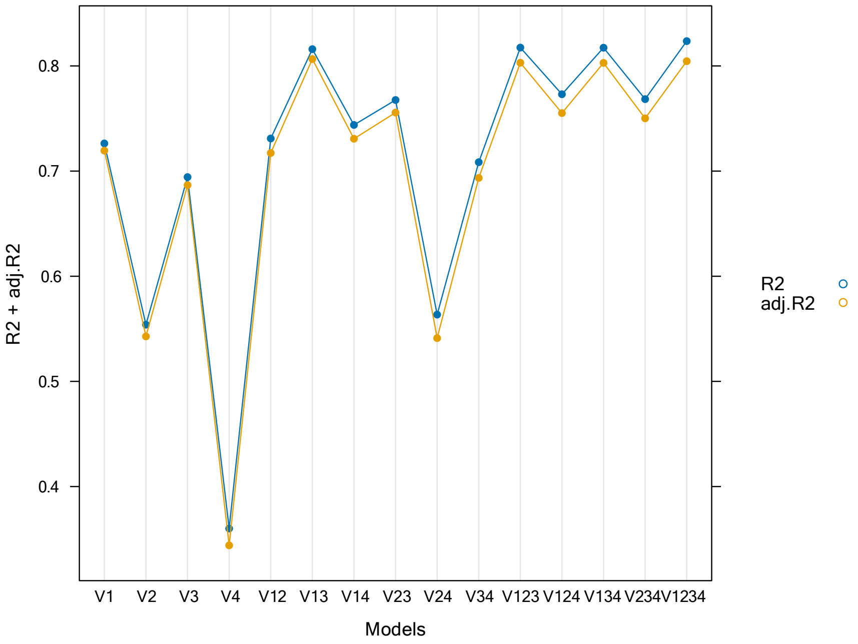
Coefficient of determination (R^2)

```
R2 <- sapply(models, function(fit) summary(fit)$r.squared)
dotplot(R2 ~ mnames, type = "o", pch = 16,auto.key=list(space="right"),xlab="Models",ylab=expressi
```

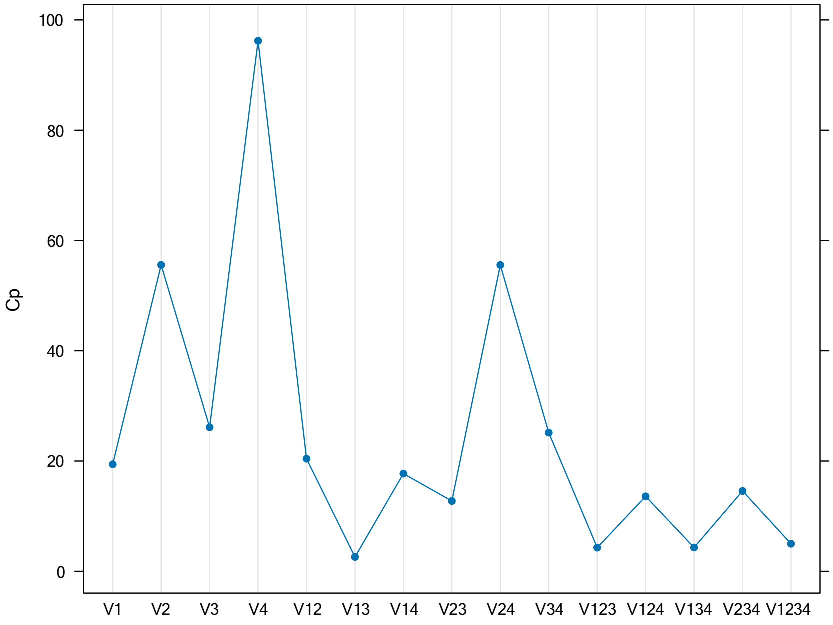


R^2 & R^2_{adj}

```
adj.R2 <- sapply(models, function(fit) summary(fit)$adj.r.squared)
dotplot(R2 + adj.R2 ~ mnames, type = "o", pch = 16,auto.key=list(space="right"),xlab="Models")
```

