Using random variables to solve geometric problems

0.1 Level 1

This exercise should be revision: Use the block below to generate 20 pairs of uniform random variables X_i and Y_i that each lie between 0 and 1. Then plot each of these pairs of random variables at (X_i, Y_i). All the points you show should have $0 < X_i < 1$ and $0 < Y_i < 1$.

0.2 Level 2

Now modify the code that you have just written so that instead of plotting the values of X_i and Y_i directly you calculate the value of a third random variable Z_i. Z_i should be equal to one if the point (X_i, Y_i) is within a circle of radius one centered on the origin and it should be zero otherwise. You should use the blocks provided to plot each of the values that you obtain for Z_i at the point (i, Z_i).

0.3 Level 3

Now use what you have learnt in other exercise about the central limit theorem to show how the sample mean for the random variable Z that you generated in the last exercise changes as the number of indepedent random variables in the sample (the number of Z_i values) increases. In other words, plot a graph with points at (i, μ_i) for i values between 0 and 20. Describe the shape of this curve in your notes.

0.4 Level 4

At the end of the last exercise you should have found that the sample mean converged to a value of $\frac{\pi}{4}$. The reason this happens in this case is connected to the fact that the area of the region where points can be generated is equal to one. The area of the quarter circle in which points have $X_i^2 + Y_i^2 < 1$ and which are thus inside the circle of radius 1 is by contrast $\frac{\pi}{4}$. The probability of generating a point in this circle is (in this case) thus simply the area of the quarter circle divided by the area of the square and is thus $\frac{\pi}{4}$.

It is tempting to assume that we can use this geometric extension of the classical interpretation of probability in all cases. What we will see in the remaining exercises here is that life is not that simple. For this purpose we are going to study a random variable whose value is determined based on a construction which consists of an equilateral triangle inside a circle. As a first step in solving this problem generate 10 uniformly distributed points that lie on the circumference of a circle, which has a radius of one and plot these points using the plotting tools.

0.5 Level 5

We are now going to use what you learnt in the previous exercise about generating uniformly distributed points on the circumference of a circle to estimate the probability that a chord that connects two points on the circumference of the circle and that is generated at random has a length longer than one of the sides of the equililateral triangle. We will do this by generating 50 random variables. Each of these random variables will be equal to one when the corresponding chord has a length longer than side of the triangle and will be zero otherwise. Furthermore, the length of each chord will be calculated by generating two (uniformly distributed) random points on the circumference and by calculating the distance between these two points using Pythagoras theorem. Use the blocks below to calculate the length of the side of the equililateral triangle using trigonometry and then generate 50 independent values for the random variable Z. Plot a graph showing how the sample mean μ_i that you estimate from these random variables changes as you increase the number of points in your sample. Describe the shape of the curve you obtain in your notes on this exercise.

0.6 Level 6

At the end of the previous exercise you should have seen that the sample mean for Z converges to a value of $\frac{1}{3}$. In this exercise we are going to check this estimate that the probability that a randomly-generated chord has a length longer than one of the sides of the equililateral triangle. Once again we will do this by generating 50 random variables, Z'. Once again each of these random variables will be equal to one when the corresponding chord has a length longer than side of the triangle and will be zero otherwise. The difference will be how we generate the chords. This time we will generate only one point on the circumference of the circle and a second point will generated at some point on the radius that connects the point on the circumference to the center of the circle. The chord will then be drawn perpendicular to this radius so that it connects two points on the circumference of the circle. You should be able to calculate the length of the chord generated in this way using Pythagoras theorem. Therefore, use the blocks below to calculate the length of the side of the equililateral triangle using trigonometry and then generate 50 independent values for the random variable Z'. Plot a graph showing how the sample mean μ_i that you estimate from these random variables changes as you increase the number of points in your sample. Describe the shape of the curve you obtain in your notes on this exercise.