Programming the ehrenfest urn problem

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We can construct a Markov chain by doing the following experiment:

> Suppose that you have two cups a pink one and a blue one and that you also have six numbered balls. Suppose that three of these balls are in the blue cup and that the remaining three balls are in the pink cup. You next roll a roll a fair dice to generate a random number X that is between 1 and 6. Having rolled an X you then take the Xth ball and if it is in the blue cup you move it to the pink cup. If by contrast the Xth ball is in the pink cup you move it to the blue cup.

Draw the transition graph for this Markov chain and write a program to simulate how the number of balls in the blue cup changes with time. Use your program to calculate the probablity mass function for the random variable $X(n)$ that tells you how many balls are in the blue cup on step n by calculating a histogram based on the behavior of the chain. Try to calculate confidence limits on the elements of your probablity mass function using resampling and explain why these confidence limits cannot be calculated using the methods based on bootstrapping or the central limit theorem that you have learnt about in the other parts of the course.

Next calculate the eigenvalues and eigenvectors of the one step transition matrix by using the linear algebra tools that form part of the numpy package. Discuss, given the eigenvalues and eigenvectors you obtain, whether the probablity mass function that you calculated is a limiting stationary distribution for this chain and how the chain behave in the long time limit? Discuss how you can modify the chain and thereby ensure that the distribution has a limiting stationary distribution and confirm that your modified chain does indeed have a limiting stationary distribution. In doing this exercise you may find it useful to consider the periodicity of each of the states in the Markov chain and to learn something about the eigenvalues of bipartite graphs.