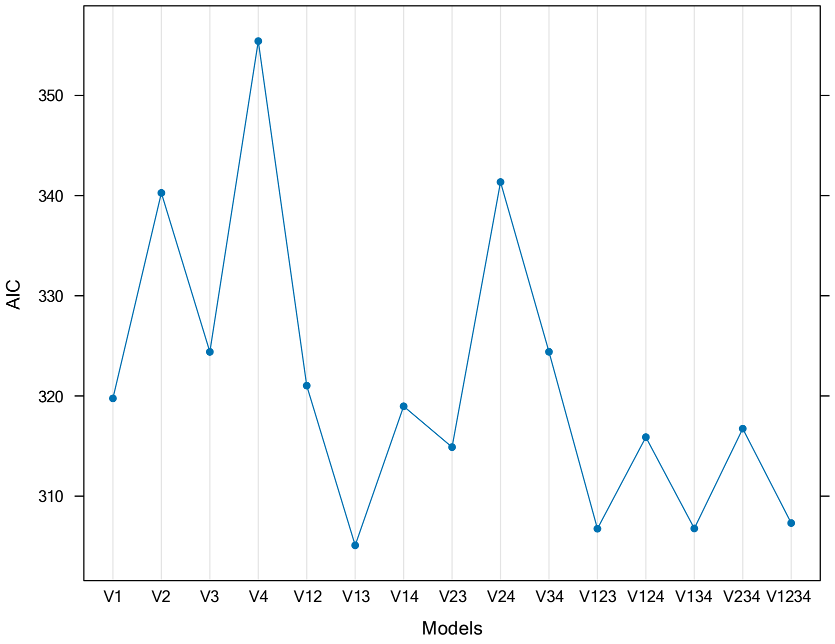


Mallow's C_p



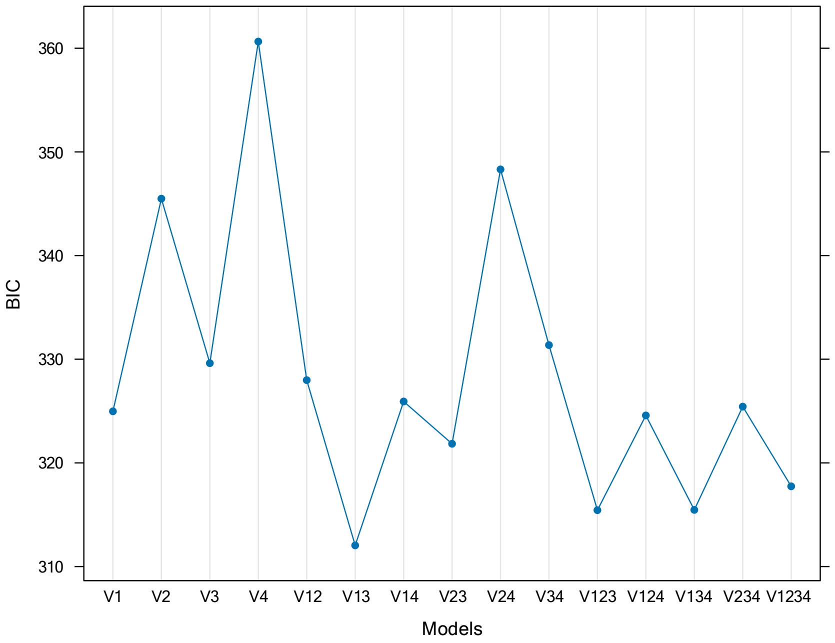
Akaike information criterion (AIC)

```
AIC <- sapply(models, function(fit) AIC(fit))
dotplot(AIC ~ mnames, type = "o", pch = 16,xlab="Models")
```



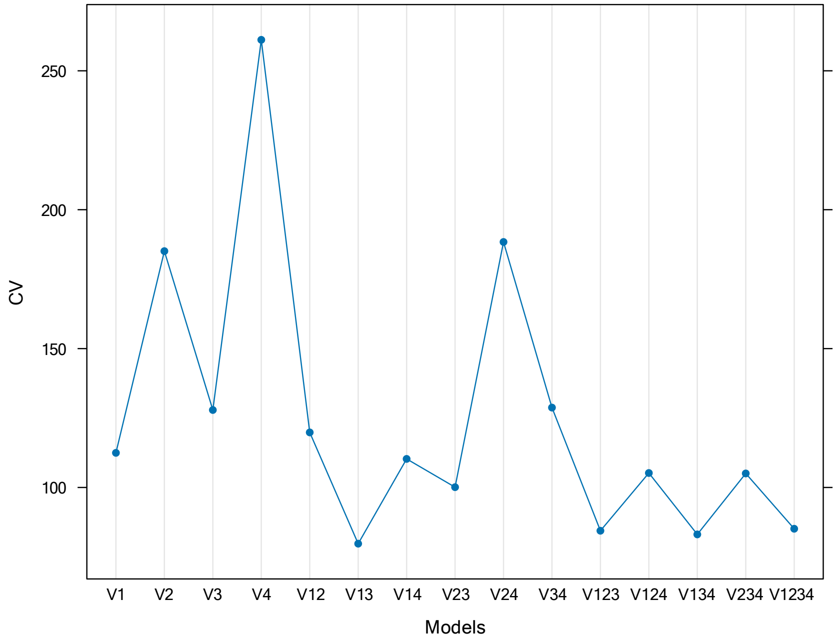
Bayesian information criterion (BIC)

```
BIC <- sapply(models, function(fit) BIC(fit))
dotplot(BIC ~ mnames, type = "o", pch = 16,xlab="Models")
```



Leave-One-Out CV

```
CV = NULL
for(i in 1:15)
{
  X_mdl_mat = model.matrix(models[[i]])
  head(X_mdl_mat)
  Y_vec  = X1$V5
  H = X_mdl_mat%*%solve(t(X_mdl_mat)%*%X_mdl_mat)%*%t(X_mdl_mat)
  h = diag(H)
  n_h = nrow(X1)
  CV[i] = (1/n_h)*sum((Y_vec-H%*%Y_vec)^2/(1-h)^2)
}
dotplot(CV ~ mnames, type = "o", pch = 16,xlab="Models")
```



