Gamblers ruin I: Probability of ruin

Gareth Tribello
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The essence of many gambling games is as follows: you place a stake of x pounds on something or other. This might be the spin of a wheel, a horse winning a race or some other event in the future. Regardless this event occurs with a probablilty of p. If the event transpires you win back double your stake - $2x$ pounds - you will thus have $a + x$ pounds in total, where a was your initial holding. If it does not transpire you loose your stake and are thus left with $a - x$ pounds.

The ideas in this first paragraph have been covered in the videos on the gamblers video and in the programming exercise. We have shown how we can describe the above process using a Markov chain, what the transition graph is for this chain and what the associated one-step transition probability matrix is for this discrete Markov chain. **Before attempting this exercise make sure you are familiar with these ideas and that you can thus do the following items:**

1. You must be able to explain why the gamblers ruin problem can be descibed using a Markov chain.

2. You must be able to write out the transition graph for the gamblers ruin problem.

3. You must be able to write out the one step transition probability matrix for the gamblers ruin problem.

4. You must be able to use the partition theorem to derive the homogeneous difference equation $\pi_k = \pi_{k+1}p + \pi_{k-1}q$, where π_k is the conditional probability of ruin given that you start with k pounds and where p and q are the probabiltiy of winning when you place each stake.

If you cannot do all of the above things watch the video on gamblers ruin again. If you are unable to do the above you will not understand the remainder of this exercise.

The purpose of this exercise is to find an exact expression for the conditional probability of ruin given that you start with exactly k pounds, π_k.

Soluton guidelines

1. We would like to determine the probablity of loosing all our money. We could do this by partitioning the transition probability matrix for the gamblers ruin problem into $\mathbf{Q}$ and $\mathbf{R}$ parts and then substituting these into the formula $\mathbf{H} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R}$ that we learnt previously. However, for this particular problem we generally adopt a different strategy. We are instead going to use the fact that we know - from the parition theorem - that: allows us to write:

$$\pi_k = \pi_{k+1}p + \pi_{k-1}q$$

This equation is a homogenous difference equation, which we can rewrite as follows:

$$\pi_k - \pi_{k+1} p - \pi_{k-1} q = 0$$

To solve homogenous difference equations we introduce a trial solution $\pi_k = \theta^k$ and find values of θ that satisfy the above equality. π is then a linear combination of the solutions we find - so for examples if we find two solutions, θ_1 and θ_2, we could write π_k as follows:

$$\phi_k = A\theta_1^k + B\theta_2^k$$

Insert the trial solution $\pi_k = \theta^k$ into the homogenous difference equation above remebering that $\phi_{k+1} = \theta^{k+1}$. Factorise the resulting equation and hence show that:

$$\pi_k = A + B\left(\frac{q}{p}\right)^k$$

where A and B are as yet unknown parameters.

2. Think about what the quantity π_k represents. This is the probability of ruin given that you start with exactly k pounds to your name. Given the meaning of this quantity, π_k, **what are the values of π_0 and π_n.** Notice that here n is the target amount of money the gambler wants to win.

3. Notice that we now have a simulaltaneous equation with two unknowns A and B as well as the values of π_0 and π_n. We can thus use what we have to construct two linear equations in A and B. **Solve this set of simultaneous equations and find values for A and B.** Hence, show that:

$$\pi_k = \frac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^n}{1 - \left(\frac{q}{p}\right)^n}$$

4. Now use everything that you have derived above show that the conditional probability, s_k, that the gambler wins the n pounds he desires given that he starts with exactly k pounds is given by:

$$s_k = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^n}$$

5. Suppose that $p = 0.5$. Why is it not possible to use the equations above to calculate π_k and s_k in this limit? Use l'Hopital's rule to determine show that when $p = 0.5$ π_k and s_k are given by the expressions shown below:

$$\pi_k = \frac{n-k}{n} \qquad s_k = \frac{k}{n}$$