2017 NBA Hack-A-Thon
Team Application

Proposed Solutions to Question #1

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Trust the Process

Background

In the 2016 offseason, Kevin Durant announced his decision to leave the Oklahoma City Thunder in his unrestricted free agency for the purpose of signing a 2-year contract with the Golden State Warriors. Analysts and reporters from around the world were baffled by this decision, especially in regards to how dominant the Warriors lineup would be in the regular season now that they added a former MVP and 4x scoring champion in Durant.

Some fans speculated that the new Warriors would not lose any consecutive games in the entire regular season. In the investigation presented here, we will assume that the independent probability of the Warriors winning any single regular-season game is 80%.

Solution Method 1 for Part (a)

To answer this question, we invoke the law of total probability. Let NCL denote the event that the Warriors have no consecutive losses. The space of outcomes for the season of games can be completely partitioned into disjoint scenarios of losing anywhere from 0 to n games.

Let K represent the number of losses. The law of total probability says that

$$\mathbb{P}(NCL) = \sum_{k=0}^{n} \mathbb{P}(NCL|K = k)\mathbb{P}(K = k)$$

The notation $\mathbb{P}(NCL|K = k)$ is read as "the probability of no consecutive losses given k losses". First, we will look at $\mathbb{P}(K = k)$, or the probability of getting exactly k losses. Let the probability of winning be p_w and the probability of losing be p_l. Since these probabilities are the same for each game, we can consider the sequence of games as a binomial distribution. So, the probability of exactly k losses is given by:

$$\mathbb{P}(K = k) = \binom{n}{k}(p_l)^k(p_w)^{n-k}$$

The value $\binom{n}{k}$ gives the number of ways to choose a subset of k items from a set of n. Now, if we are given that there are k losses, then we can calculate the probability of no consecutive losses through a combinatorial argument. Stars and bars will be utilized as a combinatorial aid to represent the ideal outcome of games for the Warriors such that they avoid consecutive losses.

Let a star ($\star$) represent a win and let a bar ($|$) represent a possible loss. Therefore, the representation below depicts the desired locations of potential losses in between wins as that guarantees that the Warriors will never lose consecutive games in their schedule.

$$| \star | \star | \star | \star | \star | \star | \star | \star | \star | \star | \ldots\ldots\ldots\ldots | \star | \star | \star |$$

Clearly, if there are n stars present, then there are $n + 1$ bars present, which is the total number of possible locations where losses can occur in the Warriors schedule.

Given k losses, there are $\binom{n-k+1}{k}$ ways to distribute the total number of losses in the total number of possible loss locations. Furthermore, the total number of arrangements of wins and losses over the course of an n game season is given by $\binom{n}{k}$.

Therefore, the probability of the Warriors not losing consecutive games in the regular season is given by:

$$\mathbb{P}(NCL|K = k) = \frac{\binom{n-k+1}{k}}{\binom{n}{k}}$$

Substituting these expressions in gives us:

$$\mathbb{P}(NCL) = \sum_{k=0}^{n} \binom{n}{k}(p_l)^k(p_w)^{n-k}\left(\frac{\binom{n-k+1}{k}}{\binom{n}{k}}\right) = \sum_{k=0}^{n}(p_l)^k(p_w)^{n-k}\binom{n-k+1}{k}$$

Note that when the values of k exceed $\frac{n}{2}$, then there is no way to have no consecutive losses. So, in the cases where $k > n - k + 1$, there is no possible way to make a subset of items.

So, this expression evaluates to zero. This makes sense because if there are more losses than wins, there must be some consecutive losses by the Pigeonhole Principle. Now, using $n = 82$, $p_w = .8$, and $p_l = .2$ as per the problem specifications, we get:

$$\mathbb{P}(NCL) = \sum_{k=0}^{82} (.2)^k (.8)^{82-k} \binom{83-k}{k} = 0.0588$$

$\therefore$ the Warriors had a **5.88**% chance of not losing any consecutive games in the regular season.

Soultion Method 2 for Part (a)

An alternative solution to this question utilizes linear recursion.

For k games played in the regular season by the Warriors, let P_k represent the probability that they do not lose consecutive games in k games played.

Let $P(W) = 0.8$ & $P(L) = 0.2$ represent the independent probabilities of the Warriors winning and losing any given game respectively.

We need to start by considering the outcome of the first game:

1. If the first game is a win, with probability P(W), then the probability of not losing consecutive games is changed to P_{k-1}, since the first game essentially acts as a reset when checking for consecutive losses.
2. Losing the first game makes the second game very important to consider. If the first game is a loss and the second game is a win, with probability $P(W) * P(L)$, then the probability of not losing consecutive games is changed to P_{k-2}, since the second game acts as a reset in this case, despite losing the first game of the season.
3. The first two games could end up in losses, with a probability of $P(L)^2$.

From the above criteria, the following linear system can be constructed:

$$\begin{aligned} P_k &= P(W) * P_{k-1} + P(L) * P(W) * P_{k-2} \\ P_{k-1} &= P_{k-1} + 0 * P_{k-2} \end{aligned}$$

Our initial conditions are $P_0 = P_1 = 1$ since it is impossible to experience consecutive losses in less than 2 games played. Subsequent conversion of the linear system yields the following matrix expression:

$$\begin{pmatrix} P_k \\ P_{k-1} \end{pmatrix} = \begin{pmatrix} P(W) & P(L) * P(W) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_{k-1} \\ P_{k-2} \end{pmatrix}$$

Let $X_k = \begin{pmatrix} P_k \\ P_{k-1} \end{pmatrix}$ and $C = \begin{pmatrix} P(W) & P(L) * P(W) \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.16 \\ 1 & 0 \end{pmatrix}$.