Values of different measures for all the models

We list down the values of R^2, R^2_{adj} , Mallow's C_p, AIC, BIC, CV (1) values in one table for all the 15 models for better comparison :-

```
            R2      adj.R2         Cp       AIC      BIC          CV
V1    0.7262947 0.7194521 19.406493 319.7608 324.9738 112.46225
V2    0.5539177 0.5427657 55.560548 340.2757 345.4888 185.08170
V3    0.6942925 0.6866498 26.118573 324.4050 329.6180 127.88888
V4    0.3600579 0.3440593 96.220397 355.4325 360.6455 261.15078
V12   0.7309995 0.7172046 20.419718 321.0326 327.9832 119.82317
V13   0.8159665 0.8065289  2.598883 305.0896 312.0402  79.76883
V14   0.7438759 0.7307413 17.719047 318.9724 325.9231 110.28385
V23   0.7675782 0.7556591 12.747757 314.8939 321.8445 100.10949
V24   0.5635089 0.5411247 55.548917 341.3629 348.3135 188.38242
V34   0.7084529 0.6935017 25.148610 324.4131 331.3638 128.75523
V123  0.8174541 0.8030425  4.286879 306.7487 315.4370  84.42043
V124  0.7730732 0.7551579 13.595257 315.8890 324.5773 105.20886
V134  0.8173399 0.8029194  4.310820 306.7749 315.4633  83.10493
```

```
V234   0.7684411 0.7501601 14.566777 316.7376 325.4260 105.04886
V1234 0.8235897 0.8045183   5.000000 307.3127 317.7387   85.14023
```

Conclusion

Now, we write down the optimals models we get from different model selection criterions with corresponding values :-

Criterions	Optimum Model	Value
R^2_{adj}	$Y = \beta_0 + \beta_1 A1 + \beta_3 A3$	0.8065
Mallow's C_p	$Y = \beta_0 + \beta_1 A1 + \beta_3 A3$	2.598
AIC	$Y = \beta_0 + \beta_1 A1 + \beta_3 A3$	305.089
BIC	$Y = \beta_0 + \beta_1 A1 + \beta_3 A3$	312.0402
CV (1)	$Y = \beta_0 + \beta_1 A1 + \beta_3 A3$	79.76883

Hence, clearly this indicates among all the linear models, $Y = \beta_0 + \beta_1 A1 + \beta_3 A3$ is optimum based on several criterions.

This is also intuitive from the fact that here we are removing the covariates which had linear dependence.

In terms of terminology of the given dataset, the optimum predictors of Y = Cirrhosis death rate are A1 = Urban population & A3 = Wine consumption per capita.

So the optimum fitted model can be written as :-

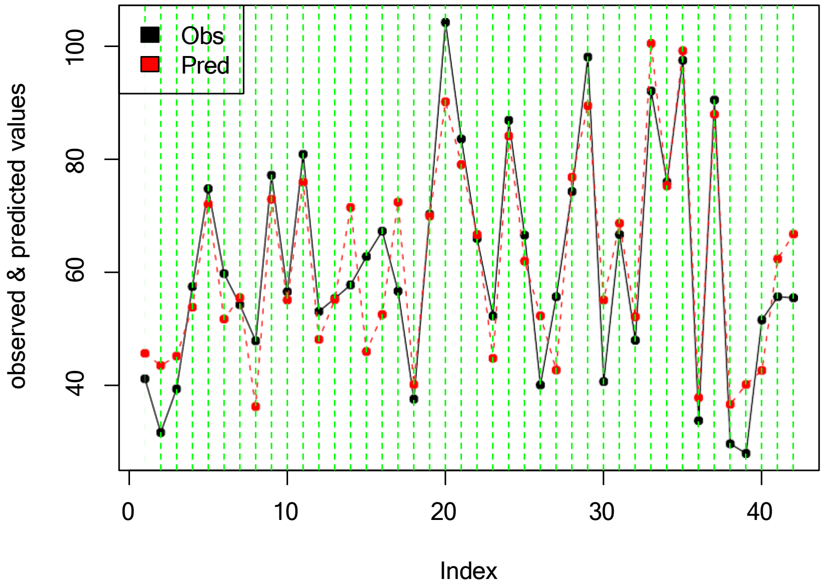
$$Y = 9.9241 + 0.6397 A1 + 1.5159 A3$$

Also we observed that all the covariates in the model are signifcant.

Obs vs Fitted Values

We plot the observed vs fitted values obtained using this model :-

```r
X1 = X[-c(12,20,30,36),]
mod_opt = lm(Y ~ A1+A3, data = X[-c(12,20,30,36),])
plot(1:nrow(X1),X1$Y,type = "o",pch = 20,ylab = "observed & predicted values",xlab = "Index")
lines(1:nrow(X1),mod_opt$fitted.values,type = "o",pch = 20,col = "red",lty = 2)
abline(v = 1:nrow(X1),lty = 2,col = rgb(0,1,0,alpha = 0.3))
legend("topleft",legend = c("Obs","Pred"),fill = c("black","red"))
```



Here we can see the prediction is accurate compared to the full model.

Conclusion

Again we perform all the diagonistic tests for this final model and find the following :-

Tests	p-values
Shapiro-Wilk	0.6282
Durbin-Watson	0.645
Breusch-Pagan	0.2723
Breusch-Godfrey	0.9345

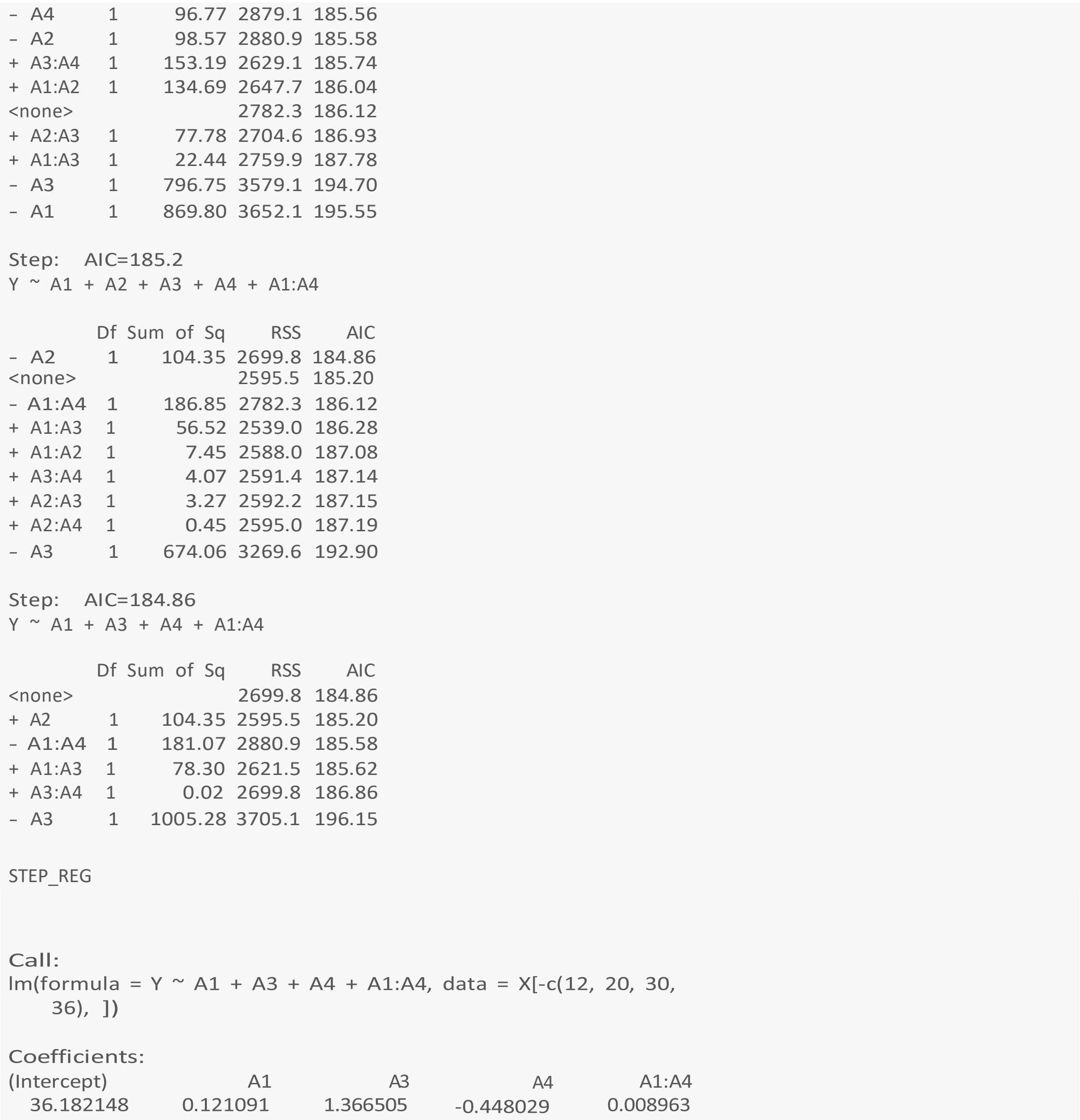
All the assumptions seem to be satishfied here. Hence we can really consider this to be a good model.

Models with Interaction Terms

One class of models that we have not considered yet are those with interaction terms (upto second order). Since there can be too many of them, we will not perform best subset selection here. Rather we again perform stepwise regression for choosing an optimal one among them.