The above expression can then be condensed to:

$$X_k = C * X_{k-1}$$

Using the initial values of P_0 & P_1 dictated above:

$$X_2 = C * X_1 = \begin{pmatrix} 0.8 & 0.16 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.96 \\ 1 \end{pmatrix}$$

To write this recursion in terms of X_2 rather than X_{k-1}, it suffices to express X_k intuitively as:

$$X_k = C^{k-2} * X_2$$

Substituting k=82 will yield $X_{82} = \begin{pmatrix} P_{82} \\ P_{81} \end{pmatrix}$, where the first element P_k equals the probability we desire:

$$\begin{aligned} X_{82} = \begin{pmatrix} P_{82} \\ P_{81} \end{pmatrix} &= C^{80} * X_2 \\ &= \begin{pmatrix} 0.8 & 0.16 \\ 1 & 0 \end{pmatrix}^{80} * \begin{pmatrix} 0.96 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0.0588 \\ 0.0609 \end{pmatrix} \end{aligned}$$

Based on the above determination of the matrix X_{82}, the first element $P_{82} = 0.0588$.

$\therefore$ the Warriors had a **5.88%** chance of not losing any consecutive games in the regular season.

Extension of Part (b)

It is of interest to consider another interpretation of the total set of 82 games that the Warriors play: the expected number of games played by the Warriors from the start of the season in which they will experience their first consecutive losses.

Within the context of probability theory, the expected value $\mathbb{E}(X)$ for a discrete random variable X and its corresponding probability function $P(X)$ is given by:

$$\mathbb{E}(X) = \sum_{i=1}^{n} x_i P(x_i)$$

where the summation accounts for all of the distinct outcomes x_i that can be undertaken by

X and:

$$\sum_{i=1}^{n} P(x_i) = 1$$

by definition of a random variable.

Let X represent the discrete random variable that governs the occurrence of consecutive losses in the Warriors schedule.

Let g represent the expected number of games that the Warriors will play such that they will experience their first consecutive losses and $g = \mathbb{E}(X) < 82$.

Furthermore, let $P(W) = 0.8$ represent the probability that the Warriors win any given game, clearly implying that $P(L) = 1 - P(W) = 0.2$ where $P(L)$ represents the probability that the Warriors lose any given game.

There are three distinct probabilistic cases to consider for X:

1. The Warriors win their first game, meaning that it will take them $g + 1$ total games to experience their first consecutive losses.
   $\implies x_1 = g + 1$ and $P(x_1) = P(W)$.
2. The Warriors lose their first game and win their second game, meaning that it will take them $g + 2$ total games to experience their first consecutive losses.
   $\implies x_2 = g + 2$ and $P(x_2) = P(W)P(L)$.
3. The Warriors lose their first two games, meaning that it will take them 2 games total to experience their first consecutive losses.
   $\implies x_3 = 2$ and $P(x_3) = P(L)P(L) = P(L)^2$.

Before any further progress can be made, it suffices to show that X is random by satisfying the following:

$$\begin{aligned}
\sum_{i=1}^{n} P(x_i) &= 1 \\
\sum_{i=1}^{3} P(x_i) &= P(x_1) + P(x_2) + P(x_3) \\
&= P(W) + P(W)P(L) + P(L)^2 \\
&= 0.8 + (0.8)(0.2) + 0.2^2 \\
&= 0.8 + 0.16 + 0.04 \\
&= 1
\end{aligned}$$

$\therefore X$ satisfies the requirement of being random.

Given that X is a random discrete variable, g can be calculated as follows:

$$
\begin{aligned}
g &= \sum_{i=1}^{n} x_i P(x_i) \\
g &= \sum_{i=1}^{3} x_i P(x_i) \\
&= x_1 P(x_1) + x_2 P(x_2) + x_3 P(x_3) \\
&= (g+1)[P(W)] + (g+2)[P(W)P(L)] + 2[P(L)^2] \\
&= 0.8(g+1) + (0.8)(0.2)(g+2) + 2(0.2^2) \\
&= 0.8g + 0.8 + 0.16g + 0.32 + 0.08 \\
&= 0.96g + 1.20 \\
g - 0.96g &= 1.20 \\
0.04g &= 1.20 \\
g &= \frac{1.20}{0.04} \\
g &= 30
\end{aligned}
$$

Therefore, it was probabilistically expected that the Warriors would experience their first consecutive losses within their first **30** games of the regular season, losing both their 29^{th} and 30^{th} games.

Solution to Part (c)

From our first solution method in Part (a), we see that the Warriors would need a higher win percentage in order to raise the probability of no consecutive losses. So, we simply need to find the probability, P, at which $\mathbb{P}(NCL)$ exceeds 0.5. This was done with the aid of the following Python program, where values of P were exhaustively searched until the desired value was found. From the program, the value was

$$P = 0.9038$$

So, the Warriors would need to win **90.38**% of their games to have a 50% chance of not experiencing consecutive losses at any point in the regular season.

The Python code used to find this answer is given as follows:

```
from sympy import binomial #import the combinatorial choose

def calculate(n,p):
    total_prob = 0
    for i in range(n+1): #iterate to n+1 because 0 through n inclusive
        total_prob += (p**i)*((1-p)**(n-i))*binomial(n-i+1,i)
    return total_prob

win = .8
games = 82
ind = 0 #indicator variable to help stop the following loop
while (ind < .5):
    win += .0001
    ind = calculate(games, 1-win) #p above is loss probability
print(win)
```