In this class, we get the following model :-

```
mod_wt_out = lm(Y ~ A1+A2+A3+A4, data = X[-c(12,20,30,36),])
STEP_REG = stepAIC(mod_wt_out,scope = list(upper = ~(A1+A2+A3+A4)^2, lower = ~1),trace = TRUE)

Start:   AIC=186.12
Y ~ A1 + A2 + A3 + A4

         Df Sum of Sq    RSS    AIC
+ A1:A4   1    186.85 2595.5 185.20
+ A2:A4   1    173.31 2609.0 185.42
```



```
- A4      1      96.77 2879.1 185.56
- A2      1      98.57 2880.9 185.58
+ A3:A4   1     153.19 2629.1 185.74
+ A1:A2   1     134.69 2647.7 186.04
<none>                 2782.3 186.12
+ A2:A3   1      77.78 2704.6 186.93
+ A1:A3   1      22.44 2759.9 187.78
- A3      1     796.75 3579.1 194.70
- A1      1     869.80 3652.1 195.55

Step:  AIC=185.2
Y ~ A1 + A2 + A3 + A4 + A1:A4

        Df Sum of Sq    RSS    AIC
- A2     1    104.35 2699.8 184.86
<none>               2595.5 185.20
- A1:A4  1    186.85 2782.3 186.12
+ A1:A3  1     56.52 2539.0 186.28
+ A1:A2  1      7.45 2588.0 187.08
+ A3:A4  1      4.07 2591.4 187.14
+ A2:A3  1      3.27 2592.2 187.15
+ A2:A4  1      0.45 2595.0 187.19
- A3     1    674.06 3269.6 192.90

Step:  AIC=184.86
Y ~ A1 + A3 + A4 + A1:A4

        Df Sum of Sq    RSS    AIC
<none>               2699.8 184.86
+ A2     1    104.35 2595.5 185.20
- A1:A4  1    181.07 2880.9 185.58
+ A1:A3  1     78.30 2621.5 185.62
+ A3:A4  1      0.02 2699.8 186.86
- A3     1   1005.28 3705.1 196.15

STEP_REG


Call:
lm(formula = Y ~ A1 + A3 + A4 + A1:A4, data = X[-c(12, 20, 30,
    36), ])

Coefficients:
(Intercept)          A1          A3          A4       A1:A4
  36.182148    0.121091    1.366505   -0.448029    0.008963
```

This model has an adjusted R^2 value equal to 0.8299. But the main problem is this model has very high vif values and many of the predictors are not signifcant. So we don’t consider these type of models.

Conclusion

Hence we conclude our final multiple linear regression model is :-

$$Y = 9.9241 + 0.6397A1 + 1.5159A3$$

Obviously this model also has some drawback and there is no such “best” model that we can have but this performs more or less better than most of the models hence, it is a good one.

Next we use other types of regression models with different interpretations.

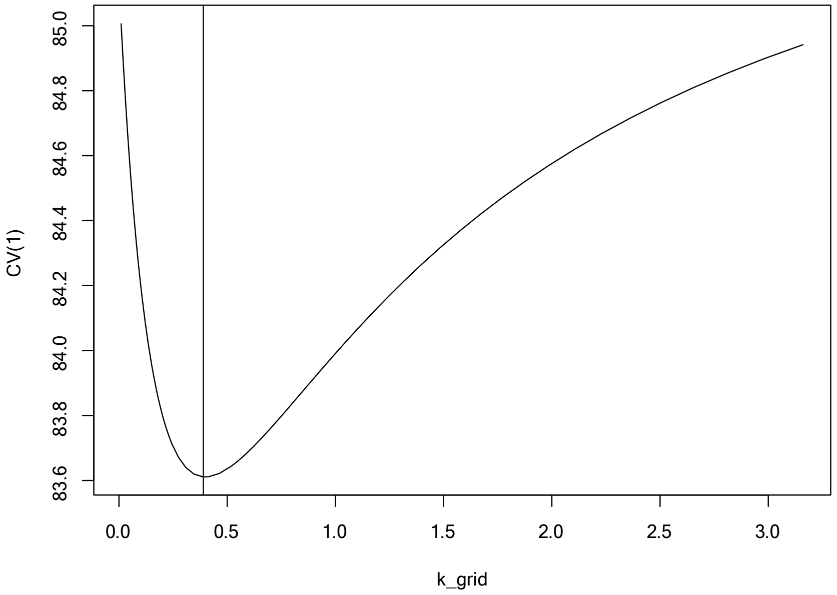
3.8 Shrinkage Methods

Ridge Regression

An alternate approach to deal with collinearity is fitting a ridge regression model as it can improve the accuracy of the predictions. The ridge estimate of the model parameters is $\hat{\boldsymbol{\beta}}(k) = \left(\boldsymbol{X}^T\boldsymbol{X} + k\boldsymbol{I}\right)^{-1}\boldsymbol{X}^T\boldsymbol{Y}$ where k is the ridge parameter. For an optimal choice of k, we calculate estimates of prediction errors of the ridge predictors for different choices of k over a set of trial values. This can be expressed as $CV_k(1) = \frac{1}{n}\sum_{i=1}^{n}\frac{\left[Y_i - \boldsymbol{x}_i'\hat{\boldsymbol{\beta}}(k)\right]^2}{[1-a_{ii}(k)]^2}$ and choose the k_{opt} for which this quantity is minimum.

We plot the $CV_k(1)$ values for different choices of k :-

```
k_grid = 10^seq(-2,1/2,length.out = 100)
PE = NULL
X_R = as.matrix(X[-c(12,20,30,36),c(2,3,4,5,6)])
Y_R = as.matrix(X[-c(12,20,30,36),c(7)])
n = nrow(X_R)
for(i in 1:length(k_grid))
{
  k = k_grid[i]
  beta_k = solve(t(X_R)%*%X_R + k*diag(rep(1,5)))%*%t(X_R)%*%Y_R
  A_k = X_R%*%solve(t(X_R)%*%X_R + k*diag(rep(1,5)))%*%t(X_R)
  Y_ft_R = X_R%*%beta_k
  A_K_diag = diag(A_k)
  PE[i] = (1/n)*sum((Y_R-Y_ft_R)^2/(1-A_K_diag)^2)
}

plot(k_grid,PE,type = "l",ylab = "CV(1)",xBlab = "k")
k_opt = k_grid[which(PE == min(PE))]
abline(v = k_opt)
```

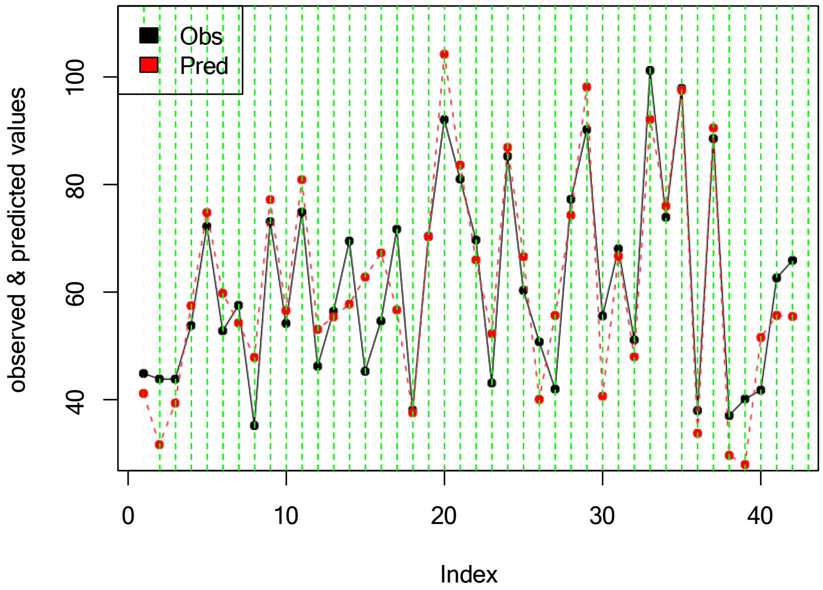


We find that the $CV_k(1)$ is minimum for $k \approx 0.3898$ hence, we calculate the corresponding ridge estimates.

The fitted model then becomes :-

$$Y = 11.777 + 0.767A1 + -0.322A2 + 1.324A3 + 0.114A4$$

This model has estimated prediction error ≈ 83.612.
This is close to the optimum OLS model that we have fitted.
We plot the observed & fitted values with the same index in the x-axis and get the following output :-



Lasso Regression

Another efficient way of model selection is using the Lasso Regression method. Here we minimize the sum of squares $||\boldsymbol{Y} - \boldsymbol{X\beta}||^2$ subject to the constraint $\sum_j |\beta_j| \leq \lambda$ for some $\lambda > 0$.

Using R, we find the Lasso Estimates of $\boldsymbol{\beta}$ where the value of λ is chosen using k-fold cross-validation criteria.

The optimum value of λ chosen by the criteria approximately equals≈ 0.501 and the model is :-

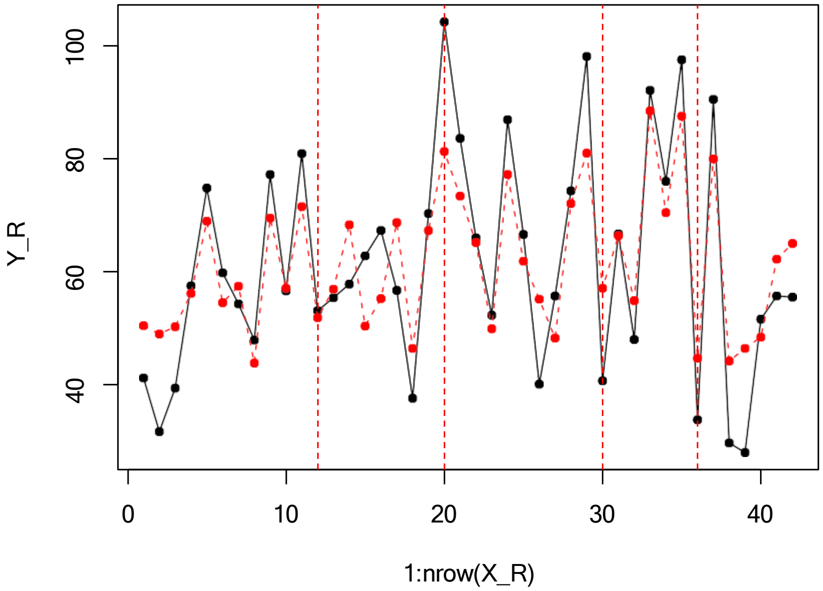
```
library(glmnet)
x <- as.matrix(X[-c(12,20,30,36),c(3,4,5,6)])
y <- X[-c(12,20,30,36),7]
lambdas <- 10^seq(-1, 5, by = 0.1)

lasso_reg <- cv.glmnet(x,y, alpha = 1, lambda = lambdas, standardize = TRUE, nfolds = 10)

lambda_best <- lasso_reg$lambda.min

lasso_model <- glmnet(x,y, alpha = 1, lambda = 5, standardize = TRUE)
c(lasso_model$a0,t(lasso_model$beta))


$s0
[1]  25.2888

[[2]]
```

```
1 x 4 sparse Matrix of class "dgCMatrix"
          A1  A2        A3 A4
s0 0.4579335  . 1.013119  .
```

Hence the Lasso Estimates of the parameters $\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} 25.28 \\ 0.457 \\ 0 \\ 1.013 \\ 0 \end{pmatrix}$.

As we can see from the output, here also, the variables “A2”, “A4” has been dropped and this also gives strong evidence in favour of the optimum linear model.

Here also make the observed and fitted plot for different index values :-



3.9 Robust Regression Methods

We have detected influential points in our dataset, and also removed them to get better models.

Now, we demonstrate using different robust regression methods how they can be used even if we have outliers in our dataset.

So we perform the rest of the methods using the full dataset, without removing any observation.

Least Absolute Deviation

Here we minimize the quantity $\sum_i |e_i(\boldsymbol{b})|$ i.e. $\hat{\boldsymbol{\beta}}_{LAD} = \underset{\boldsymbol{b}}{\text{argmin}} \sum_i |e_i(\boldsymbol{b})|$where $e_i(\boldsymbol{b}) = Y_i - \boldsymbol{x}_i^T\boldsymbol{b}$.

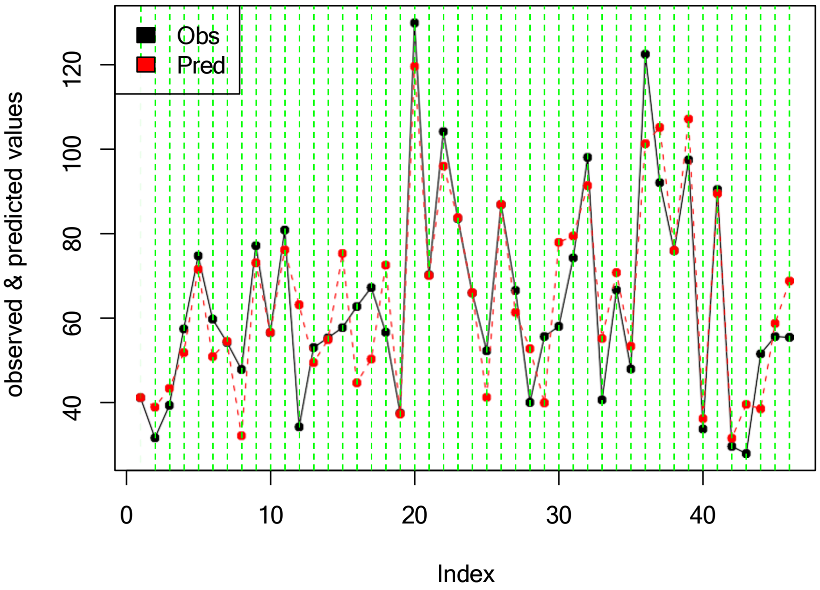
The estimated values of $\hat{\boldsymbol{\beta}}_{LAD}$ equals :-

```
library("L1pack")
rmodel_l <- lad(formula = Y~A1+A2+A3+A4,data = X)
rmodel_l$coefficients
```

```
(Intercept)          A1          A2          A3          A4
-7.19977780  0.42727435  0.61749765  1.96154792 -0.02363183
```

We plot the observed vs fitted values obtained using this model :-

```
plot(1:nrow(X),X$Y,type = "o",pch = 20,ylab = "observed & predicted values",xlab = "Index")
lines(1:nrow(X),rmodel_l$fitted.values,type = "o",pch = 20,col = "red",lty = 2)
abline(v = 1:nrow(X),lty = 2,col = rgb(0,1,0,alpha = 0.3))
legend("topleft",legend = c("Obs","Pred"),fill = c("black","red"))
```



The plot shows here the predicted values are more or less accurate for all the observations.

Least Median Square

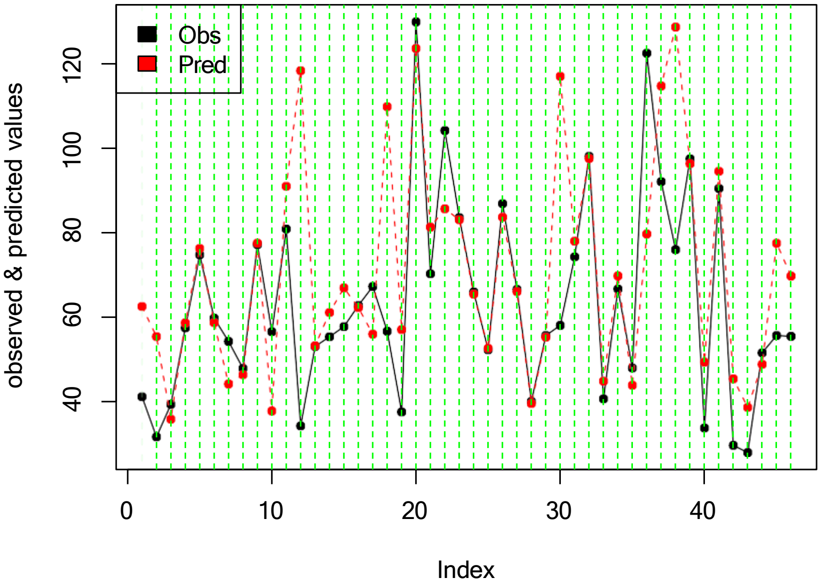
Here we minimize the median of the squared residuals $\hat{\beta}_{LMS} = \underset{b}{\text{argmin}}\ \underset{i}{\text{med}}\ e^2_i(\boldsymbol{b})$
.

Using R, we get the estimated value of $\hat{\beta}_{LMS}$ as :-

```
rmodel_l <- lqs(Y~A1+A2+A3+A4,data = X,method = "lms")
rmodel_l$coefficients

 (Intercept)           A1           A2           A3           A4
100.67646579   1.43011898  -3.48160404   3.15085397  -0.04186281
```

We plot the observed vs fitted values obtained using this model :-

```
par(mfrow = c(1,1))
plot(1:nrow(X),X$Y,type = "o",pch = 20,ylab = "observed & predicted values",xlab = "Index")
lines(1:nrow(X),rmodel_l$fitted.values,type = "o",pch = 20,col = "red",lty = 2)
abline(v = 1:nrow(X),lty = 2,col = rgb(0,1,0,alpha = 0.3))
legend("topleft",legend = c("Obs","Pred"),fill = c("black","red"))
```



We can see the prediction is quite accurate at some places where as it is bad at some others possibly due to the presence of outliers.

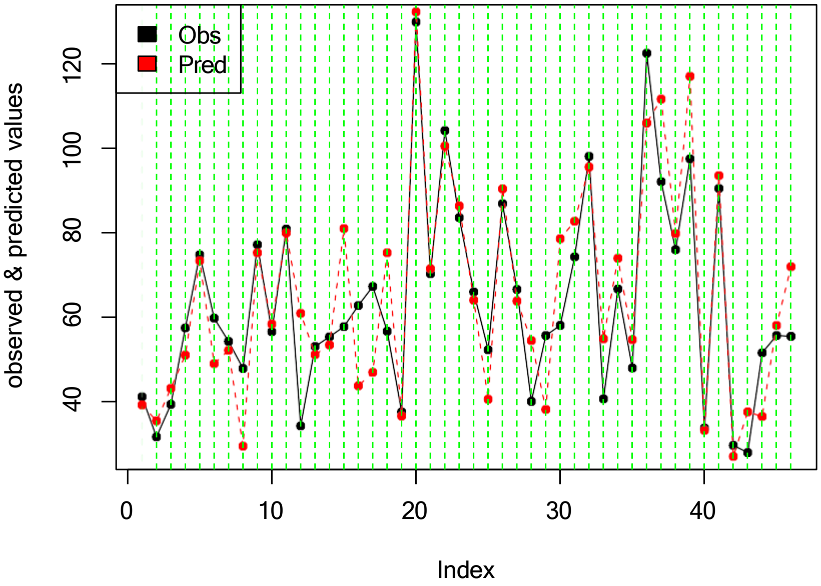
Least Trimmed Squares Estimate

Lastly, we compute the LTS estimates of $\boldsymbol{\beta}$ where we minimize the trimmed mean of the squared residuals $\hat{\boldsymbol{\beta}}_{LMS} = \underset{\boldsymbol{b}}{\operatorname{argmin}} \frac{1}{h} \sum_{i=1}^{h} e_i^2(\boldsymbol{b})$ for some appropriate choice of h. (Here we choose $h = [n/2] + 1$)

```
rmodel_l<-lqs(Y~A1+A2+A3+A4,data = X,method = "lts")
rmodel_l$coefficients

(Intercept)           A1           A2           A3           A4
-21.7304500    0.2861684    1.2215938    2.4980685   -0.1567162
```

We plot the observed vs fitted values obtained using this model :-

```
plot(1:nrow(X),X$Y,type = "o",pch = 20,ylab = "observed & predicted values",xlab = "Index")
lines(1:nrow(X),rmodel_l$fitted.values,type = "o",pch = 20,col = "red",lty = 2)
abline(v = 1:nrow(X),lty = 2,col = rgb(0,1,0,alpha = 0.3))
legend("topleft",legend = c("Obs","Pred"),fill = c("black","red"))
```



We can see that this method performs better in terms of prediction compared to LMS estimates.

Comparative Study Of All Models

Finally, as a measure of comparison of different models, we use the root mean square error $RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\hat{y}_i - y_i)^2}$ where $\hat{y}_i$ denotes the fitted values using different regression models. We calculate this measure for all the "good" models we have found so far :-

Methods	$RMSE$ values
OLS model with "A1" & "A3"	8.313178
Ridge Model	8.195655
Lasso Model	9.889319
LAD Model	10.59494
LMS Model	11.6656
LTS Model	11.0904

Hence, we can conclude the Ridge and the OLS model with influential points removed with covariates "A1" & "A3" performs more or less better than the others in terms of prediction accuracy.